Precalculus Workbook Solutions

Complex numbers

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MATH

COMPLEX NUMBERS

1. Simplify the imaginary number.

$$i^{437}$$

Solution:

We need to look for the largest number less than or equal to 437 that's divisible by 4. 437 isn't divisible by 4, so we try 436. 436 is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{437}$$

$$i^{436+1}$$

$$i^{436}i^1$$

Rewrite i^{436} as a power of 4.

$$(i^4)^{109}i^1$$

We know that i^4 is always 1, so

$$(1)^{109}i^1$$

$$1i^1$$

$$i^1$$

$$i$$

■ 2. Simplify the imaginary number.

$$i^{2,314}$$

Solution:

We need to look for the largest number less than or equal to $2{,}314$ that's divisible by 4. $2{,}314$ isn't divisible by 4, so we try $2{,}313$, then $2{,}312$. $2{,}312$ is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{2,314}$$

$$i^{2,312+2}$$

$$i^{2,312}i^2$$

Rewrite $i^{2,312}$ as a power of 4.

$$(i^4)^{578}i^2$$

We know that i^4 is always 1, so

$$(1)^{578}i^2$$

$$1i^2$$

$$i^2$$

-1

■ 3. Name the real and imaginary parts of the complex number.

$$z = -5 + 17i$$

Solution:

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = -5 + 17i$, -5 is the real part, and 17 is the imaginary part.

■ 4. Name the real and imaginary parts of the complex number.

$$z = \sqrt{7} - 4\pi i$$

Solution:

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = \sqrt{7} - 4\pi i$, $\sqrt{7}$ is the real part, and -4π is the imaginary part.

■ 5. How can the numbers be classified?

$$z = -3 + 9i$$

$$z = 0 - 15i$$

$$z = 6 + 0i$$

Solution:

For the number $z = -3 + 9i$, both the real part and the imaginary part are non-zero. So, this is a complex number.

For the number $z = 0 - 15i$, the real part is 0 and the imaginary part is non-zero. Because the real-number part of the complex number is 0, we end up with

$$z = 0 - 15i$$

$$z = -15i$$

This is a pure imaginary number. Because every pure imaginary number is also a complex number, we can call this a complex number as well.

For the number $z = 6 + 0i$, the real part is non-zero and the imaginary part is 0. Because the imaginary part of the complex number is 0, we end up with

$$z = 6 + 0i$$

$$z = 6$$

This is a real number. Because every real number is also a complex number, we can call this a complex number as well.

■ 6. How can the numbers be classified?

$$z = 0 - \pi i$$

$$z = -\sqrt{5} + 0i$$

$$z = -11 + \frac{2}{3}i$$

Solution:

For the number $z = 0 - \pi i$, the real part is 0 and the imaginary part is non-zero. Because the real-number part of the complex number is 0, we end up with

$$z = 0 - \pi i$$

$$z = -\pi i$$

This is a pure imaginary number. Because every pure imaginary number is also a complex number, we can call this a complex number as well.

For the number $z=-\sqrt{5}+0i$, the real part is non-zero and the imaginary part is 0. Because the imaginary part of a complex number is 0, we end up with

$$z=-\sqrt{5}+0i$$

$$z=-\sqrt{5}$$

This is a real number. Because every real number is also a complex number, we can call this a complex number as well.

For the number $z=-11+(2/3)i$, both the real part and the imaginary part are non-zero. So, this is a complex number.

COMPLEX NUMBER OPERATIONS

■ 1. Find the sum and difference of the complex numbers.

$$\frac{7}{5}-\frac{2}{3}i$$

$$\frac{7}{2}-\frac{8}{3}i$$

Solution:

The sum of the complex numbers is

$$\left(\frac{7}{5}-\frac{2}{3}i\right)+\left(\frac{7}{2}-\frac{8}{3}i\right)$$

$$\left(\frac{7}{5}+\frac{7}{2}\right)+\left(-\frac{2}{3}i-\frac{8}{3}i\right)$$

$$\left(\frac{14}{10}+\frac{35}{10}\right)+\left(-\frac{10}{3}i\right)$$

$$\frac{49}{10}-\frac{10}{3}i$$

The difference of the complex numbers is

$$\left(\frac{7}{5}-\frac{2}{3}i\right)-\left(\frac{7}{2}-\frac{8}{3}i\right)$$

$$\frac{7}{5}-\frac{2}{3}i-\frac{7}{2}+\frac{8}{3}i$$

$$\frac{7}{5}-\frac{7}{2}-\frac{2}{3}i+\frac{8}{3}i$$

$$\frac{14}{10}-\frac{35}{10}-\frac{2}{3}i+\frac{8}{3}i$$

$$-\frac{21}{10}+\frac{6}{3}i$$

$$-\frac{21}{10}+2i$$

■ 2. Find the product of the complex numbers.

$$-7i$$

$$-5+9i$$

Solution:

Use the distributive property to find the product of the complex numbers.

$$-7i(-5+9i)$$

$$(-7i)(-5)+(-7i)(9i)$$

$$35i-63i^2$$

Using $i^2 = -1$ in the last term, we get

$$35i - 63(-1)$$

$$35i + 63$$

$$63 + 35i$$

■ 3. Find the product of the complex numbers.

$$5 - 2i$$

$$6 - 11i$$

Solution:

Use FOIL to find the product of the complex numbers.

$$(5 - 2i)(6 - 11i)$$

$$(5)(6) + (5)(-11i) + (-2i)(6) + (-2i)(-11i)$$

$$30 - 55i - 12i + 22i^2$$

$$30 - 67i + 22i^2$$

Using $i^2 = -1$ in the last term, we get

$$30 - 67i + 22(-1)$$

$$30 - 67i - 22$$

$$8 - 67i$$

■ 4. Divide the complex number $-4 + 15i$ by the imaginary number $5i$.

Solution:

Set up the division.

$$\frac{-4 + 15i}{5i}$$

$$\frac{-4}{5i} + \frac{15i}{5i}$$

$$-\frac{4}{5}i^{-1} + 3$$

We know that i^{-1} is equal to $-i$.

$$-\frac{4}{5}(-i) + 3$$

$$\frac{4}{5}i + 3$$

$$3 + \frac{4}{5}i$$

■ 5. Find the complex conjugate of each complex number.

$9 - 9i$

$-3 + 13i$

$11 - 22i$

Solution:

For each of these, we keep the real part (9, -3, or 11) and change the sign of the imaginary part (from -9 to 9, from 13 to -13, or from -22 to 22). So the complex conjugates are:

The complex conjugate of $9 - 9i$ is $9 + 9i$.

The complex conjugate of $-3 + 13i$ is $-3 - 13i$.

The complex conjugate of $11 - 22i$ is $11 + 22i$.

■ 6. Express the fraction in the form $a + bi$ where a and b are real numbers.

$$\frac{-3 + 7i}{4 - 5i}$$

Solution:

Multiply by the conjugate of the denominator.

$$\left(\frac{-3+7i}{4-5i}\right)\left(\frac{4+5i}{4+5i}\right)$$

$$\frac{(-3+7i)(4+5i)}{(4-5i)(4+5i)}$$

Use FOIL to expand the numerator and denominator.

$$\frac{-12-15i+28i+35i^2}{16+20i-20i-25i^2}$$

$$\frac{-12+13i+35i^2}{16-25i^2}$$

Using $i^2=-1$ gives

$$\frac{-12+13i+35(-1)}{16-25(-1)}$$

$$\frac{-12+13i-35}{16+25}$$

$$\frac{-47+13i}{41}$$

$$-\frac{47}{41}+\frac{13}{41}i$$

GRAPHING COMPLEX NUMBERS

■ 1. Graph $-3+5i$, $2-4i$, and 5 in the complex plane.

Solution:

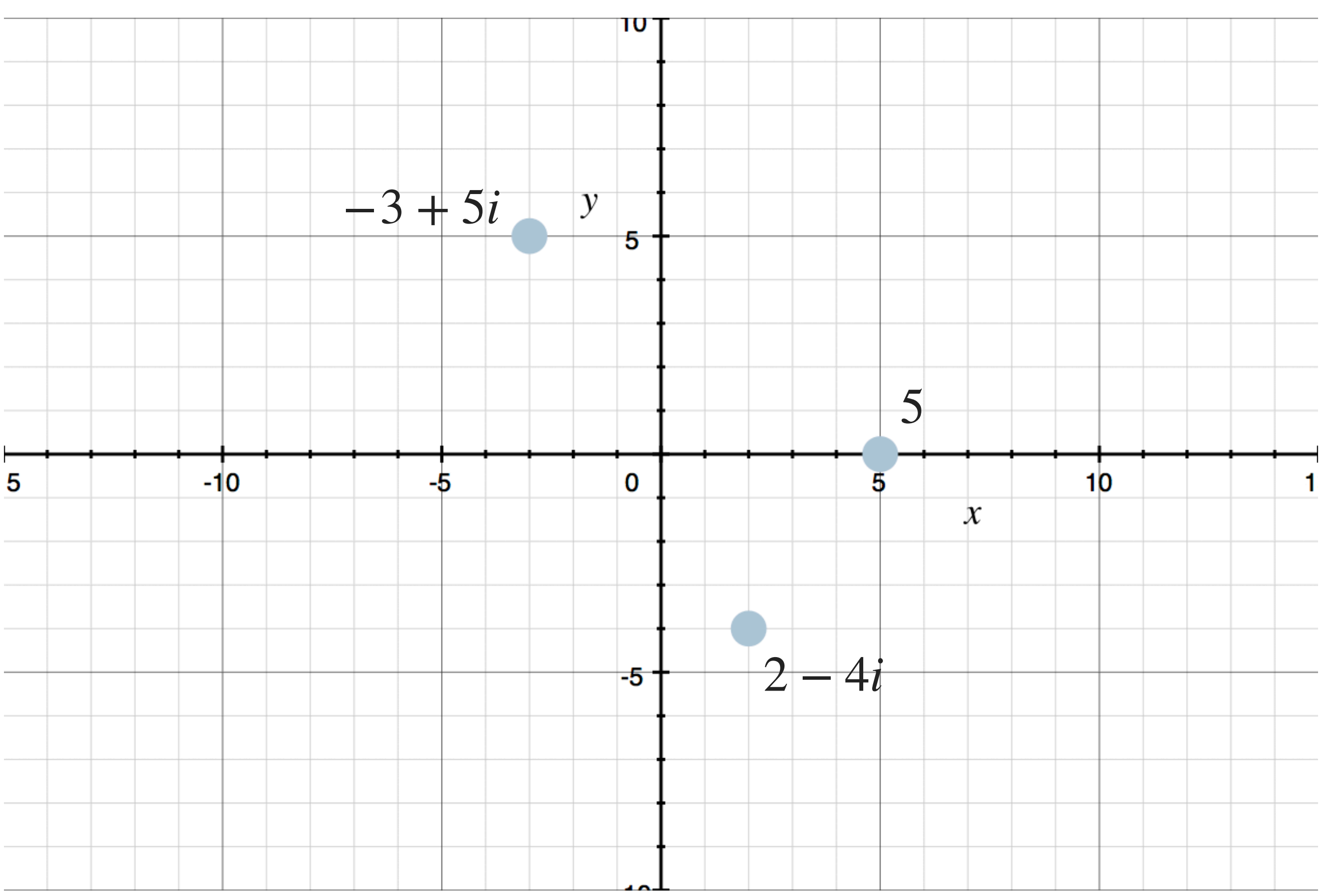
Let's break down each complex number.

$-3+5i$ has real part $a=-3$ and imaginary part $b=5$.

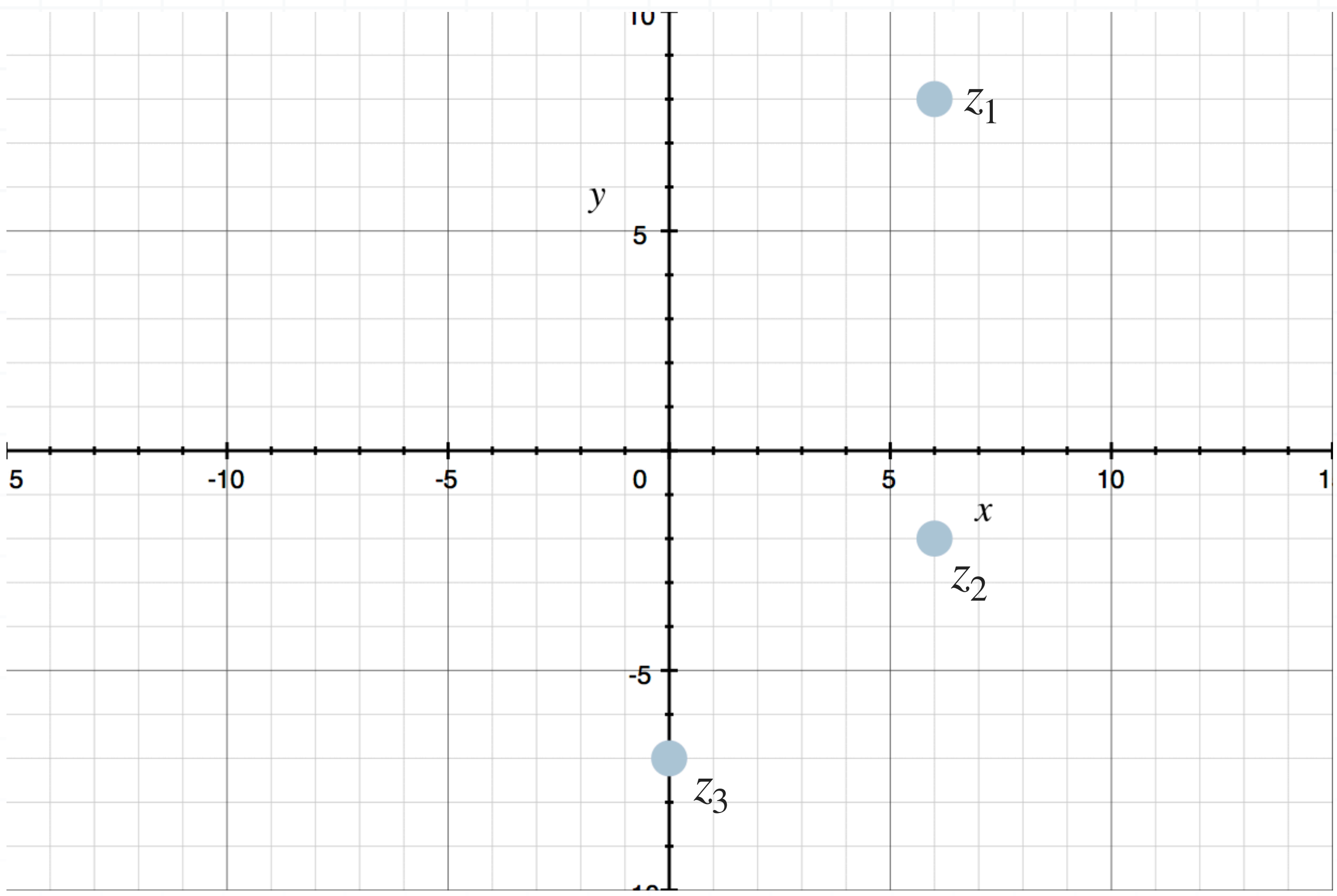
$2-4i$ has real part $a=2$ and imaginary part $b=-4$.

5 has real part $a=5$ and imaginary part $b=0$.

Now we can graph all of them together.



■ 2. Which three complex numbers are represented in the graph?



Solution:

The point z_1 is 6 units to the right of the vertical axis and 8 units above the horizontal axis, so it's the complex number $6+8i$.

The point z_2 is 6 units to the right of the vertical axis and 2 units below the horizontal axis, so it's the complex number $6-2i$.

The point z_3 is 0 units to the left or right of the vertical axis and 7 units below the horizontal axis, so it's the complex number $-7i$.

■ 3. Graph the sum of the complex numbers $5-4i$ and $-1+10i$.

Solution:

First, find the sum.

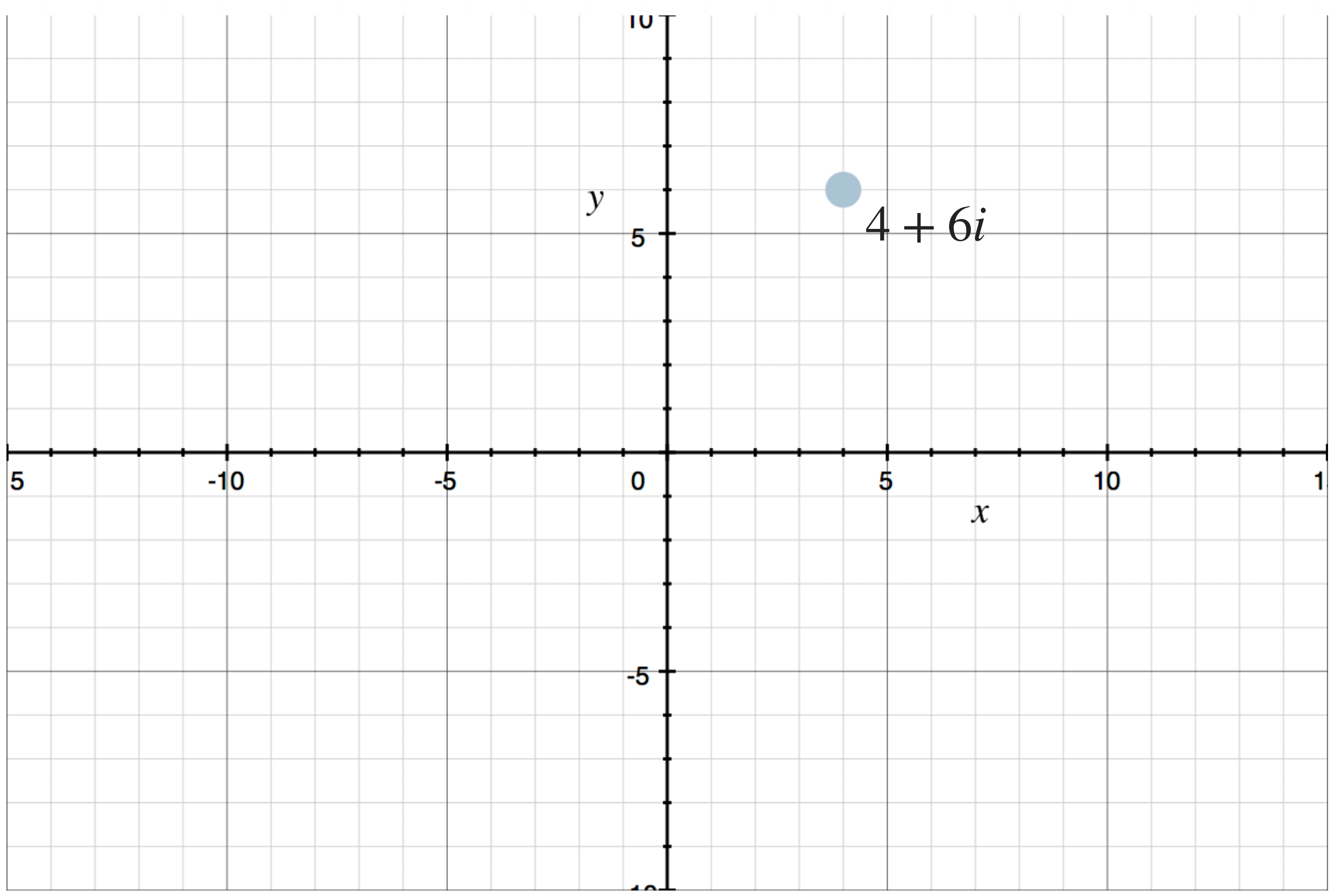
$$(5-4i)+(-1+10i)$$

$$(5+(-1))+(-4+10)i$$

$$(5-1)+(-4+10)i$$

$$4+6i$$

Now graph the complex number $4+6i$, which has a real part 4 and an imaginary part 6.



■ 4. Graph the difference of the complex numbers $8 - 7i$ and $13 - 4i$.

Solution:

First, find the difference.

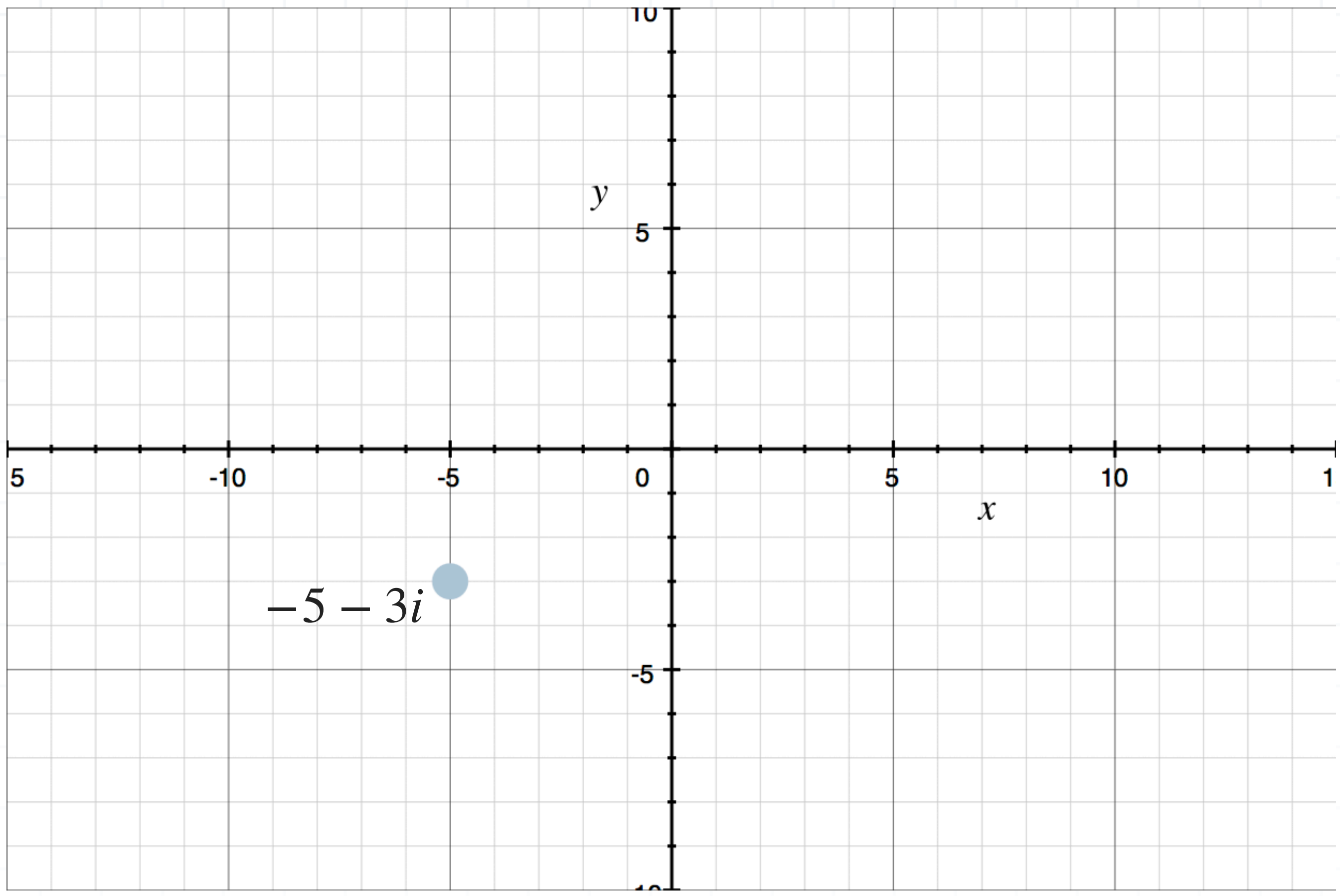
$$(8 - 7i) - (13 - 4i)$$

$$(8 - 13) + (-7 - (-4))i$$

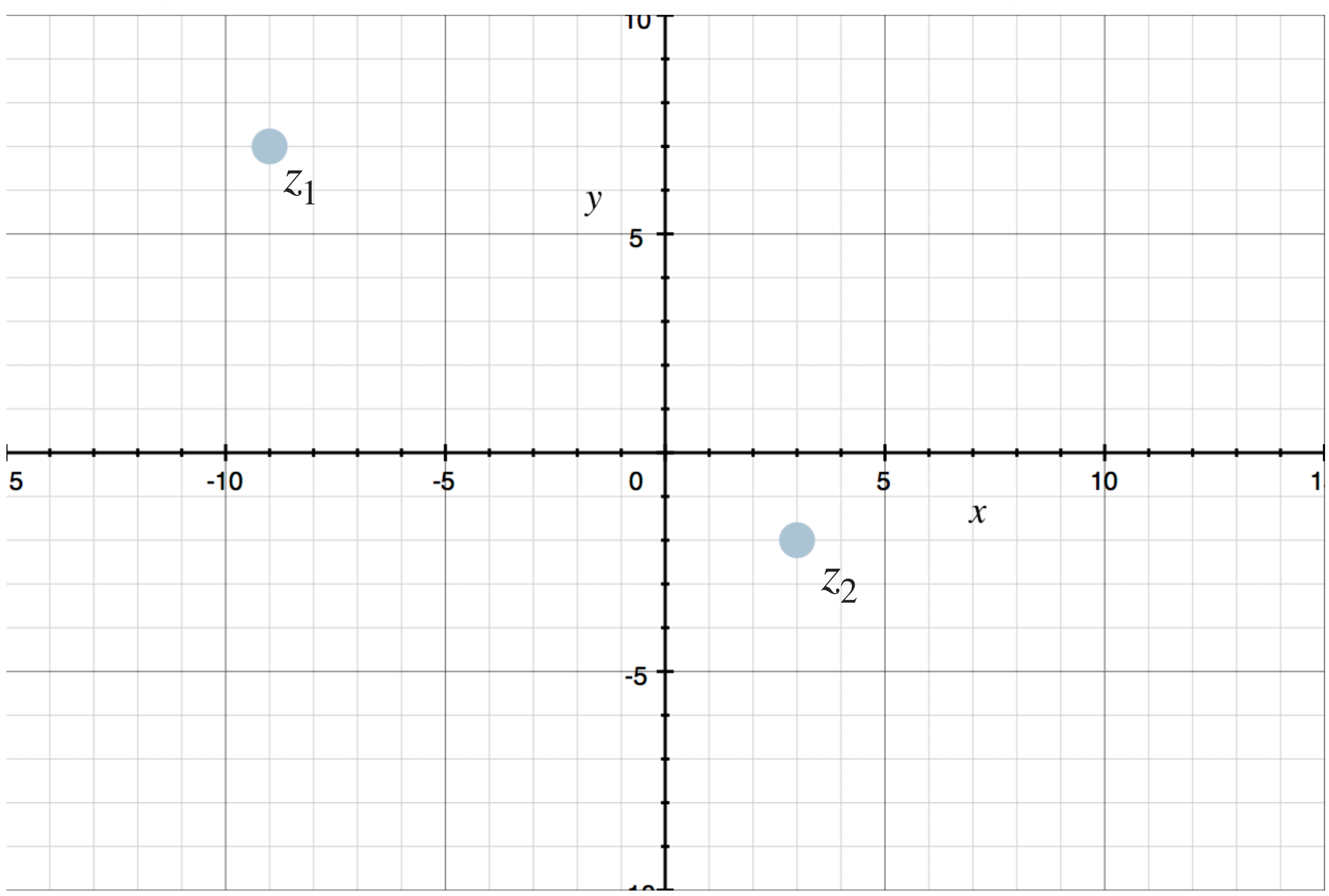
$$(8 - 13) + (-7 + 4)i$$

$$-5 - 3i$$

Now graph the complex number $-5 - 3i$, which has a real part -5 and an imaginary part -3.



■ 5. Graph the sum of the complex numbers z_1 and z_2.



Solution:

The point z_1 is 9 units to the left of the vertical axis and 7 units above the horizontal axis, which means that complex number is $z_1 = -9 + 7i$.

The point z_2 is 3 units to the right of the vertical axis and 2 units below the horizontal axis, which means that complex number is $z_2 = 3 - 2i$.

The sum of z_1 and z_2 is

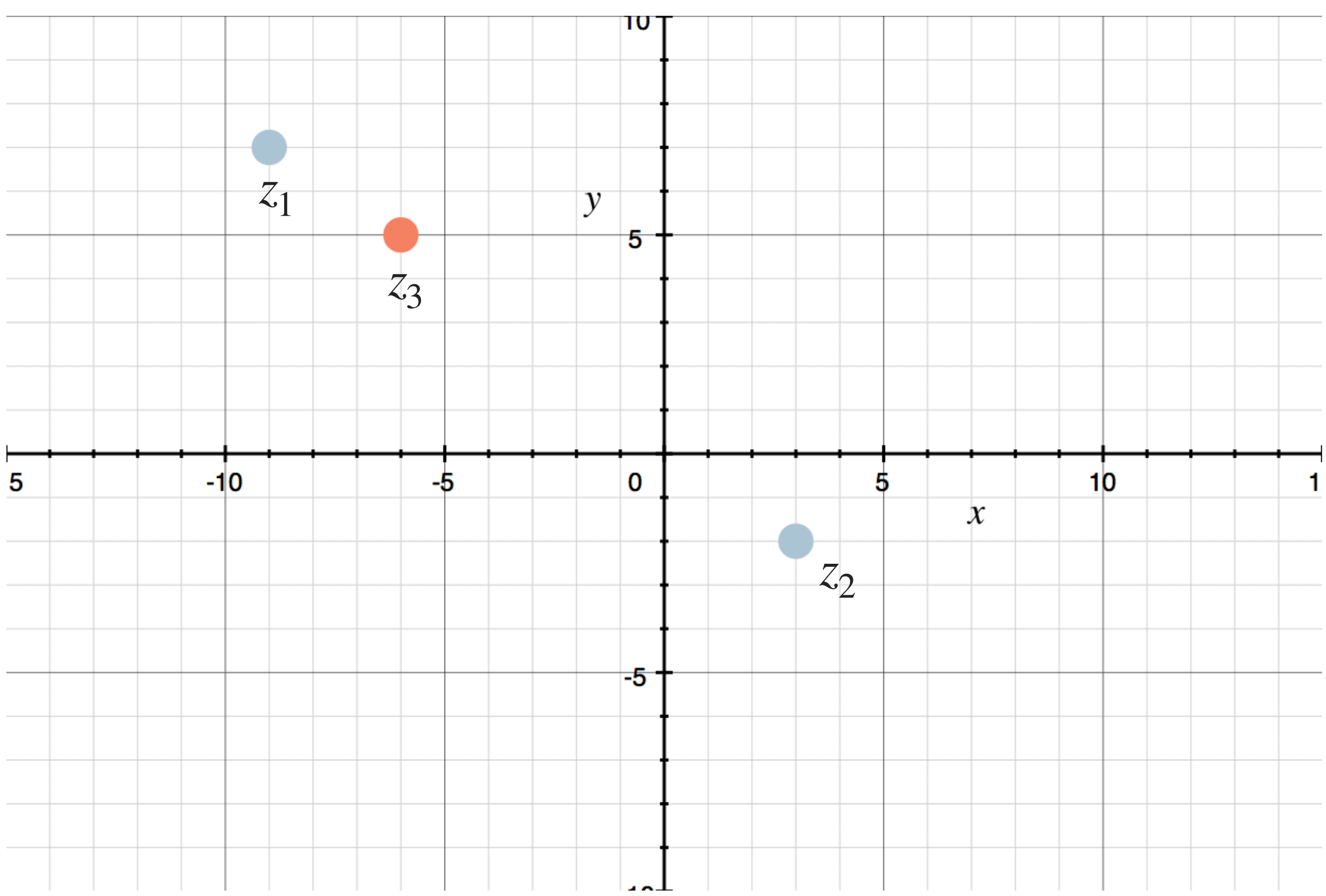
$$z_1 + z_2 = (-9 + 7i) + (3 - 2i)$$

$$z_1 + z_2 = (-9 + 3) + (7 + (-2))i$$

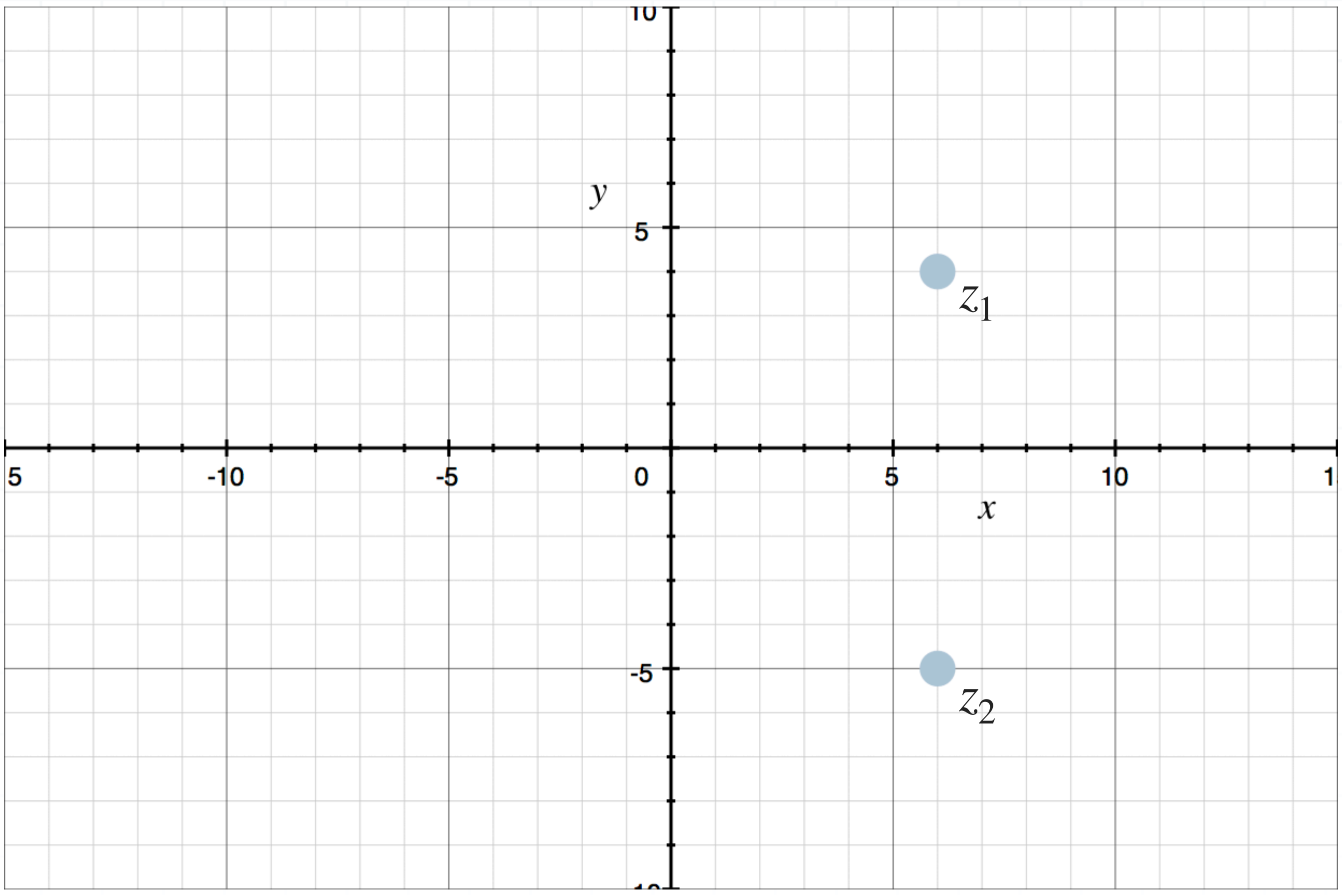
$$z_1 + z_2 = (-9 + 3) + (7 - 2)i$$

$$z_1 + z_2 = -6 + 5i$$

So if we plot the sum on the same set of axes, we get



■ 6. Graph the difference of the complex numbers z_1 and z_2.



Solution:

The point z_1 is 6 units to the right of the vertical axis and 4 units above the horizontal axis, which means that complex number is $z_1 = 6 + 4i$.

The point z_2 is 6 units to the right of the vertical axis and 5 units below the horizontal axis, which means that complex number is $z_2 = 6 - 5i$.

The difference of z_1 and z_2 is

$$z_1 - z_2 = (6 + 4i) - (6 - 5i)$$

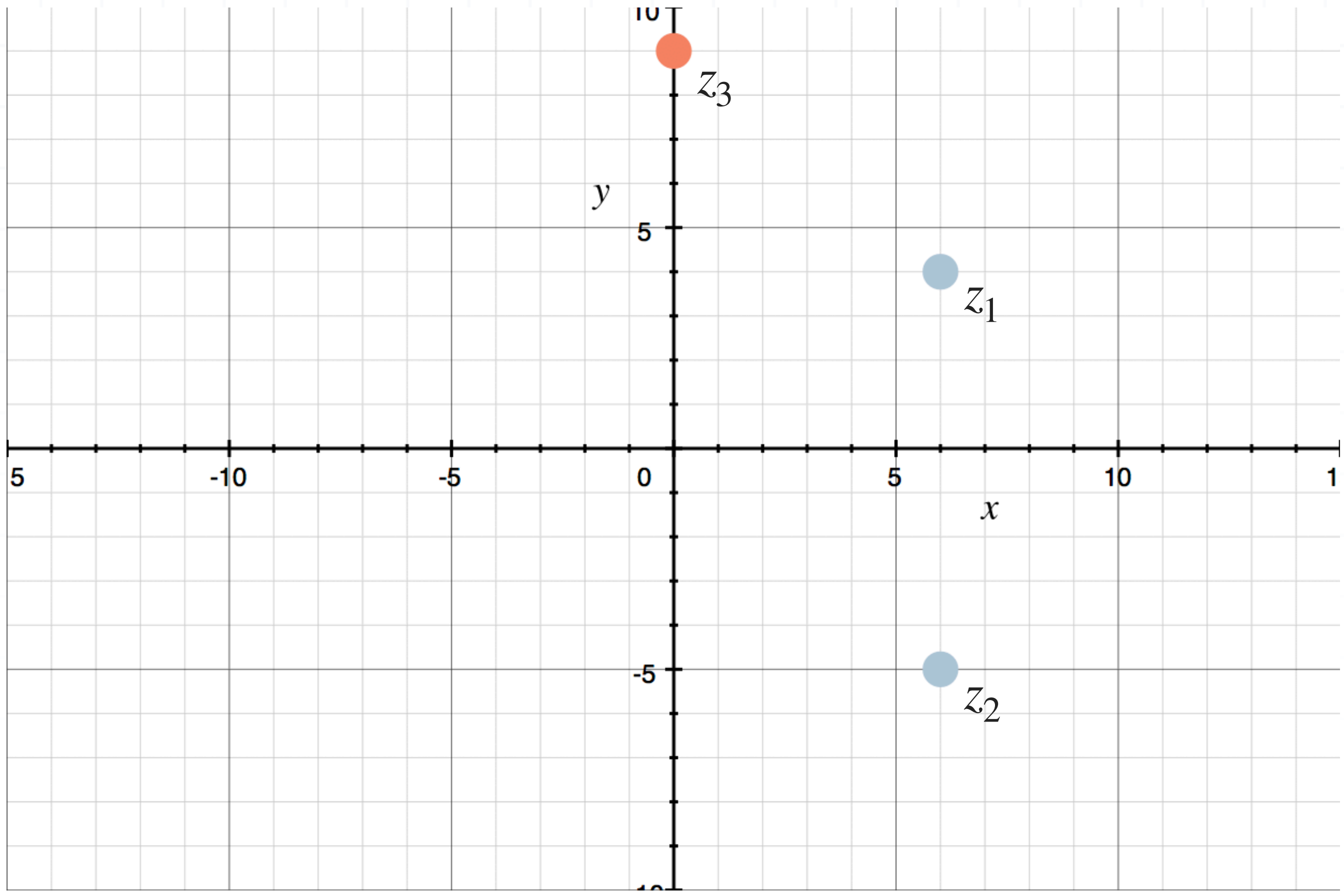
$$z_1 - z_2 = (6 - 6) + (4 - (-5))i$$

$$z_1 - z_2 = (6 - 6) + (4 + 5)i$$

$$z_1 - z_2 = 0 + 9i$$

$$z_1 - z_2 = 9i$$

So if we plot the difference on the same set of axes, we get



DISTANCES AND MIDPOINTS

■ 1. Find the distance between $s = 5 + 3i$ and $t = 1 - i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(5-1)^2 + (3-(-1))^2}$$

$$d = \sqrt{4^2 + 4^2}$$

$$d = \sqrt{16 + 16}$$

$$d = \sqrt{32}$$

$$d = 4\sqrt{2}$$

■ 2. Find the distance between $u = -5 - 3i$ and $v = 4 + 2i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(-5-4)^2 + (-3-2)^2}$$

$$d = \sqrt{(-9)^2 + (-5)^2}$$

$$d = \sqrt{81 + 25}$$

$$d = \sqrt{106}$$

■ 3. Find the distance between $w = 2 + 6i$ and $z = -2 - 6i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(2 - (-2))^2 + (6 - (-6))^2}$$

$$d = \sqrt{(2 + 2)^2 + (6 + 6)^2}$$

$$d = \sqrt{4^2 + 12^2}$$

$$d = \sqrt{16 + 144}$$

$$d = \sqrt{160}$$

$$d = 4\sqrt{10}$$

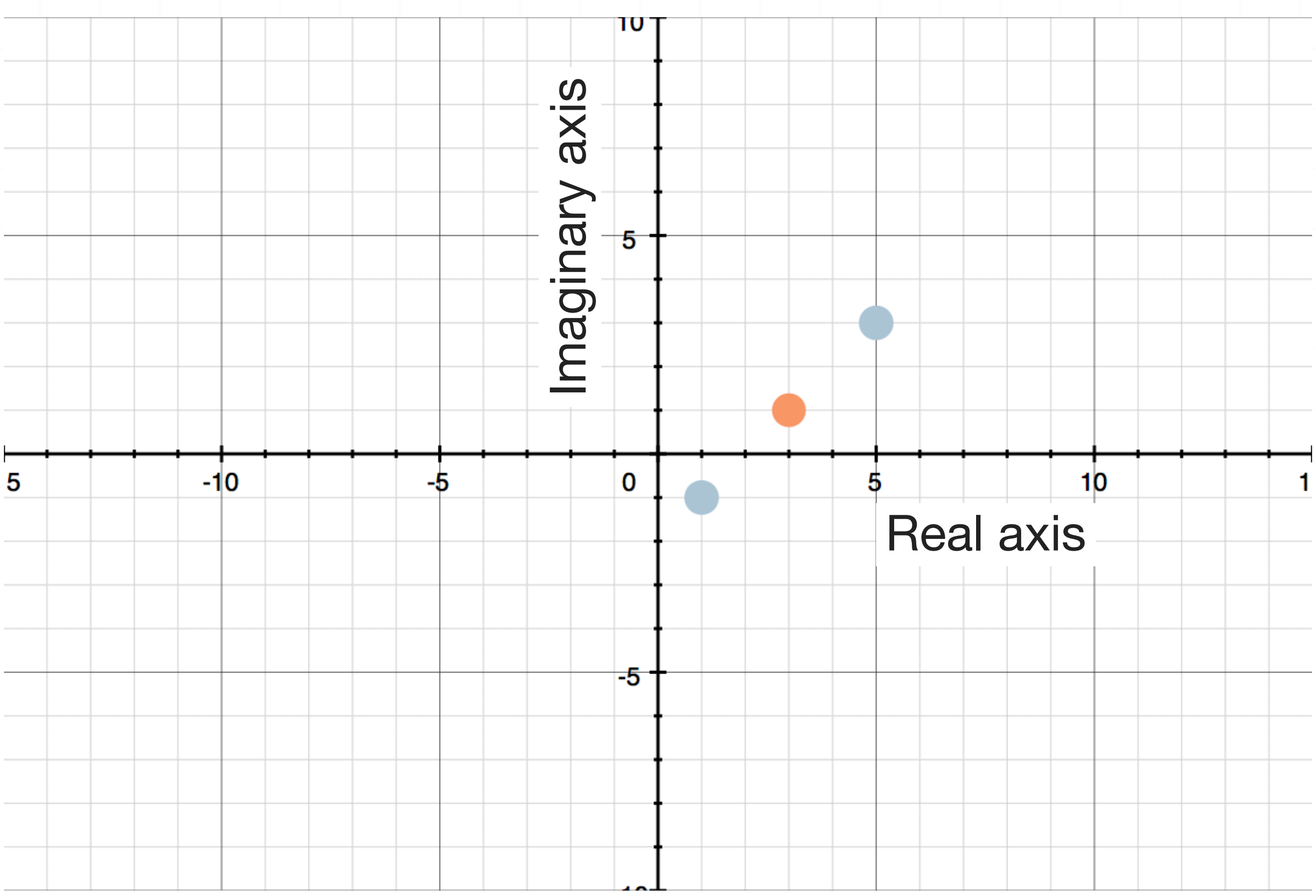
■ 4. Find the midpoint between $s = 5 + 3i$ and $t = 1 - i$.

Solution:

The distance between the real parts is $5-1=4$, and half of that distance is $4/2=2$. The value that's 2 units from 5 and 2 units from 2 is 3. The midpoint of the real parts is 3.

The distance between the imaginary parts is $3-(-1)=3+1=4$, and half of that distance is $4/2=2$. The value that's 2 units from 3 and 2 units from -1 is 1. The midpoint of the imaginary parts is 1.

The midpoint between $s=5+3i$ and $t=1-i$ is $m=3+i$.



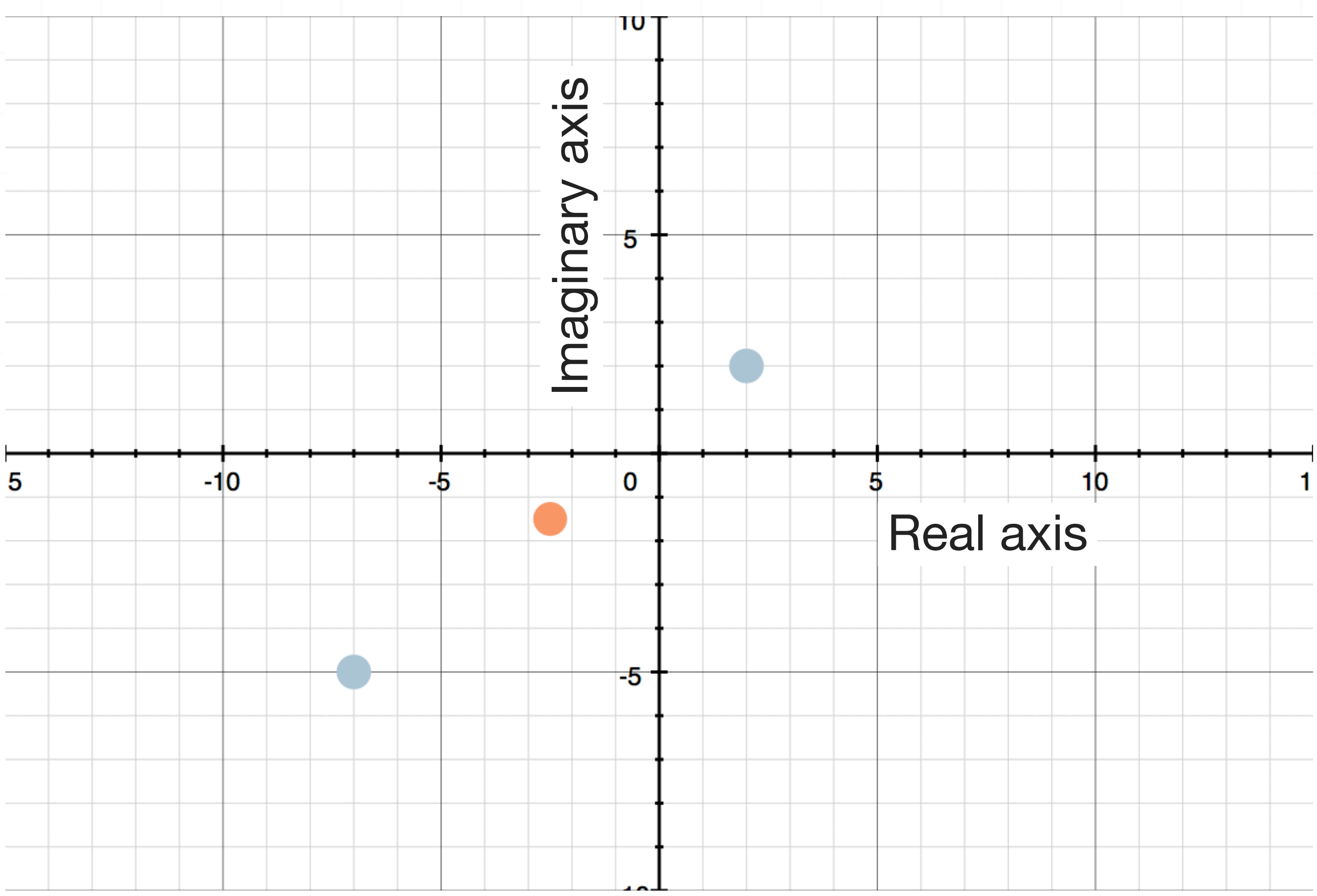
■ 5. Find the midpoint between $u=-7-5i$ and $z=2+2i$.

Solution:

The distance between the real parts is $-7 - 2 = -9$, and half of that distance is $-9/2 = -4.5$. The value that's -4.5 units from -7 and -4.5 units from 2 is -2.5. The midpoint of the real parts is -2.5.

The distance between the imaginary parts is $-5 - 2 = -7$, and half of that distance is $-7/2 = -3.5$. The value that's -3.5 units from -7 and -3.5 units from 2 is -1.5. The midpoint of the imaginary parts is -1.5.

The midpoint between $u = -7 - 5i$ and $z = 2 + 2i$ is $z = -2.5 - 1.5i$.



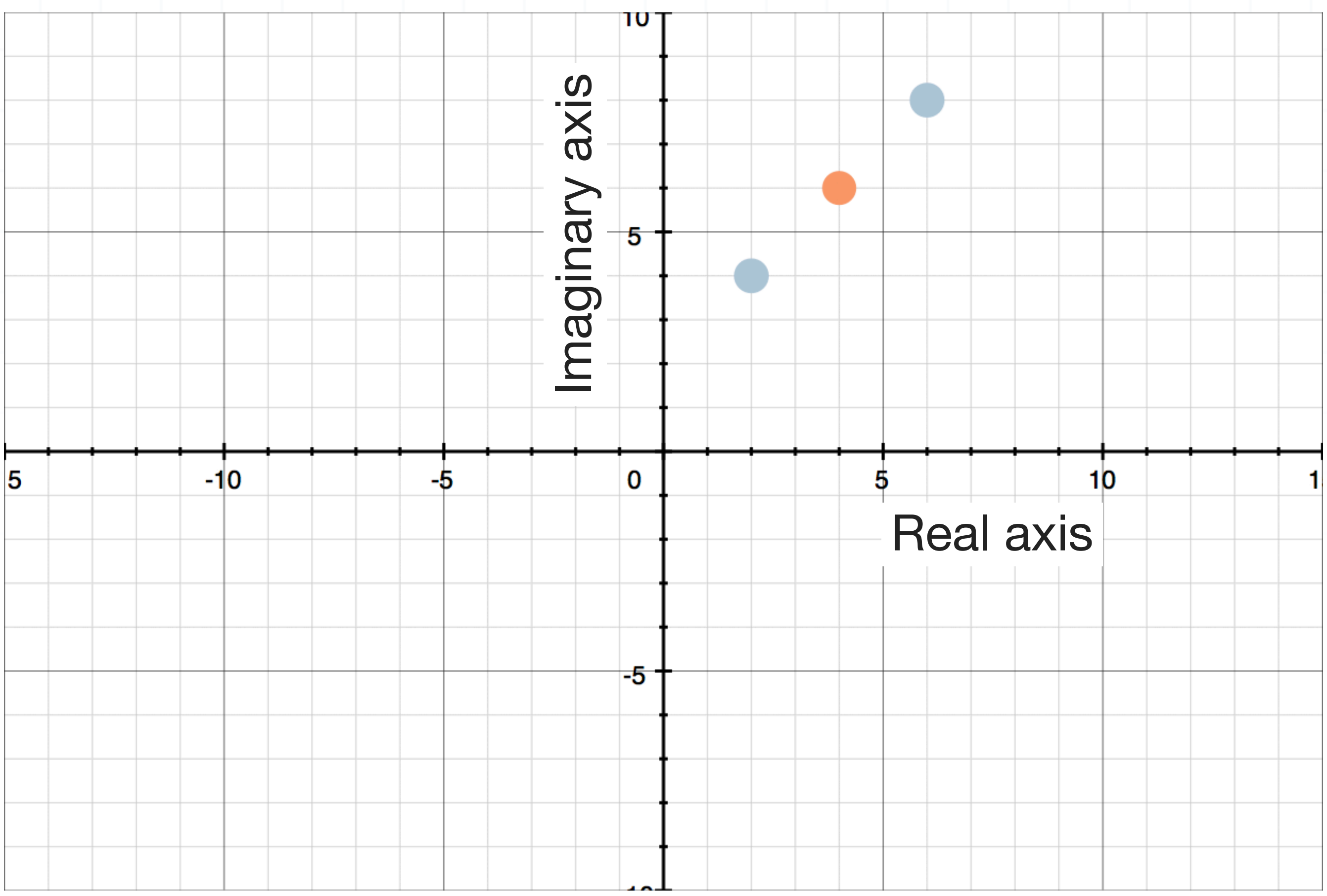
■ 6. Graph the midpoint between $w = 6 + 8i$ and $z = 2 + 4i$.

Solution:

The distance between the real parts is $6 - 2 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 6 and 2 units from 2 is 4. The midpoint of the real parts is 4.

The distance between the imaginary parts is $8 - 4 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 8 and 2 units from 4 is 6. The midpoint of the imaginary parts is 6.

The midpoint between $w = 6 + 8i$ and $z = 2 + 4i$ is $m = 4 + 6i$.



COMPLEX NUMBERS IN POLAR FORM

■ 1. If the complex number $6-2i$ is expressed in polar form, which quadrant contains the angle θ?

Solution:

If we set the complex number equal to its polar form, we get

$$6-2i=r(\cos\theta+i\sin\theta)$$

$$6-2i=r\cos\theta+ri\sin\theta$$

From this equation, we know that

$$6=r\cos\theta$$

$$\cos\theta=\frac{6}{r}$$

The value of r is always positive, since r represents a distance, so $6/r$ has to be greater than 0, which means $\cos\theta$ has to be positive.

We also know from $6-2i=r\cos\theta+ri\sin\theta$ that

$$-2=r\sin\theta$$

$$\sin\theta=-\frac{2}{r}$$

Because the value of r is always positive, $-2/r$ has to be less than 0, which means $\sin\theta$ has to be negative.

Angles with a positive cosine and negative sine are always in the fourth quadrant.

■ 2. Find r for the complex number.

$$-9 - 3i$$

Solution:

In the complex number, the real part is $a = -9$ and the imaginary part is $b = -3$, so the value of r will be

$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{(-9)^2 + (-3)^2}$$

$$r = \sqrt{81 + 9}$$

$$r = \sqrt{90}$$

$$r = 3\sqrt{10}$$

■ 3. What is the polar form of the complex number?

$$5 + 12i$$

Solution:

If we match up $5 + 12i$ with the standard form $a + bi$, we get $a = 5$ and $b = 12$, so

$$r = \sqrt{a^2 + b^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

The value of θ is

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{12}{5}$$

$$\arctan(\tan\theta) = \arctan\frac{12}{5}$$

$$\theta = \arctan\frac{12}{5}$$

$$\theta \approx 1.18$$

Then the complex number written in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z \approx 13[\cos(1.18) + i\sin(1.18)]$$

■ 4. Write the complex number in polar form.

$$11i$$

Solution:

The complex number $11i$ can be written as $0 + 11i$, so its real part is 0, which means the number is located on the imaginary axis. Because $a = 0$ and $b = 11$, the distance of $0 + 11i$ from the origin is

$$r = \sqrt{a^2 + b^2} = \sqrt{0^2 + 11^2} = \sqrt{0 + 121} = \sqrt{121} = 11$$

Since the imaginary part of $0 + 11i$ is 11, which is positive, $0 + 11i$ is located on the positive imaginary axis, so $\theta = \pi/2$. In polar form, we get

$$r(\cos\theta + i\sin\theta)$$

$$11\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

■ 5. What is the polar form of the complex number?

$$z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Solution:

In the complex number, the real part is $a = -\sqrt{3}/2$ and the imaginary part is $b = -1/2$, so the value of r will be

$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$r = \sqrt{\frac{3}{4} + \frac{1}{4}}$$

$$r = \sqrt{1}$$

$$r = 1$$

The value of θ is

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \left(-\frac{1}{2}\right)\left(-\frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\arctan(\tan\theta) = \arctan\frac{\sqrt{3}}{3}$$

$$\theta = \arctan\frac{\sqrt{3}}{3}$$

$\theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{7\pi}{6}$

Because the complex number is in quadrant III, we use $\theta = 7\pi/6$. Then the complex number written in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 1\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

■ 6. Write the complex number in polar form.

$$-5$$

Solution:

The complex number -5 can be written as $-5 + 0i$, so its imaginary part is 0, which means the number is located on the real axis. Because $a = -5$ and $b = 0$, the distance of $-5 + 0i$ from the origin is

$$r = \sqrt{a^2 + b^2} = \sqrt{(-5)^2 + 0^2} = \sqrt{25 + 0} = \sqrt{25} = 5$$

Since the real part of $-5 + 0i$ is -5, which is negative, $-5 + 0i$ is located on the negative real axis, so $\theta = \pi$. In polar form, we get

$$r(\cos\theta + i\sin\theta)$$

$$5(\cos\pi + i\sin\pi)$$

MULTIPLYING AND DIVIDING POLAR FORMS

■ 1. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = (5\sqrt{2})\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right]$$

Simplify.

$$z_1z_2 = 5\sqrt{2}\left[\cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) + i\sin\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)\right]$$

$$z_1z_2 = 5\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

■ 2. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = \sqrt{3}\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$$

$$z_2 = \frac{\sqrt{5}}{3}\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = \left(\sqrt{3}\frac{\sqrt{5}}{3}\right)\left[\cos\left(\frac{4\pi}{5} + \frac{11\pi}{8}\right) + i\sin\left(\frac{4\pi}{5} + \frac{11\pi}{8}\right)\right]$$

Simplify.

$$z_1z_2 = \frac{\sqrt{15}}{3}\left[\cos\left(\frac{32\pi}{40} + \frac{55\pi}{40}\right) + i\sin\left(\frac{32\pi}{40} + \frac{55\pi}{40}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{15}}{3}\left(\cos\frac{87\pi}{40} + i\sin\frac{87\pi}{40}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract 2π from the angle, we get

$$z_1z_2 = \frac{\sqrt{15}}{3}\left[\cos\left(\frac{87\pi}{40} - 2\pi\right) + i\sin\left(\frac{87\pi}{40} - 2\pi\right)\right]$$

$$z_1z_2 = \frac{\sqrt{15}}{3}\left[\cos\left(\frac{87\pi}{40} - \frac{80\pi}{40}\right) + i\sin\left(\frac{87\pi}{40} - \frac{80\pi}{40}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{15}}{3}\left(\cos\frac{7\pi}{40} + i\sin\frac{7\pi}{40}\right)$$

■ 3. What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1 = 12\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$z_2 = 15\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Solution:

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{12}{15}\left[\cos\left(\frac{7\pi}{6} - \frac{\pi}{2}\right) + i\sin\left(\frac{7\pi}{6} - \frac{\pi}{2}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2}=\frac{4}{5}\left[\cos\left(\frac{7\pi}{6}-\frac{3\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}-\frac{3\pi}{6}\right)\right]$$

$$\frac{z_1}{z_2}=\frac{4}{5}\left(\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}\right)$$

$$\frac{z_1}{z_2}=\frac{4}{5}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

■ 4. What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1=\sqrt{7}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

$$z_2=\frac{1}{\sqrt{2}}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

Solution:

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$$

$$\frac{z_1}{z_2} = \frac{\sqrt{7}}{\frac{1}{\sqrt{2}}}\left[\cos\left(\frac{\pi}{12} - \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{12} - \frac{2\pi}{3}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2} = \sqrt{14}\left[\cos\left(\frac{\pi}{12} - \frac{8\pi}{12}\right) + i\sin\left(\frac{\pi}{12} - \frac{8\pi}{12}\right)\right]$$

$$\frac{z_1}{z_2} = \sqrt{14}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we add 2π to the angle, we get

$$\frac{z_1}{z_2} = \sqrt{14}\left[\cos\left(\frac{-7\pi}{12} + 2\pi\right) + i\sin\left(\frac{-7\pi}{12} + 2\pi\right)\right]$$

$$\frac{z_1}{z_2} = \sqrt{14}\left[\cos\left(\frac{-7\pi}{12} + \frac{24\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12} + \frac{24\pi}{12}\right)\right]$$

$$z_1z_2 = \sqrt{14}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

■ 5. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = \frac{\sqrt{15}}{4}\left(\cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}\right)$$

$$z_2 = \frac{1}{\sqrt{5}}\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = \left(\frac{\sqrt{15}}{4}\cdot\frac{1}{\sqrt{5}}\right)\left[\cos\left(\frac{7\pi}{2} + \frac{6\pi}{5}\right) + i\sin\left(\frac{7\pi}{2} + \frac{6\pi}{5}\right)\right]$$

Simplify.

$$z_1z_2 = \frac{\sqrt{15}}{4\sqrt{5}}\left[\cos\left(\frac{35\pi}{10} + \frac{12\pi}{10}\right) + i\sin\left(\frac{35\pi}{10} + \frac{12\pi}{10}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{3}}{4}\left(\cos\frac{47\pi}{10} + i\sin\frac{47\pi}{10}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract $2\cdot 2\pi = 4\pi$ from the angle, we get

$$z_1z_2 = \frac{\sqrt{3}}{4}\left[\cos\left(\frac{47\pi}{10} - 4\pi\right) + i\sin\left(\frac{47\pi}{10} - 4\pi\right)\right]$$

$$z_1z_2=\frac{\sqrt{3}}{4}\left[\cos\left(\frac{47\pi}{10}-\frac{40\pi}{10}\right)+i\sin\left(\frac{47\pi}{10}-\frac{40\pi}{10}\right)\right]$$

$$z_1z_2=\frac{\sqrt{3}}{4}\left(\cos\frac{7\pi}{10}+i\sin\frac{7\pi}{10}\right)$$

■ 6. Suppose that a complex number z is the product $z_1 \cdot z_2$ of the given complex numbers. If z is expressed in polar form, $r(\cos\theta+i\sin\theta)$, where is θ located?

$$z_1=3\sqrt{5}\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)$$

$$z_2=6\left(\cos\frac{7\pi}{10}+i\sin\frac{7\pi}{10}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$$

$$z_1z_2=(3\sqrt{5}\cdot 6)\left[\cos\left(\frac{2\pi}{5}+\frac{7\pi}{10}\right)+i\sin\left(\frac{2\pi}{5}+\frac{7\pi}{10}\right)\right]$$

Simplify.

$$z_1z_2 = 18\sqrt{5}\left[\cos\left(\frac{4\pi}{10}+\frac{7\pi}{10}\right)+i\sin\left(\frac{4\pi}{10}+\frac{7\pi}{10}\right)\right]$$

$$z_1z_2 = 18\sqrt{5}\left(\cos\frac{11\pi}{10}+i\sin\frac{11\pi}{10}\right)$$

The fraction $11/10$ is equal to 1.1, so the angle is 1.1π, which is in the third quadrant.

POWERS OF COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

■ 1. Find z^5 in polar form.

$$z = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Solution:

Plug $r = 2$, $\theta = \pi/12$, and $n = 5$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^5 = 2^5\left[\cos\left(5 \cdot \frac{\pi}{12}\right) + i\sin\left(5 \cdot \frac{\pi}{12}\right)\right]$$

$$z^5 = 32\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

■ 2. Find z^7 in polar form.

$$z = \sqrt{5}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$$

Solution:

Plug $r = \sqrt{5}$, $\theta = 2\pi/5$, and $n = 7$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^7 = (\sqrt{5})^7\left[\cos\left(7 \cdot \frac{2\pi}{5}\right) + i\sin\left(7 \cdot \frac{2\pi}{5}\right)\right]$$

$$z^7 = 125\sqrt{5}\left(\cos\frac{14\pi}{5} + i\sin\frac{14\pi}{5}\right)$$

We could leave the answer this way, but the angle $14\pi/5$ is larger than 2π, so we can reduce the angle to one that's coterminal with $14\pi/5$, but within the interval $[0,2\pi)$.

$$\frac{14\pi}{5} - 2\pi = \frac{14\pi}{5} - 2\pi\left(\frac{5}{5}\right) = \frac{14\pi}{5} - \frac{10\pi}{5} = \frac{4\pi}{5}$$

So the complex number z^4 in polar form can be written as

$$z^7 = 125\sqrt{5}\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$$

■ 3. Find z^6 in rectangular form $a + bi$.

$$z = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Solution:

Plug $r = \sqrt{2}/2$, $\theta = \pi/8$, and $n = 6$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^6 = \left(\frac{\sqrt{2}}{2}\right)^6 \left[\cos\left(6 \cdot \frac{\pi}{8}\right) + i\sin\left(6 \cdot \frac{\pi}{8}\right)\right]$$

$$z^6 = \frac{1}{8}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$z^6 = \frac{1}{8}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$z^6 = \frac{1}{8}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{8}\left(\frac{\sqrt{2}}{2}i\right)$$

$$z^6 = -\frac{\sqrt{2}}{16} + \frac{\sqrt{2}}{16}i$$

■ 4. Find z^3 in rectangular form $a + bi$.

$$z = 2\sqrt{6}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Solution:

Plug $r = 2\sqrt{6}$, $\theta = 5\pi/3$, and $n = 3$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^3 = (2\sqrt{6})^3\left[\cos\left(3\cdot\frac{5\pi}{3}\right) + i\sin\left(3\cdot\frac{5\pi}{3}\right)\right]$$

$$z^3 = 8\cdot 6\sqrt{6}(\cos 5\pi + i\sin 5\pi)$$

$$z^3 = 48\sqrt{6}(\cos 5\pi + i\sin 5\pi)$$

$$z^3 = 48\sqrt{6}(-1 + i(0))$$

$$z^3 = -48\sqrt{6}$$

■ 5. Find z^5 in polar form.

$$z = -4 - 4i$$

Solution:

First, convert $z = -4 - 4i$ to polar form by finding the modulus $|z|$ and the angle θ. The distance from the origin is

$$|z| = r = \sqrt{a^2 + b^2}$$

$$|z| = r = \sqrt{(-4)^2 + (-4)^2}$$

$$|z| = r = \sqrt{16 + 16}$$

$$|z| = r = \sqrt{32}$$

$$|z| = r = 4\sqrt{2}$$

and the angle is

$$\theta = \arctan\frac{b}{a} = \arctan\frac{-4}{-4} = \arctan(1) = \frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

Because the complex number $z = -4 - 4i$ is in quadrant III, we use $\theta = 5\pi/4$. Then $z = -4 - 4i$ in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

To find z^5, plug $r = 4\sqrt{2}$, $\theta = 5\pi/4$, and $n = 5$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^5 = (4\sqrt{2})^5\left[\cos\left(5\cdot\frac{5\pi}{4}\right) + i\sin\left(5\cdot\frac{5\pi}{4}\right)\right]$$

$$z^5 = 4{,}096\sqrt{2}\left(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}\right)$$

We could leave the answer this way, but the angle $25\pi/4$ is larger than 2π, so we can reduce the angle to one that's coterminal with $25\pi/4$, but within the interval $[0,2\pi)$.

$$\frac{25\pi}{4} - 3(2\pi) = \frac{25\pi}{4} - 6\pi\left(\frac{4}{4}\right) = \frac{25\pi}{4} - \frac{24\pi}{4} = \frac{\pi}{4}$$

So the complex number z^5 in polar form can be written as

$$z^5 = 4{,}096\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

■ 6. Find z^4 in rectangular form $a + bi$.

$$z = \sqrt{6} - \sqrt{2}i$$

Solution:

First, convert $z = \sqrt{6} - \sqrt{2}i$ to polar form by finding the modulus $|z|$ and the angle θ. The distance from the origin is

$$|z| = r = \sqrt{a^2 + b^2}$$

$$|z| = r = \sqrt{(\sqrt{6})^2 + (-\sqrt{2})^2}$$

$$|z| = r = \sqrt{6 + 2}$$

$$|z| = r = \sqrt{8}$$

$$|z| = r = 2\sqrt{2}$$

and the angle is

$$\theta = \arctan\frac{b}{a} = \arctan\frac{-\sqrt{2}}{\sqrt{6}} = \arctan\frac{-\sqrt{3}}{3} = -\frac{\pi}{6}$$

Then $z = \sqrt{6} - \sqrt{2}i$ in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

To find z^4, plug $r = 2\sqrt{2}$, $\theta = -\pi/6$, and $n = 4$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^4 = (2\sqrt{2})^4\left[\cos\left(4 \cdot -\frac{\pi}{6}\right) + i\sin\left(4 \cdot -\frac{\pi}{6}\right)\right]$$

$$z^4 = 64\left[\cos\left(-\frac{4\pi}{6}\right) + i\sin\left(-\frac{4\pi}{6}\right)\right]$$

$$z^4 = 64\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$$

$$z^4 = 64\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$z^4 = 64\left(-\frac{1}{2}\right) + 64\left(-\frac{\sqrt{3}}{2}i\right)$$

$$z^4 = -32 - 32\sqrt{3}i$$

COMPLEX NUMBER EQUATIONS

■ 1. Find the solutions of the complex equation.

$$z^2 = 49$$

Solution:

Rewrite z^2 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^2 = r^2[\cos(2\theta) + i\sin(2\theta)]$$

Rewrite 49 as the complex number $49 + 0i$. The modulus and angle of $49 + 0i$ are

$$r = \sqrt{49^2 + 0^2}$$

$$r = \sqrt{49^2}$$

$$r = 49$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{49} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 49[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots)]$$

$$z = 49[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 49[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^2 = 49$, we can start making substitutions.

$$z^2 = 49$$

$$r^2[\cos(2\theta) + i\sin(2\theta)] = 49$$

$$r^2[\cos(2\theta) + i\sin(2\theta)] = 49[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^2 = 49$$

$$2\theta = 360^\circ k$$

From these equations, we get

$r^2 = 49$, so $r = 7$

$2\theta = 360^\circ k$, so $\theta = 180^\circ k$

To $\theta = 180^\circ k$, if we plug in $k = 0,\ 1,\ \ldots$, we get

For $k = 0$, $\theta = 180^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 180^\circ(1) = 180^\circ$

...

We could keep going for $k = 2, 3, 4, 5, \ldots$, but $k = 2$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ,\ 180^\circ$.

Plugging these two angles and $r = 7$ into the formula for polar form of a complex number, we'll get the solutions to $z^2 = 49$.

$$z_1 = 7[\cos(0^\circ) + i\sin(0^\circ)] = 7[1 + i(0)] = 7$$

$$z_2 = 7[\cos(180^\circ) + i\sin(180^\circ)] = 7[-1 + i(0)] = -7$$

■ 2. Find the solution of the complex equation that lies in the third quadrant.

$$z^3 = 216$$

Solution:

Rewrite z^3 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^3 = r^3[\cos(3\theta) + i\sin(3\theta)]$$

Rewrite 216 as the complex number $216 + 0i$. The modulus and angle of $216 + 0i$ are

$$r = \sqrt{216^2 + 0^2}$$

$$r = \sqrt{216^2}$$

$$r = 216$$

and

$$\theta = \arctan \frac{b}{a} = \arctan \frac{0}{216} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 216[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 216[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 216[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^3 = 216$, we can start making substitutions.

$$z^3 = 216$$

$$r^3[\cos(3\theta) + i\sin(3\theta)] = 216$$

$$r^3[\cos(3\theta) + i\sin(3\theta)] = 216[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we’ve got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^3 = 216$$

$$3\theta = 360^\circ k$$

From these equations, we get

$r^3 = 216$, so $r = 6$

$3\theta = 360^\circ k$, so $\theta = 120^\circ k$

To $\theta = 120^\circ k$, if we plug in $k = 0, 1, 2, \ldots$, we get

For $k = 0$, $\theta = 120^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 120^\circ(1) = 120^\circ$

For $k = 2$, $\theta = 120^\circ(2) = 240^\circ$

...

We could keep going for $k = 3, 4, 5, 6, \ldots$, but $k = 3$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 120^\circ, 240^\circ$.

Plugging these three angles and $r = 6$ into the formula for polar form of a complex number, we'll get the solutions to $z^3 = 216$.

$$z_1 = 6[\cos(0^\circ) + i\sin(0^\circ)] = 6[1 + i(0)] = 6$$

$$z_2 = 6[\cos(120^\circ) + i\sin(120^\circ)] = 6\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -3 + 3\sqrt{3}i$$

$$z_3 = 6[\cos(240^\circ) + i\sin(240^\circ)] = 6\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -3 - 3\sqrt{3}i$$

Roots in the third quadrant will have a negative real part and a negative imaginary part. So, z_3 is the solution in the third quadrant.

■ 3. Find the solutions of the complex equation.

$$z^4 = 256$$

Solution:

Rewrite z^4 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^4 = r^4[\cos(4\theta) + i\sin(4\theta)]$$

Rewrite 256 as the complex number $256 + 0i$. The modulus and angle of $256 + 0i$ are

$$r = \sqrt{256^2 + 0^2}$$

$$r = \sqrt{256^2}$$

$$r = 256$$

and

$$\theta = \arctan \frac{b}{a} = \arctan \frac{0}{256} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 256[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i \sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 256[\cos(2\pi k) + i \sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 256[\cos(360^\circ k) + i \sin(360^\circ k)]$$

Starting again with $z^4 = 256$, we can start making substitutions.

$$z^4 = 256$$

$$r^4[\cos(4\theta) + i \sin(4\theta)] = 256$$

$$r^4 \left[\cos(4\theta) + i \sin(4\theta)\right] = 256[\cos(360^\circ k) + i \sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^4 = 256$$

$$4\theta = 360^\circ k$$

From these equations, we get

$r^4 = 256$, so $r = 4$

$4\theta = 360^\circ k$, so $\theta = 90^\circ k$

To $\theta = 90^\circ k$, if we plug in $k = 0, 1, 2, 3, \ldots$, we get

For $k = 0$, $\theta = 90^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 90^\circ(1) = 90^\circ$

For $k = 2$, $\theta = 90^\circ(2) = 180^\circ$

For $k = 3$, $\theta = 90^\circ(3) = 270^\circ$

...

We could keep going for $k = 4, 5, 6, 7, \ldots$, but $k = 4$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ$.

Plugging these four angles and $r = 4$ into the formula for polar form of a complex number, we'll get the solutions to $z^4 = 256$.

$$z_1 = 4[\cos(0^\circ) + i\sin(0^\circ)] = 4[1 + i(0)] = 4$$

$$z_2 = 4[\cos(90^\circ) + i\sin(90^\circ)] = 4[0 + i(1)] = 4i$$

$$z_3 = 4[\cos(180^\circ) + i\sin(180^\circ)] = 4[-1 + i(0)] = -4$$

$$z_4 = 4[\cos(270^\circ) + i\sin(270^\circ)] = 4[0 + i(-1)] = -4i$$

■ 4. Find the solutions of the complex equation.

$$z^6 = 729$$

Solution:

Rewrite z^6 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^6 = r^6[\cos(6\theta) + i\sin(6\theta)]$$

Rewrite 729 as the complex number $729 + 0i$. The modulus and angle of $729 + 0i$ are

$$r = \sqrt{729^2 + 0^2}$$

$$r = \sqrt{729^2}$$

$$r = 729$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{729} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 729[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 729[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 729[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^6 = 729$, we can start making substitutions.

$$z^6 = 729$$

$$r^6[\cos(6\theta) + i\sin(6\theta)] = 729$$

$$r^6[\cos(6\theta) + i\sin(6\theta)] = 729[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^6 = 729$$

$$6\theta = 360^\circ k$$

From these equations, we get

$r^6 = 729$, so $r = 3$

$6\theta = 360^\circ k$, so $\theta = 60^\circ k$

To $\theta = 60^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, \ldots$, we get

For $k = 0$, $\theta = 60^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 60^\circ(1) = 60^\circ$

For $k = 2$, $\theta = 60^\circ(2) = 120^\circ$

For $k = 3$, $\theta = 60^\circ(3) = 180^\circ$

For $k = 4$, $\theta = 60^\circ(4) = 240^\circ$

For $k = 5$, $\theta = 60^\circ(5) = 300^\circ$

...

We could keep going for $k = 6, 7, 8, 9, \ldots$, but $k = 6$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$.

Plugging these six angles and $r = 3$ into the formula for polar form of a complex number, we'll get the solutions to $z^6 = 729$.

$$z_1 = 3[\cos(0^\circ) + i\sin(0^\circ)] = 3[1 + i(0)] = 3$$

$$z_2 = 3[\cos(60^\circ) + i\sin(60^\circ)] = 3\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$z_3 = 3[\cos(120^\circ) + i\sin(120^\circ)] = 3\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$z_4 = 3[\cos(180^\circ) + i\sin(180^\circ)] = 3\left[-1 + i(0)\right] = -3$$

$$z_5 = 3[\cos(240^\circ) + i\sin(240^\circ)] = 3\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

$$z_6 = 3[\cos(300^\circ) + i\sin(300^\circ)] = 3\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

■ 5. Find the solutions of the complex equation.

$$z^5 = 32$$

Solution:

Rewrite z^5 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^5 = r^5[\cos(5\theta) + i\sin(5\theta)]$$

Rewrite 32 as the complex number $32 + 0i$. The modulus and angle of $32 + 0i$ are

$$r = \sqrt{32^2 + 0^2}$$

$$r = \sqrt{32^2}$$

$$r = 32$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{32} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 32[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 32[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 32[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^5 = 32$, we can start making substitutions.

$$z^5 = 32$$

$$r^5[\cos(5\theta) + i\sin(5\theta)] = 32$$

$$r^5[\cos(5\theta) + i\sin(5\theta)] = 32[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^5 = 32$$

$$5\theta = 360^\circ k$$

From these equations, we get

$r^5 = 32$, so $r = 2$

$5\theta = 360^\circ k$, so $\theta = 72^\circ k$

To $\theta = 72^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, \ldots$, we get

For $k = 0$, $\theta = 72^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 72^\circ(1) = 72^\circ$

For $k = 2$, $\theta = 72^\circ(2) = 144^\circ$

For $k = 3$, $\theta = 72^\circ(3) = 216^\circ$

For $k = 4$, $\theta = 72^\circ(4) = 288^\circ$

...

We could keep going for $k = 5, 6, 7, 8, 9, \ldots$, but $k = 5$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 72^\circ, 144^\circ, 216^\circ, 288^\circ$.

Plugging these five angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^5 = 32$. Because these angles are not on the unit circle, we will need to use decimal approximations.

$$z_1 = 2[\cos(0^\circ) + i\sin(0^\circ)] = 2[1 + i(0)] = 2$$

$$z_2 \approx 2[\cos(72^\circ) + i\sin(72^\circ)] \approx 2[0.309 + 0.951i] \approx 0.618 + 1.902i$$

$$z_3 \approx 2[\cos(144^\circ) + i\sin(144^\circ)] \approx 2[-0.809 + 0.588i] \approx -1.618 + 1.176i$$

$$z_4 \approx 2[\cos(216^\circ) + i\sin(216^\circ)] \approx 2[-0.809 - 0.588i] \approx -1.618 - 1.176i$$

$$z_5 \approx 2[\cos(288^\circ) + i\sin(288^\circ)] \approx 2[0.309 - 0.951i] \approx 0.618 - 1.902i$$

■ 6. How many solutions of the complex equation lie in the second quadrant?

$$z^8 = 256$$

Solution:

Rewrite z^8 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^8 = r^8[\cos(8\theta) + i\sin(8\theta)]$$

Rewrite 256 as the complex number $256 + 0i$. The modulus and angle of $256 + 0i$ are

$$r = \sqrt{256^2 + 0^2}$$

$$r = \sqrt{256^2}$$

$$r = 256$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{256} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 256[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots)]$$

$$z = 256[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 256[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^8 = 256$, we can start making substitutions.

$$z^8 = 256$$

$$r^8[\cos(8\theta) + i\sin(8\theta)] = 256$$

$$r^8[\cos(8\theta) + i\sin(8\theta)] = 256[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^8 = 256$$

$$8\theta = 360^\circ k$$

From these equations, we get

$r^8 = 256$, so $r = 2$

$8\theta = 360^\circ k$, so $\theta = 45^\circ k$

To $\theta = 45^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, 6, 7, ...,$ we get

For $k = 0$, $\theta = 45^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 45^\circ(1) = 45^\circ$

For $k = 2$, $\theta = 45^\circ(2) = 90^\circ$

For $k = 3$, $\theta = 45^\circ(3) = 135^\circ$

For $k = 4$, $\theta = 45^\circ(4) = 180^\circ$

For $k = 5$, $\theta = 45^\circ(5) = 225^\circ$

For $k = 6$, $\theta = 45^\circ(6) = 270^\circ$

For $k = 7$, $\theta = 45^\circ(7) = 315^\circ$

...

We could keep going for $k=8, 9, 10, 11, \ldots$, but $k=8$ gives 360°, which is coterminal with the 0° value we already found for $k=0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ, 225^\circ, 270^\circ, 315^\circ$.

Plugging these eight angles and $r=2$ into the formula for polar form of a complex number, we'll get the solutions to $z^8=256$.

$$z_1 = 2[\cos(0^\circ) + i\sin(0^\circ)] = 2[1 + i(0)] = 2$$

$$z_2 = 2[\cos(45^\circ) + i\sin(45^\circ)] = 2\left[\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right] = \sqrt{2} + \sqrt{2}i$$

$$z_3 = 2[\cos(90^\circ) + i\sin(90^\circ)] = 2\left[0 + i(1)\right] = 2i$$

$$z_4 = 2[\cos(135^\circ) + i\sin(135^\circ)] = 2\left[-\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right] = -\sqrt{2} + \sqrt{2}i$$

$$z_5 = 2[\cos(180^\circ) + i\sin(180^\circ)] = 2\left[-1 + i(0)\right] = -2$$

$$z_6 = 2[\cos(225^\circ) + i\sin(225^\circ)] = 2\left[-\frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2}\right)\right] = -\sqrt{2} - \sqrt{2}i$$

$$z_7 = 2[\cos(270^\circ) + i\sin(270^\circ)] = 2\left[0 + i(-1)\right] = -2i$$

$$z_8 = 2[\cos(315^\circ) + i\sin(315^\circ)] = 2\left[\frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2}\right)\right] = \sqrt{2} - \sqrt{2}i$$

Roots in the second quadrant will have a negative real part and a positive imaginary part. That's only z_4, so there's one solution in the second quadrant.

ROOTS OF COMPLEX NUMBERS

■ 1. Find the cube roots of the complex number.

$$z = 27\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Solution:

We're looking for the third (or cube) roots of z, which means there will be 3 of them, given by $k = 0,\ 1,\ 2$. And since the complex number is given in radians, we'll plug $n = 3$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[3]{z} = \sqrt[3]{r}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$$

With $r = 27$ and $\theta = \pi/4$ from the complex number, we get

$$\sqrt[3]{z} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right)\right]$$

Now we'll find values for $k = 0,\ 1,\ 2$.

For $k = 0$:

$$\sqrt[3]{z}_{k=0} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(0)}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\pi(0)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=0} = 3\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

For $k=1$:

$$\sqrt[3]{z}_{k=1} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(1)}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\pi(1)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=1} = 3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

For $k=2$:

$$\sqrt[3]{z}_{k=2} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(2)}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\pi(2)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=2} = 3\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)$$

The roots are

$$\sqrt[3]{z}_{k=0} = 3\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

$$\sqrt[3]{z}_{k=1} = 3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$\sqrt[3]{z}_{k=2} = 3\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)$$

■ 2. Find the 4th roots of the complex number.

$$z = 256(\cos 60^\circ + i \sin 60^\circ)$$

Solution:

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k = 0, 1, 2, 3$. And since the complex number is given in degrees, we'll plug $n = 4$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i \sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[4]{z} = \sqrt[4]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{4}\right) + i \sin\left(\frac{\theta + 360^\circ k}{4}\right)\right]$$

With $r = 256$ and $\theta = 60^\circ$ from the complex number, we get

$$\sqrt[4]{z} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ k}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ k}{4}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3$.

For $k = 0$:

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(0)}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0} = 4[\cos(15^\circ) + i\sin(15^\circ)]$$

For $k = 1$:

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(1)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = 4[\cos(105^\circ) + i\sin(105^\circ)]$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(2)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = 4[\cos(195^\circ) + i\sin(195^\circ)]$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(3)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = 4[\cos(285^\circ) + i\sin(285^\circ)]$$

The roots are

$$\sqrt[4]{z}_{k=0} = 4[\cos(15^\circ) + i\sin(15^\circ)]$$

$$\sqrt[4]{z}_{k=1} = 4[\cos(105^\circ) + i\sin(105^\circ)]$$

$$\sqrt[4]{z}_{k=2} = 4[\cos(195^\circ) + i\sin(195^\circ)]$$

$$\sqrt[4]{z}_{k=3} = 4[\cos(285^\circ) + i\sin(285^\circ)]$$

■ 3. Find the 5th roots of the complex number that lies in the first quadrant of the complex plane.

$$z = 25(\cos 80^\circ + i\sin 80^\circ)$$

Solution:

We're looking for the 5th roots of z, which means there will be 5 of them, given by $k = 0, 1, 2, 3, 4$. And since the complex number is given in degrees, we'll plug $n = 5$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[5]{z} = \sqrt[5]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{5}\right) + i\sin\left(\frac{\theta + 360^\circ k}{5}\right)\right]$$

With $r = 25$ and $\theta = 80^\circ$ from the complex number, we get

$$\sqrt[5]{z} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ k}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ k}{5}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4$.

For $k = 0$:

$$\sqrt[5]{z}_{k=0} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ(0)}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ(0)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=0} = \sqrt[5]{25}[\cos(16^\circ) + i\sin(16^\circ)]$$

For $k = 1$:

$$\sqrt[5]{z}_{k=1} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ(1)}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ(1)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=1} = \sqrt[5]{25}[\cos(88^\circ) + i\sin(88^\circ)]$$

For $k = 2$:

$$\sqrt[5]{z}_{k=2} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ(2)}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ(2)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=2} = \sqrt[5]{25}[\cos(160^\circ) + i\sin(160^\circ)]$$

For $k = 3$:

$$\sqrt[5]{z}_{k=3} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ(3)}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ(3)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=3} = \sqrt[5]{25}[\cos(232^\circ) + i\sin(232^\circ)]$$

For $k = 4$:

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ(4)}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ(4)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}[\cos(304^\circ) + i\sin(304^\circ)]$$

The roots are

$$\sqrt[5]{z}_{k=0} = \sqrt[5]{25}[\cos(16^\circ) + i\sin(16^\circ)]$$

$$\sqrt[5]{z}_{k=1} = \sqrt[5]{25}[\cos(88^\circ) + i\sin(88^\circ)]$$

$$\sqrt[5]{z}_{k=2} = \sqrt[5]{25}[\cos(160^\circ) + i\sin(160^\circ)]$$

$$\sqrt[5]{z}_{k=3} = \sqrt[5]{25}[\cos(232^\circ) + i\sin(232^\circ)]$$

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}[\cos(304^\circ) + i\sin(304^\circ)]$$

Anything in the first quadrant will fall in the interval $(0^\circ,90^\circ)$. In this case, the angles for $k = 0$ and $k = 1$ are in the first quadrant.

■ 4. Find the 4th roots of the complex number.

$$z = 34\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$

Solution:

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k = 0, 1, 2, 3$. And since the complex number is given in radians, we'll plug $n = 4$ into the formula for nth roots in radians.

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

$$\sqrt[4]{z}=\sqrt[4]{r}\left[\cos\left(\frac{\theta+2\pi k}{4}\right)+i\sin\left(\frac{\theta+2\pi k}{4}\right)\right]$$

With $r=34$ and $\theta=3\pi/5$ from the complex number, we get

$$\sqrt[4]{z}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi k}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi k}{4}\right)\right]$$

Now we'll find values for $k=0,\ 1,\ 2,\ 3.$

For $k=0$:

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi(0)}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{34}\left(\cos\frac{3\pi}{20}+i\sin\frac{3\pi}{20}\right)$$

For $k=1$:

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi(1)}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{34}\left(\cos\frac{13\pi}{20}+i\sin\frac{13\pi}{20}\right)$$

For $k=2$:

$$\sqrt[4]{z}_{k=2}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi(2)}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2}=\sqrt[4]{34}\left(\cos\frac{23\pi}{20}+i\sin\frac{23\pi}{20}\right)$$

For $k=3$:

$$\sqrt[4]{z}_{k=3}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi(3)}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3}=\sqrt[4]{34}\left(\cos\frac{33\pi}{20}+i\sin\frac{33\pi}{20}\right)$$

The roots are

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{34}\left(\cos\frac{3\pi}{20}+i\sin\frac{3\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{34}\left(\cos\frac{13\pi}{20}+i\sin\frac{13\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=2}=\sqrt[4]{34}\left(\cos\frac{23\pi}{20}+i\sin\frac{23\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=3}=\sqrt[4]{34}\left(\cos\frac{33\pi}{20}+i\sin\frac{33\pi}{20}\right)$$

■ 5. Find the 6th roots of the complex number that lies in the second quadrant of the complex plane.

$$z = 11\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Solution:

We're looking for the 6th roots of z, which means there will be 6 of them, given by $k = 0, 1, 2, 3, 4, 5$. And since the complex number is given in radians, we'll plug $n = 6$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[6]{z} = \sqrt[6]{r}\left[\cos\left(\frac{\theta + 2\pi k}{6}\right) + i\sin\left(\frac{\theta + 2\pi k}{6}\right)\right]$$

With $r = 11$ and $\theta = 5\pi/6$ from the complex number, we get

$$\sqrt[6]{z} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi k}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi k}{6}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5$.

For $k = 0$:

$$\sqrt[6]{z}_{k=0} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi(0)}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi(0)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=0} = \sqrt[6]{11}\left(\cos\frac{5\pi}{36} + i\sin\frac{5\pi}{36}\right)$$

For $k = 1$:

$$\sqrt[6]{z}_{k=1} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi(1)}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi(1)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=1} = \sqrt[6]{11}\left(\cos\frac{17\pi}{36} + i\sin\frac{17\pi}{36}\right)$$

For $k = 2$:

$$\sqrt[6]{z}_{k=2} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi(2)}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi(2)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=2} = \sqrt[6]{11}\left(\cos\frac{29\pi}{36} + i\sin\frac{29\pi}{36}\right)$$

We can start to see how we're just adding $12\pi/36$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[6]{z}_{k=0} = \sqrt[6]{11}\left(\cos\frac{5\pi}{36} + i\sin\frac{5\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=1} = \sqrt[6]{11}\left(\cos\frac{17\pi}{36} + i\sin\frac{17\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=2} = \sqrt[6]{11}\left(\cos\frac{29\pi}{36} + i\sin\frac{29\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=3} = \sqrt[6]{11}\left(\cos\frac{41\pi}{36} + i\sin\frac{41\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=4} = \sqrt[6]{11}\left(\cos\frac{53\pi}{36} + i\sin\frac{53\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=5} = \sqrt[6]{11}\left(\cos\frac{65\pi}{36} + i\sin\frac{65\pi}{36}\right)$$

If we find the decimal approximations of these angles, we get

For $k = 0$, $(5/36)\pi \approx 0.14\pi$

For $k = 1$, $(17/36)\pi \approx 0.47\pi$

For $k = 2$, $(29/36)\pi \approx 0.81\pi$

For $k = 3$, $(41/36)\pi \approx 1.14\pi$

For $k = 4$, $(53/36)\pi \approx 1.47\pi$

For $k = 5$, $(65/36)\pi \approx 1.81\pi$

Anything in the second quadrant will fall in the interval $(0.5\pi, 1.0\pi)$, which in this case is the angle for $k = 2$.

■ 6. Find the 7th roots of the complex number.

$$z = 20(\cos 120^\circ + i \sin 120^\circ)$$

Solution:

We're looking for the 7th roots of z, which means there will be 7 of them, given by $k = 0, 1, 2, 3, 4, 5, 6$. And since the complex number is given in degrees, we'll plug $n = 7$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i \sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[7]{z} = \sqrt[7]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{7}\right) + i \sin\left(\frac{\theta + 360^\circ k}{7}\right)\right]$$

With $r = 20$ and $\theta = 120^\circ$ from the complex number, we get

$$\sqrt[7]{z} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ k}{7}\right) + i \sin\left(\frac{120^\circ + 360^\circ k}{7}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5, 6$.

For $k = 0$:

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ(0)}{7}\right) + i \sin\left(\frac{120^\circ + 360^\circ(0)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{20}\left[\cos\left(\frac{120}{7}\right)^{\circ}+i\sin\left(\frac{120}{7}\right)^{\circ}\right]$$

For $k=1$:

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{20}\left[\cos\left(\frac{120^{\circ}+360^{\circ}(1)}{7}\right)+i\sin\left(\frac{120^{\circ}+360^{\circ}(1)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{20}\left[\cos\left(\frac{480}{7}\right)^{\circ}+i\sin\left(\frac{480}{7}\right)^{\circ}\right]$$

For $k=2$:

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{20}\left[\cos\left(\frac{120^{\circ}+360^{\circ}(2)}{7}\right)+i\sin\left(\frac{120^{\circ}+360^{\circ}(2)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{20}\left[\cos\left(\frac{840}{7}\right)^{\circ}+i\sin\left(\frac{840}{7}\right)^{\circ}\right]$$

We can start to see how we're just adding $360/7$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{20}\left[\cos\left(\frac{120}{7}\right)^{\circ}+i\sin\left(\frac{120}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{20}\left[\cos\left(\frac{480}{7}\right)^{\circ}+i\sin\left(\frac{480}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{20}\left[\cos\left(\frac{840}{7}\right)^\circ+i\sin\left(\frac{840}{7}\right)^\circ\right]$$

$$=\sqrt[7]{20}[\cos(120)^\circ+i\sin(120)^\circ]$$

$$\sqrt[7]{z}_{k=3}=\sqrt[7]{20}\left[\cos\left(\frac{1{,}200}{7}\right)^\circ+i\sin\left(\frac{1{,}200}{7}\right)^\circ\right]$$

$$\sqrt[7]{z}_{k=4}=\sqrt[7]{20}\left[\cos\left(\frac{1{,}560}{7}\right)^\circ+i\sin\left(\frac{1{,}560}{7}\right)^\circ\right]$$

$$\sqrt[7]{z}_{k=5}=\sqrt[7]{20}\left[\cos\left(\frac{1{,}920}{7}\right)^\circ+i\sin\left(\frac{1{,}920}{7}\right)^\circ\right]$$

$$\sqrt[7]{z}_{k=6}=\sqrt[7]{20}\left[\cos\left(\frac{2{,}280}{7}\right)^\circ+i\sin\left(\frac{2{,}280}{7}\right)^\circ\right]$$

W W W . K R I S T A K I N G M A T H . C O M