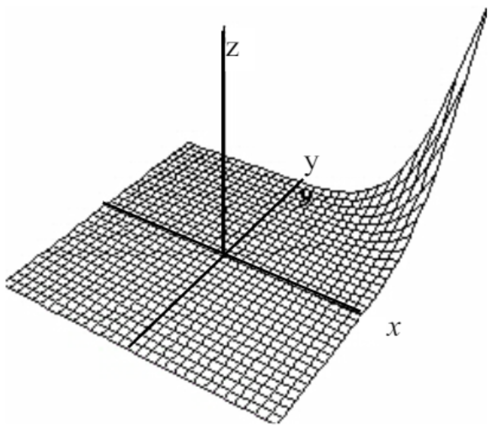
Multivariate function: A function of more than one variable is called a multivariate function.

Example 0-6:

$z = f(x, y) = e^{x+y}$ is a function of two variables, the domain is the entire 2-dimensional plane (the set $\{(x, y)|$ x,y are both real numbers$\}$), and the range is the set of strictly positive real numbers. The function could be graphed in 3-dimensional x-y-z space. The domain would be the (horizontal) x-y plane, and the range would be the (vertical) z-dimension.

The 3-dimensional graph is shown below.



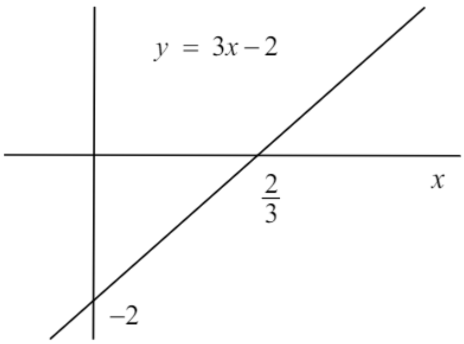
□

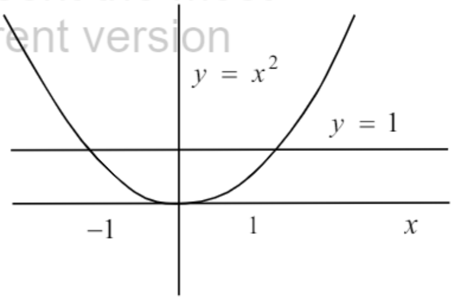
The concept of the inverse of a function is important when formulating the distribution of a transformed random variable. A preliminary concept related to the inverse of a function is that of a one-to-one function.

One-to-one function: The function f is called a one-to-one if the equation $f(x) = y$ has at most one solution x for each y (or equivalently, different x-values result in different $f(x)$ values). If a graph is drawn of a one-to-one function, any horizontal line crosses the graph in at most one place.

Example 0-7:

The function $f(x) = 3x - 2$ is one-to-one, since for each value of y, the relation $y = 3x - 2$ has exactly one solution for x in terms of y; $x = \frac{y+2}{3}$. The function $g(x) = x^2$ with the whole set of real numbers as its domain is not one-to-one, since for each $y > 0$, there are two solutions for x in terms of y for the relation $y = x^2$ (those two solutions are $x = \sqrt{y}$ and $x = -\sqrt{y}$; note that if we restrict the domain of $g(x) = x^2$ to the positive real numbers, it becomes a one-to-one function). The graphs are below.





□

Inverse of function f: The inverse of the function f is denoted f^{-1}. The inverse exists only if f is one-to-one, in which case, $f^{-1}(y) = x$ is the (unique) value of x which satisfies $f(x) = y$ (finding the inverse of $y = f(x)$ means that we solve for x in terms of y, $x = f^{-1}(y)$). For instance, for the function $y = 2x^3 = f(x)$, if $x = 1$ then $y = f(1) = 2(1^3) = 2$, so that $1 = f^{-1}(2) = (2/2)^{1/3}$. For the example just considered, the inverse function applied to $y = 2$ is the value of x for which $f(x) = 2$, or equivalently, $2x^3 = 2$, from which we get $x = 1$.

Example 0-8:

(i) The inverse of the function $y = 5x - 1 = f(x)$ is the function $x = \frac{y+1}{5} = f^{-1}(y)$ (we solve for x in terms of y).

(ii) Given the function $y = x^2 = f(x)$, solving for x in terms of y results in $x = \pm\sqrt{y}$, so there are two possible values of x for each value of y; this function does not have an inverse. However, if the function is defined to be $y = x^2 = f(x)$ **for $x \geq 0$ only**, then $x = +\sqrt{y} = f^{-1}(y)$ would be the inverse function, since f is one-to-one on its domain which consists of non-negative numbers.

□

Quadratic functions and equations:

A quadratic function is of the form $p(x) = ax^2 + bx + c$.

The roots of the quadratic equation $ax^2 + bx + c = 0$ are $r_1, r_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

The quadratic equation has:

(i) distinct real roots if $b^2 - 4ac > 0$,

(ii) distinct complex roots if $b^2 - 4ac < 0$, and

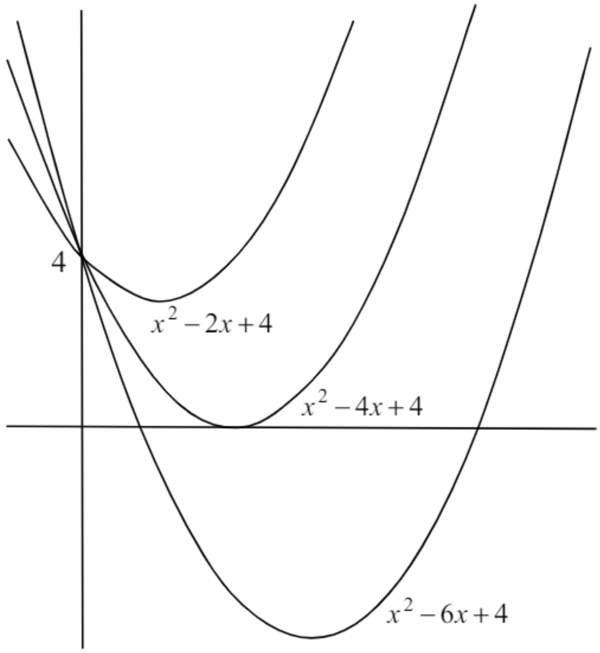
(iii) equal real roots if $b^2 - 4ac = 0$.

Example 0-9:

The quadratic equation $x^2 - 6x + 4 = 0$ has two distinct real solutions: $x = 3 \pm \sqrt{5}$.

The quadratic equation $x^2 - 4x + 4 = 0$ has both roots equal: $x = 2$.

The quadratic equation $x^2 - 2x + 4 = 0$ has two distinct complex roots: $x = 1 \pm i\sqrt{3}$.



$\square$

Exponential and logarithmic functions: Exponential functions are of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, and the inverse of this function is denoted $log_b(y)$.

Thus $y = b^x \Leftrightarrow log_b(y) = x$. The log function with base e is the **natural logarithm**, $log_e(y) = ln\, y$ Some important properties of these functions are:

$b^0 = 1$	$log_b(1) = 0$
$domain(f) = \mathbb{R} = range(f^{-1})$	$range(f) = (0, +\infty) = domain(f^{-1})$
$b^{log_b(y)} = y$ for $y > 0$	$log_b(b^x) = x$ for all x
$b^x = e^{x \cdot ln\, b}$	$log_b(y) = \frac{ln\, y}{ln\, b}$
$(b^x)^y = b^{xy}$	$log_b(y^k) = k \cdot log_b(y)$
$b^x b^y = b^{x+y}$	$log_b(yz) = log_b(y) + log_b(z)$
$b^x / b^y = b^{x-y}$	$log_b(y/z) = log_b(y) - log_b(z)$

For the function e^x, we have $e^{ln\, y} = y$ for an $y > 0$, and for the natural log function, we have $ln\, e^x = x$ for any real number x.

LIMITS AND CONTINUITY

Intuitive definition of limit: The expression $\lim_{x \to c} f(x) = L$ means that as x gets close to (approaches) the number c, the value of $f(x)$ gets close to L.

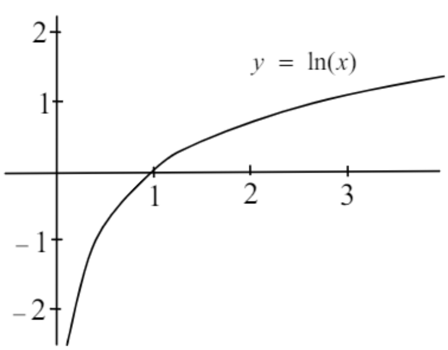
Example 0-10:
$\lim_{x \to 1}(x+3) = 4$, $\lim_{x \to +\infty} e^{-x} = 0$ and $\lim_{x \to 1} \frac{x^2+2x-3}{x-1} = \lim_{x \to 1}(x+3) = 4$ (for this last limit, note that $\frac{x^2+2x-3}{x-1} = \frac{(x+3)(x-1)}{x-1} = x+3$ if $x \neq 1$, but in taking this limit we are only concerned with what happens "near" $x = 1$, that fact that $\frac{x^2+2x-3}{x-1} = \frac{0}{0}$ at $x = 1$ does not mean that the limit does not exist; it means that the function does not exist at the point $x = 1$). □

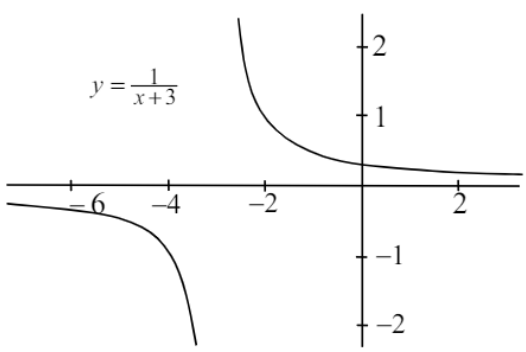
Continuity: The function $\boldsymbol{f}$ is continuous at the point $\boldsymbol{x = c}$ if there is no "break" or "hole" in the graph of $y = f(x)$, or equivalently, if $\boldsymbol{\lim_{x \to c} f(x) = f(c)}$. In Example 0-10 above, the third function is not continuous at $x = 1$ because $f(1) = \frac{0}{0}$ is not defined. Another reason for a discontinuity in $f(x)$ occurring at $x = c$ is that the limit of $f(x)$ is different from the left than it is from the right.

Example 0-11:

(i) If $f(x) = ln\, x$ and $c = 0$ then f is discontinuous at $c = 0$ since the function $ln\, x$ is not defined at the point $x = 0$ (this would also be the case for the function $f(x) = \frac{1}{x+3}$ and $c = -3$).

(ii) If $f(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$, then $f(x)$ is discontinuous at $x = 0$ since even though $f(0)$ is defined, $\lim_{x \to 0} f(x) \neq f(0)$ ($\lim_{x \to 0} f(x)$ doesn't exist).





□

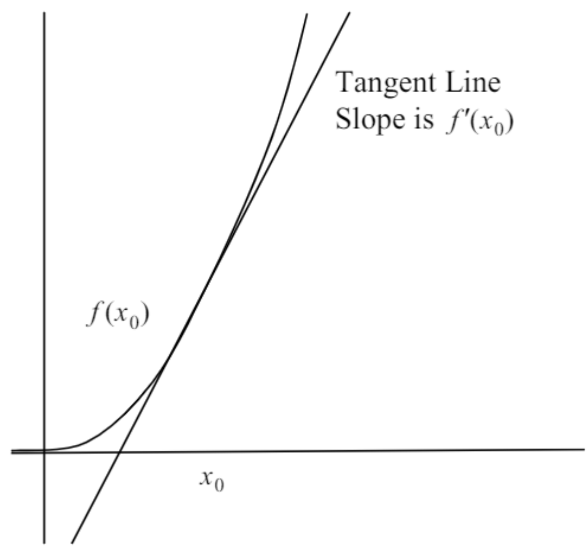
DIFFERENTATION

Geometric interpretation of derivative: The derivative of the function $f(x)$ at the point $x = x_0$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$. The derivative of $f(x)$ at $x = x_0$ is denoted $f'(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$.

This is also referred to as the derivative of f with respect to x at the point $x = x_0$.

The algebraic definition of $f'(x_0)$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$



The second derivative of f at x_0 is the derivative of $f'(x)$ at the point x_0. It is denoted $f''(x_0)$ or $f^{(2)}(x_0)$ or $\left.\frac{d^2f}{dx^2}\right|_{x=x_0}$. The n-th order derivative of f at x_0 (n repeated applications of differentiation) is denoted $f^{(n)}(x_0) = \left.\frac{d^nf}{dx^n}\right|_{x=x_0}$.

The derivative as a rate of change: Perhaps the most important interpretation of the derivative $f'(x_0)$ is as the **"instantaneous" rate at which the function is increasing or decreasing as x increases (if $f' > 0$, the graph of $y = f(x)$ is rising, with the tangent line to the graph having positive slope, and if $f' < 0$, the graph of $y = f(x)$ is falling)**, and if $f'(x_0) = 0$ then the tangent line at that point is horizontal (has slope 0). This interpretation is the one most commonly used when analyzing physical, economic or financial processes.

The following is a summary of some important differentiation rules.

Rules of differentiation:	$f(x)$	$f'(x)$
	c (a constant)	0
Power rule -	cx^n $(n \in \mathbb{R})$	cnx^{n-1}
	$g(x) + h(x)$	$g'(x) + h'(x)$
Product rule -	$g(x) \cdot h(x)$	$g'(x) \cdot h(x) + g(x) \cdot h'(x)$
	$u(x)v(x)w(x)$	$u'vw + uv'w + uvw'$
Quotient rule -	$\frac{g(x)}{h(x)}$	$\frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$
Chain rule -	$g(h(x))$	$g'(h(x)) \cdot h'(x)$
	$e^{g(x)}$	$g'(x) \cdot e^{g(x)}$
	$ln(g(x))$	$\frac{g'(x)}{g(x)}$
	a^x $(a > 0)$	$a^x \, ln \, a$
	e^x	e^x
	$ln \, x$	$\frac{1}{x}$
	$log_b x$	$\frac{1}{x \, lnb}$
	$sin \, x$	$cos \, x$
	$cos \, x$	$- \, sin \, x$

Example 0-12:

What is the derivative of $f(x) = 4x(x^2+1)^3$?

Solution:

We apply the product rule and chain rule:

$$f(x) = g(x) \cdot h(x),$$

where

$$g(x) = 4x \ , \ h(x) = (x^2+1)^3 \ , \ g'(x) = 4 \ , \ h'(x) = 3(x^2+1)^2 \cdot 2x \ ,$$
$$f'(x) = 4x \times 3(x^2+1)^2 \times 2x + 4(x^2+1)^3 = 4(x^2+1)^2(7x^2+1).$$

Notice that $h(x) = (x^2+1)^3 = [w(x)]^3 = h(w(x))$, where $h(w) = w^3$ and $w(x) = x^2+1$.
The chain rule tells us that $h'(x) = h'(w) \times w'(x) = 3w^2 \times (2x) = 3(x^2+1)^2 \times (2x)$. $\square$

L'Hospital's rules for calculating limits: A limit of the form $\lim_{x \to c} \frac{f(x)}{g(x)}$ is said to be in indeterminate form if both the numerator and denominator go to 0, or if both the numerator and denominator go to $\pm\infty$. L'Hospital's rules are:

1. **IF** $\begin{cases} \text{(i) } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f'(c) \text{ exists, } \textbf{and} \\ \text{(iii) } g'(c) \text{ exists and is } \neq 0 \end{cases}$ **THEN** $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$

2. **IF** $\begin{cases} \text{(i) } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f \text{ and } g \text{ are differentiable near } c, \textbf{ and} \\ \text{(iii) } \lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists} \end{cases}$ **THEN** $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

In 1 or 2, the conditions $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$ can be replaced by the conditions $\lim_{x \to c} f(x) = \pm\infty$ and $\lim_{x \to c} g(x) = \pm\infty$, and the point c can be replaced by $\pm\infty$ with the conclusions remaining valid.

Example 0-13: Find $\lim_{x \to 2} \frac{3^{x/2}-3}{3^x-9}$.

Solution:

The limits in both the numerator and denominator are 0, so we apply l'Hospital's rule. $\frac{d}{dx}3^x = 3^x \, ln\, 3$, and $\frac{d}{dx}3^{x/2} = 3^{x/2} \cdot \frac{1}{2}\, ln\, 3$, so that $\lim_{x \to 2} \frac{3^{x/2}-3}{3^x-9} = \lim_{x \to 2} \frac{3^{x/2} \cdot \frac{1}{2}\, ln\, 3}{3^x\, ln\, 3} = \frac{1}{6}$. This limit can also be found by factoring the denominator into $3^x - 9 = (3^{x/2}-3)(3^{x/2}+3)$, and then canceling out the factor $3^{x/2}-3$ in the numerator and denominator. □

Differentiation of functions of several variables - partial differentiation:

Given the function $f(x,y)$, a function of two variables, the partial derivative of f with respect to x at the point (x_0, y_0) is found by differentiating f with respect to x and regarding the variable y as constant - then substitute in the values $x = x_0$ and $y = y_0$. The partial derivative of f with respect to x is usually denoted $\frac{\partial f}{\partial x}$. The partial derivative with respect to y is defined in a similar way: "Higher order" partial derivatives can be defined - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$; and "mixed partial" derivatives can be defined (the order of partial differentiation does not usually matter): $\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$.

Example 0-14:

If $f(x,y) = x^y$ for $x, y > 0$ then find $\left.\frac{\partial f}{\partial x}\right|_{(4,\frac{1}{2})}$ and $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(4,\frac{1}{2})}$.

Solution:

$$\frac{\partial f}{\partial x} = \left. yx^{y-1}\right|_{(4,\frac{1}{2})} = (\tfrac{1}{2})(4)^{-1/2} = \tfrac{1}{4}, \text{ and}$$

$$\frac{\partial f}{\partial y} = x^y\,(ln\, x) \text{ and } \frac{\partial^2 f}{\partial y^2} = \left. x^y\,(ln\, x)^2\right|_{(4,\frac{1}{2})} = 4^{1/2}\,(ln\, 4)^2 = 2\,(ln\, 4)^2. \quad \square$$

INTEGRATION

Geometric interpretation of the "definite integral" - the area under the curve:

Given a function $f(x)$ on the interval $[a, b]$, the definite integral of $f(x)$ over the interval is denoted $\int_a^b f(x)\,dx$, and is equal to the "signed" area between the graph of the function and the x-axis from $x = a$ to $x = b$. Signed area is positive when $f(x) > 0$ and is negative when $f(x) < 0$. What is meant by signed area here is the area from the interval(s) where $f(x)$ is positive minus the area from the intervals where $f(x)$ is negative.

Integration is related to the antiderivative of a function. Given a function $f(x)$, an antiderivative of $f(x)$ is any function $F(x)$ which satisfies the relationship $F'(x) = f(x)$. According to the Fundamental Theorem of Calculus, the definite integral for $f(x)$ can be found by first finding $F(x)$, an antiderivative of $f(x)$. The basic relationships relating integration and differentiation are:

(i) If $F'(x) = f(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

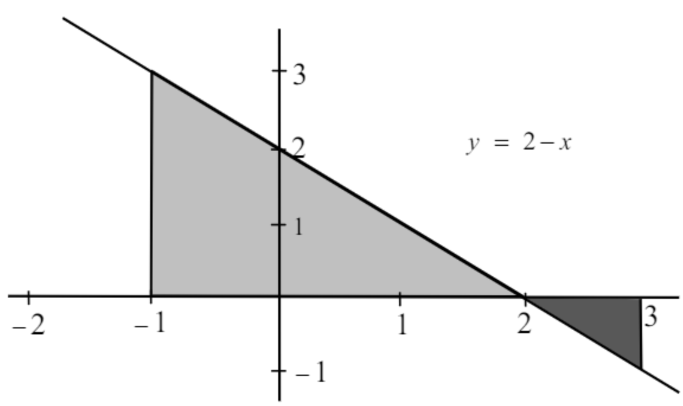
(ii) If $G(x) = \int_a^x g(t)\,dt$, then $G'(x) = g(x)$

Example 0-15:

Find the definite integral of the function $f(x) = 2 - x$ on the interval $[-1, 3]$.

Solution:

The graph of the function is given below. It is clear that $f(x) > 0$ for $x < 2$, and $f(x) < 0$ for $x > 2$. An antiderivative for $f(x)$ is $F(x) = 2x - \frac{x^2}{2}$. The definite integral will be $\int_{-1}^{3}(2-x)\,dx = F(3) - F(-1) = (6 - \frac{3^2}{2}) - (-2 - \frac{(-1)^2}{2}) = 4$. Note that the area between the graph and the x-axis from $x = -1$ to $x = 2$ is $\frac{1}{2}(3)(3) = \frac{9}{2}$, and the signed area between the graph and the x-axis from $x = 2$ to $x = 3$ is $-\frac{1}{2} \times 1 \times 1 = -\frac{1}{2}$. The total signed area is $\frac{9}{2} - \frac{1}{2} = 4$.



□

Antiderivatives of some frequently used functions:

$f(x)$	$\int f(x)dx$ (antiderivative)
$g(x)+h(x)$	$\int g(x)dx+\int h(x)dx+c$
$x^n\ (n\neq -1)$	$\frac{x^{n+1}}{n+1}+c$
$\frac{1}{x}$	$ln\,x\ +c$
e^x	e^x+c
$a^x\ (a>0)$	$\frac{a^x}{ln\,a}+c$
xe^{ax}	$\frac{xe^{ax}}{a}-\frac{e^{ax}}{a^2}+c$.
$sin\,x$	$-\,cos\,x\ +c$
$cos\,x$	$sin\,x\ +c$

Integration of f on $[a,b]$ when f is not defined at a or b, or when a or b is $\pm\infty$:

Integration over an infinite interval (an "improper integral") is defined by taking limits:
$\int_a^\infty f(x)\,dx=\lim\limits_{b\to\infty}\int_a^b f(x)\,dx$, with a similar definition applying to $\int_{-\infty}^b f(x)\,dx$,
and $\int_{-\infty}^\infty f(x)\,dx=\lim\limits_{a\to+\infty}\int_{-a}^a f(x)\,dx$.

If f is not defined at $x=a$ (also called an improper integral), or if f is discontinuous at $x=a$, then $\int_a^b f(x)$ $dx=\lim\limits_{c\to a^+}\int_c^b f(x)\,dx$.

A similar definition applies if f is not defined at $x=b$, or if f is discontinuous at $x=b$.

If $f(x)$ has a discontinuity at the point $x=c$ in the interior of $[a,b]$, then
$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$.

Example 0-16:

(a) $\int_0^1\frac{1}{\sqrt{x}}\,dx=\lim\limits_{c\to 0^+}\int_c^1 x^{-1/2}dx=\lim\limits_{c\to 0^+}\left[2x^{1/2}\Big|_{x=c}^{x=1}\right]=\lim\limits_{c\to 0^+}[2-2\sqrt{c}]=2$,

$\int_1^\infty\frac{1}{\sqrt{x}}\,dx=\lim\limits_{c\to\infty}\int_1^c x^{-1/2}dx=\lim\limits_{c\to\infty}\left[2x^{1/2}\Big|_{x=1}^{x=c}\right]=\lim\limits_{c\to\infty}[2\sqrt{c}-2]=+\infty$.

(b) $\int_1^{+\infty}\frac{1}{x^2}\,dx=\lim\limits_{b\to\infty}\left[-\frac{1}{x}\Big|_{x=1}^{x=b}\right]=\lim\limits_{b\to\infty}[-\frac{1}{b}-(-1)]=1$

(c) $\int_{-\infty}^1\frac{1}{x^2}\,dx$. Note that $\frac{1}{x^2}$ has a discontinuity at $x=0$, so that
$\int_{-\infty}^1\frac{1}{x^2}\,dx=\int_{-\infty}^0\frac{1}{x^2}\,dx+\int_0^1\frac{1}{x^2}\,dx$. The second integral is
$\int_0^1\frac{1}{x^2}\,dx=\lim\limits_{a\to 0^+}\int_a^1\frac{1}{x^2}\,dx=\lim\limits_{a\to 0^+}[-1+\frac{1}{a}]=+\infty$, thus, the second improper integral does not exist (when $\lim\limits_{\to}\int$ is infinite or does not exist, the integral is said to "diverge"). □

A few other useful integration rules are:

(i) for integer $n \geq 0$ and real number $c > 0$ $\int_0^\infty x^n \, e^{-cx} \, dx = \frac{n!}{c^{n+1}}$,

(ii) if $G(x) = \int_a^{h(x)} f(u) \, du$, then $G'(x) = f[h(x)] \times h'(x)$,

(iii) if $G(x) = \int_x^b f(u) \, du$, then $G'(x) = -f(x)$,

(iv) if $G(x) = \int_{g(x)}^b f(u) \, du$, then $G'(x) = -f[g(x)] \times g'(x)$,

(v) if $G(x) = \int_{g(x)}^{h(x)} f(u) \, du$, then $G'(x) = f[h(x)] \times h'(x) - f[g(x)] \times g'(x)$.

Double integral: Given a continuous function of two variables, $f(x, y)$ on the rectangular region bounded by $x = a$, $x = b$, $y = c$ and $y = d$, it is possible to define the definite integral of f over the region. It can be expressed in one of two equivalent ways:

$$\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

The interpretation of the first expression is $\int_a^b [\int_c^d f(x, y) \, dy] \, dx$, in which the "inside integral" is $\int_c^d f(x, y) \, dy$, and it is calculated assuming that the value of x is constant (it is an integral with respect to the variable y). When this definite "inside integral" has been calculated, it will be a function of x alone, which can then be integrated with respect to x from $x = a$ to $x = b$. The second equivalent expression has a similar interpretation; $\int_a^b f(x, y)$ dx is calculated assuming that y is constant; this results in a function of y alone which is then integrated with respect to y from $y = c$ to $y = d$. Double integration arises in the context of finding probabilities for a joint distribution of continuous random variables.

Example 0-17:

Find $\int_0^1 \int_1^2 \frac{x^2}{y} \, dy \, dx$.

Solution:

First we assume that x is constant and find $\int_1^2 \frac{x^2}{y} \, dy = x^2 (\ln y)\Big|_{y=1}^{y=2} = x^2 (\ln 2)$. Then we find

$$\int_0^1 [x^2 (\ln 2)] \, dx = (\ln 2) \cdot \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{\ln 2}{3}.$$

We can also write the integral as $\int_1^2 \int_0^1 \frac{x^2}{y} \, dx \, dy$, and first find

$\int_0^1 \frac{x^2}{y} \, dx = \frac{x^3}{3y}\Big|_{x=0}^{x=1} = \frac{1}{3y}$. Then, $\int_1^2 \frac{1}{3y} \, dy = \frac{1}{3} (\ln y)\Big|_{y=1}^{y=2} = \frac{1}{3} (\ln 2)$. □

For double integration over the rectangular two-dimensional region $a \leq x \leq b$, $c \leq y \leq d$, as the expression $\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$ indicates, it is possible to calculate the double integral by integrating with respect to the variables in either order (y first and x second for the integral on the left, and x first and y second for the integral on the right of the " $=$ " sign).

Formulations of probabilities and expectations for continuous joint distributions sometimes involve integrals over a non-rectangular two-dimensional region. It will still be possible to arrange the integral for integration in either order ($dy\,dx$ or $dx\,dy$), but care must be taken in setting up the limits of integration. If the limits of integration are properly specified, then the double integral will be the same whichever order of integration is used. Note also that in some situations, it may be more efficient to formulate the integration in one order than in the other.

Example 0-18:

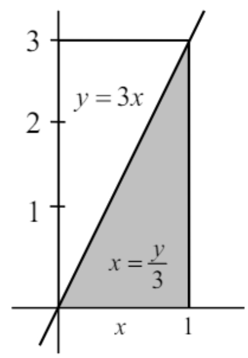
Which of the following integrals is equal to $\int_0^1 \int_0^{3x} f(x,y)\,dy\,dx$ for every function for which the integral exists?

A) $\int_0^3 \int_0^{y/3} f(x,y)\,dx\,dy$ B) $\int_0^1 \int_{3x}^3 f(x,y)\,dx\,dy$ C) $\int_0^3 \int_{3y}^1 f(x,y)\,dx\,dy$

D) $\int_0^1 \int_0^{x/3} f(x,y)\,dx\,dy$ E) $\int_0^3 \int_{y/3}^1 f(x,y)\,dx\,dy$

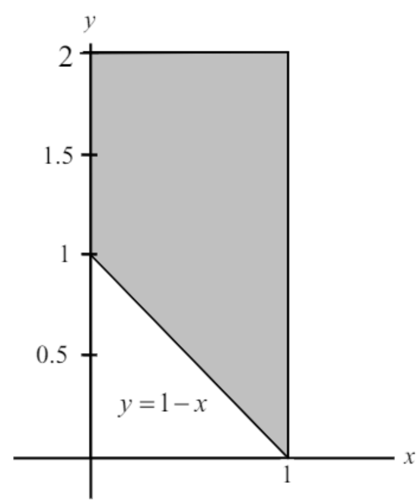
Solution:

The graph at the right illustrates the region of integration. The region is $0 \le x \le 1$, $0 \le y \le 3x$. Writing $y = 3x$ as $x = \frac{y}{3}$, we see that the inequalities translate into $0 \le y \le 3$, and $\frac{y}{3} \le x \le 1$. Answer: E □



Example 0-19: The function $f(x,y)$ is to be integrated over the two-dimensional region defined by the following constraints: $0 \le x \le 1$ and $1 - x \le y \le 2$. Formulate the double integration in the $dy\,dx$ order and then in the $dx\,dy$ order.

Solution: The graph at the right illustrates the region of integration. The region is $0 \le x \le 1$, $1 - x \le y \le 2$. The integral can be formulated in the $dy\,dx$ order as $\int_0^1 \int_{1-x}^2 f(x.y)\,dy\,dx$; for each x, the integral in the vertical direction starts on the line $y = 1 - x$ and continues to the upper boundary $y = 2$. To use the $dx\,dy$ order, we must split the integral into two double integrals; $\int_0^1 \int_{1-y}^1 f(x,y)\,dx\,dy$ to cover the triangular area below $y = 1$, and $\int_1^2 \int_0^1 f(x,y)\,dx\,dy$ to cover the square area above $y = 1$. □



There are a few integration techniques that are useful to know. The integrations that arise on Exam P are usually straightforward, but knowing a few additional techniques of integration are sometimes useful in simplifying an integral in an efficient way.

The Method of Substitution: Substitution is a basic technique of integration that is used to rewrite the integral in a standard form for which the antiderivative is well known. In general, to find $\int f(x)\,dx$ we may make the substitution $u = g(x)$ for an "appropriate" function $g(x)$.
We then define the "differential" du to be $du = g'(x)\,dx$, and we try to rewrite $\int f(x)\,dx$ as an integral with respect to the variable u.

For example, to find $\int (x^3-1)^{4/3}x^2dx$, we let $u = x^3 - 1$, so that $du = 3x^2dx$, or equivalently, $\frac{1}{3} \times du = x^2dx$; then the integral can be written as $\int u^{4/3} \cdot \frac{1}{3}\,du$, which has antiderivative $\int u^{4/3} \times \frac{1}{3}\,du = \frac{1}{3} \times \int u^{4/3}du = \frac{1}{3} \times \frac{u^{7/3}}{7/3} = \frac{1}{7}u^{7/3}(+c)$.
We can then write the antiderivative in terms of the original variable x -
$\int (x^3-1)^{4/3}x^2dx = \frac{1}{7}u^{7/3} = \frac{1}{7}(x^3-1)^{7/3}$.

The main point to note in applying the substitution technique is that the choice of $u = g(x)$ should result in an antiderivative which is easier to find than was the original antiderivative.

Example 0-20:
Find $\int_0^1 x\sqrt{1-x^2}dx$.
Solution:
Let $u = 1 - x^2$ Then $du = -2x\,dx$, so that $-\frac{1}{2} \cdot du = x\,dx$, and the antiderivative can be written as $\int u^{1/2} \times (-\frac{1}{2})du = -\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2}$.
The definite integral is then $\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{3}(1-x^2)^{3/2}\Big|_{x=0}^{x=1} = -0-(-\frac{1}{3}) = \frac{1}{3}$.
Note that once the appropriate substitution has been made, the definite integral may be calculated in terms of the variable u: $u(0) = 1$ and $u(1) = 0$ -

$$\int_0^1 x\sqrt{1-x^2}dx = \int_{u(0)=1}^{u(1)=0} u^{1/2} \times (-\tfrac{1}{2})du = -\tfrac{1}{3}u^{3/2}\Big|_{u=1}^{u=0} = -0-(-\tfrac{1}{3}) = \tfrac{1}{3}. \qquad \square$$

Integration by parts:
This technique of integration is based upon the product rule
$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \times g'(x) + f'(x) \times g(x)$. This can be rewritten as
$f(x) \times g'(x) = \frac{d}{dx}[f(x) \times g(x)] - f'(x) \times g(x)$, which means that the antiderivative of $f(x) \times g'(x)$ can be written as $\int f(x) \times g'(x)\,dx = f(x) \times g(x) - \int f'(x) \times g(x)\,dx$.

This technique is useful if $f'(x) \times g(x)$ has an easier antiderivative to find than $f(x) \times g'(x)$. Given an integral, it may not be immediately apparent how to define $f(x)$ and $g(x)$ so that the integration by parts technique applies and results in a simplification. It may be necessary to apply integration by parts more than once to simplify an integral.

Example 0-21:
Find $\int xe^{ax}\,dx$, where a is a constant.

Solution:
If we define $f(x) = x$ and $g(x) = \frac{e^{ax}}{a}$, then $g'(x) = e^{ax}$, and
$\int xe^{ax}\,dx = \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$.
Since $f'(x) = 1$, it follows that $\int f'(x)g(x)\,dx = \int \frac{e^{ax}}{a}\,dx = \frac{e^{ax}}{a^2}$, and therefore
$\boldsymbol{\int xe^{ax}\,dx = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + c}$.

An alternative to integration by parts is the following approach:
$\boldsymbol{\frac{d}{da}\int e^{ax}\,dx = \int xe^{ax}\,dx}$ and $\boldsymbol{\frac{d}{da}\int e^{ax}\,dx = \frac{d}{da}\,\frac{e^{ax}}{a} = \frac{axe^{ax} - e^{ax}}{a^2}}$
so it follows that $\boldsymbol{\int xe^{ax}\,dx = \frac{axe^{ax} - e^{ax}}{a^2} = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}}$.

This integral has appeared a number of times on the exam, usually with $a < 0$ (it is valid for any $a \neq 0$) and it is important to be familiar with it. □

An alternative way to apply integration by parts is via "Tabular Integration". Suppose that we wish to find the integral $\int u(x) \times v(x)\,dx$. We create two columns, a column of the successive derivatives of $u(x)$ and a column of the successive antiderivatives of $v(x)$. In order for the method to work efficiently, we try to choose $u(x)$ to be a polynomial which will eventually have a derivative of 0. The integral in Example 0-21 will be used to illustrate tabular integration. We make the following choices for $u(x)$ and $v(x)$: $u(x) = x$, $v(x) = e^{ax}$. The two columns are

Row	Derivatives of $u(x) = x$	Antiderivatives of $v(x) = e^{ax}$
0	x	e^{ax}
1	1	e^{ax}/a
2	0	e^{ax}/a^2

We pair up entries in each row of the "Derivatives of $u(x)$" column with the following row of the "Antiderivatives of $v(x)$" column and multiply the pairs, alternative " $+$ " and " $-$ " for each pair, and add them up. In this example, we pair x (Row 0 of "Derivatives") with e^{ax}/a (Row 1 of "Antiderivatives"), then we pair 1 (Row 1 of "Derivatives") with e^{ax}/a^2 (Row 2 of "Antiderivatives") and apply " $-$ " to this pair.

For this example, we can stop at this point, since all higher order derivatives of $u(x)$ are 0 in Row 2 and higher. The integral $\int xe^{ax}\,dx$ is $x \times e^{ax}/a - 1 \cdot e^{ax}/a^2 = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}$, as in Example 0-21.

NOTE: An extension of Example 0-21 shows that for integer $n \geq 0$ and $c > 0$ $\int_0^\infty x^n e^{-cx}\,dx = \frac{n!}{c^{n+1}}$. This is another useful identity for the exam.

GEOMETRIC AND ARITHMETIC PROGRESSIONS

Geometric progression: $a,\ ar,\ ar^2,\ ar^3,\ \ldots,$ sum of the first n terms is
$a + ar + ar^2 + \cdots + ar^{n-1} = a[1 + r + r^2 + \cdots + r^{n-1}] = a \cdot \frac{r^n - 1}{r-1} = a \times \frac{1-r^n}{1-r}$,
if $-1 < r < 1$ then the infinite series can be summed, $a + ar + ar^2 + \cdots = \frac{a}{1-r}$.

Arithmetic progression: $a,\ a+d,\ a+2d,\ a+3d,\ \ldots,$
sum of the first n terms of the series is $na + d \times \frac{n(n-1)}{2}$,
a special case is the sum of the first n integers - $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Example 0-22:
A product sold 10,000 units last week, but sales are forecast to decrease 2% per week if no advertising campaign is implemented. If an advertising campaign is implemented immediately, the sales will decrease by 1% of the previous week's sales but there will be 200 new sales for the week (starting with this week). Under this model, calculate the number of sales for the 10-th week, 100-th week and 1000-th week of the advertising campaign (last week is week 0, this week is week 1 of the campaign).

Solution:
Week 1 sales: $(.99)(10,000) + 200$,
Week 2 sales: $(.99)[(.99)(10,000) + 200] + 200 = (.99)^2(10,000) + (200)[1 + .99]$
Week 3 sales: $(.99)[(.99)^2(10,000) + (200)[1 + .99]] + 200$
$$= (.99)^3(10,000) + (200)[1 + .99 + .99^2]$$
$$\vdots$$
Week 10 sales: $(.99)^{10}(10,000) + (200)[1 + .99 + .99^2 + \cdots + .99^9]$
$$= (.99)^{10}(10,000) + (200)[\tfrac{1-.99^{10}}{1-.99}] = 10,956.2.$$
Week 100 sales: $(.99)^{100}(10,000) + (200)[\frac{1-.99^{100}}{1-.99}] = 16,339.7.$
Week 1000 sales: $(.99)^{1000}(10,000) + (200)[\frac{1-.99^{1000}}{1-.99}] = 19,999.6.$ □

PROBLEM SET 0

Review of Algebra and Calculus

1. The manufacturer of a certain product is conducting a market survey. The manufacturer started a major marketing campaign at the end of last year and is trying to determine the effect of that campaign on consumer use of the product. 15,000 individuals are surveyed. The following information is obtained.

    - 4,500 used the product last year,
    - 7,500 used the product this year, and
    - 4,000 used the product both last year and this year.

    Of those surveyed, determine:

    (i) the number who used the product either last year or this year, or both years,

    (ii) the number that did not use the product either last year or this year,

    (iii) the number who used the product last year but not this year, and

    (iv) the number who used the product this year but not last year.

2. A group of 5000 undergraduate college students were surveyed regarding the following characteristics:

    - participate in extracurricular activities,
    - have a double major, and
    - have a part-time job.

    The following data was obtained.

    2600 participated in extracurricular activities,

    1200 had a double major,

    2500 had a part time-job ,

    400 both participated in extracurricular activities and had a double major,

    1000 both participated in extracurricular activities and had a part-time job,

    300 both had a double major and had a part-time job,

    200 participated in extracurricular activities and had a double major and had a part-time job.

    Determine each of the following.

    (i) The number who had a double major but did not participate in extra-curricular activities and did not have a part-time job.

    (ii) The number who had a double major and either participated in extra-curricular activities or had a part-time job , but not both.

    (iii) The number who neither participated in extracurricular activities nor had a part-time job.

3. A group of 1000 patients each diagnosed with a certain disease is being analyzed with regard to the disease symptoms present. The symptoms are labeled A, B and C, and each patient has at least one symptom. The following information has also been determined:

- 900 have either symptom A or B (or both),
- 900 have either symptom A or C (or both),
- 800 have either symptom B or C (or both),
- 650 have symptom A,
- 500 have symptom B,
- 550 have symptom C.

Determine each of the following.

(i) The number who had both symptoms A and B.

(ii) The number who had either symptom A or B (or both) but not C.

(iii) The number who had all three symptoms.

4. An insurance agent offers his clients auto insurance, homeowners insurance and renters insurance. The purchase of homeowners insurance and the purchase of renters insurance are mutually exclusive. The profile of the agent's clients is as follows:

i) 17% of the clients have none of these three products.

ii) 64% of the clients have auto insurance.

iii) Twice as many of the clients have homeowners insurance as have renters insurance.

iv) 35% of the clients have two of these three products.

v) 11% of the clients have homeowners insurance, but not auto insurance.

Calculate the percentage of the agent's clients that have both auto and renters insurance.

A) 7% B) 10% C) 16% D) 25% E) 28%

5. A survey of 100 TV watchers revealed that over the last year:

i) 34 watched CBS.

ii) 15 watched NBC.

iii) 10 watched ABC.

iv) 7 watched CBS and NBC.

v) 6 watched CBS and ABC.

vi) 5 watched NBC and ABC.

vii) 4 watched CBS, NBC, and ABC.

viii) 18 watched HGTV and of these, none watched CBS, NBC, or ABC.

Calculate how many of the 100 TV watchers did not watch any of the four channels (CBS, NBC, ABC or HGTV).

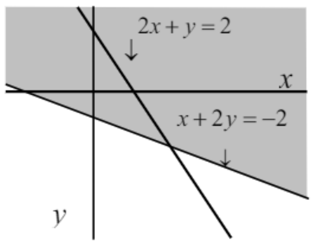
A) 1 B) 37 C) 45 D) 55 E) 82

6. $\lim_{N \to \infty} \frac{5^N}{N!} =$

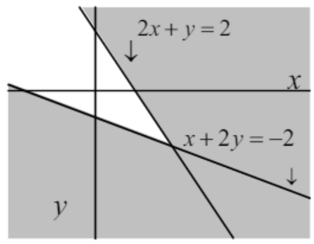
A) 0 B) $\frac{1}{2}$ C) $5\,ln\,5$ D) $+\infty$ E) None of A,B,C,D

7. Which region of the plane represents the solution set to the inequalities
$\{(x,y) : 2x + y > 2\} \cup \{(x,y) : x + 2y < -2\}$?

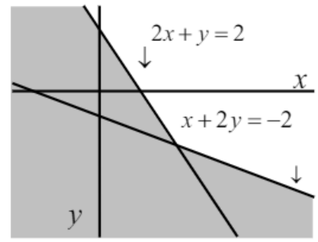
A)



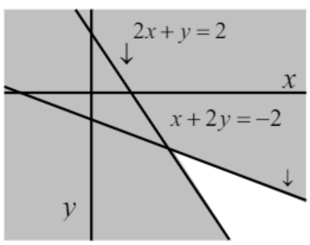
B)



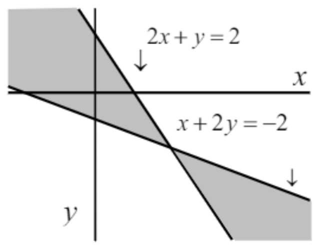
C)



D)



E)



8. For what real values of k are the roots of $kx^2 + 3x - 2 = 0$ not real numbers?
A) $k < -\frac{9}{8}$ B) $k < \frac{9}{8}$ C) $k > -\frac{9}{8}$ D) $k < -\frac{8}{9}$ E) $k < \frac{8}{9}$

9. If $f(x) = f^{-1}(x)$ then $f(x)$ may equal:
A) 2^x B) $\frac{1}{2^x}$ C) 1 D) $\frac{1}{x}$ E) $\frac{1}{\sqrt{x}}$

10. Let $f(x) = x^4 e^x$. Determine the nth derivative of f at $x = 0$.
A) 0 B) 1 C) $n(n-1)(n-2)$ D) $n(n-1)(n-2)(n-3)$ E) $n!$

11. A model for world population assumes a population of 6 billion at reference time 0, with population increasing to a limiting population of 30 billion. The model assumes that the rate of population growth at time $t \geq 0$ is $\frac{Ae^t}{(.02A+e^t)^2}$ billion per year, where t is regarded as a continuous variable. According to this model, at what time will the population reach 10 billion (nearest .1)?

A) .3 B) .4 C) .5 D) .6 E) .8

12. Calculate the area of the closed region in the xy-plane bounded by $y = x - 5$ and $y^2 = 2x + 5$.

A) 8 B) $\frac{74}{3}$ C) $\frac{98}{3}$ D) $\frac{122}{3}$ E) $\frac{128}{3}$

13. Let $F(x) = \int_0^{x^{1/3}} \sqrt{1+t^4}\, dt$. $F'(0) =$

A) 0 B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) 1 E) Does not exist

14. Let f be a continuous function on $\mathbb{R}^2$ and let $I = \int_0^2 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\, dy\, dx$.

Which of the following expressions is equal to I with the order of integration reversed?

A) $\int_{-2}^{2}\int_{y^2}^{2} f(x,y)\, dx\, dy$ B) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{y^2}^{2} f(x,y)\, dx\, dy$ C) $\int_{-2}^{2}\int_{y}^{\sqrt{2}} f(x,y)\, dx\, dy$

D) $\int_0^2 \int_2^{y^2} f(x,y)\, dx\, dy$ E) $\int_0^2 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx\, dy$

15. Calculate $\int_0^1 \int_x^1 \frac{1}{1+y^2}\, dy\, dx$.

A) $1 - ln\,2$ B) $\frac{1}{2}\, ln\,2$ C) $\frac{\pi}{4}$ D) $\frac{1}{2} + ln\,2$ E) $\frac{\pi}{2} - \frac{1}{2}\, ln\,2$

Question 16 and 17 relate to the following information. Smith begins a new job at a salary of 100,000 Smith expects to receive a 5% raise every year until he retires.

16. Suppose that Smith works for 35 years. Determine the total salary earned over Smith's career (nearest million).

A) 5 B) 6 C) 7 D) 8 E) 9

17. At the end of each year, Smith's employer deposits 3% of Smith's salary (for the year just finished) into a fund earning 4% per year compounded each year. Find the value of the fund just after the final deposit at the end of Smith's 35th year of employment (nearest 10,000)

A) 450,000 B) 460,000 C) 470,000 D) 480,000 E) 490,000

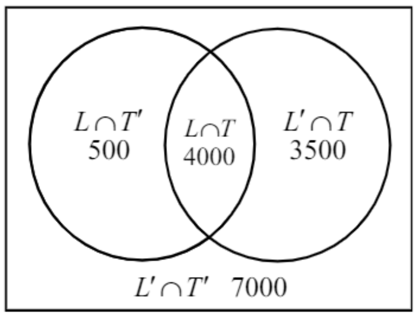
PROBLEM SET 0 SOLUTIONS

1. The "total set" is the set of all those who were surveyed and consists of 15,000 individuals. We define the following sets:

   L - those who used the product last year,

   T - those who used the product this year.

   We are given $n(L) = 4,500$, $n(T) = 7,500$ and $n(L \cap T) = 4,000$.
   This can be represented in Venn diagram form as follows (not to scale):



   (i) The number who used the product either last year or this year, or both years is $n(L \cup T) = 500 + 4000 + 3500 = 8000$.

   (ii) The number that did not use the product either last year or this year is $n[(L \cup T)'] = n(L' \cap T') = n(\text{total set}) - n(L \cup T) = 15,000 - 8,000 = 7,000$.

   (iii) The number who used the product last year but not this year is $n(L \cap T') = 500$

   (iv) The number who used the product this year but not last year is $n(L' \cap T) = 3500$.

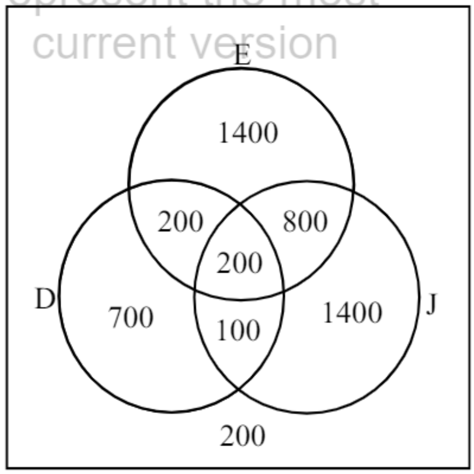
2. Following the same method applied in Example 0-3 of the notes of this study material, we get the Venn diagram entries below for the numbers in the various combinations.

   $E =$ participated in extracurricular activities,

   $D =$ had a double major, and

   $J =$ had a part-time job.

   An example illustrating the calculation of one of the entries is the following. Since there are 1000 who both participated in extra-curricular activities and had a part-time job, and since 200 were in all three groups, the number who both participated in extra-curricular activities and had a part-time job but didn't have a double major is $1000 - 200 = 800$ (this would be $n(E \cap D' \cap J)$).



(i) The number who had a double major but did not participate in extra-curricular activities and did not have a part-time job is $n(D \cap E' \cap J') = 700$.

(ii) The number who had a double major and either participated in extra-curricular activities or had a part-time job , but not both is $n(D \cap E \cap J') + n(D \cap E' \cap J) = 200 + 100 = 300$.

(iii) The number who neither participated in extracurricular activities nor had a part-time job is $n(E' \cap J') = n[(E \cup J)'] = 700 + 200 = 900$ (those in $(E \cup J)'$ with double major and those in $(E \cup J)'$ without double major).

3. This is quite similar to the previous problem, but the information is based on different combinations of sets. We are given

$n(A) = 650$, $n(B) = 500$, $n(C) = 550$, $n(A \cup B) = 900$, $n(A \cup C) = 900$,
$n(B \cup C) = 800$ and $n(A \cup B \cup C) = 1000$.

(i) We are asked for $n(A \cap B)$. We can use the relationship
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ to get $n(A \cap B) = 650 + 500 - 900 = 250$.

In order to solve (ii) and (iii) we can use the given information to fill in the numbers for the component subsets. We have $n(A \cap C) = n(A) + n(C) - n(A \cup C) = 300$,
$n(B \cap C) = 250$. We then use

$$n(A \cup B \cup C) = n(A) + n(B) + n(C)$$
$$- n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

so that $1000 = 650 + 500 + 550 - 250 - 300 - 250 + n(A \cap B \cap C)$,
from which we get $n(A \cap B \cap C) = 100$.