12. A manufacturing company ships 10,000 units of a product per shipment. In any given shipment there are a proportion of units that are defective. The company has determined that in 25% of the shipments, a unit of product has a probability of 0.2 of being defective, and in the other 75% of shipments, any unit of product has a prob. of 0.1 of being deficctive. A shipment is selected at random and 10 units of the product are chosen at random from that shipment. Find the probability that at least 2 of the units in that sample are defective (nearest 0.05).

A) 0.30 B) 0.35 C) 0.40 D) 0.45 E) 0.50

13. A population of insured individuals consists of a% low risk, b% medium risk and c% high risk. The number of claims in a year for a low risk individual has a Poisson distribution with a mean of 1 claim, for a medium risk individual the number of claims in a year is Poisson with a mean of 2, and for a high risk individual the number of annual claims is Poisson with a mean of 3.
An individual is picked at random from the population and it is found that the mean number of claims for the year is 2.1 and the variance of the number of claims for the year is 2.59. For a randomly chosen member of the population find the probability of no claims occurring in the year.

A) 0.088 B) 0.116 C) 0.156 D) 0.198 E) 0.228

14. An insurer is considering taking over a group of policies. The policies in the group are identically distributed and mutually independent of one another. Each policy in the group has a claim distribution which is exponentially distributed with mean 100. The premium for each policy is 120. The insurer wants a probability of at least 95% that the premium received will be enough to cover the claims. Using the normal approximation, determine the minimum number of policies needed in the group in order for the insurer's criterion to be met.

A) 60 B) 62 C) 64 D) 66 E) 68

15. X has a distribution with pdf $f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$, $x > 0$.
Y has an exponential distribution with mean $\frac{1}{\alpha}$.
Which of the following is the correct transformation linking Y and X?

A) $Y = ln(X+\theta)$ B) $Y = ln(\frac{X}{\theta})$ C) $Y = ln(\frac{X+\theta}{\theta})$
D) $Y = ln(\frac{X+\theta}{X})$ E) $Y = ln(\frac{X}{X+\theta})$

16. Binary digits are transmitted over a communication system. If a 1 is sent, it will be received as a 1 with probability 0.95 and as a 0 with probability 0.05. If a 0 is sent, it will be received as a 0 with probability 0.99 and as a 1 with probability 0.01. A series of 0's and 1's is sent in random order, with 0's and 1's each being equally likely. If a digit is received as a 1 find the probability that it was sent as a 1.

A) less than 0.96 B) at least 0.96 but less than 0.97 C) at least 0.97 but less than 0.98
D) at least 0.98 but less than 0.99 E) at least 0.99

17. The model chosen for a discrete, integer-valued, non-negative random variable N with mean 2 is the binomial distribution with n trials and probability of success p on each trial. Various combinations of n and p are considered, and $P[N=0]$ is calculated. Find $\lim_{n\to\infty} P[N=0]$.

A) e^{-2} B) e^{-1} C) 0 D) $\frac{1}{2}$ E) 1

18. When a fire occurs, the model for fire damage on a particular property is based on the following joint distribution for X (structural damage) and Y (damage to contents):
$f(x,y) = ax + by$, $0 < x < 2$, $0 < y < 1$ (scaled to appropriately sized units). The probability that X is greater than Y is $\frac{5}{6}$. Find the total expected damage if a fire occurs, $E[X+Y]$.

A) $\frac{10}{9}$ B) $\frac{16}{9}$ C) $\frac{22}{9}$ D) $\frac{28}{9}$ E) $\frac{34}{9}$

19. A machine requires the continual operation of two independent devices in order to keep functioning. The machine breaks down as soon as the first device stops operating.
The time until failure of Device 1 is uniformly distributed between time 0 and time 1, and the time until failure of Device 2 has pdf $f(t) = 2t$, $0 < t < 1$. Find the expected time until the machine breaks down.

A) $\frac{1}{12}$ B) $\frac{1}{6}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{5}{12}$

20. X and Y have a bivariate normal distribution with $E[X] = 0$ and $E[Y] = 1$. It is also known that $E[X|Y=9] = 8$ and $E[Y|X=9] = 2$. Find ρ, the coefficient of correlation between X and Y.

A) $-\frac{1}{2}$ B) $-\frac{1}{3}$ C) 0 D) $\frac{1}{3}$ E) $\frac{1}{2}$

21. Pick the correct relationship:

A) $P[A \cap B] \leq P[A] \times P[B]$ for any events A and B

B) $P[A \cup B] \geq P[A] + P[B]$ for any events A and B

C) $P[A \cap B'] \geq P[A] - P[B]$ for any events A and B

D) $P[A \cup B|C] = P[A|C] + P[B|C]$ for independent events A and B and any event C

E) $P[A|B] = P[B|A]$ for any events A and B

22. A city lotto is held each week. The lotto ticket costs \$1, and the lotto prize is \$10 and there is a $\frac{1}{30}$ chance of winning the prize. Smith decides to try his luck at the lotto, and decides to buy 1 ticket each week until he wins, at which time he will stop. Find Smith's expected gain for his lotto-ticket enterprise.

A) -20 B) -15 C) -10 D) -5 E) 0

23. When a fire occurs, the model for fire damage on a particular property is based on a joint distribution for X (structural damage) and Y (damage to contents). The marginal distribution of X has density function $f_X(x) = 2 - 2x$ for $0 < x < 1$. If the amount structural damage is x, then the distribution of damage to contents is uniform on the interval $(0, \frac{x}{2})$. Find the expected amount of damage to contents when a fire occurs.

    A) 1 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{6}$ E) $\frac{1}{12}$

24. A loss has a distribution which is uniform between 0 and 1. An insurer issues a policy in this loss which pays the amount of the loss above a deductible of amount d, where $0 < d < 1$. The expected claim on the insurer is c, where $0 < c < \frac{1}{2}$

    Find the amount of the deductible.

    A) $\sqrt{2c}$ B) $1 - \sqrt{2c}$ C) $1 + \sqrt{2c}$ D) $\frac{1}{\sqrt{2c}} + 1$ E) $\frac{1}{\sqrt{2c}} - 1$

25. X has an exponential distribution with a mean of 1. Y is defined to be the conditional distribution of $X - 2$ given that $X > 2$, so for instance, for $c > 0$, we have $P[Y > c] = P[X - 2 > c | X > 2]$ What is the distribution of Y?

    A) Exponential with mean 1 B) Exponential with mean 2 C) Exponential with mean $\frac{1}{2}$

    D) Exponential with mean e E) Exponential with mean e^2

26. A die is being loaded so that the probability of tossing a 1 is p and the probability of tossing a 6 is $\frac{1}{3} - p$. The probabilities of tossing a 2, 3, 4 or 5 are all $\frac{1}{6}$. X is the outcome from tossing the die once. Find the value of p for which the variance of X is maximized.

    A) 0 B) $\frac{1}{12}$ C) $\frac{1}{6}$ D) $\frac{1}{4}$ E) $\frac{1}{3}$

27. The daily high temperature in Toronto in January is normally distributed with a mean of -5 degrees Celsius and a standard deviation of 4 degrees Celsius. The daily high temperature in Winnipeg in January is normally distributed with a mean of -10 degrees Celsius and a standard deviation of 8 degrees Celsius. Assuming that daily high temperatures in Toronto and Winnipeg are independent of one another, find the probability that on a given day in January, the high temperatures for that day in Toronto and Winnipeg are within 1 degree Celsius of each other (nearest 0.025).

    A) 0.025 B) 0.050 C) 0.075 D) 0.100 E) 0.125

28. The joint distribution of random variables X and Y has pdf
$f(x,y) = x + y\,,\ 0 < x < 1\,, 0 < y < 1.$
The joint distribution of random variables Y and Z has pdf
$g(y,z) = 3(y + \frac{1}{2})z^2\,,\ 0 < y < 1\,, 0 < z < 1$
Which of the following could be the pdf of the joint distribution of X and Z?
A) $x + \frac{3}{2}z^2\,,\ 0 < x < 1\,, 0 < z < 1$
B) $x + \frac{1}{2} + 3z^2\,,\ 0 < x < 1\,, 0 < z < 1$
C) $3(x + \frac{1}{2})z^2\,,\ 0 < x < 1\,, 0 < z < 1$
D) $x + z\,,\ 0 < x < 1\,, 0 < z < 1$
E) $4xz\,,\ 0 < x < 1\,, 0 < z < 1$

29. A small company wishes to insure against losses incurred in the case of a strike by the company's employees. An insurer agrees to pay $100,000 for each strike that occurs within the next year, up to a maximum payment of $300,000. The distribution used to model strike behavior is
$P[n \text{ strikes within the next year}] = (0.8)(0.2)^n,\ n \geq 0.$
The small company estimates that it will lose $150,000 for each strike that occurs.
For the year, find the company's expected loss that is not covered by the insurance.
A) 12,100 B) 12,300 C) 12,500 D) 12,700 E) 12,900

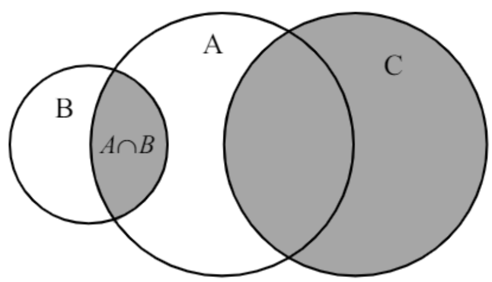
30. X and Y have a joint distribution with pdf $f(x,y) = e^{-(x+y)}\,,\ x > 0\,, y > 0.$
The random variable U is defined to be equal to $U = e^{-(x+y)}$.
Find the pdf of U, $f_U(u)$.
A) $f_U(u) = 1$ for $0 < u < 1$
B) $f_U(u) = \frac{1}{u^2}$ for $u > 1$
C) $f_U(u) = -\ln u$ for $0 < u < 1$
D) $f_U(u) = 2u$ for $0 < u < 1$
E) $f_U(u) = e^{-u}$ for $u > 0$

PRACTICE EXAM 5 - SOLUTIONS

1. Since $\int_0^\infty f(x)\,dx = 1$, it follows that $c = 1/\int_0^\infty \frac{1}{(x+\theta)^{\alpha+1}}\,dx = \alpha\theta^\alpha$,
so that $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$. Then $F(x) = \int_0^x f(t)\,dt = \int_0^x \frac{\alpha\theta^\alpha}{(t+\theta)^{\alpha+1}}\,dt = 1 - (\frac{\theta}{x+\theta})^\alpha$.
This is called a Pareto distribution with parameters α and θ. Answer: B

2. In this case, for a particular policy, the probability of a claim occurring is
$q = \frac{1}{6}$, and if a claim occurs the (conditional) distribution of the amount of the claim B is defined by the given density function $f(y) = 2(1-y)$ for $0 < y < 1$. Then, the expected claim from any one policy is
$E[X] = q \times E[B] = \frac{1}{6} \times \int_0^1 y \times 2(1-y)\,dy = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$
The expected value of total benefits paid (claims) is the sum of the individual policy expected values for the 32 policies; this is $32 \times \frac{1}{18} = \frac{16}{9}$. Answer: A

3. $A \cap B \subset B$ and since B and C are mutually exclusive, $A \cap B$ is disjoint from C.
It then follows that $C \subset (A \cap B)'$, and thus $(A \cap B)' \cup C = (A \cap B)'$.
This can be seen in the diagram below.



Then $P[(A \cap B)' \cup C] = P[(A \cap B)'] = 1 - P[A \cap B]$.
Since A and B are independent, $P[A \cap B] = P[A] \times P[B] = \frac{1}{24}$.
Thus, $P[(A \cap B)' \cup C] = \frac{23}{24}$. Answer: D

4. $M(t) = E[e^{tX}] = E[1 + tX + \frac{t^2X^2}{2} + \cdots] = E[1] + t \cdot E[X] + \frac{t^2}{2} \times E[X^2] + \cdots$
$E[1] = 1$, $E[X]$ is given as 2, and $Var[X] = E[X^2] - (E[X])^2$ is given to be 8,
so that $E[X^2] = 12$. Then the first 3 terms of the expansion of $M(t)$ are $1 + 2t + 6t^2$.
Answer: B

5. The number of passengers who show up is a random variable N, which has a binomial distribution with $n = 32$, $p = 0.9$ (each of the 32 ticket holders can be regarded as a trial experiment that can end in success - showing up - prob. 0.9, or failure - not showing up - prob. 0.1). In order for more passengers show up for the flight than there are seats available, N must be 31 or 32. Therefore, the probability that more passengers show for the flight than there are seats available is $P[N = 31 \text{ or } 32] = P[N = 31] + P[N = 32]$.
Using the formulation for binomial distribution probabilities we have
$P[N = 31] = \binom{32}{31}(0.9)^{31}(0.1)^1 = .12209$, and $P[N = 32] = \binom{32}{32}(0.9)^{32}(0.1)^0 = .03434$.
The probability in question is $P[N = 31] + P[N = 32] = 0.1564$. Answer: E

6. $M_X(t) = Ae^t + Be^{2t}$. For any random variable, $M(0) = 1 \rightarrow A + B = 1$
$\rightarrow M_X(t) = Ae^t + (1-A)e^{2t}$.
$E[X] = M_X'(0) = A + 2(1-A) = 2 - A$.
$E[X^2] = M_X''(0) = A + 4(1-A) = 4 - 3A$.
$Var[X] = E[X^2] - (E[X])^2 = 4 - 3A - (2-A)^2 = A - A^2 = \frac{2}{9}$
$\rightarrow A^2 - A + \frac{2}{9} = 0 \rightarrow A = \frac{1}{3}$ or $\frac{2}{3}$.
Since we are given that $A < \frac{1}{2}$, it follows that $A = \frac{1}{3}$ and $E[X] = 2 - \frac{1}{3} = \frac{5}{3}$. Answer: E

7. Y is the discretized distribution. $P[Y = k + \frac{1}{2}] = P[k < X \leq k+1] = e^{-k} - e^{-k-1}$ for $k = 0, 1, 2, \ldots$

$$E[Y] = \sum_{k=0}^{\infty}(k + \tfrac{1}{2})(e^{-k} - e^{-k-1}) = \sum_{k=0}^{\infty} k(e^{-k} - e^{-k-1}) + \tfrac{1}{2}\sum_{k=0}^{\infty}(e^{-k} - e^{-k-1}).$$

$$\sum_{k=0}^{\infty}(e^{-k} - e^{-k-1}) = (e^0 - e^{-1}) + (e^{-1} - e^{-2}) + \cdots = 1$$

$$\sum_{k=0}^{\infty} ke^{-k} = \frac{e^{-1}}{(1-e^{-1})^2} \;,\; \sum_{k=0}^{\infty} ke^{-k-1} = \frac{e^{-2}}{(1-e^{-1})^2}$$

$$\rightarrow \sum_{k=0}^{\infty} k(e^{-k} - e^{-k-1}) = \frac{e^{-1}}{(1-e^{-1})^2} - \frac{e^{-2}}{(1-e^{-1})^2} = \frac{e^{-1}}{1-e^{-1}}$$

$E[Y] = \frac{e^{-1}}{1-e^{-1}} + \frac{1}{2} = 1.08$ Answer: E

8. The distribution function of T is
$F_T(t) = P[T \leq t] = 1 - P[T > t] = 1 - P[\text{first play occurs after time } t]$
$= 1 - P[\text{no one plays from time 0 to } t] = 1 - \binom{10}{0}(0.5t)^0(1-0.5t)^{10} = 1 - (1-0.5t)^{10}$.
The pdf of T is $f_T(t) = F_T'(t) = 5(1-0.5t)^9$. Answer: B

9. This year's expected insurance payment is

$\int_0^{500} x \times (0.001e^{-0.001x})\,dx + 500 \times P[X > 500]$

$= \int_0^{500} (1 - F(x))\,dx = \int_0^{500} e^{-0.001x}\,dx \;=\; 1000(1 - e^{-0.5}) = 393.47.$

We have used the identity from page 314 on expected payment on a loss with a policy limit:

$\int_0^{u} x \times f_X(x)\,dx + u \times [1 - F_X(u)] = \int_0^{u} [1 - F_X(x)]\,dx.$

This identity can be proven using integration by parts. Next year's expected insurance payment is

$\int_0^{u} x \times 0.0008\,dx + u \times P[X > u] = \int_0^{u} (1 - F_1(x))\,dx \;= \int_0^{u} (1 - 0.0008x)\,dx = u - 0.0004u^2$

We want $u - .0004u^2 = (1.25)(393.47) = 491.84.$

Solving the quadratic equation results in $u = 673$ or $1827.$

We ignore the root $u = 1827$, since it is above the maximum loss.

Answer: C

10. I. True.

II. False. The sum of independent exponentials, each with the same mean, has a gamma distribution.

III. False.

IV. True.

Answer: C

11. $A =$ "YES" to (i) , $B =$ "YES" to (ii)

$P[A \cup B] = 0.8$, $P[A' \cup B'] = 0.8 \rightarrow P[A \cap B] = P[(A' \cup B')'] = 1 - 0.8 = 0.2$

$\rightarrow P$["YES' to exactly one] $= P[A \cap B'] + P[A' \cap B]$

$= P[A \cup B] - P[A \cap B] = 0.8 - 0.2 = 0.6.$ Answer: E

12. N, number of defective in a sample of size 10, has a binomial distribution with probability p of any one being defective.

$P[N \geq 2|p] = 1 - P[N0 \text{ or } 1|p] = 1 - \binom{10}{0}(1-p)^{10} - \binom{10}{1}(1-p)^9 \times p.$

For shipment with .2 defective,

$P[N \geq 2|p = 0.2] = 1 - (.8)^{10} - (10)(0.8)^9(0.2) = 0.6242.$

For shipment with 0.1 defective,

$P[N \geq 2|p = 0.1] = 1 - (0.9)^{10} - (10)(0.9)^9(0.1) = 0.2639.$

$P[N \geq 2] = P[N \geq 2|p = 0.2] \times P[p = 0.2] + P[N \geq 2|p = 0.1] \times P[p = 0.1]$

$= (0.6242)(0.25) + (0.2639)(0.75) = 0.354.$ Answer: B

13. With mixing weight a applied to the Poisson with mean 1 and mixing weight b applied to the Poisson with mean 2, the mean of X is $E(X) = a + 2b + 3(1 - a - b) = 2.1$.
The second moment of X is $E(X^2) = 2a + 6b + 12(1 - a - b) = 2.59 + 2.1^2 = 7.0$.
We get the two equations $2a + b = .9$ and $10a + 6b = 5$.
Solving these two equations results in $a = 0.2$ and $b = 0.5$.
Then $P(X = 0) = 0.2e^{-1} + 0.5e^{-2} + 0.3e^{-3} = 0.156$. Answer: C

14. Claim on policy i is X_i. With n policies, the aggregate claim is $C = \sum_{i=1}^{n} X_i$, and the mean of the aggregate claim is $E[C] = E[X_1 + X_2 + \cdots + X_n] = nE[X] = 100n$,
and the variance is $Var[C] = Var[X_1 + X_2 + \cdots + X_n] = nVar[X] = 10,000n$
(since the X_i's are independent, the variance of the sum is the sum of the variances, and variance of the exponential distribution is the square of the mean). Total premium collected is $120n$.
Using the normal approximation the probability that total premium exceeds total claims is

$$P[C < 120n] = P[\frac{C-E[C]}{Var[C]} < \frac{120n-100n}{\sqrt{10,000n}}].$$

$\frac{C-E[C]}{Var[C]}$ is approximately normal, so in order for this probability to be $\geq$ 0.95, we must have $\frac{120n-100n}{\sqrt{10,000n}} \geq 1.645$ (from the normal table). This is equivalent to $0.2\sqrt{n} \geq 1.645$, or equivalently, $n \geq 67.7$. Answer: E

15. If $Y = g(X)$ with inverse function $X = k(Y)$, then $f_Y(y) = f_X(k(y)) \times |k'(y)|$.
Also, we can think of X as a function of Y, so that $f_X(x) = f_Y(g(x)) \times |g'(x)|$.
But we are given that Y is exponential with mean $\frac{1}{\alpha}$, so that $f_Y(y) = \alpha e^{-\alpha y}$.
Therefore, $f_X(x) = \alpha e^{-\alpha g(x)} \times |g'(x)| = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$.
We can try each of the transformations $Y = g(X)$ to see which one satisfies the relationship.

A) $g(x) = ln(x + \theta) \rightarrow g'(x) = \frac{1}{x+\theta}$ and
$\alpha e^{-\alpha g(x)} \times |g'(x)| = \alpha e^{-\alpha\, ln(x+\theta)} \cdot |\frac{1}{x+\theta}| = \frac{\alpha}{(x+\theta)^{\alpha+1}}$. Incorrect.

B) $g(x) = ln(\frac{x}{\theta}) \rightarrow g'(x) = \frac{1}{x}$ and
$\alpha e^{-\alpha g(x)} \times |g'(x)| = \alpha e^{-\alpha\, ln(x/\theta)} \times |\frac{1}{x}| = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$. Incorrect.

C) $g(x) = ln(\frac{x+\theta}{\theta}) \rightarrow g'(x) = \frac{1}{x+\theta}$ and
$\alpha e^{-\alpha g(x)} \times |g'(x)| = \alpha e^{-\alpha\, ln(\frac{x+\theta}{\theta})} \times |\frac{1}{x+\theta}| = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$. Correct.

Answer: C

16. $A = 1$ was sent , $B = 1$ was received.

$P[A] = P[A'] = 0.5$. $P[B|A] = 0.95$, $P[B|A'] = 0.01$.

$P[B|A] = \frac{P[B \cap A]}{P[A]} \to P[B \cap A] = 0.95 \times 0.5 = 0.475$,

$P[B|A'] = \frac{P[B \cap A']}{P[A']} \to P[B \cap A'] = 0.01 \times 0.5 = 0.005$.

$P[B] = P[B \cap A] + P[B \cap A'] = 0.475 + 0.005 = 0.48$.

$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.475}{0.48} = 0.9896$. Answer: D

17. $P[N = 0] = (1-p)^n$. Since $E[N] = np = 2$, it follows that $p = \frac{2}{n}$.

Then $\lim_{n\to\infty} P[N=0] = \lim_{n\to\infty} (1-p)^n = \lim_{n\to\infty} (1 - \frac{2}{n})^n$.

We consider $log[(1-\frac{2}{n})^n] = n\,log(1-\frac{2}{n}) = \frac{log(1-\frac{2}{n})}{\frac{1}{n}}$.

Applying l'Hopital's rule, we get

$$\lim_{n\to\infty} log[(1-\frac{2}{n})^n] = \lim_{n\to\infty} \frac{log(1-\frac{2}{n})}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{2/n^2}{1-\frac{2}{n}}}{-\frac{1}{n^2}} = -2.$$

Therefore, $\lim_{n\to\infty} (1-\frac{2}{n})^n = e^{-2} = \lim_{n\to\infty} P[N=0]$. Answer: A

18. In order to be a proper density function, $f(x,y)$ must satisfy the relationship

$\int_0^2 \int_0^1 (ax+by)\, dy\, dx = 1$, so that $2a + b = 1$.

From the given probability, we have $\int_0^1 \int_y^2 (ax+by)\, dx\, dy = \frac{5}{6}$,

so that $\int_0^1 [\frac{a}{2}(4-y^2) + b(2y-y^2)]\, dy = \frac{11a}{6} + \frac{2b}{3} = \frac{5}{6} \to 11a + 4b = 5$.

Solving the two equations $2a + b = 1$, $11a + 4b = 5$ results in $a = b = \frac{1}{3}$.

Then, $E[X+Y] = \int_0^2 \int_0^1 (x+y)(\frac{1}{3}x + \frac{1}{3}y)\, dy\, dx = \frac{1}{3}\int_0^2 \int_0^1 (x^2 + 2xy + y^2)\, dy\, dx$

$= \frac{1}{3}\int_0^2 (x^2 + x + \frac{1}{3})\, dx = \frac{1}{3}(\frac{16}{3}) = \frac{16}{9}$. Answer: B

19. $f_1(s) = 1$, $0 < s < 1$, $f_2(t) = 2t$, $0 < t < 1$. Time to breakdown is $X = min(S, T)$.

The distribution function of X is

$F_X(x) = P[X \le x] = 1 - P[X > x] = 1 - P[(S > x) \cap (T > x)]$.

Since S and T are independent, we have

$P[(S > x) \cap (T > x)] = P[S > x] \times P[T > x]$.

$P[S > x] = \int_x^1 1\, ds = 1 - x$, and $P[T > x] = \int_x^1 2t\, dt = 1 - x^2$.

Therefore, $F_X(x) = 1 - (1-x)(1-x^2) = x + x^2 - x^3$, and the

pdf of X is $f_X(x) = F_X'(x) = 1 + 2x - 3x^2$.

The mean of X is $E[X] = \int_0^1 x \times f_X(x)\, dx = \int_0^1 (x + 2x^2 - 3x^3)\, dx = \frac{5}{12}$.

Note that since $X \ge 0$, the mean of X can be formulated as

$E[X] = \int_0^\infty [1 - F_X(x)]\, dx = \int_0^\infty S(x) \times T(x)\, dx = \int_0^1 (1-x)(1-x^2)\, dx = \frac{5}{12}$. Answer: E

20. For a bivariate normal distribution, we have the following.
$E[X|Y=y] = \mu_X + \rho \times \frac{\sigma_X}{\sigma_Y} \cdot (y - \mu_Y) = 0 + \rho \times \frac{\sigma_X}{\sigma_Y} \cdot (y-1) \rightarrow 8 = \rho \times \frac{\sigma_X}{\sigma_Y} \times 8$.
$E[Y|X=x] = \mu_Y + \rho \times \frac{\sigma_Y}{\sigma_X} \times (x - \mu_X) = 1 + \rho \times \frac{\sigma_Y}{\sigma_X} \times (x-0) \rightarrow 2 = 1 + \rho \times \frac{\sigma_Y}{\sigma_X} \times 9$.
Therefore, $\rho \times \frac{\sigma_X}{\sigma_Y} = 1$ and $\rho \times \frac{\sigma_Y}{\sigma_X} = \frac{1}{9}$, from which we get
$\rho \times \frac{\sigma_X}{\sigma_Y} \times \rho \times \frac{\sigma_Y}{\sigma_X} = \rho^2 = 1 \times \frac{1}{9} = \frac{1}{9}$.
Since $8 = E[X|Y=9] = E[X] + \rho \times \frac{\sigma_X}{\sigma_Y} \times 8 > E[X] = 0$, it follows that $\rho > 0$ (because $\frac{\sigma_X}{\sigma_Y} > 0$ always). Therefore, ρ is the positive square root of $\frac{1}{9}$, which is $\frac{1}{3}$. Answer: D

21. $P[A \cap B'] = P[A] - P[A \cap B] \geq P[A] - P[B]$, since $P[B] \geq P[A \cap B]$.
Answer: C

22. The probability of first win occurring in week n is $(\frac{29}{30})^{n-1} \times \frac{1}{30}$.
If week n is Smith's first week then his net gain is $10 - n$ dollars (10 dollar prize minus n weeks with cost of 1 per week). Smith's expected net gain is
$\sum_{n=1}^{\infty} (10-n)(\frac{29}{30})^{n-1} \times \frac{1}{30} = 10 \sum_{n=1}^{\infty} (\frac{29}{30})^{n-1} \times \frac{1}{30} - \sum_{n=1}^{\infty} n \times (\frac{29}{30})^{n-1} \times \frac{1}{30}$
The week number in which the first win occurs is the form of the geometric distribution on the integers 1, 2, 3, ... with $p = \frac{1}{30}$. The mean is $\frac{1}{p} = 30 = \sum_{n=1}^{\infty} n \times (\frac{29}{30})^{n-1} \times \frac{1}{30}$.
Also, $\sum_{n=1}^{\infty} (\frac{29}{30})^{n-1} \times \frac{1}{30} = 1$. Smith's net gain is then $10(1) - 30 = -20$. Answer: A

23. $f_{Y|X}(y|x) = \frac{2}{x}$ for $0 < y < \frac{x}{2}$. The joint density of X and Y is
$f(x,y) = f_{Y|X}(y|x) \times f_X(x) = \frac{2}{x} \times (2 - 2x) = \frac{4}{x} - 4$, $0 < x < 1$, $0 < y < \frac{x}{2}$.
$E[Y] = \int_0^1 \int_0^{x/2} y \times (\frac{4}{x} - 4)\, dy\, dx = \int_0^1 (\frac{x}{2} - \frac{x^2}{2})\, dx = \frac{1}{12}$.
A second approach is somewhat more work. The region of joint density can also be described as
$0 < y < \frac{1}{2}$, $0 < 2y < x < 1$.
The density function of the marginal distribution of Y is
$f_Y(y) = \int_{2y}^1 (\frac{4}{x} - 4)\, dx = (4\,ln\,x - 4x)\Big|_{x=2y}^{x=1} = (-4) - (4\,ln\,2y - 8y)$
$= 8y - 4\,ln\,2y - 4$ for $0 < y < \frac{1}{2}$.
The mean of Y is $E[Y] = \int_0^{1/2} y \times (8y - 4\,ln\,2y - 4)\, dy$
$= \int_0^{1/2} (8y^2 - 4y\,ln\,2y - 4y)\, dy$. This approach requires finding the antiderivative of $y\,ln\,2y$. This can be done by integration by parts.
$\int y\,ln\,2y\; dy = \int ln\,2y\, d(\frac{1}{2}y^2) = \frac{1}{2}y^2 \times ln\,2y) - \int \frac{1}{2}y^2\, d(ln\,2y)$
$= \frac{1}{2}y^2 \times ln\,2y - \int \frac{1}{2}y^2 \times \frac{1}{y}\, dy = \frac{1}{2}y^2 \times (ln\,2y) - \frac{y}{4}$. Then,
$E[Y] = \int_0^{1/2} (8y^2 - 4y\,ln\,2y - 4y)\, dy = \frac{8}{3}y^3 - 4(\frac{1}{2}y^2 \cdot (ln\,2y) - \frac{y^2}{4}) - 2y^2\Big|_{y=0}^{y=1/2}$
$= \frac{1}{3} - 4(\frac{1}{8}\,ln\,1 - \frac{1}{16}) - \frac{1}{2} = \frac{1}{12}$. Answer: E

24. The expected amount paid by the insurer is
$\int_d^1 (x-d)\, f_X(x)\, dx = \int_d^1 (x-d)\, dx = \frac{1}{2}(1-d)^2$.
Then, $\frac{1}{2}(1-d)^2 = c \rightarrow d = 1-\sqrt{2c}$. Answer: B

25. For $c > 0$ we have
$P[Y > c] = P[X-2 > c|X > 2] = P[X > c+2|X > 2] = \frac{P[(X>c+2)\cap(X>2)]}{P[X>2]}$.
$P[X > 2] = e^{-2}$ ($= \int_2^\infty e^{-x}\, dx$) and since $c > 0$,
$P[(X > c+2)\cap(X > 2)] = P[X > c+2] = e^{-(c+2)}$.
Then, $P[Y > c] = 1 - F_Y(c) = \frac{e^{-c-2}}{e^{-2}} = e^{-c}$, so that $f_Y(c) = F'_Y(c) = e^{-c}$.
Therefore, Y has an exponential distribution with mean 1. Answer: A

26. $E[X] = 1\times p + (2+3+4+5)\times\frac{1}{6} + 6\times(\frac{1}{3}-p) = \frac{13}{3} - 5p$.
$E[X^2] = 1^2\times p + (2^2+3^2+4^2+5^2)\times\frac{1}{6} + 6^2\times(\frac{1}{3}-p) = 21 - 35p$.
$Var[X] = E[X^2] - (E[X])^2 = 21 - 35p - (\frac{13}{3} - 5p)^2 = \frac{20}{9} + \frac{25}{3}p - 25p^2$.
$Var[X]$ is maximized where $\frac{d}{dp}(\frac{20}{9} + \frac{25}{3}p - 25p^2) = \frac{25}{3} - 50p = 0$, so that $p = \frac{1}{6}$.
Answer: C

27. $X =$ Toronto high temp. $N(-5, 16)$, $Y =$ Winnipeg high temp. $N(-10, 64)$.
High temperature difference is $X - Y \sim N(5, 80)$.
$P[\text{high temp. diff.} \leq 1] = P[|X-Y| \leq 1] = P[-1 \leq X-Y \leq 1]$
$= P[\frac{-6}{\sqrt{80}} \leq Z \leq \frac{-4}{\sqrt{80}}] = \Phi(-.45) - \Phi(-.67) = [1-\Phi(.45)] - [1-\Phi(.67)]$
$= \Phi(.67) - \Phi(.45) = .748 - .673 = .075$. Answer: C

28. Marginal distribution of X has pdf $f_X(x) = \int_0^1 f(x,y)\, dy = x + \frac{1}{2}$, $0 < x < 1$.
Marginal distribution of Y has pdf $f_Y(y) = \int_0^1 f(x,y)\, dx = y + \frac{1}{2}$, $0 < y < 1$,
or $f_Y(y) = \int_0^1 g(y,z)\, dz = y + \frac{1}{2}$, $0 < y < 1$.
Marginal distribution of Z has pdf $f_Z(z) = \int_0^1 g(y,z)\, dy = 3z^2$, $0 < z < 1$.
Only C has marginal distributions for X and Z that are correct. Answer: C

29. The number of strikes in the year has a geometric distribution with mean $\frac{0.2}{0.8} = 0.25$. The expected loss to the company in the year (before any insurance coverage) is $150,000\times 0.25 = 37,500$.
The expected amount paid by the insurance company in the year is
$0\times P[N=0] + 100,000\times P[N=1] + 200,000\times P[N=2] + 300,000\times P[N\geq 3]$
$= 100,000[0.8\times 0.2 + 2\times 0.8\times(.2)^2 + 3(1-.8-(.8)(.2)-(.8)(.2)^2] = 24,800$.
The expected loss to the company during the year not covered by insurance is
$37,500 - 24,800 = 12,700$. Answer: D

30. Using the transformations $U = e^{-(X+Y)} = u(X,Y)$ and $V = X = v(X,Y)$, we have inverse transformations $X = V = h(U,V)$ and $Y = -\ln U - V = k(U,V)$.

Applying the "Jacobian" method to find the joint distribution of transformed random variables U and V, we have

$$g(u,v) = f(h(u,v),k(u,v)) \times \left|\frac{\partial h}{\partial u} \times \frac{\partial k}{\partial v} - \frac{\partial h}{\partial v} \times \frac{\partial k}{\partial u}\right|$$
$$= f(v, -\ln u - v) \times |0 \times (-1) - 1 \times (-\tfrac{1}{u})| = e^{-(v - \ln u - v)} \times \frac{1}{u} = 1\,.$$

Since $-\ln U = X + Y > X = V$, the region of joint density is $0 < V < -\ln U$ and $0 < U < 1$. The marginal pdf of U is $f_U(u) = \int_0^{-\ln u} 1\,dv = -\ln u$, on the region $0 < u < 1$.

Answer: C

PRACTICE EXAM 6

1. A survey of the public determines the following about the "Lord of the Rings" trilogy (3 movies).

Have Seen #1	Have Seen #2	Have Seen #3	Percentage of Public
No	No	No	50%
Yes	?	?	35%
?	Yes	?	33%
?	?	Yes	31%
Yes	No	No	8%
Yes	Yes	No	4%
Yes	Yes	Yes	20%

Based on this information, determine the percentage of the public that has seen exactly one of the three "Lord of the Rings" movies.

A) 15 B) 17 C) 19 D) 21 E) 23

2. Suppose that events A and B are independent and suppose that $A \subseteq B$. Which of the following pairs of values is impossible?

A) $P(A) = \frac{1}{3}$ and $P(B) = 1$ B) $P(A) = \frac{1}{2}$ and $P(B) = 1$ C) $P(A) = 0$ and $P(B) = \frac{1}{2}$
D) $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$ E) $P(A) = 1$ and $P(B) = 1$

3. For a particular disease, it is found that 1% of the population will develop the disease and 2% of the population has a family history of having the disease. A genetic test is devised to predict whether or not the individual will develop the disease. For those with a family history of the disease, 20% of the time the genetic test predicts that the individual will develop the disease and for those with no family history of the disease, 1% of the time the genetic test predicts that the individual will develop the disease. The genetic test is not perfect, and individuals are followed to determine whether or not they actually develop the disease. It is found that for those who have a family history of the disease and for whom the genetic test predicts the disease will develop, 80% actually develop the disease. It is also found that for those who have a family history of the disease and for whom the genetic test does not predict the disease will develop, 10% actually develop the disease. Find the probability that someone with a family history of the disease will develop the disease.

A) 0.20 B) 0.22 C) 0.24 D) 0.26 E) 0.28

4. You are given the following information: $P(A|B) = 0.4 = P(A'|B')$ and $P(A) = 0.5$
Find $P(B)$.

A) 0.4 B) 0.5 C) 0.6 D) 0.7 E) 0.8

5. If with each new birth, boys and girls are equally likely to be born, find the probability that in a family with three children, exactly one is a girl.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

6. Six digits from $2, 3, 4, 5, 6, 7, 8$ are chosen and arranged in a row without replacement to create a 6-digit number. Find the probability that the resulting number is divisible by 2.

A) $\frac{5}{14}$ B) $\frac{3}{7}$ C) $\frac{1}{2}$ D) $\frac{4}{7}$ E) $\frac{9}{14}$

7. A bag contains 3 red balls, 2 white balls and 3 blue balls. Three balls are selected randomly from the bag **with replacement**. Given that no blue ball has been selected, calculate the probability that the number of red balls exceeds the number of white balls chosen.

A) $\frac{3}{8}$ B) $\frac{3}{5}$ C) $\frac{7}{10}$ D) $\frac{81}{512}$ E) $\frac{81}{125}$

8. A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions (each question has the same chance of being selected). To pass, he needs to answer at least two questions correctly. What is the probability that the student will pass the examination if he only knows the answers to 90 questions on the list?

A) Less than 0.96 B) At least 0.96 but less than 0.97 C) At least 0.97 but less than 0.98
D) At least 0.98 but less than 0.99 E) At least 0.99

9. A random variable X has a probability mass of 0.2 at $X = 0$ and a probability mass of 0.1 at $X = 1$. For all other values, X has the following density function:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 2x & 1 < x < c \\ 0 & x \geq c \end{cases}, \text{ where } c \text{ is a constant.}$$

Find $P(X < 1 | X > 0.5)$

A) Less than 0.6 B) At least 0.6 but less than 0.7 C) At least 0.7 but less than 0.8
D) At least 0.8 but less than 0.9 E) At least 0.9

10. An ordinary fair die is tossed independently until two consecutive tosses result in the same face turning up. X denotes the toss number on which this happens, so $X \geq 2$. Which of the following is $F(x)$, the CDF of X for $x \geq 2$?

A) $1 - (\frac{5}{6})^{x-1}$ B) $1 - (\frac{5}{6})^{x}$ C) $1 - (\frac{1}{6})^{x-1}$ D) $1 - (\frac{1}{6})^{x}$ E) $(\frac{5}{6})^{x}$

11. A discrete random variable X has the following probability function

x :	10	20	30	40	50
$f(x)$:	0.1	0.1	0.4	0.3	0.1

Denote by μ_X and σ_X the mean and the standard deviation of X. Find $P(|X - \mu_X| \leq \sigma_X)$.

A) 1 B) 0.8 C) 0.7 D) 0.5 E) 0.4

12. The continuous random variable X has pdf $f(x) = \frac{(k+1)x^k}{c^{k+1}}$ for $0 < x < c$, where $k > 0$.
The coefficient of variation of a random variable is defined to be $\frac{\sqrt{Variance}}{Mean}$.
Find the coefficient of variation of X.
A) $\frac{1}{\sqrt{(k+1)(k+2)}}$ B) $\frac{1}{\sqrt{(k+2)(k+3)}}$ C) $\frac{1}{\sqrt{(k+1)(k+3)}}$ D) $\frac{1}{k+1}$ E) $\frac{1}{k+3}$

13. X and Y are discrete random variables on the integers $\{0, 1, 2\}$, with moment generating functions $M_X(t)$ and $M_Y(t)$. You are given the following:
$M_X(t) + M_Y(t) = \frac{3}{4} + \frac{3}{4}e^t + \frac{1}{2}e^{2t}$ and $M_X(t) - M_Y(t) = \frac{1}{4} - \frac{1}{4}e^t$.
Find $P(X = 1)$.
A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

14. X is a continuous random variable for the density function $f(x) = \frac{|x|}{10}$ for $-4 \le x \le 2$, and $f(x) = 0$ otherwise. Find $|E(X) - m|$, where m is the median of X.
A) Less than .2 B) At least .2 but less than .4 C) At least .4 but less than .6
D) At least .6 but less than .8 E) At least .8

15. Smith plays a gambling game in which his probability of winning on any given play of the game is 0.4. If Smith bets 1 and wins, the amount he wins is 1, and if he loses, then he loses the amount of 1 that he bet. Smith devises the following strategy. If he loses a game, he doubles the amount that he bets on the next play of the game. He continues this strategy of doubling after each loss until he wins for the first time. He stops as soon as he wins for the first time. Smith has a limited amount of money to gamble, say $\$c$. If he loses all $\$c$ he goes broke and stops playing the game. Find the minimum amount c that Smith needs in order for him to have a probability of at least .95 of eventually winning before he goes broke.
A) 7 B) 15 C) 31 D) 63 E) 127

16. A production process for electronic components has a followup inspection procedure. Inspectors assign a rating of high, medium or low to each component inspected. Long run inspection data have yielded the following probabilities for component ratings:
$P(\text{high}) = 0.5$, $P(\text{medium}) = 0.4$, $P(\text{low}) = 0.1$.
Find the probability that in the next batch of 5 components inspected, at least 3 are rated high, and at most 1 is rated low.
A) Less than 0.10 B) At least 0.10 but less than 0.20 C) At least 0.20 but less than 0.30
D) At least 0.30 but less than 0.40 E) At least 0.40

17. A teacher in a high school class of 25 students must pick 5 students from the class for a school board math test. The students must be chosen randomly from the class. According to the teacher's assessment, there are 3 exceptional math students in the class and all the rest are average. Find the probability that at least 2 of the exceptional students are chosen for the test.

A) 0.01 B) 0.03 C) 0.05 D) 0.07 E) 0.09

18. Smith is a quality control analyst who uses the exponential distribution with a mean of 10 years as the model for the exact time until failure for a particular machine. Smith is really only interested in the integer number of years, say X, until the machine fails, so if failure is within the first year, Smith regards that as 0 (integer) years until failure, and if the machine does not fail during the first year but fails in the second year, the Smith regards that as 1 (integer) year until failure, etc. Smith's colleague Jones, who is also a quality control analyst reviews Smith's model for the random variable X and has two comments:

I. X has a geometric distribution.

II. The mean of X is 10.

Determine which, if any, of the statements made by Jones are true?

A) Neither are true B) Only Statement I is true C) Only Statement II is true

D) Both are true E) None of A, B, C or D is correct

19. X_1, X_2, X_3, X_4 and X_5 are independent normal random variables with $E(X_i) = Var(X_i) = i$ for $i = 1, 2, 3, 4, 5$.

We define Y to be $Y = \frac{1}{5}\sum_{i=1}^{5} X_i$. What is the 50th percentile of $Y - 3$? A) 0 B) 1.645 C) 3.84

D) 11.07 E) 19.21

20. According to the definition of the beta distribution X on the interval (0,1) with integer parameters $a \geq 1$ and $b \geq 1$, the pdf is $f(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} \cdot x^{a-1}(1-x)^{b-1}$.

Which of the following statements are true?

I. If $a = b$ then $E(X) = \frac{1}{2}$.

II. If $a = b$ then $Var(X) = \frac{1}{8a+1}$.

III. As k increases, $E(X^k)$ increases.

A) I only B) II only C) III only D) All but II E) All but III

21. The joint pdf of X and Y is $f(x,y) = kx^{-3}e^{-y/3}$ for $1 < x < \infty$ and $1 < y < \infty$.

Find $E(X)$ (the mean of the marginal distribution of X).

A) 4 B) 2 C) 1 D) $\frac{1}{2}$ E) $\frac{1}{4}$

22. Smith and Jones are financial analysts who enter a stock picking contest. Smith picks the stock of Company A and Jones picks the stock of Company B. Over the next week, the gain in the price of stock A will be X and the gain in the price of stock B will be Y, where X and Y have the joint density

$$f(x,y) = \frac{2}{3}(x+2y) \text{ for } 0 < x < 1 \text{ and } 0 < y < 1.$$

If the gain in Smith's stock for the week is greater than the gain in Jones' stock, Jones will pay Smith \$1000. How much should Smith pay Jones if the gain in Smith's stock is less than that of Jones' stock in order that Smith's expected return in this contest is 0?

A) 400 B) 800 C) 1200 D) 1600 E) 2000

23. X and Y are normal random variables with means μ_X and μ_Y, standard deviations σ_X and σ_Y, and correlation coefficient ρ. $F_X(t)$ and $F_Y(t)$ denote the cdf's of X and Y respectively.

If $\sigma_X = 2\sigma_Y$, for what values of t is it true that $F_X(t) \geq F_Y(t)$?

A) $t \leq 2\mu_Y - \mu_X$ B) $t \leq 2\mu_X - \mu_Y$ C) $t \leq \rho(\mu_X + \mu_Y)$

D) $t \leq \rho(\mu_X - \mu_Y)$ E) All real numbers t

24. X and Y are continuous random variables with pdf $f(x,y) = 2$ for $0 \leq x \leq y \leq 1$, and $f(x,y) = 0$ otherwise. Find the conditional expectation of Y given $X = x$.

A) $\frac{1}{2}$ B) $\frac{x}{2}$ C) $\frac{x+1}{2}$ D) $\frac{1-x}{2}$ E) x

25. $X_1, X_2, X_3, \ldots.$ is a sequence of independent random variables, each with mean 0 and variance 1. We define Y_k to be $X_1 + X_2 + \cdots + X_k$.

If $k < j$, what is the coefficient of correlation between Y_k and Y_j?

A) kj B) $\sqrt{kj}$ B) $\frac{k}{j}$ D) $\sqrt{\frac{k}{j}}$ E) $\frac{j}{k}$

26. X has an exponential distribution with mean 1 and $Y = X^2 - 1$. Find $F_Y(3)$.

A) $\frac{e^{-3}}{3}$ B) $\frac{e^{-2}}{4}$ C) $1 - 2e^{-2}$ D) $1 - e^{-2}$ E) $1 - e^{-3}$

27. A financial analyst uses the following model for the daily change in the price of a certain stock: $ln(\frac{X_{k+1}}{X_k})$ has a distribution with a mean of .01 and a variance of .0009 , where X_i is the stock closing price for trading day i. Assuming that the daily changes in price are independent from one day to the next, and assuming that the stock price closed at 1 on day 1, use the normal approximation to find the probability that the stock price is at least 4 at the close of trading day 101.

A) 0.05 B) 0.1 C) 0.5 D) 0.9 E) 0.95

28. An insurance policy has a deductible of 1 and pays a maximum of 1. The underlying loss random variable being insured by the policy has an exponential distribution with a mean of 1. Find the expected amount paid by the insurer on this policy.

A) $2e^{-2}$ B) e^{-1} C) $e^{-1} - e^{-2}$ D) $e^{-1} - 2e^{-2}$ E) $2(e^{-1} - e^{-2})$

29. X has a uniform distribution on the interval $(0, 2)$, and $Y = max\{X, 1\}$.

Find $Var(Y)$.

A) .05 B) .10 C) .15 D) .20 E) .25

30. X has a distribution with the following cdf $F(x) = 1 - e^{-(x/\theta)^\tau}$,

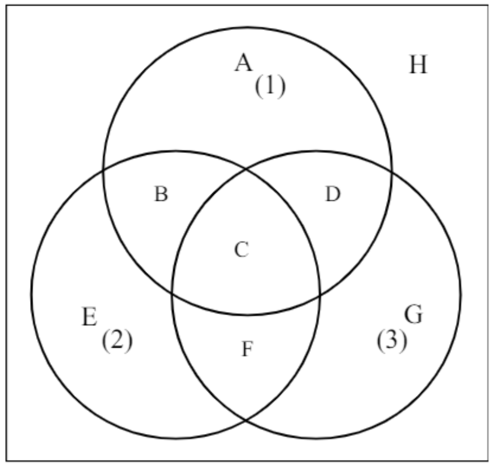
where $\tau > 0$ and $\theta > 0$. The random variable Y is defined to be $Y = g(X)$.

Which of the following transformations results in a distribution of Y which is exponential with a mean of 1?

A) $g(x) = x^\tau$ B) $g(x) = x^{1/\tau}$ C) $g(x) = (\frac{x}{\theta})^\tau$ D) $g(x) = (\frac{x}{\theta})^{1/\tau}$ E) $g(x) = \frac{\theta}{x^\tau}$

PRACTICE EXAM 6 - SOLUTIONS

1. We can represent the events in the following diagram:



The top circle, $A \cup B \cup C \cup D$ represents the event of having seen #1 of the movie series, the lower left circle, $E \cup B \cup C \cup F$ represents the event of having seen #2 of the movie series, the lower right circle, $G \cup F \cup C \cup D$ represents the event of having seen #3 of the movie series, and H represents the event of having seen none of the three movies.

From the given information, we know that the percentage for event H is $h = 50$.
The second line of the information table indicates that 35% of the public has seen movie #1 but we don't know about movies #2 and #3 for this group. This is interpreted as the percentage for $A \cup B \cup C \cup D$ is $a + b + c + d = 35$.

Similarly, the percentage for $E \cup B \cup C \cup F$ is $e + b + c + f = 33$, and the percentage for $G \cup F \cup C \cup D$ is $g + f + c + d = 31$.

The 5th line of the table indicates that 8% have seen movie #1 and not movies #2 or #3. Therefore, the percentage for event A is $a = 8$.

Event B is the event of having seen both #1 and #2 but not #3 and this has percentage $b = 4$, and event C is the event of have seen all three, and this has percentage $c = 20$.

The event of having seen exactly one of the three movies is the combination $A \cup E \cup G$.
This will be $a + e + g$.

We know that $a + b + c + d + e + f + g + h = 100$ percent, since everyone either sees a movie or doesn't. This leads to the following 8 equations:
$h = 50$ (1) , $a + b + c + d = 35$ (2) , $e + b + c + f = 33$ (3) , $g + f + c + d = 31$ (4),
$a = 8$ (5) , $b = 4$ (6) , $c = 20$ (7) , $a + b + c + d + e + f + g + h = 1$ (8) .

From equations (3), (6) and (7) we get $e+f=9$ (9).

From equations (1), (2) and (8) we get $e+f+g=15$ (10).

From equations (9) and (10) we get $g=6$ (11).

From equations (2), (5) and (6) we get $c+d=23$ (12).

From equations (11), (12) and (4) we get $f=2$ (13).

From equations (9) and (13) we get $e=7$.

Then $a+e+g=8+7+6=21$ is the percentage that has seen exactly one of the three movies.

Once we have determined the individual values of a, b, c, d, e, f, g, h, we can find the percentage for any combination. For instance, the percentage of people who have seen #1 and #3 but not #2 is $d=3$.

Answer: D

2. Since A and B are independent, we must have $P(A \cap B) = P(A) \cdot P(B)$.

Since $A \subseteq B$, it follows that $A \cap B = A$.

Therefore, we must have $P(A) = P(A) \times P(B)$.

The only way this can be true is if $P(A)$ is 0 or 1, or if $P(B)$ is 1.

Only D does not satisfy one of these conditions. Answer: D

3. We denote events as follows:

F - an individual has a family history of the disease

T - the genetic test indicates that an individual will develop the disease

D - an individual will develop the disease

We are given the probabilities $P(D) = 0.01$ and $P(F) = 0.02$.

The language "for those with a family history of the disease, 20% of the time the genetic test predicts that the individual will develop the disease" describes the conditional probability

$P($the genetic test indicates that the individual will develop the disease$|$the individual has a family history of the disease$) = P(T|F) = 0.20$.

In a similar way $P(T|F') = 0.01$, $P(D|T \cap F) = 0.80$ and $P(D|T' \cap F) = 0.10$.

We are asked to find $P(D|F) = \frac{P(D \cap F)}{P(F)}$.

We are given $P(F) = .02$ so that $P(F') = 0.98$.

Then, since $.20 = P(T|F) = \frac{P(T \cap F)}{P(F)}$, we get $P(T \cap F) = P(T|F) \times P(F) = 0.2 \times 0.02 = 0.004$.

Then, since $P(F) = P(T \cap F) + P(T' \cap F)$ it follows that $P(T' \cap F) = 0.02 - 0.004 = 0.016$.

Also, $P(D \cap T \cap F) = P(D|T \cap F) \cdot P(T \cap F) = 0.80 \times 0.004 = 0.0032$.

Similarly $P(D \cap T' \cap F) = P(D|T' \cap F) \times P(T' \cap F) = 0.10 \times 0.016 = 0.0016$.

Then $P(D \cap F) = P(D \cap T \cap F) + P(D \cap T' \cap F) = 0.0032 + 0.0016 = 0.0048$.

Finally, $P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{0{,}0048}{0.02} = 0.24$. Answer: C