Computational Topology
Homework 1

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January 1, 2023

1 Theoretical problems

1.1 Homology

The simplicial complex is named X.

(a)

$$C_2 = \langle AFD, FBE, DEC \rangle$$
$$C_1 = \langle AF, AD, FD, FB, FE, BE, DE, DC, EC \rangle$$
$$C_0 = \langle A, B, C, D, E, F \rangle$$

(b) The boundary homomorphisms connect the chain groups into a sequence:

$$\langle 0 \rangle \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} \langle 0 \rangle$$

We have

$$\delta_2(AFD) = AF + FD - AD$$
$$\delta_2(FBE) = FB + BE - FE$$
$$\delta_2(DEC) = DE + EC - DC$$

$$\delta_1(AF) = F - A$$
$$\delta_1(AD) = D - A$$
$$\delta_1(FD) = D - F$$
$$\delta_1(FB) = B - F$$
$$\delta_1(FE) = E - F$$
$$\delta_1(BE) = E - B$$
$$\delta_1(DE) = E - D$$
$$\delta_1(DC) = C - D$$

$$\delta_1(EC) = C - E$$

$$\delta_0(A) = 0$$
$$\delta_0(B) = 0$$
$$\delta_0(C) = 0$$
$$\delta_0(D) = 0$$
$$\delta_0(E) = 0$$
$$\delta_0(F) = 0$$

(c) There are no cycles for $n = 2$ because X has three 2-simplices and there needs to be at least four, to form a 2-cycle. So,

$$Z_2 = \ker\delta_2 = \langle 0 \rangle$$

There are four linearly independent cycles for $n = 1$, namely $AF + FD - AD$, $FB + BE - FE$, $DE + EC - DC$ and $FD - FE + DE$, so:

$$Z_1 = \ker\delta_1 = \langle AF + FD - AD, FB + BE - FE, DE + EC - DC, FD - FE + DE \rangle$$

Since δ_0 maps all vertices to 0, for $n = 0$ I have:

$$Z_0 = \ker\delta_0 = \langle A, B, C, D, E, F \rangle$$

(d) Boundaries for $n = 2, 1, 0$ are:

$$B_2 = \mathrm{im}\delta_3 = \langle 0 \rangle$$
$$B_1 = \mathrm{im}\delta_2 = \langle AF + FD - AD, FB + BE - FE, DE + EC - DC \rangle$$
$$B_0 = \mathrm{im}\delta_1 = \langle F - A, D - A, D - F, B - F, E - F, E - B, E - D, C - D \rangle$$

(e) In this part of the exercise I calculated the homology groups with $\mathbb{Z}$ coefficients $H_n(X, \mathbb{Z}) = \frac{Z_n}{B_n}$ for $n = 2, 1, 0$:

$$H_2 = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$$

$$H_1 = \frac{\langle AF + FD - AD, FB + BE - FE, DE + EC - DC, FD - FE + DE \rangle}{\langle AF + FD - AD, FB + BE - FE, DE + EC - DC \rangle} = \langle FD - FE + DE \rangle = \mathbb{Z}$$

$$H_0 = \frac{Z_0}{B_0} = \frac{\langle A, B, C, D, E, F \rangle}{\langle F - A, D - A, D - F, B - F, E - F, E - B, E - D, C - D \rangle} = \langle A \rangle = \mathbb{Z}$$

(e) In this part of the exercise I calculated the homology groups with $\mathbb{Z}_2$ coefficients $H_n(X, \mathbb{Z}_2) = \frac{Z_n}{B_n}$ for $n = 2, 1, 0$. The calculation does not change much, because in $\mathbb{Z}_2$ it

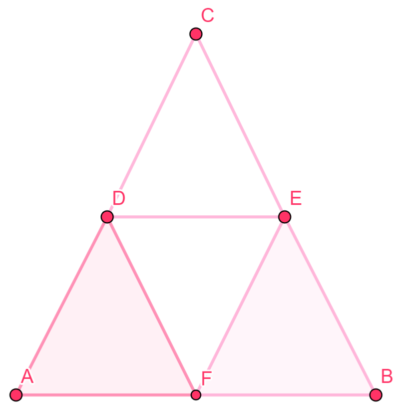

Figure 1: Drawing of a simplicial complex X from exercise 2.

holds $A = -A$ and consequently $B - A = B + A$ and so on. The equality $B + A = 0$ still implies $B = A$ so for $n = 0$ the elements still cancel each other out. The same also holds for other n. So I got:

$$H_2 = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$$

$$H_1 = \frac{\langle AF + FD + AD, FB + BE + FE, DE + EC + DC, FD + FE + DE \rangle}{\langle AF + FD + AD, FB + BE + FE, DE + EC + DC \rangle} = \langle FD{+}FE{+}DE \rangle = \mathbb{Z}_2$$

$$H_0 = \frac{Z_0}{B_0} = \frac{\langle A, B, C, D, E, F \rangle}{\langle F + A, D + A, D + F, B + F, E + F, E + B, E + D, C + D \rangle} = \langle A \rangle = \mathbb{Z}_2$$

(f) The Betti numbers of X are $b_2 = 0$, $b_1 = 1$ and $b_0 = 1$.
(g) The Euler characteristic of X is $\chi(X) = 1 - 1 + 0 = 0$.

1.2 Homology

The simplicial complex is named X.

(a) X as a planar 2-dimensional simplicial complex can be seen on Figure 1.
(b)

$$C_2 = \langle AFD, FBE \rangle$$

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Boundaries for $n = 2, 1, 0$ are:

$$B_2 = \text{im}\delta_3 = \langle 0 \rangle$$

$$B_1 = \text{im}\delta_2 = \langle AF + FD - AD, FB + BE - FE \rangle$$

$$B_0 = \text{im}\delta_1 = \langle F - A, D - A, D - F, B - F, E - F, E - B, E - D, C - D \rangle$$

(e) In this part of the exercise I calculated the homology groups with $\mathbb{Z}$ coefficients $H_n(X, \mathbb{Z}) = \frac{Z_n}{B_n}$ for $n = 2, 1, 0$:

$$H_2 = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$$

$$H_1 = \frac{\langle AF + FD - AD, FB + BE - FE, DE + EC - DC, FD - FE + DE \rangle}{\langle AF + FD - AD, FB + BE - FE \rangle} =$$

$$= \langle DE + EC - DC, FD - FE + DE \rangle = \mathbb{Z} \oplus \mathbb{Z}$$

$$H_0 = \frac{Z_0}{B_0} = \frac{\langle A, B, C, D, E, F \rangle}{\langle F - A, D - A, D - F, B - F, E - F, E - B, E - D, C - D \rangle} = \langle A \rangle = \mathbb{Z}$$

(f) In this part of the exercise I calculated the homology groups with $\mathbb{Z}_2$ coefficients $H_n(X, \mathbb{Z}_2) = \frac{Z_n}{B_n}$ for $n = 2, 1, 0$. The calculation does not change much, because in $\mathbb{Z}_2$ it holds $A = -A$ and consequently $B - A = B + A$ and so on. The equality $B + A = 0$ still implies $B = A$ so for $n = 0$ the elements still cancel each other out. The same also holds for other n. So I got:

$H_n(X, \mathbb{Z}_2) = \frac{Z_n}{B_n}$ for $n = 2, 1, 0$:

$$H_2 = \frac{Z_2}{B_2} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$$

$$H_1 = \frac{\langle AF + FD + AD, FB + BE + FE, DE + EC + DC, FD + FE + DE \rangle}{\langle AF + FD + AD, FB + BE + FE \rangle} =$$

$$= \langle DE + EC + DC, FD + FE + DE \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$H_0 = \frac{Z_0}{B_0} = \frac{\langle A, B, C, D, E, F \rangle}{\langle F + A, D + A, D + F, B + F, E + F, E + B, E + D, C + D \rangle} = \langle A \rangle = \mathbb{Z}_2$$

(g)
The Betti numbers of X are $b_2 = 0$, $b_1 = 2$ and $b_0 = 1$.
The Euler characteristic of X is $\chi(X) = 1 - 2 + 0 = -1$.

2 Programming problems

References