Calculus 1 Workbook Solutions

Solving limits

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SOLVING WITH SUBSTITUTION

1. What is the value of the limit?

$$\lim_{x\to 3}(-x^4 + x^3 + 2x^2)$$

Solution:

Use substitution and plug $x = 3$ into the function.

$$\lim_{x\to 3}(-x^4 + x^3 + 2x^2)$$

$$-(3)^4 + (3)^3 + 2(3)^2$$

$$-36$$

2. What is the value of the limit?

$$\lim_{x\to 7}\frac{x^2 - 5}{x^2 + 5}$$

Solution:

Use substitution and plug $x = 7$ into the function.

$$\lim_{x \to 7} \frac{x^2 - 5}{x^2 + 5}$$

$$\frac{7^2 - 5}{7^2 + 5}$$

$$\frac{44}{54} = \frac{22}{27}$$

■ 3. What is the value of the limit.

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 4x - 6}{x^2 + 7x + 6}$$

Solution:

Use substitution and plug $x = -2$ into the function.

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 4x - 6}{x^2 + 7x + 6}$$

$$\frac{(-2)^3 - 5(-2)^2 + 4(-2) - 6}{(-2)^2 + 7(-2) + 6}$$

$$\frac{21}{2}$$

■ 4. Evaluate the limit.

$$\lim_{y \to -2} \frac{|y-5|}{y+1}$$

Solution:

Use substitution and plug $y = -2$ into the function.

$$\lim_{y \to -2} \frac{|y-5|}{y+1}$$

$$\frac{|-2-5|}{-2+1}$$

$$\frac{|-7|}{(-1)}$$

$$\frac{7}{(-1)}$$

$$-7$$

5. Evaluate the limit.

$$\lim_{x \to 2} \left(\sin\left(\frac{\pi x}{4}\right) + \ln\left(\frac{2e}{x}\right) \right)$$

Solution:

Use substitution and plug $x = 2$ into the function.

$$\lim_{x\to 2}\left(\sin\left(\frac{\pi x}{4}\right)+\ln\left(\frac{2e}{x}\right)\right)$$

$$\sin\left(\frac{2\pi}{4}\right)+\ln\left(\frac{2e}{2}\right)$$

$$\sin\left(\frac{\pi}{2}\right)+\ln e$$

$$1+1$$

$$2$$

■ 6. Evaluate the limits $\lim_{x\to -1} f(x)$ and $\lim_{x\to 2} f(x)$.

$$f(x)=\begin{cases} -3x+5 & x<-1 \\ \frac{1}{2}x^2-3x+1 & x\geq -1 \end{cases}$$

Solution:

Since -1 is the value where the definition of the function changes, the substitution rule is not valid. We inspect each one-sided limit.

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}(-3x+5)=-3(-1)+5=3+5=8$$

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+}\left(\frac{1}{2}x^2 - 3x + 1\right) = \frac{1}{2}(-1)^2 - 3(-1) + 1 = \frac{1}{2} + 3 + 1 = \frac{9}{2}$$

Since $\lim_{x\to-1^-} f(x) \neq \lim_{x\to-1^+} f(x)$, the general limit $\lim_{x\to-1} f(x)$ does not exist at $x = -1$.

To find the limit as x approaches 2, we'll substitute $x = 2$ into the second piece of the function, since that's the piece that defines the function when $x \geq -1$, which includes $x = 2$.

$$\lim_{x\to2} f(x) = \lim_{x\to2}\left(\frac{1}{2}x^2 - 3x + 1\right) = \frac{1}{2}(2)^2 - 3(2) + 1 = \frac{1}{2}(4) - 6 + 1 = 2 - 5 = -3$$

SOLVING WITH FACTORING

■ 1. What is the value of the limit?

$$\lim_{x \to -7} \frac{6x^3 + 42x^2}{2x^2 + 26x + 84}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to -7} \frac{6x^3 + 42x^2}{2x^2 + 26x + 84}$$

$$\lim_{x \to -7} \frac{6x^2(x + 7)}{2(x + 6)(x + 7)}$$

$$\lim_{x \to -7} \frac{6x^2}{2(x + 6)}$$

Now we can evaluate the limit at $x = -7$ using substitution.

$$\frac{6(-7)^2}{2\left(-7 + 6\right)}$$

$$-147$$

2. What is the value of the limit?

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}$$

$$\lim_{x\to 1}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}$$

Now we can evaluate the limit at $x=1$ using substitution.

$$\frac{1}{\sqrt[3]{1^2}+\sqrt[3]{1}+1}$$

$$\frac{1}{1+1+1}$$

$$\frac{1}{3}$$

3. What is the value of the limit?

$$\lim_{x \to 0} \frac{(x+3)^2 - 9}{x}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to 0} \frac{(x+3)^2 - 9}{x}$$

$$\lim_{x \to 0} \frac{x^2 + 6x + 9 - 9}{x}$$

$$\lim_{x \to 0} \frac{x^2 + 6x}{x}$$

$$\lim_{x \to 0} \frac{x(x+6)}{x}$$

$$\lim_{x \to 0} (x+6)$$

Now we can evaluate the limit at $x = 0$ using substitution.

$$0 + 6$$

$$6$$

4. What is the value of the limit?

$$\lim_{x \to 7} \frac{x^3 - x^2 - 42x}{2x^2 - 20x + 42}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to 7} \frac{x^3 - x^2 - 42x}{2x^2 - 20x + 42}$$

$$\lim_{x \to 7} \frac{x(x-7)(x+6)}{2(x-3)(x-7)}$$

$$\lim_{x \to 7} \frac{x(x+6)}{2(x-3)}$$

Now we can evaluate the limit at $x = 7$ using substitution.

$$\frac{7(7+6)}{2(7-3)}$$

$$\frac{91}{8}$$

5. What is the value of the limit?

$$\lim_{x \to 8} \frac{x^2 + 2x - 80}{2x^3 - 24x^2 + 64x}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to 8} \frac{x^2 + 2x - 80}{2x^3 - 24x^2 + 64x}$$

$$\lim_{x \to 8} \frac{(x + 10)(x - 8)}{2x(x - 8)(x - 4)}$$

$$\lim_{x \to 8} \frac{x + 10}{2x(x - 4)}$$

Now we can evaluate the limit at $x = 8$ using substitution.

$$\frac{8 + 10}{2(8)(8 - 4)}$$

$$\frac{18}{64} = \frac{9}{32}$$

■ 6. What is the value of the limit?

$$\lim_{x \to 0} \frac{1}{x} \left(1 - \frac{16}{(x - 4)^2} \right)$$

Solution:

If we use substitution, we get an undefined value. But we can factor the function instead.

$$\lim_{x \to 0} \frac{1}{x}\left(1 - \frac{16}{(x-4)^2}\right)$$

$$\lim_{x \to 0} \frac{1}{x}\left(\frac{(x-4)^2 - 16}{(x-4)^2}\right)$$

$$\lim_{x \to 0} \frac{1}{x}\left(\frac{x^2 - 8x + 16 - 16}{(x-4)^2}\right)$$

$$\lim_{x \to 0} \frac{1}{x}\left(\frac{x^2 - 8x}{(x-4)^2}\right)$$

$$\lim_{x \to 0} \frac{x-8}{(x-4)^2}$$

Now we can evaluate the limit at $x = 0$ using substitution.

$$\frac{0-8}{(0-4)^2}$$

$$\frac{-8}{(-4)^2}$$

$$\frac{-8}{16}$$

$$-\frac{1}{2}$$

SOLVING WITH CONJUGATE METHOD

1. Use conjugate method to evaluate the limit.

$$\lim_{x \to 16} \frac{3(x-16)}{\sqrt{x}-4}$$

Solution:

We'll apply conjugate method by multiplying both the numerator and denominator by the conjugate of the denominator.

$$\lim_{x \to 16} \frac{3(x-16)(\sqrt{x}+4)}{(\sqrt{x}-4)(\sqrt{x}+4)}$$

$$\lim_{x \to 16} \frac{3(x-16)(\sqrt{x}+4)}{x-16}$$

$$\lim_{x \to 16} 3(\sqrt{x}+4)$$

Then use substitution to evaluate the limit.

$$3(\sqrt{16}+4)$$

$$3(4+4)$$

$$24$$

■ 2. What is the value of the limit?

$$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \left(\frac{\sqrt{x+3} + \sqrt{3}}{\sqrt{x+3} + \sqrt{3}} \right)$$

$$\lim_{x \to 0} \frac{x + 3 - \sqrt{3}\sqrt{x+3} + \sqrt{3}\sqrt{x+3} - 3}{x(\sqrt{x+3} + \sqrt{3})}$$

$$\lim_{x \to 0} \frac{x + 3 - 3}{x(\sqrt{x+3} + \sqrt{3})}$$

$$\lim_{x \to 0} \frac{x}{x(\sqrt{x+3} + \sqrt{3})}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{x+3} + \sqrt{3}}$$

Then use substitution to evaluate the limit.

$$\frac{1}{\sqrt{0+3} + \sqrt{3}}$$

$$\frac{1}{\sqrt{3}+\sqrt{3}}$$

$$\frac{1}{2\sqrt{3}}$$

■ 3. What is the value of the limit?

$$\lim_{x \to 0} \frac{\sqrt{x^2+16}-4}{x^2}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x \to 0} \frac{\sqrt{x^2+16}-4}{x^2}$$

$$\lim_{x \to 0} \frac{\sqrt{x^2+16}-4}{x^2}\left(\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4}\right)$$

$$\lim_{x \to 0} \frac{(\sqrt{x^2+16})^2-4^2}{x^2(\sqrt{x^2+16}+4)}$$

$$\lim_{x \to 0} \frac{x^2+16-16}{x^2(\sqrt{x^2+16}+4)}$$

$$\lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2+16}+4)}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{x^2+16}+4}$$

Then use substitution to evaluate the limit.

$$\frac{1}{\sqrt{0^2+16}+4}$$

$$\frac{1}{\sqrt{16}+4}$$

$$\frac{1}{8}$$

■ 4. Use conjugate method to evaluate the limit.

$$\lim_{x \to 49} \frac{x-49}{3(\sqrt{x}-7)}$$

Solution:

We'll apply conjugate method by multiplying both the numerator and denominator by the conjugate of the denominator.

$$\lim_{x\to 49} \frac{(x-49)(\sqrt{x}+7)}{3(\sqrt{x}-7)(\sqrt{x}+7)}$$

$$\lim_{x\to 49} \frac{(x-49)(\sqrt{x}+7)}{3(x-49)}$$

$$\lim_{x\to 49} \frac{\sqrt{x}+7}{3}$$

Then use substitution to evaluate the limit.

$$\frac{\sqrt{49}+7}{3}$$

$$\frac{14}{3}$$

■ 5. What is the value of the limit?

$$\lim_{x\to 1} \frac{4-\sqrt{x+15}}{2(x-1)}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x\to 1} \frac{4-\sqrt{x+15}}{2(x-1)}$$

$$\lim_{x\to 1} \frac{4-\sqrt{x+15}}{2(x-1)}\left(\frac{4+\sqrt{x+15}}{4+\sqrt{x+15}}\right)$$

$$\lim_{x\to 1} \frac{4^2-(\sqrt{x+15})^2}{2(x-1)(4+\sqrt{x+15})}$$

$$\lim_{x\to 1} \frac{16-(x+15)}{2(x-1)(4+\sqrt{x+15})}$$

$$\lim_{x\to 1} \frac{1-x}{2(x-1)(4+\sqrt{x+15})}$$

$$\lim_{x\to 1} -\frac{1}{2(4+\sqrt{x+15})}$$

Then use substitution to evaluate the limit.

$$-\frac{1}{2(4+\sqrt{1+15})}$$

$$-\frac{1}{2(4+\sqrt{16})}$$

$$-\frac{1}{2(4+4)}$$

$$-\frac{1}{2(8)}$$

$$-\frac{1}{16}$$

■ 6. What is the value of the limit?

$$\lim_{x\to 2}\frac{\sqrt{11-x}-3}{\sqrt{6-x}-2}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, apply conjugate method, first using the conjugate of the numerator,

$$\lim_{x\to 2}\frac{\sqrt{11-x}-3}{\sqrt{6-x}-2}$$

$$\lim_{x\to 2}\frac{\sqrt{11-x}-3}{\sqrt{6-x}-2}\left(\frac{\sqrt{11-x}+3}{\sqrt{11-x}+3}\right)$$

$$\lim_{x\to 2}\frac{(\sqrt{11-x})^2-3^2}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{11-x-9}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{2-x}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}$$

and then the conjugate of the denominator.

$$\lim_{x\to 2}\frac{2-x}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}\left(\frac{(\sqrt{6-x}+2)}{(\sqrt{6-x}+2)}\right)$$

$$\lim_{x\to 2}\frac{(2-x)(\sqrt{6-x}+2)}{((\sqrt{6-x})^2-2^2)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{(2-x)(\sqrt{6-x}+2)}{(6-x-4)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{(2-x)(\sqrt{6-x}+2)}{(2-x)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{\sqrt{6-x}+2}{\sqrt{11-x}+3}$$

Then use substitution to evaluate the limit.

$$\frac{\sqrt{6-2}+2}{\sqrt{11-2}+3}$$

$$\frac{\sqrt{4}+2}{\sqrt{9}+3}$$

$$\frac{4}{6}$$

$$\frac{2}{3}$$

INFINITE LIMITS AND VERTICAL ASYMPTOTES

■ 1. What is the value of the limit?

$$\lim_{x \to 2} \frac{x^2 - x - 6}{-3x^2 - 3x + 18}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to 2} \frac{x^2 - x - 6}{-3x^2 - 3x + 18}$$

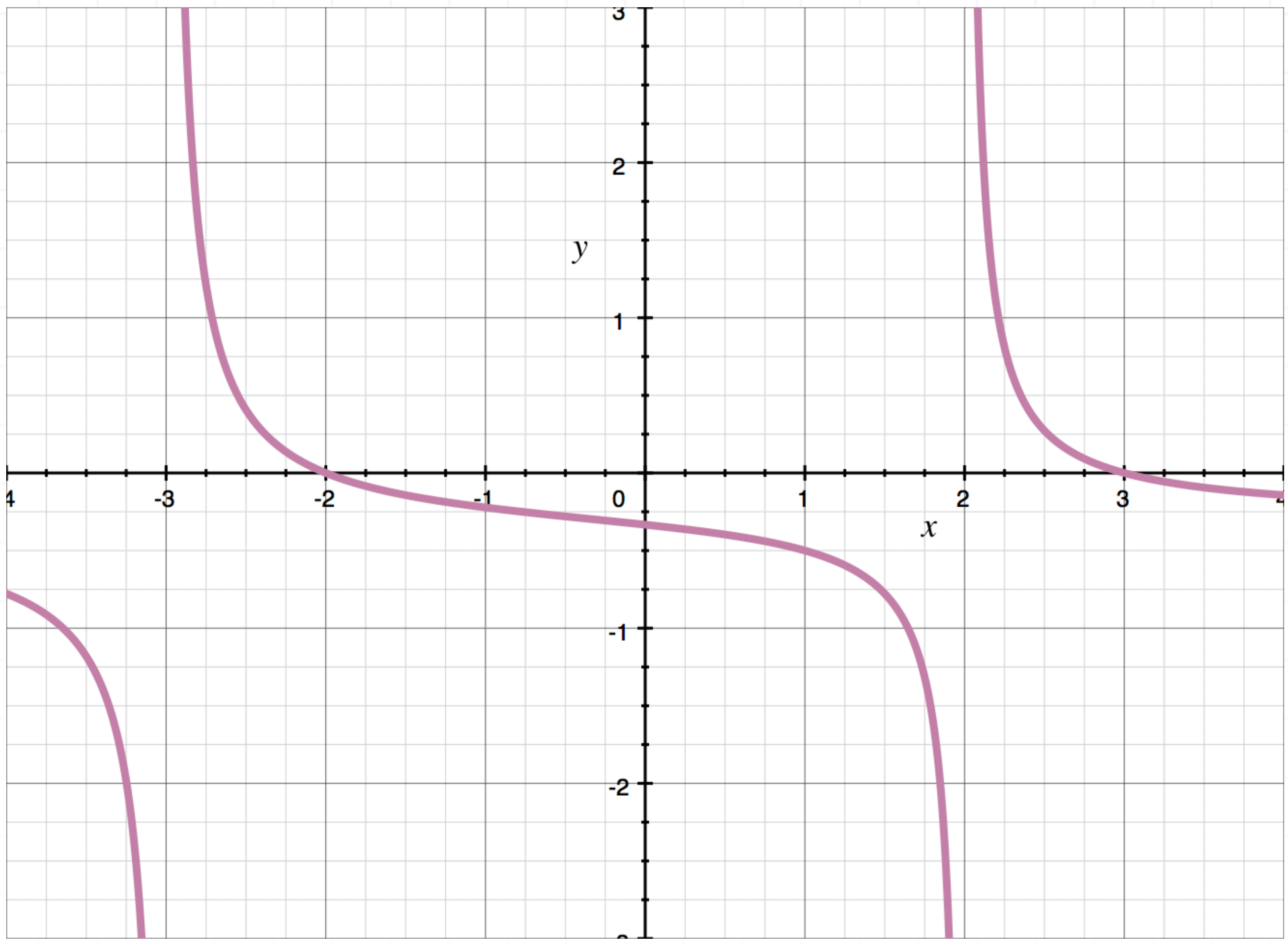
$$\lim_{x \to 2} \frac{(x-3)(x+2)}{-3(x+3)(x-2)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to 2^-} \frac{(x-3)(x+2)}{-3(x+3)(x-2)} = -\infty$$

$$\lim_{x \to 2^+} \frac{(x-3)(x+2)}{-3(x+3)(x-2)} = \infty$$

and they are not the same. Therefore, the limit does not exist (DNE). The graph is shown below.



2. What is the value of the limit?

$$\lim_{x\to -1}\frac{x^2+x-6}{4x^2+16x+12}$$

Solution:

Factor to simplify the limit.

$$\lim_{x\to -1}\frac{x^2+x-6}{4x^2+16x+12}$$

$$\lim_{x\to -1}\frac{(x+3)(x-2)}{4(x+3)(x+1)}$$

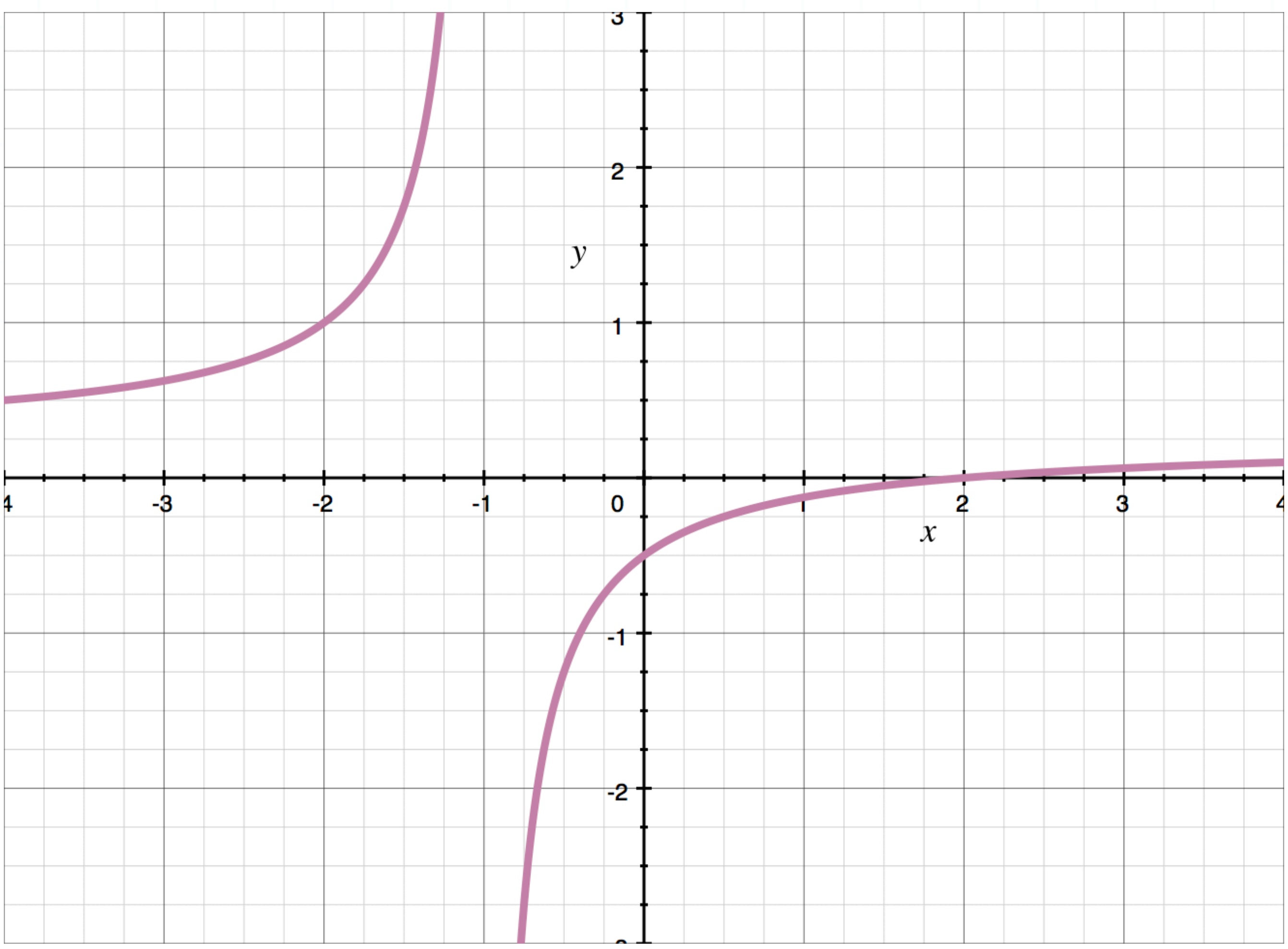
$$\lim_{x \to -1} \frac{x-2}{4(x+1)}$$

No other factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to -1^-} \frac{x-2}{4(x+1)} = \infty$$

$$\lim_{x \to -1^+} \frac{x-2}{4(x+1)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



■ 3. What is the value of the limit?

$$\lim_{x \to 3} \frac{1}{|x-3|}$$

Solution:

No factors can be cancelled. The left- and right-hand limits are

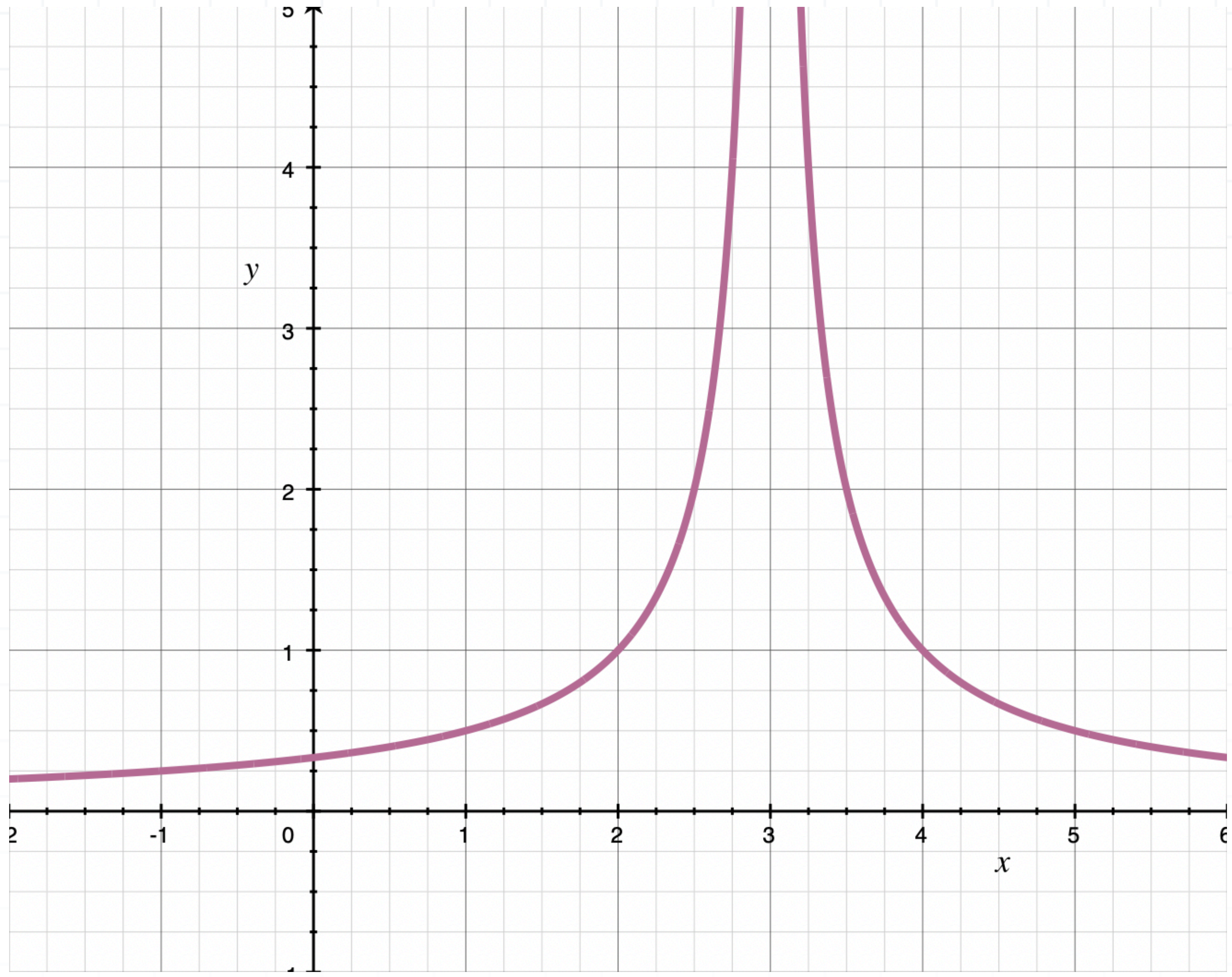
$$\lim_{x \to 3^-} \frac{1}{|x-3|} = \infty$$

$$\lim_{x \to 3^+} \frac{1}{|x-3|} = \infty$$

and they are the same. Therefore,

$$\lim_{x \to 3} \frac{1}{|x-3|} = \infty$$

The graph is shown below.



4. What is the value of the limit?

$$\lim_{x \to -4} \frac{\sqrt{x^2 - 1}}{x + 4}$$

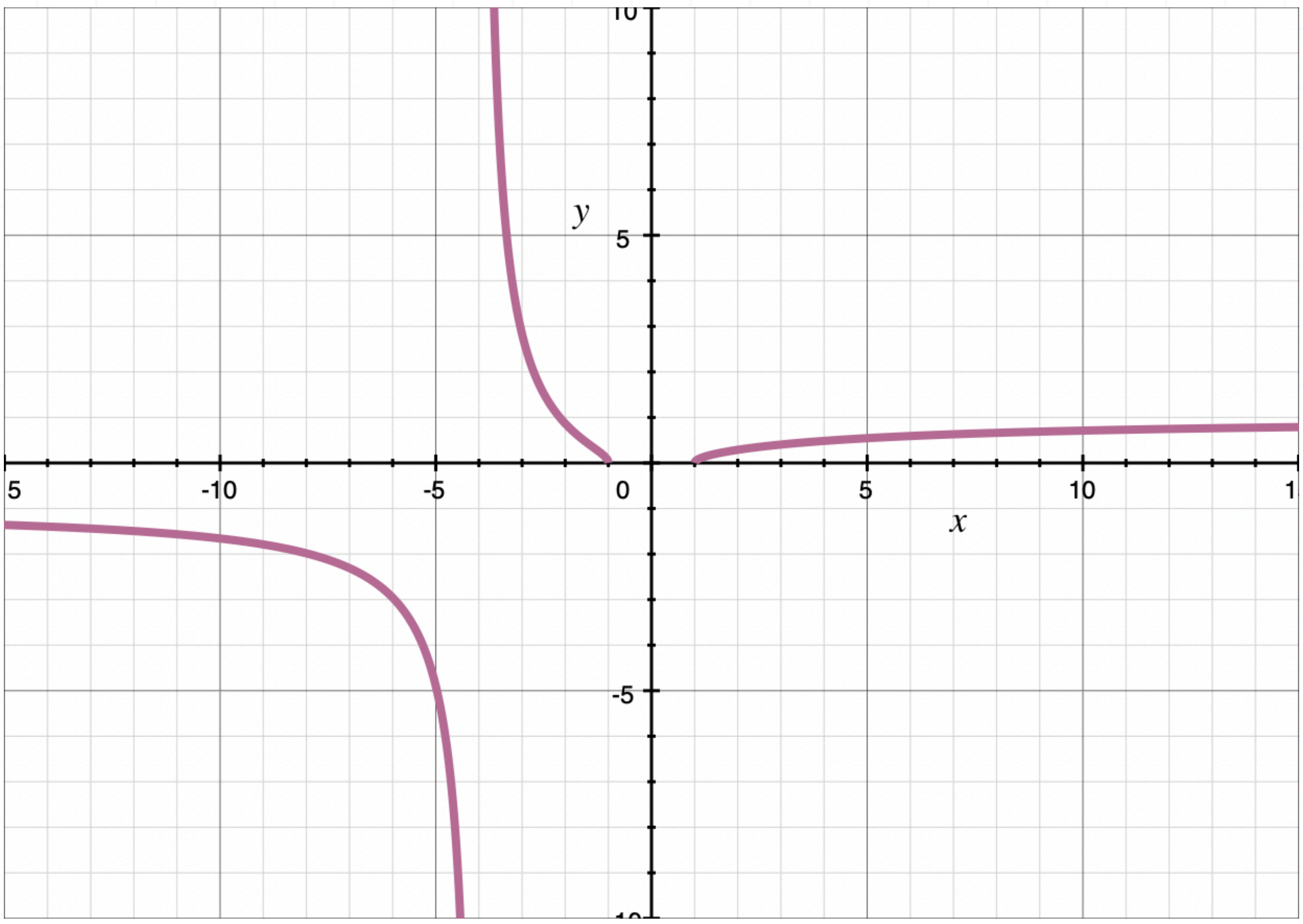
Solution:

No factors can be canceled. The left- and right-hand limits are

$$\lim_{x \to -4^-} \frac{\sqrt{x^2 - 1}}{x + 4} = -\infty$$

$$\lim_{x \to -4^+} \frac{\sqrt{x^2 - 1}}{x + 4} = \infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



5. What is the value of the limit?

$$\lim_{x \to 3} \frac{x^2 - 4x}{x^2 - 2x - 3}$$

Solution:

Factor to simplify the limit.

$$\lim_{x\to 3}\frac{x^2-4x}{x^2-2x-3}$$

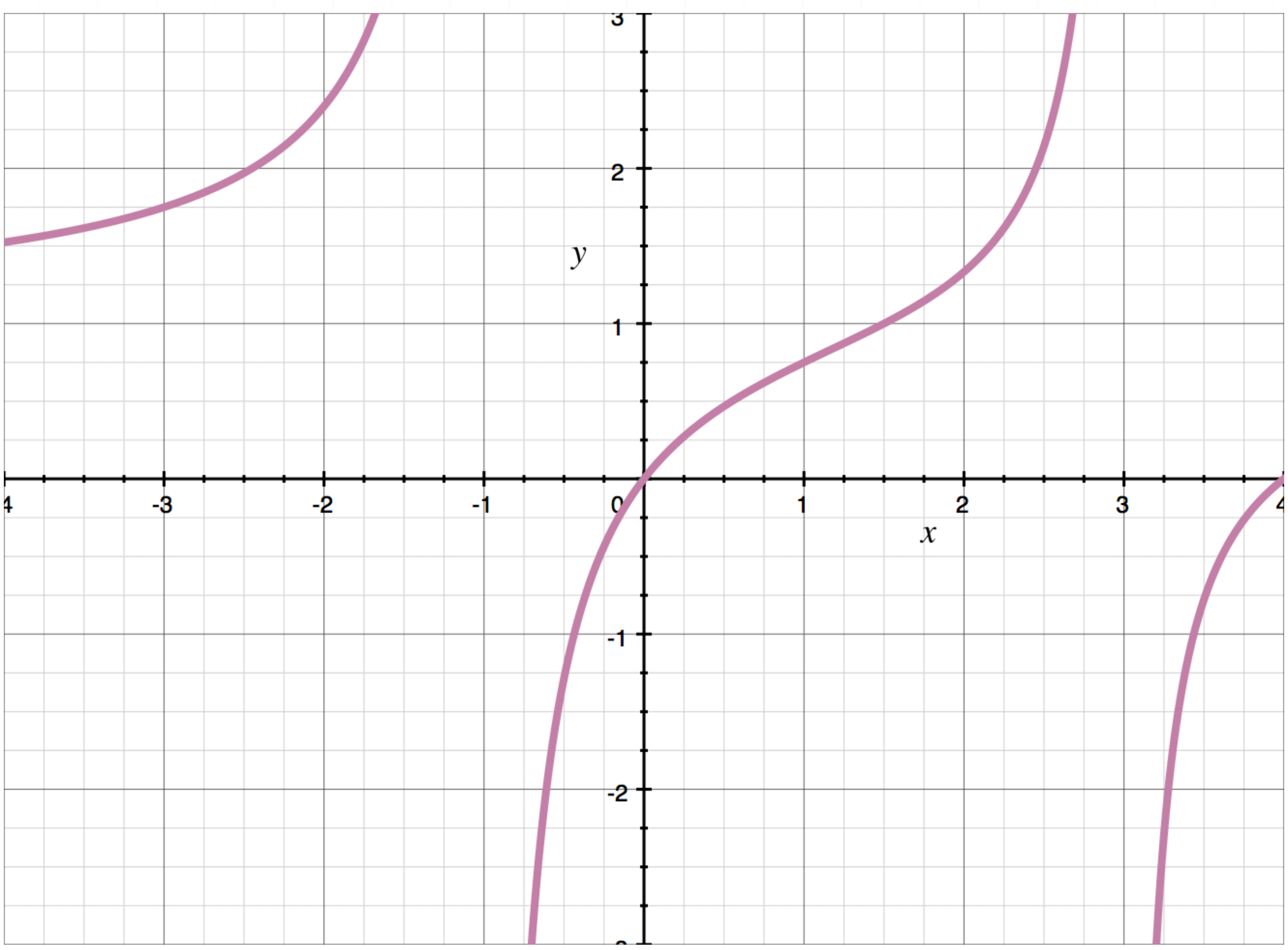
$$\lim_{x\to 3}\frac{x(x-4)}{(x-3)(x+1)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x\to 3^-}\frac{x(x-4)}{(x-3)(x+1)}=\infty$$

$$\lim_{x\to 3^+}\frac{x(x-4)}{(x-3)(x+1)}=-\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



6. What is the value of the limit?

$$\lim_{x \to -2} \frac{x^2 - 16}{-x^2 + x + 6}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to -2} \frac{x^2 - 16}{-x^2 + x + 6}$$

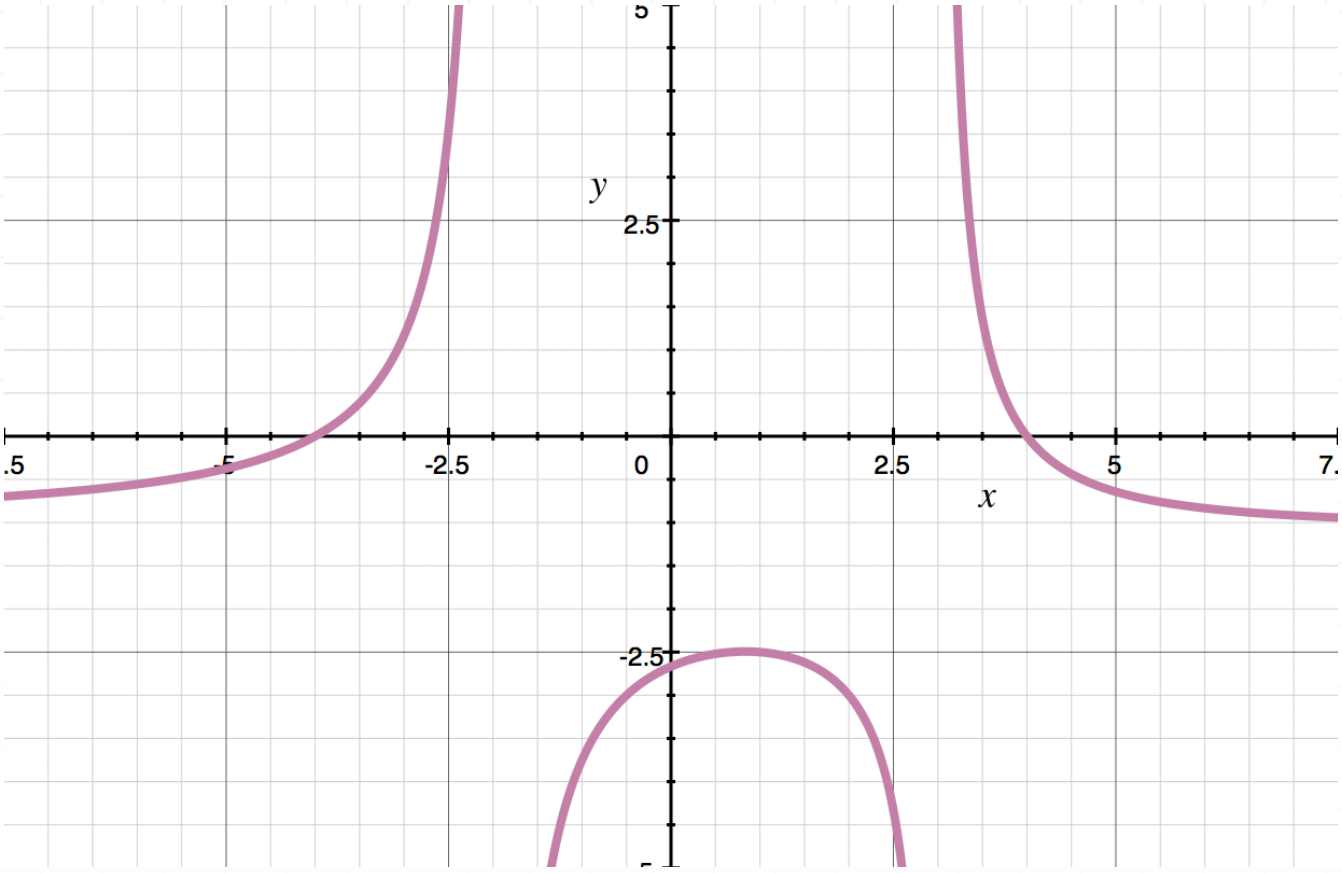
$$\lim_{x \to -2} \frac{(x+4)(x-4)}{-(x-3)(x+2)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to -2^-} \frac{(x+4)(x-4)}{-(x-3)(x+2)} = \infty$$

$$\lim_{x \to -2^+} \frac{(x+4)(x-4)}{-(x-3)(x+2)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



y
x
2.5
-2.5
5
-5
-7.5
-2.5
0
2.5
5
7.5

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

1. What is the value of the limit?

$$\lim_{x\to\infty}\frac{3x^3-5x+2}{9x^3+7x^2-x}$$

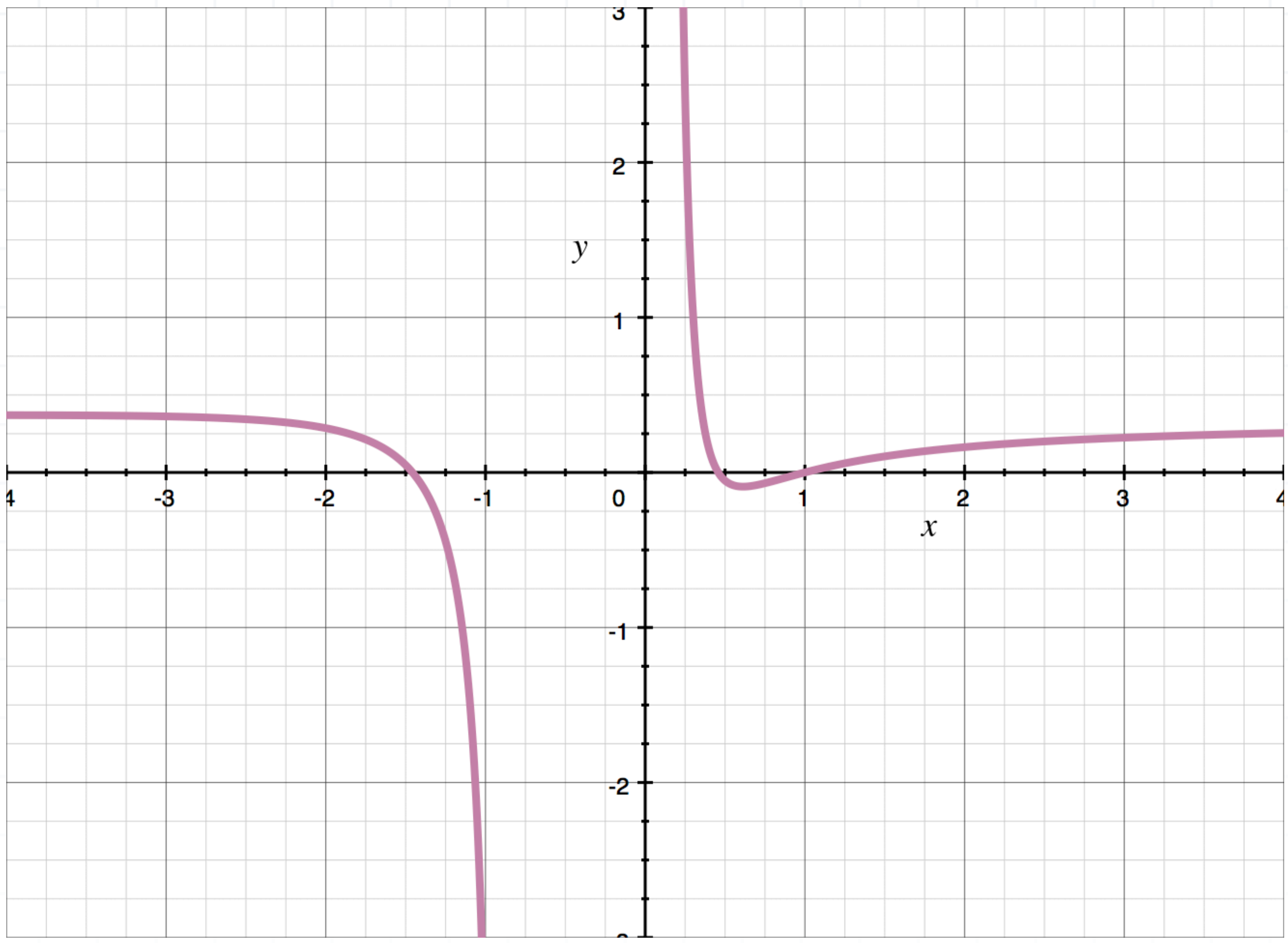
Solution:

Since this is a limit as $x\to\infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behavior of the function.

If the highest power in the numerator is the same as the highest power in the denominator, then the limit as $x\to\infty$ is the ratio of the leading coefficients.

$$\lim_{x\to\infty}\frac{3x^3-5x+2}{9x^3+7x^2-x}=\lim_{x\to\infty}\frac{3x^3}{9x^3}=\lim_{x\to\infty}\frac{3}{9}=\frac{3}{9}=\frac{1}{3}$$

The graph of the function shows this end behavior.



■ 2. What is the value of the limit?

$$\lim_{x \to -\infty} \frac{\sqrt[3]{x} + 1}{\sqrt{x} - 1}$$

Solution:

Since this is a limit as $x \to -\infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behavior of the function.

The highest-degree term in the numerator is $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$, which has a degree of $1/3$. The highest-degree term in the denominator is $\sqrt{x}$ or $x^{\frac{1}{2}}$, which has a degree of $1/2$.

If the highest power in the numerator is smaller than the highest power in the denominator, then the limit as $x \to -\infty$ is 0.

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x}+1}{\sqrt{x}-1} = \lim_{x\to-\infty} \frac{\sqrt[3]{x}}{\sqrt{x}} = 0$$

■ 3. What is the value of the limit?

$$\lim_{x\to\infty} \frac{x^3+x^2+1}{1+2x}$$

Solution:

Since this is a limit as $x \to \infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behavior of the function.

If the highest power in the numerator is greater than the highest power in the denominator, then the limit as $x \to \infty$ does not exist.

We can divide the numerator and the denominator by the x with the greatest power. Let's divide all terms by x^3.

$$\lim_{x\to\infty} \frac{x^3+x^2+1}{1+2x} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{1}{x^3}}{\frac{1}{x^3}+\frac{2x}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^3}}{\frac{1}{x^3}+\frac{2}{x^2}}$$

Evaluating the limit gives

$$\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{1}{x^3}}{\frac{1}{x^3}+\frac{2}{x^2}}=\frac{1+0+0}{0+0}=\frac{1}{0}$$

Division by zero is undefined, so this limit does not exist.

■ 4. What is the value of the limit?

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+5}}{x+4}$$

Solution:

We can divide the numerator and the denominator by the greatest power of x that we find in the fraction, which in this case is x.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+5}}{x+4}=\lim_{x\to-\infty}\frac{\frac{\sqrt{x^2+5}}{x}}{\frac{x+4}{x}}=\lim_{x\to-\infty}\frac{\pm\sqrt{\frac{x^2+5}{x^2}}}{\frac{x+4}{x}}=\lim_{x\to-\infty}\frac{\pm\sqrt{1+\frac{5}{x^2}}}{1+\frac{4}{x}}$$

Evaluating the limit gives

$$\lim_{x\to-\infty}\frac{\pm\sqrt{1+\frac{5}{x^2}}}{1+\frac{4}{x}}=\frac{\pm\sqrt{1+0}}{1+0}=\pm 1$$

Now we could pick test values, like $x=-100$ and $x=100$, and we'd find that the function approaches 1 as $x \to \infty$, and that the function approaches -1 as $x \to -\infty$.

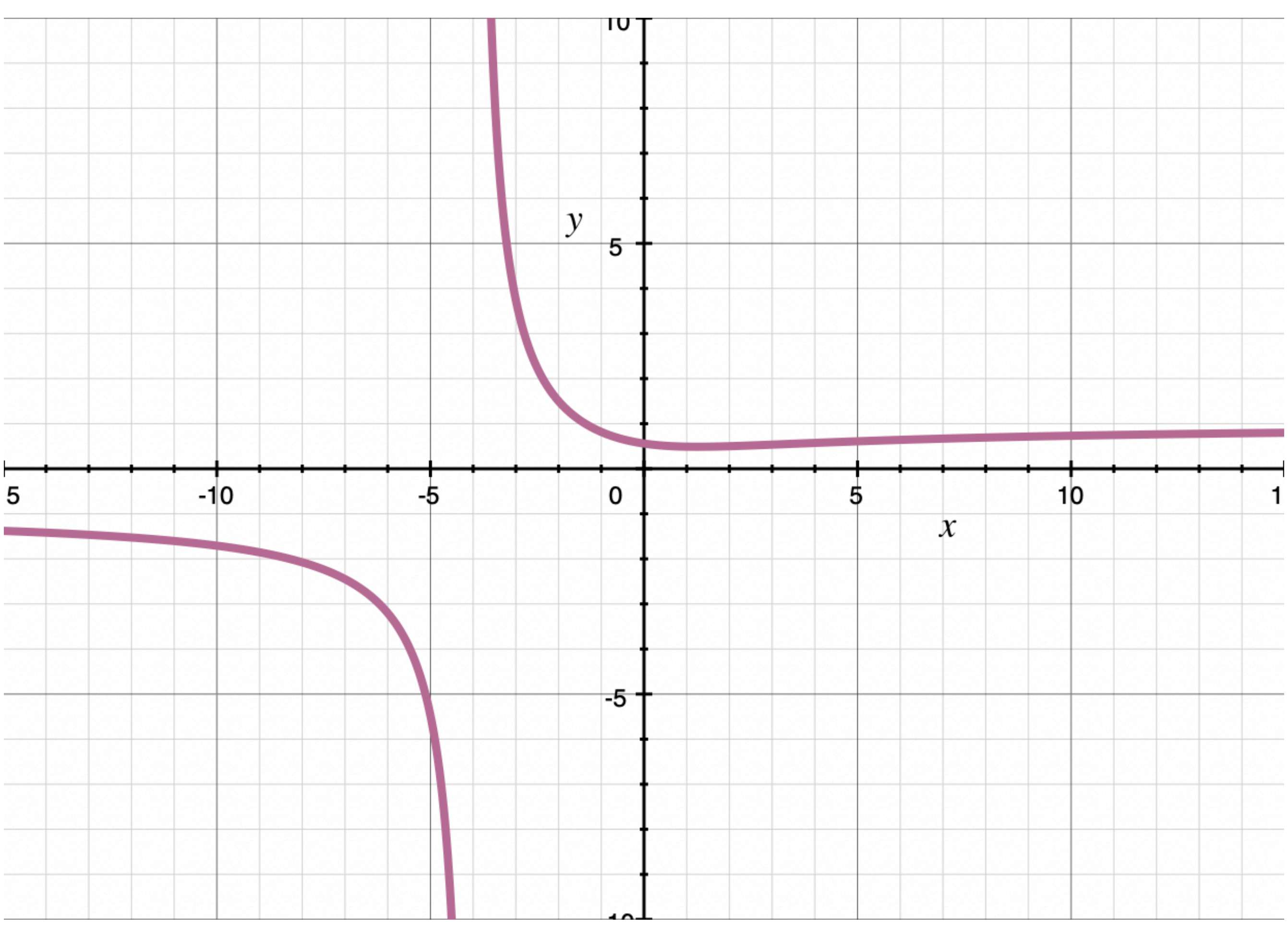
$$f(-100)=\frac{\sqrt{(-100)^2+5}}{-100+4}=\frac{\sqrt{10{,}005}}{-96}\approx -1.04$$

$$f(100)=\frac{\sqrt{100^2+5}}{100+4}=\frac{\sqrt{10{,}005}}{104}\approx 0.96$$

Putting these two tests together, we can verify that

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+5}}{x+4}=-1$$

The graph of the function shows this end behavior.



5. What is the value of the limit?

$$\lim_{x \to -\infty} \frac{19x + 21}{x^3 + 15x + 11}$$

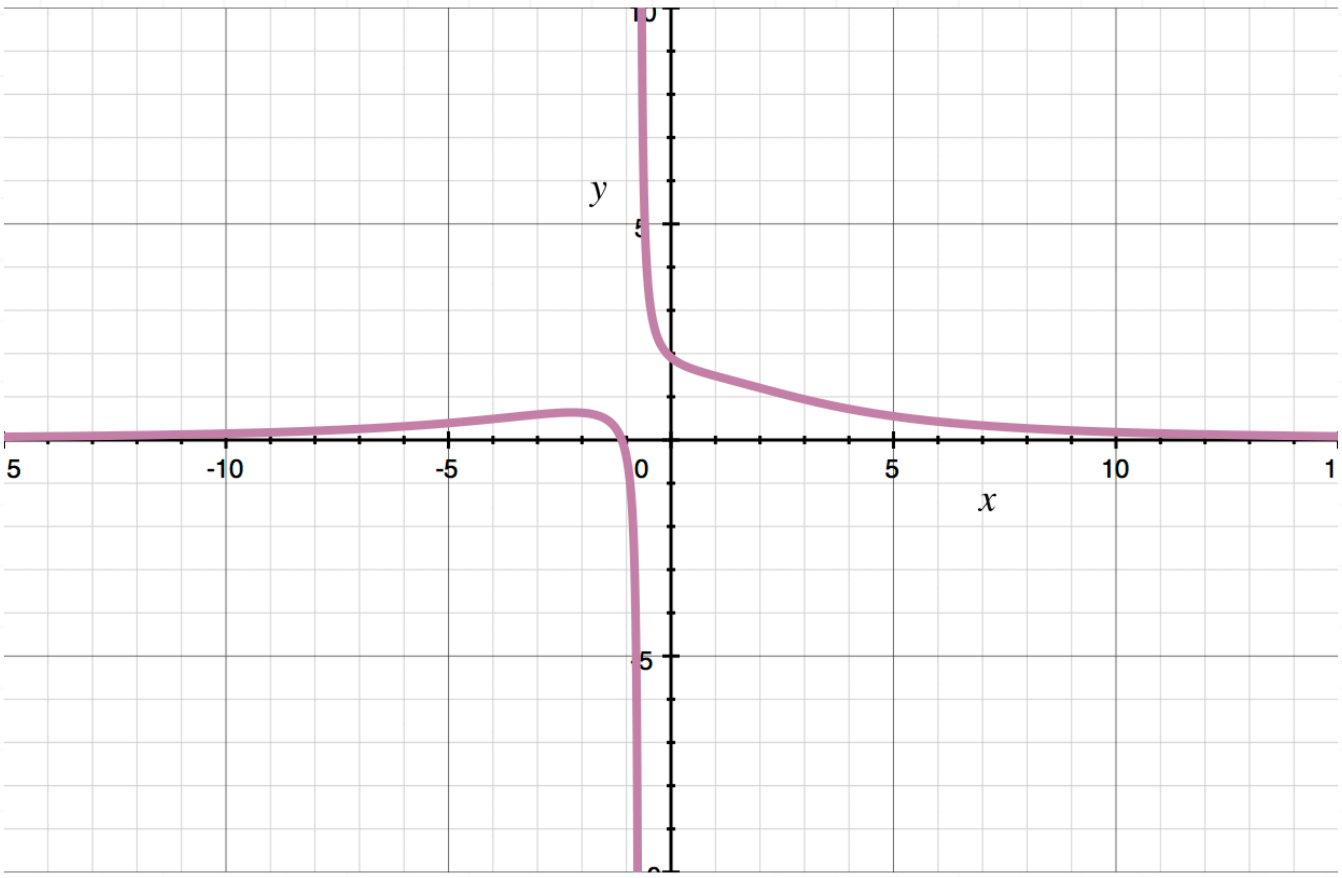
Solution:

Since this is a limit as $x \to -\infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behaviors.

If the highest power in the numerator is smaller than the highest power in the denominator, then the limit as $x \to -\infty$ is 0.

$$\lim_{x \to -\infty} \frac{19x + 21}{x^3 + 15x + 11} = \lim_{x \to -\infty} \frac{19x}{x^3} = \lim_{x \to -\infty} \frac{19}{x^2} = 0$$

The graph of the function shows this end behavior.



■ 6. What is the value of the limit?

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 2x} - x\right)$$

Solution:

To find the limit we need to apply conjugate method.

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 2x} - x\right)\left(\frac{\sqrt{x^2 + 2x} + x}{\sqrt{x^2 + 2x} + x}\right)$$

$$\lim_{x \to \infty} \frac{x^2 + 2x + x\sqrt{x^2 + 2x} - x\sqrt{x^2 + 2x} - x^2}{\sqrt{x^2 + 2x} + x}$$

$$\lim_{x\to\infty} \frac{x^2 + 2x - x^2}{\sqrt{x^2 + 2x} + x}$$

$$\lim_{x\to\infty} \frac{2x}{\sqrt{x^2 + 2x} + x}$$

Now we can divide the numerator and the denominator by the greatest power of x. Let's divide all terms by x.

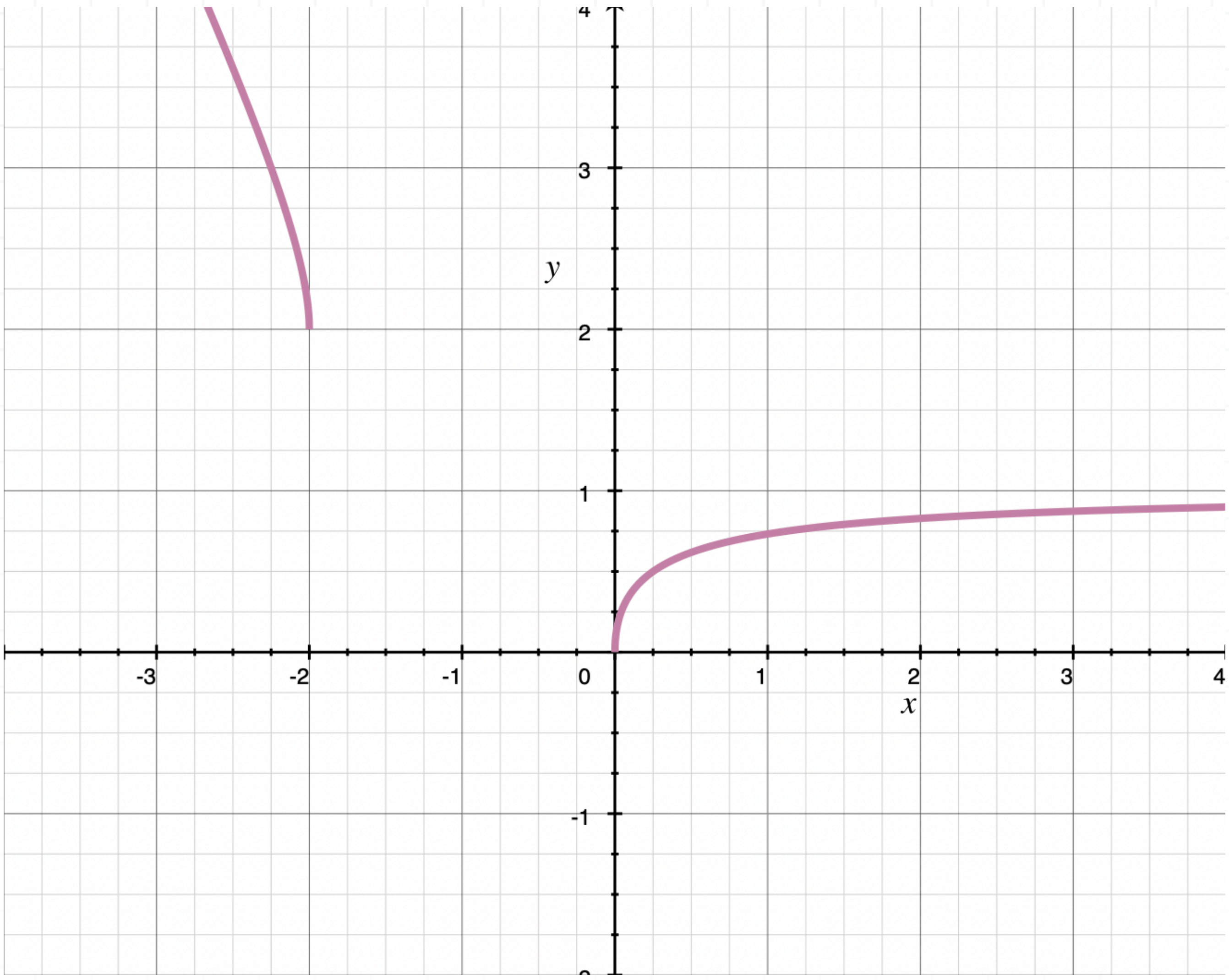
$$\lim_{x\to\infty} \frac{\frac{2x}{x}}{\frac{\sqrt{x^2+2x}+x}{x}} = \lim_{x\to\infty} \frac{2}{\frac{\sqrt{x^2+2x}}{x} + \frac{x}{x}} = \lim_{x\to\infty} \frac{2}{\frac{\sqrt{x^2+2x}}{\sqrt{x^2}} + 1} = \lim_{x\to\infty} \frac{2}{\sqrt{\frac{x^2+2x}{x^2}} + 1}$$

$$\lim_{x\to\infty} \frac{2}{\sqrt{\frac{x^2}{x^2} + \frac{2x}{x^2}} + 1} = \lim_{x\to\infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1}$$

Evaluating the limit gives

$$\frac{2}{\sqrt{1+0}+1} = \frac{2}{1+1} = \frac{2}{2} = 1$$

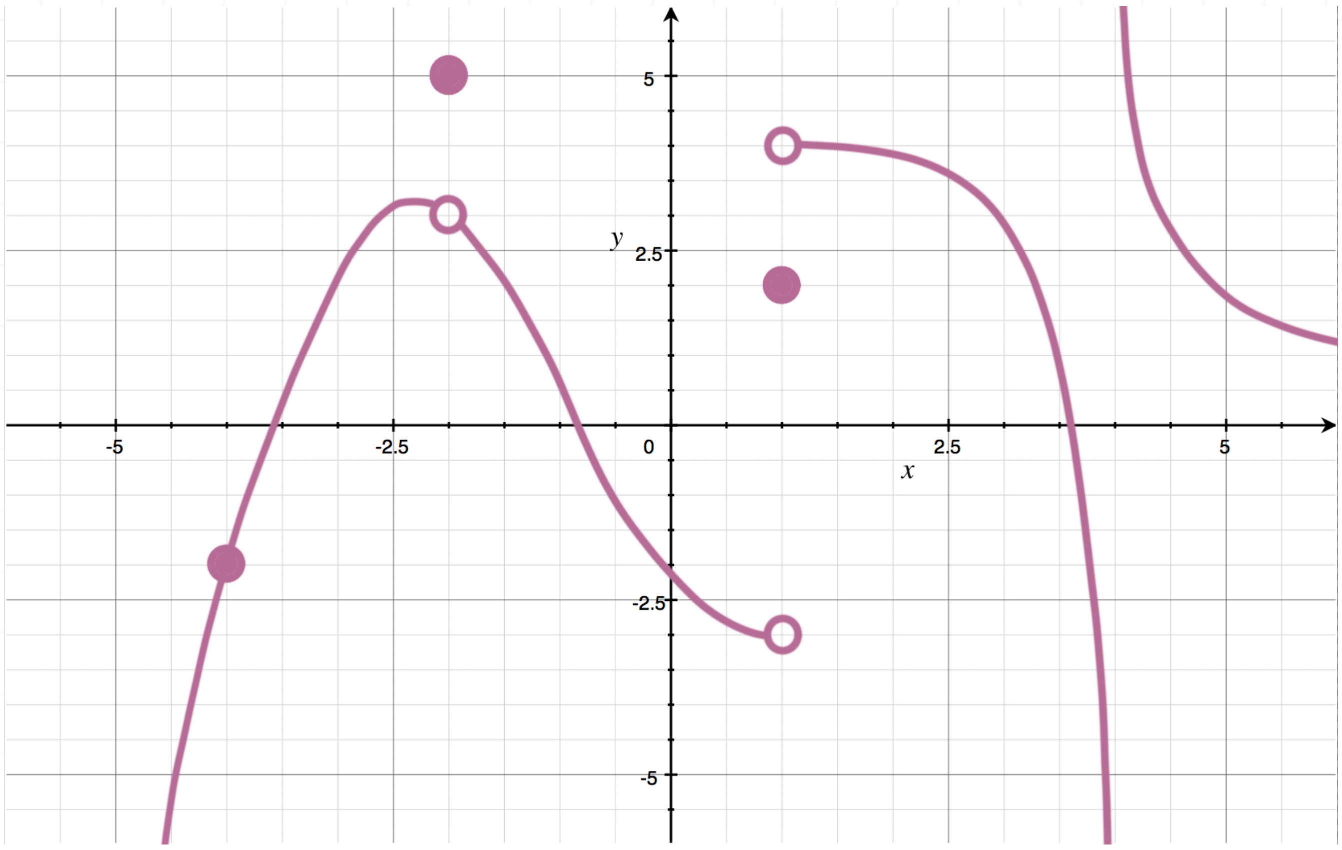
The graph of the function shows this end behavior.



y
x
4
3
2
1
0
-1
-3
-2
-1
1
2
3
4

CRAZY GRAPHS

■ 1. Use the graph to find the value of $\lim_{x\to 1} f(x)$.



Solution:

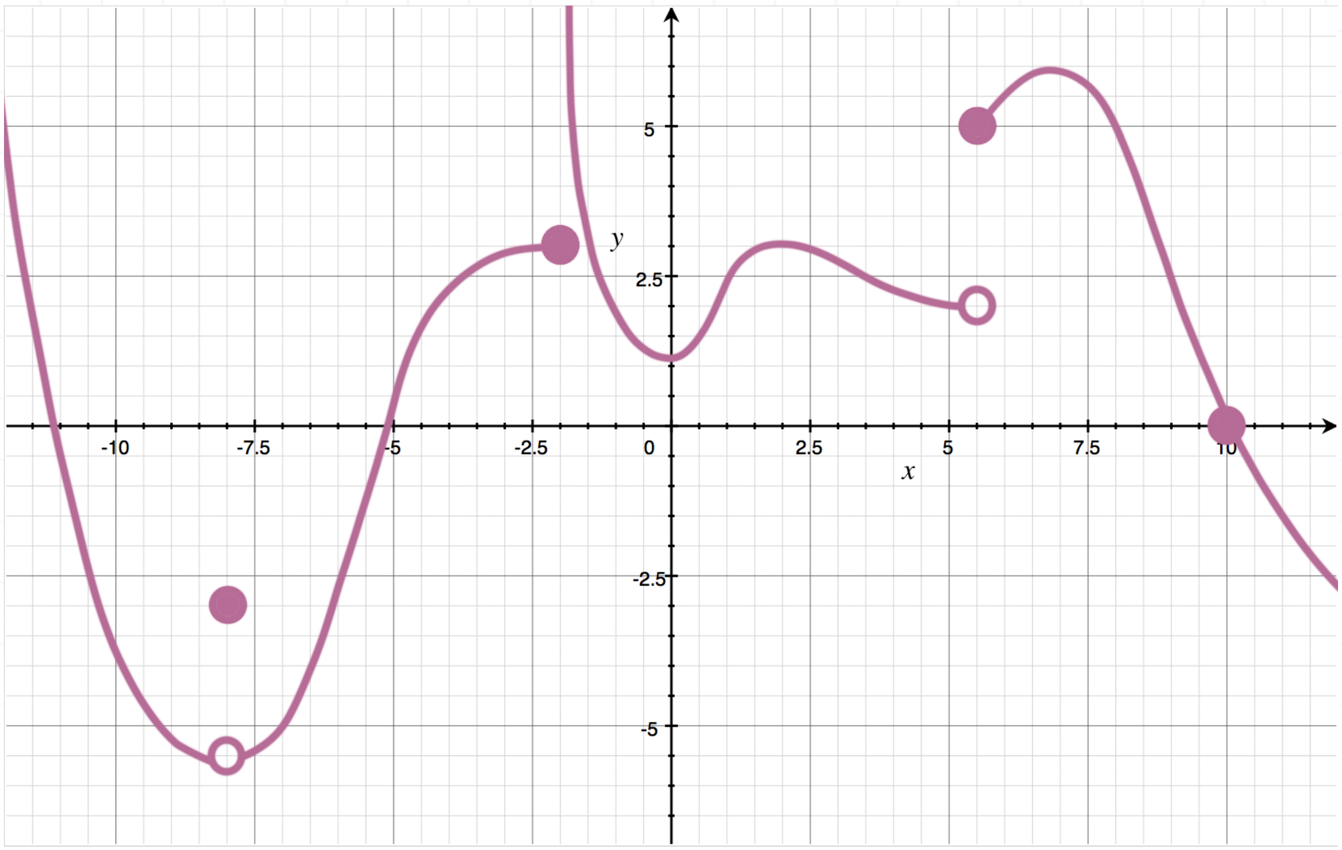
Notice in the graph that $f(x)$ has a jump discontinuity at $x = 1$.

$$\lim_{x\to 1^-} f(x) = -3$$

$$\lim_{x\to 1^+} f(x) = 4$$

Because these limits are unequal, the limit does not exist (DNE).

■ 2. Use the graph to find the value of $\lim_{x \to 5.5} g(x)$.



Solution:

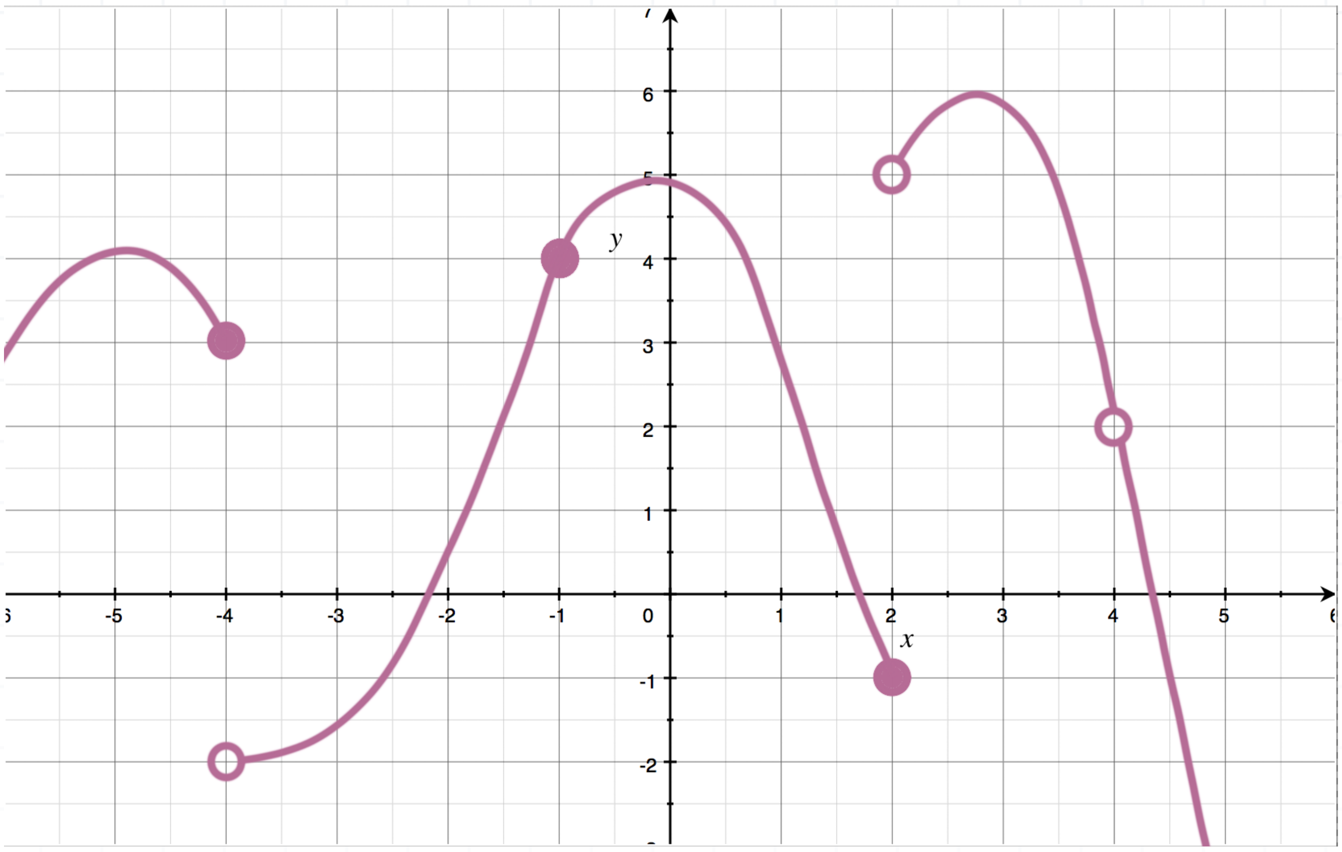
Notice in the graph that $g(x)$ has a jump discontinuity at $x = 5.5$.

$$\lim_{x \to 5.5^-} g(x) = 2$$

$$\lim_{x \to 5.5^+} g(x) = 5$$

Because these limits are unequal, the limit does not exist (DNE).

■ 3. Use the graph to find the value of $\lim_{x \to 4} h(x)$.



Solution:

Notice in the graph that $h(x)$ has a discontinuity at $x = 4$.

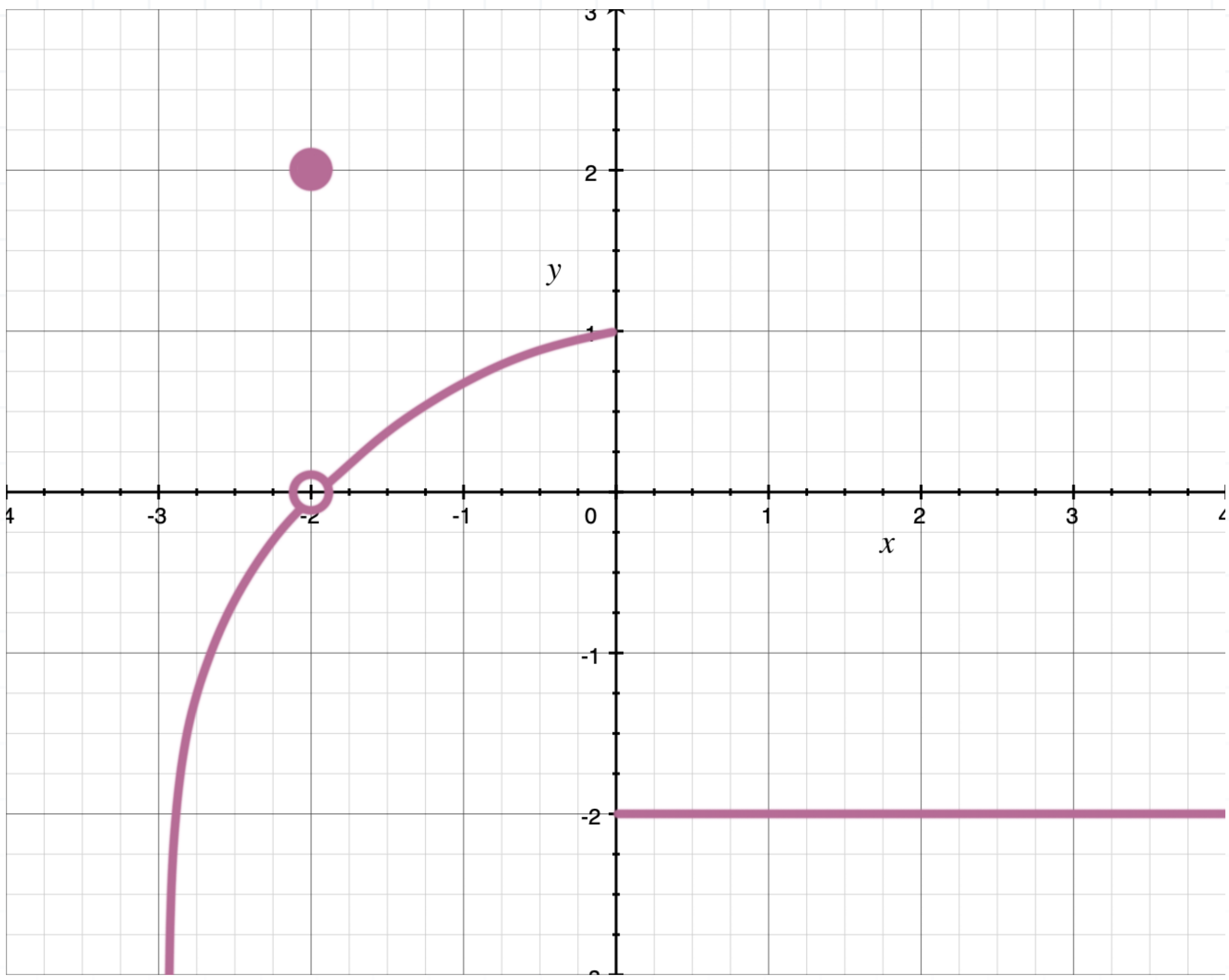
$$\lim_{x \to 4^-} h(x) = 2$$

$$\lim_{x \to 4^+} h(x) = 2$$

These limits are the same, which means

$$\lim_{x \to 4} h(x) = 2$$

■ 4. Use the graph to determine whether or not the limit exists at $x = 0$.



Solution:

At $x = 0$, the function is approaching 1 from the left side. But from the right side, the function is approaching -2. So if we say that the graph represents the function $f(x)$, then the one-sided limits are

$$\lim_{x\to 0^-} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = -2$$

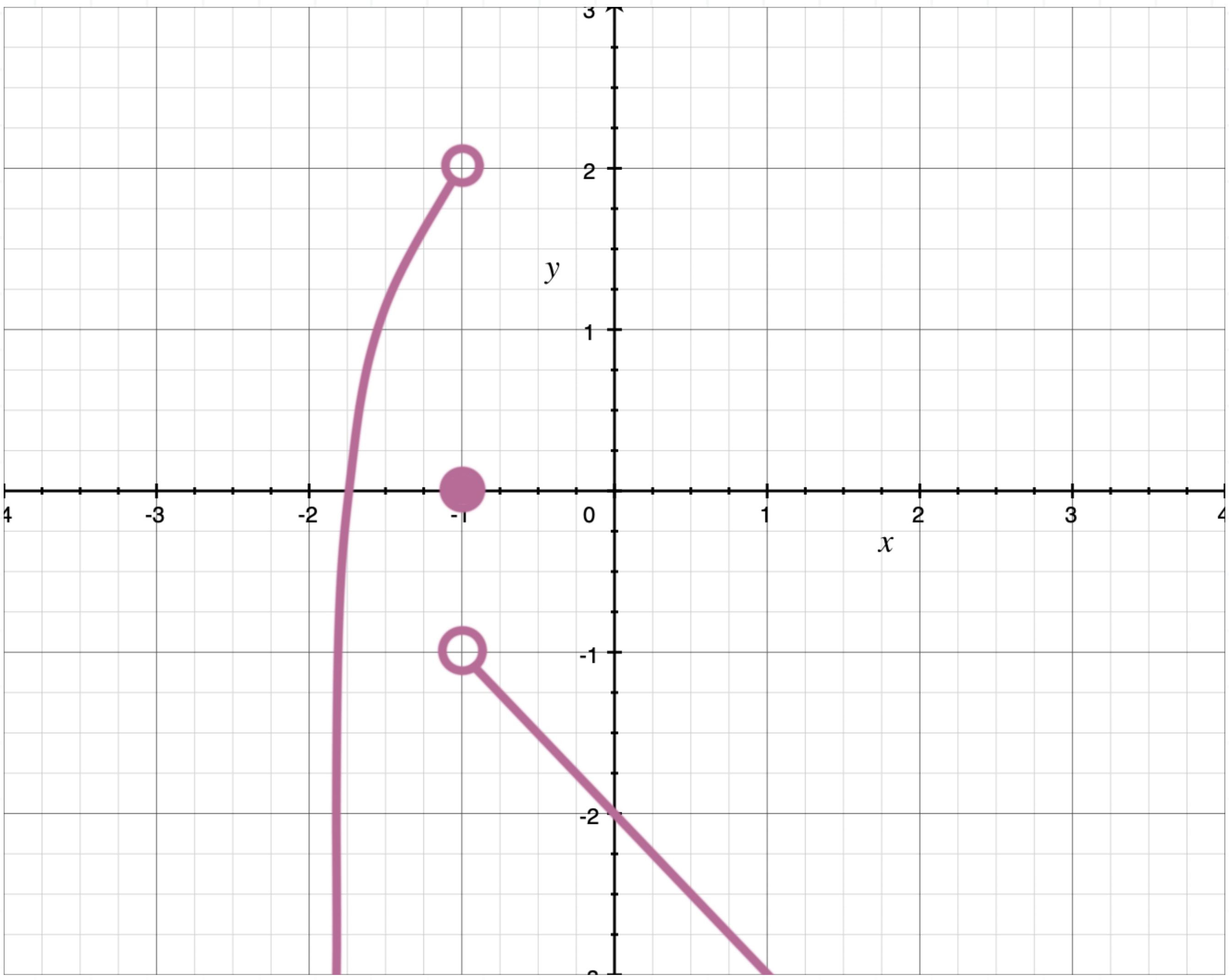
Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at $x = 0$.

5. Sketch the graph of a function that satisfies each of the following conditions.

$$\lim_{x\to -1^-} f(x) = 2 \qquad \lim_{x\to -1^+} f(x) = -1 \qquad f(-1) = 0$$

Solution:

There are an infinite number of graphs that could satisfy these conditions, but the function's value at $x = -1$ is 0, so we plot a point at $(-1,0)$. Then we need another piece of the function that approaches 2 as x gets close to -1 from the negative side, and finally another piece of the function that approaches -1 as x gets close to -1 from the positive side.



■ 6. Sketch the graph of a function that satisfies each of the following conditions.

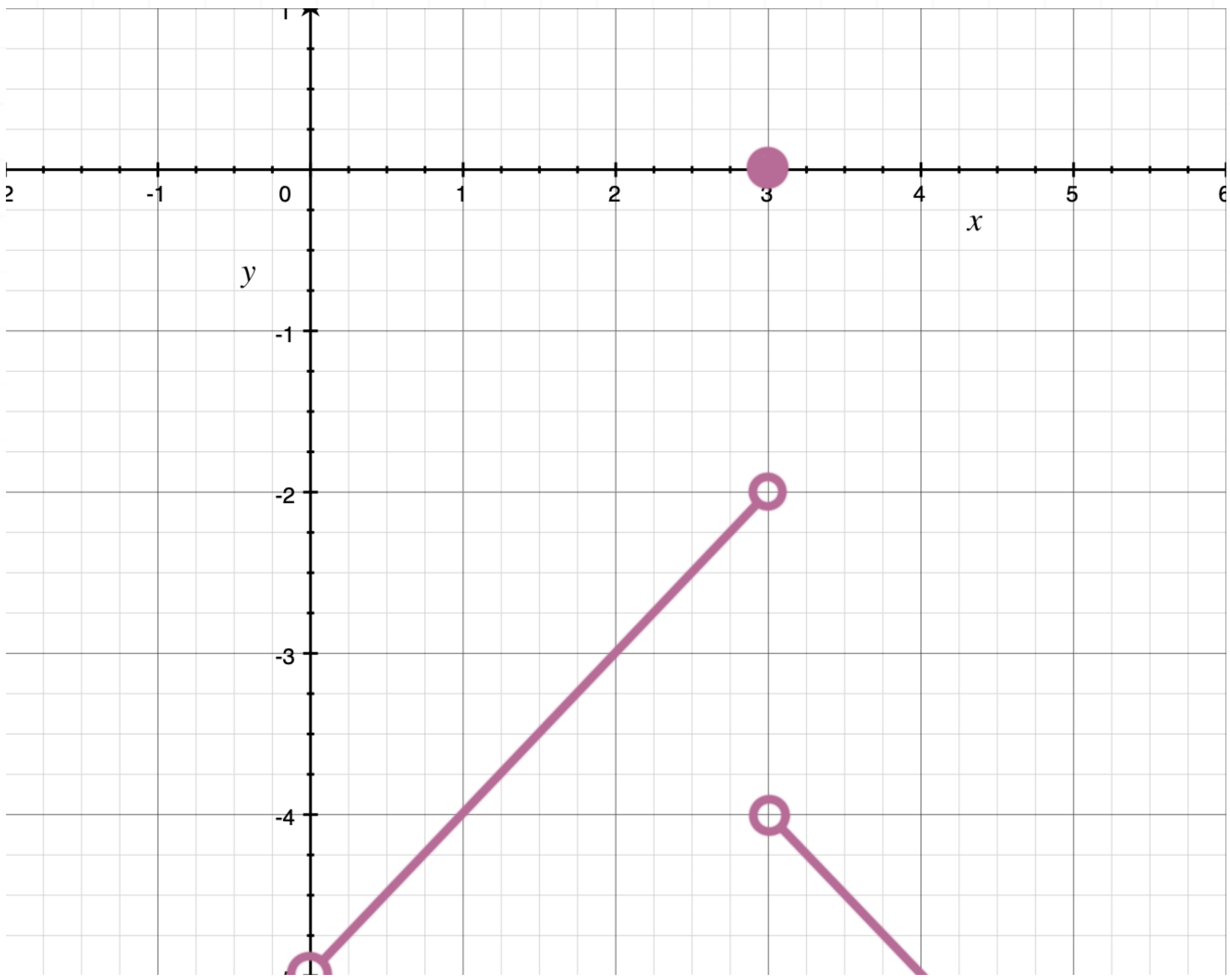
$$\lim_{x \to 0} f(x) = -5 \qquad f(0) \text{ does not exist}$$

$$\lim_{x \to 3^-} f(x) = -2 \qquad \lim_{x \to 3^+} f(x) = -4 \qquad f(3) = 0$$

Solution:

There are an infinite number of graphs that could satisfy these conditions, but the function's value at $x = 3$ is 0, so we plot a point at $(3,0)$. Then we need another piece of the function that approaches -2 as x gets close to 3 from the negative side, and then another piece of the function that approaches -4 as x gets close to 3 from the positive side.

Because the function does not exist when $x = 0$, we'll add a point discontinuity there, and then make sure that the function's limit is -5 as x gets close to 0.



TRIGONOMETRIC LIMITS

■ 1. Find $\lim_{x\to\pi} f(x)$ if $f(x) = 3\cos x - 2$.

Solution:

The one-sided limits are

$$\lim_{x\to\pi^-} 3\cos x - 2 = -5$$

$$\lim_{x\to\pi^+} 3\cos x - 2 = -5$$

Therefore, $\lim_{x\to\pi} f(x) = -5$.

■ 2. Evaluate the limit.

$$\lim_{x\to 0} \frac{\sin(8x)}{x}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{\sin(8 \cdot 0)}{0}$$

$$\frac{0}{0}$$

But if we rewrite the limit as

$$\lim_{x \to 0} \frac{\sin 8x}{x} = \lim_{x \to 0} \frac{\sin 8x}{x\frac{8}{8}} = \lim_{x \to 0} \frac{\sin 8x}{8x \cdot \frac{1}{8}} = \lim_{x \to 0} \frac{\sin 8x}{8x} \cdot 8 = 8 \lim_{x \to 0} \frac{\sin 8x}{8x}$$

and we know the value of the trig limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

then we can evaluate the limit as

$$8 \cdot 1$$

$$8$$

■ 3. Evaluate the limit.

$$\lim_{x \to \frac{\pi}{4}} \frac{1 - \cot x}{\cos x - \sin x}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{1-\cot\frac{\pi}{4}}{\cos\frac{\pi}{4}-\sin\frac{\pi}{4}}$$

$$\frac{1-1}{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}$$

$$\frac{0}{0}$$

We can rewrite the limit as

$$\lim_{x\to\frac{\pi}{4}}\frac{1-\cot x}{\cos x-\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{1-\frac{\cos x}{\sin x}}{\cos x-\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{\frac{\sin x-\cos x}{\sin x}}{\cos x-\sin x}$$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{(\cos x-\sin x)\sin x}=\lim_{x\to\frac{\pi}{4}}\left(-\frac{\cos x-\sin x}{(\cos x-\sin x)\sin x}\right)=\lim_{x\to\frac{\pi}{4}}\left(-\frac{1}{\sin x}\right)$$

Now we can evaluate the limit using direct substitution.

$$-\frac{1}{\sin\frac{\pi}{4}}$$

$$-\frac{1}{\frac{\sqrt{2}}{2}}$$

$$-\sqrt{2}$$

4. Evaluate the limit.

$$\lim_{x \to 0} \frac{\tan(4x)}{\sin(2x)}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{\tan(0)}{\sin(0)}$$

$$\frac{0}{0}$$

We can rewrite the limit as

$$\lim_{x \to 0} \frac{\tan(4x)}{\sin(2x)} = \lim_{x \to 0} \frac{4\frac{\tan(4x)}{4x}}{2\frac{\sin(2x)}{2x}} = 2 \lim_{x \to 0} \frac{\frac{\tan(4x)}{4x}}{\frac{\sin(2x)}{2x}}$$

We know the value of the trig limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Therefore,

$$2 \lim_{x \to 0} \frac{\frac{\tan(4x)}{4x}}{\frac{\sin(2x)}{2x}} = 2 \lim_{x \to 0} \frac{\frac{\tan(4x)}{4x}}{1} = 2 \lim_{x \to 0} \frac{\tan(4x)}{4x}$$

$$2\lim_{x\to 0}\frac{\tan(4x)}{4x}=2\lim_{x\to 0}\frac{\frac{\sin(4x)}{\cos(4x)}}{4x}=2\lim_{x\to 0}\frac{\sin(4x)}{4x\cos(4x)}=2\lim_{x\to 0}\frac{\sin(4x)}{4x}\cdot\lim_{x\to 0}\frac{1}{\cos(4x)}$$

We know the value of the trig limit

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

We get

$$2\cdot 1\cdot\lim_{x\to 0}\frac{1}{\cos(4x)}$$

Now we can evaluate the limit using direct substitution.

$$2\cdot 1\cdot\frac{1}{\cos(4\cdot 0)}$$

$$2\cdot 1\cdot\frac{1}{1}$$

$$2$$

■ 5. Evaluate the limit.

$$\lim_{x\to 0}\frac{x}{\sin\frac{x}{3}}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\lim_{x \to 0} \frac{0}{\sin 0}$$

$$\frac{0}{0}$$

We can rewrite the limit as

$$\lim_{x \to 0} \frac{x}{\sin \frac{x}{3}} = \lim_{x \to 0} \frac{\frac{x}{x}}{\frac{\sin \frac{x}{3}}{3\frac{x}{3}}} = 3 \lim_{x \to 0} \frac{1}{\frac{\sin \frac{x}{3}}{\frac{x}{3}}}$$

We know the value of the trig limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

We get

$$3 \cdot \frac{1}{1}$$

$$3$$

■ 6. Evaluate the limit.

$$\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{\sin^2 0}{1 - \cos 0}$$

$$\frac{0}{1 - 1}$$

$$\frac{0}{0}$$

Substitution doesn't work, and there's nothing to factor, but since we have exactly two terms in the numerator, we can actually use the conjugate method for the first step of this problem.

$$\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x}\left(\frac{1 + \cos x}{1 + \cos x}\right)$$

$$\lim_{x \to 0} \frac{\sin^2 x(1 + \cos x)}{1 - \cos^2 x}$$

Applying the Pythagorean identity $1 - \cos^2 x = \sin^2 x$ to the denominator gives

$$\lim_{x \to 0} \frac{\sin^2 x(1 + \cos x)}{\sin^2 x}$$

$$\lim_{x \to 0}(1 + \cos x)$$

Now we can evaluate the limit using direct substitution.

$$1 + \cos 0$$

$$1 + 1$$

$$2$$

MAKING THE FUNCTION CONTINUOUS

1. What value of c makes the function $h(x)$ continuous if c is a constant?

$$h(x) = \begin{cases} x^2 & x \le 4 \\ 3x + c & x > 4 \end{cases}$$

Solution:

Since $h(x)$ is defined as a piecewise function, $x^2 = 3x + c$ at $x = 4$.

$$4^2 = 3(4) + c$$

$$16 = 12 + c$$

$$c = 4$$

2. What value of k makes the function $f(x)$ continuous if k is a constant?

$$f(x) = \begin{cases} kx^2 - 2x + 1 & x \le 3 \\ kx + 1 & x > 3 \end{cases}$$

Solution:

Since $f(x)$ is defined as a piecewise function, the function will be continuous at $x = 3$ when $kx^2 - 2x + 1 = kx + 1$.

$$kx^2 - 2x + 1 = kx + 1$$

$$k(3)^2 - 2(3) + 1 = k(3) + 1$$

$$9k - 6 + 1 = 3k + 1$$

$$6k = 6$$

$$k = 1$$

■ 3. What values of a and b make the function $g(x)$ continuous if a and b are constant?

$$g(x) = \begin{cases} 3 & x \leq -2 \\ ax - b & -2 < x < 2 \\ -2 & x \geq 2 \end{cases}$$

Solution:

Since $g(x)$ is defined as a piecewise function, the function will be continuous at $x = -2$ when $3 = ax - b$, and continuous at $x = 2$ when $ax - b = -2$. So we need to solve these two equations as a system.

$$3 = -2a - b$$

$$2a - b = -2$$

Add the equations.

$$3 + (-2) = -2a - b + (2a - b)$$

$$3 - 2 = -2a - b + 2a - b$$

$$-2b = 1$$

$$b = -\frac{1}{2}$$

Substitute $b = -1/2$ into the first equation to solve for a.

$$3 = -2a - \left(-\frac{1}{2}\right)$$

$$\frac{5}{2} = -2a$$

$$a = -\frac{5}{4}$$

So the function is continuous when $a = -5/4$ and $b = -1/2$.

■ 4. What value of c makes the function $f(x)$ continuous if c is a constant?

$$f(x) = \begin{cases} 2x^3 - 6x^2 + 8x + 3 & x \le 1 \\ cx + 9 & x > 1 \end{cases}$$

Solution:

Since $f(x)$ is defined as a piecewise function, $2x^3 - 6x^2 + 8x + 3 = cx + 9$ at $x = 1$.

$$2(1)^3 - 6(1)^2 + 8(1) + 3 = c(1) + 9$$

$$2 - 6 + 8 + 3 = c + 9$$

$$7 = c + 9$$

$$c = -2$$

5. What value of c makes the function $g(x)$ continuous if c is a constant?

$$g(x) = \begin{cases} \sqrt{x} + 18 & x \le 16 \\ x - 2c & x > 16 \end{cases}$$

Solution:

Since $g(x)$ is defined as a piecewise function, the function will be continuous at $x = 16$ when $\sqrt{x} + 18 = x - 2c$.

$$\sqrt{16} + 18 = 16 - 2c$$

$$4 + 18 = 16 - 2c$$

$$22 = 16 - 2c$$

$$6 = -2c$$

$$c = -3$$

■ 6. What values of a and b make the function $h(x)$ continuous if a and b are constant?

$$h(x) = \begin{cases} ax^2 & x \leq -1 \\ ax + b & -1 < x < 3 \\ bx + 2 & x \geq 3 \end{cases}$$

Solution:

Since $h(x)$ is defined as a piecewise function, the function will be continuous at $x = -1$ when $ax^2 = ax + b$, and continuous at $x = 3$ when $ax + b = bx + 2$. So we need to solve these two equations as a system.

$$a = -a + b$$

$$3a + b = 3b + 2$$

Simplify the system.

$$2a = b$$

$$3a - 2b = 2$$

Substitute the first equation into the second equation to solve for a.

$$3a - 2(2a) = 2$$

$$3a - 4a = 2$$

$$-a = 2$$

$$a = -2$$

Then, $b = 2(-2) = -4$. So the function is continuous when $a = -2$ and $b = -4$.

FOR THE LOVE OF MATH

W W W . K R I S T A K I N G M A T H . C O M