27. Suppose that X_1 is the amount of Jim's loss and X_2 is Bob's loss. Since there is a .6 chance of no loss for an individual, the pdf of loss amount X is $f(x) = \begin{cases} 0.6 & X_2 = 0 \\ 0.0001 & 1000 \le X_2 \le 5000 \end{cases}$, and $f(x) = 0$ otherwise.

We wish to find $P[X_1 + X_2 > 8000 | X_1 > 2000]$. This is equal to $\frac{P[(X_1+X_2>8000)\cap(X_1>2000)]}{P[X_1>2000]}$.

From the pdf for X we have
$P[X_1 > 2000] = 0.0001 \times (5000 - 2000) = 0.3$.

If $X_1 \le 3000$, then it is impossible for $X_1 + X_2$ to be > 8000. Then,
$P[(X_1 + X_2 > 8000) \cap (X_1 > 2000)] = \int_{3000}^{5000} \int_{8000-x_1}^{5000} (0.0001)^2 \, dx_2 \, dx_1$
$\int_{3000}^{5000} (0.0001)^2 (x_1 - 3000) \, dx_1 = \frac{(0.0001)^2 (x_1-3000)^2}{2} \Big|_{x_1=3000}^{x_2=5000} = 0.02$.

The conditional probability in question is then
$\frac{P[(X_1+X_2>8000)\cap(X_1>2000)]}{P[X_1>2000]} = \frac{0.02}{0.3} = \frac{1}{15}$. Answer: B

28. Let us denote the 75th percentile of Y by c. Thus, $F(c) = 0.75$, so that
$1 - e^{-\frac{1}{2}(c-a)^2} = 0.75$. Solving this equation for c results in $e^{-\frac{1}{2}(c-a)^2} = 0.25$,
or equivalently, $\frac{1}{2}(c-a)^2 = ln\, 4 \rightarrow c = a + \sqrt{2\, ln\, 4} = a + 2\sqrt{ln\, 2}$. Answer: E

29. The transformation $Y = u(X) = \frac{1}{X-1}$ (for $X \ge 2$) is a decreasing function, and therefore is invertible, $X = u^{-1}(Y) = v(Y) = \frac{1}{Y} + 1$. Then using the standard method for finding the density of a transformed random variable, we have the density function of Y is
$g(y) = f(v(y)) \times \left| \frac{d}{dy} v(y) \right| = 2(\frac{1}{y} + 1)^{-2} \times \left| -\frac{1}{y^2} \right| = \frac{2}{(y+1)^2}$.

Alternatively, $F_Y(y) = P[Y \le y] = P[\frac{1}{X-1} \le y] = P[X \ge 1 + \frac{1}{y}]$
$= \int_{(y+1)/y}^{\infty} 2x^{-2} dx = \frac{2y}{y+1} \rightarrow f_Y(y) = F_Y'(y) = \frac{2}{(y+1)^2}$. Answer: B

30. $P[X + Y < 2] = P[(X = 0) \cap (Y < 2)] + P[(X = 1) \cap (Y < 1)]$
$P[(X = 0) \cap (Y < 2)] = \int_0^2 f(0, y) \, dy = \int_0^2 \frac{1}{12} e^{-y/2} dy = \frac{1}{6}[1 - e^{-1}]$,
$P[(X = 1) \cap (Y < 1)] = \int_0^1 f(1, y) \, dy = \int_0^1 \frac{2}{12} e^{-y/2} dy = \frac{1}{3}[1 - e^{-1/2}]$
$\Rightarrow P[X + Y < 2] = \frac{1}{6}[1 - e^{-1}] + \frac{1}{3}[1 - e^{-1/2}] = \frac{1}{6}[3 - 2e^{-1/2} - e^{-1}]$ Answer: A

PRACTICE EXAM 4

1. Event C is a subevent of $A \cup B$. Which of the following must be true.

   I. $A \cup C = B \cup C$

   II. $P(C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

   III. $A' \cap C \subseteq B$

   A) All but I B) All but II C) All but III D) II only E) III only

2. A dental insurance policy covers three procedures: orthodontics, fillings and extractions. During the life of the policy, the probability that the policyholder needs:

   · orthodontic work is 1/2

   · orthodontic work or a filling is 2/3

   · orthodontic work or an extraction is 3/4

   · a filling and an extraction is 1/8

   The need for orthodontic work is independent of the need for a filling and is also independent of the need for an extraction. Calculate the probability that the policyholder will need a filling or an extraction during the life of the policy.

   A) 7/24 B) 3/8 C) 2/3 D) 17/24 E) 5/6

3. A loss random variable has a continuous uniform distribution between 0 and \$100.
   An insurer will insure the loss amount above a deductible c. The variance of the amount that the insurer will pay is 69.75. Find c.

   A) 65 B) 70 C) 75 D) 80 E) 85

4. The joint probability of the three discrete random variables X, Y, Z is
   $f_{X,Y,Z}(x,y,z) = \frac{xy+xz^2}{24}$ for $x = 1, 2$, $y = 1, 2$, $z = 0, 1$.
   How many of the following statements are true?

   I. X and Y are independent

   II. X and Z are independent

   II. Y and Z are independent

   A) 0 B) 1 C) 2 D) 3

5. Let A and B be events such that $P[A] = 0.7$ and $P[B] = 0.9$.
   Calculate the largest possible value of $P[A \cup B] - P[A \cap B]$.

   A) 0.20 B) 0.34 C) 0.40 D) 0.60 E) 1.60

6. Bob has a fair die and tosses it until a "1" appears. Doug also has a fair die, and he tosses it until a "1" appears. Joe also has a fair die and he tosses it until a "1" appears. They each stop tossing as soon as a "1" turns up on their die. We define the random variable X to be the total number of tosses that Bob, Doug and Joe made (including the first "1" that each of them tossed).
Find $Var[X]$.
A) 50 B) 60 D) 70 D) 80 E) 90

7. Let X and Y be independent continuous random variables with common density function

$$f(t) = \begin{cases} 1 \text{ for } 0<t<1 \\ 0, \text{otherwise} \end{cases}.$$

What is $P[X^2 \geq Y^3]$?
A) $\frac{1}{3}$ B) $\frac{2}{5}$ C) $\frac{3}{5}$ D) $\frac{2}{3}$ E) 1

8. An auto insurer's portfolio of policies is broken into two classes - low risk, which make up 75% of the policies, and high risk, which make up 25% of the policies. The number of claims per year that occur from a policy in the low risk group has a Poisson distribution with a mean of 0.2, and the number of claims per year that occur from a policy in the high risk group has a Poisson distribution with a mean of 1.5. A policy is chosen at random from the insurer's portfolio. Find the probability that there will be exactly one claim during the year on that policy.
A) 0.21 B) 0.25 C) 0.29 D) 0.33 E) 0.37

9. Every member of an insured group has an annual claim amount distribution that is exponentially distributed. The expected claim amount of a randomly chosen member of the group is $\frac{1}{c}$, where c is uniformly distributed between 1 and 2. Find the probability that a randomly chosen member of the group has annual claim less than 1.
A) Less than 0.4 B) At least 0.4 but less than 0.5 C) At least 0.5 but less than 0.6
D) At least 0.6 but less than 0.7 E) At least 0.7

10. X has pdf $f(x) = e^{-x}$ for $x > 0$. If $a > 0$ and A is the event that $X > a$, find $f_{X|A}(x|x > a)$ the density of the conditional distribution of X given that $X > a$.
A) e^{-x} B) $e^{-(x-a)}$ C) e^{-x-a} D) e^{-ax} E) ae^{-ax}

11. A fair coin is tossed. If a head occurs, 1 fair die is rolled; if a tail occurs, 2 fair dice are rolled. If Y is the total on the die or dice, then $P[Y = 6] =$
A) $\frac{1}{9}$ B) $\frac{5}{36}$ C) $\frac{11}{72}$ D) $\frac{1}{6}$ E) $\frac{11}{36}$

12. A company is planning to begin a production process on a certain day. Government approval is needed in order to begin production, and it is possible that the approval might not be granted until after the planned starting day. For each day that the start of the process is delayed the company will incur a cost of 100,000. The company has determined that the number of days that the process will be delayed is a random variable with probability function

$p(n) = \frac{1}{(n+1)(n+2)}$ for $n = 0, 1, 2, \ldots$.

The company purchases "delay insurance" which will have the insurer pay the company's cost up to a maximum of 500,000. Find the expected cost to the insurer.

A) 100,000 B) 115,000 C) 130,000 D) 145,000 E) 160,000

13. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} 2 \text{ for } 0<x<y<1 \\ 0, \text{otherwise} \end{cases}.$$

Determine the density function of the conditional distribution of Y given $X = x$, where $0 < x < 1$.

A) $\frac{1}{1-x}$ for $x < y < 1$ B) $2(1-x)$ for $x < y < 1$ C) 2 for $x < y < 1$
D) $\frac{1}{y}$ for $x < y < 1$ E) $\frac{1}{1-y}$ for $x < y < 1$

14. A carnival sharpshooter game charges \$25 for 25 shots at a target. If the shooter hits the bullseye fewer than 5 times then he gets no prize. If he hits the bullseye 5 times he gets back \$10.

For each additional bullseye over 5 he gets back an additional \$5. The shooter estimates that he has a 0.2 probability of hitting the bullseye on any given shot. What is the shooter's expected gain if he plays the game (nearest \$1)?

A) -15 B) -10 C) -5 D) 0 E) 5

15. Let X and Y be random losses with joint density function

$f(x,y) = e^{-(x+y)}$ for $x > 0$ and $y > 0$.

An insurance policy is written to reimburse $X + Y$.

Calculate the probability that the reimbursement is less than 1.

A) e^{-2} B) e^{-1} C) $1 - e^{-1}$ D) $1 - 2e^{-1}$ E) $1 - 2e^{-2}$

16. Two components in an electrical circuit have continuous failure times X and Y. Both components will fail by time 1, but the circuit is designed so that the combined times until failure is also less than 1, so that the joint distribution of failure times satisfies the requirements $0 < x + y < 1$. Suppose that the joint density is constant on the probability space. Find the probability that both components will fail by time $\frac{1}{2}$.

A) $\frac{1}{16}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) 1

17. A company has annual losses that can be described by the continuous random variable X, with density function $f(x)$. The company wishes to obtain insurance coverage that covers annual losses above a deductible. The company is trying to choose between deductible amounts d_1 and d_2, where $d_1 < d_2$. With deductible d_1 the expected annual losses that would not be covered by insurance is E_1, and with deductible d_2 the expected annual losses that would not be covered by insurance is E_2. Which of the following is the correct expression for $E_2 - E_1$?

A) $\int_{d_1}^{d_2} x f(x)\,dx$ B) $(d_2 - d_1) \times [F(d_2) - F(d_1)]$

C) $\int_{d_1}^{d_2} x f(x)\,dx + (d_2 - d_1) \times [F(d_2) - F(d_1)]$ D) $\int_{d_1}^{d_2} x f(x)\,dx + (d_2 - d_1)$

E) $\int_{d_1}^{d_2} x f(x)\,dx + (d_2 - d_1) - [d_2 F(d_2) - d_1 F(d_1)]$

18. A discrete integer valued random variable has the following probability function: $P[X = n] = a_n - a_{n+1}$, where the a's are numbers which satisfy the following conditions:

(i) $a_0 = 1$

(ii) $a_0 > a_1 > a_2 > \cdots > a_k > a_{k+1} > \cdots > 0$

Find the probability $P[X \le 5 | X > 1]$.

A) $1 - \frac{a_5}{a_2}$ B) $1 - \frac{a_5}{a_1}$ C) $a_1 - a_5$ D) $\frac{a_2}{a_1} - \frac{a_5}{a_2}$ E) $\frac{a_2 - a_6}{a_2}$

19. In a carnival sharpshooter game the shooter pays \$10 and takes successive shots at a target until he misses. Each time he hits the target he gets back \$3. The game is over as soon as he misses a target. The sharpshooter estimates his probability of hitting the target on any given shot as p. According to this estimate he expects to gain \$2 on the game. Find p.

A) 0.5 B) 0.6 C) 0.7 D) 0.8 E) 0.9

20. In a model for hospital room charges X and hospital surgical charges Y for a particular type of hospital admission, the region of probability (after scaling units) is $0 \le y \le 2x + 1 \le 3$ and $x \ge 0$. The joint density function of X and Y is $f(x, y) = 0.3 \times (x + y)$.

Find $E[Y - X]$, the expected excess of surgical charges over room charges for an admission.

A) $-\frac{3}{4}$ B) $-\frac{1}{4}$ C) 0 D) $\frac{1}{4}$ E) $\frac{3}{4}$

21. Auto claim amounts, in thousands, are modeled by a random variable with density function $f(x) = xe^{-x}$ for $x > 0$. The company expects to pay 100 claims if there is no deductible.

How many claims does the company expect to pay if the company decides to introduce a deductible of 1000?

A) 26 B) 37 C) 50 D) 63 E) 74

22. Let X have a uniform distribution on the interval $(1, 3)$. What is the probability that the sum of 2 independent observations of X is greater than 5?

A) $\frac{1}{18}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

23. Let X_1, X_2 and X_3 be three independent continuous random variables each with density function $f(x) = \begin{cases} \sqrt{2}-x & \text{for } 0<x<\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$.

What is the probability that exactly 2 of the 3 random variables exceeds 1?

A) $\frac{3}{2} - \sqrt{2}$ B) $3 - 2\sqrt{2}$ C) $3(\sqrt{2}-1)(2-\sqrt{2})^2$

D) $(\frac{3}{2} - \sqrt{2})^2(\sqrt{2} - \frac{1}{2})$ E) $3(\frac{3}{2} - \sqrt{2})^2(\sqrt{2} - \frac{1}{2})$

24. An insurer has a portfolio of 1000 independent one-year term insurance policies. For any one policy, there is a probability of .01 that there will be a claim. Use the normal approximation to find the probability that the insurer will experience at least 15 claims using the integer correction.

A) 0.08 B) 0.10 C) 0.12 D) 0.14 E) 0.16

25. The number of claims occurring in a period has a Poisson distribution with mean λ.
The insurer determines the conditional expectation of the number of claims in the period given that at least one claim has occurred, say $e(\lambda)$. Find $\lim_{\lambda \to 0} e(\lambda)$.

A) 0 B) e^{-1} C) 1 D) e E) ∞

26. An insurer notices that for a particular class of policies, whenever the claim amount is over 1000, the average amount by which the claim exceeds 1000 is 500. The insurer assumes that the claim amount distribution has a uniform distribution on the interval $[0, c]$, where $c > 1000$. Find the value of c that is consistent with the observation of the insurer.

A) 1500 B) 2000 C) 2500 D) 4000 E) 5000

27. Let X and Y be discrete random variables with joint probability function

$$f(x,y) = \begin{cases} \frac{(x+1)(y+2)}{54} & \text{for } x=0,1,2\,;\ y=0,1,2 \\ 0, \text{otherwise} \end{cases}.$$

What is $E[Y|X=1]$?

A) $\frac{11}{27}$ B) 1 C) $\frac{11}{9}$ D) $\frac{y+2}{9}$ E) $\frac{y^2+2y}{9}$

28. The following probabilities of three events in a sample space are given:

$P[A] = 0.6\,,\ P[B] = 0.5\,,\ P[C] = 0.4\,,$

$P[A \cup B] = 1\,,\ P[A \cup C] = 0.7\,,\ P[B \cup C] = 0.7$

Find $P[A \cap B \cap C | (A \cap B) \cup (A \cap C) \cup (B \cap C)]$.

A) 0.1 B) 0.15 C) 0.2 D) 0.25

E) Cannot be determined from the given information

29. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 1-|x| \text{ for } -1<x<1 \\ 0, \text{otherwise} \end{cases}.$$

Determine the density function of $Y = X^2$ where it is nonzero.

A) $\frac{1}{\sqrt{y}} - 1$ B) $2\sqrt{y} - y$ C) $2\sqrt{y}$ D) $1 - \sqrt{y}$ E) $\frac{1}{\sqrt{y}}$

30. The claim amount random variable B has the following distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2,000 & 0 \le x < 1000 \\ 0.75 & x = 1000 \\ (x+11,000)/16,000 & 1000 < x < 5000 \\ 1 & x \ge 5000 \end{cases}.$$

What is $E[B] + \sqrt{Var(B)}$?

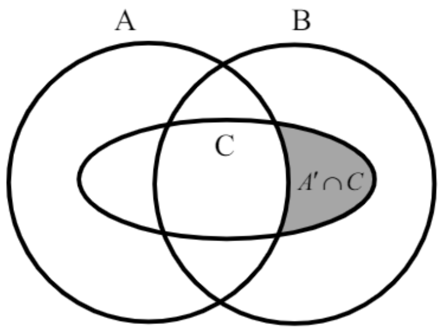
A) 2400 B) 2450 C) 2500 D) 2550 E) 2600

PRACTICE EXAM 4 - SOLUTIONS

1. I. False.

   II. $C = (A \cap C) \cup (B \cap C) \rightarrow P(C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$. True.

   III. The following diagram explains this situation.



$C = (A \cap C) \cup (A' \cap C)$, and $C = (A \cap C) \cup (B \cap C)$

$\rightarrow (A' \cap C) \subseteq (A \cap C) \cup (B \cap C)$

$\rightarrow (A' \cap C) = [(A' \cap C) \cap (A \cap C)] \cup [(A' \cap C) \cap (B \cap C)]$

$= \emptyset \cup (A' \cap B \cap C) \subseteq B$. True. Answer: A

2. We identify the events:

   O - orthodontic work will be needed during the lifetime of the policy

   F - a filling will be needed during the lifetime of the policy

   E - an extraction will be needed during the lifetime of the policy

   We wish to find $P[F \cup E]$.

   We are given: $P[O] = \frac{1}{2}$, $P[O \cup F] = \frac{2}{3}$, $P[O \cup E] = \frac{3}{4}$, $P[F \cap E] = \frac{1}{8}$.

   Using rules of probability, we have $P[O \cup F] = P[O] + P[F] - P[O \cap F]$,
   and since O and F are independent, we have $P[O \cap F] = P[O] \cdot P[F]$, so that
   $P[O \cup F] = \frac{2}{3} = P[O] + P[F] - P[O] \times P[F] = \frac{1}{2} + P[F] - \frac{1}{2} \times P[F]$,
   from which it follows that $P[F] = \frac{1}{3}$.
   In a similar way, since O and E are independent,
   $P[O \cup E] = \frac{3}{4} = P[O] + P[E] - P[O] \times P[E] = \frac{1}{2} + P[E] - \frac{1}{2} \times P[E]$,
   from which it follows that $P[E] = \frac{1}{2}$.
   Now, $P[F \cup E] = P[F] + P[E] - P[F \cap E] = \frac{1}{3} + \frac{1}{2} - \frac{1}{8} = \frac{17}{24}$. Answer: D

3. The insurer pays $Y = \begin{cases} 0, \text{ if } x<c \\ x-c, \text{ if } c<x\leq 100 \end{cases}$ with constant density 0.01.

Then $E[Y] = \int_c^{100} \frac{x-c}{100}\,dx = \frac{(100-c)^2}{200}$ and $E[Y^2] = \int_c^{100} \frac{(x-c)^2}{100}\,dx = \frac{(100-c)^3}{300}$

so that $Var[Y] = \frac{(100-c)^3}{300} - [\frac{(100-c)^2}{200}]^2$. Substituting the possible answers, we see that with $c = 70$, the variance of Y is 69.75. Answer: B

4. The joint probability table is

x	y	z	$f_{X,Y,Z}(x,y,z)$
1	1	0	$1/24$
1	1	1	$2/24$
1	2	0	$2/24$
1	2	1	$3/24$
2	1	0	$2/24$
2	1	1	$4/24$
2	2	0	$4/24$
2	2	1	$6/24$

The marginal probability functions are

$f_X(1) = \frac{1+2+2+3}{24} = \frac{1}{3}$, $f_X(2) = \frac{2+4+4+6}{24} = \frac{2}{3}$,

$f_Y(1) = \frac{1+2+2+4}{24} = \frac{3}{8}$, $f_Y(2) = \frac{2+3+4+6}{24} = \frac{5}{8}$,

$f_Z(0) = \frac{1+2+2+4}{24} = \frac{3}{8}$, $f_Z(1) = \frac{2+3+4+6}{24} = \frac{5}{8}$.

The joint distribution of X and Y is

$f_{X,Y}(1,1) = \frac{1+2}{24} = \frac{1}{8} = \frac{1}{3} \times \frac{3}{8} = f_X(1) \times f_Y(1)$,

$f_{X,Y}(1,2) = \frac{2+3}{24} = \frac{5}{24} = \frac{1}{3} \times \frac{5}{8} = f_X(1) \times f_Y(2)$,

$f_{X,Y}(2,1) = \frac{2+4}{24} = \frac{1}{4} = \frac{2}{3} \times \frac{3}{8} = f_X(2) \times f_Y(1)$,

$f_{X,Y}(2,2) = \frac{4+6}{24} = \frac{5}{12} = \frac{2}{3} \times \frac{5}{8} = f_X(2) \times f_Y(2)$.

Since $f_{X,Y}(x,y) = f_X(x) \times f_Y(y)$ for all x, y it follows that X and Y are independent.

The joint distribution of X and Z is

$f_{X,Z}(1,0) = \frac{1}{8} = \frac{1}{3} \times \frac{3}{8} = f_X(1) \times f_Z(0)$,

$f_{X,Z}(1,1) = \frac{5}{24} = \frac{1}{3} \times \frac{5}{8} = f_X(1) \times f_Z(1)$,

$f_{X,Z}(2,0) = \frac{1}{4} = \frac{2}{3} \times \frac{3}{8} = f_X(2) \times f_Z(0)$,

$f_{X,Z}(2,1) = \frac{5}{12} = \frac{2}{3} \times \frac{5}{8} = f_X(2) \times f_Z(1)$.

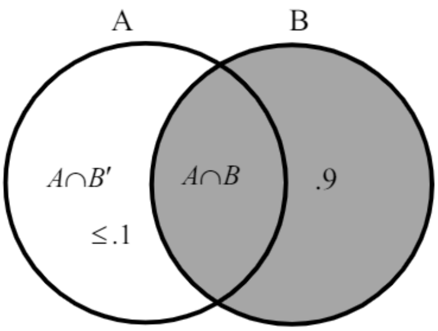
Since $f_{X,Z}(x,z) = f_X(x) \times f_Z(z)$ for all x, z it follows that X and Z are independent.

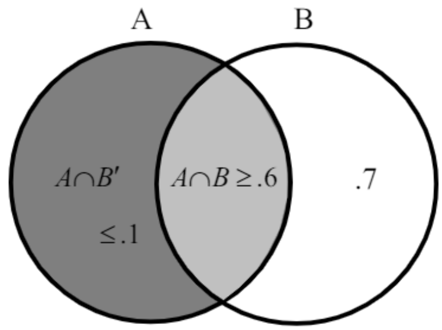
For the joint distribution of Y and Z we have

$f_{Y,Z}(1,0) = \frac{1}{8} \neq \frac{3}{8} \times \frac{3}{8} = f_Y(1) \times f_Z(0)$.

It follows that Y and Z are not independent. Answer: C

5. $P[A \cup B] - P[A \cap B] = P[A] + P[B] - 2P[A \cap B] = 1.6 - 2P[A \cap B]$.
This will be maximized if $P[A \cap B]$ is minimized.
But $0.7 = P[A] = P[A \cap B'] + P[A \cap B]$, and the maximum possible value of $P[A \cap B']$ is 0.1 (since $P[B]$ is 0.9, it follows that $P[A \cap B'] \leq P[B'] = 0.1$), so that the minimum possible value for $P[A \cap B]$ is 0.6, and then the maximum of $P[A \cup B] - P[A \cap B]$ is $1.6 - 2 \times 0.6 = 0.4$.





Answer: C

6. If we consider Doug first and define "success" to be tossing a "1" and "failure" to be a toss that is not a "1" then the number of tosses Doug makes, say X, before his first 1 is the number of failures before the first success. X has a geometric distribution with $p = \frac{1}{6}$ (p is the probability of success on a single trial).
Doug's total number of tosses is $X + 1$, and

$$Var(X+1) = Var(X) = \frac{1-p}{p^2} = \frac{1-\frac{1}{6}}{(\frac{1}{6})^2} = 30.$$

Bob and Joe have the same distribution for the number of tosses until a first "1", and since they toss independently of Doug and each other, the variance of the total number of tosses is just the sum of the three variances, which is $30 + 30 + 30 = 90$. Answer: E

7. Since both X and Y are between 0 and 1, the event $X^2 > Y^3$ is equivalent to $X > Y^{3/2}$. Since X and Y are independent, their joint density is $f(x,y) = f_X(x) \cdot f_Y(y) = 1$. Then,
$P[X^2 > Y^3] = \int_0^1 \int_{y^{3/2}}^1 1\, dx\, dy = \int_0^1 (1 - y^{3/2})\, dy = \frac{3}{5}$. Answer: C

8. $P[N=1] = P[N=1|\text{low risk}] \times P[\text{low risk}] + P[N=1|\text{high risk}] \times P[\text{high risk}]$
$P[N=1|\text{low risk}] = e^{-.2} \times 0.2$, $P[N=1|\text{high risk}] = e^{-1.5} \times 1.5$,
$P[\text{low risk}] = 0.75$, $P[\text{high risk}] = 0.25$
$\rightarrow P[N=1] = 0.75 \times e^{-.2} \times 0.2 + 0.25 \times e^{-1.5} \times 1.5 = 0.206$. Answer: A

9. $P[X < 1|c] = 1 - e^{-c}$

$P[X < 1] = \int_1^2 P[X < 1|c] \times f(c)\,dc = \int_1^2 (1 - e^{-c})\,dc = 0.767$ Answer: E

10. $P(A) = P[X > a] = \int_a^\infty e^{-x}\,dx = e^{-a}$

$f_{X|A}(x|X > a) = \frac{f_X(x)}{P(A)} = \frac{e^{-x}}{e^{-a}} = e^{-(x-a)}$, for $x > a$, and $f_{X|A}(x|X > a) = 0$ for $x \le a$

Answer: B

11. If one fair die is rolled, the probability of rolling a 6 is $\frac{1}{6}$, and if two fair dice are rolled, the probability of rolling a 6 is $\frac{5}{36}$ (of the 36 possible rolls from a pair of dice, the rolls 1-5, 2-4, 3-3, 4-2 and 5-1 result in a total of 6). Since the coin is fair, the probability of rolling a head or tail is .5. Thus, the probability that $Y = 6$ is $0.5 \times \frac{1}{6} + 0.5 \times \frac{5}{36} = \frac{11}{72}$. Answer: C

12. The amount covered by the insurance (in 100,000's) is

Days of delay	0	1	2	3	4	5	6	$\cdots$
Insurer Cost	0	1	2	3	4	5	5	$\cdots$

The expected insurer cost is

$0 \times p(0) + 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + 4 \times p(4) + 5 \times P[N \ge 5]$

$= 0 \times \frac{1}{2} + 1 \times \frac{1}{6} + 2 \times \frac{1}{12} + 3 \times \frac{1}{20} + 4 \times \frac{1}{30} + 5 \times [1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{12} - \frac{1}{20} - \frac{1}{30}] = 1.45.$

Answer: D

13. The region of joint density is the triangular region above the line $y = x$ and below the horizontal line $y = 1$ for $0 < x < 1$. The conditional density of y given $X = x$ is

$f(y\,|\,X = x) = \frac{f(x,y)}{f_X(x)}$, where $f_X(x)$ is the marginal density function of x.

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_x^1 2\,dy = 2(1 - x)$, so that $f(y\,|\,X = x) = \frac{2}{2(1-x)} = \frac{1}{1-x}$

and the region of density for the conditional distribution of Y given $X = x$ is $x < y < 1$.

It is true in general that if a joint distribution is uniform (has constant density in a region) then any conditional (though not necessarily marginal) distribution will be uniform on it restricted region of probability - the conditional distribution of Y given $X = x$ is uniform on the interval $x < y < 1$, with constant density $\frac{1}{1-x}$. Answer: A

14.

No. of bullseyes	0 to 4	5	6	7	8	9	10 ...	23	24	25
Prize	0	10	15	20	25	30	35 ...	100	105	110
Gain	– 25	– 15	– 10	– 5	0	5	10 ...	75	80	85

Let $X =$ number of bullseyes. X has a binomial distribution with $n = 25$, $p = 0.2$, and $E[X] = 5$. $p(x) = \binom{25}{x}(0.2)^x(0.8)^{25-x}$.

Note that for 5 bullseyes or more the prize is $5X - 15$.

We can find the expected prize by first finding $E[5X - 15]$ and adjusting for the factors corresponding to $X = 0, 1, 2, 3, 4$. Therefore,

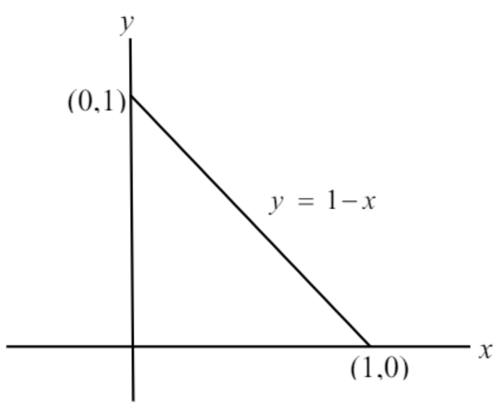
Expected prize

$$= E[5X - 15] + 15 \times p(0) + 10 \times p(1) + 5 \times p(2) + 0 \times p(3) - 5 \times p(4)$$
$$= 5E[X] - 15 + 15\binom{25}{0}(0.2)^0(0.8)^{25} + 10\binom{25}{1}(0.2)(0.8)^{24} + 5\binom{25}{2}(0.2)^2(0.8)^{23}$$
$$+ (0)\binom{25}{3}(0.2)^3(.8)^{22} - 5\binom{25}{4}(02)^4(0.8)^{21} = 9.71.$$

The expected gain is $9.71 - 25 = -15.29$. Answer: A

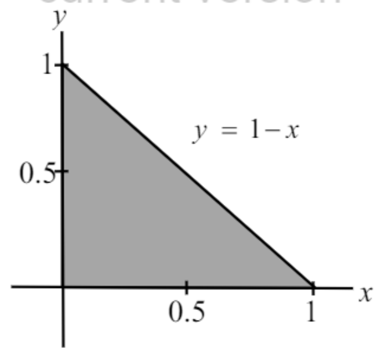
15. The probability in question is found by integrating the joint density function $f(x, y)$ over the two-dimensional region that represents the event. This two-dimensional region is $\{(x, y) : x + y < 1,\ x > 0\ ,\ y > 0\} = \{(x, y) : y < 1 - x,\ x > 0\ ,\ y > 0\}$. This region is represented in the shaded area in the graph below. The probability is

$$\int_0^1 \int_0^{1-x} f(x, y)\, dy\, dx = \int_0^1 \int_0^{1-x} e^{-(x+y)}\, dy\, dx = \int_0^1 e^{-x} \times \left[-e^{-y}\Big|_{y=0}^{y=1-x} \right] dx$$
$$= \int_0^1 e^{-x}[1 - e^{x-1}]\, dx = \int_0^1 [e^{-x} - e^{-1}]\, dx = \left[-e^{-x}\Big|_{x=0}^{x=1} \right] - e^{-1} = 1 - 2e^{-1}.$$



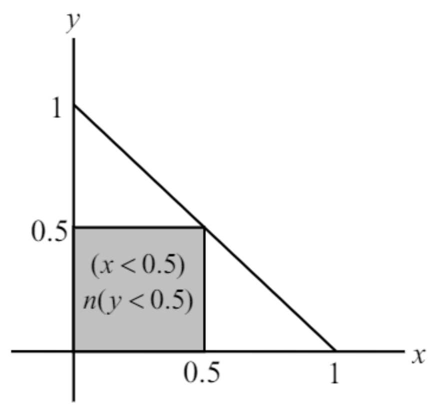
Answer: D

16. The region of probability is the shaded area below.



The joint density is $f(x,y) = \frac{1}{Area\,of\,probability\,space} = \frac{1}{1/2} = 2$.

Since the joint density is constant, the probability that both components will fail by time $\frac{1}{2}$ is $\frac{area\,of\,region\,A}{total\,area\,of\,probability\,space}$, where A is the region representing the event that both components fail by time $\frac{1}{2}$, $(X < \frac{1}{2}) \cap (Y < \frac{1}{2})$. This area is $\frac{1}{4}$, so that $P[(X < \frac{1}{2}) \cap (Y < \frac{1}{2})] = \frac{1/4}{1/2} = \frac{1}{2}$.



Alternatively, we can also formulate the probability as
$\int_0^{1/2}\int_0^{1/2} f(x,y)\,dy\,dx = \int_0^{1/2}\int_0^{1/2} 2\,dy\,dx = \frac{1}{2}$. Answer: D

17. $E_1 = \int_0^{d_1} xf(x)\,dx + \int_{d_1}^{\infty} d_1 \times f(x)\,dx = \int_0^{d_1} xf(x)\,dx + d_1 \times [1 - F(d_1)]$,
and similarly, $E_2 = \int_0^{d_2} xf(x)\,dx + d_2 \times [1 - F(d_2)]$. Answer: E

18. $P[X \le 5 | X > 1] = \frac{P[1<X\le 5]}{P[X>1]}$

$P[X > 1] = 1 - P[X = 0,1] = 1 - (1 - a_1) - (a_1 - a_2) = a_2$

$P[1 < X \le 5] = P[X = 2,3,4,5]$

$= (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) + (a_5 - a_6) = a_2 - a_6$

$P[X \le 5 | X > 1] = \frac{a_2 - a_6}{a_2}$ Answer: E

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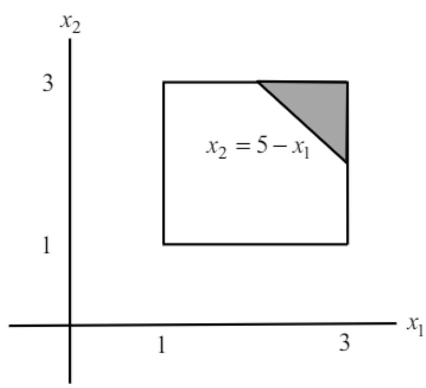
19. Let X be the number of targets he hits until the first miss. Then the probability function for X is $P[X=k]=p^k(1-p)$. This is the form of the geometric distribution in which we count the number of failures until the first success ("failure" in this context means hitting the target, and "success" means missing the target); however in this case the probability of success is $1-p$.
Therefore, the expected value of X is $E[X]=\frac{1-(1-p)}{1-p}=\frac{p}{1-p}$.
The expected gain from the game is $3E[X]-10$ (3 dollars for each hit minus the initial cost).
To have an expected gain of 2, we have $3(\frac{p}{1-p})-10=2 \rightarrow p=0.8$.
Answer: D

20. $E[Y-X]=\int_0^1\int_0^{2x+1}(y-x)(.3)(x+y)\,dy\,dx=\int_0^1(0.2x^3+0.9x^2+0.6x+.1)\,dx=\frac{3}{4}$.
Answer: E

21. There are 100 expected claims (each claim will result in a payment if there is no deductible).
The probability of a claim being above 1000 (one thousand) is
$\int_1^\infty xe^{-x}\,dx=-xe^{-x}-e^{-x}\Big|_{x=1}^{\infty}=2e^{-1}=0.7358$. Of the 100 policies, the expected number that will have claim amounts over 1000 is $(1000)(.7358)=73.6 \rightarrow 74$. Answer: E

22. The probability $P[X_1+X_2>5]$ is the integral of the joint density of $X_1$and X_2 over the shaded region at the right. This region is $2\leq x_1\leq 3$ and $5-x_1\leq x_2\leq 3$. The probability is
$\int_2^3\int_{5-x_1}^3\frac{1}{2}\times\frac{1}{2}dx_2\,dx_1=\frac{1}{8}$.
Answer: B



23. $P[X\leq 1]=\int_0^1(\sqrt{2}-x)\,dx=\sqrt{2}-\frac{1}{2}$, $P[X>1]=1-P[X\leq 1]=\frac{3}{2}-\sqrt{2}$.
With 3 independent random variables, X_1, X_2 and X_3, there are 3 ways in which exactly 2 of the X_i's exceed 1 (either X_1,X_2 or X_1,X_3 or X_2,X_3). Each way has probability $(P[X>1])^2\cdot P[X\leq 1]=(\frac{3}{2}-\sqrt{2})^2(\sqrt{2}-\frac{1}{2})$ for a total probability of $3\cdot(\frac{3}{2}-\sqrt{2})^2(\sqrt{2}-\frac{1}{2})$. Answer: E

24. The total number of claims follows a binomial distribution with $n = 1000$ trials and $q = .01$ probability of "success" (claim) for each trial. Since the binomial distribution is the sum of independent Bernoulli trials, the normal approximation applies to the total number of claims N. $E[N] = 1000 \times 0.01 = 10$ and $Var[N] = 1000 \times 0.01 \times 0.99 = 9.9$.
Then $P[N \geq 15] = P[N \geq 14.5] = P\left[\frac{N - E[N]}{\sqrt{Var[N]}} \geq \frac{14.5 - 10}{\sqrt{9.9}}\right] = P[Z \geq 1.43]$
$= 1 - \Phi(1.43) = 1 - 0.9234 = 0.0766$ (interpolation between $\Phi(1.4)$ and $\Phi(1.5)$ in the normal tables provided with the exam). Answer: A

25. $E[N|N \geq 1] = \sum_{n=1}^{\infty} n \times f(n|N \geq 1) = \sum_{n=1}^{\infty} n \times \frac{f(n)}{1-f(0)} = \sum_{n=0}^{\infty} n \times \frac{f(n)}{1-f(0)}$
$= \frac{1}{1-f(0)} \times \sum_{n=0}^{\infty} n \times f(n) = \frac{E[N]}{1-f(0)} = \frac{\lambda}{1-e^{-\lambda}}$.
$\lim_{\lambda \to 0} E[N|N \geq 1] = \lim_{\lambda \to 0} \frac{\lambda}{1-e^{-\lambda}} =$ (by l'Hospital's rule) $\lim_{\lambda \to 0} \frac{1}{e^{-\lambda}} = 1$. Answer: C

26. If the claim amount X is uniform on the interval $[0, c]$, then the conditional density
$f(x|X > 1000) = \frac{f(x)}{P[X>1000]} = \frac{1/c}{(c-1000)/c} = \frac{1}{c-1000}$, for $1000 < x < c$.
Then the density of $X - 1000$ given $X > 1000$ is uniform on the interval $[0, c - 1000]$, and has a mean of $\frac{c-1000}{2}$. In order for this to be 500, we must have $\frac{c-1000}{2} = 500$, so that $c = 2000$. Answer: B

27. $f_X(1) = P[X = 1] = \sum_{y=-\infty}^{\infty} f(1, y) = f(1, 0) + f(1, 1) + f(1, 2) = \frac{1}{3}$
Then we have conditional probabilities $P[Y = 0|X = 1] = \frac{f(1,0)}{P[X=1]} = \frac{4/54}{1/3} = \frac{2}{9}$,
and similarly, $P[Y = 1|X = 1] = \frac{1}{3}$ and $P[Y = 2|X = 1] = \frac{4}{9}$.
Then, $E[Y|X = 1] = 0 \times \frac{2}{9} + 1 \times \frac{1}{3} + 2 \times \frac{4}{9} = \frac{11}{9}$. Answer: C

28. $P[A \cap B \cap C|(A \cap B) \cup (A \cap C) \cup (B \cap C)] = \frac{P[A \cap B \cap C]}{P[(A \cap B) \cup (A \cap C) \cup (B \cap C)]}$.
$P[A \cap B] = P[A] + P[B] - P[A \cup B] = .1$, and similarly, $P[A \cap C] = 0.3$,
$P[B \cap C] = 0.2$. Since $P[A \cup B] = 1$, it follows that $P[A \cup B \cup C] = 1$.
Then, since $1 = P[A \cup B \cup C]$
$= P[A] + P[B] + P[C] - (P[A \cap B] + P[A \cap C] + P[B \cap C]) + P[A \cap B \cap C]$
it follows that $P[A \cap B \cap C] = 0.1$. Also,
$P[(A \cap B) \cup (A \cap C) \cup (B \cap C)]$
$= P[A \cap B] + P[A \cap C] + P[B \cap C] - 3P[A \cap B \cap C] + P[A \cap B \cap C] = 0.4$,
so that $\frac{P[A \cap B \cap C]}{P[(A \cap B) \cup (A \cap C) \cup (B \cap C)]} = 0.25$. Answer: D

29. For $0 \le y < 1$, $F_Y(y) = P[Y \le y] = P[X^2 \le y] = P[|X| \le \sqrt{y}]$
$= \int_{-\sqrt{y}}^{\sqrt{y}} (1 - |x|)\, dx = \int_{-\sqrt{y}}^{0} (1 + x)\, dx + \int_0^{\sqrt{y}} (1 - x)\, dx = 2\sqrt{y} - y.$
Then, $f_Y(y) = F_Y'(y) = \frac{1}{\sqrt{y}} - 1$ for $0 \le y < 1$.

Note that in this case the transformation $u(x) = x^2$ is not one-to-one on the region of probability of X $(-1 < x < 1)$, we cannot use the $f_Y(y) = f_X(v(y)) \times \left|\frac{d}{dy} v(y)\right|$ approach.

Answer: A

30. The pdf of B is $f(x) = 0$ for $x < 0$ and $f(x) = 0$ for $x \ge 5000$,
it is $f(x) = 0.0005$ for $0 \le x \le 1000$, and it is $f(x) = 0.0000625$ for $1000 < x < 5000$.
There is a point mass of probability with $f(x) = 0.25$ at $x = 1000$ (B has a mixed distribution).
$E[B] = \int_0^{1000} x \times 0.0005\, dx + 1000 \times 0.25 + \int_{1000}^{5000} x \times 0.0000625\, dx = 1250,$
$E[B^2] = \int_0^{1000} x^2 \times 0.0005\, dx + 1000^2 \times 0.25 + \int_{1000}^{5000} x^2 \times 0.0000625\, dx = 3,000,000$
$Var[B] = E[B^2] - (E[B])^2 = 1,437,500.$ $E[B] + \sqrt{Var[B]} = 2449.$ Answer: B.

PRACTICE EXAM 5

1. X has pdf $f(x) = \frac{c}{(x+\theta)^{\alpha+1}}$, defined for $x > 0$, where α and θ are both > 0.
   Find $F(x)$ for $x > 0$.
   A) $(\frac{\theta}{x+\theta})^{\alpha}$ B) $1 - (\frac{\theta}{x+\theta})^{\alpha}$ C) $(\frac{\theta}{x+\theta})^{\alpha+1}$ D) $1 - (\frac{\theta}{x+\theta})^{\alpha+1}$ E) $(\frac{\theta}{x+\theta})^{\alpha-1}$

2. Micro Insurance Company issued insurance policies to 32 independent risks. For each policy, the probability of a claim is 1/6. The benefit amount given that there is a claim has probability density function
   $$f(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$
   Calculate the expected value of total benefits paid.
   A) $\frac{16}{9}$ B) $\frac{8}{3}$ C) $\frac{32}{9}$ D) $\frac{16}{3}$ E) $\frac{32}{3}$

3. Let A, B and C be events such that A and B are independent, B and C are mutually exclusive, $P[A] = \frac{1}{4}$, $P[B] = \frac{1}{6}$, and $P[C] = \frac{1}{2}$. Find $P[(A \cap B)' \cup C]$.
   A) $\frac{11}{24}$ B) $\frac{3}{4}$ C) $\frac{5}{6}$ D) $\frac{23}{24}$ E) 1

4. If the mean and variance of random variable X are 2 and 8, find the first three terms in the Taylor series expansion of the moment generating function of X about the point $t = 0$.
   A) $2t + 2t^2$ B) $1 + 2t + 6t^2$ C) $1 + 2t + 2t^2$ D) $1 + 2t + 4t^2$ E) $1 + 2t + 12t^2$

5. A small commuter plane has 30 seats. The probability that any particular passenger will not show up for a flight is 0.10, independent of other passengers. The airline sells 32 tickets for the flight. Calculate the probability that more passengers show up for the flight than there are seats available.
   A) 0.0042 B) 0.0343 C) 0.0382 D) 0.1221 E) 0.1564

6. The moment generating function for the random variable X is $M_X(t) = Ae^t + Be^{2t}$.
   You are given that $Var[X] = \frac{2}{9}$ and $A < \frac{1}{2}$. Find $E[X]$.
   A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) 1 D) $\frac{4}{3}$ E) $\frac{5}{3}$

7. The exponential distribution with mean 1 is being used as the model for a loss distribution. An actuary attempts to "discretize" the distribution by assigning a probability to $k + \frac{1}{2}$ for $k = 0, 1, 2, \ldots$. The probability assigned to $k + \frac{1}{2}$ is $P[k < X \leq k+1]$, where X is the exponential random variable with mean 1. Find the mean of the discretized distribution.

A) 1.00 B) 1.02 C) 1.04 D) 1.06 E) 1.08

8. A casino manager creates a model for the number of customers who play on a particular gambling machine during a 2-hour period. If $0 \leq t \leq 2$ (in hours), then the probability that k people play on the machine during the time interval from time 0 to time t is $\binom{10}{k}(.5t)^k(1-.5t)^{10-k}$ (binomial). T denotes the time (measured from time 0) at which the first person plays on the machine. Find the pdf of T.

A) $10(1-0.5t)^9$ B) $5(1-0.5t)^9$ C) $5.5(1-0.5t)^{10}$
D) $10(0.5t)^{-9}$ E) $5(0.5t)^9$

9. A loss distribution this year is exponentially distributed with mean 1000. An insurance policy pays the loss amount up to a maximum of 500. As a result of inflation, the loss distribution next year will be uniformly distributed between 0 and 1250. The insurer increases the maximum amount of payment to u so that the insurer's expected payment is 25% higher next year. Find u.

A) 525 B) 559 C) 673 D) 707 E) 779

10. Of the following statements regarding the sums of independent random variables, how many are true?
I. The sum of independent Poisson random variables has a Poisson distribution.
II. The sum of independent exponential random variables has an exponential distribution.
III. The sum of independent geometric random variables has a geometric distribution.
IV. The sum of independent normal random variables has a normal distribution.

A) 0 B) 1 C) 2 D) 3 E) 4

11. Among the questions asked in a marketing study were the following:
(i) Are you a member of a group health insurance plan?
(ii) Are you a member of a fitness club?
It was found that 80% of the respondents answered "YES" to at least one of those two questions and 80% answered "NO" to at least one of those two questions. Find the percentage that answered "YES" to exactly one of those two questions.

A) 0.2 B) 0.3 C) 0.4 D) 0.5 E) 0.6