Note that for each policy in Portfolio A, $Var[B_i] = 0$. When the loss random variable is given in this form, we have for policy i, $E[X_i] = q_i \times E[B_i]$, and $Var[X_i] = q_i \times (1 - q_i) \times (E[B_i])^2 + q_i \times Var[B_i]$, and for a portfolio of independent policies, $E[S] = \sum E[X_i]$ and $Var[S] = \sum Var[X_i]$.

For Portfolio A, any policy in Class 1 has $Var[X_i] = q_i \times (1 - q_i) \times (E[B_i])^2 + q_i \times Var[B_i] = 0.05 \times .95 \times 1^2 + 0.05 \times 0 = 0.0475$ and any policy in Class 2 has $Var[X_i] = 0.10 \times 0.90 \times 2^2 + 0.10 \times 0 = 0.36$, so that

$$Var[S_A] = 2000 \times 0.0475 + 500 \times 0.36 = 275.$$

For Portfolio B, any policy in Class 1 has

$$Var[X_i] = 0.05 \times .95 \times 1^2 + 0.05 \times 1 = .0975$$

and any policy in Class 2 has

$$Var[X_i] = 0.10 \times 0.90 \times 2^2 + 0.10 \times 4 = .76$$

so that

$$Var[S_B] = 2000 \times 0.0975 + 500 \times 0.76 = 575$$

Then, $\frac{Var[S_A]}{Var[S_B]} = 0.478$. □

The normal approximation to aggregate claims: For an aggregate claims distribution S, if the mean and variance of S are known ($E[S]$ and $Var[S]$), it is possible to approximate probabilities for S by using the normal distribution. The 95-th percentile of aggregate claims is the number Q for which $P[S \le Q] = 0.95$. If S is assumed to have a distribution which is approximately normal, then by standardizing S we have

$$P[S \le Q] = P\left[\frac{S - E[S]}{\sqrt{Var[S]}} \le \frac{Q - E[S]}{\sqrt{Var[S]}}\right] = 0.95 ,$$

so that $\frac{Q - E[S]}{\sqrt{Var[S]}}$ is equal to the 95-th percentile of the standard normal distribution (which is found to be 1.645 when referring to the standard normal table), so that Q can be found;

$$Q = E[S] + 1.645 \times \sqrt{Var[S]}$$

If the insurer collects total premium of amount Q, then (assuming that it is reasonable to use the approximation) there is a 95% chance (approximately) that aggregate claims will be less than the premium collected, and there is a 5% chance that aggregate claims will exceed the premium. Since S is a sum of many independent individual policy loss random variables, the Central Limit Theorem suggests that the normal approximation to S is not unreasonable. **The normal approximation has come up frequently on past exams.** The normal approximation and the integer correction when applied to integer-valued random variables was discussed earlier in Section 7 of this study guide.

Example 10-9:

An insurance company provides insurance to three classes of independent insureds with the following characteristics:

			For Each Insured	
	Number	Probability	Expected	Variance of
Class	in Class	of a Claim	Claim Amount	Claim Amount
1	500	0.05	5	5
2	1000	0.10	10	10
3	500	0.15	5	5

For each class, the amount of premium collected is $(1+\theta)$(expected claims), where θ is the same for all three classes. Using the normal approximation to aggregate claims, find θ so that the probability that total claims exceed the amount of premium collected is 0.05.

Solution:

We wish to find $Q = (1+\theta)E[S]$, so that $P[S > Q] = 0.05$,
or equivalently, $P[S \leq Q] = 0.95$.

Applying the normal approximation and standardizing S, this can be written in the form $P[S \leq Q] = P\left[\frac{S-E[S]}{\sqrt{Var[S]}} \leq \frac{Q-E[S]}{\sqrt{Var[S]}}\right] = 0.95$, so that $\frac{Q-E[S]}{\sqrt{Var[S]}} = 1.645$ (the 95-th percentile of the standard normal distribution). Thus, once $E[S]$ and $Var[S]$ are found, we can find $Q = (1+\theta)E[S]$ and then find θ.

$$E[S] = \Sigma E[X_i] = \Sigma(q_i \times E[B_i]) = 500 \times 0.05 \times 5 + 1000 \times 0.1 \times 10 + 500 \times 0.15 \times 5 = 1500$$

since there are 500 policies in class 1, each with expected claim 0.05×5, and similarly for classes 2 and 3. The policies are independent so that the variance of the sum of all policy claims is the sum of the variances (no covariances when independence is assumed). The variance of a claim for a policy from class 1 is
$q \times (1-q) \times (E[B])^2 + q \times Var[B] = 0.05 \times 0.95 \times 5^2 + 0.05 \times 5$,
and there are 500 of those policies, and similarly for classes 2 and 3.

$$\begin{aligned} Var[S] &= \Sigma Var[X_i] = \Sigma[q_i \times (1-q_i) \times (E[B_i])^2 + q_i \times Var[B_i]] \\ &= 500 \times [0.05 \times 0.95 \times 5^2 + 0.05 \times 5] + 1000 \times [0.1 \times 0.9 \times 10^2 + 0.1 \times 10] \\ &\quad + 500 \times [0.15 \times 0.85 \times 5^2 + 0.15 \times 5] = 12,687.5 \end{aligned}$$

Then, $Q = 1685.29$, and $\theta = 0.1235$. □

Example 10-10:

Suppose that a multiple choice exam has 40 questions, each with 5 possible answers. A well prepared student feels that he has a probability of 0.5 of getting any particular question correct, with independence from question to question. Apply the normal approximation to X, the number of correct answers out of 40 to determine the probability of getting at least 25 correct. Find the probability with the integer correction, and then without the correction.

Solution:

The number of questions answered correctly, say X, has a binomial distribution with mean $40 \times 0.5 = 20$ and variance $40 \times 0.5 \times 0.5 = 10$. Applying the normal approximation to X, with integer correction since the binomial distribution is a discrete integer-valued random variable to find the probability of answering at least 25 correct, we get

$$P[X \geq 25] = P[X \geq 24.5] = P\left[\frac{X-20}{\sqrt{10}} \geq \frac{24.5-20}{\sqrt{10}}\right] = P[Z \geq 1.42] = 1 - \Phi(1.42) = 0.078\,.$$

Without the integer correction, the probability is

$$P[X \geq 25] = P\left[\frac{X-20}{\sqrt{10}} \geq \frac{25-20}{\sqrt{10}}\right] = P[Z \geq 1.58] = 1 - \Phi(1.58) = 0.057.$$

There is a noticeable difference in these numerical values. If X has a much larger standard deviation, then the difference is not as noticeable. □

Mixture of Loss Distributions

A portfolio of policies might consist of two or more classes of policyholders, as in the previous example. In the previous example, the number of policies in each class was known. It may be the case that the number of policies in each class is not known but the proportion of policies in each class is known. In such a situation, we might be asked to describe the distribution of the loss for a randomly chosen policy from the portfolio of policies. This will be a mixture of the distributions representing the different classes of policyholders. Mixtures of distributions were considered near the end of Section 9 of the study guide. The following example illustrates this idea.

Example 10-11:

The insurer of a portfolio of automobile insurance policies classifies each policy as either high risk, medium risk or low risk. The portfolio consists of 10% high risk policies, 30% medium risk and 60% low risk. The claim means and variances for the three risk classes are

	mean	variance
high risk	10	16
medium risk	4	4
low risk	1	1

A policy is chosen at random from the portfolio. Find the mean and variance of this policy.

Solution:

The distribution of the randomly chosen policy is the mixture of the three risk class claim distributions, using the percentages as the mixing factors. If X denotes the claim for the randomly chosen policy, then all moments of X (pdf and cdf also) are the weighted averages of the moments for the component distributions in the mixture.

$E[X] = 0.1 \times 10 + 0.3 \times 4 + 0.6 \times 1 = 2.8$ is the mean.

Since $Var[X] = E[X^2] - (E[X])^2$ we need $E[X^2]$ in order to find the variance of X.

Let X_H denote the claim random variable for a high risk policy. Then

$16 = Var[X_H] = E[X_H^2] - (E[X_H])^2 = E[X_H^2] - 10^2$ from which we get $E[X_H^2] = 116$.

In a similar way we get $E[X_M^2] = 4 + 4^2 = 20$ and $E[X_L^2] = 1 + 1^2 = 2$.

Then $E[X^2] = 0.1 \times 116 + 0.3 \times 20 + 0.6 \times 2 = 18.8$, and $Var[X] = 18.8 - 2.8^2 = 10.96$.

Note that the variance of X is not the weighted average of the variances of X_H , X_M and X_L.
We can use the conditional variance formula $\boldsymbol{Var[X] = E[\,Var[X|Y]\,] + Var[\,E[X|Y]\,]}$ to get the variance of X. If we define Y to be the 3-point random variable $Y = \{H, M, L\}$ with probabilities $P(Y = H) = 0.1$ (high risk) , $P(Y = M) = 0.3$ and $P(Y = L) = 0.6$, then from the information given, we have $E(X|Y = H) = 10$, $Var(X|Y = H) = 16$,
$E(X|Y = M) = 4$, $Var(X|Y = M) = 4$, and $E(X|Y = L) = 1$, $Var(X|Y = L) = 1$.

Then, from the double expectation rule, we have

$$E[X] = E[\,E[X|Y]\,] = 10 \times 0.1 + 4 \times 0.3 + 1 \times 0.6 = 2.8$$

We can think of $Var[X|Y]$ as a 3-point random variable

$$Var(X|Y) = \begin{cases} Var(X|Y = H) = 16 & \text{prob. } 0.1 \\ Var(X|Y = M) = 4 & \text{prob. } 0.3 \\ Var(X|Y = L) = 1 & \text{prob. } 0.6 \end{cases}$$

And then we get the expected value of that 3-point random variable

$$E[Var[X|Y]] = 16 \times 0.1 + 4 \times 0.3 + 1 \times 0.6 = 3.4$$

To find $Var[E[X|Y]]$, we think of $E(X|Y)$ as a 3-point random variable,

$$E(X|Y) = \begin{cases} E(X|Y = H) = 10 & \text{prob. } 0.1 \\ E(X|Y = M) = 4 & \text{prob. } 0.3 \\ E(X|Y = L) = 1 & \text{prob. } 0.6 \end{cases}.$$

We then find the variance of this 3-point random variable:

$$10^2 \times 0.1 + 4^2 \times 0.3 + 1^2 \times 0.6 - (10 \times 0.1 + 4 \times 0.3 + 1 \times 0.6)^2 = 15.4 - 2.8^2 = 7.56$$

This is $Var[E[X|Y]]$. Then $Var[X] = E[Var[X|Y]] + Var[E[X|Y]] = 3.4 + 7.56 = 10.96$. □

Loss Distribution Formulated By Conditioning

A loss distribution may be presented in a conditional format in the following way. We may be given the conditional distribution of the loss variable X given some other variable. For instance, in Example 10-11, we were presented with the distribution of claim for three types of policies, high, medium and low risk. What we are given is actually the conditional claim X given risk type. We are given the conditional mean $E(X|\text{risk type})$ and the conditional variance $Var(X|\text{risk type})$. If we denote "risk type" as Y, we can think of a randomly chosen policy as coming from the distribution of Y where $Y = \{H, M, L\}$ with probabilities $P(Y = H) = 0.1$ (high risk) , $P(Y = M) = 0.3$ and $P(Y = L) = 0.6$. Then we can use the double expectation rule and the conditional variance rule to find the overall or total mean claim $E(X)$ and overall claim variance $Var(X)$.

A special case that arises when a loss distribution is formulated by conditioning in this way is the following. Suppose that the number of losses, say N, in specified period of time has a Poisson distribution with mean λ. Suppose that the severity of each loss is a random variable X, and suppose that the number of losses and the amount of each loss are mutually independent. The mean and variance of the aggregate loss S can be found as follows.

$$E[S] = E[E[S|N]] = E[N \times E[X]] = E[N] \times E[X] = \lambda \times E[X] \text{ , and}$$

$$\begin{aligned} Var[S] &= E[Var[S|N]] + Var[E[S|N]] = E[N \times Var[X]] + Var[N \times E[X]] \\ &= E[N] \times Var[X] + Var[N] \times (E[X])^2 = \lambda \times (Var[X] + E[X]^2) = \lambda \times E[X^2] \end{aligned}$$

Example 10-12 (SOA):

The number of hurricanes that will hit a certain house in the next ten years is Poisson distributed with mean 4. Each hurricane results in a loss that is exponentially distributed with mean 1000. Losses are mutually independent of the number of hurricanes. Calculate the mean and variance of the total loss due to hurricanes hitting the house in the next ten years.

Solution:

The number of hurricanes in 10 years is N, which is Poisson with a mean of 4. The size of each hurricane loss X is exponential with mean 1000. The average loss in 10 years is

$E[S] = E[N] \times E[X] = 4 \times 1000 = 4000$. The variance of the loss in 10 years is

$$\begin{aligned} Var[S] &= E[N] \times Var[X] + Var[N] \times (E[X])^2 \\ &= 4 \times 1000^2 + 4 \times 1000^2 = 8,000,000 \end{aligned}$$

Alternatively, since N has a Poisson distribution, we also have

$$Var[S] = E[N[\times E[X^2] = 4 \times (2 \times 1000^2) = 8,000,000$$ □

PROBLEM SET 10

Loss Distributions and Insurance

1. (SOA) An insurance policy pays an individual 100 per day for up to 3 days of hospitalization and 25 per day for each day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability function

$$P(X=k)=\begin{cases}\frac{6-k}{15} & \text{for } k=1,2,3,4,5\\ 0 & \text{otherwise.}\end{cases}$$

Calculate the expected payment for hospitalization under this policy.

A) 85 B) 163 C) 168 D) 213 E) 255

2. An insurance company issues insurance contracts to two classes of independent lives, as shown below.

Class	Probability of Death	Benefit Amount	Number in Class
A	0.01	200,000	500
B	0.05	100,000	300

The company wants to collect an amount, in total, equal to the 95-th percentile of the distribution of total claims. The company will collect an amount from each life insured that is proportional to that life's expected claim. That is, the amount for life j with expected claim $E[X_j]$ would be $kE[X_j]$. Calculate k.

A) 1.30 B) 1.32 C) 1.34 D) 1.36 E) 1.38

3. (SOA) An insurance policy reimburses a loss up to a benefit limit of 10 . The policyholder's loss, Y, follows a distribution with density function:

$$f(y)=\begin{cases}\frac{2}{y^3} & \text{for } y>1\\ 0 & \text{otherwise.}\end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

A) 1.0 B) 1.3 C) 1.8 D) 1.9 E) 2.0

4. (SOA) A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X=\max(T,2)$. Determine $E[X]$.

A) $2+\frac{1}{3}e^{-6}$ B) $2-2e^{-2/3}+5e^{-4/3}$ C) 3 D) $2+3e^{-2/3}$ E) 5

5. (SOA) The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X has density function

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 0 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

Let Y be the age of the machine at the time of replacement. Determine the variance of Y.

A) 1.3 B) 1.4 C) 1.7 D) 2.1 E) 7.5

6. (SOA) An insurance policy is written to cover a loss, X, where X has a uniform distribution on [0, 1000]. At what level must a deductible be set in order for the expected payment to be 25% of what it would be with no deductible?

A) 250 B) 375 C) 500 D) 625 E) 750

7. A derivative investment speculator identifies certain technical financial conditions which, when they arise, allow her to place an investment whose return will be normally distributed with a mean return of \$100,000 and a standard deviation of \$20,000. Experience indicates that the number of times per year these specific financial conditions arise has a Poisson distribution with a mean of 3. Assuming that the financial conditions arise independently of one another and that the speculator places the investment each time it arises, find the probability that the speculator earns less than \$100,000 in a year in total on her investments.

A) 0.12 B) 0.18 C) 0.24 D) 0.30 E) 0.36

8. (SOA) An insurance company sells a one-year automobile policy with a deductible of 2 The probability that the insured will incur a loss is 0.05. If there is a loss, the probability of a loss of amount N is $\frac{K}{N}$, for $N = 1, \ldots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur. Determine the net premium for this policy.

A) 0.031 B) 0.066 C) 0.072 D) 0.110 E) 0.150

9. A company's dental plan pays the annual dental expenses above a deductible of \$100 for each of 50 employees. The distribution of annual dental expenses X for an individual employee is

$$X = \begin{cases} 0, \text{ prob. } 0.1 \\ 100, \text{ prob. } 0.2 \\ 200, \text{ prob. } 0.4 \\ 500, \text{ prob. } 0.2 \\ 1000, \text{ prob. } 0.1 \end{cases}.$$

Using the normal approximation, find the 95th percentile of the aggregate annual claims distribution that the company pays (nearest \$10).

A) 11,640 B) 12,640 C) 13,640 D) 14,640 E) 15,640

10. A portfolio of independent one-year insurance policies has three classes of policies:

Class	Numberof in Class	Probability of Claim per Policy	Claim Amount
1	1000	0.01	1
2	2000	0.02	1
3	500	0.04	2

Find the standard deviation of the aggregate one-year claims distribution.

A) 10.0 B) 10.4 C) 10.8 D) 11.2 E) 11.6

11. A loss random variable X has the following (cumulative) distribution function:

$$F(x) = \begin{cases} 0, \text{ if } x < 0 \\ 0.2 + 0.3x, \text{ if } 0 \le x < 2 \\ 1, \text{ if } x \ge 2 \end{cases}$$

An insurer will provide proportional insurance on this loss, covering fraction α of the loss ($0 < \alpha < 1$). The expected claim on the insurer is 0.5. Find α.

A) 0.25 B) 0.3 C) 0.45 D) 0.5 E) 0.65

12. If a loss occurs, the amount of loss will be uniform between \$1000 and \$2,000.

The probability of the loss occurring is 0.2. An insurance policy pays the total loss, if a loss occurs. Find the standard deviation of the amount paid by the insurer.

A) 584 B) 614 C) 634 D) 654 E) 674

13. X and Y are random losses with joint density function

$f(x,y) = \frac{x}{500{,}000}$ for $0 < x < 100$ and $0 < y < 100$

An insurance policy on the losses pays the total of the two losses to a maximum payment of 100. Find the expected payment the insurer will make on this policy (nearest 1).

A) 90 B) 92 C) 94 D) 96 E) 98

14. The number of claims N that can result from a small group insurance policy is 0, 1 or 2, each with probability $\frac{1}{3}$. Information about the aggregate loss S incurred by the insurer is available in conditional form:

$$E[S|N=0] = 0\,,\ Var[S|N=0] = 0\,,$$
$$E[S|N=1] = 10\,,\ Var[S|N=1] = 5\,,$$
$$E[S|N=2] = 20\,,\ Var[S|N=2] = 8\,.$$

Find the unconditional variance of the aggregate loss S.

A) 13/3 B) 6.5 C) 13 D) 200/3 E) 71

15. In modeling the behavior of insurance claims, a risk manager uses an exponential distribution with mean μ as the distribution describing the claim size random variable. The risk manager forecasts that claim sizes will increase next year, with the average claim size increasing by 10% from this year to next. The risk manager plans to continue using the exponential distribution as the model for claim amounts next year. The risk manager calculates the median of the claim size distribution this year, M_0 and for next year, M_1.
Find M_1/M_0 .

A) 1 B) $1+ln\,1.1$ C) 1.1 D) $e^{0.1}$ E) $e^{1.1}$

16. (SOA) An auto insurance company insures an automobile worth 15,000 for one year under a policy with a 1,000 deductible. During the policy year there is a 0.04 chance of partial damage to the car and a 0.02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function

$$f(x) = \begin{cases} 0.5003e^{-x/2} & \text{for } \ 0 < x < 15 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected claim payment?

A) 320 B) 328 C) 352 D) 380 E) 540

17. An insurer finds that for automobile drivers classified as high risk, the number of accidents in one year has a binomial distribution with $n = 2$ and $p = 0.02$, and for drivers classified as low risk, the number of accidents in one year has a Bernoulli distribution with $n = 1$ and $p = .01$. The insurer's portfolio is made up of 25% policies on high risk drivers and 75% low risk drivers. Suppose that a driver has had no accidents in the past year. Find the probability that the same driver will have no accidents in the next year.

A) 0.980 B) 0.983 C) 0.986 D) 0.991 E) 0.994

18. (SOA) A manufacturer's annual losses follow a distribution with density function

$$f(x) = \begin{cases} \frac{2.5(0.6)^{2.5}}{x^{3.5}} & \text{for } \ x > 0.6 \\ 0 & \text{otherwise.} \end{cases}$$

To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2. What is the mean of the manufacturer's annual losses not paid by the insurance policy?

A) 0.84 B) 0.88 C) 0.93 D) 0.95 E) 1.00

19. Let X and Y be random variables with joint density function

$$f(x,y) = 2x \text{ for } \ 0 < x < 1 \text{ and } \ 0 < y < 1$$

An insurance policy is written to cover the loss $X+Y$. The policy has a deductible of 1.
Calculate the expected payment under the policy.

A) 1/4 B) 1/3 C) 1/2 D) 7/12 E) 5/6

20. (SOA) An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than or equal to 0.5 is 0.64. Calculate C.

A) 0.1 B) 0.3 C) 0.4 D) 0.6 E) 0.8

21. (SOA) The owner of an automobile insures it against damage by purchasing an insurance policy with a deductible of 250. In the event that the automobile is damaged, repair costs can be modeled by a uniform random variable on the interval $(0, 1500)$. Determine the standard deviation of the insurance payment in the event that the automobile is damaged.

A) 361 B) 403 C) 433 D) 464 E) 521

22. An insurer models the claim random variable X for a certain insurance policy as follows:

x	$P[X = x]$
0	0.30
50	0.10
200	0.10
500	0.20
$1,000$	0.20
$10,000$	0.10

The insurer wishes to summarize the claim amount distribution with the parameters:

$q =$ probability a non-zero claim occurs

$B =$ conditional distribution of claim amount given that a claim occurs.

Find the standard deviation of B (nearest 100).

A) 3200 B) 3300 C) 3400 D) 3500 E) 3600

23. (SOA) A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and 10,000 for each one thereafter, until the end of the year. The number of major snowstorms per year that shut down business is assumed to have a Poisson distribution with mean 1.5 . What is the expected amount paid to the company under this policy during a one-year period?

A) 2,769 B) 5,000 C) 7,231 D) 8,347 E) 10,578

24. A loss random variable has density function $f(x) = xe^{-x}$ for $x > 0$. An insurance policy on the loss has a deductible of 2. Find the expected insurance payment.

A) 0.46 B) 0.50 C) 0.54 D) 0.58 E) 0.62

25. A loss random variable has density function $f(x) = 2 - 2x$ for $0 \le x \le 1$.
At what level should a policy limit be set so that the expected insurer payment is one-half of the overall expected loss?
A) 0.11 B) 0.16 C) 0.21 D) 0.26 E) 0.31

26. (SOA) A baseball team has scheduled its opening game for April 1. If it rains on April l, the game is postponed and will be played on the next day that it does not rain. The team purchases insurance against rain. The policy will pay 1000 for each day, up to 2 days, that the opening game is postponed. The insurance company determines that the number of consecutive days of rain beginning on April 1 is a Poisson random variable with mean 0.6. What is the standard deviation of the amount the insurance company will have to pay?
A) 668 B) 699 C) 775 D) 817 E) 904

27. (SOA) An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100. Losses incurred follow an exponential distribution with mean 300. What is the 95th percentile of actual losses that exceed the deductible?

A) 600 B) 700 C) 800 D) 900 E) 1000

Problems 28 and 29 are based on the following information:
A portfolio of independent insurance policies is divided into three classes:

Class	Numberof Claim in Class	Probability per Policy	Claim Amount
1	1000	0.01	1
2	2000	0.02	1
3	500	0.04	2

28. The insurer charges a premium Q that is the 95-th percentile of the aggregate claims distribution based on the normal approximation to the aggregate claims distribution. Find Q (nearest 5).
A) 90 B) 95 C) 100 D) 105 E) 110

29. The insurer calculates the variance of the aggregate claims random variable. The insurer then changes the assumptions regarding the claims and now supposes that the individual policy claim amounts are also random variables, and that the claim amount listed in the table above is the expected claim amount for each of the policies, and the variance of the claim amount per policy is σ^2. The insurer recalculates the variance of the aggregate claims and finds that it is 67% larger than the initial calculation. Find σ^2.
A) 1.0 B) 1.2 C) 1.4 D) 1.6 E) 1.8

30. An insurer with aggregate claim distribution S charges a premium which includes a relative security loading of θ (i.e., the premium is $Q=(1+\theta)E[S]$). The insurer purchases proportional reinsurance, in which the reinsurer will pay the fraction $\alpha \times$ S of aggregate claims. The reinsurer charges a premium that includes a relative security loading of θ'. After reinsurance, the ceding insurer's resulting effective relative security loading is θ''. Which is the correct expression for α in terms of θ, θ', and θ''?

A) $\frac{\theta-\theta'}{\theta''-\theta'}$ B) $\frac{\theta-\theta''}{\theta'-\theta''}$ C) $\frac{\theta'-\theta}{\theta''-\theta'}$ D) $\frac{\theta'-\theta''}{\theta''-\theta}$ E) $\frac{\theta''-\theta'}{\theta'-\theta}$

31. (SOA) An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is:

$$f(x)=\begin{cases} ce^{-0.004x} & \text{for } x\geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant. Calculate the median benefit for this policy.

(A) 161 B) 165 C) 173 D) 182 E) 250

32. For a certain insurance, individual losses in 2000 were uniformly distributed over $(0\,,\,1000)$. A deductible of 100 is applied to each loss. In 2001, individual losses have increased 5%, and are still uniformly distributed. A deductible of 100 is still applied to each loss. Determine the percentage increase in the standard deviation of amount paid.

A) Less than 5.0% B) At least 5.0% but less than 5.5% C) At least 5.5% but less than 6.0%
D) At least 6.0% but less than 6.5% E) At least 6.5%

33. Losses follow a distribution with density function $f(x)=\frac{1}{1000}e^{-x/1000}$, $0<x<\infty$.
There is a policy deductible of 500 and 10 losses are expected to exceed the deductible each year. Determine the expected number of losses that would exceed the deductible each year if all loss amounts doubled, but the deductible remained at 500.

A) Less than 10 B) At least 10, but less than 12 C) At least 12, but less than 14
D) At least 14, but less than 16 E) At least 16

34. You are given the following:

- the underlying distribution for year 2000 losses is given by $f(x)=e^{-x}$, $x>0$, where losses are expressed in millions of dollars
- inflation of 5% impacts all claims uniformly from 2000 to 2001
- under a basic limits policy, payments on individual losses are capped at 1 million per year

Find the inflation rate from 2000 to 2001 on expected payments.

A) Less than 1.5% B) At least 1.5%, but less than 2.5% C) At least 2.5%, but less than 3.5%
D) At least 3.5%, but less than 4.5% E) At least 4.5%

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35. An insurer issues a one-year malpractice liability insurance policy to a medical clinic. If a malpractice suit is brought against the clinic, the distribution of the total of legal plus settlement costs to the clinic is assumed to be uniformly distributed between \$100,000 and \$1,000,000. The number of malpractice suits brought against the clinic in one year, N, is assumed to have the following distribution $P[N=0]=0.96$, $P[N=1]=0.03$, $P[N=2]=0.01$. The insurer charges a premium which is equal to $E[Y]+\sqrt{Var[Y]}$ where Y is the annual total of the clinic's claims. It is assumed that the number of malpractice suits and the costs arising from each suit are mutually independent. Find the premium charged by the insurer (nearest 25,000).

A) 100,000 B) 125,000 C) 150,000 D) 175,000 E) 200,000

36. For a certain insurance, individual losses last year were uniformly distributed over the interval $(0, 1000)$. A deductible of 100 is applied to each loss (the insurer pays the loss in excess of the deductible of 100). This year, individual losses are uniformly distributed over the interval $(0, 1050)$ and a deductible of 100 is still applied. Determine the percentage increase in the expected amount paid by the insurer from last year to this year.

A) 2 B) 4 C) 6 D) 8 E) 10

37. A life insurance company covers 16,000 mutually independent lives for 1-year term life insurance:

Benefit Class	Amount	Number Covered	Probability of Claim
1	1	8000	0.025
2	2	3500	0.025
3	4	4500	0.025

The insurance company's retention limit is 2 units per life (the insurer only covers an individual life up to a payment of 2). Reinsurance is purchased for 0.03 per benefit unit. The ceding insurer collects a premium of $Q=E[S]+2\sqrt{Var[S]}+R$, where S denotes the distribution of retained claims and R is the cost of reinsurance. Find Q.

A) Less than 900 B) At least 900 but less than 910 C) At least 910 but less than 920
D) At least 920 but less than 930 E) At least 930

38. (SOA) The cumulative distribution function for health care costs experienced by a policyholder is modeled by the function $F(x)=\begin{cases}1-e^{-x/100} & \text{for } x>0\\ 0 & \text{otherwise,}\end{cases}$

The policy has a deductible of 20. An insurer reimburses the policyholder for 100% of health care costs between 20 and 120 less the deductible. Health care costs above 120 are reimbursed at 50%. Let G be the cumulative distribution function of reimbursements given that the reimbursement is positive. Calculate $G(115)$.

A) 0.683 B) 0.727 C) 0.741 D) 0.757 E) 0.777

39. (SOA) The amount of a claim that a car insurance company pays out follows an exponential distribution. By imposing a deductible of d, the insurance company reduces the expected claim payment by 10%. Calculate the percentage reduction on the variance of the claim payment.

A) 1% B) 5% C) 10% D) 20% E) 25%

40. (SOA) A motorist makes three driving errors, each independently resulting in an accident with probability 0.25. Each accident results in a loss that is exponentially distributed with mean 0.80. Losses are mutually independent and independent of the number of accidents. The motorist's insurer reimburses 70% of each loss due to an accident. Calculate the variance of the total unreimbursed loss the motorist experiences due to accidents resulting from these driving errors.

A) 0.0432 B) 0.0756 C) 0.1782 D) 0.2520 E) 0.4116

41. (SOA) Automobile losses reported to an insurance company are independent and uniformly distributed between 0 and 20,000. The company covers each such loss subject to a deductible of 5000. Calculate the probability that the total payout on 200 reported losses is between 1,000,000 and 1,200,000.

A) 0.0803 B) 0.1051 C) 0.1799 D) 0.8201 E) 0.8575

42. (SOA) An automobile insurance company issues a one-year policy with a deductible of 500. The probability is 0.8 that the insured automobile has no accident and 0.0 that the automobile has more than one accident. If there is an accident, the loss before the application of the deductible is exponentially distributed with mean 3000. Calculate the 95th percentile of the insurance company payout on this policy.

A) 3466 B) 3659 C) 4159 D) 8487 E) 8987

PROBLEM SET 10 SOLUTIONS

1. The probability function is

x :	1	2	3	4	5
$P[X=x]$:	$\frac{5}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
Amount paid :	100	200	300	325	350

Expected amount paid

$= 100 \times \frac{5}{15} + 200 \times \frac{4}{15} + 300 \times \frac{3}{15} + 325 \times \frac{2}{15} + 350 \times \frac{1}{15} = 213.3$ Answer: D

2. This is an example of the "individual risk model" for total claims. When there are a large number of policies, it is generally assumed that the distribution of total claims, S, can be approximated by a normal distribution. The total claim random variable is $S = X_1 + X_2 + \cdots + X_{800}$ (there are a total of 800 policies), where X_i denotes the claim from policy i (which may be 0). We wish to find the amount needed, say G, so that $P[S \le G] = 0.95$ (G is the 95-th percentile of the distribution of total claims S). If we knew the mean and variance of S, then we can write

$$P[S \le G] = P\left[\frac{S-E[S]}{\sqrt{Var[S]}} \le \frac{G-E[S]}{\sqrt{Var[S]}}\right] = 0.95$$

We have "standardized" S, meaning that $\frac{S-E[S]}{\sqrt{Var[S]}}$ has a standard normal distribution, and, therefore $\frac{G-E[S]}{\sqrt{Var[S]}}$ is the 95-th percentile of the standard normal, which is 1.645.

Thus, $\frac{G-E[S]}{\sqrt{Var[S]}} = 1.645$. So, if we know the numerical values of $E[S]$ and $Var[S]$, then we can find G, the 95-th percentile of S (under the normal approximation).

Since $S = X_1 + X_2 + \cdots + X_{800}$, it follows that

$E[S] = \sum_{i=1}^{800} E[X_i]$, and $Var[S] = \sum_{i=1}^{800} Var[X_i]$ (it is assumed that the X_i's are independent, and therefore the variance of the sum is the sum of the variances with no covariance factors).

In the notes on risk topics earlier in this study material, the following comments were made.

Suppose that the probability of a non-negative loss occurring is specified (usually denoted q, with $1-q=p$ being the probability no loss occurs), and the **conditional distribution of the loss amount given that a loss occurs** is specified, say random variable B. The random variable B might be described in detail, or only the mean and variance of B might be given.

In this case, $X = 0$ if no loss occurs (probability p) and $X = B$ if a loss does occur (probability q). It is possible to use a mixture of distributions formulation $E[X] = q \times E[B]$ and $E[X^2] = q \times E[B^2]$, so that $Var[X] = q \times E[B^2] - (q \times E[B])^2$.

In this problem there are 500 policies with $q = 0.01$, $E[B_i] = 200,000$ and $E[B_i^2] = 200,000^2$, and there are 300 policies with $q = 0.05$, $E[B_i] = 100,000$ and $E[B_i^2] = 100,000^2$.

According to the comments above, for each of the first 500 policies,

$E[X_i] = q \cdot E[B_i] = 0.01 \times 200,000 = 2,000$,

$Var[X_i] = 0.01 \times 200,000^2 - [0.01 \times 200,000]^2 = 396,000,000$,

and for each of the next 300 policies, $E[X_i] = 0.05 \times 100,000 = 5,000$,

$Var[X_i] = 0.05 \times 100,000^2 - [0.05 \times 100,000]^2 = 475,000,000$.

Then, $E[S] = 500 \times 2,000 + 300 \times 5,000 = 2,500,000$ and

$Var[S] = 500 \times 396,000,000 + 300 \times 475,000,000 = 3.405 \times 10^{11}$.

Then, $G = E[S] + 1.645 \times \sqrt{Var[S]} = 3,460,000$ (nearest 1,000).

The company wishes to collect $kE[X_i]$ from policyholder i, for a total amount collected equal to $k \times E[S]$. Thus, $k \times 2,500,000 = 3,460,000$, so that $k = 1.38$. Answer: E

3. Amount paid $= \begin{cases} y & 1 < y \le 10 \\ 10 & y > 10 \end{cases}$

Expected amount paid $= \int_1^{10} y \times f(y)\, dy + (10)P(Y > 10)$

$P[Y > 10] = \int_{10}^{\infty} f(y)\, dy = \int_{10}^{\infty} \frac{2}{y^3}\, dy = \frac{1}{100}$

Expected amount paid $= \int_1^{10} y \times \frac{2}{y^3}\, dy + 10 \times 0.01 = 1.8 + 0.1 = 1.9$. Answer: D

4. The pdf of T is $f(t) = \frac{1}{3}e^{-t/3}$ for $t > 0$.

$$X = Max(T, 2) = \begin{cases} 2 & T \le 2 \\ T & T > 2 \end{cases}$$

This random variable has a discrete point,

$$P[X = 2] = P[T < 2] = \int_0^2 \frac{1}{3} e^{-t/3}\, dt = 1 - e^{-2/3}$$

$$E[X] = 2 \times (1 - e^{-2/3}) + \int_2^{\infty} t \times \frac{1}{3} e^{-t/3}\, dt$$

Integration by parts gives us

$$\int_2^{\infty} t \times \frac{1}{3}^{-t/3}\, dt = \int_2^{\infty} t\, d(-e^{-t/3}) = -te^{-t/3}\Big|_2^{\infty} - \int_2^{\infty} -e^{-t/3} dt = 2e^{-2/3} + 3e^{-2/3} = 5e^{-2/3}.$$

Recall that in general, the antiderivative of xe^{ax} is $\frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}$ (valid for any $a \ne 0$).

Then, $E[X] = 2 \times (1 - e^{-2/3}) + 5e^{-2/3} = 2 + 3e^{-2/3}$. Answer: D

5. $Y = Min\{X, 4\}$ (Y is the smaller of X and 4, since the machine will be replaced at time 4 if it is still operating). Y is a combination of a continuous distribution and one discrete point. The density function of Y is the same as that of X for $Y < 4$, and the probability that $Y = 4$ is the probability that $X \geq 4$.
$P[X \geq 4] = \int_4^5 \frac{1}{5}\, dx = \frac{1}{5}$.
Therefore, $f_Y(y) = \begin{cases} \frac{1}{5} & Y < 4 \\ \frac{1}{5} & Y = 4 \end{cases}$.
We use the following formulation for variance of Y, $Var[Y] = E[Y^2] - (E[Y])^2$.
$E[Y] = \int_0^4 y \times \frac{1}{5}\, dy + 4 \times \frac{1}{5} = \frac{12}{5}$, $E[Y^2] = \int_0^4 y^2 \times \frac{1}{5}\, dy + 4^2 \times \frac{1}{5} = \frac{112}{15}$.
$Var[Y] = \frac{112}{15} - (\frac{12}{5})^2 = 1.71$ Answer: C

6. Expected payment with no deductible is 500 (mean of a uniform distribution on interval from 0 to 1000). With deductible d, the amount paid on a loss of amount x is
$\begin{cases} 0 & x \leq d \\ x - d & x > d \end{cases}$, and the expected payment is
$\int_d^{1000} (x - d) \times 0.001\, dx = 0.0005 \times (1000 - d)^2$
In order for this to be 25% of the expected payment with no deductible, we must have
$0.0005 \times (1000 - d)^2 = 0.25 \times 500 \rightarrow d = 500$. Answer: C

7. Let N denote the Poisson random variable, the number of times in one year the financial conditions arise for the investment to be made, and let X denote the aggregate return on the investments for the year. Then

$$P[X < 100,000] = P[X < 100,000|N = 0] \times P[N = 0] \\ + P[X < 100,000|N = 1] \times P[N = 1] + \cdots$$

If $N = 0$, then no investments were made and $P[X < 100,000|N = 0] = 1$, and if $N = 1$, then one investment was made whose return is normal with mean $100,000$ so that $P[X < 100,000|N = 1] = 0.5$. If $N = 2$, then X is the sum of two independent normal random variables with total mean $200,000$ and variance $2 \times 20,000^2 = 28,284^2$, so that
$P[X < 100,000|N = 2] = P\left[\frac{X-200,000}{28,284} < \frac{100,000-200,000}{28,284}\right] = P[Z < -3.54]$
where Z has a standard normal distribution - this probability is essentially 0, and so will be $P[X < 100,000|N = 3, 4, ...]$. Since $P[N = 0] = e^{-3} = 0.0498$ and $P[N = 1] = e^{-3} \times 3 = 0.1494$, it follows that
$P[X < 100,000] = 1 \times 0.0498 + 0.5 \times 0.1494 = 0.12$. Answer: A

8. In order for the loss random variable to be properly defined, the probabilities must add to 1: $\frac{K}{1}+\frac{K}{2}+\frac{K}{3}+\frac{K}{4}+\frac{K}{5}=1 \rightarrow K=\frac{1}{2.2833}=.4380.$

The net premium for the policy is the expected amount paid by the policy.

The amount paid by the policy can be regarded as a mixture of the outcome 0 (amount paid if no loss occurs) with probability 0.95, and the outcome Y (amount paid after deductible if a loss occurs) with probability 0.05. The expected amount paid by the policy is $0 \times 0.95 + E(Y) \times 0.05$. Y is the amount paid after the deductible is applied, given that a loss has occurred.

Therefore, $P(Y=0)=P(N=1 \text{ or } 2)=\frac{K}{1}+\frac{K}{2}=0.657$,

$P(Y=1)=P(N=3)=\frac{K}{3}=0.146$, $P(Y=2)=P(N=4)=\frac{K}{4}=0.110$,

$P(Y=3)=P(N=5)=0.088$. Then $E(Y)=1\times 0.146+2\times 0.110+3\times 0.088=.63$ and the expected amount paid by the policy is $0.05\times 0.63=0.0315$. Answer: A

9. For an individual employee, the distribution of the amount the company pays above the deductible of \$100 is $Y=\begin{cases} 0, \text{ prob. } 0.3 \\ 100, \text{ prob. } 0.4 \\ 400, \text{ prob. } 0.2 \\ 900, \text{ prob. } 0.1 \end{cases}$, with $E[Y]=210$ and $E[Y^2]=117,000$

so that $Var[Y]=72,900$. If S is the aggregate amount paid by the company in one year, then $E[S]=50E[Y]=10,500$ and $Var[S]=50Var[Y]=3,645,000$.

The 95-th percentile if S is a, where $P[S \le a]=0.95$. This probability can be rewritten as $P4\left[\frac{S-E[S]}{\sqrt{Var[S]}} \le \frac{a-E[S]}{\sqrt{Var[S]}}\right]=0.95$, and then applying the normal approximation to S, $\frac{S-E[S]}{\sqrt{Var[S]}}$ has an approximately standard normal distribution.

Then, $\frac{a-E[S]}{\sqrt{Var[S]}}=\frac{a-10,500}{\sqrt{3,645,000}}$ is equal to the 95-th percentile of the standard normal distribution, which is 1.645. Thus, $\frac{a-10,500}{\sqrt{3,645,000}}=1.645 \rightarrow a=13,641$. Answer: C

10. A policy with probability of claim p and claim amount C has a two-point claim distribution $X=\begin{cases} 0\,, \text{ prob. } 1-p \\ C\,, \text{ prob. } p \end{cases}$. Then, $E[X]=Cp$, $E[X^2]=C^2p$ and $Var[X]=C^2p-(Cp)^2=C^2p(1-p)$. Since the policies are mutually independent, the variance of the aggregate claim S is the sum of the variances of the individual policy claims:

$Var[S]=1000\times 1^2\times 0.01\times 0.99+2000\times 1^2\times 0.02\times 0.98+500\times 2^2\times 0.04\times 0.96=125.9$

and the standard deviation of S is $\sqrt{Var[S]}=\sqrt{125.9}=11.22$. Answer: D

11. The expected loss is $E[X]=\int_0^\infty [1-F(x)]\,dx=\int_0^2 (0.8-.3x)\,dx=1$.

The expected claim on the insurer is $0.5=E[\alpha X]=\alpha E[X]=\alpha$. Answer: D

12. The insurance payout Y has a mixed distribution - there is a .8 probability that $Y=0$, and the conditional density of Y given that a claim has occurred is

$f_Y(y|\text{claim}) = \frac{f_Y(y)}{prob.\, claim\, occurs} = 0.001$ for $1000 \le y \le 2000$, so that

$f_Y(y) = .0002$ for $1000 \le y \le 2000$: $f(y) = \begin{cases} 0.8\,, \text{ if } y=0 \\ 0.0002, \text{ if } 1000 \le y \le 2000 \end{cases}$.

Then $E[Y] = (.8)(0) + \int_{1000}^{2000} 0.0002 \times y\, dy = 300$, and

$E[Y^2] = 0.8 \times 0^2 + \int_{1000}^{2000} 0.0002 \times y^2\, dy = \frac{1{,}400{,}000}{3} \rightarrow Var[Y] = \frac{1{,}130{,}000}{3} \approx 614^2$

Alternatively,

$E[Y] = E[Y|\text{no claim occurs}] \times P[\text{no claim}] + E[Y|\text{claim occurs}] \times P[\text{claim}]$

$= 0 \times 0.8 + 1500 \times 0.2 = 300$, and

$E[Y^2] = E[Y^2|\text{no claim occurs}] \times P[\text{no claim}] + E[Y^2|\text{claim occurs}] \times P[\text{claim}]$

$= 0^2 \times 0.8 + \frac{7{,}000{,}000}{3} \times 0.2 = \frac{1{,}400{,}000}{3}$. Answer: B

13. The maximum payment on the policy occurs when $X+Y \ge 100$. The expected payment is

$$\int_{R_1}\int (x+y) \frac{x}{500{,}000}\, dy\, dx \quad + \int_{R_2}\int (100) \frac{x}{500{,}000}\, dy\, dx$$

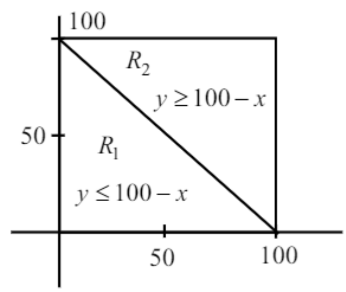
where R_1 and R_2 are the regions in the square $\{(x,y)|0 \le x \le 100,\ 0 \le y \le 100$ represented in the graph below:

$R_1 = \{(x,y)|x+y \le 100\}$, $R_2 = \{(x,y)|x+y \ge 100\}$.

The expected payment is

$$\int_0^{100}\int_0^{100-x} (x+y) \times \frac{x}{500{,}000}\, dydx + \int_0^{100}\int_{100-x}^{100} 100 \times \frac{x}{500{,}000}\, dy\, dx$$

$$= \frac{1}{500{,}000}\int_0^{100} [x^2(100-x) + \tfrac{1}{2}x(100-x)^2]\, dx + \int_0^{100} \frac{x^2}{5000}\, dx = 25 + 66.7 \approx 92$$



Answer: B

14. We use the conditional variance approach: $Var[S] = E[Var[S|N]] + Var[E[S|N]]$

$Var[S|N]$ is a 3-point random variable - $Var[S|N] = \begin{cases} 0, \text{ if } N=0\,, \text{ prob. } 1/3 \\ 5, \text{ if } N=1\,, \text{ prob. } 1/3 \\ 8, \text{ if } N=2\,, \text{ prob. } 1/3 \end{cases}$,

so that $E[Var[S|N]] = (0+5+8) \times \frac{1}{3} = \frac{13}{3}$

$E[S|N]$ is also a 3-point random variable - $E[S|N] = \begin{cases} 0, \text{ if } N=0\,, \text{ prob. } 1/3 \\ 10, \text{ if } N=1\,, \text{ prob. } 1/3 \\ 20, \text{ if } N=2\,, \text{ prob. } 1/3 \end{cases}$,

so that $E[E[S|N]] = (0+10+20) \times \frac{1}{3} = 10$, and

$E[(E[S|N])^2] = (0+10^2+20^2) \times \frac{1}{3} = \frac{500}{3}$, and then

$Var[E[S|N]] = E[(E[S|N])^2] - (E[E[S|N]])^2 = \frac{500}{3} - 10^2 = \frac{200}{3}$

Then, $Var[S] = \frac{13}{3} + \frac{200}{3} = \frac{213}{3} = 71$. Answer: E