(viii) $P[(x_1 < X \le x_2) \cap (y_1 < Y \le y_2)] = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$

(ix) $P[(X \le x) \cup (Y \le y)] = F_X(x) + F_Y(y) - F(x, y) \le 1$

(x) For any jointly distributed random variables X and Y, $-1 \le \rho_{XY} \le 1$

(xi) $M_{X,Y}(t_1, 0) = E[e^{t_1 X}] = M_X(t_1)$ and $M_{X,Y}(0, t_2) = E[e^{t_2 Y}] = M_Y(t_2)$

(xii) $\frac{\partial}{\partial t_1} M_{X,Y}(t_1, t_2)\Big|_{t_1=t_2=0} = E[X]$, $\frac{\partial}{\partial t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=t_2=0} = E[Y]$

$\frac{\partial^{r+s}}{\partial^r t_1 \, \partial^s t_2} M_{X,Y}(t_1, t_2)\Big|_{t_1=t_2=0} = E[X^r \times Y^s]$

(xiii) If $M(t_1, t_2) = M(t_1, 0) \times M(0, t_2)$ for t_1 and t_2 in a region about $(0,0)$, then X and Y are independent.

(xiv) If $Y = aX + b$ then $M_Y(t) = e^{bt} M_X(at)$.

(xv) If X and Y are jointly distributed, then for any y, $E[X|Y=y]$ depends on y, say $E[X|Y=y] = h(y)$. It can then be shown that $E[h(Y)] = E[X]$; this is more usually written in the form $\boldsymbol{E[\,E[X|Y]\,] = E[X]}$. This is referred to as the **Law of Total Expectation**, and may also be called the Double Expectation Rule or the Tower Rule.

It can also be shown that $\boldsymbol{Var[X] = E[\,Var[X|Y]\,] + Var[\,E[X|Y]\,]}$.
This relationship has come up on several questions on recent exams.

Example 8-20:

For random variables X, Y and Z, you are given that $Var[X] = Var[Y] = 1$, $Var[Z] = 2$, $Cov(X,Y) = -1$, $Cov(X,Z) = 0$, $Cov(Y,Z) = 1$.

Find the covariance between $X + 2Y$ and $Y + 2Z$.

Solution: $Cov(X+2Y, Y+2Z) = Cov(X,Y) + Cov(X,2Z) + Cov(2Y,Y) + Cov(2Y,2Z)$

$= -1 + 2Cov(X,Z) + 2Cov(Y,Y) + 4Cov(Y,Z)$

$= -1 + 2 \times 0 + 2Var[Y] + 4 \times 1 = -1 + 2(1) + 4 = 5.$ □

Example 8-21:

Random variables X and Y are jointly distributed on the region $0 \le y \le x$, $y \le 1$, $x \le 2$.

The joint distribution has a constant density over the entire region.

(i) Find the joint density.

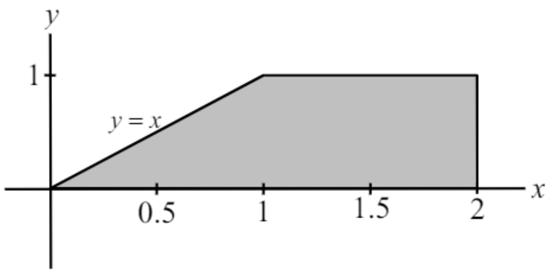
(ii) Find the marginal densities of X and Y.

(iii) Find the conditional densities of X given $Y = y$, and Y given $X = x$.

(iv) Find the probabilities $P[X > 1]$, $P[Y \le \frac{1}{2}]$, and $P[X + Y \le 1]$.

Solution:

The region of probability is outlined below.



(i) Since we are told that the joint density is constant on the region of probability that constant must be equal to $\frac{1}{\text{area of region of probability}}$. The area of the region of probability is 1.5, so that $f(x, y) = \frac{1}{1.5} = \frac{2}{3}$ for (x, y) points in the region of probability.

(ii) The marginal density function of X is $f_X(x) = \int_0^1 f(x, y)\, dy$.
If $0 \le x \le 1$, the probability region is $0 \le y \le x$, so that $f_X(x) = \int_0^x \frac{2}{3}\, dy = \frac{2x}{3}$.
If $1 \le x \le 2$, the probability region is $0 \le y \le 1$, so that $f_X(x) = \int_0^1 \frac{2}{3}\, dy = \frac{2}{3}$.

The marginal density of Y is $f_Y(y) = \int_y^2 \frac{2}{3}\, dx = \frac{2}{3}(2 - y)$ for $0 \le y \le 1$.

(iii) The conditional density of X given $Y = y$ is $f_{X|Y}(x|Y = y) = \frac{f(x,y)}{f_Y(y)} = \frac{2/3}{2(2-y)/3} = \frac{1}{2-y}$.
Note that the interval of probability for the conditional distribution of X given $Y = y$ is $y \le x \le 2$. The conditional density of X given $Y = y$ is uniform on that interval.

The conditional density of Y given $X = x$ is $f_{Y|X}(y|X = x) = \frac{f(x,y)}{f_X(x)}$.
For $0 \le x \le 1$ this is $\frac{2/3}{2x/3} = \frac{1}{x}$, a uniform distribution for $0 \le y \le x$.
For $1 \le x \le 2$ this is $\frac{2/3}{2/3} = 1$, a uniform distribution for $0 \le y \le 1$.
Note that in all cases the conditional distributions are uniform. This will always be the case if the joint distribution is uniform on its probability space.

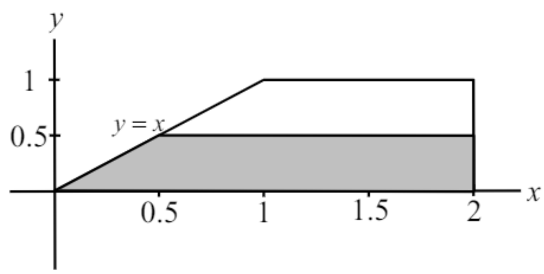
(iv) $P[X > 1]$ can be found in two ways. Since we have the marginal density for X, we can use it to find $P[X > 1] = \int_1^2 f_X(x)\,dx = \int_1^2 \frac{2}{3}\,dx = \frac{2}{3}$.

Alternatively, since the joint distribution is uniform, the probability of any event A is the proportion $\frac{Area\,of\,A}{Area\,of\,R}$. The area of the full probability space R was already found to be 1.5.

The area of the region $X > 1$ is 1. Therefore, $P[X > 1] = \frac{1}{1.5} = \frac{2}{3}$.

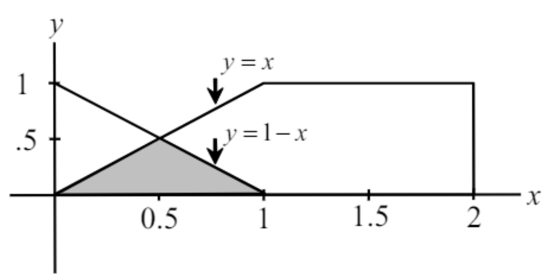
The same comment applies to $P[Y \leq \frac{1}{2}]$. From the marginal density of Y we have $P[Y \leq \frac{1}{2}] = \int_0^{1/2} \frac{2}{3}(2-y)\,dy = \frac{7}{12}$.

Alternatively, the area of the region $Y \leq \frac{1}{2}$ is $\frac{7}{8}$ (outlined in the diagram below).



The probability $P[Y \leq \frac{1}{2}]$ is the proportion $\frac{7/8}{3/2} = \frac{7}{12}$.

The region $X + Y \leq 1$ is outlined in the diagram below (the region below $y = 1 - x$ within the original probability space).



The area of the region $X + Y \leq 1$ is $\frac{1}{4}$, so the probability is $P[X + Y \leq 1] = \frac{1/4}{3/2} = \frac{1}{6}$.

We could also find the probability by integrating the joint density over the appropriate region, $P[X + Y \leq 1] = \int_0^{.5}\int_y^{1-y} \frac{2}{3}\,dx\,dy = \int_0^{.5} \frac{2}{3}(1-2y)\,dy = \frac{1}{6}$. □

Example 8-22:

Suppose that W is the 3-point discrete uniform random variable $\{1, 2, 3\}$, with

$$P[W = 1] = P[W = 2] = P[W = 3] = \tfrac{1}{3},$$

and suppose that the conditional distribution of Y given $W = w$ is exponential with mean w.

Find the (unconditional) mean and (unconditional) variance of Y.

Solution:

The conditional pdf of Y given $W = w$ is $f_{Y|W}(y|W = w) = \frac{1}{w}e^{-y/w}$.

Also, we are given $E[Y|W = w] = w$, so that $E[Y|W] = \begin{cases} 1 & \text{if } W = 1, \text{prob. } \frac{1}{3} \\ 2 & \text{if } W = 2, \text{prob. } \frac{1}{3} \\ 3 & \text{if } W = 3, \text{prob. } \frac{1}{3} \end{cases}$.

We see that $\boldsymbol{E[Y|W]}$ **is a 3-point random variable.**

Let us use the notation $Z = E[Y|W]$ for this 3-point random variable.

Then $E[Z] = E[\,E[Y|W]\,] = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = 2$.

This is $E[Y]$ according to the double expectation rule cited above.

Also, as a random variable, $Z = E[Y|W]$ has a variance:

$$\begin{aligned} Var[Z] &= Var[\,E[Y|W]\,] = E[Z^2] - (E[Z])^2 = E[(E[Y|W])^2] - (E[\,E[Y|W]\,])^2 \\ &= [(1^2)(\tfrac{1}{3}) + (2^2)(\tfrac{1}{3}) + (3^2)(\tfrac{1}{3})] - 2^2 = \tfrac{2}{3}. \end{aligned}$$

The variance of an exponential random variable is the square of the mean, so since the conditional distribution of Y given W is exponential with mean w, the conditional variance of Y given $W = w$ is $Var[Y|W = w] = w^2$. As with the conditional mean of $Y|W$, **the conditional variance of $\boldsymbol{Y|W}$ is a random variable dependent on the outcome of $\boldsymbol{W}$.**

$$Var[Y|W] = \begin{cases} Var[Y|W = 1] = 1 & \text{if } W = 1, \text{prob. } \frac{1}{3} \\ Var[Y|W = 2] = 4 & \text{if } W = 2, \text{prob. } \frac{1}{3} \\ Var[Y|W = 3] = 9 & \text{if } W = 3, \text{prob. } \frac{1}{3} \end{cases}.$$

$Var[Y|W]$ is a three-point random variable. Let us use the notation $U = Var[Y|W]$.

Then $$\begin{aligned} E[U] &= E[\,Var[Y|W]\,] \\ &= (Var[Y|w=1]) \times \tfrac{1}{3} + (Var[Y|w=2]) \times \tfrac{1}{3} + (Var[Y|w=3]) \times \tfrac{1}{3} \\ &= 1 \times \tfrac{1}{3} + 4 \times \tfrac{1}{3} + 9 \times \tfrac{1}{3} = \tfrac{14}{3}. \end{aligned}$$

Using the variance rule cited above in (xv) above, we have

$$Var[Y] = E[\,Var[Y|W]\,] + Var[\,E[Y|W]\,] = E[U] + Var[Z] = \tfrac{14}{3} + \tfrac{2}{3} = \tfrac{16}{3}. \qquad \square$$

PROBLEM SET 8

Joint, Marginal and Conditional Distributions

1. A wheel is spun with the numbers 1, 2 and 3 appearing with equal probability of $\frac{1}{3}$ each. If the number 1 appears, the player gets a score of 1.0; if the number 2 appears, the player gets a score of 2.0; if the number 3 appears, the player gets a score of X, where X is a normal random variable with mean 3 and standard deviation 1. If W represents the player's score on 1 spin of the wheel, then what is $P[W \leq 1.5]$?
A) 0.13 B) 0.33 C) 0.36 D) 0.40 E) 0.64

2. Let X and Y be discrete loss random variables with joint probability function
$$f(x,y) = \begin{cases} \frac{y}{24x} \text{ for } x=1,2,4;\ y=2,4,8;\ x \leq y \\ 0, \text{otherwise} \end{cases}.$$
An insurance policy pays the full amount of loss X and half of loss Y. Find the probability that the total paid by the insurer is no more than 5.
A) $\frac{1}{8}$ B) $\frac{7}{24}$ C) $\frac{3}{8}$ D) $\frac{5}{8}$ E) $\frac{17}{24}$

3. Let X and Y be continuous random variables with joint density function
$$f(x,y) = \begin{cases} .75x \text{ for } 0<x<2 \text{ and } 0<y \leq 2-x \\ 0, \text{otherwise} \end{cases}.$$
What is $P[X > 1]$?
A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{3}{4}$

4. Let X and Y be independent random variables with $\mu_X = 1$, $\mu_Y = -1$, $\sigma_X^2 = \frac{1}{2}$, and $\sigma_Y^2 = 2$. Calculate $E[(X+1)^2(Y-1)^2]$.
A) 1 B) $\frac{9}{2}$ C) 16 D) 17 E) 27

5. (SOA) Let T_1 be the time between a car accident and reporting a claim to the insurance company. Let T_2 be the time between the report of the claim and payment of the claim. The joint density function of T_1 and T_2, $f(t_1, t_2)$, is constant over the region $0 < t_1 < 6$, $0 < t_2 < 6$, $t_1 + t_2 < 10$, and zero otherwise. Determine $E[T_1 + T_2]$, the expected time between a car accident and payment of the claim.
A) 4.9 B) 5.0 C) 5.7 D) 6.0 E) 6.7

6. (SOA) A diagnostic test for the presence of a disease has two possible outcomes: 1 for disease present and 0 for disease not present. Let X denote the disease state of a patient, and let Y denote the outcome of the diagnostic test. The joint probability function of X and Y is given by:

$$P(X=0,Y=0)=0.800\ ,\ P(X=1,Y=0)=0.050$$
$$P(X=0,Y=1)=0.025\ ,\ P(X=1,Y=1)=0.125$$

Calculate $Var(Y|X=1)$.

A) 0.13 B) 0.15 C) 0.20 D) 0.51 E) 0.71

7. (SOA) A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold.

$$P(X=0,Y=0)=\tfrac{1}{6}\ ,\ P(X=1,Y=0)=\tfrac{1}{12}\ ,\ P(X=1,Y=1)=\tfrac{1}{6}$$
$$P(X=2,Y=0)=\tfrac{1}{12}\ ,\ P(X=2,Y=1)=\tfrac{1}{3}\ ,\ P(X=2,Y=2)=\tfrac{1}{6}$$

What is the variance of X?

A) 0.47 B) 0.58 C) 0.83 D) 1.42 E) 2.58

8. (SOA) Once a fire is reported to a fire insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint density function

$$f(x,y)=\frac{2}{x^2(x-1)}y^{-(2x-1)/(x-1)}\qquad x>1,\ y>1$$

Given that the initial claim estimated by the company is 2, determine the probability that the final settlement amount is between 1 and 3.

A) $\frac{1}{9}$ B) $\frac{2}{9}$ C) $\frac{1}{3}$ D) $\frac{2}{3}$ E) $\frac{8}{9}$

9. (SOA) An auto insurance policy will pay for damage to both the policyholder's car and the other driver's car in the event that the policyholder is responsible for an accident. The size of the payment for damage to the policyholder's car, X, has a marginal density function of 1 for $0<x<1$. Given $X=x$, the size of the payment for damage to the other driver's car, Y, has conditional density of 1 for $x<y<x+1$. If the policyholder is responsible for an accident, what is the probability that the payment for damage to the other driver's car will be greater than 0.5?

A) $\frac{3}{8}$ B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) $\frac{7}{8}$ E) $\frac{15}{16}$

10. (SOA) The future lifetimes (in months) of two components of a machine have the following joint density function:

$$f(x,y) = \begin{cases} \frac{6}{125{,}000}(50-x-y) & \text{for } 0 < x < 50-y < 50 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that both components are still functioning 20 months from now?

A) $\frac{6}{125{,}000}\int_0^{20}\int_0^{20}(50-x-y)\,dy\,dx$

B) $\frac{6}{125{,}000}\int_{20}^{30}\int_{20}^{50-x}(50-x-y)\,dy\,dx$

C) $\frac{6}{125{,}000}\int_{20}^{30}\int_{20}^{50-x-y}(50-x-y)\,dy\,dx$

D) $\frac{6}{125{,}000}\int_{20}^{50}\int_{20}^{50-x}(50-x-y)\,dy\,dx$

E) $\frac{6}{125{,}000}\int_{20}^{50}\int_{20}^{50-x-y}(50-x-y)\,dy\,dx$

11. (SOA) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{8}{3}xy & \text{for } 0 \le x \le 1 \text{ , } x \le y \le 2x \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the covariance of X and Y.

A) 0.04 B) 0.25 C) 0.67 D) 0.80 E) 1.24

12. (SOA) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 15y & \text{for } x^2 \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

Let g be the marginal density function of Y. Which of the following represents g?

A) $g(y) = \begin{cases} 15y & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

B) $g(y) = \begin{cases} \frac{15y^2}{2} & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$

C) $g(y) = \begin{cases} \frac{15y^2}{2} & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

D) $g(y) = \begin{cases} 15y^{3/2}(1-y^{1/2}) & \text{for } x^2 < y < x \\ 0 & \text{otherwise} \end{cases}$

E) $g(y) = \begin{cases} 15y^{3/2}(1-y^{1/2}) & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

13. (SOA) An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f(x,y) = \begin{cases} \frac{2x+2-y}{4} & \text{for } 0 < x < 1 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that the total loss is at least 1?

A) 0.33 B) 0.38 C) 0.41 D) 0.71 E) 0.75

14. (SOA) Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval (0, 12). Given $X = x$, Y is uniformly distributed on the interval $(0, x)$. Determine $Cov(X, Y)$ according to this model.

A) 0 B) 4 C) 6 D) 12 E) 24

15. (SOA) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 24xy & \text{for } 0 < x < 1 \text{ and } 0 \le y \le 1-x \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $P[Y < X | X = \frac{1}{3}]$.

A) $\frac{1}{27}$ B) $\frac{2}{27}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{4}{9}$

16. (SOA) A joint density function is given by

$$f(x,y) = \begin{cases} kx & \text{for } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{where } k \text{ is a constant.}$$

What is $\text{Cov}(X, Y)$?

A) $-\frac{1}{6}$ B) 0 C) $\frac{1}{9}$ D) $\frac{1}{6}$ E) $\frac{2}{3}$

17. (SOA) An actuary determines that the annual numbers of tornadoes in counties P and Q are jointly distributed as follows:

		Annual number of tornadoes in county Q			
		0	1	2	3
Annual number	0	0.12	0.06	0.05	0.02
of tornadoes	1	0.13	0.15	0.12	0.03
in county P	2	0.05	0.15	0.10	0.02

Calculate the conditional variance of the annual number of tornadoes in county Q, given that there are no tornadoes in county P .

A) 0.51 B) 0.84 C) 0.88 D) 0.99 E) 1.76

18. (SOA) A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s,t)$, where $0 < s < 1$ and $0 < t < 1$. What is the probability that the device fails during the first half hour of operation?

A) $\int_0^{0.5}\int_0^{0.5} f(s,t)\,ds\,dt$ B) $\int_0^1\int_0^{0.5} f(s,t)\,ds\,dt$

C) $\int_{0.5}^1\int_{0.5}^1 f(s,t)\,ds\,dt$ D) $\int_0^{0.5}\int_0^1 f(s,t)\,ds\,dt + \int_0^1\int_0^{0.5} f(s,t)\,ds\,dt$

E) $\int_0^{0.5}\int_{0.5}^1 f(s,t)\,ds\,dt + \int_0^1\int_0^{0.5} f(s,t)\,ds\,dt$

19. (SOA) A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy. Let X denote the proportion of employees who purchase the basic policy, and Y the proportion of employees who purchase the supplemental policy. Let X and Y have the joint density function $f(x,y) = 2(x+y)$ on the region where the density is positive. Given that 10% of the employees buy the basic policy, what is the probability that fewer than 5% buy the supplemental policy?

A) 0.010 B) 0.013 C) 0.108 D) 0.417 E) 0.500

20. (SOA) The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y) = \begin{cases} 2x & \text{for } 0 < x < 1,\ x < y < x+1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the conditional variance of Y given that $X = x$?

A) $\frac{1}{12}$ B) $\frac{7}{6}$ C) $x + \frac{1}{2}$ D) $x^2 - \frac{1}{6}$ E) $x^2 + x + \frac{1}{3}$

21. Let X and Y be continuous random variables having a bivariate normal distribution with means μ_X and μ_Y, common variance σ^2, and correlation coefficient ρ_{XY}. Let F_X and F_Y be the cumulative distribution functions of X and Y respectively. Determine which of the following is a necessary and sufficient condition for $F_X(t) \geq F_Y(t)$ for all t.

A) $\mu_X \geq \mu_Y$ B) $\mu_X \leq \mu_Y$ C) $\mu_X \geq \rho_{XY}\mu_Y$ D) $\mu_X \leq \rho_{XY}\mu_Y$ E) $\rho_{XY} \geq 0$

22. If the joint probability density function of X_1, X_2 is $f(x_1, x_2) = 1$, for $0 < x_1 < 1$ and $0 < x_2 < 1$, and 0 otherwise, then the moment generating function $M(t_1, t_2)$, $t_1, t_2 \neq 0$ of the joint distribution is

A) $\frac{e^{t_1}-1}{t_1}$ B) $\frac{(e^{t_1}-1)(e^{t_2}-1)}{t_1 t_2}$ C) $\frac{(e^{t_1}+1)(e^{t_2}+1)}{t_1 t_2}$ D) $\frac{1}{t_1 t_2}$ E) $e^{t_1+t_2} - 1$

23. The moment generating function for the joint distribution of random variables X and Y is $M_{X,Y}(t_1, t_2) = \frac{1}{3(1-t_2)} + \frac{2}{3}e^{t_1} \times \frac{2}{(2-t_2)}$, for $t_2 < 1$. Find $Var[X]$.

A) $\frac{1}{18}$ B) $\frac{1}{9}$ C) $\frac{1}{6}$ D) $\frac{2}{9}$ E) $\frac{1}{3}$

24. (SOA) A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f(x,y) = \begin{cases} 6[1-(x+y)] & \text{for } x>0,\ y>0,\ x+y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that the portion of a claim representing damage to the house is less than 0.2.

A) 0.360 B) 0.480 C) 0.488 D) 0.512 E) 0.520

25. The distribution of Smith's future lifetime is X, an exponential random variable with mean α, and the distribution of Brown's future lifetime is Y, an exponential random variable with mean β. Smith and Brown have future lifetimes that are independent of one another. Find the probability that Smith outlives Brown.

A) $\frac{\alpha}{\alpha+\beta}$ B) $\frac{\beta}{\alpha+\beta}$ C) $\frac{\alpha-\beta}{\alpha}$ D) $\frac{\beta-\alpha}{\beta}$ E) $\frac{\alpha}{\beta}$

26. X and Y are continuous losses with joint distribution

$$f(x,y) = \begin{cases} \frac{3}{4}(2-x-y) & \text{for } 0<x<2,\ 0<y<2, \text{ and } x+y<2 \\ 0, & \text{otherwise} \end{cases}.$$

An insurance policy pays the total $X+Y$. Find the expected amount the policy will pay.

A) 0 B) .5 C) 1 D) 1.5 E) 2

27. A pair of loss random variables X and Y have joint density function

$$f(x,y) = \begin{cases} 6xy+3x^2 & \text{for } 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that the loss Y is no more than 0.5.

A) 0.015625 B) 0.03125 C) 0.0625 D) 0.125 E) 0.25

28. The joint density function of two random losses X and Y is

$$f(x,y) = \begin{cases} x+y, & \text{for } 0<x<1 \text{ and } 0<y<1 \\ 0, \text{ elsewhere} \end{cases}.$$

Find the probability that loss X is less than double the loss Y.

A) $\frac{7}{32}$ B) $\frac{1}{4}$ C) $\frac{3}{4}$ D) $\frac{19}{24}$ E) $\frac{7}{8}$

29. (SOA) A device contains two circuits. The second circuit is a backup for the first, so the second is used only when the first has failed. The device fails when and only when the second circuit fails. Let X and Y be the times at which the first and second circuits fail, respectively. X and Y have joint probability density function

$$f(x,y) = \begin{cases} 6e^{-x}e^{-2y} & \text{for } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected time at which the device fails?

A) 0.33 B) 0.50 C) 0.67 D) 0.83 E) 1.50

30. (SOA) Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident. X and Y have joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{64}(10 - xy^2) & \text{for } 2 \le x \le 10 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected age of an insured automobile involved in an accident.

A) 4.9 B) 5.2 C) 5.8 D) 6.0 E) 6.4

31. A health insurance policy for a family of three covers up to two claims per person during a year. The joint probability function for the number of claims by the three family members is $f(x,y,z) = \frac{6-x-y-z}{81}$, where x, y, z can each be 0, 1 or 2, and X, Y and Z are the number of claims for person 1, 2 and 3 in the family. Find the probability that the total number of claims for the family in the year is 2 given that person 1 has no claims for the year.

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{6}$ D) $\frac{1}{9}$ E) $\frac{1}{12}$

32. (SOA) Let T_1 and T_2 represent the lifetimes in hours of two linked components in an electronic device. The joint density function for T_1 and T_2 is uniform over the region defined by $0 \le t_1 \le t_2 \le L$ where L is a positive constant. Determine the expected value of the sum of the squares of T_1 and T_2.

A) $\frac{L^2}{3}$ B) $\frac{L^2}{2}$ C) $\frac{2L^2}{3}$ D) $\frac{3L^2}{4}$ E) L^2

33. (SOA) Two insurers provide bids on an insurance policy to a large company. The bids must be between 2000 and 2200. The company decides to accept the lower bid if the two bids differ by 20 or more. Otherwise, the company will consider the two bids further. Assume that the two bids are independent and are both uniformly distributed on the interval from 2000 to 2200. Determine the probability that the company considers the two bids further.

A) 0.10 B) 0.19 C) 0.20 D) 0.41 E) 0.60

34. The distribution of loss due to fire damage to a warehouse is:

Amount of Loss	Probability
0	0.900
500	0.060
1,000	0.030
10,000	0.008
50,000	0.001
100,000	0.001

Given that a loss is greater than zero, calculate the expected amount of the loss.

A) 290 B) 322 C) 1,704 D) 2,900 E) 32,222

35. (SOA) A family buys two policies from the same insurance company. Losses under the two policies are independent and have continuous uniform distributions on the interval from 0 to 10. One policy pays the amount of the loss in excess of 1 (if the loss is in excess of 1) and the other pays the loss in excess of 2 (if the loss is in excess of 2). The family experiences exactly one loss under each policy. Find the probability that the total benefit paid to the family does not exceed 5.

A) 0.13 B) 0.25 C) 0.30 D) 0.32 E) 0.42

36. Let X and Y be discrete random variables with joint probability function

$$p(x,y) = \begin{cases} \frac{2x+y}{12} & \text{for } (x,y) = (0,1), (0,2), (1,2), (1,3) \\ 0 & \text{otherwise} \end{cases}$$

Determine the marginal probability function for X.

A) $p(x) = \begin{cases} \frac{1}{6} & \text{for } x = 0 \\ \frac{5}{6} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$ B) $p(x) = \begin{cases} \frac{1}{4} & \text{for } x = 0 \\ \frac{3}{4} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$

C) $p(x) = \begin{cases} \frac{1}{3} & \text{for } x = 0 \\ \frac{2}{3} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$ D) $p(x) = \begin{cases} \frac{2}{9} & \text{for } x = 1 \\ \frac{3}{9} & \text{for } x = 2 \\ \frac{4}{9} & \text{for } x = 3 \\ 0 & \text{otherwise} \end{cases}$

E) $p(x) = \begin{cases} \frac{y}{12} & \text{for } x = 0 \\ \frac{2+y}{12} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$

37. (SOA) An insurance policy pays a total medical benefit consisting of two parts for each claim. Let X represent the part of the benefit that is paid to the surgeon, and let Y represent the part that is paid to the hospital. The variance of X is 5000, the variance of Y is 10,000, and the variance of the total benefit, $X+Y$, is 17,000. Due to increasing medical costs, the company that issues the policy decides to increase X by a flat amount of 100 per claim and to increase Y by 10% per claim. Calculate the variance of the total benefit after these revisions have been made.

A) 18,200 B) 18,800 C) 19,300 D) 19,520 E) 20,670

38. (SOA) Let X denote the size of a surgical claim and let Y denote the size of the associated hospital claim. An actuary is using a model in which $E(X)=5$, $E(X^2)=27.4$, $E(Y)=7$, $E(Y^2)=51.4$. and $Var(X+Y)=8$. Let $C_1=X+Y$ denote the size of the combined claims before the application of a 20% surcharge on the hospital portion of the claim, and let C_2 denote the size of the combined claims after the application of that surcharge. Calculate $Cov(C_1,C_2)$.
A) 8.80 B) 9.60 C) 9.76 D) 11.52 E) 12.32

39. In reviewing some data on smoking (X, number of packages of cigarettes smoked per year), income (Y, in thousands per year) and health (Z, number of visits to the family physician per year) for a sample of males, it is found that $E[X]=10$, $Var[X]=25$, $E[Y]=50$, $Var[Y]=100$, $E[Z]=6$, $Var[Z]=4$, and $Cov(X,Y)=-10$, $Cov(X,Z)=2.5$ (covariances).
Dr. N.A. Ively, a young statistician, attempts to describe the variable Z in terms of X and Y by the relation $Z=X+cY$, where c is a constant to be determined. Dr. Ively's methodology for determining c is to find the value of c for which $Cov(X,Z)$ remains equal to 2.5 when Z is replaced by $X+cY$. What value of c does Dr. Ively find?
A) 2.00 B) 2.25 C) 2.50 D) -2.00 E) -2.25

40. (SOA) An insurance policy is written to cover a loss X where X has density function

$$f(x)=\begin{cases} \frac{3}{8}x^2 & \text{for } 0\le x\le 2 \\ 0 & \text{otherwise.} \end{cases}$$

The time (in hours) to process a claim of size x, where $0\le x\le 2$, is uniformly distributed on the interval from x to $2x$. Calculate the probability that a randomly chosen claim on this policy is processed in three hours or more.
A) 0.17 B) 0.25 C) 0.32 D) 0.58 E) 0.83

41. (SOA) An insurance company sells two types of auto insurance policies: Basic and Deluxe. The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days. What is the probability that the next claim will be a Deluxe Policy claim?
A) 0.172 B) 0.223 C) 0.400 D) 0.487 E) 0.500

42. (SOA) The joint probability density for X and Y is

$$f(x,y)=\begin{cases} 2e^{-(x+2y)} & \text{for } x>0,\ y>0 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the variance of Y given that $X>3$ and $Y>3$.
A) 0.25 B) 0.50 C) 1.00 D) 3.25 E) 3.50

43. (SOA) The definition of Y, given X, is uniform on the interval $[0, X]$. The marginal density of X is

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the conditional density of X, given $Y = y$, where positive.

A) 1 B) 2 C) $2x$ D) $\frac{1}{y}$ E) $\frac{1}{1-y}$

44. (SOA) A man purchases a life insurance policy on his 40th birthday, The policy will pay 5000 only if he dies before his 50th birthday and will pay 0 otherwise. The length of lifetime, in years, of a male born the same year as the insured has the cumulative distribution function

$$F(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1 - e^{(1-1.1^t)/1000} & \text{otherwise} \end{cases}$$

Calculate the expected payment to the man under this policy.

A) 333 B) 348 C) 421 D) 549 E) 574

45. (SOA) The number of workplace injuries, N, occurring in a factory on any given day is Poisson distributed with mean λ. The parameter λ is a random variable that is determined my the level of activity in the factory, and is uniformly distributed on the interval $[0, 3]$.

Calculate $Var[N]$.

A) λ B) 2λ C) 0.75 D) 1.50 E) 2.25

46. (SOA) A fair die is rolled repeatedly. Let X be the number of rolls neede to obtain a 5 and Y the number of rolls needed to obtain a 6. Calculate $E(X|Y = 2)$.

A) 5.0 B) 5.2 C) 6.0 D) 6.6 E) 6.8

47. (SOA) Let X and Y be identically distributed independent random variables such that the moment generating function of $X + Y$ is

$$M(t) = 0.09e^{-2t} + 0.24e^{-t} + 0.34 + 0.24e^{t} + 0.09e^{2t} \text{ for } -\infty < t < \infty.$$

Calculate $P[X \le 0]$.

A) 0.33 B) 0.34 C) 0.50 D) 0.67 E) 0.70

48. (SOA) New dental and medical plan options will be offered to state employees next year. An actuary uses the following density function to model the joint distribution of the proportion X of state employees who will choose Dental Option 1 and the proportion Y who will choose Medical Option 1 under the new plan options:

$$f(x,y)=\begin{cases} 0.50, & \text{for } 0<x<0.5 \text{ and } 0<y<0.5 \\ 1.25, & \text{for } 0<x<0.5 \text{ and } 0.5<y<1 \\ 1.50, & \text{for } 0.5<x<1 \text{ and } 0<y<0.5 \\ 0.75, & \text{for } 0.5<x<1 \text{ and } 0.5<y<1 \end{cases}$$

Calculate $Var(Y|X=0.75)$ 0.

A) 0.00 B) 0.061 C) 0.076 D) 0.083 E) 0.141

49. The joint density function for the pair of random variables X and Y is

$$f(x,y)=\frac{1}{2}e^{-x}\cdot \sin y\ ,\ 0<x<\infty\ ,\ 0<y<\pi$$

Find $P[(X<1)\cap(Y<\frac{\pi}{2})]$.

A) $\frac{1-e^{-1}}{2}$ B) $\frac{e-1}{2}$ C) $\frac{2}{e-1}$ D) $\frac{2}{1-e^{-1}}$ E) $\frac{e}{\pi}$

50. (SOA) Let N_1 and N_2 represent the numbers of claims submitted to a life insurance company in April and May, respectively, The joint probability function of N_1 and N_2 is

$$p(n_1,n_2)=\begin{cases} \frac{3}{4}(\frac{1}{4})^{n_1-1}e^{-n_1}(1-e^{-n_1})^{n_2-1}, & \text{for } n_1=1,2,3,\ldots \text{ and } n_2=1,2,3,\ldots \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the expected number of claims that will be submitted to the company in May if exactly 2 claims were submitted in April.

A) $\frac{3}{16}(e^2-1)$ B) $\frac{3}{16}e^2$ C) $\frac{3e}{4-e}$ D) e^2-1 E) e^2

51. (SOA) A machine consists of two components, whose lifetimes have the joint density function

$$f(x,y)=\begin{cases} \frac{1}{50}, & \text{for } x>0,\ y>0,\ x+y<10 \\ 0, & \text{otherwise.} \end{cases}$$

The machine operates until both components fail.

Calculate the expected operational time of the machine.

A) 1.7 B) 2.5 C) 3.3 D) 5.0 E) 6.7

52. (SOA) A client spends X minutes in an insurance agent's waiting room and Y minutes meeting with the agent. The joint density function of X and Y can be modeled by

$$f(x,y)=\begin{cases}\frac{1}{800}\,e^{-\frac{x}{40}}e^{-\frac{y}{20}}, & \text{for } x>0,\ y>0\\ 0, & \text{otherwise.}\end{cases}$$

Which of the following expressions represents that probability that a client spends less than 60 minutes in the agent's office?

A) $\frac{1}{800}\int_0^{40}\int_0^{20} e^{-\frac{x}{40}}e^{-\frac{y}{20}}\,dy\,dx$ B) $\frac{1}{800}\int_0^{40}\int_0^{20-x} e^{-\frac{x}{40}}e^{-\frac{y}{20}}\,dy\,dx$

C) $\frac{1}{800}\int_0^{20}\int_0^{40-x} e^{-\frac{x}{40}}e^{-\frac{y}{20}}\,dy\,dx$ D) $\frac{1}{800}\int_0^{60}\int_0^{60} e^{-\frac{x}{40}}e^{-\frac{y}{20}}\,dy\,dx$

E) $\frac{1}{800}\int_0^{60}\int_0^{60-x} e^{-\frac{x}{40}}e^{-\frac{y}{20}}\,dy\,dx$

PROBLEM SET 8 SOLUTIONS

1. Let N denote the number that appears on the wheel, so that $P[N=1]=P[N=2]=P[N=3]=\frac{1}{3}$. Then, conditioning over N,

$P[W \leq 1.5]=P[W \leq 1.5|N=1] \times P[N=1]+]P[W \leq 1.5|N=2] \times P[N=2]$
$\quad + P[W \leq 1.5|N=3] \times P[N=3]$

If $N=1$ then $W=1$, so that $P[W \leq 1.5|N=1]=1$ and

if $N=2$ then $W=2$, so that $P[W \leq 1.5|N=2]=0$.

If $N=3$ then $W \sim N(3,1)$ so that

$P[W \leq 1.5|N=3]=P\left[\frac{W-3}{1} \leq \frac{1.5-3}{1}|N=3\right]=P[Z \leq -1.5]=0.07$

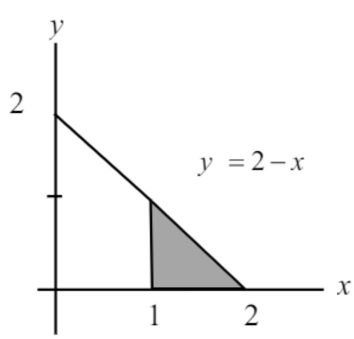
(Z has a standard normal distribution - the probability is found from the table).

Then, $P[W \leq 1.5]=1 \times \frac{1}{3}+0 \times \frac{1}{3}+0.07 \times \frac{1}{3}=0.357.$ Answer: C

2. This discrete distribution has the following 8 points and probabilities:

$(1,2)\,,\,\frac{1}{12}\,;\ (1,4)\,,\,\frac{1}{6}\,;\ (1,8)\,,\,\frac{1}{3}\,;\ (2,2)\,,\,\frac{1}{24}\,;\ (2,4)\,,\,\frac{1}{12}\,;\ (2,8)\,,\,\frac{1}{6}\,;$
$(4,4)\,,\,\frac{1}{24}\,;\ (4,8)\,,\,\frac{1}{12}$. The event $X+\frac{Y}{2} \leq 5$ occurs at the points
$(1,2)\,,\,(1,4)\,,\,(1,8)\,,\,(2,2)$ and $(2,4)$. The total probability of this event occurring is
$\frac{1}{12}+\frac{1}{6}+\frac{1}{3}+\frac{1}{24}+\frac{1}{12}=\frac{17}{24}$. Answer: E

3. $P[X>1] = \int_1^2 \int_0^{2-x} \frac{3}{4}x\,dy\,dx$
$= \int_1^2 \frac{3}{4}x(2-x)\,dx$
$= \int_1^2 \frac{3}{4}(2x-x^2)\,dx=\frac{1}{2}.$



Answer: D

4. It follows from the independence of X and Y that

$E[(X+1)^2(Y-1)^2] = E[(X+1)^2] \times E[(Y-1)^2]$

$E[(X+1)^2] = E[X^2+2X+1] = E[X^2]+2E[X]+1$ and since

$\sigma_X^2 = E[X^2]-(E[X])^2$, we have $E[X^2] = \sigma_X^2 + (E[X])^2 = \frac{3}{2}$ and then

$E[(X+1)^2] = E[X^2+2X+1] = \frac{3}{2}+2(1)+1 = \frac{9}{2}$.

In a similar way, $E[Y^2] = \sigma_Y^2 + (E[Y])^2 = 3$ and

$E[(Y-1)^2] = E[Y^2-2Y+1] = 3-2(-1)+1 = 6$

so that $E[(X+1)^2(Y-1)^2] = E[(X+1)^2] \times E[(Y-1)^2] = \frac{9}{2} \times 6 = 27$.

Note that we could also find $E[(X+1)^2]$ in the following way:

$(X+1)^2 = X^2+2X+1 = X^2-2X+1+4X = (X-1)^2+4X$, and then

$E[(X+1)^2] = E[(X-1)^2]+4E[X] = \sigma_X^2 + 4\mu_X = \frac{9}{2}$

(since $Var[X] = E[(X-\mu_X)^2]$, and $\mu_X = 1$). $E[(Y-1)^2]$ can be found in a similar way.

Answer: E

5. Since the joint density is a constant, say c, over the probability region, and since the total probability in the region must be 1, it follows that $c \times (\text{Region Area}) = 1$, so that

$c = \frac{1}{Region\ Area}$. The area of the region is the area of the 6 by 6 square minus the area of the upper right triangle. This is $36 - \frac{1}{2} \times 2 \times 2 = 34$, so that $c = \frac{1}{34}$.

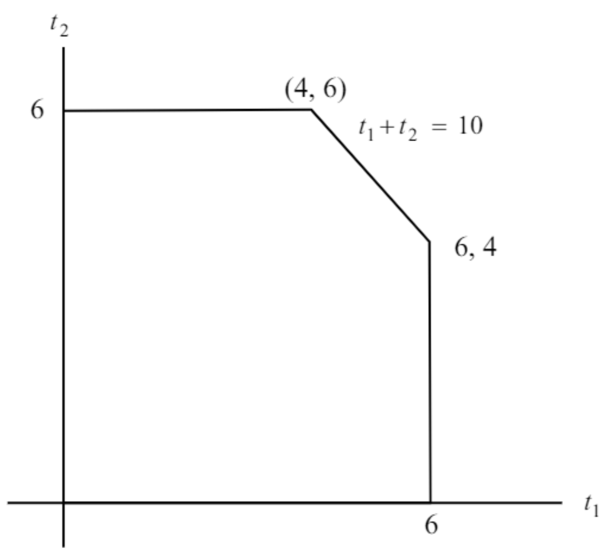
Then, $E[T_1+T_2] = \int_0^4 \int_0^6 (t_1+t_2) \times \frac{1}{34}\, dt_2\, dt_1 + \int_4^6 \int_0^{10-t_1} (t_1+t_2) \times \frac{1}{34}\, dt_2\, dt_1$

$\int_0^6 (t_1+t_2) \times \frac{1}{34}\, dt_2 = (6t_1+18) \times \frac{1}{34} \Rightarrow \int_0^4 (6t_1+18) \times \frac{1}{34}\, dt_1 = (48+72) \times \frac{1}{34}$

$\int_0^{10-t_1} (t_1+t_2) \times \frac{1}{34}\, dt_2 = \left[(10-t_1)t_1 + \frac{(10-t_1)^2}{2}\right] \times \frac{1}{34} = (50-\frac{1}{2}t_1^2) \times \frac{1}{34}$

$\rightarrow \int_4^6 (50-\frac{1}{2}t_1^2) \times \frac{1}{34}\, dt_1 = \frac{224}{3} \times \frac{1}{34}$.

Then, $E[T_1+T_2] = (48+72) \times \frac{1}{34} + \frac{224}{3} \times \frac{1}{34} = 5.73$.



Answer: C