4. $P(A \cap B) = P(A|B) \times P(B) = 0.4 \times P(B)$.
$P(A|B') = 1 - P(A'|B') = 0.6$ and $P(A \cap B') = P(A|B') \times P(B') = 0.6 \times [1 - P(B)]$.
Then $0.5 = P(A) = P(A \cap B) + P(A \cap B')$
$= 0.4 \times P(B) + 0.6 \times [1 - P(B)] = 0.6 - 0.2 \times P(B)$.
Solving for $P(B)$ results in $P(B) = 0.5$. Answer: B

5. There are 8 possible orderings of births in a family of three children:
GGG , GGB , GBG , GBB , BGG , BGB , BBG , BBB.
# of these orderings result in a family with exactly one girl: GBB , BGB , BBG.
Each of the orderings has a probability of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ of occurring, so the probability of exactly one girl is $\frac{3}{8}$. Answer: C

6. To find the probability that a certain type of combination or arrangement occurs, the probability is usually formulated as $\frac{\text{number of combinations or arrangement of the specific type required}}{\text{total number of all combinations or arrangements}}$.
For these problems, the denominator is the total number of all 6 digit numbers that can be created by choosing 6 digits without replacement from 2 , 3 , 4 , 5 , 6 , 7 , 8. The total number of 6-digit numbers is $7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5,040$ since the first digit can be any one of the 7 integers, the second digit can be any one of the remaining 6 integers, etc.
The number is even if it ends in 2 , 4 , 6 or 8 . For each of these 4 cases, there are $6 \times 5 \times 4 \times 3 \times 2 = 720$ arrangements of the first 5 digits in the number, since the other 5 digits are chosen from the 6 remaining integers. The numerator of the probability is $4 \times 720 = 2880$, and the probability is $\frac{2880}{5040} = \frac{4}{7}$.
An alternative solution is to note that there are 7 possible equally likely final digits for the 6-digit number, and 4 of them make the number even. The probability is $\frac{4}{7}$. Answer: D

7. We wish to find $P(A|B) = \frac{P(A \cap B)}{P(B)}$, where
$A =$ more red than white are chosen, and $B =$ no blue are chosen.
Out of 8 balls in the bag, 5 are not blue, so for each of the three trials, the probability of picking a ball that is not blue is $\frac{5}{8}$; since the ball chosen is always replaced, the probability remains the same on each subsequent choice. Then, $P(B) = \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8} = \frac{125}{512}$.
On any given trial, the probability of a red ball being chosen is $\frac{3}{8}$ and the probability of a white ball being chosen is $\frac{2}{8}$. $A \cap B =$ no blue are chosen and more red than white are chosen.
The following sequences of choices result in $A \cap B$ occurring: RRR, RRW, RWR, WRR.
The probabilities of these sequences is $P(RRR) = \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{27}{512}$, $P(RRW) = \frac{3}{8} \times \frac{3}{8} \times \frac{2}{8} = \frac{18}{512}$, $P(RWR) = \frac{3}{8} \times \frac{2}{8} \times \frac{3}{8} = \frac{18}{512}$, $P(WRR) = \frac{2}{8} \times \frac{3}{8} \times \frac{3}{8} = \frac{18}{512}$.
Then, $P(A \cap B) = \frac{27+18+18+18}{512} = \frac{81}{512}$. Finally, $P(A|B) = \frac{81/512}{125/512} = \frac{81}{125}$. Answer: E

8. The student will pass if at least two of the three questions chosen come from the 90 questions that the student knows. The number of ways of choosing 3 questions from 100 is

$$\binom{100}{3} = \frac{100!}{97!\times 3!} = \frac{100\times 99\times 98}{6} = 161,700.$$

The student will pass if either

(i) all three questions are chosen from the 90 he knows the answers to, or

(ii) exactly two of the three questions are chosen from the 90 he knows the answers to and the other is chosen from the other 10.

The number of ways of (i) occurring is $\binom{90}{3} = \frac{90!}{87!\times 3!} = \frac{90\times 89\times 88}{6} = 117,480$, and the number of ways of (ii) occurring is $\binom{90}{2}\times\binom{10}{1} = \frac{90!}{88!\times 2!}\times\frac{10!}{9!\times 1!} = \frac{90\times 89}{2}\times 10 = 40,050$.

The probability that the student gets at least 2 of the 3 questions right is $\frac{117{,}480+40{,}050}{161{,}700} = 0.974$. Answer: C

9. $P(.5 < X < 1) = \int_{0.5}^{1} x\,dx = 0.375$.
$P(X < 1|X > 0.5) = \frac{P(0.5<X<1)}{P(X>0.5)}$.

Although we have not determined the value of c, we know that

$$P(X > 0.5) = 1 - P(X \le 0.5) = 1 - [P(X = 0) + P(0 < X \le 0.5)]$$
$$= 1 - [0.2 + \int_0^{0.5} x\,dx] = 1 - 0.325 = 0.675.$$

Then $P(X < 1|X > 0.5) = \frac{P(0.5<X<1)}{P(X>0.5)} = \frac{0.375}{0.675} = 0.556$. Answer: A

10. $F(x) = P(X \le x)$.
$P(X = 2) = \frac{1}{6}$, since $X = 2$ if the second toss is the same as the first, and there is a $\frac{1}{6}$ chance of that. There is a $\frac{5}{6}$ chance that the 2nd toss is not the same as the first, and then a $\frac{1}{6}$ chance that the 3rd toss is the same as the 2nd, so $P(X = 3) = \frac{5}{6}\times\frac{1}{6}$. There is a $\frac{5}{6}$ chance that the 3rd toss is not the same as the 2nd, and then a $\frac{1}{6}$ chance that the 4th toss is the same as the 3rd, so $P(X = 4) = (\frac{5}{6})^2\times\frac{1}{6}$. Continuing in this way, we see that $P(X = n) = (\frac{5}{6})^{n-2}\times\frac{1}{6}$.
Then $F(x) = \sum_{n=2}^{x} P(X = n) = \sum_{n=2}^{x} (\frac{5}{6})^{n-2}\times\frac{1}{6} = [1 + \frac{5}{6} + (\frac{5}{6})^2 + \cdots + (\frac{5}{6})^{x-2}]\times\frac{1}{6}$.

We use the geometric series expression $1 + a + a^2 + \cdots + a^k = \frac{1-a^{k+1}}{1-a}$, to get

$$F(x) = \frac{1-(\frac{5}{6})^{x-1}}{1-\frac{5}{6}}\times\frac{1}{6} = 1 - (\frac{5}{6})^{x-1}.$$ Answer: A

11. $\mu_X = (10 \times 0.1) + (20 \times 0.1) + (30 \times 0.4) + (40 \times 0.3) + (50 \times 0.1) = 32.$

$$\sigma_X^2 = E[(X - \mu_X)^2]$$
$$= (10-32)^2 \times 0.1 + (20-32)^2 \times 0.1 + (30-32)^2 \times 0.4$$
$$+ (40-32)^2 \times 0.3 + (50-32)^2 \times 0.1 = 116$$

Alternatively, $\sigma_X^2 = E(X^2) - (E[X])^2$.

$E(X^2) = (10^2 \times .1) + (20^2 \times .1) + (30^2 \times .4) + (40^2 \times .3) + (50^2 \times .1) = 1140$,

so that $\sigma_X^2 = 1140 - 32^2 = 116$, and $\sigma_X = \sqrt{116} = 10.77$.

Then, $P(|X - \mu_X| \le \sigma_X) = P(|X - 32| \le 10.77) = P(-10.77 \le X - 32 \le 10.77)$

$= P(21.23 \le X \le 42.77) = P(X = 30 \text{ or } 40) = 0.7.$ Answer: C

12. $E(X) = \int_0^c x \times f(x)\,dx = \int_0^c \frac{(k+1)x^{k+1}}{c^{k+1}}\,dx = \frac{k+1}{k+2} \times c$.

$E(X^2) = \int_0^c x^2 \times f(x)\,dx = \int_0^c \frac{(k+1)x^{k+2}}{c^{k+1}}\,dx = \frac{k+1}{k+3} \times c^2$.

$Var(X) = E(X^2) - (E[X])^2 = \frac{k+1}{k+3} \cdot c^2 - (\frac{k+1}{k+2} \cdot c)^2 = [\frac{k+1}{k+3} - (\frac{k+1}{k+2})^2]c^2 = \frac{k+1}{(k+3)(k+2)^2} \cdot c^2.$

The coefficient of variation of X is $\sqrt{\frac{k+1}{(k+3)(k+2)^2} \cdot c^2} \Big/ (\frac{k+1}{k+2} \cdot c) = \frac{1}{\sqrt{(k+1)(k+3)}}$.

Answer: C

13. We denote the probability function of X by $p_0^X = P(X=0)$, $p_1^X = P(X=1)$, and $p_2^X = P(X=2)$. With similar notation for Y.

Then $M_X(t) = E[e^{tX}] = e^0 \times p_0^X + e^t \times p_1^X + e^{2t} \times p_2^X$ and
$M_Y(t) = E[e^{tY}] = e^0 \times p_0^Y + e^t \times p_1^Y + e^{2t} \times p_2^Y$.

Then $M_X(t) + M_Y(t) = p_0^X + e^t \times p_1^X + e^{2t} \times p_2^X + p_0^Y + e^t \times p_1^Y + e^{2t} \times p_2^Y$
$= p_0^X + p_0^Y + (p_1^X + p_1^Y)e^t + (p_2^X + p_2^Y)e^{2t}$,

and it follows that $p_0^X + p_0^Y = \frac{3}{4}$, $p_1^X + p_1^Y = \frac{3}{4}$ and $p_2^X + p_2^Y = \frac{1}{2}$.

In a similar way, we have

$M_X(t) - M_Y(t) = p_0^X + e^t \times p_1^X + e^{2t} \times p_2^X - p_0^Y - e^t \times p_1^Y - e^{2t} \times p_2^Y$
$= p_0^X - p_0^Y + (p_1^X - p_1^Y)e^t + (p_2^X - p_2^Y)e^{2t}$,

and it follows that $p_0^X - p_0^Y = \frac{1}{4}$, $p_1^X - p_1^Y = -\frac{1}{4}$ and $p_2^X - p_2^Y = 0$.

From these equations, we see that $p_1^X + p_1^Y + p_1^X - p_1^Y = 2\,p_1^X = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$,
and therefore $P(X=1) = p_1^X = \frac{1}{4}$. Answer: B

14. $E(X) = \int_{-4}^{2} x\,f(x)\,dx = \int_{-4}^{0} x \times \frac{-x}{10}\,dx + \int_{0}^{2} x \times \frac{x}{10}\,dx = -\frac{28}{15}$.

The median m satisfies the relationship $\int_{-4}^{m} f(x)\,dx = \frac{1}{2}$.

We know that $\int_{-4}^{0} f(x)\,dx = \int_{-4}^{0} (\frac{-x}{10})\,dx = \frac{4}{5}$, so we must have $-4 \le m \le 0$.

Then $\int_{-4}^{m} (\frac{-x}{10})\,dx = \frac{16-m^2}{20}$. Setting this equal to $\frac{1}{2}$, we get $m = -\sqrt{6}$ (we take the negative square root because we know that the median is between -4 and 0).

Then $|E(X) - m| = |-\frac{28}{15} - (-\sqrt{6})| = 0.583$. Answer: C

15. We can approach this problem by trial and error.

If Smith has $c = 1$, then Smith will go broke if he loses the first game (prob. .6), so there is a .4 chance that Smith wins before going broke.

If Smith has $c = 1 + 2 = 3$, then Smith will go broke if he loses the first two games (prob. $0.6 \times 0.6 = 0.36$), so there is a 0.64 chance that Smith wins before going broke.

If Smith has $c = 1 + 2 + 4 = 7$, then Smith will go broke if he loses the first three games (prob. $(0.6)^3 = 0.216$), so there is a 0.784 chance that Smith wins before going broke.

If Smith has $c = 1 + 2 + 4 + 8 = 15$, then Smith will go broke if he loses the first four games (prob. $(0.6)^4 = 0.1296$), so there is a 0.8704 chance that Smith wins before going broke.

If Smith has $c = 1 + 2 + 4 + 8 + 16 = 31$, then Smith will go broke if he loses the first five games (prob. $(0.6)^5 = 0.0778$), so there is a 0.9222 chance that Smith wins before going broke.

If Smith has $c = 1 + 2 + 4 + 8 + 16 + 32 = 63$, then Smith will go broke if he loses the first six games (prob. $(0.6)^6 = 0.0467$), so there is a 0.9533 chance that Smith wins before going broke.

Answer: D

16. We can use the multinomial distribution. For a batch of n components tested, the probability that a are rated high, b are rated medium and c are rated low is $\frac{n!}{a!\,b!\,c!}(0.5)^a(0.4)^b(0.1)^c$.

In a batch of 5, in order to have at least 3 high and at most 1 low, the following combinations are possible 5H, 4H and 1M, 4H and 1L, 3H and 1M and 1L.

The probabilities of these combinations are

$$P(5H) = \frac{5!}{5!\,0!\,0!}(0.5)^5(0.4)^0(0.1)^0 = 0.03125\ ,$$
$$P(4H \text{ and } 1M) = \frac{5!}{4!\,1!\,0!}(0.5)^4(0.4)^1(0.1)^0 = 0.125\ ,$$
$$P(4H \text{ and } 1L) = \frac{5!}{4!\,0!\,1!}(0.5)^4(0.4)^0(0.1)^1 = 0.03125\ ,$$
$$P(3H \text{ and } 1M \text{ and } 1L) = \frac{5!}{3!\,1!\,1!}(0.5)^3(0.4)^1(0.1)^1 = 0.1\ .$$
$$P(3H \text{ and } 2M) = \frac{5!}{3!\,2!\,0!}(0.5)^3(0.4)^2(0.1)^0 = 0.2\ .$$

The total probability is $0.03125 + 0.125 + 0.03125 + 0.1 + 0.2 = 0.4875$. Answer: E

17. The number of exceptional students chosen, say X, can be described as having a hypergeometric distribution. For $k = 0, 1, 2, 3$ we have $P(X = k) = \frac{\binom{3}{k}\binom{22}{5-k}}{\binom{25}{5}}$

(k of the 3 exceptional students are chosen, and $22 - k$ of the average students are chosen).

Then the probability that at least 2 exceptional students are chosen for the test is

$P(X = 2 \text{ or } 3) = \frac{\binom{3}{2}\binom{22}{3}}{\binom{25}{5}} + \frac{\binom{3}{3}\binom{22}{2}}{\binom{25}{5}} = \frac{(3)(1540)+(1)(231)}{53{,}130} = 0.0913$. Answer: E

18. We denote by W the exponential distribution with a mean of 10, so the pdf of W is $f_W(w) = \frac{e^{-w/10}}{10}$.

Then the distribution of X can be found from the distribution of W.

X is a discrete integer-valued random variable ≥ 0.

$X = 0$ if $0 < W \leq 1$ (if failure is in the first year), and the probability is

$P(X = 0) = P(0 < W \leq 1) = \int_0^1 f_W(w)\, dw = \int_0^1 0.1e^{-0.1w}\, dw = 1 - e^{-0.1} = 0.095163.$

$X = 1$ if $1 < W \leq 2$ (if failure is in the second year), and the probability is

$P(X = 1) = P(1 < W \leq 2) = \int_1^2 0.1e^{-0.1w}\, dw = e^{-0.1} - e^{0.2} = 0.086107.$

$X = k$ if $k < W \leq k+1$ (if failure is in the first year), and the probability is

$P(X = k) = P(k < W \leq k+1) = \int_k^{k+1} 0.1e^{-0.1w}\, dw$

$= e^{-0.1k} - e^{-0.1(k+1)} = (e^{-0.1})^k(1 - e^{-0.1})$.

The commonly used definition of the geometric distribution is as follows.

Suppose that $0 < p < 1$ and suppose that Z is an integer-valued random variable ≥ 0 with probability function $P(Z = j) = (1-p)^k \cdot p$ for $j = 0, 1, 2, \ldots$.

Z is said to have a geometric distribution with parameter p, and the mean of Z is

$E[Z] = \frac{1-p}{p}$ (and the variance of Z is $Var[Z] = \frac{1-p}{p^2}$).

From the probability function of X described above, if we let $p = 1 - e^{-0.1}$, then

$1 - p = e^{-0.1}$ and $P(X = k) = (e^{-0.1})^k(1 - e^{-0.1}) = (1-p)^k \times p$.

We see that X has a geometric distribution, with $p = 1 - e^{-.1}$.

The mean of X is $\frac{1-p}{p} = \frac{e^{-0.1}}{1-e^{-0.1}} = 9.508.$

Jones is correct about the distribution of X being geometric, but is wrong about the mean of X.

Answer: B

19. The sum of normal random variables is normal, and a constant multiple of a normal random variable is normal. Therefore Y is normal and the mean of Y is $\frac{1}{5}\sum_{i=1}^{5}E(X_i) = \frac{1}{5}(1+2+3+4+5) = 3$. Then the mean of $Y-3$ is 0.

The mean and median (50th percentile) of a normal random variable are the same, so that the median of $Y-3$ is equal to 0. Answer: A

20. Since $f(x)$ is a pdf, it follows that $\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}$,

and this is valid for any integers a and b.

Then $E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}\,x^a(1-x)^{b-1}dx$

$= \frac{(a+b-1)!}{(a-1)!(b-1)!}\int_0^1 x^a(1-x)^{b-1}dx = \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{a!(b-1)!}{(a+b)!} = \frac{a}{a+b}$

(this follows by using $a+1$ instead of a for the pdf).

Therefore, if $a=b$ then $E(X) = \frac{1}{2}$, so statement I is true.

$$E(X^2) = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}\,x^{a+1}(1-x)^{b-1}dx$$
$$= \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{(a+1)!(b-1)!}{(a+b+1)!} = \frac{a(a+1)}{(a+b)(a+b+1)}\,.$$

If $a=b$ then $E(X^2) = \frac{a(a+1)}{(2a)(2a+1)} = \frac{a+1}{2(2a+1)}$.

Then with $a=b$ we have $Var(X) = \frac{a+1}{2(2a+1)} - (\frac{1}{2})^2 = \frac{a+1}{4a+2} - \frac{1}{4} = \frac{2}{4(4a+2)} = \frac{1}{8a+4}$.

Statement II is false.

$$E(X^k) = \int_0^1 x^k\,f(x)\,dx = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!}\,x^{a+k-1}(1-x)^{b-1}dx\,.$$

Since $0<x<1$, as k increases x is raised to a higher power, so x^{a+k-1} becomes smaller numerically, and so does the integral. Statement III is false. Answer: A

21. In order to be a properly defined pdf we must have $\int_1^\infty\int_1^\infty f(x,y)\,dy\,dx = 1$. Therefore

$$\int_1^\infty\int_1^\infty kx^{-3}e^{-y/3}\,dy\,dx = k\int_1^\infty 3e^{-1/3}x^{-3}dx = k\cdot 3e^{-1/3}\times\frac{1}{2} = \frac{3}{2}ke^{-1/3} = 1\,,$$

and it follows that $k = \frac{2e^{1/3}}{3}$.

The marginal density of X is

$$f_X(x) = \int_1^\infty f(x,y)\,dy = \int_1^\infty \frac{2e^{1/3}}{3}\times x^{-3}e^{-y/3}\,dy = \frac{2e^{1/3}}{3}\times x^{-3}\times 3e^{-1/3} = 2x^{-3}$$

for $1<x<\infty$.

Then $E(X) = \int_1^\infty x\,f_X(x)\,dx = \int_1^\infty x\times 2x^{-3}\,dx = \int_1^\infty 2x^{-2}\,dx = 2$. Answer: B

22. Smith wins 1000 if $X > Y$. The probability of this is $\int_0^1 \int_0^x \frac{2}{3}(x+2y)\,dy\,dx = \frac{4}{9}$.
Therefore, the probability that $X < Y$ is $\frac{5}{9}$.
Suppose that Smith pays Jones $\$C$ if $X < Y$.
Then Smith's expected return is $1000(\frac{4}{9}) - C(\frac{5}{9}) = \frac{4000-5C}{9}$.
In order for this to be 0 we must have $C = 800$. Answer: B

23. Standardizing X and Y, we have $\frac{X-\mu_X}{\sigma_X}$ has a standard normal distribution, as does $\frac{Y-\mu_Y}{\sigma_Y}$.
Then $F_X(t) = P(X \le t) = P(\frac{X-\mu_X}{\sigma_X} \le \frac{t-\mu_X}{\sigma_X}) = \Phi(\frac{t-\mu_X}{\sigma_X})$, and similarly, $F_Y(t) = \Phi(\frac{t-\mu_Y}{\sigma_Y})$.
Then $F_X(t) \ge F_Y(t)$ if $\Phi(\frac{t-\mu_X}{\sigma_X}) \ge \Phi(\frac{t-\mu_Y}{\sigma_Y})$, which occurs if $\frac{t-\mu_X}{\sigma_X} \ge \frac{t-\mu_Y}{\sigma_Y}$.
This inequality can be written as $t - \mu_X \ge (t - \mu_Y) \cdot \frac{\sigma_X}{\sigma_Y} = 2(t - \mu_Y)$
(since we were given that $\sigma_X = 2\sigma_Y$).
The inequality can be rewritten as $t \le 2\mu_Y - \mu_X$. Answer: A

24. The conditional density of Y given $X = x$ is $f(y|x) = \frac{f(x,y)}{f_X(x)}$, where $f_X(x)$ is the marginal density of X. The marginal density of X is
$f_X(x) = \int_x^1 f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x)$ on the interval $x \le y \le 1$.
The conditional density of Y given $X = x$ is $f(y|x) = \frac{2}{2(1-x)} = \frac{1}{1-x}$ for $x \le y \le 1$.
The conditional mean of Y given $X = x$ is $\int_x^1 y \cdot \frac{1}{1-x}\,dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$.
We also could have noted that $f(y|x) = \frac{1}{1-x}$ is a uniform density on the interval $x \le y \le 1$.
so the mean is the midpoint of the interval, $\frac{x+1}{2}$. Answer: C

25. The coefficient of correlation is $\frac{Cov(Y_k,Y_j)}{\sqrt{Var(Y_k) \times Var(Y_j)}}$.
$Var(Y_k) = Var(\sum_{i=1}^{k} X_i) = \sum_{i=1}^{k} Var(X_i)$ (because of independence) $= 1 + 1 + \cdots + 1 = k$,
and similarly, $Var(Y_j) = j$.

Since $k < j$, it follows that $Y_j = Y_k + X_{k+1} + \cdots + X_j = Y_k + (Y_j - Y_k)$
and therefore $Cov(Y_k, Y_j) = Cov(Y_k, Y_k) + Cov(Y_k, Y_j - Y_k) = Var(Y_k) + 0 = k$.

Since $Y_j - Y_k = X_{k+1} + \cdots + X_j$, we see that $Y_j - Y_k$ is independent of Y_k, and therefore $Cov(Y_k, Y_j - Y_k) = 0$.
The coefficient of correlation is $\frac{k}{\sqrt{kj}} = \sqrt{\frac{k}{j}}$. Answer: D

26. $F_Y(3) = P(Y \le 3) = P(X^2 - 1 \le 3) = P(X^2 \le 4) = P(-2 \le X \le 2) = F_X(2)$.
The pdf of X *is* $f(x) = e^{-x}$ since X is exponential with mean 1. Then,
$F_X(2) = \int_0^2 e^{-x}\,dx = 1 - e^{-2}$. Answer: D

27. From the properties of natural log, we have
$ln(\frac{X_2}{X_1}) + ln(\frac{X_3}{X_2}) + \cdots + ln(\frac{X_{101}}{X_{100}}) = ln(\frac{X_{101}}{X_1}) = ln(X_{101})$.
This is the sum of 100 independent rv's each with mean 0.01 and variance 0.0009, so
$ln(X_{101})$ has a distribution which is approximately normal with mean $100(0.01) = 1$ and variance
$100(0.0009) = 0.09$. The probability that X_{101} is at least 4 is
$P(X_{101} \geq 4) = P[ln(X_{101}) \geq ln4] = P[\frac{ln(X_{101})-1}{\sqrt{0.09}} \geq \frac{ln\,4-1}{\sqrt{0.09}}] = 1 - \Phi(1.29) = 0.10$.
Answer: B

28. The maximum payment will occur if the loss is 2 or more, since the deductible would bring the payment to 1. If Y is the amount paid by the insurance and X is the underlying loss, then

$$Y = \begin{cases} 0 & X \leq 1 \\ X-1 & 1 < X \leq 2 \\ 1 & X > 2 \end{cases} .$$

The pdf of X is e^{-x}, so the expected value of Y is $\int_1^2 (x-1)e^{-x}\,dx + P(X > 2)$.
$\int_1^2 (x-1)e^{-x}\,dx = \int_1^2 xe^{-x}\,dx - \int_1^2 e^{-x}\,dx = (-xe^{-x} - e^{-x}\Big|_{x=1}^{x=2}) - (e^{-1} - e^{-2})$
$= e^{-1} - 2e^{-2}$, and
$P(X > 2) = \int_2^\infty e^{-x}\,dx = e^{-2}$.
Then $E(Y) = e^{-1} - 2e^{-2} + e^{-2} = e^{-1} - e^{-2}$. Answer: C

29. $Y = \begin{cases} 1 & X \leq 1 \\ X & X > 1 \end{cases}$. The pdf of X is $f(x) = \frac{1}{2}$.
$E(Y) = P(X < 1) + \int_1^2 x \times \frac{1}{2}\,dx = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$ and
$E(Y^2) = P(X < 1) + \int_1^2 x^2 \times \frac{1}{2}\,dx = \frac{1}{2} + \frac{7}{6} = \frac{5}{3}$.
$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{5}{3} - (\frac{5}{4})^2 = 0.1042$. Answer: B

30. For the exponential random variable Y with mean 1, we have $P(Y > y) = e^{-y}$.
For this distribution, we have $P(X > x) = e^{-(x/\theta)^\tau}$.
With transformation A we have
$P(Y > y) = P(X^\tau > y) = P(X > y^{1/\tau}) = e^{-(y^{1/\tau}/\theta)^\tau} = e^{-y/\theta^\tau} \neq e^{-y}$.
With transformation B we have
$P(Y > y) = P(X^{1/\tau} > y) = P(X > y^\tau) = e^{-(y^\tau/\theta)^\tau} = e^{-y^{\tau^2}/\theta^\tau} \neq e^{-y}$.
With transformation C we have
$P(Y > y) = P((\frac{X}{\theta})^\tau > y) = P(X > \theta y^{1/\tau}) = e^{-(\theta y^{1/\tau}/\theta)^\tau} = e^{-y}$. This is the correct probability for the exponential distribution with mean 1. Answer: C

PRACTICE EXAM 7

1. A study of the relationship between blood pressure and cholesterol level showed the following results for people who took part in the study:

(a) of those who had high blood pressure, 50% had a high cholesterol level, and

(b) of those who had high cholesterol level, 80% had high blood pressure.

Of those in the study who had at least one of the conditions of high blood pressure or high cholesterol level, what is the proportion who had both conditions?

A) $\frac{1}{3}$ B) $\frac{4}{9}$ C) $\frac{5}{9}$ D) $\frac{2}{3}$ E) $\frac{7}{9}$

2. A study of international athletes shows that of the two performance-enhancing steroids Dianabol and Winstrol, 5% of athletes use Dianabol and not Winstrol, 2% use Winstrol and not Dianabol, and 1% use both. A breath test has been developed to test for the presence of the these drugs in an athlete. Past use of the test has resulted in the following information regarding the accuracy of the test. Of the athletes that are using both drugs, the test indicates that 75% are using both drugs, 15% are using Dianabol only and 10% are using Winstrol only. In 80% of the athletes that are using Dianabol but not Winstrol, the test indicates they are using Dianabol but not Winstrol, and for the other 20% the test indicates they are using both drugs. In 60% of the athletes that are using Winstrol but not Dianabol, the test indicates that they are using Winstrol only, and for the other 40% the test indicates they are using both drugs. For all athletes that are using neither Dianabol nor Winstrol, the test always indicates that they are using neither drug. Of those athletes who test positive for Dianabol but not Winstrol, find the percentage that are using both drugs.

A) 1.2% B) 2.4% C) 3.6% D) 4.8% E) 6.0%

3. The random variable N has the following characteristics:

(i) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.

(ii) With probability $1 - p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.

Which of the following is a correct expression for $\text{Prob}(N = 2)$?

A) $0.125p^2$ B) $0.375 + 0.125p$ C) $0.375 + 0.125p^2$

D) $0.375 - 0.125p^2$ E) $0.375 - 0.125p$

4. An insurance company does a study of claims that arrive at a regional office. The study focuses on the days during which there were at most 2 claims. The study finds that for the days on which there were at most 2 claims, the average number of claims per day is 1.2. The company models the number of claims per day arriving at that office as a Poisson random variable. Based on this model, find the probability that at most 2 claims arrive at that office on a particular day.

A) 0.62 B) 0.64 C) 0.66 D) 0.68 E) 0.70

5. An actuarial trainee working on loss distributions encounters a special distribution. The student reads a discussion of the distribution and sees that the density of X is $f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$ on the region $X > \theta$, where α and θ must both be > 0, and the mean is $\frac{\alpha\theta}{\alpha-1}$ if $\alpha > 1$.
The student is analyzing loss data that is assumed to follow such a distribution, but the values of α and θ are not specified, although it is known that $\theta < 200$. The data shows that the average loss for all losses is 180, and the average loss for all losses that are above 200 is 300.
Find the median of the loss distribution.
A) Less than 100 B) At least 100, but less than 120 C) At least 120, but less than 140
D) At least 140, but less than 160E) At least 160

6. An insurance claims administrator verifies claims for various loss amounts.
For a loss claim of amount x, the amount of time spent by the administrator to verify the claim is uniformly distributed between 0 and $1 + x$ hours. The amount of each claim received by the administrator is uniformly distributed between 1 and 2. Find the average amount of time that an administrator spends on a randomly arriving claim.
A) 1.125 B) 1.250 C) 1.375 D) 1.500 E) 1.625

7. A husband and wife have a health insurance policy. The insurer models annual losses for the husband separately from the wife. X is the annual loss for the husband and Y is the annual loss for the wife. X has a uniform distribution on the interval $(0, 5)$ and Y has a uniform distribution on the interval $(0, 7)$, and X and Y are independent. The insurer applies a deductible of 2 to the combined annual losses, and the insurer pays a maximum of 8 per year. Find the expected annual payment made by the insurer for this policy.
A) 2 B) 3 C) 4 D) 5 E) 6

8. X has a Poisson distribution with a mean of 2.
Y has a geometric distribution on the integers 0,1,2,..., also with mean 2.
X and Y are independent.
Find $P(X = Y)$.
A) $\frac{e^{-2/3}}{3}$ B) $\frac{e^{-1/3}}{3}$ C) $\frac{1}{3}$ D) $\frac{e^{1/3}}{3}$ E) $\frac{e^{2/3}}{3}$

9. X has a uniform distribution on the interval $(0, 1)$.
The random variable Y is defined by $Y = X^{-k}$, where $k > 0$.
Find the mean of Y, assuming that it is finite.
A) $\frac{1}{k}$ B) $\frac{1}{1-k}$ C) k D) $\frac{k}{1-k}$ E) $\frac{1}{k-1}$

10. The marginal distributions of X and Y are both normal with mean 0, but X has a variance of 1, and Y has a variance of 4.

X and Y have a bivariate normal distribution with the following joint pdf:
$f(x,y) = \frac{.3125}{\pi} \cdot e^{-.78125(x^2-.6xy+.25y^2)}$.

Find the coefficient of correlation between $X+Y$ and $X-Y$.

A) Less than -0.6 B) At least -0.6, but less than -0.2 C) At least -0.2, but less than 0.2

D) At least 0.2, but less than 0.6 E) At least 0.6

11. An insurer is considering insuring two independent risks. The loss for each risk has an exponential distribution with a mean of 1. The insurer is considering issuing two separate insurance policies, one for each risk, each of which has a policy limit (maximum payment) of 2. The insurer is also considering issuing a single policy covering the combined loss on both risks, with a policy limit of 4. We denote by A the expected insurance payment for each of the two se prate policies, and we denote by B the expected insurance payment for the single policy covering the combined loss. Find B/A.

A) 1.4 B) 1.8 C) 2.2 D) 2.6 E) 3.0

12. At the start of a year, Smith is presented with an investment proposal. Smith's payoff from the investment is related to the closing value of an international financial stock index on the last day of the year. If the closing value of the index on the last day of the year is X, Smith's payoff will be $Y = Min\{\,Max\{X, 20\}\,,\,50\,\}$.
At the start of the year, when Smith is considering this proposal, Smith's model for X is that X has a continuous uniform distribution on the interval $(0, 100)$.
Based on Smith's model, find the expected payoff.

A) Less than 30 B) At least 30, but less than 32 C) At least 32, but less than 34

D) At least 34, but less than 36 E) At least 36

13. In the 2006 World Cup of soccer, according to an online ranking service, Brazil, England and Germany are the three most highly ranked teams to win the tournament. A survey of soccer fans asks the fans to rank from most likely to least likely the chance of each those country's teams winning the world cup. The survey found that 50% of the fans ranked Brazil first, 30% ranked Brazil second, 30% ranked England second, 50% ranked England third, and 20% ranked Brazil first and England second. Of the fans surveyed who ranked England first, find the proportion who ranked Brazil last.

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

14. In the 2006 World Cup of soccer, according to an online ranking service, Brazil, England and Germany are the three most highly ranked teams to win the tournament. A survey of soccer fans asks the fans to rank from most likely to least likely the chance of each those country's teams winning the world cup. The survey found the following:

- 2/3 of those who ranked Germany first ranked Brazil second ,
- 1/7 of those who didn't rank Germany first ranked Brazil second ,
- 30% of those surveyed ranked Brazil second.

Of those surveyed who ranked Brazil second, find the proportion that ranked Germany third.

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

15. In the Canadian national lottery called "6-49", a ticket consists of 6 distinct numbers from 1 to 49 chosen by the player. The lottery chooses 6 distinct numbers at random from 1 to 49. If a player's ticket matches at least 3 of the 6 numbers chosen at random, then the player wins a prize. The next lottery is next Wednesday. A lottery player buys the following two tickets for next Wednesday's lottery:

Ticket 1 - 1 , 2 , 3 , 4 , 5 , 6
Ticket 2 - 7 , 8 , 9 , 10 , 11 , 12

Find the player's chance of not matching any of the 6 random numbers chosen on either of her two tickets.

A) Less than 0.1 B) At least 0.1, but less than 0.15 C) At least 0.15, but less than 0.2
D) At least 0.2, but less than 0.25 E) At least 0.25

16. A loaded six-sided die has the following probability function:

$$P(X=1)=P(X=3)=P(X=5)=\frac{1}{9}\,,$$
$$P(X=2)=P(X=4)=P(X=6)=\frac{2}{9}.$$

The die is tossed repeatedly until the outcome is 1, 2 or 3.
The first 1, 2 or 3 is the random variable Y. Find the variance of Y.

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{3}{4}$

17. X has a discrete non-negative integer valued distribution with a mean of 5 and a variance of 10. Two new distributions are created from X.

Y has the same probability function as X for $Y = 2, 3, 4, \ldots$, but $P(Y = 0) = 0$ and $P(Y = 1) = P(X = 0) + P(X = 1)$.

Z has the same probability function as X for $Z = 3, 4, \ldots$, but $P(Z = 0) = P(Z = 1) = 0$ and $P(Z = 2) = P(X = 0) + P(X = 1) + P(X = 2)$

You are given that the mean of Y is 5.1 and the mean of Z is 5.3. Find the variance of Z.

A) 7.0 B) 7.2 C) 7.4 D) 7.6 E) 7.8

18. The time until failure of a machine is modeled as an exponential distribution with a mean of 3 years. A warranty on the machine provides the following schedule of refunds:

- if the machine fails within 1 year, the full purchase price is refunded,
- if the machine fails after 1 year but before 2 years, 3/4 of the purchase price is refunded,
- if the machine fails after 2 years but before 4 years, 1/2 of the purchase price is refunded, and
- if the machine fails after 4 years, 1/4 of the purchase price is refunded.

Find the expected fraction of the purchase price that will be refunded under the warranty.

A) Less than 0.2 B) At least 0.2, but less than 0.4 C) At least 0.4, but less than 0.6
D) At least 0.6, butt less than 0.8 E) At least 0.8

19. A loss random variable X is uniformly distributed on the interval $[0\,,\ 1000]$.

An insurance policy on the loss pays the following amount:

(i) 0 if the loss is below 200,
(ii) one-half of the loss in excess of 200 if the loss is between 200 and 500, and
(iii) 150 plus one-quarter of the loss in excess of 500 if the loss is at least 500.

Y is the amount paid by the insurer when a loss occurs. Find the coefficient of variation of Y.

A) Less than 0.2 B) At least 0.2, but less than 0.4 C) At least 0.4, but less than 0.6
D) At least 0.6, butt less than 0.8 E) At least 0.8

20. A fair 6-sided die with faces numbered 1 to 6 is tossed successively and independently until the total of the faces is at least 14. Find the probability that at least 4 tosses are needed.

A) Less than 0.2 B) At least 0.2, but less than 0.4 C) At least 0.4, but less than 0.6
D) At least 0.6, butt less than 0.8 E) At least 0.8

21. A loss random variable has an exponential distribution with mean 800. If an insurer imposes a policy limit of u on the loss, the insurer will pay a maximum of u when a loss occurs. The expected payment by the insurer with a policy limit of u is A. If instead the insurer imposes a policy limit of $2u$ on the loss, the expected payment by the insurer will be $1.2865A$ when a loss occurs. Find u.

A) 250 B) 500 C) 1000 D) 2000 E) 4000

22. An insurer is insuring 800 independent losses. 400 of the losses each have an exponential distribution with mean 1, and the other 400 losses each have an exponential distribution with mean 2. The insurer applies the normal approximation to find each of the following:

(a) the 95-th percentile of the aggregate of the first 400 losses with mean 1 each, say A,

(b) the 95-th percentile of the aggregate of the second 400 losses with mean 2 each, say B , and

(c) the 95-th percentile of the aggregate of all 800 losses, say C.

Find $\frac{C}{A+B}$.

A) Less than 0.2 B) At least 0.2, but less than 0.4 C) At least 0.4, but less than 0.6

D) At least 0.6, but less than 0.8 E) At least 0.8

23. A model describes the time until a loss occurs, X, and the size of the loss, Y.

X has pdf $f_X(x) = \frac{1}{x^2}$ for $x > 1$.

The conditional distribution of Y given $X = x$ has pdf $f_{Y|X}(y|X = x) = \frac{1}{x}$ for $x < y < 2x$.

Find pdf of the marginal distribution of Y, $f_Y(y)$.

A) $\begin{cases} \frac{1}{2} - \frac{1}{2y^2} & 1 < y < 2 \\ \frac{3}{2y^2} & y \geq 2 \end{cases}$ B) $\begin{cases} \frac{1}{2} - \frac{1}{3y^3} & 1 < y < 2 \\ \frac{2}{3y^2} & y \geq 2 \end{cases}$ C) $\begin{cases} \frac{1}{2} + \frac{1}{2y^2} & 1 < y < 2 \\ \frac{3}{2y^2} & y \geq 2 \end{cases}$

D) $\begin{cases} \frac{1}{2} + \frac{1}{3y^3} & 1 < y < 2 \\ \frac{3}{2y^2} & y \geq 2 \end{cases}$ E) $\begin{cases} \frac{1}{3} - \frac{1}{4y^3} & 1 < y < 2 \\ \frac{2}{3y^2} & y \geq 2 \end{cases}$

24. X and Y have a bivariate normal distribution, and X and Y each have marginal distributions that are standard normal (mean 0, variance 1).

You are given $P(X > Y + 1) = 0.2119$. Find $P(X > Y + 2)$.

A) .050 B) .055 C) .060 D) .065 E) .070

25. The Toronto Blue Jays baseball team holds a Children's Hospital Day. The Blue Jays will donate \$100,000 for each home run hit after the 2nd home run in the game. The team's model for the number of home runs hit in the game is Poisson with a mean of 4. Find the expected amount that the Blue Jays will donate.

A) Less than 150,000 B) At least 150,000, but less than 175,000

C) At least 175,000, but less than 200,000 D) At least 200,000, but less than 225,000

E) At least 225,000

26. In the Texas Hold'em poker game, each person is dealt two cards before any betting begins.

For an ordinary deck of cards (spades, hearts, diamonds, clubs, 4 aces, 4 kings, etc), find the probability that a randomly chosen player has a pair in the first two cards received.

A) 0.0188 B) 0.0288 C) 0.0388 D) 0.0488 E) 0.0588

27. X has a mean of 2 and a variance of 4. $aX+b$ has a mean of 5 and a variance of 1. What is ab assuming that $a>0$?

A) 1 B) 2 C) 3 D) 4 E) 5

28. X and Y have the following joint distribution:

		X	
		1	2
Y	1	c	$2c$
	2	$c/2$	c

Find $COV(X,Y)$.

A) $-\frac{4}{3}$ B) $-\frac{2}{3}$ C) 0 D) $\frac{2}{3}$ E) $\frac{4}{3}$

29. X and Y are independent continuous random variables, with X uniformly distributed on the interval $[0,\theta]$ and Y uniformly distributed on the interval $[0,2\theta]$. Find $P(Y<3X)$.

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{5}{6}$

30. An insurance company is considering insuring a loss. The amount of the loss is uniformly distributed on the interval $[0\,,\,1000]$. The insurer considers two possible insurance policies.

Policy 1 - The insurer applies a deductible of 100 to the loss, and if the loss is above 100, the insurer limits the payment to a maximum payment amount of 500.

Policy 2 - If the loss is above 500 the insurer pays 400. If the loss is below 500, there is no deductible.

Find the ratio $\frac{\text{Expected insurance payment with Policy 1}}{\text{Expected insurance payment with Policy 2}}$.

A) $\frac{4}{5}$ B) $\frac{5}{6}$ C) 1 D) $\frac{6}{5}$ E) $\frac{5}{4}$

PRACTICE EXAM 7 - SOLUTIONS

1. We will use B to denote the event that a randomly chosen person in the study has high blood pressure, and C will denote the event high cholesterol level.

   The information given tells us that $P(C|B) = 0.50$ and $P(B|C) = 0.80$.

   We wish to find $P(B \cap C|B \cup C)$. This is

   $$\frac{P[(B\cap C)\cap(B\cup C)]}{P(B\cup C)} = \frac{P[B\cap C)]}{P(B)+P(C)-P(B\cap C)} = \frac{1}{\frac{P(B)+P(C)-P(B\cap C)}{P(B\cap C)}}$$

   $$= \frac{1}{[\frac{1}{P(C|B)}+\frac{1}{P(C|B)}-1]} = \frac{1}{\frac{1}{0.5}+\frac{1}{0.8}-1} = \frac{1}{2.25} = \frac{4}{9}.$$

   Answer: B

2. We define the following events:

   D - the athlete uses Dianabol , W- the athlete uses Winstrol

   TD - the test indicates that the athlete uses Dianabol , TW - the test indicates that the athlete uses Winstrol

   We are given the following probabilities

   $P(D \cap W') = 0.05$, $P(D' \cap W) = 0.02$, $P(D \cap W) = 0.01$,

   $P(TD \cap TW|D \cap W) = 0.75$, $P(TD \cap TW'|D \cap W) = 0.15$, $P(TD' \cap TW|D \cap W) = 0.1$,

   $P(TD \cap TW|D \cap W') = 0.2$, $P(TD \cap TW'|D \cap W') = 0.8$,

   $P(TD \cap TW|D' \cap W) = 0.4$, $P(TD' \cap TW|D' \cap W) = 0.6$

   We wish to find $P(D \cap W|TD \cap TW') = \frac{P(D\cap W\cap TD\cap TW')}{P(TD\cap TW')}$

   The numerator is

   $P(D \cap W \cap TD \cap TW') = P(TD \cap TW'|D \cap W) \times P(D \cap W) = 0.15 \times 0.01 = 0.0015$

   The denominator is

   $$P(TD \cap TW') = P(TD \cap TW' \cap D \cap W) + P(TD \cap TW' \cap D' \cap W)$$
   $$+ P(TD \cap TW' \cap D \cap W') + P(TD \cap TW' \cap D' \cap W')$$

   We have used the rule $P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots$, where $B_1, B_2, ...$ forms a partition. The partition in this case is $B_1 = D \cap W$, $B_2 = D' \cap W$, $B_3 = D \cap W'$, $B_4 = D' \cap W'$, since an athlete must be using both, one or neither of the drugs.

   We have just seen that $P(TD \cap TW' \cap D \cap W) = .0015$.

   In a similar way, we have

   $P(TD \cap TW' \cap D' \cap W) = P(TD \cap TW'|D' \cap W) \cdot P(D' \cap W) = 0 \times 0.02 = 0$, and

   $P(TD \cap TW' \cap D \cap W') = P(TD \cap TW'|D \cap W') \cdot P(D \cap W') = 0.8 \times 0.05 = 0.04$, and

   $P(TD \cap TW' \cap D' \cap W') = P(TD \cap TW'|D' \cap W') \cdot P(D' \cap W') = 0 \times 0.92 = 0$

   (note: $P(D' \cap W') = 1 - P(D \cup W) = 1 - P(D \cap W') - P(D' \cap W) - P(D \cap W) = 0.92$)

   Then, $P(D \cap W|TD \cap TW') = \frac{0.0015}{0.0015+0+0.04+0} = 0.036$, 3.6%.

   Answer: C

For trial use only.
Content may not
represent the most
current version

3. $P(N=2) = pP(N_1=2) + (1-p)P(N_2=2) = p(.5)^2 + (1-p)\times 6\times (0.5)^4$
$= 0.375 - 0.125p$. We have used the binomial probabilities $\binom{m}{k}q^k(1-q)^{m-k}$. Answer: E

4. Suppose that the mean number of claims per day arriving at the office is λ.
Let X denote the number of claims arriving in one day.
Then the probability of at most 2 claims in one day is $P(X\le 2) = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2}$.

The conditional probability of 0 claims arriving on a day given that there are at most 2 for the day is
$$P(X=0|X\le 2) = \frac{P(X=0)}{P(X\le 2)} = \frac{e^{-\lambda}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2 e^{-\lambda}}{2}} = \frac{1}{1+\lambda+\frac{\lambda^2}{2}} .$$

The conditional probability of 1 claim arriving on a day given that there are at most 2 for the day is
$$P(X=1|X\le 2) = \frac{P(X=1)}{P(X\le 2)} = \frac{\lambda e^{-\lambda}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2 e^{-\lambda}}{2}} = \frac{\lambda}{1+\lambda+\frac{\lambda^2}{2}} .$$

The conditional probability of 2 claims arriving on a day given that there are at most 2 for the day is
$$P(X=2|X\le 2) = \frac{P(X=2)}{P(X\le 2)} = \frac{\frac{\lambda^2 e^{-\lambda}}{2}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2 e^{-\lambda}}{2}} = \frac{\frac{\lambda^2}{2}}{1+\lambda+\frac{\lambda^2}{2}} .$$

The expected number of claims per day, given that there were at most 2 claims per day is
$$(0)\left(\frac{1}{1+\lambda+\frac{\lambda^2}{2}}\right) + (1)\left(\frac{\lambda}{1+\lambda+\frac{\lambda^2}{2}}\right) + (2)\left(\frac{\frac{\lambda^2}{2}}{1+\lambda+\frac{\lambda^2}{2}}\right) = \frac{\lambda+\lambda^2}{1+\lambda+\frac{\lambda^2}{2}} .$$

We are told that this is 1.2.

Therefore $\lambda+\lambda^2 = 1.2\times(1+\lambda+\frac{\lambda^2}{2})$, which becomes the quadratic equation $0.4\lambda^2 - 0.2\lambda - 1.2 = 0$
Solving the equation results in $\lambda = 2$ or -1.5 , but we ignore the negative root. The probability of at most 2 claims arriving at the office on a particular day is $P(X\le 2) = e^{-2} + 2e^{-2} + \frac{2^2 e^{-2}}{2} = 0.6767$.
Answer: D

5. The distribution function will be $F(y) = \int_\theta^y f(x)\,dx = \int_\theta^y \frac{\alpha\theta^\alpha}{x^{\alpha+1}}\,dx = 1 - \frac{\theta^\alpha}{y^\alpha}$.
The median m occurs where $F(m) = \frac{1}{2}$. If α and θ were known, we could find the median.
The average loss for all losses is $\frac{\alpha\theta}{\alpha-1} = 180$, but both θ and α are not known.
The conditional distribution of loss amount x given that $X > 200$ is
$f(x|X>200) = \frac{f(x)}{P(X>200)} = \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \Big/ \frac{\theta^\alpha}{200^\alpha} = \frac{\alpha 200^\alpha}{x^{\alpha+1}}$.
This random variable has a mean of $\frac{200\alpha}{\alpha-1}$. We are given that this mean is 300,
so $\frac{200\alpha}{\alpha-1} = 300$, and therefore $\alpha = 3$.
Then, from $\frac{\alpha\theta}{\alpha-1} = 180$, we get $\frac{3\theta}{2} = 180$, so that $\theta = 120$.
The median m satisfies the relation $\frac{1}{2} = F(m) = 1 - \frac{\theta^\alpha}{m^\alpha} = 1 - \left(\frac{120}{m}\right)^3$, so that $m = 151.2$.
Answer: D

6. X = amount of loss claim, uniformly distributed on $(1,2)$, so $f_X(x)=1$ for $1<x<2$.
Y = amount of time spent verifying claim.
We are given that the conditional distribution of Y given $X=x$ is uniform on $(0,1+x)$,
so $f(y|x)=\frac{1}{1+x}$ for $0<y<1+x$.

We wish to find $E[Y]$. The joint density of X and Y is
$f(x,y)=f(y|x)\times f_X(x)=\frac{1}{1+x}$ for $0<y<1+x$ and $1<x<2$.

There are three ways to find $E[Y]$:

(i) $E[Y]=\int\int y\,f(x,y)\,dy\,dx$ or $E[Y]=\int\int y\,f(x,y)\,dx\,dy$, with careful setting of the integral limits, or

(ii) $E[Y]=\int y\,f_Y(y)\,dy$, where $f_Y(y)$ is the pdf of the marginal distribution of Y.

(iii) The double expectation rule, $E[Y]=E[\,E[Y|X]\,]$.

If we apply the first approach for method (i), we get
$E[Y]=\int_1^2\int_0^{1+x} y\times\frac{1}{1+x}\,dy\,dx=\int_1^2\frac{(1+x)^2}{2(1+x)}\,dy=\int_1^2\frac{1+x}{2}\,dx=\frac{5}{4}$.

If we apply the second approach for method (i), we must split the double integral into
$E[Y]=\int_0^2\int_1^2 y\times\frac{1}{1+x}\,dx\,dy\;+\;\int_2^3\int_{y-1}^2 y\times\frac{1}{1+x}\,dx\,dy$
The first integral becomes $\int_0^2 y\,ln(\frac{3}{2})\,dy\;=2\,ln(\frac{3}{2})$.
The second integral becomes $\int_2^3 y\,[ln3\;-\;ln\,y]\,dy=\frac{5}{2}\,ln3-\int_2^3 y\,ln\,y\,dy$.
The integral $\int_2^3 y\,ln\,y\,dy$ is found by integration by parts.

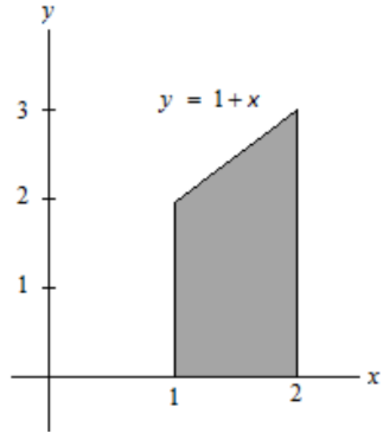
Let $\int y\,ln\,y\,dy\;=\;A$.
Let $u=y$ and $dv=\;ln\,y\,dy$, then $v=y\,lny\;-\;y$ (antiderivative of $ln\,y$), and then
$A=\int y\,ln\,y\,dy=y(y\,lny\;-\;y)-\int(y\,ln\,y-y)\,dy=y^2\,ln\,y\;-\;y^2-A+\frac{y^2}{2}$,
so that $A=\int y\,ln\,y\,dy=\frac{1}{2}y^2\,ln\,y\;-\;\frac{y^2}{4}$.
Then $\int_2^3 y\,ln\,y\,dy=\frac{1}{2}y^2\,ln\,y\;-\;\frac{y^2}{4}\Big|_2^3=\frac{9}{2}\,ln3\;-\;\frac{9}{4}-(\frac{4}{2}\,ln\,2\;-\;1)=\frac{9}{2}\,ln3\;-2\,ln2-\frac{5}{4}$.
Finally, $E[Y]=2\,ln(\frac{3}{2})+\frac{5}{2}\,ln3-\int_2^3 y\,ln\,y\,dy$
$=2\,ln\,3\;-\;2\,ln\,2\;+\;\frac{5}{2}\,ln3-(\frac{9}{2}\,ln3\;-2\,ln2-\frac{5}{4})=\frac{5}{4}$.
The first order of integration for method (i) was clearly the more efficient one.

(ii) This method is equivalent to the second approach in method (i) because we find $f_Y(y)$ from the relationship $f_Y(y)=\int f(x,y)\,dx$. The two-dimensional region of probability for the joint distribution is $1<x<2$ and $0<y<1+x$. For $0<y<2$, $f_Y(y)=\int_1^2 f(x,y)\,dx=\int_1^2\frac{1}{1+x}\,dx=ln(\frac{3}{2})$,

and for $2\le y<3$, $f_Y(y)=\int_{y-1}^2 f(x,y)\,dx=\int_{y-1}^2\frac{1}{1+x}\,dx=ln\,3\;-\;ln\,y$.

Then $E[Y]=\;\int_0^2 y\,ln(\frac{3}{2})\,dy\;+\;\int_2^3 y\,[ln3\;-\;ln\,y]\,dy$, which is the same as the second part of method (i).

This is illustrated in the graph below

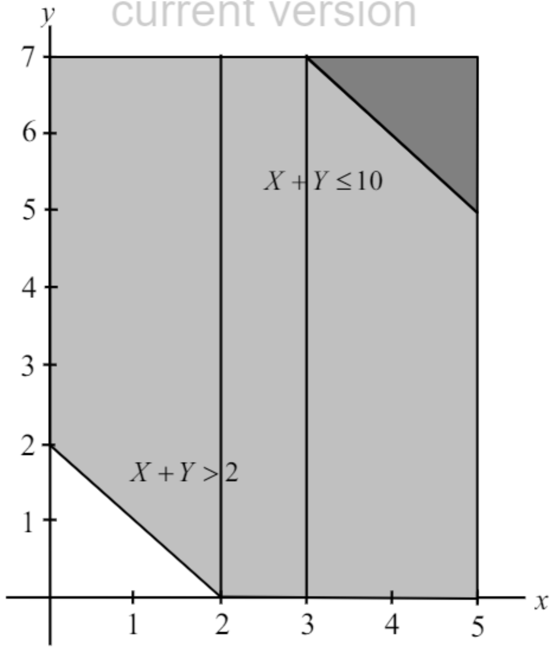


For $0<y<2$, $f_Y(y)=\int_1^2 f(x,y)\,dx=\int_1^2 \frac{1}{1+x}\,dx=ln(\frac{3}{2})$,
and for $2\le y<3$, $f_Y(y)=\int_{y-1}^2 f(x,y)\,dx=\int_{y-1}^2 \frac{1}{1+x}\,dx=ln\,3\,-\,ln\,y$.
Then $E[Y]=\int_0^2 y\,ln(\frac{3}{2})\,dy\;+\;\int_2^3 y\,[ln3\,-\,ln\,y]\,dy$, which is the same as the second part of method (i).

(iii) According to the double expectation rule, for any two random variables U and W, we have $E[U]=E[\,E[U|W]\,]$. Therefore, $E[Y]\;=\;E[\,E[Y|X]\,]$.
We are told that the conditional distribution of Y given $X=x$ is uniform on the interval $(0,1+x)$, so $E[Y|X]=\frac{1+X}{2}$.
Then $E[\,E[Y|X]\,]=E[\frac{1+X}{2}]=\frac{1}{2}+\frac{1}{2}E[X]=\frac{1}{2}+\frac{1}{2}\times\frac{3}{2}=\frac{5}{4}$, since X is uniform on $(1,2)$ and X has mean $\frac{3}{2}$. Answer: B

7. The joint distribution of X and Y has pdf $f(x,y)=\frac{1}{5}\times\frac{1}{7}=\frac{1}{35}$ on the rectangle $0<x<5$ and $0<y<7$. The insurer pays $X+Y-2$ if the combined loss $X+Y$ is >2.
The maximum payment of 8 is reached if $X+Y-2\ge 8$, or equivalently, if $X+Y\ge 10$.
Therefore, the insurer pays $X+Y-2$ if $2<X+Y\le 10$ (the lighter shaded region in the diagram below), and the insurer pays 8 if $X+Y>10$ (the darker shaded region in the diagram below). The expected amount paid by the insurer is a combination of two integrals:
$\int\int(x+y-2)\times\frac{1}{35}\,dy\,dx$, where the integral is taken over the region $2<x+y\le 10$ (the lightly shaded region), plus $\int\int 8\times\frac{1}{35}\,dy\,dx$, where the integral is taken over the region $X+Y>10$ (the darker region). The second integral is $\frac{8}{35}\times 2=\frac{16}{35}$, since the area of the darkly shaded triangle is 2 (it is a 2×2 right triangle). The first integral can be broken into three integrals:

$$\int_0^2\int_{2-x}^7(x+y-2)\times\frac{1}{35}\,dy\,dx+\int_2^3\int_0^7(x+y-2)\times\frac{1}{35}\,dy\,dx+\int_3^5\int_0^{10-x}(x+y-2)\times\frac{1}{35}\,dy\,dx$$
$$=\frac{1}{35}\times[\int_0^2\frac{(x+5)^2}{2}\,dx\;+\;\int_2^3\frac{7(2x+3)}{2}\,dx\;+\;\int_3^5\frac{60+4x-x^2}{2}\,dx\,]$$
$$=\frac{1}{35}\times[\,\frac{109}{3}+28+\frac{179}{3}]=\frac{124}{35}.$$



The total expected insurance payment is $\frac{16}{35}+\frac{124}{35}=\frac{140}{35}=4.$ Answer: C

8. The probability function of X is $P(X=k)=\frac{e^{-2}\cdot 2^k}{k!}$.

The general probability function of a geometric distribution on $0,1,2,\ldots$ is of the form $P(Y=k)=p(1-p)^k$ for $k=0,1,2,\ldots$ and the mean is $\frac{1-p}{p}$.

Since the mean is 2, we have $\frac{1-p}{p}=2$, from which we get $p=\frac{1}{3}$,

so the probability function of Y is $P(Y=k)=\frac{1}{3}\times\left(\frac{2}{3}\right)^k$.

$P(X=Y)=P(X=Y=0)+P(X=Y=1)+\cdots=\sum_{k=0}^{\infty}P(X=Y=k)$.

Since X and Y are independent, we have

$P(X=Y=k)=P(X=k)\times P(Y=k)=\frac{e^{-2}\cdot 2^k}{k!}\times\frac{1}{3}\times\left(\frac{2}{3}\right)^k=\frac{e^{-2}}{3}\times\frac{(4/3)^k}{k!}$.

Then, $P(X=Y)=\sum_{k=0}^{\infty}P(X=Y=k)=\sum_{k=0}^{\infty}\frac{e^{-2}}{3}\times\frac{(4/3)^k}{k!}=\frac{e^{-2}}{3}\times\sum_{k=0}^{\infty}\frac{(4/3)^k}{k!}$.

The Taylor series expansion for e^x is $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$, so it follows that $\sum_{k=0}^{\infty}\frac{(4/3)^k}{k!}=e^{4/3}$.

Then, $P(X=Y)=\frac{e^{-2}}{3}\times e^{4/3}=\frac{e^{-2/3}}{3}$. Answer: A

9. $X=Y^{-1/k}=h(Y)$. According to the method by which we find the density of a transformed random variable, the pdf of Y is $g(y)=f(h(y))\times|h'(y)|$, where f is the pdf of X.

Since X is uniform on $(0,1)$, we know that $f(x)=1$. Therefore, $g(y)=\left|-\frac{y^{-(k+1)/k}}{k}\right|=\frac{y^{-(k+1)/k}}{k}$.

Since $y=x^{-k}$, it follows that $y>1$, since $0<x<1$.

The mean of Y will be $\int_1^{\infty}y\times g(y)\,dy=\int_1^{\infty}y\times\frac{y^{-(k+1)/k}}{k}\,dy=\int_1^{\infty}\frac{y^{-1/k}}{k}\,dy=\frac{y^{(k-1)/k}}{k-1}\Big|_{y=1}^{y=\infty}$.

This will be ∞ if $k\geq 1$. If $k<1$, then $E[Y]=-\frac{1}{k-1}=\frac{1}{1-k}$. Answer: B