Computational Topology
Homework 1

Bernarda Petek

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1 Theoretical problems

1.1 Exploring different metrics

(a) I determined the distances between points $(1,2),(2,4)$ and $(2,-1)$ in all three metrics.

First, I determined the distances in metric α. No two given points were the same so I always used the second part of the α distance definition:

$$\alpha((1,2),(2,4)) = \sqrt{1^2+2^2} + \sqrt{2^2+4^2} = \sqrt{5} + \sqrt{20} = \sqrt{5} + \sqrt{4\cdot 5} = \sqrt{5} + 2\cdot\sqrt{5} = 3\cdot\sqrt{5}$$

$$\alpha((1,2),(2,-1)) = \sqrt{1^2+2^2} + \sqrt{2^2+(-1)^2} = \sqrt{5} + \sqrt{5} = 2\cdot\sqrt{5}$$

$$\alpha((2,4),(2,-1)) = \sqrt{2^2+4^2} + \sqrt{2^2+(-1)^2} = \sqrt{20} + \sqrt{5} = 2\cdot\sqrt{5} + \sqrt{5} = 3\cdot\sqrt{5}$$

Next, I determined the distances in metric β. Pair of points $(1,2)$ and $(2,4)$ satisfied the first condition (i.e. $1\cdot 4 = 2\cdot 2$), but the other two pairs of points $(1,2),(2,-1)$ and $(2,4),(2,-1)$ did not. ($1\cdot(-1) \neq 2\cdot 2$ and $2\cdot(-1) \neq 2\cdot 4$, respectively):

$$\beta((1,2),(2,4)) = \sqrt{(1-2)^2+(2-4)^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}$$

$$\beta((1,2),(2,-1)) = \sqrt{1^2+2^2} + \sqrt{2^2+(-1)^2} = \sqrt{5} + \sqrt{5} = 2\cdot\sqrt{5}$$

$$\beta((2,4),(2,-1)) = \sqrt{2^2+4^2} + \sqrt{2^2+(-1)^2} = \sqrt{20} + \sqrt{5} = 2\cdot\sqrt{5} + \sqrt{5} = 3\cdot\sqrt{5}$$

Lastly, I determined the distances in metric γ. The first pair of points $(1,2),(2,4)$ did not satisfy the first condition: $1 \neq 2$. The second pair of points $(1,2),(2,-1)$ also did not satisfy the first condition: $1 \neq 2$. However, the last pair of points $(2,4),(2,-1)$ did satisfy the first condition: $2 = 2$.

$$\gamma((1,2),(2,4)) = |2| + |2-1| + |4| = |2| + |1| + |4| = 7$$

$$\gamma((1,2),(2,-1)) = |2| + |2-1| + |-1| = |2| + |1| + |-1| = 2+1+1 = 4$$

$$\gamma((2,4),(2,-1)) = |-1-4| = |-5| = 5$$

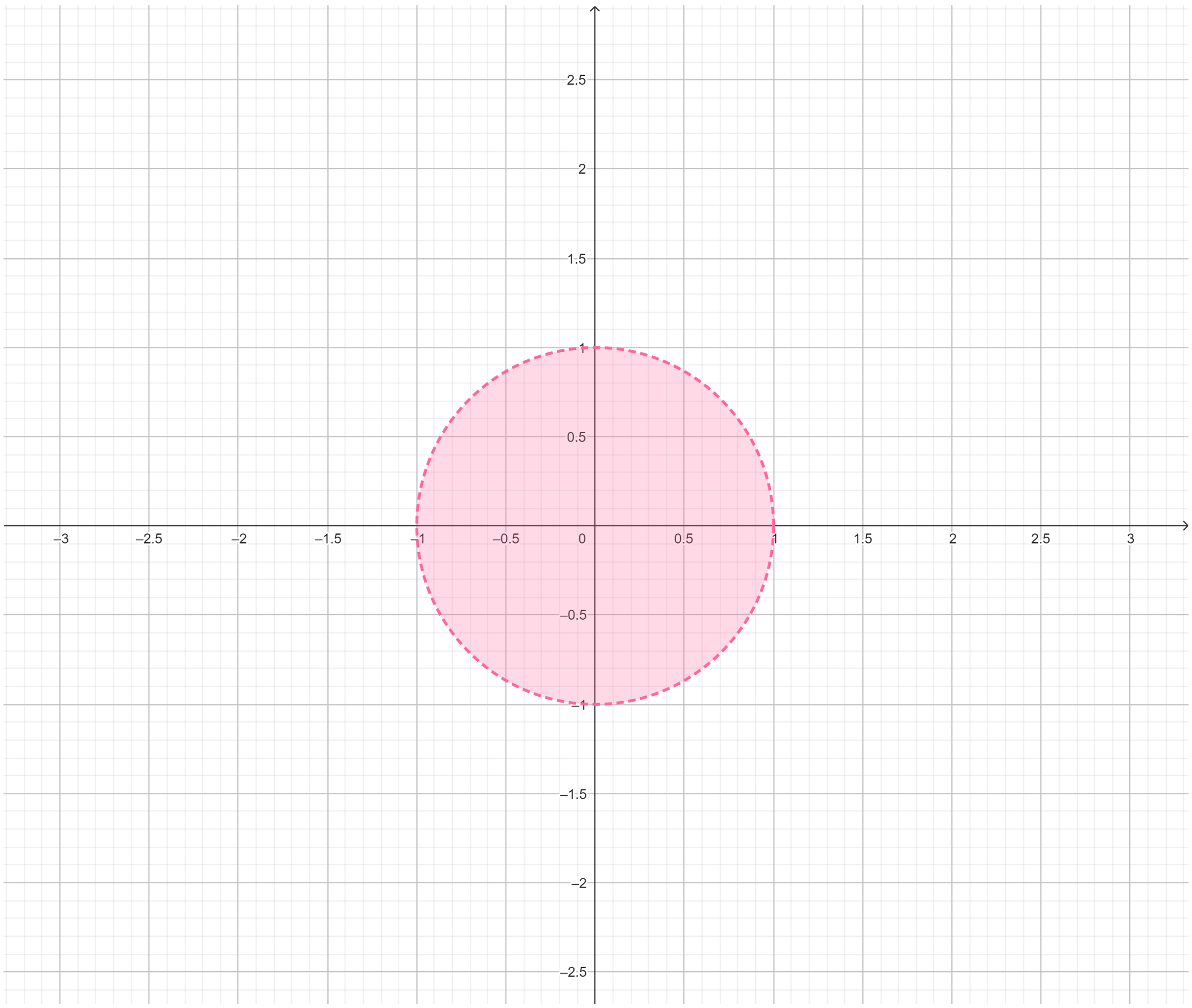

Figure 1: Open ball $B((0,0),1)$ in α metric

(b) I drew the open balls $B((0,0),1)$, $B((0,1),2)$ and $B((1,2),1+\sqrt{5})$ in α metric. The centre of the ball is always contained in it because the α distance from and to itself is always 0 (first condition) and 0 is always smaller than the radius $r>0$ of the open ball. The drawings can be seen on Figures 1, 2 and 3. The calculations I made were:

$$\begin{aligned} B((0,0),1) &= \{(x_1,x_2)\in\mathbb{R};\ \alpha((0,0),(x_1,x_2))<1\} \\ &= \{(x_1,x_2)\in\mathbb{R};\ x_1=0\wedge x_2=0\wedge 0<1\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\vee x_2\neq 0\wedge\sqrt{x_1^2+x_2^2}<1 \\ &= \{(0,0)\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\vee x_2\neq 0\wedge x_1^2+x_2^2<1\} \\ &= \{(x_1,x_2)\in\mathbb{R};\ x_1^2+x_2^2<1\} \end{aligned}$$

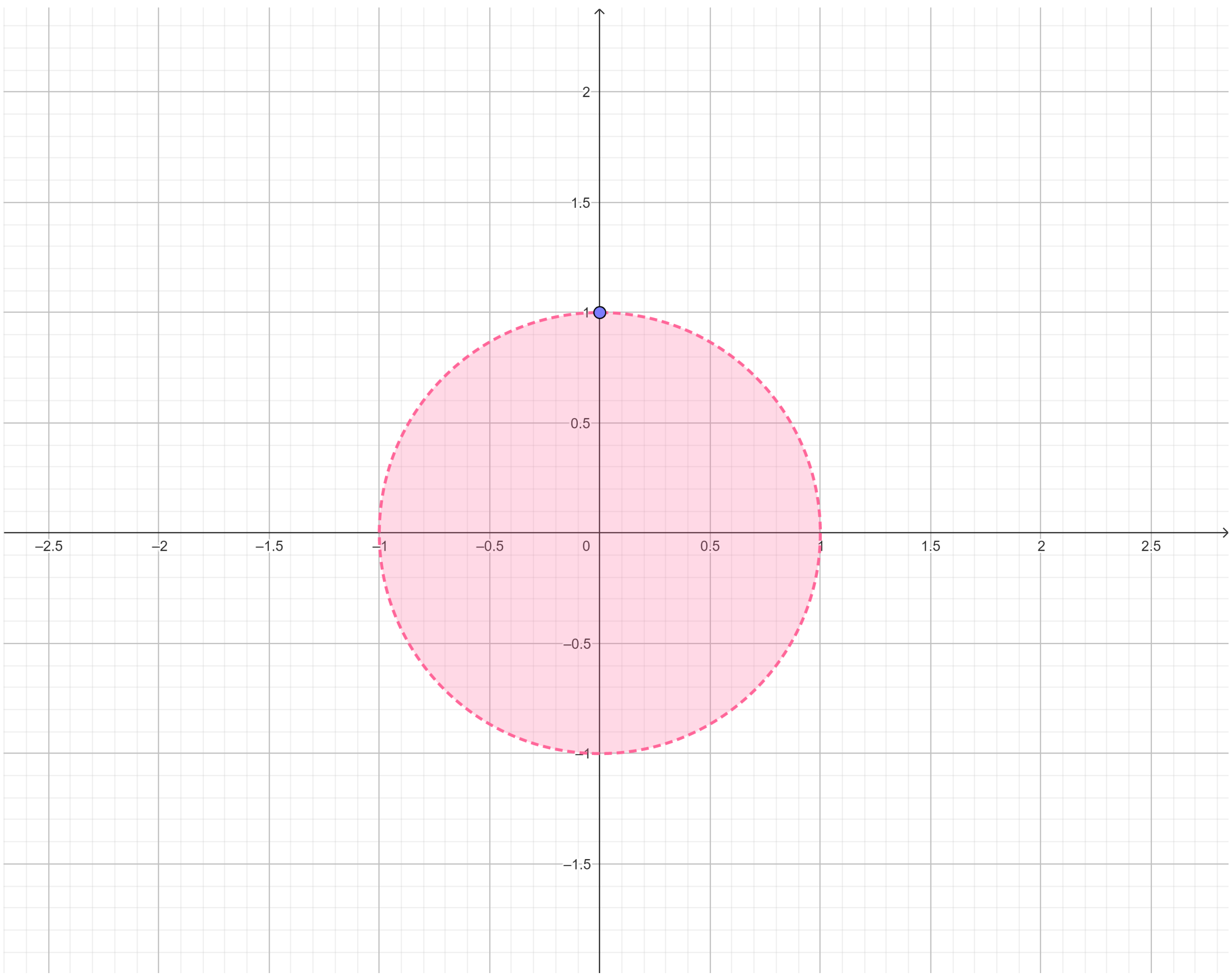

Figure 2: Open ball $B((0,1),2)$ in α metric

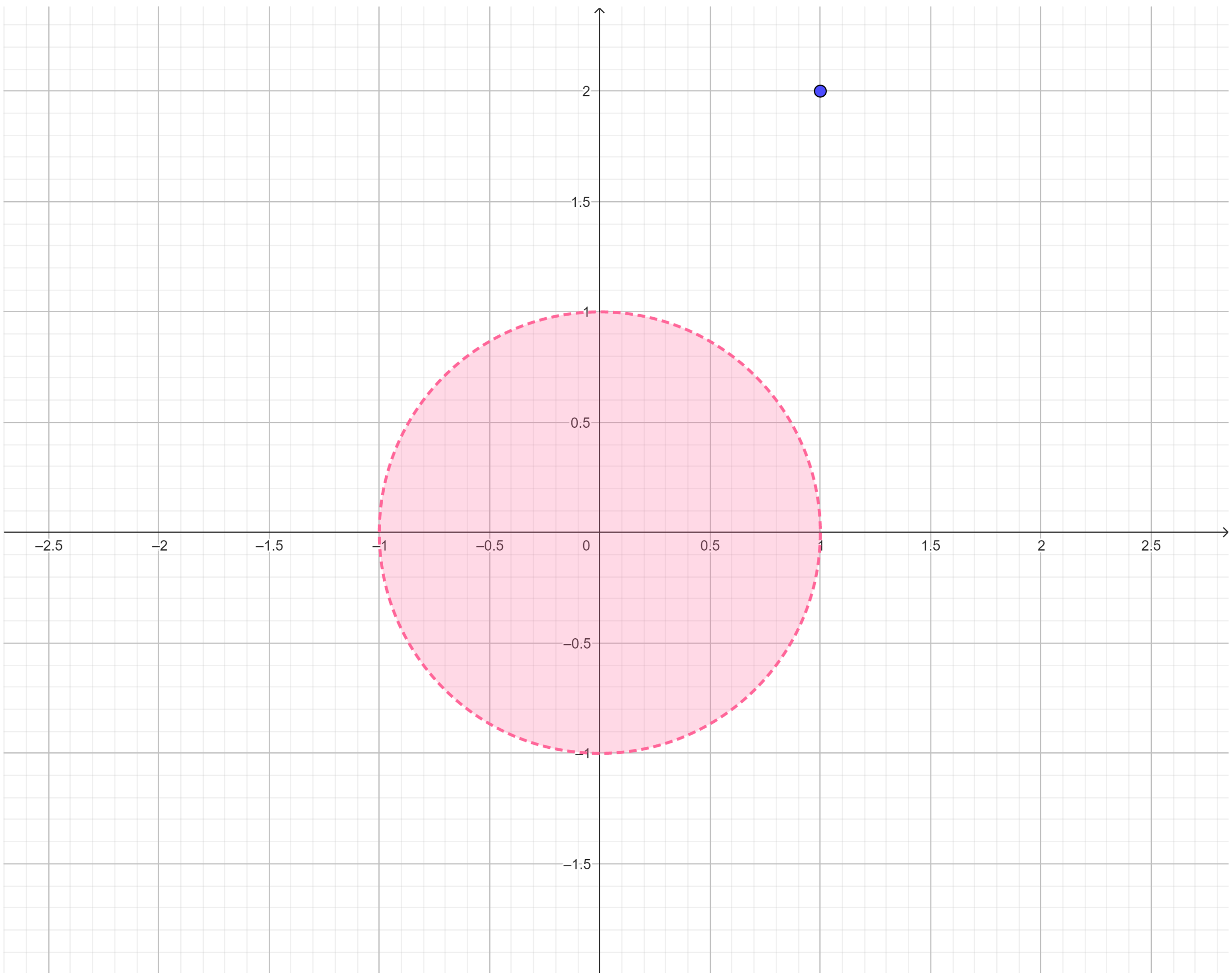

Figure 3: Open ball $B((1,2), 1+\sqrt{5})$ in α metric

$$
\begin{aligned}
B((0,1),2) &= \{(x_1,x_2) \in \mathbb{R};\ \alpha((0,1),(x_1,x_2)) < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 0 \wedge x_2 = 1 \wedge 0 < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \vee x_2 \neq 1 \wedge \sqrt{0^2+1^2} \\
&+ \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(0,1)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \vee x_2 \neq 1 \wedge \sqrt{1} + \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(0,1)\} \cup \{(x_1,x_2) \in \mathbb{R};\ \sqrt{x_1^2+x_2^2} < 1\} \\
&= \{(0,1)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1^2+x_2^2 < 1\}
\end{aligned}
$$

$$
\begin{aligned}
B((1,2),1+\sqrt{5}) &= \{(x_1,x_2) \in \mathbb{R};\ \alpha((1,2),(x_1,x_2)) < 1+\sqrt{5}\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 1 \wedge x_2 = 2 \wedge 0 < 1+\sqrt{5}\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 1 \vee x_2 \neq 2 \\
&\wedge \sqrt{1^2+2^2} + \sqrt{x_1^2+x_2^2} < 1+\sqrt{5}\} \\
&= \{(1,2)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 1 \vee x_2 \neq 2 \wedge \sqrt{5} + \sqrt{x_1^2+x_2^2} < 1+\sqrt{5}\} \\
&= \{(1,2)\} \cup \{(x_1,x_2) \in \mathbb{R};\ \sqrt{x_1^2+x_2^2} < 1\} \\
&= \{(1,2)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1^2+x_2^2 < 1\}
\end{aligned}
$$

(c) I drew the open balls $B((0,0),1)$, $B((0,1),2)$ and $B((2,2),\sqrt{2})$ in β metric. The drawings can be seen on Figures 4,5 and 6. The calculations I made were:

$$
\begin{aligned}
B((0,0),1) &= \{(x_1,x_2) \in \mathbb{R};\ \beta((0,0),(x_1,x_2)) < 1\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 = 0 \cdot x_1 \wedge \sqrt{(0-x_1)^2+(0-x_2)^2} < 1\} \cup \\
&\{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 \neq 0 \cdot x_1 \wedge \sqrt{0^2+0^2} + \sqrt{x_1^2+x_2^2} < 1 \\
&= \{(x_1,x_2) \in \mathbb{R};\ \sqrt{(-x_1)^2+(-x_2)^2} < 1\} \cup \emptyset \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1^2+x_2^2 < 1\}
\end{aligned}
$$

$$
\begin{aligned}
B((0,1),2) &= \{(x_1,x_2) \in \mathbb{R};\ \beta((0,1),(x_1,x_2)) < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 = 1 \cdot x_1 \wedge \sqrt{(0-x_1)^2+(1-x_2)^2} < 2\} \cup \\
&\{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 \neq 1 \cdot x_1 \wedge \sqrt{0^2+1^2} + \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 0 \wedge \sqrt{(1-x_2)^2} < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \wedge \sqrt{1} + \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 0 \wedge |1-x_2| < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \wedge \sqrt{x_1^2+x_2^2} < 1\} \\
&= \{(0,x_2) \in \mathbb{R}; |1-x_2| < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \wedge x_1^2+x_2^2 < 1\}
\end{aligned}
$$

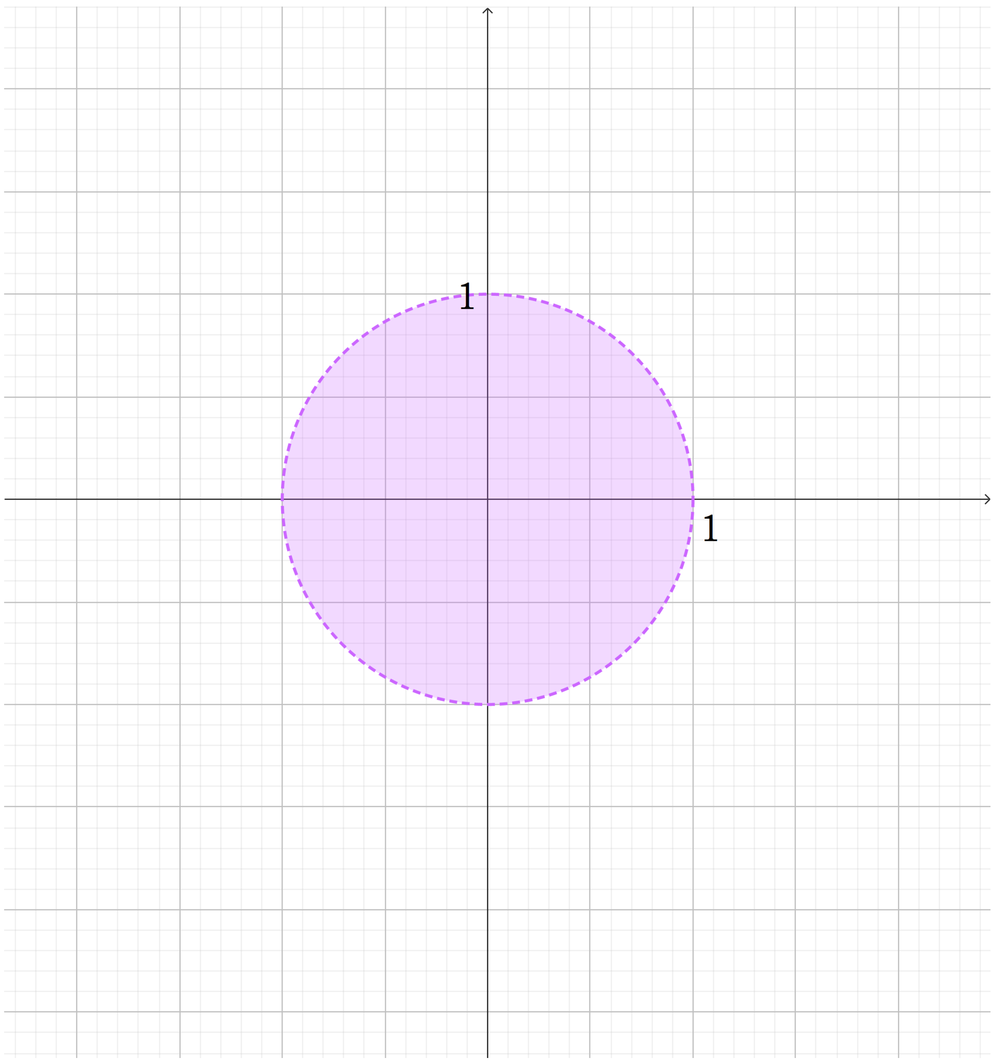

Figure 4: Open ball $B((0,0),1)$ in β metric

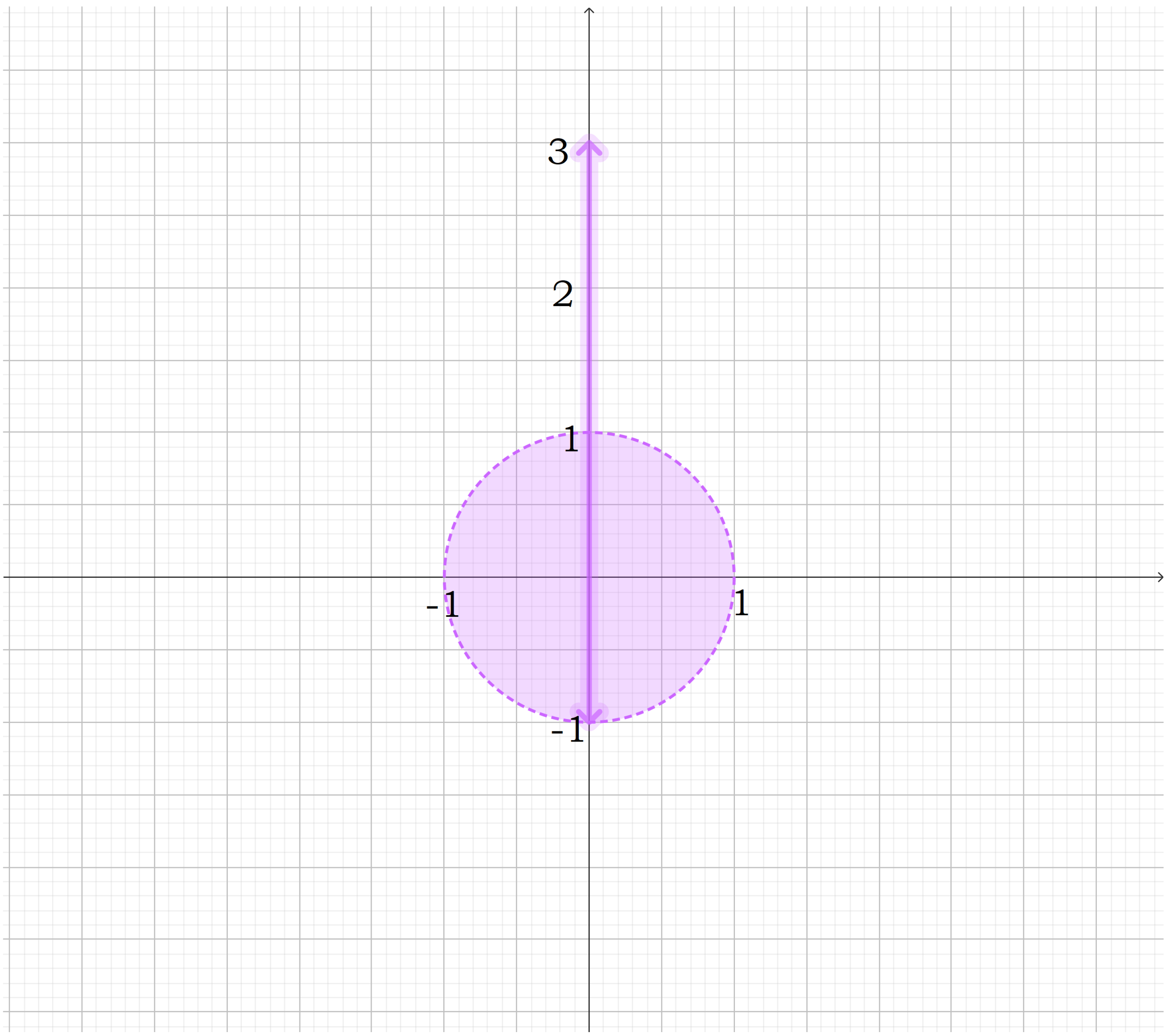

Figure 5: Open ball $B((0,1),2)$ in β metric

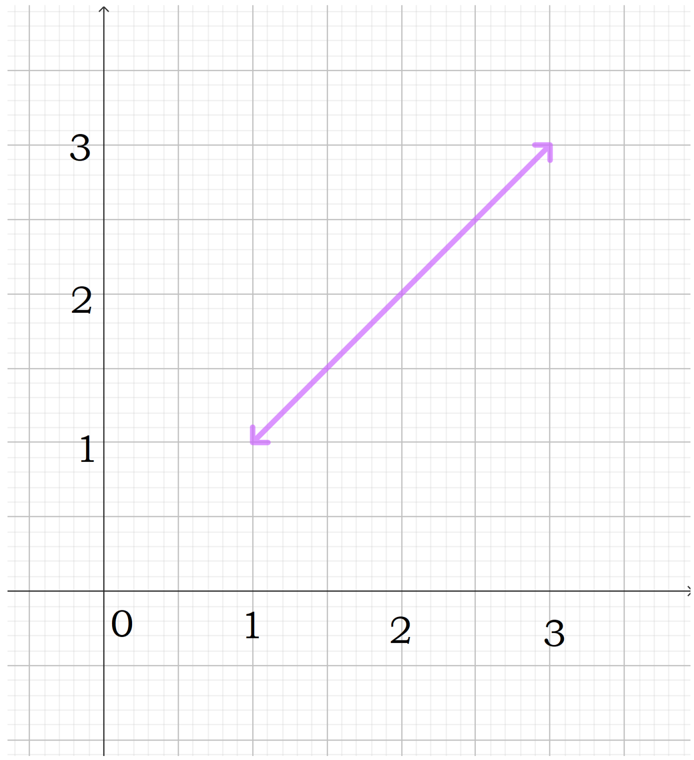

Figure 6: Open ball $B((1,2),\sqrt{2})$ in β metric

$$\begin{aligned}
B((2,2),\sqrt{2}) &= \{(x_1,x_2)\in\mathbb{R};\ \beta((2,2),(x_1,x_2)) < \sqrt{2}\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ 2\cdot x_2 = 2\cdot x_1 \wedge \sqrt{(2-x_1)^2+(2-x_2)^2} < \sqrt{2}\} \cup\\
&\{(x_1,x_2)\in\mathbb{R};\ 2\cdot x_2 \neq 2\cdot x_1 \wedge \sqrt{2^2+2^2} + \sqrt{x_1^2+x_2^2} < \sqrt{2}\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ \cdot x_1 = x_2 \wedge \sqrt{2\cdot(2-x_2)^2} < \sqrt{2}\}\\
&\cup\{(x_1,x_2)\in\mathbb{R};\ x_1 \neq x_2 \wedge \sqrt{8} + \sqrt{x_1^2+x_2^2} < \sqrt{2}\}\\
&= \{(x_1,x_1)\in\mathbb{R};\ |2-x_1| < 2\}\ \cup\emptyset
\end{aligned}$$

(d) I drew the open balls $B((0,0),1)$, $B((0,1),2)$ and $B((2,2),\sqrt{2})$ in γ metric. The drawings can be seen on Figures 7,8 and 9. The calculations I made were:

$$\begin{aligned}
B((0,0),1) &= \{(x_1,x_2)\in\mathbb{R};\ \gamma((0,0),(x_1,x_2)) < 1\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ x_1 = 0 \wedge |x_2-0| < 1\} \cup \{(x_1,x_2)\in\mathbb{R};\ x_1 \neq 0 \wedge |0| + |x_1-0| + |x2| < 1\}\\
&= \{(0,x_2)\in\mathbb{R};\ \wedge |x_2| < 1\} \cup \{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0 \wedge |x_1| + |x_2| < 1\}
\end{aligned}$$

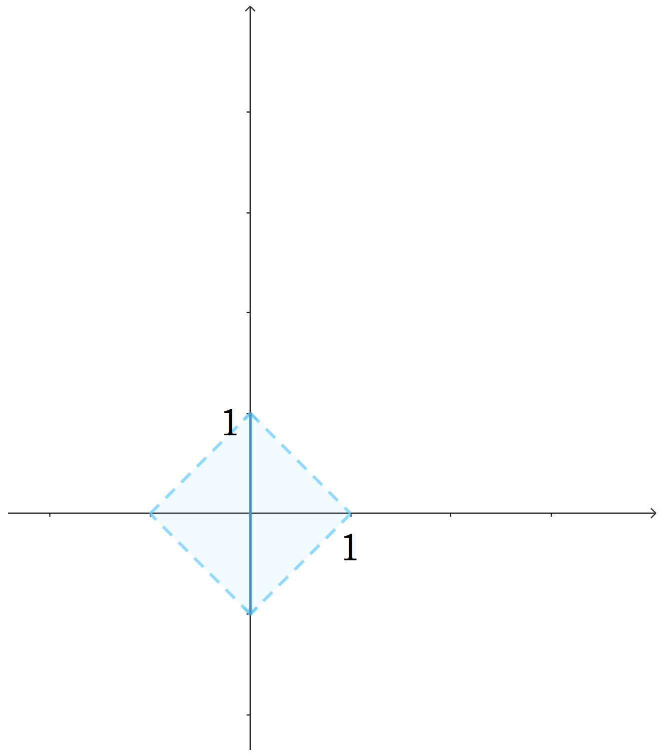

Figure 7: Open ball $B((0,0),1)$ in γ metric

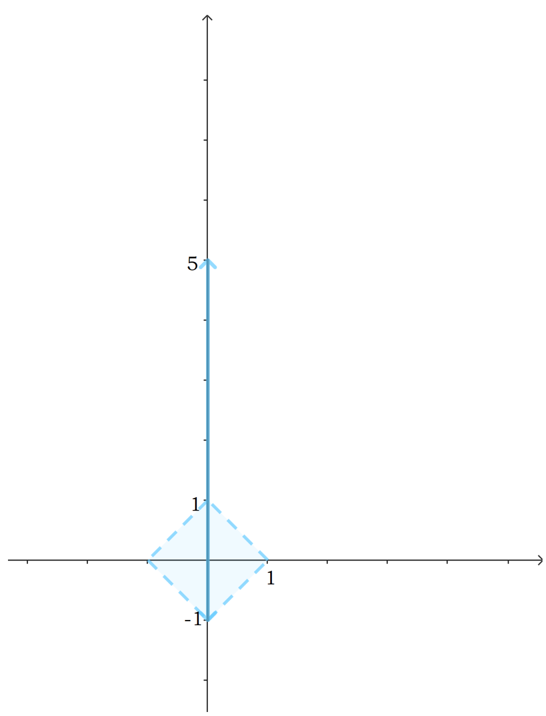

Figure 8: Open ball $B((0,2),3)$ in γ metric

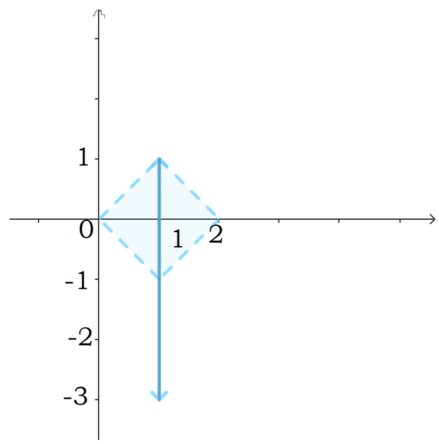

Figure 9: Open ball $B((1,-1),2)$ in γ metric

$$
\begin{aligned}
B((0,2),3) &= \{(x_1,x_2)\in\mathbb{R};\ \gamma((0,2),(x_1,x_2))<3\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ x_1=0\wedge|x_2-2|<3\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|2|+|x_1-0|+|x2|<3\}\\
&= \{(0,x_2)\in\mathbb{R};\ \wedge|x_2-2|<3\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|x_1|+|x_2|<1\}
\end{aligned}
$$

$$
\begin{aligned}
B((1,-1),2) &= \{(x_1,x_2)\in\mathbb{R};\ \gamma((1,-1),(x_1,x_2))<2\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ x_1=1\wedge|x_2+1|<2\}\\
&\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|-1|+|x_1-1|+|x2|<2\}\\
&= \{(1,x_2)\in\mathbb{R};\ \wedge|x_2+1|<2\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|x_1-1|+|x_2|<1\}
\end{aligned}
$$

1.2 Discrete metric

(a)

$$
\begin{aligned}
B(1,\frac{1}{2}) &= \{x\in\mathbb{N}; d(1,x)<\frac{1}{2}\}\\
&= \{x\in\mathbb{N}; x=1\wedge 0<\frac{1}{2}\}\cup\{x\in\mathbb{N}; x\neq 1\wedge d(1,x)<\frac{1}{2}\}\\
&= \{1\}\cup\emptyset\\
&= \{1\}
\end{aligned}
$$

$$
\begin{aligned}
B(2,1) &= \{x\in\mathbb{N}; d(2,x)<1\}\\
&= \{x\in\mathbb{N}; x=2\wedge 0<1\}\cup\{x\in\mathbb{N}; x\neq 2\wedge d(2,x)<2\}\\
&= \{2\}\cup\emptyset\\
&= \{2\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{B}(3, \frac{1}{2}) &= \{x \in \mathbb{N}; d(3, x) \leq \frac{1}{2}\} \\
&= \{x \in \mathbb{N}; x = 3 \wedge 0 \leq \frac{1}{2}\} \cup \{x \in \mathbb{N}; x \neq 3 \wedge d(3, x) \leq \frac{1}{2}\} \\
&= \{3\} \cup \emptyset \\
&= \{3\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{B}(4, 1) &= \{x \in \mathbb{N}; d(4, x) \leq 1\} \\
&= \{x \in \mathbb{N}; x = 4 \wedge 0 \leq 1\} \cup \{x \in \mathbb{N}; x \neq 4 \wedge 1 \leq 1\} \\
&= \{4\} \cup \mathbb{N} \setminus \{4\}\} \\
&= \mathbb{N}
\end{aligned}
$$

In general, this discrete metric on space X induces a metric space (X, d) where the distance between any two pairwise distinct elements is 1. Open and closed balls in this space contain one element or all of them:

$$
B(x_0, r) = \begin{cases} \{x_0\} & 0 < r \leq 1 \\ \mathbb{N} & r > 1 \end{cases}
$$

$$
\overline{B}(x_0, r) = \begin{cases} \{x_0\} & 0 < r < 1 \\ \mathbb{N} & r \geq 1 \end{cases}
$$

(b) Because the distance between any two pairwise distinct elements is 1, every triangle abc with pairwise distinct elements a, b and c is equilateral (all three edges have the same lenght 1).

1.3 Homeomorphic spaces

In this exercise I was working with spaces $X_n = S^{n-1} \times [0, 1] \subset \mathbb{R}^{n+1}$ and $Y_n = \{(x_1, ..., x_n) \in \mathbb{R}^n; 1 \leq x_1^2 + ... + x_n^2 \leq 4\}$.

Spaces X_1, Y_1, X_2, Y_2 that I've drawn can be seen on Figures 10, 11, 12 and 13.

I got the inspiration for the proof for a general n from the solved problems book by asist. dr. Aleksandra Franc (2) on spletna učilnica where lies the proof for $n = 2$. I generalized the idea for $n \in \mathbb{N}$.

In my proof I used the following theorem from the lectures:

Theorem Let X and Y be topological spaces. If we can find continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$, then X and Y are homeomorphic.

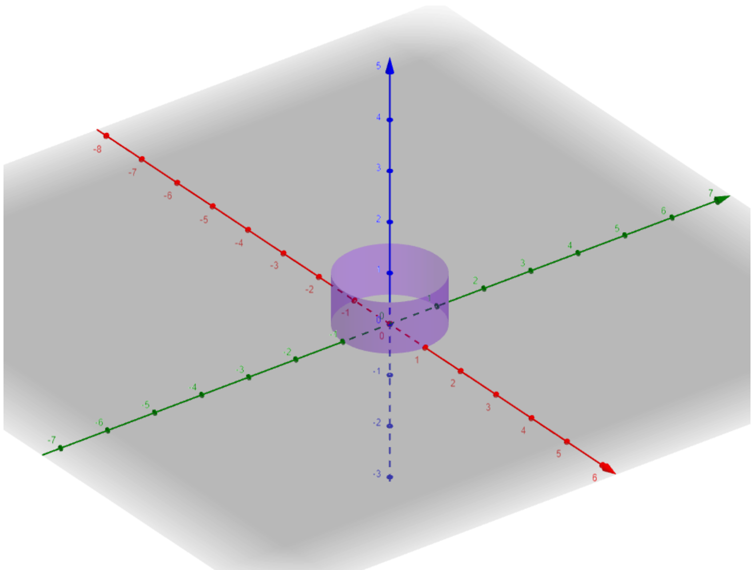

Figure 10: X_2 space

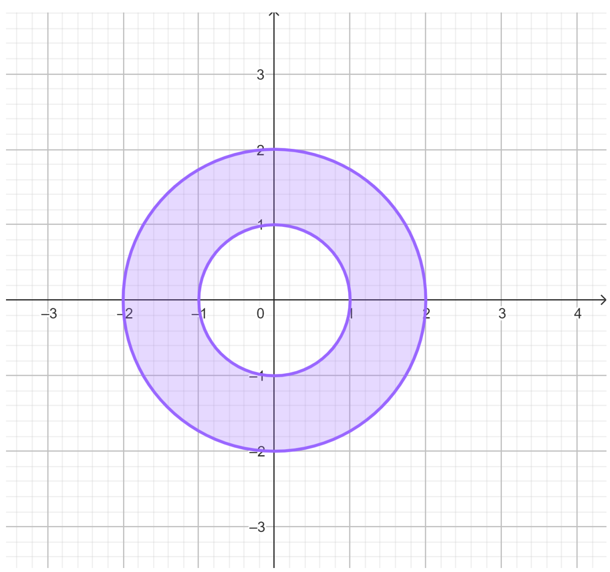

Figure 11: Y_2 space

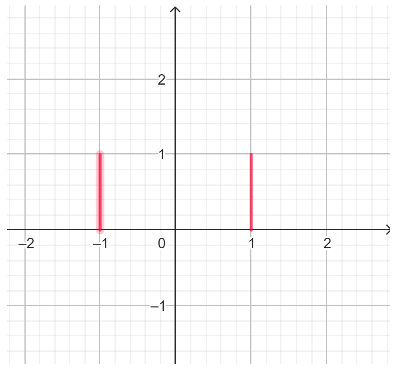

Figure 12: X_1 space

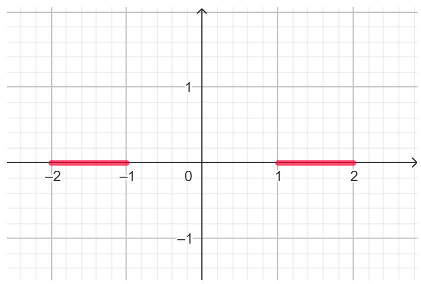

Figure 13: Y_1 space

Proof that X_n and Y_n are homeomorphic for any $n \in \mathbb{N}$:

First I rewrote X_n as:

$$X_n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1};\ x_1^2 + ... + x_n^2 = 1,\ 0 \leq x_{n+1} \leq 1\}$$

Next, I defined two functions f and g and then used the beforementioned theorem.

I defined $f : X_n \to Y_n$ as:

$$f(x_1, ..., x_{n+1}) = ((x_{n+1} + 1) \cdot x_1, ..., (x_{n+1} + 1) \cdot x_1, (x_n + 1) \cdot x_{n+1})$$

The idea was that I took the last coordinate, added 1 and multiplied it with other coordinates to lower the dimension. It is the same idea as for $n = 2$, where we mapped surface to the plane. The proof that f is well defined ($f(x_1, ..., x_{n+1}) \in Y_n$):

Let $(x_1, ..., x_{n+1}) \in X_n$. That means $x_1^2 + ... + x_n^2 = 1$ and $0 \leq x_{n+1} \leq 1$. Subsequently:

$$\begin{aligned} ((x_{n+1} + 1) \cdot x_1)^2 + ... + ((x_{n+1} + 1) \cdot x_n)^2 &= (x_{n+1} + 1)^2 \cdot x_1^2 + ... + (x_{n+1} + 1)^2 \cdot x_n^2 \\ &= (x_{n+1} + 1)^2 \cdot (x_1^2 + ... + x_n^2) \\ &= (x_{n+1} + 1)^2 \cdot 1 \\ &= (x_{n+1} + 1)^2 \end{aligned}$$

Because $0 \leq x_{n+1} \leq 1$ it holds that:

$$1 \leq (x_{n+1}+1)^2 \leq 4$$

And so: $f(x_1, ..., x_{n+1}) \in Y_n$

I also defined $g : Y_n \to X_n$ as:

$$g(x_1, ..., x_n) = (\frac{x_1}{\sqrt{x_1^2 + ... + x_n^2}}, ..., \frac{x_n}{\sqrt{x_1^2 + ... + x_n^2}}, \sqrt{x_1^2 + ... + x_n^2} - 1)$$

The idea was to make every vector from origin to point unit by dividing it with Euclidean distance $||x_1, ..., x_n||_2 = \sqrt{x_1^2 + ... + x_n^2}$ from origin to its ending point and instead express their lenght in the last coordinate. I wrote $r = ||x_1, ..., x_n||_2$. The proof that g is well defined $g(x_1, ..., x_n) \in X_n$:

Let $(x_1, ..., x_n) \in Y_n$.

$$\begin{aligned}
\left(\frac{x_1}{\sqrt{x_1^2 + ... + x_n^2}}\right)^2 \ + \ ... \ + \ \left(\frac{x_n}{\sqrt{x_1^2 + ... + x_n^2}}\right)^2 &= \frac{x_1^2}{r^2} + ... + \frac{x_n^2}{r^2} \\
&= \frac{x_1^2 + ... + x_n^2}{r^2} \\
&= \frac{x_1^2 + ... + x_n^2}{x_1^2 + ... + x_n^2} \\
&= 1
\end{aligned}$$

Because $1 \leq x_1^2 + ... + x_n^2 \leq 4$ it holds that:

$$\begin{aligned}
1 \leq \sqrt{x_1^2 + ... + x_n^2} & \qquad \leq 2 \\
0 \leq \sqrt{x_1^2 + ... + x_n^2} - 1 & \qquad \leq 1
\end{aligned}$$

And so g is well defined.

The proof that $f \circ g = id_{Y_n}$:

$$\begin{aligned}
f(g(x_1, ..., x_n)) &= f(\frac{x_1}{r}, ..., \frac{x_n}{r}, r - 1) \\
&= ((r - 1 + 1) \cdot \frac{x_1}{r}, ..., (r - 1 + 1) \cdot \frac{x_n}{r}) \\
&= (r \cdot \frac{x_1}{r}, ..., r \cdot \frac{x_n}{r}) \\
&= (x_1, ..., x_n)
\end{aligned}$$

The proof that $g \circ f = id_{X_n}$:

In my computation are used the following preconditions: $x_1^2 + ... + x_n^2 = 1,\ 0 \leq x_{n+1} \leq 1$.

$$\begin{aligned}
\sqrt{(x_{n+1}+1)^2 \cdot x_1^2 + ... + (x_{n+1}+1)^2 \cdot x_n^2} &= \sqrt{(x_{n+1}+1)^2(x_1^2 + ... + x_n^2)} \\
&= \sqrt{(x_{n+1}+1)^2 \cdot (x_1^2 + ... + x_n^2)} \\
&= \sqrt{(x_{n+1}+1)^2} \\
&= |(x_{n+1}+1)| \\
&= (x_{n+1}+1)
\end{aligned}$$

$$\begin{aligned}
g \circ f &= g(f(x_1, ..., x_n, x_{n+1})) \\
&= g((x_{n+1}+1)x_1, ..., (x_{n+1}+1)x_n) \\
&= \left(\frac{(x_{n+1}+1)x_1}{x_{n+1}+1}, ..., \frac{(x_{n+1}+1)x_n}{x_{n+1}+1}, x_{n+1}+1-1 \right) \\
&= (x_1, ..., x_n, x_{n+1})
\end{aligned}$$

□

2 Programming problems

2.1 Deciding connectivity

To find connected components in graph I used the dfs algorithm. My program graphcomponents.py consists of function findComponents. Its inputs are list of vertices V and list of edges E. Inside function there is another function, named dfs, which is defined only locally. Its inputs are vertex, graph that is a set of edges and set named visited, that contains vertices already visited by the dfs algorithm. Dfs algorithm searches through the vertices and edges and add visited vertices in visited set. Visited vertices are part of one of the connected components. In the main while loop function dfs is called on every iteration and list of components is updated with lists of visited vertices until there are no more vertices left in the graph. Vertices in components are sorted by numbers from the smallest to the greatest. I tested my program on five different test cases (one of them can be seen on Figure 14):

$V = [1, 2, 3, 4, 5, 6, 7, 8]$
$E = [(1, 2), (2, 3), (1, 3), (4, 5), (5, 6), (5, 7), (6, 7), (7, 8)]$
$findComponents(V, E) = [[1, 2, 3], [4, 5, 6, 7, 8]]$

$V = [1, 2, 3, 4, 5]$
$E = [(1, 2), (1, 3), (1, 4), (1, 5)]$
$findComponents(V, E) = [[1, 2, 3, 4, 5]]$

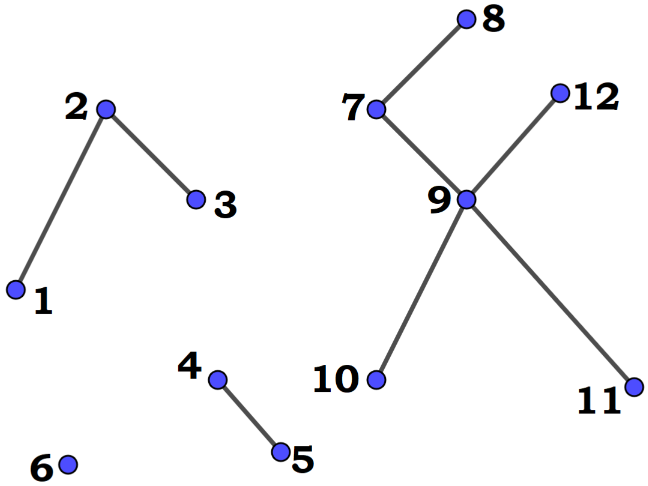

Figure 14: Testing case in Deciding connectivity problem.

$V = [1, 2, 3, 4, 5, 6, 7, 8, 9]$
$E = [(1, 2), (1, 3), (1, 8), (3, 7), (4, 5), (4, 6), (4, 9), (5, 6), (5, 9), (7, 8)]$
$findComponents(V, E) = [[1, 2, 3, 7, 8], [4, 5, 6, 9]]$

$V = [1, 2, 3, 4, 5]$
$E = []$
$findComponents(V, E) = [[1], [2], [3], [4], [5]]$

$V = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$
$E = [(1, 2), (2, 3), (4, 5), (7, 8), (7, 9), (9, 10), (9, 11), (9, 12)]$
$findComponents(V, E) = [[1, 2, 3], [4, 5], [6], [7, 8, 9, 10, 11, 12]]$

2.2 Shelling discs

References