PROBLEM SET 9 SOLUTIONS

1. We use the probability rule $Var[aX+bY+c] = a^2Var[X] + b^2Var[Y] + 2abCov[X,Y]$ where a, b, c are constants and X and Y are random variables.
Since X and Y are independent, we have $Cov[X,Y] = 0$.
Therefore, $Var[3X - Y - 5] = 3^2 \times Var[X] + (-1)^2 \times Var[Y] = 9 \times 1 + 1 \times 2 = 11$.
Answer: D

2. X_1, X_2 and X_3 are independent binomial random variables, all with the same value of p, and therefore, $S = X_1 + X_2 + X_3$ has a binomial distribution with parameters p and $n = n_1 + n_2 + n_3$. The probability function of S is
$P[S = s] = \binom{n}{s} p^s (1-p)^{n-s} = \binom{n_1+n_2+n_3}{s} p^s(1-p)^{n_1+n_2+n_3-s}$. Answer: A

3. The density function for Y is $f_Y(y)$. If we can find $F_Y(y)$, the cumulative distribution function for Y then $f_Y(y) = F'_Y(y)$. We can find $F_Y(y)$ from the relationship between Y and T and from $F_T(t)$ (the cdf of T).
$F_Y(y) = P[Y \le y] = P[T^2 \le y] = P[0 < T \le \sqrt{y}\,]$
(the description of $F_T(t)$ indicates that T is defined for only positive numbers).
Therefore, $F_Y(y) = F_T(\sqrt{y}) = 1 - (\frac{2}{\sqrt{y}})^2 = 1 - \frac{4}{y}$.
The density function for Y is $f_Y(y) = F'_Y(y) = \frac{4}{y^2}$. Answer: A

4. $M_{X+2Y}(t) = E[e^{t(X+2Y)}] = E[e^{tX} \times e^{2tY}] = E[e^{tX}] \times E[e^{2tY}] = M_X(t) \times M_Y(2t)$
$= exp(t^2 + 2t) \times exp[3(2t)^2 + 2t] = exp(13t^2 + 4t)$. Note that the equality
$E[e^{tX} \times e^{2tY}] = E[e^{tX}] \times E[e^{2tY}]$ follows from the independence of X and Y. Answer: E

5. $E[X_1] = \frac{\partial}{\partial t_1} M(t_1,t_2)\Big|_{t_1=t_2=0}$ and $E[X_2] = \frac{\partial}{\partial t_2} M(t_1,t_2)\Big|_{t_1=t_2=0}$
$\frac{\partial}{\partial t_1} M(t_1,t_2) = 0.1e^{t_1} + 0.4e^{t_1+t_2} \rightarrow E[X_1] = 0.5$,
$\frac{\partial}{\partial t_2} M(t_1,t_2) = 0.2e^{t_2} + 0.4e^{t_1+t_2} \rightarrow E[X_2] = 0.6$,
$\Rightarrow \quad E[2X_1 - X_2] = 2E[X_1] - E[X_2] = 0.4$. Answer: B

6. $F(v) = P[V \le v] = P[10{,}000e^R \le v] = P\big[R \le ln(\frac{v}{10{,}000})\big]$.
If X has a uniform distribution on the interval (a,b) then if $a < x < b$,
$P[X \le x] = \frac{x-a}{b-a}$. Since R is uniform on $(0.04, 0.08)$ it follows that
$P\big[R \le ln(\frac{v}{10{,}000})\big] = \frac{ln(\frac{v}{10{,}000}) - 0.04}{0.08-0.04} = 25\big[ln(\frac{v}{10{,}000}) - 0.04\big]$ if $0.04 < ln(\frac{v}{10{,}000}) < 0.08$
Answer: E

7. The distribution of $W = Y - X$ is discrete with possible values $0, \ -1, \ -2$, and 1.
The probabilities are $f_W(-2) = 0.2$ (this occurs only if $Y = 0$ and $X = 2$),
$f_W(-1) = 0.4$ $(Y = 0, \ X = 1)$, $f_W(0) = 0.2$ $(Y = 1, \ X = 1)$ and
$f_W(1) = 0.2$ $(Y = 1, \ X = 0)$. Then $E[W] = -0.6$, and $E[W^2] = 1.4$, so that
$Var[W] = 1.4 - (-.6)^2 = 1.04$. Answer: C

8. $W = X_1 - X_2$ has a normal distribution with a mean of $1 - 1 = 0$, and a variance of $1 + 1 = 2$. Then,
$E[c|X_1 - X_2|] = E[c|W|] = c\int_{-\infty}^{\infty} |w| \times f_W(w)\, dw$

$$= c\int_{-\infty}^{0}(-w) f_W(w)\, dw + \int_0^{\infty} w\, f_W(w)\, dw = 2c\int_0^{\infty} w\, f_W(w)\, dw$$

But $f_W(w) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \times e^{-w^2/4}$ (from the pdf for $N(\mu, \sigma^2)$), so that
$\int_0^{\infty} w\, f_W(w)\, dw = \int_0^{\infty} w \times \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \times e^{-w^2/4} dw = -\frac{1}{\sqrt{\pi}} \times e^{-w^2/4}\Big|_{w=0}^{w=\infty} = \frac{1}{\sqrt{\pi}}$
Thus, $2c \times \frac{1}{\sqrt{\pi}} = 1 \rightarrow c = \frac{\sqrt{\pi}}{2}$. Answer: E

9. As the sum of independent Poisson random variables, $W = X + Y + Z$ has a Poisson distribution with parameter $3 + 1 + 4 = 8$, so that
$P[W \leq 1] = P[W = 0] + P[W = 1] = e^{-8} + \frac{e^{-8}\cdot 8}{1!} = 9e^{-8}$. Answer: B

10. Let us denote the cumulative distribution function of Company I's monthly profit by
$F(x) = P[X \leq x]$, and let us denote Company II's density function and cumulative distribution function of monthly profit by $G(y) = P[Y \leq y]$ and $g(y)$, respectively.
Company II's monthly profit is $Y = 2X$. The cumulative distribution function for Company II's monthly profit is $G(y) = P[Y \leq y] = P[2X \leq y] = P[X \leq \frac{y}{2}] = \int_0^{y/2} f(x)\, dx$.
The density function for Company II's profit is then $G'(y) = \frac{d}{dy}\int_0^{y/2} f(x)\, dx = f(\frac{y}{2}) \times \frac{1}{2}$
(this uses the differentiation rule $\frac{d}{dy}\int_a^{h(y)} k(s)\, ds = k(h(y)) \times h'(y)$;
in this case, $h(y) = \frac{y}{2}$ and $k(s) = f(s)$).

Looking at the cdf of the new random variable is a standard method for determining the density function of a random variable that is defined in terms of or as a transformation of another random variable.
Answer: A

11. We first find the distribution function of Y, $F_Y(y)$. Then $f_Y(y) = F_Y'(y)$.
$F_Y(y) = P[Y \leq y] = P[10X^{0.8} \leq y] = P[X \leq (0.1y)^{1.25}] = 1 - e^{-(0.1y)^{1.25}}$.
Then, $f_Y(y) = \frac{d}{dy}[1 - e^{-(0.1y)^{1.25}}] = -e^{-(0.1y)^{1.25}} \times (-1.25(0.1y)^{0.25}(0.1))$

$$= 0.125(0.1y)^{0.25}e^{-(0.1y)^{1.25}}.$$

Answer: E

12. $E[3X+2Y-Z]=3E[X]+2E[Y]-E[Z]=3(1)+2(2)-1(3)=4$.
$Var[3X+2Y-Z]=9Var[X]+4Var[Y]+Var[Z]$
$+2(6Cov[X,Y]-3Cov[X,Z]-2Cov[Y,Z])=67.$ Answer: C

13. Let $W=2T_1$. Then the cdf of W is
$F_W(w)=P[W\le w]=P[2T_1\le w]=P[T_1\le\frac{w}{2}]=F_{T_1}(\frac{w}{2})=1-e^{-w/2}.$

Then the pdf of W is $f_W(w)=F'_W(w)=\frac{1}{2}e^{-w/2}.$

The density of $Y=2T_1+T_2=W+T_2$ can be found by convolution.
$f_Y(y)=\int_0^y f_W(w)\times f_{T_2}(y-w)\,dw=\int_0^y\frac{1}{2}e^{-w/2}\times e^{-(y-w)}\,dw=\frac{1}{2}e^{-y}\int_0^y e^{w/2}\,dw$
$=\frac{1}{2}e^{-y}(\frac{e^{y/2}-1}{1/2})=e^{-y/2}-e^{-y}$, $y>0.$

Alternatively, the density function of the joint distribution of T_1 and T_2 is
$f_{T_1,T_2}(s,t)=f_{T_1}(s)\times f_{T_2}(t)=e^{-s}\times e^{-t}$ (by independence).

Then with $Y=2T_1+T_2$
$P[Y\le y]=P[2T_1+T_2\le y]=\int_0^{y/2}\int_0^{y-2s}e^{-s}\times e^{-t}\,dt\,ds=\int_0^{y/2}e^{-s}\times(1-e^{-(y-2s)})\,ds$
$=\int_0^{y/2}e^{-s}\,ds-\int_0^{y/2}e^{-(y-s)})\,ds=1-e^{-y/2}-e^{-y}(e^{y/2}-1)=1+e^{-y}-2e^{-y/2}$
$\Rightarrow f_Y(y)=\frac{d}{dy}(1-e^{-y}-2e^{-y/2})=e^{-y/2}-e^{-y}$, $y>0.$ Answer: A

14. Since the second generator starts after the first one fails, the total time that the generators are working is the sum of the two separate working times: $T=T_1+T_2$.
The length of time the second generator operates is not related to how long the first generated operated, so T_1 and T_2 are independent. Therefore,

$Var[T_1+T_2]=Var[T_1]+Var[T_2]$
(in general, $Var[X+Y]=Var[X]+Var[Y]+2Cov[X,Y]$, but if X and Y are independent, then $Cov[X,Y]=0$).

Each of T_1 and T_2 has an exponential distribution with mean 10. The variance of an exponential random variable is the square of the mean, so that each of T_1 and T_2 has a variance of 100. Therefore, $Var[T]=200.$ Answer: E

15. The random variable X is defined to be $X=\frac{C}{P}$, where C (claims) has an exponential distribution with mean 1, and P (premiums) has an exponential distribution with mean 2. If we can find $F_X(x)$, the distribution function of X, then the density function is $f_X(x)=\frac{d}{dx}F_X(x)$.
$F_X(x)=P[X\le x]=P[\frac{C}{P}\le x]=P[C\le Px].$

The 2-dimensional (C,P) region described by this event (with P on the horizontal axis and C on the vertical axis) is all points (p,c) (≥ 0) below the line $c=px$.

The density function of C is $f_C(c) = e^{-c}$ (exponential with mean 1) and the density function of P is $f_P(p) = \frac{1}{2}e^{-p/2}$ (exponential with mean 2). Since P and C are independent, the density function of the joint distribution of P and C is
$f(p,c) = f_P(p) \cdot f_C(c) = \frac{1}{2}e^{-p/2} \cdot e^{-c}$. The probability $P[C \leq Px]$ is

$$\int_0^\infty \int_0^{px} \tfrac{1}{2}e^{-p/2} \cdot e^{-c}\,dc\,dp = \int_0^\infty \tfrac{1}{2}e^{-p/2} \cdot (1 - e^{-px})\,dp$$
$$= \int_0^\infty \tfrac{1}{2}e^{-p/2}\,dp - \int_0^\infty \tfrac{1}{2}e^{-p(x+\frac{1}{2})}\,dp = 1 - \tfrac{1}{2x+1}.$$

This is $F_X(x)$, so that the density function of X is
$f_X(x) = \frac{d}{dx}[1 - \frac{1}{2x+1}] = \frac{2}{(2x+1)^2}$. Answer: B

16. Since T has a uniform distribution on the interval from 8 to 12, T's distribution function is
$F_T(t) = P[T \leq t] = \frac{t-8}{12-8}$, and $P[T \geq t] = \frac{12-t}{12-8}$, for $8 \leq t \leq 12$.
$R = \frac{10}{T}$; $F_R(r) = P[R \leq r] = P[\frac{10}{T} \leq r] = P[T \geq \frac{10}{r}] = \frac{12-\frac{10}{r}}{12-8} = 3 - \frac{2.5}{r}$.
The density function of R is $f_R(r) = F_R'(r) = \frac{d}{dr}F_R(r) = \frac{d}{dr}[3 - \frac{2.5}{r}] = \frac{2.5}{r^2}$.
Alternatively, $f_T(t) = \frac{1}{4}$ for $8 \leq t \leq 12$. $R = \frac{10}{T} = g(T) \to T = \frac{10}{R} = h(R)$
$\to f_R(r) = f_T(h(r)) \times |h'(r)| = \frac{1}{4} \cdot |\frac{-10}{r^2}| = \frac{2.5}{r^2}$. Answer: E

17. The standard approximation to the sum (total) of a collection of independent random variables is the normal approximation. The total contribution is $T = C_1 + C_2 + \cdots + C_{2025}$, the sum of the 2025 contributions. C_i is the amount of the i-th contribution, the C_i's are mutually independent, and each has mean $E[C_i] = 3125$ and variance $Var[C_i] = (250)^2$.
The mean and variance of T are $E[T] = \sum_{i=1}^{2025} E[C_i] = (2025)(3125) = 6,328,125$ and
$Var[T] = \sum_{i=1}^{2025} Var[C_i] = 2025 \times 250^2 = 126,562,500$.
We will denote the 90th percentile of T by p. We find the approximate 90th percentile of T by applying the normal approximation to T. We wish to find p so that $P[T \leq p] = 0.9$.
We standardize the probability: $P[T \leq p] = P\left[\frac{T-6,328,125}{\sqrt{126,562,500}} \leq \frac{p-6,328,125}{\sqrt{126,562,500}}\right] = 0.90$.

$\frac{T-6,328,125}{\sqrt{126,562,500}}$ is approximately standard normal (mean 0, variance 1), so that
$\frac{p-6,328,125}{\sqrt{126,562,500}}$ is the 90-th percentile of the standard normal distribution. From the table for the standard normal distribution, we see that $\Phi(1.282) = 0.90$. Therefore we have
$\frac{p-6,328,125}{\sqrt{126,562,500}} = 1.282$, from which we get $p = 6,342,547.5$. Answer: C

18. For policyholder i, let X_i be the number of claims filed in the year, $i = 1, 2, ..., 1250$.
Each X_i is Poisson with a mean of 2, and therefore has variance of 2 also; $E[X_i] = 2$, $Var[X_i] = 2$. The total number of claims in the year is $T = \sum_{i=1}^{1250} X_i$. ince the X_i's are mutually independent, the distribution of T is approximately normal. The mean of T is
$E[T] = E[\sum_{i=1}^{1250} X_i] = \sum_{i=1}^{1250} E[X_i] = 1250 \times 2 = 2500$, and the variance of T is
$Var[T] = Var[\sum_{i=1}^{1250} X_i] = \sum_{i=1}^{1250} Var[X_i] = 1250 \times 2 = 2500$ (since the X_i's are independent, there are no covariances between X_i's). We wish to find $P[2450 \le T \le 2600]$, by using the normal approximation for T. Applying the normal approximation we get

$$P[2450 \le T \le 2600] = P\left[\frac{2450-E[T]}{\sqrt{Var[T]}} \le \frac{T-E[T]}{\sqrt{Var[T]}} \le \frac{2600-E[T]}{\sqrt{Var[T]}}\right]$$
$$= P\left[\frac{2450-2500}{\sqrt{2500}} \le \frac{T-2500}{\sqrt{2500}} \le \frac{2600-2500}{\sqrt{2500}}\right] = P[-1 \le Z \le 2]$$
$= \Phi(2) - [1 - \Phi(1)] = 0.9772 - (1 - 0.8413) = 0.8185$ (from the normal the table provided with the exam). Answer: B

19. The sum of independent Poisson random variables is also Poisson, so that the number of claims occurring in a 5 day week has a Poisson distribution with 6 claims expected. Then

$$P[X \ge 3] = 1 - P[X = 0] - P[X = 1] - P[X = 2]$$
$$= 1 - e^{-6} - e^{-6} \times \frac{6}{1!} - e^{-6} \times \frac{6^2}{2!} = 1 - 0.0620 = 0.938.$$
Answer: C

20. Let T_i represent the time until a catastrophe occurs on policy i, and let T represent the time until the first catastrophe occurs. Then

$$P[T > t] = P[\text{All } T_i > t] = P[(T_1 > t) \cap (T_2 > t) \cap \cdots \cap (T_n > t)]$$
$= P[T_1 > t] \times P[T_2 > t] \times \cdots \times P[T_n > t]$ (this last equality follows from the independence of the T_i's). From the exponential distribution, we have $P[T_i > t] = e^{-t/\alpha}$, so that $P[T > t] = (e^{-t/\alpha})^n = e^{-tn/\alpha}$, thus T has an exponential distribution with mean α/n. Answer: B

21. For any given round age X_i, the error E_i is uniform between -2.5 and 2.5.
Therefore, $E[E_i] = 0$ and $Var[E_i] = \frac{25}{12}$ (the variance of the uniform distribution on the interval from a to b has a mean of $\frac{a+b}{2}$ and a variance of $\frac{(b-a)^2}{12}$).
The total error in 48 independent rounded ages is $\sum_{i=1}^{48} E_i$, which has a mean of 0, and variance $48(\frac{25}{12}) = 100$. The mean of the errors in the 48 rounded ages, $\overline{E} = \frac{1}{48}\sum_{i=1}^{48} E_i$ has expected value 0 and variance $Var[\overline{E}] = (\frac{1}{48})^2 \times 100$. Using the normal approximation for the distribution of E (since it is the sum of a relatively large number of independent and identically distributed random variables) it follows that $\overline{E}$ has an approximate normal distribution, and then

$P[|\bar{E}| < 0.25] = P[-0.25 < \bar{E} < 0.25] = P\left[\frac{-.25-E(\bar{E})}{\sqrt{Var(\bar{E})}} < \frac{\bar{E}-E(\bar{E})}{\sqrt{Var(\bar{E})}} < \frac{0.25-E(\bar{E})}{\sqrt{Var(\bar{E})}}\right]$

$= P[-1.2 < Z < 1.2]$, where Z has a standard normal distribution.

$P[-1.2 < Z < 1.2] = P[Z < 1.2] - P[Z > 1.2] = 2P[Z < 1.2] - 1$

$= 2 \times 0.8849 - 1 = 0.7698.$ Answer: D

22. For a given new hire, the number of pensions N that the city will provide at retirement is either 0, 1 or 2, with probabilities $P[N=0] = 0.6$ (no longer with the police force), $P[N=1] = 0.4 \times 0.25 = 0.1$ (stays with police force and is not married),

$P[N=2] = 0.4 \times 0.75 = 0.3$ (stays with the force and is married).

The mean of N is $1 \times 0.1 + 2 \times 0.3 = 0.7$, and the variance is

$E[N^2] - (E[N])^2 = [1^2 \times 0.1 + 2^2 \times 0.3] - (0.7)^2 = 0.81.$

The number of pensions provided by the city for 100 (independent) new hires is

$T = N_1 + N_2 + \cdots + N_{100}$. We can use the normal approximation for T.

$E[T] = 100E[N] = 70$, $Var[T] = 100Var[N] = 81$. With integer correction we want

$P[T \le 90.5] = P\left[\frac{T-70}{\sqrt{81}} \le \frac{90.5-70}{\sqrt{81}}\right] = P[Z \le 2.28] = \Phi(2.28) = 0.9887.$ Answer: E

23. Let X denote the amount paid out on a particular policy. Then

$E[X] = E[X|\text{no claim}] \times P[\text{no claim}] + E[X|\text{claim occurs}] \times P[\text{claim}]$

$= 0 \times 0.99 + 2000 \times 0.01 = 20$, and

$E[X^2] = E[X^2|\text{no claim}] \times P[\text{no claim}] + E[X^2|\text{claim occurs}] \times P[\text{claim}]$

$= 0^2 \times 0.99 + E[X^2|\text{claim occurs}] \times 0.01.$

However, $1000^2 = Var[X|\text{claim occurs}] = E[X^2|\text{claim occurs}] - (E[X|\text{claim occurs}])^2$

$= E[X^2|\text{claim occurs}] - (2000)^2 \Rightarrow E[X^2|\text{claim occurs}] = 5{,}000{,}000$

so that $E[X^2] = 0^2 \times 0.99 + 5{,}000{,}000 \times 0.01 = 50{,}000$

and then $Var[X] = 50{,}000 - 20^2 = 49{,}600$. The variance on 1000 independent policies is 49,600,000, and the standard deviation is $\sqrt{49{,}600{,}000} = 7{,}043$. Answer: B

24. Suppose the 25 random claim amounts are $X_1, X_2, \ldots, X_{25}$, where each X_i has a normal distribution with mean 19,400 and standard deviation 5,000 (variance 25,000,000). Since the claims are randomly chosen, they are independent of one another. The sum of normal random variables is normal and multiplying a normal random variable by a constant results in a normal random variable. Therefore the average of the claims $A = \frac{1}{25}(X_1 + X_2 + \cdots + X_{25})$ has a normal distribution. Using the basic rules for expected value, we get the mean of A, $E[A] = \frac{1}{25} \times E[X_1 + X_2 + \cdots + X_{25}] = \frac{1}{25} \times (E[X_1] + E[X_2] + \cdots + E[X_{25}])$

$= \frac{1}{25} \times 25 \times 19{,}400 = 19{,}400.$

Since the X_i's are mutually independent, they have covariances of 0, and we get the variance of A,

$Var[A] = Var[\frac{1}{25} \times (X_1 + X_2 + \cdots + X_{25})] = (\frac{1}{25})^2 \times Var[X_1 + X_2 + \cdots + X_{25}]$

$= (\frac{1}{25})^2 \times (Var[X_1] + Var[X_2] + \cdots + Var[X_{25}]) = (\frac{1}{25})^2 \times 25 \times 25{,}000{,}000 = 1{,}000{,}000$

Therefore, A has a normal distribution with mean 19,400 and variance 1,000,000.

Then, $P[A > 20,000] = P\left[\frac{A-19,400}{\sqrt{1,000,000}} > \frac{20,000-19,400}{\sqrt{1,000,000}}\right] = P[Z > 0.6]$, where Z has a standard normal distribution. From the standard normal table distributed with the exam, we have

$P[Z > 0.6] = 1 - \Phi(0.6) = 1 - 0.7257 = 0.2743$. Answer: C

25. S is a mixture of three components:

(i) the constant 0, with probability $\frac{1}{2}$

(ii) exponential distribution X_1 with mean 5, probability $\frac{1}{3}$ and

(iii) exponential distribution X_2 with mean 8, probability $\frac{1}{6}$

For the exponential distribution with mean μ, the cdf is $F(t) = 1 - e^{-t/\mu}$. Then,

$$P[4 < S < 8] = \tfrac{1}{2} \times 0 + \tfrac{1}{3} \times P[4 < X_1 < 8] + \tfrac{1}{6} \times P[4 < X_2 < 8]$$
$$= \tfrac{1}{3} \times [e^{-4/5} - e^{-8/5}] + \tfrac{1}{6}[e^{-4/8} - e^{-8/8}] = .123.$$ Answer: C

26. (SOA) Suppose that n bulbs are purchased. Then the total lifetime of the bulbs will be $T_n = X_1 + X_2 + \cdots + X_n$, which has a normal distribution with mean $3n$ and variance n. The probability that total lifetime will be at least 40 is

$P[T_n \geq 40] = P\left[\frac{T_n - 3n}{\sqrt{n}} \quad \frac{40-3n}{\sqrt{n}}\right] = 1 - \Phi\left(\frac{40-3n}{\sqrt{n}}\right)$.

We want this probability to be at least .9772. Trial and error using the possible answers results in

A) $n = 14 \rightarrow 1 - \Phi\left(\frac{40-42}{\sqrt{14}}\right) = 1 - \Phi(-0.53) = \Phi(0.53) = 0.7107$,

B) $n = 16 \rightarrow 1 - \Phi\left(\frac{40-48}{\sqrt{16}}\right) = 1 - \Phi(-2.0) = \Phi(2.0) = 0.9772$.

The probability is reached with $n = 16$. Answer: B

27. The median of P is the midpoint of the uniform interval, 1.5,

The median of Q is m_Q, where $P[Q \leq m_Q] = 0.5$.

But $P[Q \leq m_Q] = P[10^Q \leq 10^{m_Q}] = \frac{10^{m_Q}-10}{100-10} = 0.5 \rightarrow m_Q = 1.74$.

Then, $m_P - m_Q = -0.24$. Answer: E

28. We assume that the exponential distribution used by the actuary 10 years ago for the claim amount X had a parameter λ. Then, 10 years ago, $P[X < 1000] = 1 - e^{-1000\lambda} = 0.25 \rightarrow e^{-1000\lambda} = 0.75$.

If Y denotes the random variable for a claim amount today, then $Y = 2X$, since every claim made today is twice the size of a similar claim made ten years ago. Then

$$P[Y < 1000] = P[2X < 1000] = P[X < 500] = 1 - e^{-500\lambda}$$
$$= 1 - (e^{-1000\lambda})^{1/2} = 1 - (0.75)^{1/2} = 0.134$$ Answer: C

29. C as a mixture of two distributions, based on the two groups. Good drivers are group 1 with claim distribution X_1 and bad drivers are group 2 with claim distribution X_2. The mixing factors are $\alpha_1 = .6$ (60% of drivers are classified as good) and $\alpha_2 = .4$. Then, moments of C are the "weighted" moments of X_1 and X_2. Thus, $E[C] = 0.6 \times E[X_1] + 0.4 \times E[X_2] = 0.6 \times 1400 + 0.4 \times 2000 = 1,640$.

Since $Var[X_1] = E[X_1^2] - (E[X_1])^2$, and since we are given $Var[X_1] = 40,000$ and

$E[X_1] = 1400$, it follows that $E[X_1^2] = 2,000,000$, and in a similar way we get $E[X_2^2] = 4,250,000$. Then, $E[C^2] = 0.6 \times E[X_1^2] + 0.4 \times E[X_2^2] = 2,900,000$

Finally, $Var[C] = E[C^2] - (E[C])^2 = 2,9000,000 - (1,640)^2 = 210,400$. Answer: D

30. We wish to find $P[N_2 = 1|N_1 = 1]$. Using the definition of conditional probability, we have $P[N_2 = 1|N_1 = 1] = \frac{P[N_2=1 \cap N_1=1]}{P[N_1=1]}$. We find the numerator and denominator by conditioning on the value of θ (since we don't know θ).

$$\begin{aligned} P[N_1 = 1] &= P[N_1 = 1 \cap \theta = 0.6] + P[N_1 = 1 \cap \theta = .1] \\ &= P[N_1 = 1|\theta = 0.6] \times P[\theta = 0.6] + P[N_1 = 1|\theta = 0.1] \times P[\theta = 0.1] \\ &= e^{-0.6} \times 0.6 \times 0.1 + e^{-.1} \times 0.1 \times 0.9 \end{aligned}$$

$$\begin{aligned} &P[N_1 = \ 1 \cap N_2 = 1] = P[N_1 = 1 \cap N_2 = 1 \cap \theta = 0.6] + P[N_1 = 1 \cap N_2 = 1 \cap \theta = 0.1] \\ &= P[N_1 = 1 \cap N_2 = 1|\theta = 0.6] \times P[\theta = 0.6] + P[N_1 = 1 \cap N_2 = 1|\theta = 0.1] \times P[\theta = 0.1] \\ &\quad = P[N_1 = 1|\theta = 0.6] \times P[N_2 = 1|\theta = 0.6] \times P[\theta = 0.6] \\ &\qquad + P[N_1 = 1|\theta = 0.1] \times P[N_2 = 1|\theta = 0.1] \times P[\theta = 0.1] \\ &\quad = \ e^{-0.6} \times 0.6 \times e^{-.6} \times 0.6 \times 0.1 + e^{-0.1} \times 0.1 \times e^{-0.1} \times 0.1 \times 0.9) \end{aligned}$$

Note that we have used independence of N_1 and N_2 given $\theta = 0.6$ and also given $\theta = 0.1$.

Then $P[N_2 = 1|N_1 = 1] = \frac{e^{-0.6} \times 0.6 \times e^{-0.6} \times 0.6 \times 0.1 + e^{-0.1} \times 0.1 \times e^{-0.1} \times 0.1 \times 0.9}{e^{-0.6} \times 0.6 \times 0.1 + e^{-0.1} \times 0.1 \times .9} = 0.159$. Answer: C

31. $T = X_1 + Y_1 + X_2 + Y_2 + \cdots + X_{100} + Y_{100} = T_1 + T_2 + \cdots + T_{100}$
Since the individuals are randomly selected, they are independent of one another, the T_i's are independent of one another. $E[T_i] = E[X] + E[Y] = 50 + 20 = 70$,
$Var[T_i] = Var[X] + Var[Y] + 2Cov[X, Y] = 50 + 30 + 2(10) = 100$.
$E[T] = 100E[T_i] = 7000$, $Var[T] = 100Var[T_i] = 10,000$.
We apply the normal approximation to T to get
$P[T < 7100] = P\left[\frac{T-7000}{\sqrt{10,000}} < \frac{7100-7000}{\sqrt{10,000}}\right] = P[Z < 1] = \Phi(1) = 0.8413$. Answer: B

32. If X denotes the actual loss, then the pdf of X is $f_X(t) = \frac{1}{\lambda}e^{-t/\lambda}$, $t > 0$.

The moment generating function is

$M_X(r) = E[e^{rX}] = \int_0^\infty e^{rt} \times \frac{1}{\lambda}e^{-t/\lambda}\,dt = \frac{1}{\lambda}\int_0^\infty e^{-(\frac{1}{\lambda}-r)t}dt = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}-r} = \frac{1}{1-\lambda r}$

The amount paid by the insurer is $Y = 0.75X$, and the moment generating function of Y is

$M_Y(s) = E[e^{sY}] = E[e^{s(0.75X)}] = E[e^{0.75sX}] = M_X(0.75s) = \frac{1}{1-0.75s\lambda}$. Answer: D

33. This is the pdf for an exponential distribution with a mean of 1000. The expected claim per policy is 1000, and the variance is 1000^2. The premium collected is 1100 per policy. For 100 policies, a total of 110,000 is collected in premium. The total claim is

$W = X_1 + X_2 + \cdots + X_{100}$, and $E[W] = 100E[X] = 100(1000) = 100,000$,

and $Var[W] = 100Var[X] = 100 \times 1000^2$. The central limit theorem suggests that W has an approximately normal distribution. Thus,

$$P[W > 110,000] = P\left[\frac{W-E[W]}{\sqrt{Var[W]}} > \frac{110,000-E[W]}{\sqrt{Var[W]}}\right] = P\left[Z > \frac{110,000-100,000}{\sqrt{100\cdot 1000^2}}\right]$$
$$= P[Z > 1] = 1 - 0.8413 = 0.1587$$

(Z has an approximately standard normal distribution). Answer: B

34. Current method has pdf $g(t) = 1$ for $1 < t < 2$. The cdf of this uniform distribution is

$G(t) = P[T \le t] = t - 1$ for $1 < t < 2$, and $G(t) = 1$ for $t \ge 2$.

The new method claim processing time is $U = ln\,T$, where T is uniformly distributed from 1 to 2. The range for U is $0 = ln\,1 \le u \le ln\,2$. The cdf of U is

$F(u) = P[U \le u] = P[ln\,T \le u] = P[T \le e^u] = e^u - 1$ for $0 < u < ln\,2$,

and $f(u) = 0$ for $u \ge ln\,2$. The pdf of U is $f(u) = F'(u) = e^u$ for $0 < u < ln\,2$, and $f(u) = 0$ otherwise. Answer: E

35. The correlation coefficient between X and Y is equal to $\frac{Cov[X,Y]}{\sqrt{Var[X]}\sqrt{Var[Y]}}$.

Therefore, $0.75 = \frac{Cov[X,Y]}{\sqrt{Var[X]}\times\sqrt{Var[Y]}} = \frac{Cov[X,Y]}{\sqrt{1}\times\sqrt{2}} \Rightarrow Cov[X,Y] = 0.75\sqrt{2}$.

Then, $Var[X+2Y] = Var[X] + 2^2Var[Y] + 2(2)Cov[X,Y]$
$= 1 + 8 + 3\sqrt{2} = 9 + 3\sqrt{2}$ Answer: D

36. $X_i =$ amount of claim i, $i = 1,2,3$

Largest claim is $Y = Max\{X_1, X_2, X_3\}$

The density function of Y is the derivative of the distribution function of Y,

$f_Y(y) = F'_Y(y)$. The distribution function of Y can be found as follows.

$$P[Y \le y] = P[\text{each of } X_1, X_2, X_3 \text{ is } \le y]$$
$$= P[(X_1 \le y) \cap (X_2 \le y) \cap (X_3 \le y)]$$
$$= P[(X_1 \le y)] \cdot P[(X_2 \le y) \cdot P[(X_3 \le y)]\,,\ y \text{ is measured in thousands.}$$

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The last is equality follows from the independence of X_1, X_2 and X_3
($P[A \cap B] = P[A] \cdot P[B]$ for independent events).
$P[X \leq y] = \int_1^y \frac{3}{x^4}\,dx = 1 - \frac{1}{y^3}$ (since each X must be ≥ 1, the same is true for Y).
Then $F_Y(y) = P[Y \leq y] = (1 - \frac{1}{y^3})^3$, from which we get
$f_Y(y) = 3(1 - \frac{1}{y^3})^2 \times \frac{3}{y^4} = 9\left[\frac{1}{y^4} - \frac{2}{y^7} + \frac{1}{y^{10}}\right]$.
Then, $E[Y] = \int_1^\infty y \times f_Y(y)\,dy = \int_1^\infty y \times 9\left[\frac{1}{y^4} - \frac{2}{y^7} + \frac{1}{y^{10}}\right] dy$

$$= 9\int_1^\infty \left[\frac{1}{y^3} - \frac{2}{y^6} + \frac{1}{y^9}\right] dy = 9\left[\frac{1}{2} - 2 \times \frac{1}{5} + \frac{1}{8}\right] = 2.025$$

The measurement is in thousands, so the expected value of the largest claim is 2025.
Answer: A

37. Suppose the bids are X_1, X_2, X_3 and X_4. To find the expected value of the highest bid we need to find the distribution of the highest bid. Let us call the highest bid Y. We can find the distribution function of Y as follows. In order for it to be true that $Y \leq y$, it must be true that $X_1 \leq y$ and $X_2 \leq y$ and $X_3 \leq y$ and $X_4 \leq y$. Thus, $P[Y \leq y] = P[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y) \cap (X_4 \leq y)]$
Since X_1, X_2, X_3 and X_4 are mutually independent (we are given that the bids are independent), it follows that the events $X_1 \leq y$ and $X_2 \leq y$ and $X_3 \leq y$ and $X_4 \leq y$ are mutually independent. Therefore,

$$P[(X_1 \leq y) \cap (X_2 \leq y) \cap (X_3 \leq y) \cap (X_4 \leq y)$$
$$= P[(X_1 \leq y)] \times P[(X_2 \leq y)] \times P[(X_3 \leq y)] \times P[(X_4 \leq y)]$$
$$= F(y) \times F(y) \times F(y) \times F(y) = [\tfrac{1}{2}(1 + sin\,\pi x)]^4 = P[Y \leq y] = F_Y(y)$$

The density function for Y is then $f_Y(y) = F_Y'(y) = 4[\frac{1}{2}(1 + sin\,\pi y)]^3(\frac{1}{2})(\pi\,cos\,\pi y)$.
The range for Y is between $\frac{3}{2}$ and $\frac{5}{2}$ since the maximum bid must be one of the X's, and all X's are in that range. The mean of Y is then
$E[Y] = \int_{3/2}^{5/2} y \times f_Y(y)\,dy = \int_{3/2}^{5/2} y \times \frac{1}{4} \times (1 + sin\,\pi y)^3 \times (\pi\,cos\,\pi y)\,dy$ Answer E.

38. The device fails as soon as either component fails. The probability of failure within the first hour is $P[(X \leq 1) \cup (Y \leq 1)]$. There are a couple of ways in which this can be found.
We can use the probability rule

$$P[(X \leq 1) \cup (Y \leq 1)] = P[X \leq 1] + P[Y \leq 1] - P[(X \leq 1) \cap (Y \leq 1)]$$

but this will require three separate double integrals (although the first two are equal because of the symmetry of the distribution).
Alternatively, we can use DeMorgan's rule, $P[A \cup B] = 1 - P[A' \cap B']$, so that
$P[(X \leq 1) \cup (Y \leq 1)] = 1 - P[(X > 1) \cap (Y > 1)]$.
Since both X and Y are between 0 and 3, we get
$P[(X > 1) \cap (Y > 1)] = \int_1^3\int_1^3 (\frac{x+y}{27})\,dy\,dx = \frac{1}{27} \times \int_1^3 [(xy + \frac{1}{2}y^2)\Big|_{y=1}^{y=3}]\,dx$

$$= \frac{1}{27} \times \int_1^3 (2x + 4)\,dx = \frac{1}{27} \times (x^2 + 4x)\Big|_{x=1}^{x=3} = \frac{16}{27}$$

Then, $P[(X \leq 1) \cup (Y \leq 1)] = 1 - \frac{16}{27} = \frac{11}{27} = 0.407$. Answer: B

39. $1 - P\left[Y_1 < \frac{1}{2} < Y_n\right] = P\left[Y_n \leq \frac{1}{2}\right] + P\left[Y_1 \geq \frac{1}{2}\right]$.

$P\left[Y_n \leq \frac{1}{2}\right] = P\left[\text{all } X_i\text{'s} \leq \frac{1}{2}\right] = \left(P\left[X \leq \frac{1}{2}\right]\right)^n = \frac{1}{4^n}$

Similarly, $P\left[Y_1 \geq \frac{1}{2}\right] = P\left[\text{all } X_i\text{'s} \geq \frac{1}{2}\right] = \left(P\left[X \geq \frac{1}{2}\right]\right)^n = (\frac{3}{4})^n$.

Thus, $P\left[Y_1 < \frac{1}{2} < Y_n\right] = 1 - \frac{1}{4^n} - \frac{3^n}{4^n} = \frac{4^n - 1 - 3^n}{4^n}$. Answer: C

40. Since the X_i's are independent and all X_i's have the same variance, say $Var[X]$. It follow that $Var[Y] = \sum_{i=1}^{5} Var[X_i] = 5Var[X]$, so that $Var[X] = \frac{1}{5}Var[Y]$

From the mgf of Y we get $E[Y] = M_Y'(0)$ and $E[Y^2] = M_Y''(0)$

$M_Y'(t) = e^{15e^t - 15} \times 15e^t$, $M_Y''(t) = e^{15e^t - 15} \times (15e^t)^2 + e^{15e^t - 15} \times 15e^t$

so that $E[Y] = M_Y'(0) = e^0 \times 15e^0 = 15$ and

$E[Y^2] = M_Y''(0) = e^0 \times (15e^0)^2 + e^0 \times 15e^0 = 240$

Then $Var[Y] = E[Y^2] - (E[Y])^2 = 240 - (15)^2 = 15$ and $Var[X] = \frac{1}{5}Var[Y] = 3$.

There are two alternative ways to find the variance of Y. The first alternative uses the fact that $Var[Y] = \frac{d^2}{dt^2}(ln\, M_Y(t))\Big|_{t=0}$. In this case, $ln\, M_Y(t) = 15e^t - 15$ and $\frac{d^2}{dt^2}(ln\, M_Y(t)) = \frac{d^2}{dt^2}(15e^t - 15) = 15e^t \rightarrow \frac{d^2}{dt^2}(ln\, M_Y(t))\Big|_{t=0} = 15e^0 = 15$.

The second alternative requires making the observation that if Z has a Poisson distribution with mean λ, then the mgf of Z is $M_Z(t) = e^{\lambda(e^t - 1)}$. In this case it can be seen that Y has a Poisson distribution with $\lambda = 15$. Therefore the variance (and mean) of Y is 15 and the variance of each X is $Var[X] = \frac{1}{5} \times Var[Y] = 3$. Answer: B

41. $M(t_1, t_2) = E[e^{t_1 W + t_2 Z}] = E[e^{t_1(X+Y) + t_2(Y-X)}] = E[e^{(t_1 - t_2)X + (t_1 + t_2)Y}]$

$= E[e^{(t_1 - t_2)X} \times e^{(t_1 + t_2)Y}] = E[e^{(t_1 - t_2)X}] \times E[e^{(t_1 + t_2)Y}]$

(this equality follows from the independence of X and Y)

$= M_X(t_1 - t_2) \times M_Y(t_1 + t_2) = e^{(t_1 - t_2)^2/2} \times e^{(t_1 + t_2)^2/2} = e^{t_1^2 + t_2^2}$.

Answer: E

SECTION 10 - REVIEW OF RISK MANAGEMENT CONCEPTS

LOSS DISTRIBUTIONS AND INSURANCE

Loss and insurance: When someone is subject to the risk of incurring a financial loss, the loss is generally modeled using a random variable or some combination of random variables. The loss is often related to a particular time interval. For example, an individual may own property that might suffer some damage during the following year. Someone who is at risk of a financial loss may choose some form of **insurance** protection to reduce the impact of the loss. An **insurance policy** is a contract between the party that is at risk (the **policyholder**) and an **insurer**. This contract generally calls for the policyholder to pay the insurer some specified amount, the **insurance premium**, and in return, the insurer will reimburse certain **claims** to the policyholder. A claim is all or part of the loss that occurs, depending on the nature of the insurance contract.

Modeling a loss random variable: There are a few ways of modeling a random loss/claim for a particular insurance policy, depending on the nature of the loss. Unless indicated otherwise, we will assume the amount paid to the policyholder as a claim is the amount of the loss that occurs. Once the random variable X representing the loss has been determined, the expected value of the loss, $E[X]$, is referred to as the **pure premium** for the policy. $E[X]$ is also the **expected claim** on the insurer. Note that in general, one of the outcomes of X might be 0, since it may be possible that no loss occurs. The following are the basic models used for describing the loss random variable. For a random variable X a measure of the risk is $\sigma^2 = Var[X]$. The **unitized risk** or **coefficient of variation** for the random variable X is defined to be $\frac{\sqrt{Var[X]}}{E[X]} = \frac{\sigma}{\mu}$.

Many insurance policies do not cover the full amount of the loss that occurs, but only provide partial coverage. There are a few standard types of partial insurance coverage that can be applied to a basic **ground up loss** (full loss) random variable X. These are described starting on the following page.

PARTIAL INSURANCE COVERAGE (i) - Deductible Insurance

A deductible insurance specifies a **deductible amount**, say d. If a loss of amount X occurs, the insurer pays nothing if the loss is less than d, and pays the policyholder the amount of the loss in excess of d if the loss is greater than d. The amount paid by the insurer can be described as $\boldsymbol{Y = \begin{cases} 0 \text{ if } X \le d \\ X-d \text{ if } X>d \end{cases} = Max\{X-d, 0\}}$.
This is also denoted $(X-d)_+$. The expected payment made by the insurer when a loss occurs would be $\int_d^\infty (\boldsymbol{x}-\boldsymbol{d})\, \boldsymbol{f_X(x)}\, \boldsymbol{d}x$ in the continuous case; (this is also called the *expected cost per loss*). With integration by parts, this can be shown to be equal to $\int_d^\infty [1-F_X(x)]\, dx$. This type of policy is also referred to as an ordinary deductible insurance.

Two variations on deductible insurance are the franchise deductible, and the disappearing deductible. These are less likely to appear on the exam.

(a) **Franchise deductible:** A franchise deductible of amount d refers to the situation in which the insurer pays 0 if the loss is below d but pays the full amount of loss if the loss is above d; the amount paid by the insurer can be described as $\begin{cases} 0 \text{ if } X \le d \\ X \text{ if } X>d \end{cases}$

(b) **Disappearing deductible:** A disappearing deductible with lower limit d and upper limit d' (where $d < d'$) refers to the situation in which the insurer pays 0 if the loss is below d, the insurer pays the full loss if the loss amount is above d', and the deductible amount reduces linearly from d to 0 as the loss increases from d to d'; the amount paid by the insurer can be described as $\begin{cases} 0 & X \le d \\ d' \cdot \frac{X-d}{d'-d} & d < X \le d' \\ X & X > d' \end{cases}$

Example 10-1:

A discrete loss random variable X has the following two-point distribution: $P[X=3] = P[X=12] = 0.5$. An ordinary deductible insurance policy is set up for this loss, with deductible d. It is found that the expected claim on the insurer is 3. Find d.

Solution:

The claim on the insurer is $Y = \begin{cases} 0 \text{ if } X \le d \\ X-d \text{ if } X>d \end{cases}$.

We proceed by "trial-and-error". Suppose our initial "guess" is that $d \le 3$.
Then the claim on the insurer will be either $3-d$ or $12-d$, each with probability 0.5. so that the expected claim on the insurer will be $(3-d) \times 0.5 + (12-d) \times 0.5 = 3$, which implies that $d = 4.5$. This contradicts our "guess" that $d \le 3$, which indicates that the guess was wrong. Thus, $d > 3$, so that the claim on the insurer will be 0 (if $X=3$) or $12-d$, each with probability 0.5. The expected claim on the insurer will then be $(12-d) \times .5 = 3 \rightarrow d = 6$. □

Example 10-2:

A loss random variable is uniformly distributed between 0 and 1000. A deductible of 200 is applied before any insurance payment. Find the expected amount paid by the insurer when a loss occurs.

Solution:

Expected insurance payment is

$$E[(X-200)_+] = \int_{200}^{1000} (x-200) \times \frac{1}{1000}\, dx = 320$$ □

Example 10-3:

A loss random variable is exponentially distributed with a mean of 1000. A deductible of 200 is applied before any insurance payment. Find the expected amount paid by the insurer when a loss occurs.

Solution: Expected insurance payment is

$$E[(X-200)_+] = \int_{200}^{\infty} (x-200)\left(\frac{e^{-x/1000}}{1000}\right) dx = \int_{200}^{\infty} [1-F_X(x)]\, dx$$

We can calculate this integral three ways.

(a) $\int_{200}^{\infty} (x-200) \times \frac{e^{-x/1000}}{1000}\, dx = \int_{200}^{\infty} x \times \frac{e^{-x/1000}}{1000}\, dx - 200\int_{200}^{\infty} \frac{e^{-x/1000}}{1000}\, dx$

$$= -xe^{-x/1000} - 1000e^{-x/1000} + 200e^{-x/1000}\Big|_{x=200}^{\infty}$$

$$= -0 - 0 + 200e^{-0.2} + 800e^{-0.2} = 818.73\,.$$

(b) Apply the change of variable $y = x - 200$. The integral becomes $\int_0^{\infty} y \times \frac{e^{-(y+200)/1000}}{1000}\, dy$ and we can use the general rule $\int_0^{\infty} t^n e^{-ct}\, dt = \frac{n!}{c^{n+1}}$ (when n is an integer ≥ 0 and $c > 0$) to get

$$\int_0^{\infty} y \times \frac{e^{-(y+200)/1000}}{1000}\, dy = \frac{e^{-0.2}}{1000} \times \int_0^{\infty} ye^{-y/1000}\, dy = 0.0008187 \times \frac{1}{(1/1000)^2} = 818.73$$

(c) $\int_{200}^{\infty} [1-F_X(x)]\, dx = \int_{200}^{\infty} e^{-x/1000}\, dx = 1000e^{-0.2} = 818.73$ □

PARTIAL INSURANCE COVERAGE (ii) - Policy Limit

A **policy limit of amount u** indicates that the insurer will pay a maximum amount of u when a loss occurs. Therefore, the amount paid by the insurer is $\begin{cases} X \text{ if } X \leq u \\ u \text{ if } X > u \end{cases}$.

The expected payment made by the insurer per loss would be $\int_0^u x \times f_X(x)\, dx + u \times [1 - F_X(u)]$ in the continuous case. This can be shown to be equal to $\int_0^u [1 - F_X(x)]\, dx$.

NOTE: Suppose that X is the loss random variable. An insurance policy which pays the loss amount in excess of deductible c pays $Y_1 = \begin{cases} 0 & \text{if } X \leq c \\ X-c & \text{if } X > c \end{cases}$, and an insurance policy which pays the loss up to a limit of c pays $Y_2 = \begin{cases} X \text{ if } X \leq c \\ c \text{ if } X > c \end{cases}$.

The combined payment of the two policies is $Y_1 + Y_2 = X$, since Y_2 covers any loss up to amount c and Y_1 covers the loss in excess of c. If an insurance policy with an ordinary deductible of c is purchased, then the part of the loss not paid by the insurance policy will be algebraically the same as the amount paid on a policy with policy limit c, and vice versa.

Questions on ordinary deductible and policy limit have come up regularly on the exam.

Example 10-4:

A loss random variable is uniformly distributed between 0 and 1000. An insurance policy pays the loss up to a maximum of 200. Find the expected amount paid by the insurer when a loss occurs.

Solution:

The expected amount paid by the insurer is

$$\int_0^{200} x \times \frac{1}{1000}\, dx + 200 \times [1 - F_X(200)] = 20 + (200)(1 - 0.2) = 180$$

This is the same loss random variable as in Example 10-2. In that example, with a deductible of 200, the expected amount paid by the insurer was 320. As pointed out above in the notes above, the combination of an insurance with a deductible of 200 and an insurance with a policy limit of 200 is the full loss. Therefore, it is no coincidence that the expected amounts paid by the insurer on the combination of the policy in this example and the one in Example 10-2 add up to the overall expected loss of 500. $\square$

It is possible to combine a deductible insurance with a policy limit. If a policy has a deductible of d and a limit of $u - d$ then the claim amount paid by the insurer can be described as

$$\begin{cases} 0 \text{ if } X \leq d \\ X - d \text{ if } d < X \leq u \\ u - d \text{ if } X > u \end{cases}$$

Note that "the deductible is applied after the policy limit is applied". This means that a loss of amount greater than u triggers the maximum insurance payment of amount $u - d$. The expected payment made by the insurer per loss would be $\int_d^u (x-d) \times f_X(x)\, dx + (u-d) \times [1 - F_X(u)]$ in the continuous case.

This can be shown to be equal to $\int_d^u [1 - F_X(x)]\, dx$.

Example 10-5:

A loss random variable is uniformly distributed on $(0\,,\,1000)$.

(a) Find the mean and variance of the insurance payment if a deductible of 250 is imposed.

(b) Find the mean and variance of the insurance payment if a policy limit of 500 is imposed.

(c) Find the mean and variance if a policy limit of 250 and a deductible of 250 is imposed.

Solution: Suppose the insurance payment is Y.

(a) $E[Y] = \int_{250}^{1000} (x-250) \times \frac{1}{1000}\, dx = \frac{(x-250)^2}{2(1000)}\Big|_{x=250}^{x=1000} = 281.25$

$E[Y^2] = \int_{250}^{1000} (x-250)^2 \times \frac{1}{1000}\, dx = \frac{(x-250)^3}{3(1000)}\Big|_{x=250}^{x=1000} = 140,625$

$Var[Y] = E[Y^2] - (E[Y])^2 = 140,625 - (281.25)^2 = 61,523$

(b) $E[Y] = \int_0^{500} x \times \frac{1}{1000}\, dx + 500[1 - F(500)] = 125 + 500 \times (1 - \frac{1}{2}) = 375$

$E[Y^2] = \int_0^{500} x^2 \times \frac{1}{1000}\, dx + (500)^2 \times [1 - F(500)] = 41,667 + 125,000 = 166,667$

$Var[Y] = E[Y^2] - (E[Y])^2 = 26,042$

(c) The policy limit is $u - d = u - 250 = 250$, so that $u = 500$. Then

$E[Y] = \int_{250}^{500} (x-250) \times \frac{1}{1000}\, dx + 250 \times [1 - F(500)] = 31.25 + 250 \times \frac{1}{2} = 156.25$

$E[Y^2] = \int_{250}^{500} (x-250)^2 \times \frac{1}{1000}\, dx + 250^2 \times [1 - F(500)] = 5208 + 31,250 = 36,458$

$Var[Y] = E[Y^2] - (E[Y])^2 = 12,044$ □

PARTIAL INSURANCE COVERAGE (iii) - Proportional Insurance

Proportional insurance specifies a fraction α $(0 < \alpha < 1)$, and if a loss of amount X occurs, the insurer pays the policyholder αX, the specified fraction of the full loss.

Models for describing a loss random variable X:

Case 1 (most likely): The complete description of X is given:

In this case, if X is continuous, the density function $f(x)$ or distribution function $F(x)$ is given.
If X is discrete, the probability function (or possibly the distribution function) is given. One typical (and simple) example of the discrete case is a loss random variable of the form

$$X = \begin{cases} K & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{cases}$$

(this might arise in a one- year term life insurance in which the death benefit is K, paid if the policyholder dies within the year, and probability of death within the year is q).

Another example of a discrete loss random variable (with more than two points) is the following example of dental expenses for a family over a one-year period.

Amount of Dental Expense	Probability
\$ 0	0.1
200	0.1
400	0.3
800	0.4
1500	0.1

In some problems, all that is needed is the mean and variance of X, and sometimes that is the only information about X that is given (rather than the full description of X's distribution).

Case 2 (less likely): The probability q of a non-negative loss is given, and the conditional distribution B of loss amount given that a loss has occurred is given:

The probability of no loss occurring is $1-q$, and the loss amount X is 0 if no loss occurs; thus, $P[X=0]=1-q$. If a loss does occur, the loss amount is the random variable B, so that $X=B$. The random variable B is the loss amount **given that a loss has occurred**, so that B is really the **conditional distribution of the loss amount X given that a loss occurs**. The random variable B might be described in detail, or only the mean and variance of B might be known. Note that if $E[B]$ and $Var[B]$ are given, then $E[B^2]=Var[B]+(E[B])^2$ (this is needed in the formulation of $Var[X]$).

We can formulate X as a "mixture" of two random variables, W_1 and W_2, where $W_1=0$ is the constant random variable (not really random at all), and $W_2=B$, and with mixing weights $\alpha_1=1-q$ and $\alpha_2=q$. Then the first two moments of X are

$E[X]=(1-q)\times E[W_1]+q\times E[W_2]=\boldsymbol{q\times E[B]}$, since $E[W_1]=0$ and

$E[X^2]=(1-q)\times E[W_1^2]+q\times E[W_2^2]=\boldsymbol{q\times E[B^2]}$. Then,

$\boldsymbol{Var[X]=q\times E[B^2]-(q\times E[B])^2}=q\times Var[B]+q\times(1-q)\times(E[B])^2$

(note that $Var[X]$ **is not** $q\times Var[B]$).

The mixing weight $1-q$ applies to $W_1=0$, which means that there is a probability $1-q$ the no loss will occur (loss $=0$). The mixing weight q (the probability of a loss occurring) is applied to B, the random loss amount when a loss occurs.

For example, the loss due to fire damaging a particular property might be modeled this way. Suppose that $q=.01$ is the probability that fire damage occurs, and given that fire damage occurs, the amount of damage, B, has a uniform distribution between \$10,000 and \$50,000.

Keep in mind that B is the loss amount given that a loss has occurred, whereas X is the overall loss amount. Then, $E[B]=\$30,000$ and $E[B^2]=\frac{3{,}100{,}000{,}000}{3}$ (first and second moment of the uniform distribution on $(10,000\,,\,50,000)$. Using the formulas above,

$E[X]=(.01)(30,000)=\$300$ and

$Var[X]=q\times E[B^2]-(q\times E[B])^2=.01\times\frac{3{,}100{,}000{,}000}{3}-300^2=\frac{30{,}730{,}000}{3}$.

Example 10-6:

For a one-year dental insurance policy for a family, we consider the following two models for annual claims X:

Model 1:

Amount of Dental Expense (X)	Probability
$ 0	0.1
200	0.1
400	0.3
800	0.4
1500	0.1

Model 2: There is a probability of 0.1 that no claim occurs, $P[X=0]=0.1$. If a claim does occur, the claim amount random variable B, has mean $E[B]=677.78$ and variance $Var[B]=132,839.51$.

For each loss model, find $E[X]$ and $Var[X]$.

Solution:

Model 1: In this case the complete description of X is given (Case 1 mentioned above).

$E[X]=0\times 0.1+200\times 0.1+400\times 0.3+800\times 0.4+1500\times 0.1=610$

$E[X^2]=0^2\times 0.1+200^2\times 0.1+400^2\times 0.3+800^2\times 0.4+1500^2\times 0.1=533,000$

$Var[X]=533,000-610^2=160,900$.

Model 2: In this case, the probability of a claim occurring is given ($q=.9$) along with the mean and variance of the conditional distribution B of claim amount given that a claim occurs (Case 2 mentioned above).

$E[X]=q\times E[B]=0.9\times 677.78=610$

$E[X^2]=q\times E[B^2]=0.9\times[Var[B]+(E[B])^2]=0.9\times[132,839.51+(677.78)^2]=533,003$

$Var[X]=E[X^2]-(E[X])^2=160,900$

Note that it is not a coincidence that the mean and variance of X turned out to be the same for Models 1 and 2. This is true because the mean and variance of B in Model 2 were chosen as the conditional mean and variance of the distribution in Model 1 given that a claim occurs. □

Modeling Aggregate Claims In A Portfolio Of Insurance Policies - The Individual Risk Model

The individual risk model assumes that the portfolio consists of a specific number, say n, of insurance policies, with the claim for one period on policy i being the random variable X_i. X_i would be modeled in one of the ways described above for an individual policy loss random variable. Unless mentioned otherwise, it is assumed that the X_i's are mutually independent random variables. Then the aggregate claim is the random variable

$$\boldsymbol{S=\sum_{i=1}^{n}X_i}\text{, with }\boldsymbol{E[S]=\sum_{i=1}^{n}E[X_i]}\text{ and }\boldsymbol{Var[S]=\sum_{i=1}^{n}Var[X_i]}$$

If $E[X_i]=\mu$ and $Var[X_i]=\sigma^2$ for each $i=1,2,...,n$, then the coefficient of variation of the aggregate claim distribution S is $\frac{\sqrt{Var[S]}}{E[S]}=\frac{\sqrt{nVar[X]}}{nE[X]}=\frac{\sigma}{\mu\sqrt{n}}$, which goes to 0 as $n\to\infty$.

Example 10-7:

An insurer has a portfolio of 1000 one-year term life insurance policies just issued to 1000 different (independent) individuals. Each policy will pay \$1000 in the event that the policyholder dies within the year. For 500 of the policies, the probability of death is .01 per policyholder, and for the other 500 policies the probability of death is 0.02 per policyholder. Find the expected value and the standard deviation of the aggregate claim that the insurer will pay.

Solution:

The aggregate claim random variable is $S = \sum_{i=1}^{1000} X_i$, where X_i is the claim from policy i.

Then $E[S] = \sum_{i=1}^{1000} E[X_i]$ and since the claims are independent, $Var[S] = \sum_{i=1}^{1000} Var[X_i]$.

If Y_1 is one of the 500 policies with death probability 0.01, then $Y_1 = \begin{cases} 0 & \text{prob. } 0.99 \\ 1000 & \text{prob. } 0.01 \end{cases}$

$\Rightarrow \; E[Y_1] = 1000 \times 0.01 = 10, \; Var[Y_1] = E[Y_1^2] - (E[Y_1])^2 = 9900.$

If Y_2 is one of the 500 policies with death probability .02, then $E[Y_2] = 20, \; Var[Y_2] = 19,600$.

Thus, $E[S] = 500 \times 10 + 500 \times 20 = 15,000$

$Var[S] = \sum_{i=1}^{1000} Var[X_i] = 500 \times 9900 + 500 \times 19,600 = 14,750,000 \Rightarrow \sqrt{Var[S]} = 3,841.$ □

Example 10-8:

Two portfolios of independent insurance policies have the following characteristics:

Portfolio A:

Class	Number in Classper Policy	Probability of Claim	Claim Amount
1	2,000	0.05	1
2	500	0.10	2

Portfolio B:

Class	Number in Classper Policy	Probability of Claim	Claim Amount Distribution Mean	Variance
1	2,000	0.05	1	1
2	500	0.10	2	4

The aggregate claims in the portfolios are denoted by S_A and S_B. Find $\frac{Var[S_A]}{Var[S_B]}$.

Solution:

In this example, Portfolio B information is given in the following form: for policy i, the probability of a claim occurring is given, q_i, and the mean and variance of the conditional distribution of claim amount given a claim occurs is given, $E[B_i] = \mu_i$, $Var[B_i] = \sigma_i^2$.