Precalculus
Final Exam Solutions

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MATH

Precalculus Final Exam Answer Key

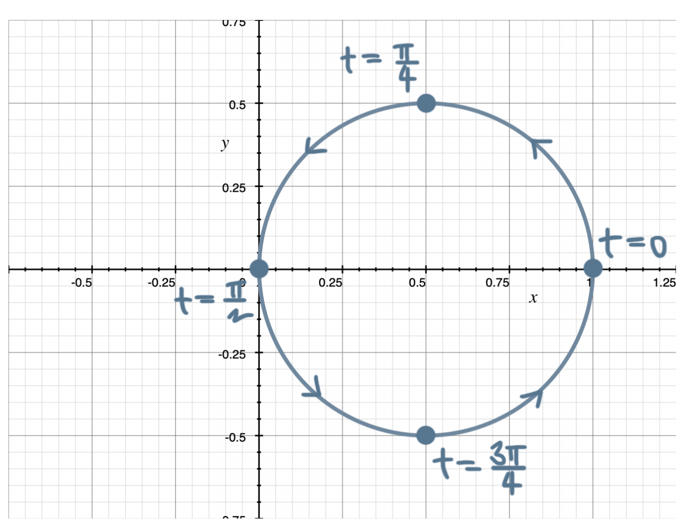
1. (5 pts)	A	B	C	■	E
2. (5 pts)	A	■	C	D	E
3. (5 pts)	■	B	C	D	E
4. (5 pts)	A	B	■	D	E
5. (5 pts)	■	B	C	D	E
6. (5 pts)	A	B	C	D	■
7. (5 pts)	A	B	C	■	E
8. (5 pts)	A	B	C	■	E

9. (15 pts) $(r, \theta) = (1, \pm \pi/3)$

10. (15 pts) $f(x) = \dfrac{2}{x} - \dfrac{1}{x+1} - \dfrac{2}{(x+1)^2}$

11. (15 pts) $y = -1 \pm \dfrac{2}{\sqrt{3}}$

12. (15 pts)



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1. D. If we use the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ to plug into the rectangular equation, we get

$$x^2 + 4y^2 = 1$$

$$r^2\cos^2\theta + 4r^2\sin^2\theta = 1$$

$$r^2(\cos^2\theta + 4\sin^2\theta) = 1$$

Now we can substitute using the trig identity $\cos^2\theta = 1 - \sin^2\theta$.

$$r^2(1 - \sin^2\theta + 4\sin^2\theta) = 1$$

$$r^2(1 + 3\sin^2\theta) = 1$$

$$r^2 = \frac{1}{1 + 3\sin^2\theta}$$

2. B. Given $z = -4 + 3i$, we can identify $a = -4$ and $b = 3$, which means the absolute value $|z|$ is

$$|z| = \sqrt{a^2 + b^2} = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$$

3. A. Given $z = 2 + i$, we need to start by finding $z - 3$.

$$z - 3 = 2 + i - 3 = -1 + i$$

Now we need to rewrite the complex number in polar form. The real part of this complex number is $a = -1$ and its imaginary part is $b = 1$, so the value of r will be

$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

The value of θ is

$$\tan\theta = \frac{1}{-1} = -1$$

$$\arctan(\tan\theta) = \arctan(-1)$$

$$\theta = \arctan(-1)$$

$$\theta = \frac{3\pi}{4}$$

Then the complex number written in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z - 3 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

To find $(z-3)^5$, we need to use $n = 5$ in De Moivre's Theorem. With $n = 5$, and $r = \sqrt{2}$ and $\theta = 3\pi/4$ from z, we plug into De Moivre's Theorem to get

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$(z-3)^5 = r^5[\cos(5\theta) + i\sin(5\theta)]$$

$$(z-3)^5 = (\sqrt{2})^5\left(\cos\left(5\cdot\frac{3\pi}{4}\right) + i\sin\left(5\cdot\frac{3\pi}{4}\right)\right)$$

$$(z-3)^5 = \sqrt{32}\left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right)$$

$$(z-3)^5 = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$(z-3)^5 = 4\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$(z-3)^5 = 4(1-i)$$

4. C. The determinant of A is

$$|A| = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = (2)(5) - (3)(3) = 10 - 9 = 1$$

To find the inverse of matrix A, we plug into the formula for the inverse matrix.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

5. A. To find the product AB of matrices A and B,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We'll multiply A by B.

$$AB = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2+0+0 & 0+0-2 \\ -2+1+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}$$

6. E. The denominator of the rational function is made of distinct linear factors, so we'll set up the decomposition as

$$\frac{6-4x}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

To solve for A, we'll remove the factor of x from the denominator of the left side, then evaluate what remains on the left at $x = 0$.

$$\frac{6-4x}{(x-1)(x-2)} \to \frac{6-4(0)}{(0-1)(0-2)} \to \frac{6}{(-1)(-2)} \to 3$$

To solve for B, we'll remove the factor of $x - 1$ from the denominator of the left side, then evaluate what remains on the left at $x = 1$.

$$\frac{6-4x}{x(x-2)} \to \frac{6-4(1)}{1(1-2)} \to \frac{6-4}{1(-1)} \to -2$$

To solve for C, we'll remove the factor of $x-2$ from the denominator of the left side, then evaluate what remains on the left at $x=2$.

$$\frac{6-4x}{x(x-1)} \to \frac{6-4(2)}{2(2-1)} \to \frac{6-8}{2(1)} \to -1$$

Therefore, the partial fractions decomposition of $f(x)$ is

$$f(x)=\frac{3}{x}-\frac{2}{x-1}-\frac{1}{x-2}$$

7. D. If we compare the equation of the conic

$$x^2-3xy+2y^2+4x-2y+3=0$$

to the standard form of a conic, we can identify $A=1$, $B=-3$, and $C=2$, so the value of the discriminant is

$$B^2-4AC=(-3)^2-4(2)(1)=1>0$$

So the conic is the equation of a hyperbola.

8. D. We can rewrite the parametric equations $x=4\cos^2 t$ and $y=9\sin^2 t$ as

$$\frac{x}{4}=\cos^2 t \text{ and } \frac{y}{9}=\sin^2 t$$

Now that the equations are in this form, we can make substitutions into the Pythagorean identity with sine and cosine.

$$\sin^2 t + \cos^2 t = 1$$

$$\frac{x}{4} + \frac{y}{9} = 1$$

$$\frac{y}{9} = 1 - \frac{x}{4}$$

$$y = -\frac{9}{4}x + 9$$

9. Because both equations $r = 2\cos\theta$ and $r = 1$ are equal to r, we can set them equal to one another.

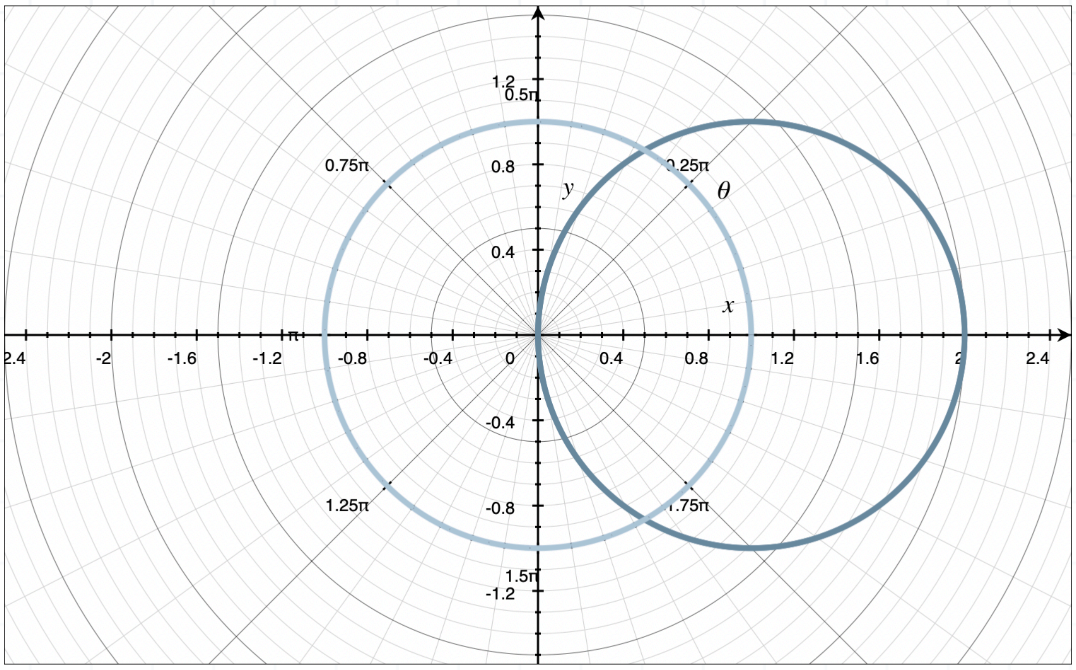
$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

If we plug these angles back into either of the polar equations, we get $r = 1$ at both angles. Therefore, $(r, \theta) = (1, \pi/3)$ and $(r, \theta) = (1, -\pi/3)$ are points of intersection of the polar curves.

Sketch the graph of the two curves in the same polar plane.



There are no hidden points of intersection, so the intersection points are only $(r, \theta) = (1, \pm \pi/3)$.

10. The denominator of the rational function is the product of a distinct linear factor and repeated linear factors, so the decomposition equation will be

$$\frac{x^2 + x + 2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

To solve for A, we'll remove the factor of x from the left side, and then evaluate what remains on the left at $x = 0$.

$$\frac{x^2 + x + 2}{(x+1)^2} \to \frac{0^2 + 0 + 2}{(0+1)^2} \to 2$$

To solve for C, we'll remove the factor of $(x+1)^2$ from the left side, and then evaluate what remains on the left at $x = -1$.

$$\frac{x^2+x+2}{x} \to \frac{(-1)^2-1+2}{-1} \to -2$$

We'll combine the fractions on the right side by finding a common denominator.

$$\frac{x^2+x+2}{x(x+1)^2} = \frac{2}{x} + \frac{B}{x+1} - \frac{2}{(x+1)^2}$$

$$\frac{x^2+x+2}{x(x+1)^2} = \frac{2x^2+4x+2+Bx^2+Bx-2x}{x(x+1)^2}$$

$$\frac{x^2+x+2}{x(x+1)^2} = \frac{(2+B)x^2+(2+B)x+2}{x(x+1)^2}$$

$$\frac{x^2+x+2}{x(x+1)^2} = \frac{(2+B)x^2+(2+B)x+2}{x(x+1)^2}$$

With the denominators equal, we can set the numerators equal to one another,

$$x^2+x+2 = (2+B)x^2+(2+B)x+2$$

and then equate coefficients on the left and right sides to get

$$2+B=1$$

$$B=-1$$

Plugging this value back into the partial fractions decomposition gives

$$f(x) = \frac{2}{x} - \frac{1}{x+1} - \frac{2}{(x+1)^2}$$

11. We need to rewrite the equation of the ellipse in standard form by completing the square with respect to both variables.

$$4x^2 + y^2 - 16x + 2y + 16 = 0$$

$$4(x^2 - 4x + 4) + (y^2 + 2y + 1) = 1$$

$$4(x - 2)^2 + (y + 1)^2 = 1$$

$$\frac{(x-2)^2}{\frac{1}{4}} + \frac{(y+1)^2}{1} = 1$$

The center of the ellipse is $(2, -1)$. The length of the major radius is $\sqrt{1} = 1$, and the length of the minor radius is $\sqrt{1/4} = 1/2$. Therefore, the focal length is

$$c = \sqrt{b^2 - a^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Therefore, the directrices are at

$$y = k \pm \frac{a^2}{c} = -1 \pm \frac{1}{\frac{\sqrt{3}}{2}} = -1 \pm \frac{2}{\sqrt{3}}$$

12. Rewrite the parametric equations using trigonometric identities.

$$x = \cos^2 t \qquad\qquad y = \sin t \cos t$$

$$x = \frac{1 + \cos(2t)}{2} \qquad y = \frac{1}{2}\sin(2t)$$

$$\cos(2t) = 2x - 1 \qquad \sin(2t) = 2y$$

If we remember the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ from Trigonometry, then we get

$$\sin^2\theta + \cos^2\theta = 1$$

$$(2x - 1)^2 + (2y)^2 = 1$$

$$4x^2 - 4x + 1 + 4y^2 = 1$$

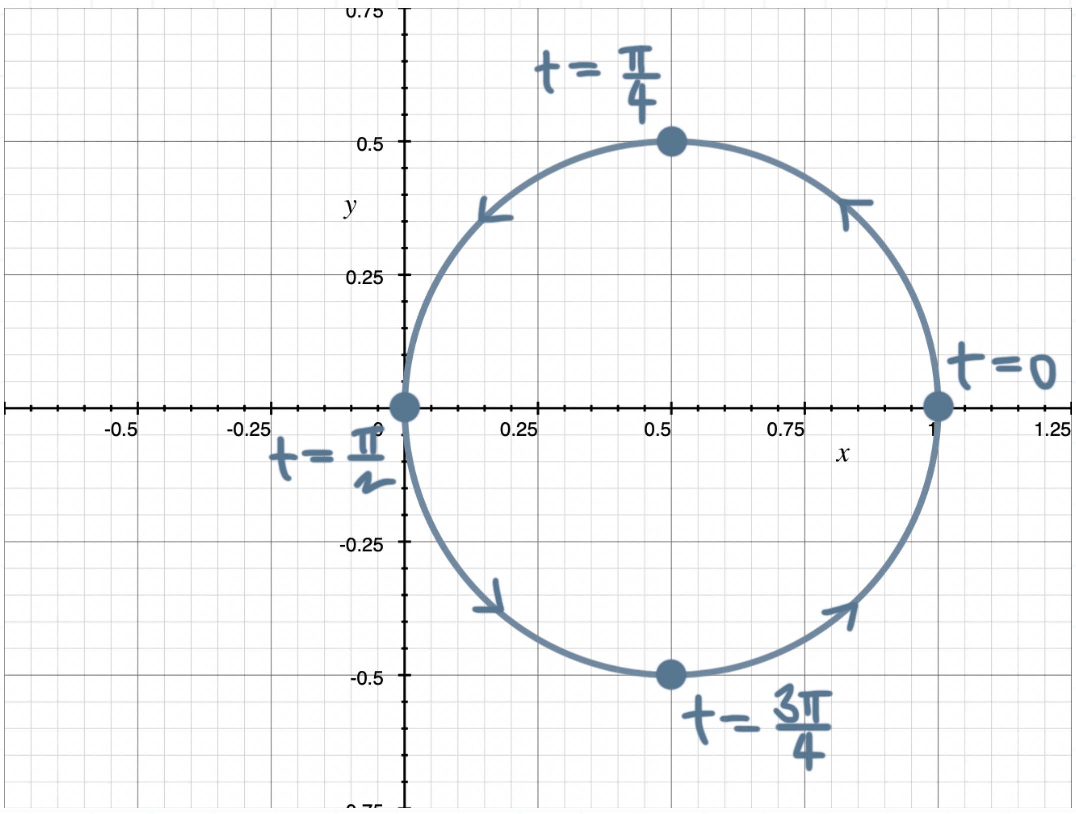
$$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

This is a circle centered at $(1/2,0)$ with radius $r = \sqrt{1/4} = 1/2$. So we can sketch the circle, but we still need to know the direction of increasing t, so we'll evaluate the parametric equations at a few values of t.

t	0	π/4	π/2	3π/4	π
x	1	1/2	0	1/2	1
y	0	1/2	0	-1/2	0

Then a sketch of the parametric equation is



0.75
0.5
y
0.25
-0.25
-0.5
-0.75
-0.5
-0.25
0
0.25
0.5
0.75
1
1.25
x
$t=\frac{\pi}{4}$
$t=0$
$t=\frac{\pi}{2}$
$t=\frac{3\pi}{4}$

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