Computational Topology
Homework 1

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November 13, 2022

1 Theoretical problems

1.1 Exploring different metrics

(a) I determined the distances between points $(1,2),(2,4)$ and $(2,-1)$ in all three metrics.

First, I determined the distances in metric α. No two given points were the same so I always used the second part of the α distance definition:

$$\alpha((1,2),(2,4)) = \sqrt{1^2+2^2} + \sqrt{2^2+4^2} = \sqrt{5} + \sqrt{20} = \sqrt{5} + \sqrt{4\cdot 5} = \sqrt{5} + 2\cdot\sqrt{5} = 3\cdot\sqrt{5}$$

$$\alpha((1,2),(2,-1)) = \sqrt{1^2+2^2} + \sqrt{2^2+(-1)^2} = \sqrt{5} + \sqrt{5} = 2\cdot\sqrt{5}$$

$$\alpha((2,4),(2,-1)) = \sqrt{2^2+4^2} + \sqrt{2^2+(-1)^2} = \sqrt{20} + \sqrt{5} = 2\cdot\sqrt{5} + \sqrt{5} = 3\cdot\sqrt{5}$$

Next, I determined the distances in metric β. Pair of points $(1,2)$ and $(2,4)$ satisfied the first condition (i.e. $1\cdot 4 = 2\cdot 2$), but the other two pairs of points $(1,2),(2,-1)$ and $(2,4),(2,-1)$ did not. ($1\cdot(-1) \neq 2\cdot 2$ and $2\cdot(-1) \neq 2\cdot 4$, respectively):

$$\beta((1,2),(2,4)) = \sqrt{(1-2)^2+(2-4)^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}$$

$$\beta((1,2),(2,-1)) = \sqrt{1^2+2^2} + \sqrt{2^2+(-1)^2} = \sqrt{5} + \sqrt{5} = 2\cdot\sqrt{5}$$

$$\beta((2,4),(2,-1)) = \sqrt{2^2+4^2} + \sqrt{2^2+(-1)^2} = \sqrt{20} + \sqrt{5} = 2\cdot\sqrt{5} + \sqrt{5} = 3\cdot\sqrt{5}$$

Lastly, I determined the distances in metric γ. The first pair of points $(1,2),(2,4)$ did not satisfy the first condition: $1 \neq 2$. The second pair of points $(1,2),(2,-1)$ also did not satisfy the first condition: $1 \neq 2$. However, the last pair of points $(2,4),(2,-1)$ did satisfy the first condition: $2 = 2$.

$$\gamma((1,2),(2,4)) = |2| + |2-1| + |4| = |2| + |1| + |4| = 7$$

$$\gamma((1,2),(2,-1)) = |2| + |2-1| + |-1| = |2| + |1| + |-1| = 2+1+1 = 4$$

$$\gamma((2,4),(2,-1)) = |-1-4| = |-5| = 5$$

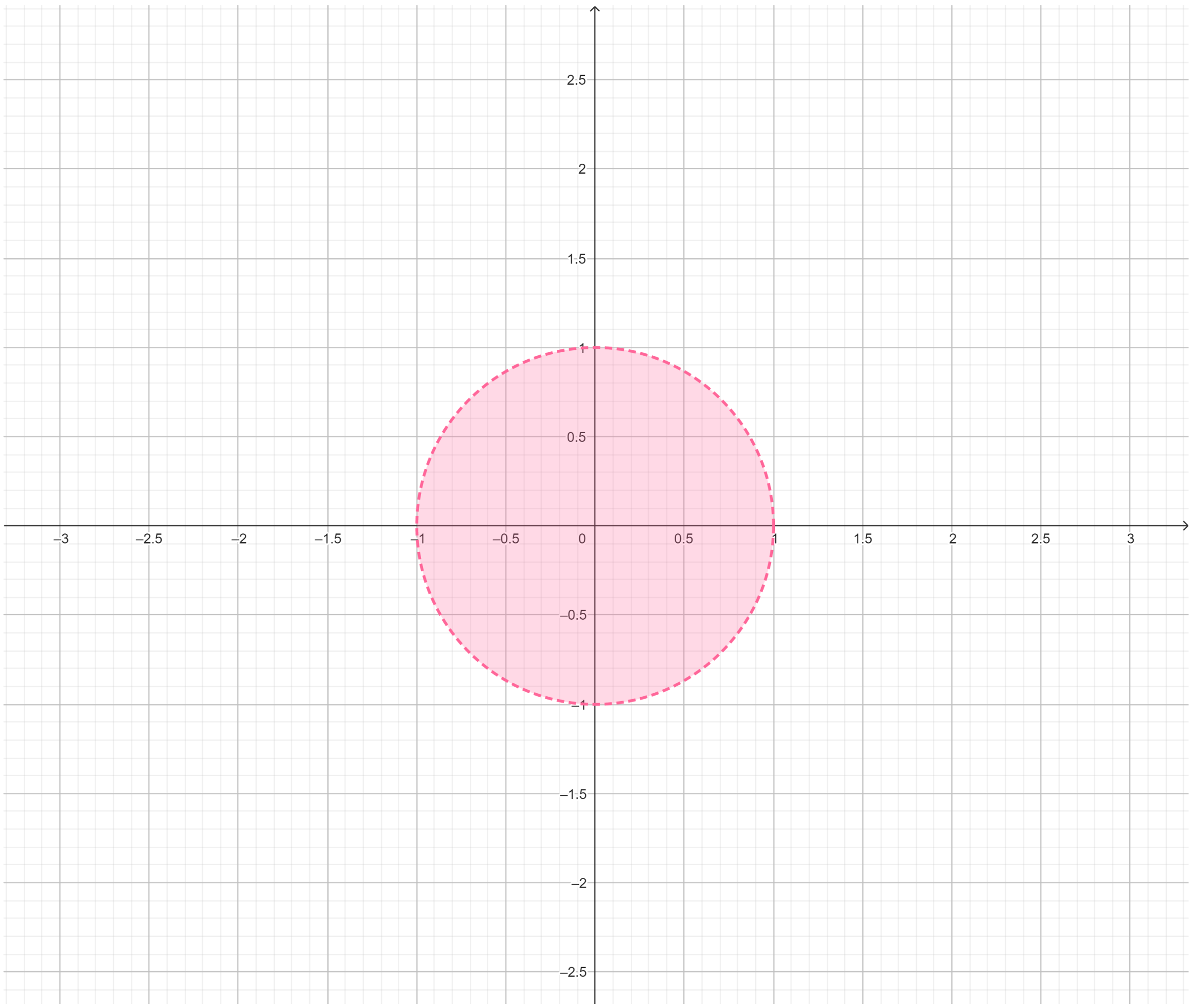

Figure 1: Open ball $B((0,0),1)$ in α metric

(b) I drew the open balls $B((0,0),1)$, $B((0,1),2)$ and $B((1,2),1+\sqrt{5})$ in α metric. The centre of the ball is always contained in it because the α distance from and to itself is always 0 (first condition) and 0 is always smaller than the radius $r > 0$ of the open ball. The drawings can be seen on Figures 1, 2 and 3. The calculations I made were:

$$
\begin{aligned}
B((0,0),1) &= \{(x_1,x_2) \in \mathbb{R};\ \alpha((0,0),(x_1,x_2)) < 1\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 0 \wedge x_2 = 0 \wedge 0 < 1\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \vee x_2 \neq 0 \wedge \sqrt{x_1^2 + x_2^2} < 1 \\
&= \{(0,0)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \vee x_2 \neq 0 \wedge x_1^2 + x_2^2 < 1\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1^2 + x_2^2 < 1\}
\end{aligned}
$$

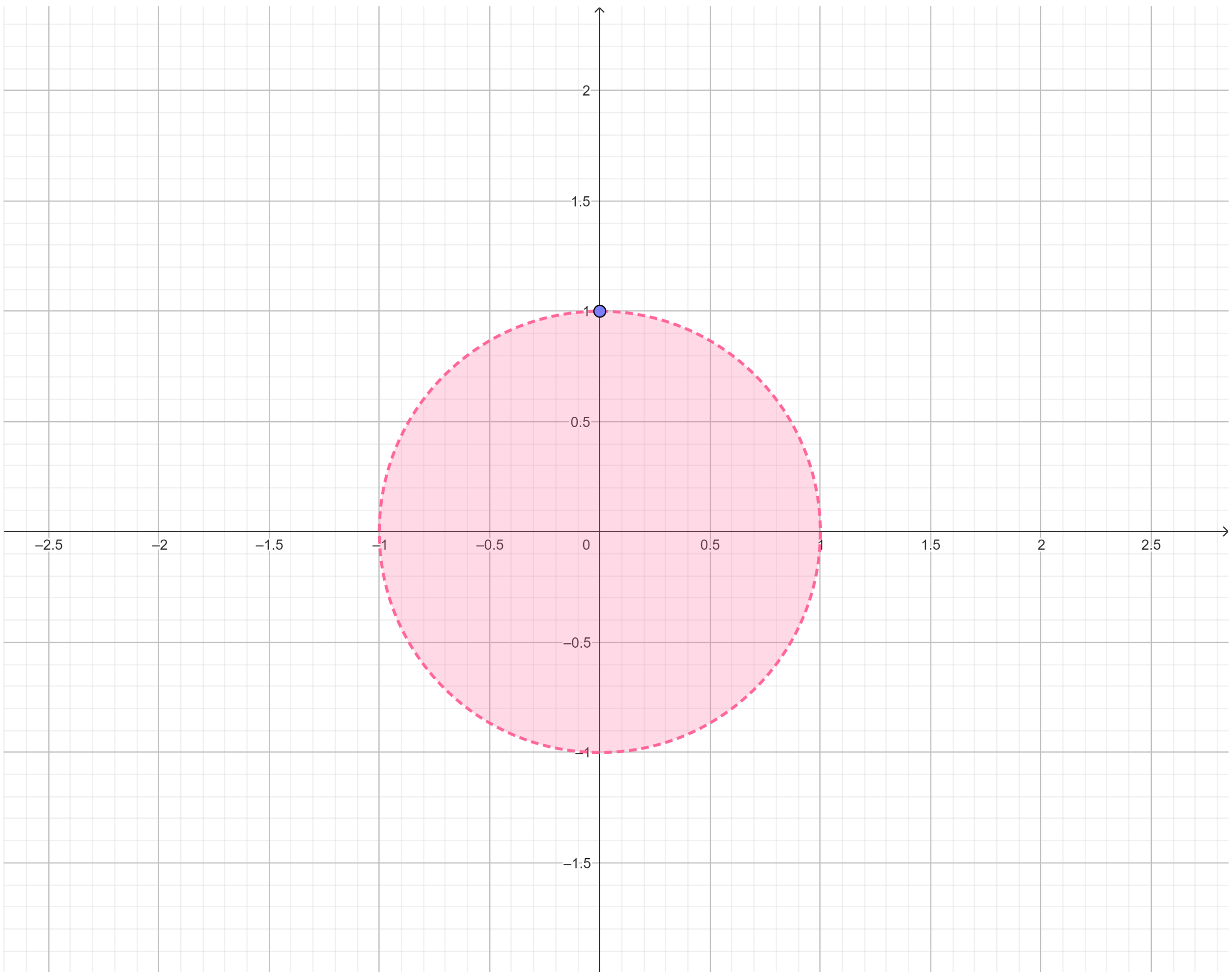

Figure 2: Open ball $B((0,1),2)$ in α metric

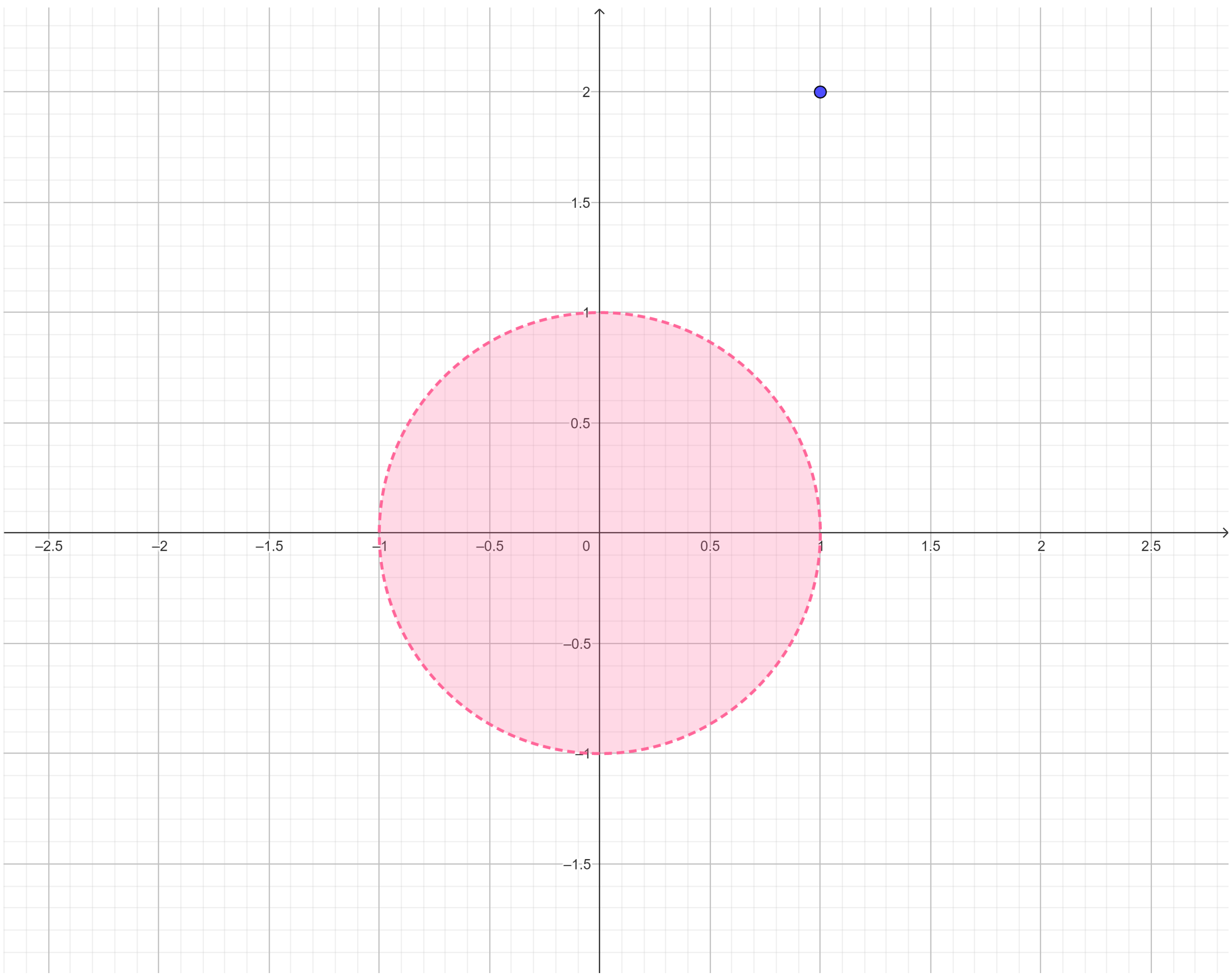

Figure 3: Open ball $B((1,2), 1+\sqrt{5})$ in α metric

$$
\begin{aligned}
B((0,1),2) &= \{(x_1,x_2) \in \mathbb{R};\ \alpha((0,1),(x_1,x_2)) < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 0 \land x_2 = 1 \land 0 < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \lor x_2 \neq 1 \land \sqrt{0^2+1^2} \\
&+ \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(0,1)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \lor x_2 \neq 1 \land \sqrt{1} + \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(0,1)\} \cup \{(x_1,x_2) \in \mathbb{R};\ \sqrt{x_1^2+x_2^2} < 1\} \\
&= \{(0,1)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1^2+x_2^2 < 1\}
\end{aligned}
$$

$$
\begin{aligned}
B((1,2),1+\sqrt{5}) &= \{(x_1,x_2) \in \mathbb{R};\ \alpha((1,2),(x_1,x_2)) < 1+\sqrt{5}\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 1 \land x_2 = 2 \land 0 < 1+\sqrt{5}\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 1 \lor x_2 \neq 2 \\
&\land \sqrt{1^2+2^2} + \sqrt{x_1^2+x_2^2} < 1+\sqrt{5}\} \\
&= \{(1,2)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 1 \lor x_2 \neq 2 \land \sqrt{5} + \sqrt{x_1^2+x_2^2} < 1+\sqrt{5}\} \\
&= \{(1,2)\} \cup \{(x_1,x_2) \in \mathbb{R};\ \sqrt{x_1^2+x_2^2} < 1\} \\
&= \{(1,2)\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1^2+x_2^2 < 1\}
\end{aligned}
$$

(c) I drew the open balls $B((0,0),1)$, $B((0,1),2)$ and $B((2,2),\sqrt{2})$ in β metric. The drawings can be seen on Figures 4,5 and 6. The calculations I made were:

$$
\begin{aligned}
B((0,0),1) &= \{(x_1,x_2) \in \mathbb{R};\ \beta((0,0),(x_1,x_2)) < 1\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 = 0 \cdot x_1 \land \sqrt{(0-x_1)^2+(0-x_2)^2} < 1\} \cup \\
&\{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 \neq 0 \cdot x_1 \land \sqrt{0^2+0^2} + \sqrt{x_1^2+x_2^2} < 1 \\
&= \{(x_1,x_2) \in \mathbb{R};\ \sqrt{(-x_1)^2+(-x_2)^2} < 1\} \cup \emptyset \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1^2+x_2^2 < 1\}
\end{aligned}
$$

$$
\begin{aligned}
B((0,1),2) &= \{(x_1,x_2) \in \mathbb{R};\ \beta((0,1),(x_1,x_2)) < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 = 1 \cdot x_1 \land \sqrt{(0-x_1)^2+(1-x_2)^2} < 2\} \cup \\
&\{(x_1,x_2) \in \mathbb{R};\ 0 \cdot x_2 \neq 1 \cdot x_1 \land \sqrt{0^2+1^2} + \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 0 \land \sqrt{(1-x_2)^2} < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \land \sqrt{1} + \sqrt{x_1^2+x_2^2} < 2\} \\
&= \{(x_1,x_2) \in \mathbb{R};\ x_1 = 0 \land |1-x_2| < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \land \sqrt{x_1^2+x_2^2} < 1\} \\
&= \{(0,x_2) \in \mathbb{R}; |1-x_2| < 2\} \cup \{(x_1,x_2) \in \mathbb{R};\ x_1 \neq 0 \land x_1^2+x_2^2 < 1\}
\end{aligned}
$$

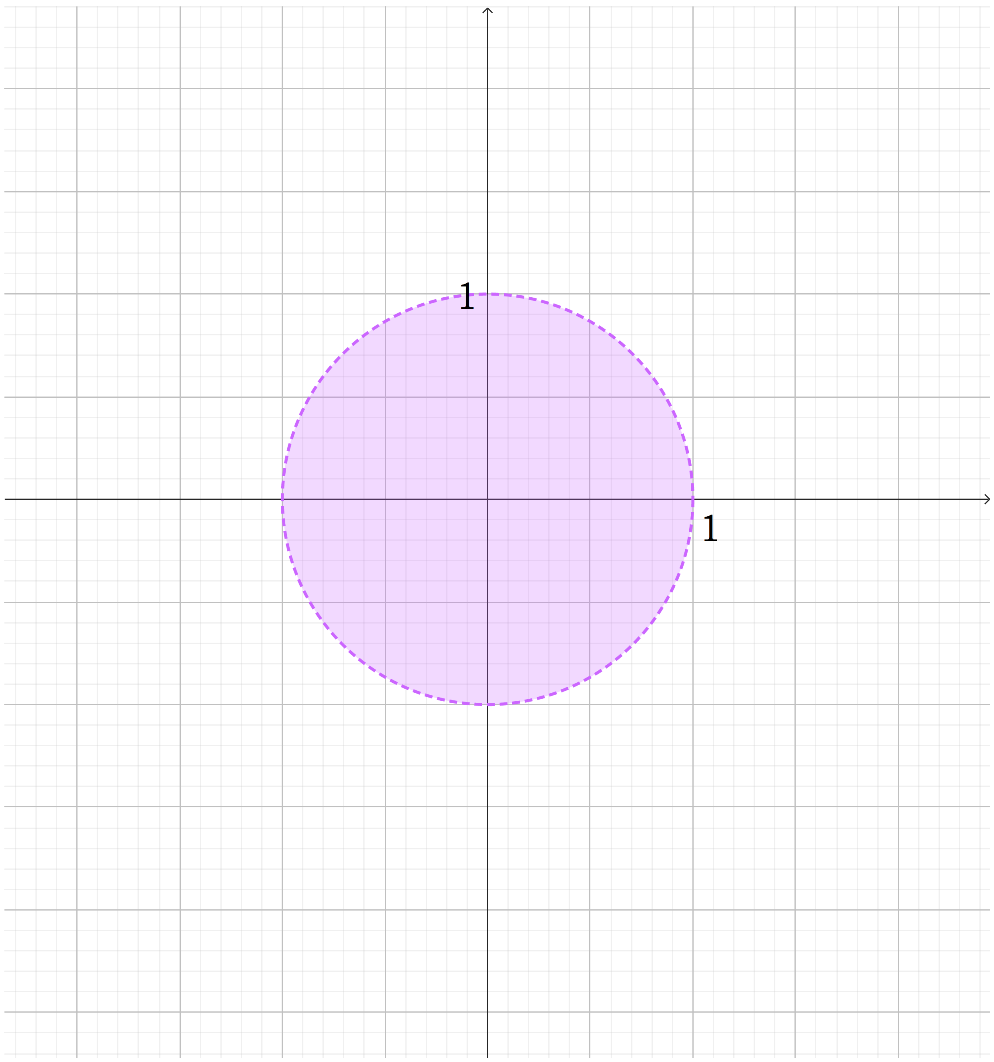

Figure 4: Open ball $B((0,0),1)$ in β metric

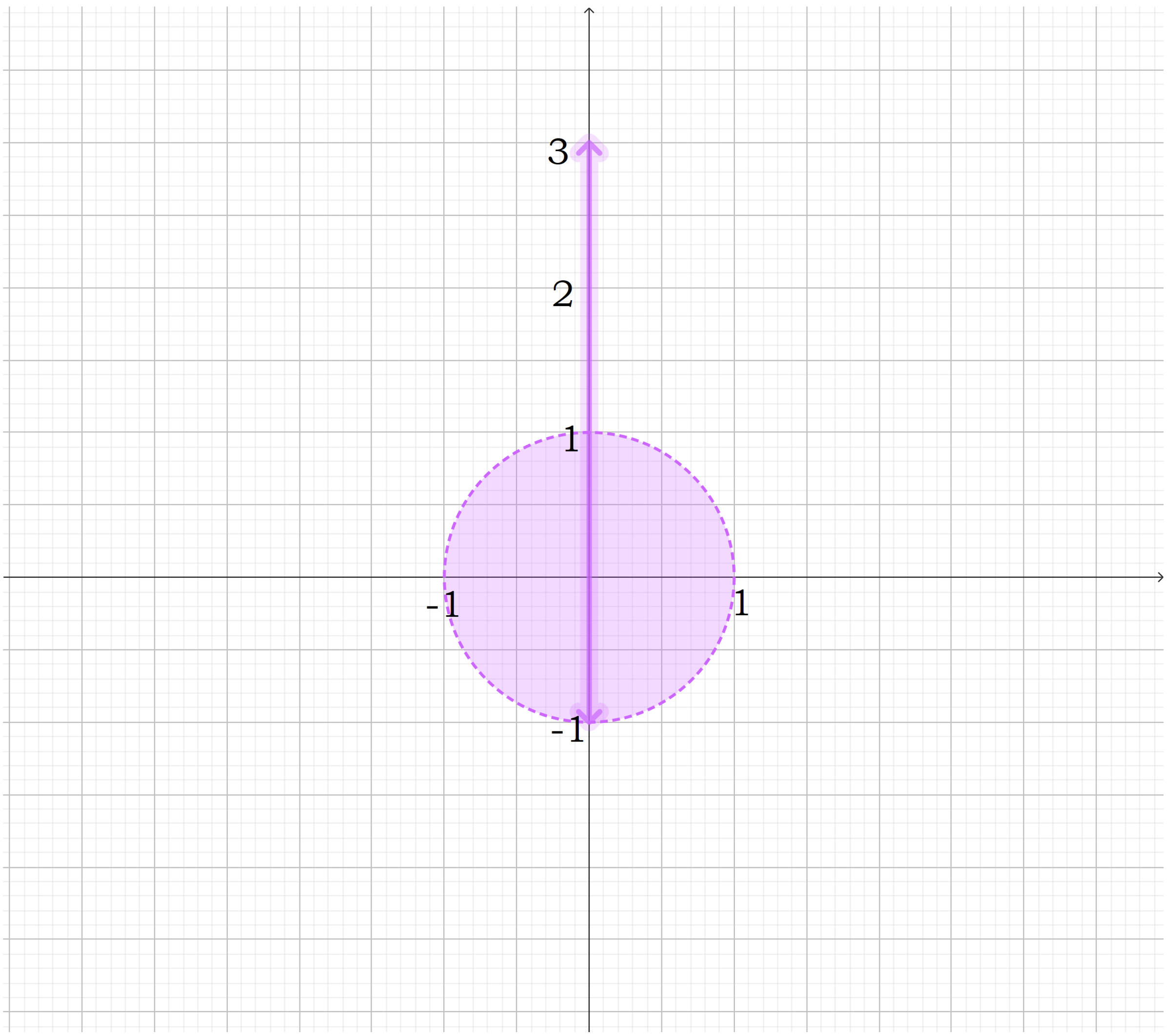

Figure 5: Open ball $B((0,1),2)$ in β metric

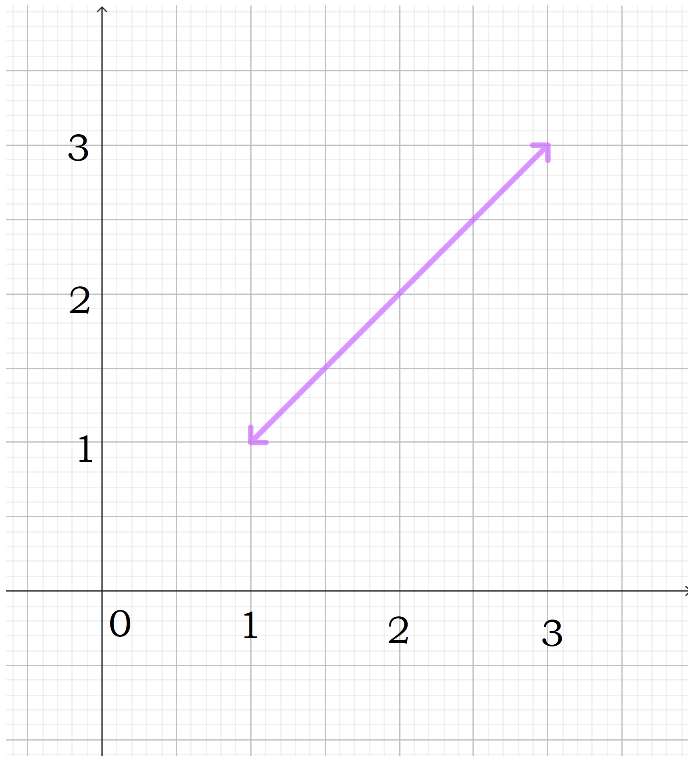

Figure 6: Open ball $B((1,2),\sqrt{2})$ in β metric

$$
\begin{aligned}
B((2,2),\sqrt{2}) &= \{(x_1,x_2)\in\mathbb{R};\ \beta((2,2),(x_1,x_2))<\sqrt{2}\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ 2\cdot x_2 = 2\cdot x_1 \wedge \sqrt{(2-x_1)^2+(2-x_2)^2}<\sqrt{2}\}\cup\\
&\{(x_1,x_2)\in\mathbb{R};\ 2\cdot x_2 \neq 2\cdot x_1 \wedge \sqrt{2^2+2^2}+\sqrt{x_1^2+x_2^2}<\sqrt{2}\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ \cdot x_1 = x_2 \wedge \sqrt{2\cdot(2-x_2)^2}<\sqrt{2}\}\\
&\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq x_2 \wedge \sqrt{8}+\sqrt{x_1^2+x_2^2}<\sqrt{2}\}\\
&= \{(x_1,x_1)\in\mathbb{R};\ |2-x_1|<2\}\ \cup\emptyset
\end{aligned}
$$

(d) I drew the open balls $B((0,0),1)$, $B((0,1),2)$ and $B((2,2),\sqrt{2})$ in γ metric. The drawings can be seen on Figures 7,8 and 9. The calculations I made were:

$$
\begin{aligned}
B((0,0),1) &= \{(x_1,x_2)\in\mathbb{R};\ \gamma((0,0),(x_1,x_2))<1\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ x_1=0\wedge|x_2-0|<1\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|0|+|x_1-0|+|x2|<1\}\\
&= \{(0,x_2)\in\mathbb{R};\ \wedge\ |x_2|<1\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|x_1|+|x_2|<1\}
\end{aligned}
$$

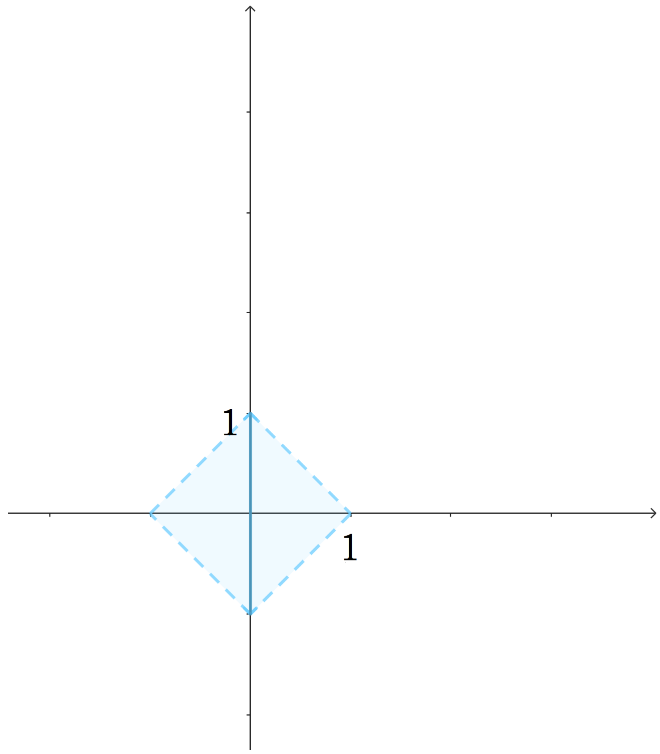

Figure 7: Open ball $B((0,0),1)$ in γ metric

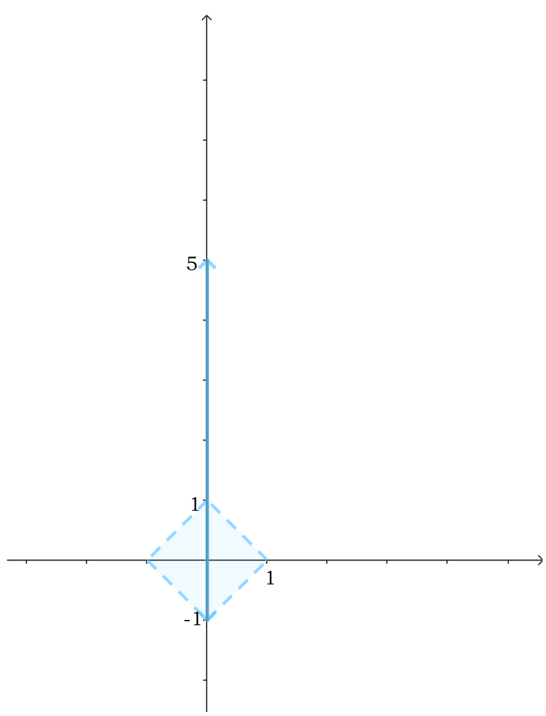

Figure 8: Open ball $B((0,2),3)$ in γ metric

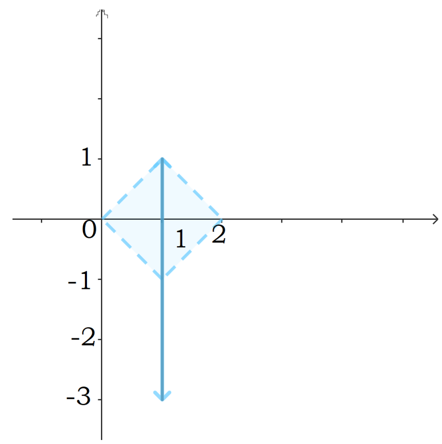

Figure 9: Open ball $B((1,-1),2)$ in γ metric

$$
\begin{aligned}
B((0,2),3) &= \{(x_1,x_2)\in\mathbb{R};\ \gamma((0,2),(x_1,x_2))<3\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ x_1=0\wedge|x_2-2|<3\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|2|+|x_1-0|+|x2|<3\}\\
&= \{(0,x_2)\in\mathbb{R};\ \wedge|x_2-2|<3\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|x_1|+|x_2|<1\}
\end{aligned}
$$

$$
\begin{aligned}
B((1,-1),2) &= \{(x_1,x_2)\in\mathbb{R};\ \gamma((1,-1),(x_1,x_2))<2\}\\
&= \{(x_1,x_2)\in\mathbb{R};\ x_1=1\wedge|x_2+1|<2\}\\
&\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|-1|+|x_1-1|+|x2|<2\}\\
&= \{(1,x_2)\in\mathbb{R};\ \wedge|x_2+1|<2\}\cup\{(x_1,x_2)\in\mathbb{R};\ x_1\neq 0\wedge|x_1-1|+|x_2|<1\}
\end{aligned}
$$

1.2 Discrete metric

(a)

$$
\begin{aligned}
B(1,\frac{1}{2}) &= \{x\in\mathbb{N}; d(1,x)<\frac{1}{2}\}\\
&= \{x\in\mathbb{N}; x=1\wedge 0<\frac{1}{2}\}\cup\{x\in\mathbb{N}; x\neq 1\wedge d(1,x)<\frac{1}{2}\}\\
&= \{1\}\cup\emptyset\\
&= \{1\}
\end{aligned}
$$

$$
\begin{aligned}
B(2,1) &= \{x\in\mathbb{N}; d(2,x)<1\}\\
&= \{x\in\mathbb{N}; x=2\wedge 0<1\}\cup\{x\in\mathbb{N}; x\neq 2\wedge d(2,x)<2\}\\
&= \{2\}\cup\emptyset\\
&= \{2\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{B}(3, \frac{1}{2}) &= \{x \in \mathbb{N}; d(3, x) \leq \frac{1}{2}\} \\
&= \{x \in \mathbb{N}; x = 3 \wedge 0 \leq \frac{1}{2}\} \cup \{x \in \mathbb{N}; x \neq 3 \wedge d(3, x) \leq \frac{1}{2}\} \\
&= \{3\} \cup \emptyset \\
&= \{3\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{B}(4, 1) &= \{x \in \mathbb{N}; d(4, x) \leq 1\} \\
&= \{x \in \mathbb{N}; x = 4 \wedge 0 \leq 1\} \cup \{x \in \mathbb{N}; x \neq 4 \wedge 1 \leq 1\} \\
&= \{4\} \cup \mathbb{N} \setminus \{4\}\} \\
&= \mathbb{N}
\end{aligned}
$$

In general, this discrete metric on space X induces a metric space (X, d) where the distance between any two pairwise distinct elements is 1. Open and closed balls in this space contain one element or all of them:

$$
B(x_0, r) = \begin{cases} \{x_0\} & 0 < r \leq 1 \\ \mathbb{N} & r > 1 \end{cases}
$$

$$
\overline{B}(x_0, r) = \begin{cases} \{x_0\} & 0 < r < 1 \\ \mathbb{N} & r \geq 1 \end{cases}
$$

(b) Because the distance between any two pairwise distinct elements is 1, every triangle abc with pairwise distinct elements a, b and c is equilateral (all three edges have the same lenght 1).

1.3 Homeomorphic spaces

In this exercise I was working with spaces $X_n = ...$ and $Y_n = ...$

Spaces X_1, Y_1, X_2, Y_2 that I've drawn can be seen on Figures 10, 11 and 12.

I got the inspiration for the proof for a general n from the solved problems book by asist. dr. Aleksandra Franc (2) on spletna učilnica where lies the proof for $n = 2$. I generalized the idea for $n \in \mathbb{N}$.

In my proof I used the following theorem from the lectures:

Theorem Let X and Y be topological spaces. If we can find continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$, then X and Y are homeomorphic.

Proof that X_n and Y_n are homeomorphic for any $n \in \mathbb{N}$:

References