Trigonometry Workbook Solutions

The law of sines and law of cosines

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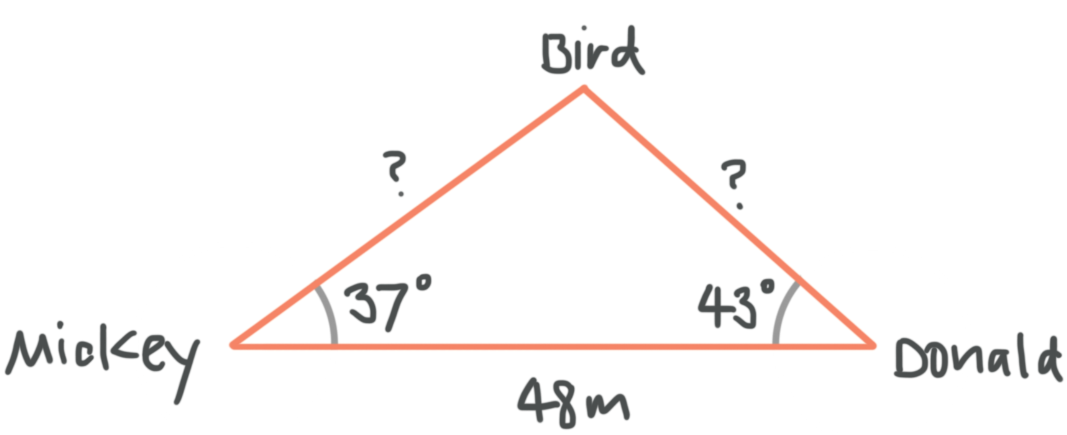
LAW OF SINES

■ 1. The interior angle measures of a triangle are 97°, 43°, and 40°. How many triangles can be made with these measurements?

Solution:

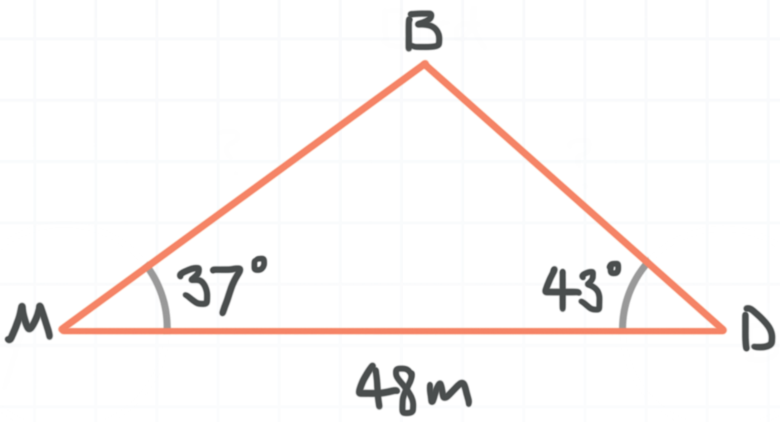
Given only the three interior angles measures of the triangle, it's impossible to determine the length of any side of the triangle. In other words, the interior angle measures determine the shape of the triangle, but not the size. Therefore, given only the angle measures, we could make an infinite number of triangles.

■ 2. Mickey and Donald stand on different sides of a tree. Each of them sees the same bird in the tree. They measure the angles of elevation from themselves to the bird, and get 37° and 43° respectively. If Mickey and Donald are 48 m apart, find the distances from Mickey and Donald to the bird.



Solution:

If M is Mickey, B is the bird, and D is Donald, then we have the triangle MBD.



The third interior angle of the triangle is

$$180^\circ - 37^\circ - 43^\circ$$

$$100^\circ$$

Then, plugging everything into the law of sines, we get

$$\frac{\overline{MD}}{\sin B} = \frac{\overline{MB}}{\sin D} = \frac{\overline{DB}}{\sin M}$$

$$\frac{48}{\sin 100^\circ} = \frac{\overline{MB}}{\sin 43^\circ} = \frac{\overline{DB}}{\sin 37^\circ}$$

Find the length of side MB using the first and second parts of this three-part equation.

$$\frac{48}{\sin 100^\circ} = \frac{\overline{MB}}{\sin 43^\circ}$$

$$\overline{MB} = \frac{48 \sin 43^\circ}{\sin 100^\circ} \approx 33 \text{ m}$$

Find the length of side DB using the first and third parts of the three-part equation.

$$\frac{48}{\sin 100^\circ} = \frac{\overline{DB}}{\sin 37^\circ}$$

$$\overline{DB} = \frac{48 \sin 37^\circ}{\sin 100^\circ} \approx 29 \text{ m}$$

Therefore, Mickey is about 33 m from the bird, and Donald is about 29 m from the bird.

3. If the measures of two interior angles of a triangle are 53° and 44°, and the length of the side opposite the 44° angle is 7, find the length b of the side opposite the 53° angle and the length c of the third side.

Solution:

We know the third interior angle has measure

$$180^\circ - 44^\circ - 53^\circ$$

$$83^\circ$$

Then the law of sines gives

$$\frac{7}{\sin 44^\circ} = \frac{b}{\sin 53^\circ} = \frac{c}{\sin 83^\circ}$$

Find b using the first two parts of this three-part equation.

$$\frac{7}{\sin 44^\circ} = \frac{b}{\sin 53^\circ}$$

$$b = \frac{7 \sin 53^\circ}{\sin 44^\circ} \approx \frac{7(0.799)}{0.695} \approx 8$$

Find c using the first and third parts of the three-part equation.

$$\frac{7}{\sin 44^\circ} = \frac{c}{\sin 83^\circ}$$

$$c = \frac{7 \sin 83^\circ}{\sin 44^\circ} \approx \frac{7(0.993)}{0.695} \approx 10$$

So the other two sides of the triangle have lengths $b \approx 8$ and $c \approx 10$.

■ 4. Solve the triangle with angle measures $A = 30^\circ$ and $C = 90^\circ$ and side length $c = 13$.

Solution:

The remaining angle of the triangle is

$$180^\circ - 30^\circ - 60^\circ$$

$$90^\circ$$

Plug all the values we have from the triangle into the law of sines.

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{13}{\sin 90^\circ}$$

Use values from the unit circle to simplify the equation.

$$\frac{a}{\frac{1}{2}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{13}{1}$$

$$2a = \frac{2b}{\sqrt{3}} = 13$$

$$2\sqrt{3}a = 2b = 13\sqrt{3}$$

Use the first and third parts of this three-part equation to find a.

$$2\sqrt{3}a = 13\sqrt{3}$$

$$2a = 13$$

$$a = \frac{13}{2}$$

Use the second and third parts of the three-part equation to find b.

$$2b = 13\sqrt{3}$$

$$b = \frac{13\sqrt{3}}{2}$$

Then the triangle has side lengths $a = 13/2$, $b = 13\sqrt{3}/2$, and $c = 13$, and angle measures $A = 30^\circ$, $B = 60^\circ$, and $C = 90^\circ$.

5. Solve the triangle with angle measures $A = 45^\circ$ and $B = 45^\circ$ and side length $c = 10$.

Solution:

The remaining angle of the triangle is

$$180^\circ - 45^\circ - 45^\circ$$

$$90^\circ$$

Plug all the values we have from the triangle into the law of sines.

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 45^\circ} = \frac{10}{\sin 90^\circ}$$

Use values from the unit circle to simplify the equation.

$$\frac{a}{\frac{\sqrt{2}}{2}} = \frac{b}{\frac{\sqrt{2}}{2}} = \frac{10}{1}$$

$$\frac{2a}{\sqrt{2}} = \frac{2b}{\sqrt{2}} = 10$$

$$2a = 2b = 10\sqrt{2}$$

$$a = b = 5\sqrt{2}$$

Then the triangle has side lengths $a = 5\sqrt{2}$, $b = 5\sqrt{2}$, and $c = 10$, and angle measures $A = 45^\circ$, $B = 45^\circ$, and $C = 90^\circ$.

■ 6. Find the lengths of the two unknown sides of a triangle with angle measures $A = 58^\circ$ and $B = 42^\circ$ and side length $a = 12$.

Solution:

The third interior angle of the triangle has measure

$$180^\circ - 58^\circ - 42^\circ$$

$$80^\circ$$

Plug the values we already have into the law of sines.

$$\frac{12}{\sin 58^\circ} = \frac{b}{\sin 42^\circ} = \frac{c}{\sin 80^\circ}$$

Use the first two parts of this three-part equation to solve for b.

$$\frac{12}{\sin 58^\circ} = \frac{b}{\sin 42^\circ}$$

$$b = \frac{12 \sin 42^\circ}{\sin 58^\circ} \approx \frac{12(0.6691)}{0.8480} \approx 9.47$$

Use the first third parts of the three-part equation to solve for c.

$$\frac{12}{\sin 58^\circ} = \frac{c}{\sin 80^\circ}$$

$$c = \frac{12 \sin 80^\circ}{\sin 58^\circ} \approx \frac{12(0.9848)}{0.8480} \approx 13.94$$

So the two unknown sides have lengths $b \approx 9.47$ and $c \approx 13.94$.

THE AMBIGUOUS CASE OF THE LAW OF SINES

1. A triangle has one side with length 15 and another with length 28. The angle opposite the side with length 15 is 128°. Complete the triangle.

Solution:

Let $a=15$ and $b=28$, and let $A=128^\circ$ be the angle opposite a. Substituting these values into the law of sines gives

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{15}{\sin 128^\circ}=\frac{28}{\sin B}=\frac{c}{\sin C}$$

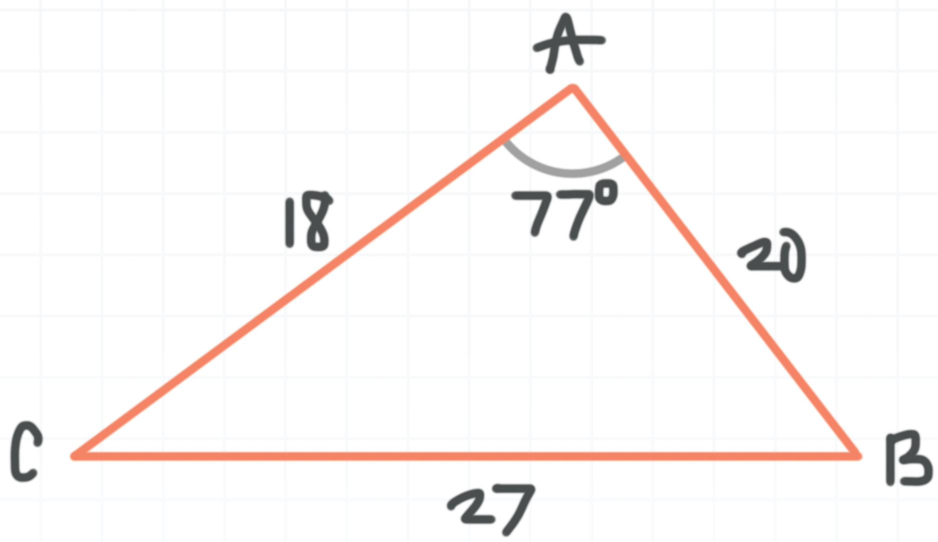
Use just the first two parts of this three-part equation in order to solve for B.

$$\frac{15}{\sin 128^\circ}=\frac{28}{\sin B}$$

$$\sin B=\frac{28\sin 128^\circ}{15}\approx\frac{28(0.788)}{15}\approx 1.47$$

Since the sine of an angle can't be greater than 1, it's impossible to build a triangle with these properties.

2. Find $\angle B$.



Solution:

Plugging what we know from the figure into the law of sines gives

$$\frac{27}{\sin 77^\circ} = \frac{18}{\sin B} = \frac{20}{\sin C}$$

Find B using the first two parts of this three-part equation.

$$\frac{27}{\sin 77^\circ} = \frac{18}{\sin B}$$

$$\sin B = \frac{18 \sin 77^\circ}{27} \approx 0.65$$

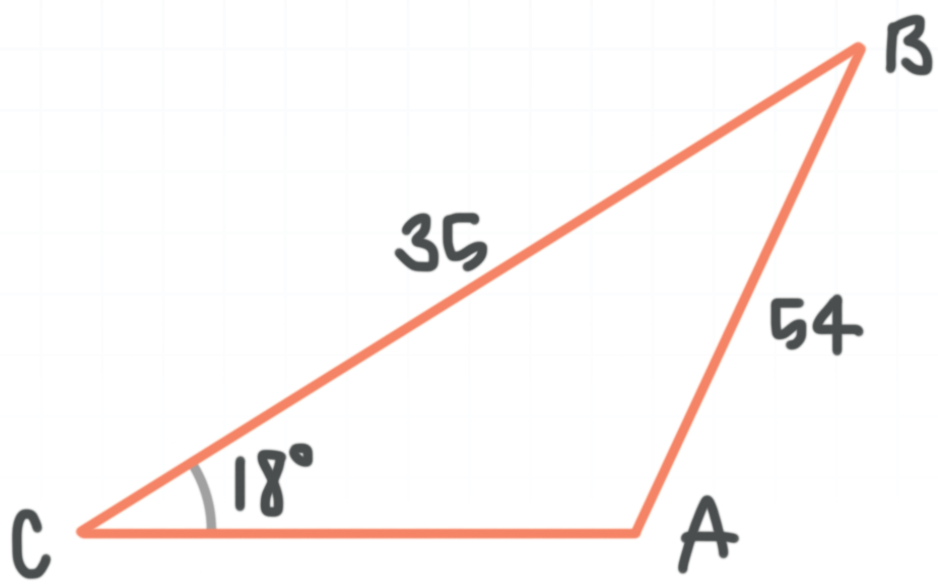
It's possible for $\sin B \approx 0.65$ in both the first and second quadrants. If we use a calculator to find $B \approx \arcsin 0.65$, we get the acute angle $B_1 \approx 40.5^\circ$. But to find $\sin B \approx 0.65$ in the second quadrant, we subtract $B_1 \approx 40.5^\circ$ from 180°.

$$B_2 \approx 180^\circ - 40.5^\circ$$

$$B_2 \approx 139.5^\circ$$

If $A = 77^\circ$ and $B_1 \approx 40.5^\circ$, then $C_1 \approx 180^\circ - 77^\circ - 40.5^\circ \approx 62.5^\circ$. If $A = 77^\circ$ and $B_2 \approx 139.5^\circ$, then $C_2 \approx 180^\circ - 77^\circ - 139.5^\circ \approx -36.5^\circ$. It's impossible to get a negative angle for C, so there's only one possible value for B, which is $B_1 \approx 40.5^\circ$.

■ 3. Find $\angle A$.



Solution:

Plugging what we know into the law of sines gives

$$\frac{35}{\sin A} = \frac{B}{\sin B} = \frac{54}{\sin 18^\circ}$$

Find A using the first and third parts of this three-part equation.

$$\frac{35}{\sin A} = \frac{54}{\sin 18^\circ}$$

$$\sin A = \frac{35 \sin 18^\circ}{54} \approx 0.2$$

It's possible for $\sin A \approx 0.2$ in both the first and second quadrants. If we use a calculator to find $A \approx \arcsin 0.2$, we get the acute angle $A_1 \approx 11.5^\circ$. But to find $\sin A \approx 0.2$ in the second quadrant, we subtract $A_1 \approx 11.5^\circ$ from 180°.

$$A_2 \approx 180^\circ - 11.5^\circ$$

$$A_2 \approx 169.5^\circ$$

If $C = 18^\circ$ and $A_1 \approx 11.5^\circ$, then $B_1 \approx 180^\circ - 18^\circ - 11.5^\circ \approx 150.5^\circ$. If $C = 18^\circ$ and $A_2 \approx 169.5^\circ$, then $B_2 \approx 180^\circ - 18^\circ - 169.5^\circ \approx -7.5^\circ$. It's impossible to get a negative angle for B, so there's only one possible value for A, which is $A_1 \approx 11.5^\circ$.

■ 4. If the lengths of two sides of a triangle are 18 and 34, and the measure of the interior angle opposite the side of length 34 is $B = 127^\circ$, find the length of the third side and the measures of angles A and C, where A is opposite the side of length 18.

Solution:

Plugging what we know into the law of sines gives

$$\frac{18}{\sin A} = \frac{34}{\sin 127^\circ} = \frac{c}{\sin C}$$

Find A using the first two parts of this three-part equation.

$$\frac{18}{\sin A} = \frac{34}{\sin 127^\circ}$$

$$\sin A = \frac{18 \sin 127^\circ}{34} \approx \frac{14.375}{34} \approx 0.423$$

If A is acute, then $A = 25^\circ$, and if angle A is obtuse, then $A = 155^\circ$. But it's impossible to have $A = 155^\circ$ because the sum of the interior angles A and B would be $155^\circ + 127^\circ = 282^\circ$, which exceeds 180°, so $A = 25^\circ$.

Then the third interior angle has measure

$$C = 180^\circ - 127^\circ - 25^\circ$$

$$C = 28^\circ$$

Find c using the second and third parts of the three-part equation.

$$\frac{34}{\sin 127^\circ} \approx \frac{c}{\sin 28^\circ}$$

$$c \approx \frac{34 \sin 28^\circ}{\sin 127^\circ} \approx \frac{34(0.469)}{0.799} \approx 20$$

So $A = 25^\circ$, $B = 127^\circ$, $C = 28^\circ$, $a = 18$, $b = 34$, and $c \approx 20$.

■ 5. A triangle has side lengths $a = 27$ and $c = 15$ and interior angle $A = 55^\circ$. Find all possible measures of the angle C to the nearest degree.

Solution:

Plugging what we know into the law of sines gives

$$\frac{27}{\sin 55^\circ} = \frac{b}{\sin B} = \frac{15}{\sin C}$$

Find C using the first and third parts of this three-part equation.

$$\frac{27}{\sin 55^\circ} = \frac{15}{\sin C}$$

$$\sin C = \frac{15 \sin 55^\circ}{27} \approx \frac{15(0.819)}{27} \approx 0.455$$

If C is acute then $C = 27^\circ$, and if C is obtuse then $C = 153^\circ$. But it's impossible to have $C = 153^\circ$ because the sum of interior angles A and C would be $153^\circ + 55^\circ = 208^\circ$, which exceeds 180°, so $C = 27^\circ$ is the only possible value of C.

■ 6. How many triangles are possible with side lengths 5 and 24, where the angle opposite the side with length 24 is 95°?

Solution:

Let $a = 24$ and $b = 5$, and let angle $A = 95^\circ$. Then, plugging what we know into the law of sines gives

$$\frac{24}{\sin 95^\circ} = \frac{5}{\sin B} = \frac{c}{\sin C}$$

Find B using the first two parts of this three-part equation.

$$\frac{24}{\sin 95^\circ} = \frac{5}{\sin B}$$

$$\sin B = \frac{5 \sin 95^\circ}{24} \approx 0.208$$

If B is acute then $B = 12^\circ$, and if B is obtuse then $B = 168^\circ$. But it's impossible to have $B = 168^\circ$ because the sum of interior angles A and B would be $168^\circ + 95^\circ = 263^\circ$, which exceeds 180°, so $B = 12^\circ$, which means only one triangle is possible.

AREA FROM THE LAW OF SINES

1. Find the area of the triangle in which two of the sides have lengths 15 and 24 and the measure of the included angle is 47°.

Solution:

Let $a = 15$ and $b = 24$. Then the included angle is $C = 47^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}(15)(24)\sin 47^\circ$$

$$\text{Area} \approx 180(0.731)$$

$$\text{Area} \approx 131.6$$

2. Find the area of the triangle with interior angles 101° and 25°, if the included side has length 23.

Solution:

The third angle in the triangle has measure

$$180^\circ - 101^\circ - 25^\circ = 54^\circ$$

Then, plugging everything we know into the law of sines, we get

$$\frac{a}{\sin 101^\circ} = \frac{b}{\sin 25^\circ} = \frac{23}{\sin 54^\circ}$$

Find a using the first and third parts of this three-part equation.

$$\frac{a}{\sin 101^\circ} = \frac{23}{\sin 54^\circ}$$

$$a = \frac{23 \sin 101^\circ}{\sin 54^\circ} \approx \frac{23(0.982)}{0.809} \approx 28$$

Use the law of sines for the area of a triangle.

$$\text{Area} = \frac{1}{2}ac \sin B$$

$$\text{Area} \approx \frac{1}{2}(28)(23)\sin 25^\circ$$

$$\text{Area} \approx \frac{644}{2}(0.423)$$

$$\text{Area} \approx 136$$

■ 3. Find the area of the triangle in which two of the sides have lengths 36 and 17 and the measure of the included angle is 90°.

Solution:

Let $a = 17$ and $b = 36$. Then the included angle is $C = 90^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

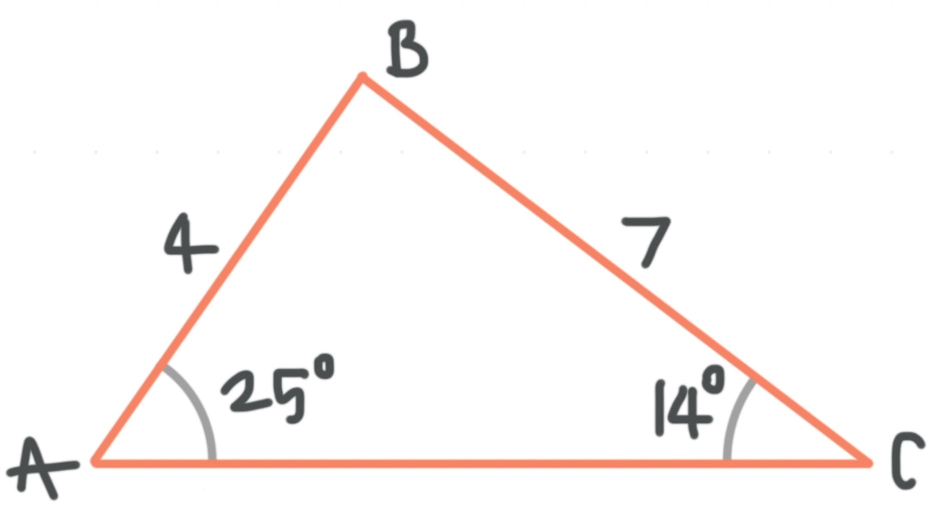
$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}(17)(36)\sin 90^\circ$$

$$\text{Area} = \frac{612}{2}(1)$$

$$\text{Area} = 306$$

■ 4. Find the area of the triangle.



Solution:

The third angle in the triangle has measure

$$180^\circ - 14^\circ - 25^\circ = 141^\circ$$

Let $a = 7$ and $c = 4$. Then the included angle is $B = 141^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}ac \sin B$$

$$\text{Area} = \frac{1}{2}(7)(4)\sin 141^\circ$$

$$\text{Area} \approx 14(0.629)$$

$$\text{Area} \approx 8.8$$

■ 5. Find the area of the triangle with interior angles 90° and 35°, if the included side has length 7.

Solution:

The third angle in the triangle has measure

$$180^\circ - 90^\circ - 35^\circ = 55^\circ$$

Then, plugging everything we know into the law of sines, we get

$$\frac{a}{\sin 35^\circ} = \frac{7}{\sin 55^\circ} = \frac{c}{\sin 90^\circ}$$

Find a using the first and second parts of this three-part equation.

$$\frac{a}{\sin 35^\circ} = \frac{7}{\sin 55^\circ}$$

$$a = \frac{7 \sin 35^\circ}{\sin 55^\circ} \approx \frac{7(0.574)}{0.819} \approx 5$$

Use the law of sines for the area of a triangle.

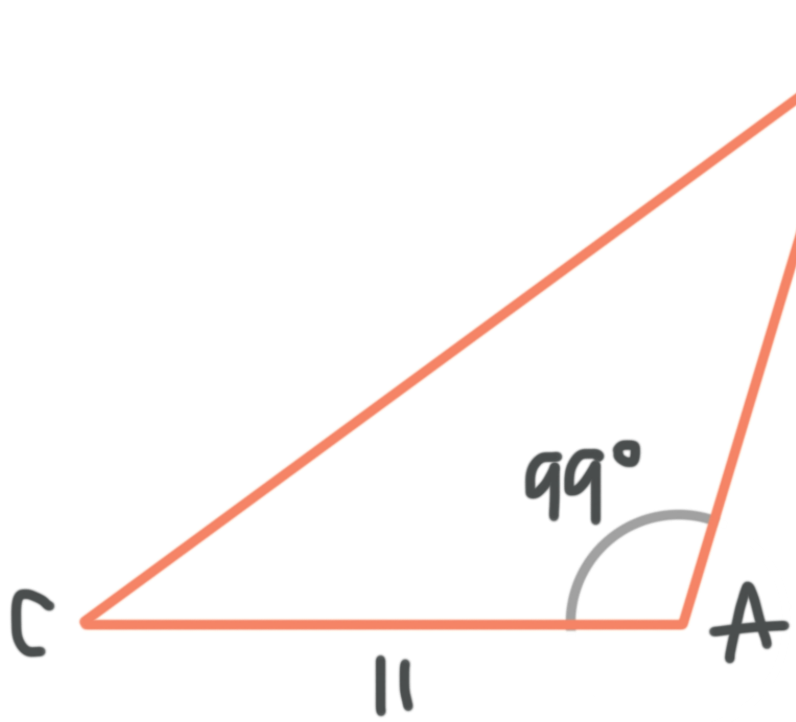
$$\text{Area} = \frac{1}{2} ab \sin C$$

$$\text{Area} \approx \frac{1}{2}(5)(7)\sin 90^\circ$$

$$\text{Area} \approx \frac{35}{2}(1)$$

$$\text{Area} \approx 17.5$$

■ 6. Find the area of a triangle.



Solution:

Let $b = 11$ and $c = 11$. Then the included angle is $A = 99^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}bc\sin A$$

$$\text{Area} = \frac{1}{2}(11)(11)\sin 99^\circ$$

$$\text{Area} \approx \frac{121}{2}(0.988)$$

$$\text{Area} \approx 60$$

LAW OF COSINES

■ 1. Solve the triangle where two of the sides are 18 and 13 and the measure of their included angle is 121°.

Solution:

Plugging what we know into the law of cosines with $\cos C$ gives

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 18^2 + 13^2 - 2(18)(13)\cos 121^\circ$$

$$c^2 = 324 + 169 - 26(18)\cos 121^\circ$$

$$c^2 = 493 - 468(-0.515)$$

$$c^2 \approx 734$$

$$c \approx 27$$

Rewrite the law of cosines with $\cos A$ by solving it for $\cos A$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{a^2 - (b^2 + c^2)}{-2bc}$$

Plug in what we know to find A.

$$\cos A \approx \frac{18^2 - (13^2 + 27^2)}{-2(13)(27)}$$

$$\cos A \approx \frac{324 - (169 + 729)}{-26(27)}$$

$$\cos A \approx \frac{324 - 898}{-702}$$

$$\cos A \approx \frac{-574}{-702}$$

$$\cos A \approx 0.818$$

$$A \approx \arccos 0.818$$

$$A \approx 35.1^\circ$$

The third angle is therefore

$$B \approx 180^\circ - 121^\circ - 35.1^\circ$$

$$B \approx 23.9^\circ$$

■ 2. If the side lengths of a triangle are $a = 15$, $b = 9$, and $c = 21$, what are the measures of its three interior angles?

Solution:

Rewrite the law of cosines with $\cos A$ by solving it for $\cos A$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{a^2 - (b^2 + c^2)}{-2bc}$$

Plugging in all three side lengths gives

$$\cos A \approx \frac{15^2 - (9^2 + 21^2)}{-2(9)(21)}$$

$$\cos A \approx \frac{225 - (81 + 441)}{-18(21)}$$

$$\cos A \approx \frac{225 - 522}{-378}$$

$$\cos A \approx \frac{-297}{-378}$$

$$\cos A = 0.786$$

$$A \approx \arccos 0.786$$

$$A \approx 38.2^\circ$$

Rewrite the law of cosines with $\cos B$ by solving it for $\cos B$.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\cos B = \frac{b^2 - (a^2 + c^2)}{-2ac}$$

Plugging in all three side lengths gives

$$\cos B \approx \frac{9^2 - (15^2 + 21^2)}{-2(15)(21)}$$

$$\cos B \approx \frac{81 - (225 + 441)}{-30(21)}$$

$$\cos B \approx \frac{81 - 666}{-630}$$

$$\cos B \approx \frac{-585}{-630}$$

$$\cos B \approx 0.929$$

$$B \approx \arccos 0.929$$

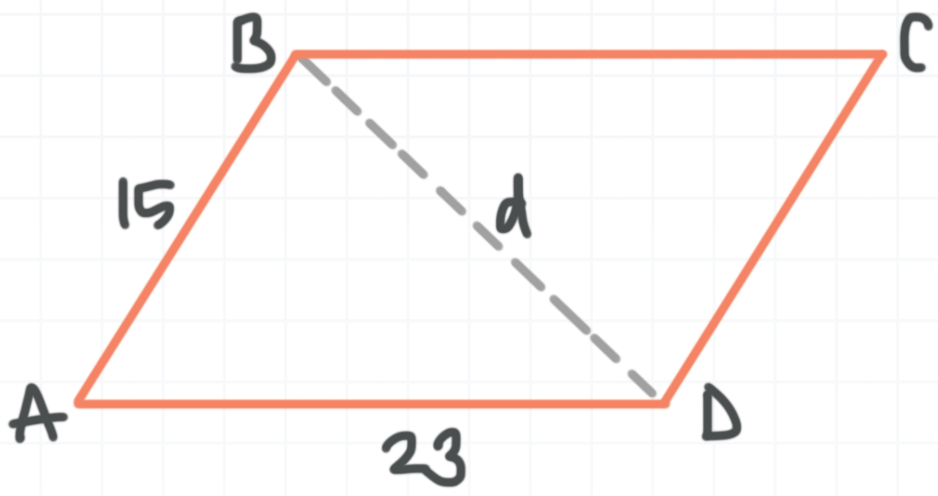
$$B \approx 21.7^\circ$$

Then the third angle is

$$C = 180^\circ - 38.2^\circ - 21.7^\circ$$

$$C = 120.3^\circ$$

■ 3. If the measure of the angle B is 56°, find the length of the parallelogram's diagonal, d, to the nearest centimeter. Hint: Consecutive angles of a parallelogram are supplementary, so $m\angle A + m\angle B = 180^\circ$.



Solution:

If consecutive angles of the parallelogram are supplementary, then

$$m\angle A + m\angle B = 180^\circ$$

$$m\angle A = 180^\circ - m\angle B$$

$$m\angle A = 180^\circ - 56^\circ$$

$$m\angle A = 124^\circ$$

Plugging what we know into the law of cosines with $\cos A$ gives

$$d^2 = \overline{AB}^2 + \overline{AD}^2 - 2(\overline{AB})(\overline{AD})\cos A$$

$$d^2 = 15^2 + 23^2 - 2(15)(23)\cos 124^\circ$$

$$d^2 = 225 + 529 - 30(23)\cos 124^\circ$$

$$d^2 = 754 - 690(-0.559)$$

$$d^2 \approx 1{,}140$$

$$d \approx 33.8$$

4. Solve the triangle where two of the sides are 27 and 14 and the measure of their included angle is 33°.

Solution:

Plugging what we know into the law of cosines with $\cos C$ gives

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 27^2 + 14^2 - 2(27)(14)\cos 33^\circ$$

$$c^2 = 729 + 196 - 54(14)\cos 33^\circ$$

$$c^2 = 925 - 756(0.839)$$

$$c^2 \approx 290.7$$

$$c \approx 17$$

Rewrite the law of cosines with $\cos A$ by solving it for $\cos A$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{a^2 - (b^2 + c^2)}{-2bc}$$

Plug in what we know to find A.

$$\cos A \approx \frac{27^2 - (14^2 + 17^2)}{-2(14)(17)}$$

$$\cos A \approx \frac{729 - (196 + 291)}{-28(17)}$$

$$\cos A \approx \frac{729 - 487}{-476}$$

$$\cos A \approx \frac{242}{-476}$$

$$\cos A \approx -0.508$$

$$A \approx \arccos(-0.508)$$

$$A \approx 120.5^\circ$$

The third angle is therefore

$$B \approx 180^\circ - 33^\circ - 120.5^\circ$$

$$B \approx 26.5^\circ$$

■ 5. If the side lengths of a triangle are $a = 17$, $b = 25$, and $c = 28$, what are the measures of its three interior angles?

Solution:

Rewrite the law of cosines with $\cos C$ by solving it for $\cos C$.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Plugging in all three side lengths gives

$$\cos C = \frac{17^2 + 25^2 - 28^2}{2(17)(25)}$$

$$\cos C = \frac{289 + 625 - 784}{34(25)}$$

$$\cos C = \frac{130}{850}$$

$$\cos C \approx 0.153$$

$$C \approx \arccos 0.153$$

$$C \approx 81.2^\circ$$

Rewrite the law of cosines with $\cos B$ by solving it for $\cos B$.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\cos B = \frac{a^2 - b^2 + c^2}{2ac}$$

Plugging in all three side lengths gives

$$\cos B = \frac{17^2 - 25^2 + 28^2}{2(17)(28)}$$

$$\cos B = \frac{289 - 625 + 784}{34(28)}$$

$$\cos B = \frac{448}{952}$$

$$\cos B \approx 0.471$$

$$B \approx \arccos 0.471$$

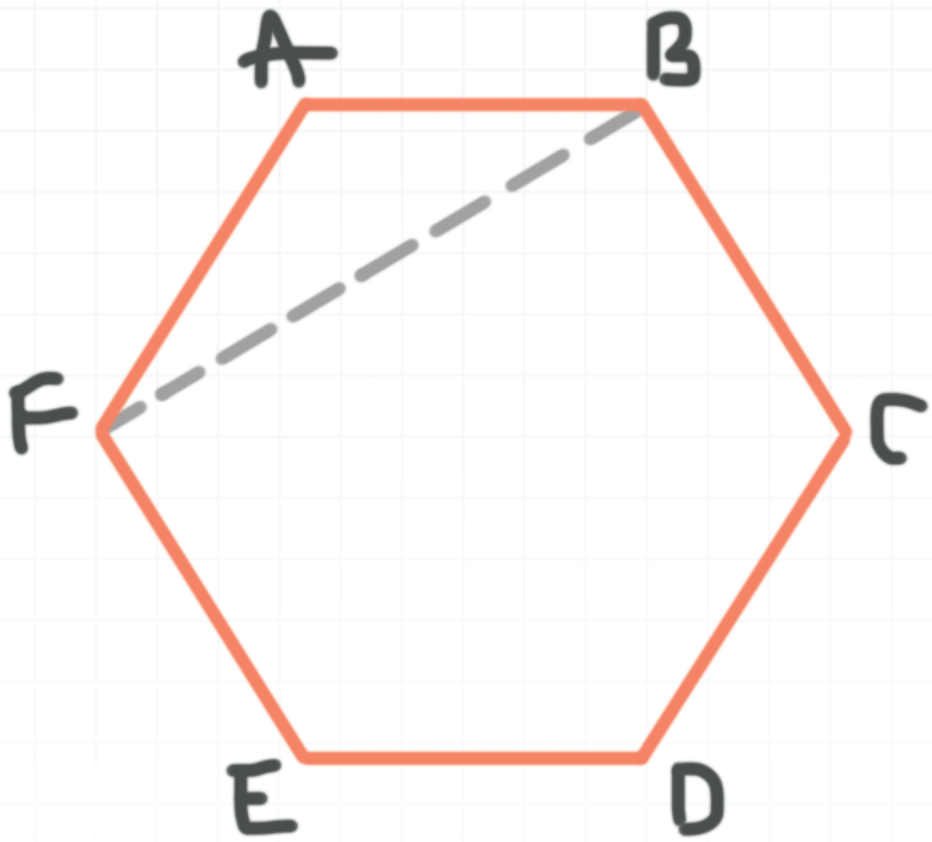
$$B \approx 61.9^\circ$$

Then the third angle is

$$C \approx 180^\circ - 81.2^\circ - 61.9^\circ$$

$$C \approx 36.9^\circ$$

■ 6. A regular hexagon (all side lengths are equal, and all interior angles are equal) has side lengths of 20 inches. Find $\overline{FB}$ to the nearest tenth. Hint: The sum of the interior angles of a hexagon is 720°.



Solution:

The sum of the interior angles of the hexagon is 720°. Since the hexagon is regular, all the interior angles are equal.

$$m\angle A = \frac{720^\circ}{6} = 120^\circ$$

Plugging what we know into the law of cosines with $\cos A$ gives

$$\overline{FB}^2 = \overline{FA}^2 + \overline{AB}^2 - 2(\overline{FA})(\overline{AB})\cos A$$

$$\overline{FB}^2 = 20^2 + 20^2 - 2(20)(20)\cos 120^\circ$$

$$\overline{FB}^2 = 400 + 400 - 40(20)\cos 120^\circ$$

$$\overline{FB}^2 = 800 - 800(-0.5)$$

$$\overline{FB}^2 \approx 1{,}200$$

$$\overline{FB} \approx 34.6$$

HERON'S FORMULA

1. The lengths of the sides of a triangle have a ratio of $5:8:12$, and the triangle's perimeter is 200 cm. Find the area of the triangle.

Solution:

Since the sides of the triangle have a ratio of $5:8:12$ and the perimeter is 200 cm, we can write

$$5x + 8x + 12x = 200$$

$$25x = 200$$

$$x = \frac{200}{25}$$

$$x = 8$$

Then the sides of the triangle will be 40 cm, 64 cm, and 96 cm.

Then find s, which is half the perimeter of the triangle.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(p) = \frac{1}{2}(200) = 100$$

By Heron's formula, the area of the triangle is

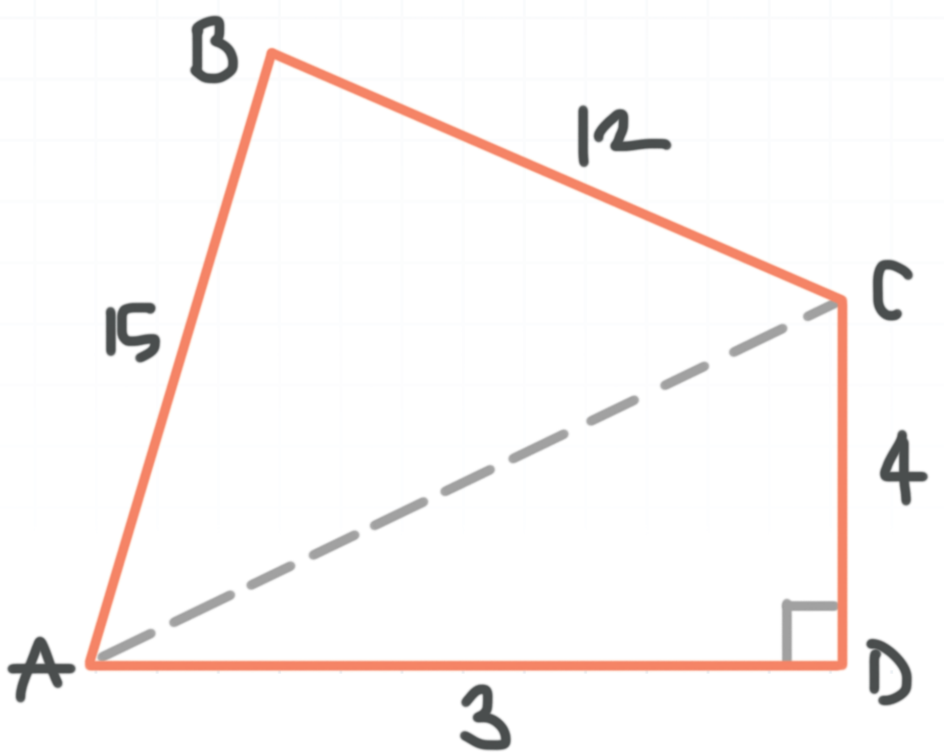
$$\text{Area} = \sqrt{100(100 - 40)(100 - 64)(100 - 96)}$$

$$\text{Area} = \sqrt{100(60)(36)(4)}$$

$$\text{Area} = \sqrt{864{,}000}$$

$$\text{Area} \approx 930 \text{ cm}^2$$

2. Find the area of the quadrilateral, given that it's made of two separate triangles.



Solution:

To find the area of the quadrilateral we need to find the sum of the area of two triangles.

The area of the right triangle ACD is

$$A_1 = \frac{1}{2}(AD)(CD)$$

$$A_1 = \frac{1}{2}(3)(4)$$

$$A_1 = 6$$

Now we need to find the hypotenuse AC. Using the Pythagorean theorem, we get

$$AC^2 = AD^2 + CD^2$$

$$AC^2 = 3^2 + 4^2$$

$$AC^2 = 9 + 16$$

$$AC^2 = 25$$

$$AC = 5$$

Now we can use Heron's formula to find the area of triangle ABC, but first we'll have to find s, which is half the perimeter of the triangle.

$$s = \frac{1}{2}(AB + BC + AC) = \frac{1}{2}(15 + 12 + 5) = \frac{1}{2}(32) = 16$$

Then by Heron's formula, the area of the triangle is

$$A_2 = \sqrt{16(16 - 15)(16 - 12)(16 - 5)}$$

$$A_2 = \sqrt{16(1)(4)(11)}$$

$$A_2 = \sqrt{704}$$

$$A_2 \approx 26.5$$

So the area of the quadrilateral is

$$\text{Area} = A_1 + A_2$$

$$\text{Area} \approx 6 + 26.5$$

$$\text{Area} \approx 32.5$$

3. A triangle and a parallelogram have the same base and the same area. If the sides of the triangle are 12 cm, 14 cm, and 16 cm, and the parallelogram has a base of 14 cm, find the height of the parallelogram. Hint: The area of a parallelogram is $A = bh$, where b is its base and h is its height.

Solution:

Find s, which is half the perimeter of the triangle.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(12 + 14 + 16) = \frac{1}{2}(26 + 16) = \frac{1}{2}(42) = 21$$

Then by Heron's formula, the area of the triangle is

$$\text{Area} = \sqrt{21(21 - 12)(21 - 14)(21 - 16)}$$

$$\text{Area} = \sqrt{21(9)(7)(5)}$$

$$\text{Area} = \sqrt{6{,}615}$$

$$\text{Area} \approx 81$$

Since the triangle and the parallelogram have the same base and the same area, and the parallelogram has a base of 14 cm,

$$14 \times h = 81$$

$$h = \frac{81}{14}$$

$$h \approx 5.8 \text{ cm}$$

■ 4. Find the area of a triangle with side lengths 34 cm and 29 cm, if half its perimeter is 62 cm.

Solution:

Since half the perimeter of the triangle is 62 cm, we can find the length of third side.

$$s = \frac{1}{2}(a + b + c)$$

$$62 = \frac{1}{2}(34 + 29 + c)$$

$$124 = 63 + c$$

$$c = 124 - 63 = 61$$

Then by Heron's formula, the area of the triangle is

$$\text{Area} = \sqrt{62(62 - 34)(62 - 29)(62 - 61)}$$

$$\text{Area} = \sqrt{62(28)(33)(1)}$$

$$\text{Area} = \sqrt{57{,}288}$$

$$\text{Area} \approx 239 \text{ cm}^2$$

■ 5. An isosceles triangle (a triangle with two equal side lengths) has a half perimeter of 48 in. Its two equal sides measure 27 in each. Find the area of the triangle.

Solution:

Since half the perimeter of the triangle is 48 in, we can find the length of third side.

$$s = \frac{1}{2}(a + b + c)$$

$$48 = \frac{1}{2}(27 + 27 + c)$$

$$96 = 54 + c$$

$$c = 42$$

Then by Heron's formula, the area of the triangle is

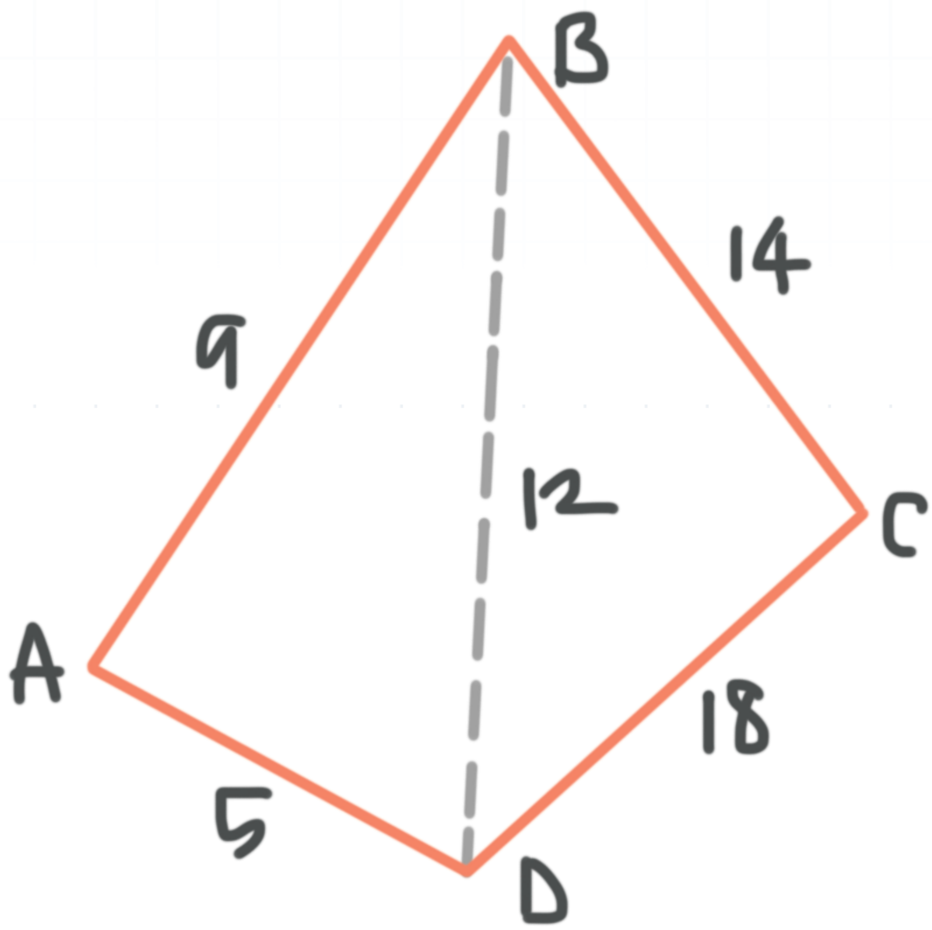
$$\text{Area} = \sqrt{48(48-27)(48-27)(48-42)}$$

$$\text{Area} = \sqrt{48(21)(21)(6)}$$

$$\text{Area} = \sqrt{127{,}008}$$

$$\text{Area} \approx 356 \text{ in}^2$$

6. Find the area of the quadrilateral by finding the sum of the areas of the triangles.



Solution:

Find s_1, which is half the perimeter of triangle ABD.

$$s_1 = \frac{1}{2}(AB + BD + AD) = \frac{1}{2}(9 + 12 + 5) = \frac{1}{2}(26) = 13$$

Then by Heron's formula, the area of triangle ABD is

$$A_1 = \sqrt{13(13 - 9)(13 - 12)(13 - 5)}$$

$$A_1 = \sqrt{13(4)(1)(8)}$$

$$A_1 = \sqrt{416}$$

$$A_1 \approx 20$$

Find s_2, which is half the perimeter of triangle BCD.

$$s_2 = \frac{1}{2}(BC + CD + BD) = \frac{1}{2}(14 + 18 + 12) = \frac{1}{2}(44) = 22$$

Then by Heron's formula, the area of triangle BCD is

$$A_2 = \sqrt{22(22 - 14)(22 - 18)(22 - 12)}$$

$$A_2 = \sqrt{22(8)(4)(10)}$$

$$A_2 = \sqrt{7{,}040}$$

$$A_2 \approx 84$$

Then the area of the quadrilateral is

$$\text{Area} = A_1 + A_2$$

$$\text{Area} = 20 + 84$$

$$\text{Area} = 104$$

W W W . K R I S T A K I N G M A T H . C O M