Precalculus Workbook Solutions

Partial fractions

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FRACTION DECOMPOSITION

1. Find the form of partial fractions decomposition of the rational function.

$$f(x) = \frac{6x + 16}{x^2 + 10x + 21}$$

Solution:

Factor the denominator of the rational function.

$$f(x) = \frac{6x + 16}{(x + 3)(x + 7)}$$

The denominator is now factored to distinct linear factors, so there should be two fractions in the decomposition that have denominators from the original denominator and numerators that are constants.

$$f(x) = \frac{A}{x + 3} + \frac{B}{x + 7}$$

2. Identify the repeated factors in the denominator of rational function.

$$f(x) = \frac{3x + 7}{(x - 1)(x^2 - 1)(x^2 + 1)^3(x^2 - 2x - 3)}$$

Solution:

The denominator of $f(x)$ can be factored further.

$$f(x)=\frac{3x+7}{(x-1)(x-1)(x+1)(x^2+1)^3(x+1)(x-3)}$$

$$f(x)=\frac{3x+7}{(x-1)^2(x+1)^2(x^2+1)^3(x-3)}$$

So the $x-1$, $x+1$, and x^2+1 factors are repeated.

■ 3. How many fractions will exist in the partial fractions decomposition of the function?

$$f(x)=\frac{1}{(x^2+1)(x^4+5x^2+6)}$$

Solution:

The denominator of the rational function can be factored further.

$$f(x)=\frac{1}{(x^2+1)(x^2+2)(x^2+3)}$$

The denominator is factored as far as it can be, so there are three fractions in the decomposition of $f(x)$.

$$f(x)=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}+\frac{Ex+F}{x^2+3}$$

■ 4. Find the form of the partial fractions decomposition of the function, without solving for the constants.

$$f(x) = \frac{1}{(x^2 + 1)(x^4 - 1)}$$

Solution:

The denominator of the function $f(x)$ can be factored further.

$$f(x) = \frac{1}{(x^2 + 1)(x^2 + 1)(x^2 - 1)}$$

$$f(x) = \frac{1}{(x^2 + 1)^2(x + 1)(x - 1)}$$

So the decomposition will be

$$f(x) = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} + \frac{E}{(x + 1)} + \frac{F}{(x - 1)}$$

■ 5. Find the form of the partial fractions decomposition of the function, without solving for the constants.

$$f(x) = \frac{x^2}{(x^2 + 2x)(x^2 + 2x + 2)}$$

Solution:

The denominator of the function $f(x)$ can be factored further.

$$f(x) = \frac{x^2}{x(x+2)(x^2+2x+2)}$$

$$f(x) = \frac{x}{(x+2)(x^2+2x+2)}$$

So the decomposition will be

$$f(x) = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+2}$$

■ 6. Find the form of the partial fractions decomposition of the function, without solving for the constants.

$$f(x) = \frac{x^2+2}{(1-x)(1-2x)(1-3x)}$$

Solution:

The denominator of the function $f(x)$ is already factored with three distinct linear factors, so the decomposition of $f(x)$ should include three fractions.

The denominator of each of the three fractions will be one of the factors from the original denominator, and the numerator of each of the three fractions will be a distinct constant, A, B, C, etc. So the decomposition will be

$$f(x) = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{1-3x}$$

DISTINCT LINEAR FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4}{(3x-1)(x+1)}$$

Solution:

These are distinct linear factors.

$$\frac{4}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}$$

To solve for A, remove $3x-1$ and then evaluate the resulting left side at $x = 1/3$.

$$\frac{4}{x+1} \to \frac{4}{\frac{1}{3}+1} \to 3$$

To solve for B, remove $x+1$ and then evaluate the resulting left side at $x = -1$.

$$\frac{4}{3x-1} \to \frac{4}{3(-1)-1} \to -1$$

Plugging $A = 3$ and $B = -1$ back into the partial fractions decomposition gives

$$f(x) = \frac{3}{3x-1} - \frac{1}{x+1}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{24}{x(x+4)(x-2)}$$

Solution:

These are distinct linear factors.

$$\frac{24}{x(x+4)(x-2)} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-2}$$

To solve for A, remove x and then evaluate the resulting left side at $x = 0$.

$$\frac{24}{(x+4)(x-2)} \to \frac{24}{(4)(-2)} \to -3$$

To solve for B, remove $x+4$ and then evaluate the resulting left side at $x = -4$.

$$\frac{24}{x(x-2)} \to \frac{24}{-4(-4-2)} \to 1$$

To solve for C, remove $x-2$ and then evaluate the resulting left side at $x = 2$.

$$\frac{24}{x(x+4)} \to \frac{24}{2(2+4)} \to 2$$

Plugging $A = -3$, $B = 1$, and $C = 2$ back into the partial fractions decomposition gives

$$f(x) = -\frac{3}{x} + \frac{1}{x+4} + \frac{2}{x-2}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{3x}{(x+2)(x+5)}$$

Solution:

These are distinct linear factors.

$$\frac{3x}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5}$$

To solve for A, remove $x+2$ and then evaluate the resulting left side at $x = -2$.

$$\frac{3x}{x+5} \to \frac{3(-2)}{-2+5} \to -2$$

To solve for B, remove $x+5$ and then evaluate the resulting left side at $x = -5$.

$$\frac{3x}{x+2} \to \frac{3(-5)}{-5+2} \to 5$$

Plugging $A = -2$ and $B = 5$ back into the partial fractions decomposition gives

$$f(x) = -\frac{2}{x+2} + \frac{5}{x+5}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{6x}{(x^2-1)(x-2)}$$

Solution:

We have to start by factoring the denominator.

$$f(x) = \frac{6x}{(x+1)(x-1)(x-2)}$$

These are distinct linear factors.

$$\frac{6x}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}$$

To solve for A, remove $x+1$ and then evaluate the resulting left side at $x = -1$.

$$\frac{6x}{(x-1)(x-2)} \to \frac{6(-1)}{(-1-1)(-1-2)} \to -1$$

To solve for B, remove $x-1$ and then evaluate the resulting left side at $x=1$.

$$\frac{6x}{(x+1)(x-2)} \to \frac{6(1)}{(1+1)(1-2)} \to -3$$

To solve for C, remove $x-2$ and then evaluate the resulting left side at $x=2$.

$$\frac{6x}{(x+1)(x-1)} \to \frac{6(2)}{(2+1)(2-1)} \to 4$$

Plugging $A=-1$, $B=-3$, and $C=4$ back into the partial fractions decomposition gives

$$f(x) = -\frac{1}{x+1} - \frac{3}{x-1} + \frac{4}{x-2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x+1}{9x^3 - x}$$

Solution:

We have to start by factoring the denominator.

$$f(x) = \frac{x+1}{x(3x+1)(3x-1)}$$

These are distinct linear factors.

$$\frac{x+1}{x(3x+1)(3x-1)}=\frac{A}{x}+\frac{B}{3x+1}+\frac{C}{3x-1}$$

To solve for A, remove x and then evaluate the resulting left side at $x=0$.

$$\frac{x+1}{(3x+1)(3x-1)}\to\frac{0+1}{(0+1)(0-1)}\to-1$$

To solve for B, remove $3x+1$ and then evaluate the resulting left side at $x=-1/3$.

$$\frac{x+1}{x(3x-1)}\to\frac{-\frac{1}{3}+1}{-\frac{1}{3}\left(3\left(-\frac{1}{3}\right)-1\right)}\to 1$$

To solve for C, remove $3x-1$ and then evaluate the resulting left side at $x=1/3$.

$$\frac{x+1}{x(3x+1)}\to\frac{\frac{1}{3}+1}{\frac{1}{3}\left(3\left(\frac{1}{3}\right)+1\right)}\to 2$$

Plugging $A=-1$, $B=1$, and $C=2$ back into the partial fractions decomposition gives

$$f(x)=-\frac{1}{x}+\frac{1}{3x+1}+\frac{2}{3x-1}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{6x - 24}{(x^2 - 1)(x^2 - 4)}$$

Solution:

We have to start by factoring the denominator.

$$f(x) = \frac{6x - 24}{(x + 1)(x - 1)(x + 2)(x - 2)}$$

These are distinct linear factors.

$$\frac{6x - 24}{(x + 1)(x - 1)(x + 2)(x - 2)} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{x + 2} + \frac{D}{x - 2}$$

To solve for A, remove $x + 1$ and then evaluate the resulting left side at $x = -1$.

$$\frac{6x - 24}{(x - 1)(x + 2)(x - 2)} \to \frac{6(-1) - 24}{(-1 - 1)(-1 + 2)(-1 - 2)} \to -5$$

To solve for B, remove $x - 1$ and then evaluate the resulting left side at $x = 1$.

$$\frac{6x - 24}{(x + 1)(x + 2)(x - 2)} \to \frac{6(1) - 24}{(1 + 1)(1 + 2)(1 - 2)} \to 3$$

To solve for C, remove $x + 2$ and then evaluate the resulting left side at $x = -2$.

$$\frac{6x - 24}{(x + 1)(x - 1)(x - 2)} \to \frac{6(-2) - 24}{(-2 + 1)(-2 - 1)(-2 - 2)} \to 3$$

To solve for D, remove $x-2$ and then evaluate the resulting left side at $x=2$.

$$\frac{6x-24}{(x+1)(x-1)(x+2)} \to \frac{6(2)-24}{(2+1)(2-1)(2+2)} \to -1$$

Plugging $A=-5$, $B=3$, $C=3$, and $D=-1$ back into the partial fractions decomposition gives

$$f(x) = -\frac{5}{x+1} + \frac{3}{x-1} + \frac{3}{x+2} - \frac{1}{x-2}$$

REPEATED LINEAR FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4}$$

Solution:

These are repeated linear factors.

$$\frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4}$$

Combine the fractions by finding the common denominator.

$$\frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4} = \frac{A(x-2)^3 + B(x-2)^2 + C(x-2) + D}{(x-2)^4}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x^3 - 2x^2 + 2x - 3 = A(x-2)^3 + B(x-2)^2 + C(x-2) + D$$

The linear factor $x - 2$ is equal to 0 when $x = 2$, so we'll evaluate this equation at $x = 2$ in order to solve for D.

$$2^3 - 2 \cdot 2^2 + 2(2) - 3 = A(2-2)^3 + B(2-2)^2 + C(2-2) + D$$

$$8 - 2 \cdot 4 + 2 \cdot 2 - 3 = D$$

$$D = 1$$

We'll plug this back into the numerator equation,

$$x^3 - 2x^2 + 2x - 3 = A(x-2)^3 + B(x-2)^2 + C(x-2) + 1$$

and then simplify the right side, collecting like terms.

$$x^3 - 2x^2 + 2x - 3 = A(x^3 - 6x^2 + 12x - 8) + B(x^2 - 4x + 4) + C(x-2) + 1$$

$$x^3 - 2x^2 + 2x - 3 = Ax^3 + (-6A + B)x^2 + (12A - 4B + C)x - 8A + 4B - 2C + 1$$

So we get the system of equations

$$A = 1$$

$$-6A + B = -2$$

$$12A - 4B + C = 2$$

$$-8A + 4B - 2C + 1 = -3$$

or $A = 1$ with

$$B = 4$$

$$-4B + C = -10$$

$$4B - 2C = 4$$

or $A = 1$ and $B = 4$ with

$$-16 + C = -10$$

$$16 - 2C = 4$$

or $A=1$, $B=4$, $C=6$, and $D=1$. Plugging these values back into the partial fractions decomposition gives

$$f(x)=\frac{1}{x-2}+\frac{4}{(x-2)^2}+\frac{6}{(x-2)^3}+\frac{1}{(x-2)^4}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{x^3+3x^2+3x+3}{(x+1)^4}$$

Solution:

These are repeated linear factors.

$$\frac{x^3+3x^2+3x+3}{(x+1)^4}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{(x+1)^4}$$

Combine the fractions by finding the common denominator.

$$\frac{x^3+3x^2+3x+3}{(x+1)^4}=\frac{A(x+1)^3+B(x+1)^2+C(x+1)+D}{(x+1)^4}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x^3+3x^2+3x+3=A(x+1)^3+B(x+1)^2+C(x+1)+D$$

The linear factor $x+1$ is equal to 0 when $x=-1$, so we'll evaluate this equation at $x=-1$ in order to solve for D.

$$(-1)^3+3(-1)^2+3(-1)+3=A(-1+1)^3+B(-1+1)^2+C(-1+1)+D$$

$$-1+3-3+3=D$$

$$D=2$$

We'll plug this back into the numerator equation,

$$x^3+3x^2+3x+3=A(x+1)^3+B(x+1)^2+C(x+1)+2$$

and then simplify the right side, collecting like terms.

$$x^3+3x^2+3x+3=A(x^3+3x^2+3x+1)+B(x^2+2x+1)+C(x+1)+2$$

$$x^3+3x^2+3x+3=Ax^3+(3A+B)x^2+(3A+2B+C)x+A+B+C+2$$

So we get the system of equations

$$A=1$$

$$3A+B=3$$

$$3A+2B+C=3$$

$$A+B+C+2=3$$

or $A=1$ with

$$B=0$$

$$2B+C=0$$

$$B + C = 0$$

or $A = 1$, $B = 0$, $C = 0$, and $D = 2$. Plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{1}{x+1} + \frac{2}{(x+1)^4}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x}{(x+2)^3}$$

Solution:

These are repeated linear factors.

$$\frac{x}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

Combine the fractions by finding the common denominator.

$$\frac{x}{(x+2)^3} = \frac{A(x+2)^2 + B(x+2) + C}{(x+2)^3}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x = A(x+2)^2 + B(x+2) + C$$

The linear factor $x+2$ is equal to 0 when $x=-2$, so we'll evaluate this equation at $x=-2$ in order to solve for C.

$$-2=A(-2+2)^2+B(-2+2)+C$$

$$C=-2$$

We'll plug this back into the numerator equation,

$$x=A(x+2)^2+B(x+2)-2$$

and then simplify the right side, collecting like terms.

$$x=A(x^2+4x+4)+B(x+2)-2$$

$$x=Ax^2+(4A+B)x+4A+2B-2$$

So we get the system of equations

$$A=0$$

$$4A+B=1$$

$$4A+2B-2=0$$

or $A=0$, $B=1$, and $C=-2$. Plugging these values back into the partial fractions decomposition gives

$$f(x)=\frac{1}{(x+2)^2}-\frac{2}{(x+2)^3}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2x+5}{x^2-2x+1}$$

Solution:

We can rewrite $f(x)$ as

$$\frac{2x+5}{x^2-2x+1} = \frac{2x+5}{(x-1)^2}$$

These are repeated linear factors.

$$\frac{2x+5}{x^2-2x+1} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Combine the fractions by finding the common denominator.

$$\frac{2x+5}{x^2-2x+1} = \frac{A(x-1)+B}{(x-1)^2}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$2x+5 = Ax - A + B$$

Then we have

$$A = 2$$

$$-A + B = 5$$

Plug $A = 2$ and $B = 7$ into the partial fractions decomposition.

$$f(x) = \frac{2}{x-1} + \frac{7}{(x-1)^2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^2 - 21x + 100}{(x-10)^3}$$

Solution:

These are repeated linear factors.

$$\frac{x^2 - 21x + 100}{(x-10)^3} = \frac{A}{x-10} + \frac{B}{(x-10)^2} + \frac{C}{(x-10)^3}$$

Combine the fractions by finding the common denominator.

$$\frac{x^2 - 21x + 100}{(x-10)^3} = \frac{A(x-10)^2 + B(x-10) + C}{(x-10)^3}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x^2 - 21x + 100 = A(x-10)^2 + B(x-10) + C$$

The linear factor $x - 10$ is equal to 0 when $x = 10$, so we'll evaluate this equation at $x = 10$ in order to solve for C.

$$10^2 - 21(10) + 100 = A(10-10)^2 + B(10-10) + C$$

$$100 - 210 + 100 = C$$

$$C = -10$$

We'll plug this back into the numerator equation,

$$x^2 - 21x + 100 = Ax^2 - 20Ax + 100 + Bx - 10B - 10$$

Then we have

$$A = 1$$

$$-21 = -20A + B$$

$$100 = 100 - 10B - 10$$

Plug $A = 1$, $B = -1$, and $C = -10$ into the partial fractions decomposition.

$$f(x) = \frac{1}{x - 10} - \frac{1}{(x - 10)^2} - \frac{10}{(x - 10)^3}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^2 + 8x}{(x + 8)^3}$$

Solution:

We can rewrite $f(x)$ as

$$\frac{x^2+8x}{(x+8)^3}=\frac{x(x+8)}{(x+8)^3}=\frac{x}{(x+8)^2}$$

These are repeated linear factors.

$$\frac{x}{(x+8)^2}=\frac{A}{x+8}+\frac{B}{(x+8)^2}$$

Combine the fractions by finding the common denominator.

$$\frac{x}{(x+8)^2}=\frac{A(x+8)+B}{(x+8)^2}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x=Ax+8A+B$$

Then we have

$$A=1$$

$$8A+B=0$$

Plug $A=1$ and $B=-8$ into the partial fractions decomposition.

$$f(x)=\frac{1}{x+8}-\frac{8}{(x+8)^2}$$

DISTINCT QUADRATIC FACTORS

1. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{3x^3+3}{(x^2+2)(x^2+5)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{3x^3+3}{(x^2+2)(x^2+5)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+5}$$

Combine the fractions by finding a common denominator.

$$\frac{3x^3+3}{(x^2+2)(x^2+5)}=\frac{(Ax+B)(x^2+5)+(Cx+D)(x^2+2)}{(x^2+2)(x^2+5)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$3x^3+3=(Ax+B)(x^2+5)+(Cx+D)(x^2+2)$$

Multiply out the right side of this numerator equation,

$$3x^3+3=Ax^3+5Ax+Bx^2+5B+Cx^3+2Cx+Dx^2+2D$$

then group like terms and factor.

$$3x^3 + 3 = (A + C)x^3 + (B + D)x^2 + (5A + 2C)x + 5B + 2D$$

Then we get

$$A + C = 3 \qquad B + D = 0$$

$$5A + 2C = 0 \qquad 5B + 2D = 3$$

Solving the two equations on the left as a system gives $A = -2$ and $C = 5$, while solving the two equations on the right as a system gives $B = 1$ and $D = -1$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{-2x + 1}{x^2 + 2} + \frac{5x - 1}{x^2 + 5}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x^2 + 4x}{(2x^2 + 2)(3x^2 + 1)}$$

Solution:

Rewrite the denominator by factoring a 2 out of $2x^2 + 2$, then canceling that 2 against the numerator.

$$f(x) = \frac{4x^2 + 4x}{(2x^2 + 2)(3x^2 + 1)} = \frac{2x^2 + 2x}{(x^2 + 1)(3x^2 + 1)}$$

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{2x^2+2x}{(x^2+1)(3x^2+1)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+5}$$

Combine the fractions by finding a common denominator.

$$\frac{2x^2+2x}{(x^2+1)(3x^2+1)}=\frac{(Ax+B)(3x^2+1)+(Cx+D)(x^2+1)}{(x^2+1)(3x^2+1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$2x^2+2x=(Ax+B)(3x^2+1)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$2x^2+2x=3Ax^3+Ax+3Bx^2+B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$2x^2+2x=(3A+C)x^3+(3B+D)x^2+(A+C)x+B+D$$

Then we get

$$3A+C=0 \qquad 3B+D=2$$

$$A+C=2 \qquad B+D=0$$

Solving the two equations on the left as a system gives $A=-1$ and $C=3$, while solving the two equations on the right as a system gives $B=1$ and $D=-1$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{-x+1}{x^2+1}+\frac{3x-1}{3x^2+1}$$

3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{6x^3 - 13x^2 + 4x - 3}{(4x^2 + 1)(x^2 + x + 1)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{6x^3 - 13x^2 + 4x - 3}{(4x^2 + 1)(x^2 + x + 1)} = \frac{Ax + B}{4x^2 + 1} + \frac{Cx + D}{x^2 + x + 1}$$

Combine the fractions by finding a common denominator.

$$\frac{6x^3 - 13x^2 + 4x - 3}{(4x^2 + 1)(x^2 + x + 1)} = \frac{(Ax + B)(x^2 + x + 1) + (Cx + D)(4x^2 + 1)}{(4x^2 + 1)(x^2 + x + 1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$6x^3 - 13x^2 + 4x - 3 = (Ax + B)(x^2 + x + 1) + (Cx + D)(4x^2 + 1)$$

Multiply out the right side of this numerator equation,

$$6x^3 - 13x^2 + 4x - 3 = Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + 4Cx^3 + Cx + 4Dx^2 + D$$

then group like terms and factor.

$$6x^3 - 13x^2 + 4x - 3 = (A + 4C)x^3 + (A + B + 4D)x^2 + (A + B + C)x + B + D$$

Then we get

$$A + 4C = 6$$

$$A + B + 4D = -13$$

$$A + B + C = 4$$

$$B + D = -3$$

Solving these as a system of equations gives $A = 2$, $B = 1$, $C = 1$, and $D = -4$. Plugging the values back into the partial fractions decomposition gives

$$f(x) = \frac{2x + 1}{4x^2 + 1} + \frac{x - 4}{x^2 + x + 1}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 1}{(x^2 + 1)(x^2 + x + 2)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{x^3 + 1}{(x^2 + 1)(x^2 + x + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + x + 2}$$

Combine the fractions by finding a common denominator.

$$\frac{x^3+1}{(x^2+1)(x^2+x+2)}=\frac{(Ax+B)(x^2+x+2)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+x+2)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^3+1=(Ax+B)(x^2+x+2)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$x^3+1=Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$x^3+1=(A+C)x^3+(A+B+D)x^2+(2A+B+C)x+2B+D$$

Then we get

$$A+C=1$$

$$A+B+D=0$$

$$2A+B+C=0$$

$$2B+D=1$$

Solving these as a system of equations gives $A=-1$, $B=0$, $C=2$, and $D=1$. Plugging the values back into the partial fractions decomposition gives

$$f(x)=-\frac{x}{x^2+1}+\frac{2x+1}{x^2+x+2}$$

5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x+3}{(x^2+x+1)(x^2+2x+2)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{x+3}{(x^2+x+1)(x^2+2x+2)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+2x+2}$$

We'll combine the fractions on the right side of the equation by finding a common denominator.

$$\frac{x+3}{(x^2+x+1)(x^2+2x+2)} = \frac{(Ax+B)(x^2+2x+2)+(Cx+D)(x^2+x+1)}{(x^2+x+1)(x^2+2x+2)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x+3 = (Ax+B)(x^2+2x+2)+(Cx+D)(x^2+x+1)$$

Multiply out the right side of this numerator equation,

$$x+3 = Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$$

then group like terms and factor.

$$x+3 = (A+C)x^3+(2A+B+C+D)x^2+(2A+2B+C+D)x+(2B+D)$$

Then we can equate coefficients to build a system of equations.

$$A + C = 0 \qquad 2A + B + C + D = 0$$

$$2B + D = 3 \qquad 2A + 2B + C + D = 1$$

Solving the two equations on the right as a system gives $B = 1$, while solving the other equations gives $A = -2$, $C = 2$, and $D = 1$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{-2x+1}{x^2+x+1} + \frac{2x+1}{x^2+2x+2}$$

6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{3x^4+3x^3+3x^2}{(x^2+1)(2x^2+5)(x^2+x+1)}$$

Solution:

We can rewrite $f(x)$ as

$$f(x) = \frac{3x^4+3x^3+3x^2}{(x^2+1)(2x^2+5)(x^2+x+1)} = \frac{3x^2(x^2+x+1)}{(x^2+1)(2x^2+5)(x^2+x+1)}$$

$$f(x) = = \frac{3x^2}{(x^2+1)(2x^2+5)}$$

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{3x^2}{(x^2+1)(2x^2+5)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{2x^2+5}$$

We'll combine the fractions on the right side of the equation by finding a common denominator.

$$\frac{3x^2}{(x^2+1)(2x^2+5)} = \frac{(Ax+B)(2x^2+5)+(Cx+D)(x^2+1)}{x^2+1}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$3x^2 = (Ax+B)(2x^2+5)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$3x^2 = 2Ax^3+5Ax+2Bx^2+5B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$3x^2 = (2A+C)x^3+(2B+D)x^2+(5A+C)x+5B+D$$

Then we can equate coefficients to build a system of equations.

$$2A+C=0 \qquad\qquad 2B+D=3$$

$$5A+C=0 \qquad\qquad 5B+D=0$$

Solving the two equations on the left as a system gives $A=0$ and $C=0$, while solving the two equations on the right as a system gives $B=-1$ and $D=5$. Plugging these back into the partial fractions decomposition gives

$$f(x) = -\frac{1}{x^2+1}+\frac{5}{2x^2+5}$$

REPEATED QUADRATIC FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{10x^3 - 7}{(5x^2 + 3)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{10x^3 - 7}{(5x^2 + 3)^2} = \frac{Ax + B}{5x^2 + 3} + \frac{Cx + D}{(5x^2 + 3)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{10x^3 - 7}{(5x^2 + 3)^2} = \frac{(Ax + B)(5x^2 + 3) + Cx + D}{(5x^2 + 3)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$10x^3 - 7 = (Ax + B)(5x^2 + 3) + Cx + D$$

Multiply out the right side of this numerator equation,

$$10x^3 - 7 = 5Ax^3 + 3Ax + 5Bx^2 + 3B + Cx + D$$

then group like terms and factor.

$$10x^3 - 7 = 5Ax^3 + 5Bx^2 + (3A + C)x + 3B + D$$

Then we get

$$5A = 10 \qquad 3A + C = 0$$

$$5B = 0 \qquad 3B + D = -7$$

Solving the two equations on the left as a system gives $A = 2$ and $B = 0$, while solving the two equations on the right as a system gives $C = -6$ and $D = -7$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{2x}{5x^2 + 3} - \frac{6x + 7}{(5x^2 + 3)^2}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{3x^3 + 2x^2 + 30x + 16}{(x^2 + 9)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{3x^3 + 2x^2 + 30x + 16}{(x^2 + 9)^2} = \frac{Ax + B}{x^2 + 9} + \frac{Cx + D}{(x^2 + 9)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{3x^3+2x^2+30x+16}{(x^2+9)^2}=\frac{(Ax+B)(x^2+9)+Cx+D}{(x^2+9)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$3x^3+2x^2+30x+16=(Ax+B)(x^2+9)+Cx+D$$

Multiply out the right side of this numerator equation,

$$3x^3+2x^2+30x+16=Ax^3+9Ax+Bx^2+9B+Cx+D$$

then group like terms and factor.

$$3x^3+2x^2+30x+16=Ax^3+Bx^2+(9A+C)x+9B+D$$

Then we get

$$A=3 \qquad\qquad 9A+C=30$$

$$B=2 \qquad\qquad 9B+D=16$$

Solving the two equations on the right as a system gives $C=3$ and $D=-2$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{3x+2}{x^2+9}+\frac{3x-2}{(x^2+9)^2}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{x^3-3x^2-11}{(x^2+x+2)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{x^3-3x^2-11}{(x^2+x+2)^2}=\frac{Ax+B}{x^2+x+2}+\frac{Cx+D}{(x^2+x+2)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{x^3-3x^2-11}{(x^2+x+2)^2}=\frac{(Ax+B)(x^2+x+2)+Cx+D}{(x^2+x+2)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^3-3x^2-11=(Ax+B)(x^2+x+2)+Cx+D$$

Multiply out the right side of this numerator equation,

$$x^3-3x^2-11=Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx+D$$

then group like terms and factor.

$$x^3-3x^2-11=Ax^3+(A+B)x^2+(2A+B+C)x+2B+D$$

Then we get

$$A=1 \qquad\qquad 2A+B+C=0$$

$$A+B=-3 \qquad\qquad 2B+D=-11$$

Solving the two equations on the left as a system gives $B=-4$, while solving the two equations on the right as a system gives $C=2$ and $D=-3$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{x-4}{x^2+x+2}+\frac{2x-3}{(x^2+x+2)^2}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{9x^4+6x^2+x+3}{(3x^2+1)^3}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{9x^4+6x^2+x+3}{(3x^2+1)^3}=\frac{Ax+B}{3x^2+1}+\frac{Cx+D}{(3x^2+1)^2}+\frac{Ex+F}{(3x^2+1)^3}$$

Combine the fractions by finding a common denominator.

$$\frac{9x^4+6x^2+x+3}{(3x^2+1)^3}=\frac{(Ax+B)(9x^4+6x^2+1)+(Cx+D)(3x^2+1)+Ex+F}{(3x^2+1)^3}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$9x^4+6x^2+x+3=(Ax+B)(9x^4+6x^2+1)+(Cx+D)(3x^2+1)+Ex+F$$

Multiply out the right side of this numerator equation,

$$9x^4 + 6x^2 + x + 3 = 9Ax^5 + 6Ax^3 + Ax + 9Bx^4 + 6Bx^2 + B$$

$$+3Cx^3 + Cx + 3Dx^2 + D + Ex + F$$

then group like terms and factor.

$$9x^4 + 6x^2 + x + 3 = 9Ax^5 + 9Bx^4 + (6A + 3C)x^3$$

$$+(6B + 3D)x^2 + (A + C + E)x + B + D + F$$

Then we get

$$9A = 0 \qquad 6A + 3C = 0 \qquad A + C + E = 1$$

$$9B = 9 \qquad 6B + 3D = 6 \qquad B + D + F = 3$$

Solving the two equations on the left as a system gives $A = 0$ and $B = 1$, solving the two equations in the middle as a system gives $C = 0$ and $D = 0$, while solving the two equations on the right as a system gives $E = 1$ and $F = 2$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{1}{3x^2 + 1} + \frac{x + 2}{(3x^2 + 1)^3}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 5x^2 + 13x + 2}{(x^2 + 4x + 6)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{x^3+5x^2+13x+2}{(x^2+4x+6)^2}=\frac{Ax+B}{x^2+4x+6}+\frac{Cx+D}{(x^2+4x+6)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{x^3+5x^2+13x+2}{(x^2+4x+6)^2}=\frac{(Ax+B)(x^2+4x+6)+Cx+D}{(x^2+4x+6)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^3+5x^2+13x+2=(Ax+B)(x^2+4x+6)+Cx+D$$

Multiply out the right side of this numerator equation,

$$x^3+5x^2+13x+2=Ax^3+4Ax^2+6Ax+Bx^2+4Bx+6B+Cx+D$$

then group like terms and factor.

$$x^3+5x^2+13x+2=Ax^3+(4A+B)x^2+(6A+4B+C)x+6B+D$$

Then we get

$$A=1 \qquad\qquad 6A+4B+C=13$$

$$4A+B=5 \qquad\qquad 6B+D=2$$

Solving the two equations on the left as a system gives $B=1$. Solving the two equations on the right as a system gives $C=3$ and $D=-4$.

Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{x+1}{x^2+4x+6} + \frac{3x-4}{(x^2+4x+6)^2}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^7}{(x^2+1)^4}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{x^7}{(x^2+1)^4} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3} + \frac{Gx+H}{(x^2+1)^4}$$

Combine the fractions by finding a common denominator.

$$\frac{x^7}{(x^2+1)^4} = \frac{(Ax+B)(x^2+1)^3 + (Cx+D)(x^2+1)^2 + (Ex+F)(x^2+1) + Gx + H}{(x^2+1)^4}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^7 = (Ax+B)(x^2+1)^3 + (Cx+D)(x^2+1)^2 + (Ex+F)(x^2+1) + Gx + H$$

Multiply out the right side of this numerator equation,

$$x^7 = (Ax+B)(x^6+3x^4+3x^2+1) + (Cx+D)(x^4+2x^2+1)$$

$$+(Ex+F)(x^2+1)+Gx+H$$

then group like terms and factor.

$$x^7=Ax^7+3Ax^5+3Ax^3+Ax+Bx^6+3Bx^4+3Bx^2+B+Cx^5+2Cx^3+Cx$$

$$+Dx^4+2Dx^2+D+Ex^3+Ex+Fx^2+F+Gx+H$$

$$x^7=Ax^7+Bx^6+(3A+C)x^5+(3B+D)x^4+(3A+2C+E)x^3+(3B+2D+F)x^2$$

$$+(A+C+E+G)x+B+D+F+H$$

Then we get

$$A=1$$

$$B=0$$

$$3A+C=0 \qquad C=-3$$

$$3B+D=0 \qquad D=0$$

$$3A+2C+E=0 \qquad E=3$$

$$3B+2D+F=0 \qquad F=0$$

$$A+C+E+G=0 \qquad G=-1$$

$$B+D+F+H=0 \qquad H=0$$

Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{x}{x^2+1}-\frac{3x}{(x^2+1)^2}+\frac{3x}{(x^2+1)^3}-\frac{x}{(x^2+1)^4}$$

MIXED FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2x^4 + 16}{x(x^2 + 2)^2}$$

Solution:

We'll set up the decomposition as

$$\frac{2x^4 + 16}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}$$

To solve for A, we'll remove the x factor from the left side, then evaluate what remains at $x = 0$.

$$\frac{0 + 16}{(0 + 2)^2} \to \frac{16}{4} \to 4 = A$$

To solve for the other constants, we'll find a common denominator on the right side, then combine the fractions.

$$\frac{2x^4 + 16}{x(x^2 + 2)^2} = \frac{4(x^4 + 4x^2 + 4) + (Bx + C)x(x^2 + 2) + (Dx + E)x}{x(x^2 + 2)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$2x^4 + 16 = 4(x^4 + 4x^2 + 4) + (Bx + C)x(x^2 + 2) + (Dx + E)x$$

Multiply out the right side of this numerator equation,

$$2x^4 + 16 = 4x^4 + 16x^2 + 16 + Bx^4 + 2Bx^2 + Cx^3 + 2Cx + Dx^2 + Ex$$

then group like terms and factor.

$$2x^4 + 16 = (4 + B)x^4 + Cx^3 + (16 + 2B + D)x^2 + (2C + E)x + 16$$

We can equate coefficients on either side of the equation to build a system of equations.

$$2 = 4 + B \qquad\qquad 16 + 2B + D = 0$$

$$C = 0 \qquad\qquad 2C + E = 0$$

Solving the two equations on the left as a system gives $B = -2$ and $C = 0$, while solving the two equations on the right as a system gives $D = -12$ and $E = 0$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{4}{x} - \frac{2x}{x^2 + 2} - \frac{12x}{(x^2 + 2)^2}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x + 8}{(x^2 - 1)(2x + 2)(2x + 1)}$$

Solution:

We can rewrite $f(x)$ as

$$\frac{4x+8}{(x^2-1)(2x+2)(2x+1)} = \frac{4x+8}{2(x-1)(x+1)(x+1)(2x+1)} = \frac{2x+4}{(x-1)(2x+1)(x+1)^2}$$

We'll set up the decomposition as

$$\frac{2x+4}{(x-1)(2x+1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{2x+1} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

To solve for A, we'll remove the $x-1$ factor from the left side, then evaluate what remains on the left side at $x=1$.

$$\frac{2+4}{(2+1)(1+1)^2} \to \frac{6}{12} \to \frac{1}{2} = A$$

To solve for B, we'll remove the $2x+1$ factor from the left side, then evaluate what remains at $x=-1/2$.

$$\frac{-1+4}{-\frac{3}{2}\cdot\frac{1}{4}} \to \frac{3}{-\frac{3}{8}} \to -8 = B$$

To solve for D, we'll remove the $(x+1)^2$ factor from the left side, then evaluate what remains at $x=-1$.

$$\frac{-2+4}{(-2)(-2+1)} = \frac{2}{2} = 1 = D$$

Now set $x=0$.

$$\frac{4}{-1\cdot 1\cdot 1^2}=\frac{\frac{1}{2}}{-1}+\frac{-8}{1}+\frac{C}{1}+\frac{1}{1}$$

$$-4=\frac{-15}{2}+C$$

$$C=\frac{7}{2}$$

Plugging the values back into the partial fractions decomposition gives

$$f(x)=\frac{1}{2(x-1)}-\frac{8}{2x+1}+\frac{7}{2(x+1)}+\frac{1}{(x+1)^2}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{2x^3+7x^2-2x+5}{x^4-1}$$

Solution:

We'll set up the decomposition as

$$\frac{2x^3+7x^2-2x+5}{(x^2+1)(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

To solve for A, we'll remove the $x-1$ factor from the left side, then evaluate what remains at $x=1$.

$$\frac{2+7-2+5}{2\cdot 2}\to\frac{12}{4}\to 3=A$$

To solve for B, we'll remove the $x+1$ factor from the left side, then evaluate what remains at $x=-1$.

$$\frac{-2+7+2+5}{2(-2)} \to \frac{12}{-4} \to -3 = B$$

Combine the fractions by finding a common denominator.

$$\frac{2x^3+7x^2-2x+5}{(x^2+1)(x-1)(x+1)} = \frac{3(x+1)(x^2+1)-3(x-1)(x^2+1)+(Cx+D)(x^2-1)}{(x^2+1)(x-1)(x+1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$2x^3+7x^2-2x+5 = 3(x+1)(x^2+1)-3(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

Multiply out the right side of this numerator equation,

$$2x^3+7x^2-2x+5 = 3x^3+3x^2+3x+3-3x^3+3x^2-3x+3+Cx^3-Cx+Dx^2-D$$

then group like terms and factor.

$$2x^3+7x^2-2x+5 = 6x^2+6+Cx^3-Cx+Dx^2-D$$

$$2x^3+7x^2-2x+5 = Cx^3+(6+D)x^2-Cx+6-D$$

Then we get

$$2 = C$$

$$5 = 6 - D$$

Plugging $C=2$ and $D=1$ back into the partial fractions decomposition gives

$$f(x) = \frac{3}{x-1} - \frac{3}{x+1} + \frac{2x+1}{x^2+1}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{36}{(x+2)(x^2-1)^2}$$

Solution:

We'll set up the decomposition as

$$\frac{36}{(x+2)(x+1)^2(x-1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

To solve for A, we'll remove the $x+2$ factor from the left side, then evaluate what remains at $x = -2$.

$$\frac{36}{1(-3)^2} \to 4 = A$$

To solve for C, we'll remove the $(x+1)^2$ factor from the left side, then evaluate what remains at $x = -1$.

$$\frac{36}{1(-2)^2} \to 9 = C$$

To solve for E, we'll remove the $(x-1)^2$ factor from the left side, then evaluate what remains at $x = 1$.

$$\frac{36}{3 \cdot 2^2} \to 3 = E$$

Now set $x = 0$,

$$\frac{36}{2 \cdot 1 \cdot 1} = \frac{4}{2} + B + 9 - D + 3$$

$$B - D = 4$$

And then set $x = 2$.

$$\frac{36}{4 \cdot 3^2 \cdot 1} = \frac{4}{4} + \frac{B}{3} + \frac{9}{9} + D + 3$$

$$\frac{B}{3} + D = -4$$

Then we get

$$B - D = 4$$

$$\frac{B}{3} + D = -4$$

Solving these as a system of equations, we get $B = 0$ and $D = -4$. Plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{4}{x+2} + \frac{9}{(x+1)^2} - \frac{4}{x-1} + \frac{3}{(x-1)^2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x^4 + 7x^3 + 4x^2 + 3x - 2}{x^3(x^2 + x + 1)}$$

Solution:

We'll set up the decomposition as

$$\frac{4x^4 + 7x^3 + 4x^2 + 3x - 2}{x^3(x^2 + x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + x + 1}$$

To solve for C, we'll remove the x^3 factor from the left side, then evaluate what remains at $x = 0$.

$$\frac{-2}{1} \to -2 = C$$

Combine the fractions by finding a common denominator.

$$\frac{4x^4 + 7x^3 + 4x^2 + 3x - 2}{x^3(x^2 + x + 1)}$$

$$= \frac{Ax^2(x^2 + x + 1) + Bx(x^2 + x + 1) + C(x^2 + x + 1) + x^3(Dx + E)}{x^3(x^2 + x + 1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$4x^4 + 7x^3 + 4x^2 + 3x - 2 = Ax^2(x^2 + x + 1) + Bx(x^2 + x + 1)$$

$$-2(x^2 + x + 1) + x^3(Dx + E)$$

Multiply out the right side of this numerator equation,

$$4x^4 + 7x^3 + 4x^2 + 3x - 2 = Ax^4 + Ax^3 + Ax^2 + Bx^3 + Bx^2 + Bx$$

$$-2x^2 - 2x - 2 + Dx^4 + Ex^3$$

then group like terms and factor.

$$4x^4 + 7x^3 + 4x^2 + 3x - 2 = (A + D)x^4 + (A + B + E)x^3$$

$$+(A + B - 2)x^2 + (B - 2)x - 2$$

Then we get

$$A + D = 4 \qquad A + B - 2 = 4$$

$$A + B + E = 7 \qquad B - 2 = 3$$

Solving these as a system of equations gives $A = 1$, $B = 5$, $D = 3$, and $E = 1$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{1}{x} + \frac{5}{x^2} - \frac{2}{x^3} + \frac{3x + 1}{x^2 + x + 1}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x + 1}{(x^2 + 1)(x^2 + x + 1)}$$

Solution:

We'll set up the decomposition as

$$\frac{x+1}{(x^2+1)(x^2+x+1)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+x+1}$$

Combine the fractions by finding a common denominator.

$$\frac{x+1}{(x^2+1)(x^2+x+1)}=\frac{(Ax+B)(x^2+x+1)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+x+1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x+1=(Ax+B)(x^2+x+1)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$x+1=Ax^3+Ax^2+Ax+Bx^2+Bx+B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$x+1=(A+C)x^3+(A+B+D)x^2+(A+B+C)x+B+D$$

Then we get

$A+C=0$

$A+B+C=1$

$A+B+D=0$

$B+D=1$

Solving these as a system of equations gives $A=-1$, $B=1$, $C=1$, and $D=0$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{-x+1}{x^2+1}+\frac{x}{x^2+x+1}$$