PROBLEM SET 5

Expectation and Other Distribution Parameters

1. If X is a random variable with density function $f(x) = \begin{cases} 1.4e^{-2x} + .9e^{-3x} & \text{for } x>0 \\ 0, & \text{elsewhere} \end{cases}$, then $E[X] =$

A) $\frac{9}{20}$ B) $\frac{5}{6}$ C) 1 D) $\frac{230}{126}$ E) $\frac{23}{10}$

2. Let X be a continuous random variable with density function $f(x) = \begin{cases} \frac{1}{30}x(1+3x) & \text{for } 1<x<3 \\ 0, \text{otherwise} \end{cases}$.

Find $E[\frac{1}{X}]$.

A) $\frac{1}{12}$ B) $\frac{7}{15}$ C) $\frac{45}{103}$ D) $\frac{11}{20}$ E) $\frac{14}{15}$

3. (SOA) Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{|x|}{10} & \text{for } \quad -2 \le x \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the expected value of X.

A) $\frac{1}{5}$ B) $\frac{3}{5}$ C) 1 D) $\frac{28}{15}$ E) $\frac{12}{5}$

4. (SOA) An insurer's annual weather related loss, X, is a random variable with density function

$$f(x) = \begin{cases} \frac{2.5(200)^{2.5}}{x^{3.5}} & \text{for } \quad x \ge 200 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the difference between the 30th and 70th percentiles of X.

A) 35 B) 93 C) 124 D) 131 E) 298

5. Let X be a continuous random variable with density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x>0 \\ 0, \text{otherwise} \end{cases}$

If the median of this distribution is $\frac{1}{3}$, then $\lambda =$

A) $\frac{1}{3}\,ln\,\frac{1}{2}$ B) $\frac{1}{3}\,ln\,2$ C) $2\,ln\,\frac{3}{2}$ D) $3\,ln\,2$ E) 3

6. Let X be a continuous random variable with density function $f(x) = \begin{cases} \frac{1}{9}x(4-x) & \text{for } 0<x<3 \\ 0, \text{otherwise} \end{cases}$.

What is the mode of X?

A) $\frac{4}{9}$ B) 1 C) $\frac{3}{2}$ D) $\frac{7}{4}$ E) 2

7. (SOA) A recent study indicates that the annual cost of maintaining and repairing a car in a town in Ontario averages 200 with a variance of 260. If a tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive), what will be the variance of the annual cost of maintaining and repairing a car?

A) 208 B) 260 C) 270 D) 312 E) 374

8. A system made up of 7 components with independent, identically distributed lifetimes will operate until any of 1 of the system's components fails. If the lifetime X of each component has density function $f(x) = \begin{cases} \frac{3}{x^4} & \text{for } 1<x \\ 0, & \text{otherwise} \end{cases}$, what is the expected lifetime until failure of the system?

A) 1.02 B) 1.03 C) 1.04 D) 1.05 E) 1.06

9. (SOA) A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim Size	20	30	40	50	60	70	80
Probability	0.15	0.10	0.05	0.20	0.10	0.10	0.30

What percentage of the claims are within one standard deviation of the mean claim size?

A) 45% B) 55% C) 68% D) 85% E) 100%

10. (SOA) An actuary determines that the claim size for a certain class of accidents is a random variable, X, with moment generating function $M_X(t) = \frac{1}{(1-2500t)^4}$.

Determine the standard deviation of the claim size for this class of accidents.

A) 1,340 B) 5,000 C) 8,660 D) 10,000 E) 11,180

11. Let X be a random variable with mean 0 and variance 4. Calculate the largest possible value of $P[|X| \geq 8]$, according to Chebyshev's inequality.

A) $\frac{1}{16}$ B) $\frac{1}{8}$ C) $\frac{1}{4}$ D) $\frac{3}{4}$ E) $\frac{15}{16}$

12. If the moment generating function for the random variable X is $M_X(t) = \frac{1}{1+t}$, find the third moment of X about the point $x = 2$.

A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) $\frac{3}{2}$ D) -38 E) $-\frac{19}{3}$

13. (SOA) A company insures homes in three cities, J, K, and L. Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are independent. The moment generating functions for the loss distributions of the cities are:

$$M_J(t) = (1-2t)^{-3} \;, \qquad M_K(t) = (1-2t)^{-2.5} \;, \qquad M_L(t) = (1-2t)^{-4.5}$$

Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

A) 1,320 B) 2,082 C) 5,760 D) 8,000 E) 10,560

14. (SOA) Let X_1 , X_2 , X_3 be a random sample from a discrete distribution with probability function

$$p(x) = \begin{cases} 1/3 & \text{for } x = 0 \\ 2/3 & \text{for } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the moment generating function, $M(t)$, of $Y = X_1 X_2 X_3$.

A) $\frac{19}{27} + \frac{8}{27}e^t$ B) $1 + 2e^t$ C) $(\frac{1}{3} + \frac{2}{3}e^t)^3$ D) $\frac{1}{27} + \frac{8}{27}e^{3t}$ E) $\frac{1}{3} + \frac{2}{3}e^{3t}$

15. Two balls are dropped in such a way that each ball is equally likely to fall into any one of four holes. Both balls may fall into the same hole. Let X denote the number of unoccupied holes at the end of the experiment. What is the moment generating function of X?

A) $\frac{7}{4} - \frac{1}{2}t$ if $t = 2$ or 3, 0 otherwise B) $\frac{9}{4}t + \frac{21}{8}t^2$ C) $\frac{1}{4}(3e^{2t} + e^{3t})$
D) $\frac{1}{4}(e^{2t} + e^{3t})$ E) $\frac{1}{4}(e^{3t/4} + 3e^{t/4})$

16. A lottery is designed so that the winning number is a randomly chosen 4-digit number (0000 to 9999). The prize is designed as follows: if your ticket matches the last 2 digits (in order) of the winning number, you win \$50, match last 3 digits (in order) and win \$500 (but not the \$50 for matching the last 2), match all 4 digits in order and win \$5000 (but not \$500 or \$50). The cost to buy a lottery ticket is \$2. Find the ticket holder's expected net gain.

A) -1.40 B) -0.60 C) 0 D) 0.60 E) 1.40

17. (SOA) An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1+x)^{-4}$, where $0 < x < \infty$. Determine the company's expected monthly claims.

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) 1 E) 3

18. The cumulative distribution function for a loss random variable X is $F(x) = \begin{cases} 0, & \text{for } x<0 \\ 1-\frac{1}{2}e^{-x}, & \text{for } x \geq 0 \end{cases}$. Find the moment generating function of X as a function of t.

A) $\frac{1}{1-t}$, $t<1$ B) $\frac{1}{2-2t}$, $t<1$ C) $\frac{2-t}{2-2t}$, $t<1$ D) $\frac{1}{2t} + \frac{1}{2(1+t)}$, $t<1$ E) Undefined

19. Suppose that the random variable X has moment generating function $M(t) = \frac{e^{at}}{1-bt^2}$ for $-1 < t < 1$. It is found that the mean and variance of X are 3 and 2 respectively.
Find $a+b$.
A) 0 B) 1 C) 2 D) 3 E) 4

20. An actuary uses the following distribution for the random variable T the time until death for a new born baby : $f(t) = \frac{t}{5000}$ for $0 < t < 100$. At the time of birth an insurance policy is set up to pay an amount of $(1.1)^t$ at time t if death occurs at that instant. Find the expected payout on this insurance policy (nearest 100).
A) 2000 B) 2200 C) 2400 D) 2600 E) 2800

21. A life insurer has created a special one year term insurance policy for a pair of business people who travel to high risk locations. The insurance policy pays nothing if neither die in the year, \$100,000 if exactly one of the two die, and $\$K > 0$ if both die. The insurer determines that there is a probability of 0.1 that at least one of the two will die during the year and a probability of 0.08 that exactly one of the two will die during the year. You are told that the standard deviation of the payout is \$74,000. Find the expected payout for the year on this policy.
A) 18,000 B) 21,000 C) 24,000 D) 27,000 E) 30,000

22. The board of directors of a corporation wishes to purchase "headhunter insurance" to cover the cost of replacing up to 3 of the corporations high-ranking executives, should they leave during the next year to take another job. The board wants the insurance policy to pay $\$1{,}000{,}000 \times K^2$, where $K = 0, 1, 2$ or 3 is the number of the three executives that leave within the next year. An actuary analyzes the past experience of the corporation's retention of executives at that level, and estimates the following probabilities for the number who will leave:
$$P[K=0] = 0.8\,, \quad P[K=1] = 0.1\,, \quad P[K=2] = P[K=3] = 0.05\,.$$
Find the expected payment the insurer will make for the year on this policy.
A) 250,000 B) 500,000 C) 750,000 D) 1,000,000 E) 2,000,000

23. (SOA) A random variable has the cumulative distribution function
$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{x^2-2x+2}{2} & \text{for } 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$
Calculate the variance of X
A) $\frac{7}{72}$ B) $\frac{1}{8}$ C) $\frac{5}{36}$ D) $\frac{4}{32}$ E) $\frac{23}{12}$

24. Smith is offered the following gamble: he is to choose a coin at random from a large collection of coins and toss it randomly. $\frac{3}{4}$ of the coins in the collection are loaded towards a head (LH) and $\frac{1}{4}$ are loaded towards a tail. If a coin is loaded towards a head, then when the coin is tossed randomly, there is a $\frac{3}{4}$ probability that a head will turn up and a $\frac{1}{4}$ probability that a tail will turn up. Similarly, if the coin is loaded towards tails, then there is a $\frac{3}{4}$ chance of tossing a tail on any given toss. If Smith tosses a head, he loses \$100, and if he tosses a tail, he wins \$200. Smith is allowed to obtain "sample information" about the gamble. When he chooses the coin at random, he is allowed to toss it once before deciding to accept the gamble with that same coin. Suppose Smith tosses a head on the sample toss. Find Smith's expected gain/loss on the gamble if it is accepted.

A) loss of 20 B) loss of 10 C) loss of 0 D) gain of 10 E) gain of 20

25. The loss amount, X, for a medical insurance policy has cumulative distribution function

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{1}{9}\left(2x^2 - \frac{x^3}{3}\right), & \text{for } 0 \le x \le 3 \\ 1, & \text{for } x > 3 \end{cases}$$

Calculate the mode of the distribution.

A) 2/3 B) 1 C) 3/2 D) 2 E) 3

26. Smith is offered the following gamble: he is to choose a coin at random from a large collection of coins and toss it randomly. The proportion of the coins in the collection that are loaded towards a head is p. If a coin is loaded towards a head, then when the coin is tossed randomly, there is a $\frac{3}{4}$ probability that a head will turn up and a $\frac{1}{4}$ probability that a tail will turn up. Similarly, if the coin is loaded towards tails, then there is a $\frac{3}{4}$ probability that a tail will turn up and a $\frac{1}{4}$ probability that a head will turn up. If Smith tosses a head, he loses \$100, and if he tosses a tail, he wins \$200. Find the proportion p for which Smith's expected gain is 0 when taking the gamble.

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{2}{3}$ E) $\frac{5}{6}$

27. The random variable X has density function $f(x) = ce^{-|x|}$ for $-\infty < x < \infty$.
Find the variance of X.

A) 0 B) .5 C) 1.0 D) 1.5 E) 2.0

28. (SOA) An auto insurance company is implementing a new bonus system. In each month, if a policyholder does not have an accident, he or she will receive a 5.00 cash-back bonus from the insurer. Among the 1,000 policyholders of the auto insurance company, 400 are classified as low-risk drivers and 600 are classified as high-risk drivers. In each month, the probability of zero accidents for high-risk drivers ins 0.80 and the probability of zero accidents for low-risk drivers is 0.90. Calculate the expected bonus payment from the insurer to the 1000 policyholders in one year.

A) 48,000 B) 50,400 C) 51,000 D) 54,000 E) 60,000

29. The non-negative integer-valued random variable X has probability generating function

$P_X(t) = \frac{1}{2-t}$ (for $t > 0$).

Whch of the following statements are correct?

I. $P[X = n] = \frac{1}{2^n}$ II. $E[X] = 1$ III. $Var[x] = 1$

A) I only B) II only C) III only D) All but I E) All but II

30. X_1 and X_2 are random variables whose probability generating functions are related in the following way:

$P_{X_2}(t) = a + (1-a) \times P_{X_1}(t)$ (where $a > 0$).

Which of the following statements are true?

I. $P[X_2 = 0] = (1-a) \times P[X_1 = 0]$

II. $E[X_2] = (1-a) \times E[X_1]$

III. $Var[X_2] = (1-a) \times Var[X_1]$

A) I only B) II only C) III only D) All but I E) All but II

PROBLEM SET 5 SOLUTIONS

1. $E[X] = \int_{-\infty}^{\infty} x \times f(x)\,dx = \int_0^{\infty} (1.4xe^{-2x} + 0.9xe^{-3x})\,dx$
$= \left(-0.7xe^{-2x} - 0.35e^{-2x} - 0.3xe^{-3x} - 0.1e^{-3x} \right)\Big|_{x=0}^{x=\infty} = 0.45$.

The integrals were found by integration by parts. Note that we could also have used $\int_0^{\infty} x^k e^{-ax} dx = \frac{k!}{a^{k+1}}$ if k is an integer ≥ 0, and $a > 0$. Answer: A

2. $E\left[\frac{1}{X}\right] = \int_1^3 \frac{1}{x} \times \frac{1}{30} x(1+3x)\,dx = \frac{7}{15}$ Answer: B

3. $E[X] = \int_{-2}^{4} x \times f(x)\,dx = \int_{-2}^{4} x \times \frac{|x|}{10}\,dx$

For $x < 0$, $|x| = -x$ and for $x > 0$, $|x| = x$

Then, $E[X] = \int_{-2}^{0} x \times \left(-\frac{x}{10}\right) dx + \int_0^4 x\left(\frac{x}{10}\right) dx = (0.1)\left(-\int_{-2}^{0} x^2\,dx + \int_0^4 x^2\,dx\right)$
$= 0.1 \times \left[-\frac{0^3-(-2)^3}{3} + \frac{4^3-0^3}{3}\right] = \frac{28}{15}$. Answer: D

4. The cdf of X is $F(y) = \int_{200}^{y} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx$.

The 30-th percentile of X, say c, is the point for which $P[X \leq c] = 0.3$.

Therefore, $0.3 = \int_{200}^{c} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx = -\frac{(200)^{2.5}}{x^{2.5}}\Big|_{x=200}^{x=c} = 1 - \left(\frac{200}{c}\right)^{2.5} = 0.3$.

Solving for c results in $c = 230.7$.

The 70-th percentile of X, say d, is the point for which $P[X \leq d] = 0.7$.

Therefore, $0.7 = \int_{200}^{d} \frac{2.5(200)^{2.5}}{x^{3.5}}\,dx = -\frac{(200)^{2.5}}{x^{2.5}}\Big|_{x=200}^{x=d} = 1 - \left(\frac{200}{d}\right)^{2.5} = 0.7$.

Solving for d results in $d = 323.7$. Then $d - c = 93$. Answer: B

5. $\int_0^{1/3} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda/3} = \frac{1}{2} \rightarrow \lambda = 3\ln 2$ Answer: D

6. The mode is the point at which $f(x)$ is maximized. $f'(x) = -\frac{1}{9}x + \frac{1}{9}(4-x) = \frac{4}{9} - \frac{2}{9}x$.

Setting $f'(x) = 0$ results in $x = 2$.

Since $f''(2) = -\frac{2}{9} < 0$, that point is a relative maximum. Answer: E

7. Let X denote the annual cost. We are given that $Var[X] = 260$.

If annual cost increases by 20% to $1.2X$, the variance is

$Var[1.2X] = (1.2)^2 \times Var[X] = 1.44 \times 260 = 374.4$. Answer: E

8. Let T be the time until failure for the system. In order for the system to not fail by time $t > 0$, it must be the case that none of the components have failed by time t. For a given component, with time until failure of W, $P[W > t] = \int_t^{\infty} \frac{3}{x^4}\, dx = \frac{1}{t^3}$. Thus, $P[T > t] = P[(W_1 > t) \cap (W_2 > t) \cap \cdots \cap (W_7 > t)]$

$= P[W_1 > t] \times P[W_2 > t] \times \cdots \times P[W_7 > t] = \frac{1}{t^{21}}$ (because of independence of the W_i's).

The cumulative distribution function for T is

$F_T(t) = P[T \le t] = 1 - P[T > t] = 1 - \frac{1}{t^{21}}$, so the density function for T is $f_T(t) = \frac{21}{t^{22}}$.

The expected value of T is then $E[T] = \int_1^{\infty} t \cdot \frac{21}{t^{22}}\, dt = \frac{21}{20} = 1.05$.

Alternatively, once the cdf of T is known, since the region of density for T is $t > 1$, the expected value of T is $E[T] = 1 + \int_1^{\infty} [1 - F_T(t)]\, dt = 1 + \int_1^{\infty} \frac{1}{t^{21}}\, dt = 1 + \frac{1}{20}$. Answer: D

9. Mean claim size $= E[X] = 20 \times 0.15 + 30 \times 0.1 + \cdots + 80 \times 0.3 = 55$;

$E[X^2] = (20)^2 \times 0.15 + (30)^2 \times 0.1 + \cdots + (80)^2 \times 0.3 = 3500$.

$Var[X] = E[X^2] - (E[X])^2 = 3500 - 55^2 = 475$.

Standard deviation $= \sqrt{Var[X]} = \sqrt{475} = 21.79$.

The claim sizes within one standard deviation of the mean claim size of 55 are those claim sizes between $55 - 21.79 = 33.21$ and $55 + 21.79 = 76.79$; those claim sizes are 40, 50, 60 and 70. The total probability of those claim sizes is $0.05 + 0.2 + 0.1 + 0.1 = 0.45$. Answer: A

10. The standard deviation is $\sqrt{Var[X]}$, and $Var[X] = E[X^2] - (E[X])^2$.

The moment generating function can be used to find the moments of X,

$E[X^k] = M^{(k)}(0)$ (k-th derivative evaluated at 0).

$M(t) = (1 - 2500t)^{-4} \rightarrow M'(t) = (-4)(1 - 2500t)^{-5}(-2500)$

$\rightarrow E[X] = M'(0) = 10,000$

$M''(t) = (-5)(-4)(1 - 2500t)^{-6}(-2500)^2$

$\rightarrow E[X^2] = M''(0) = 125 \times 10^6$

$Var[X] = 125 \times 10^6 - (10,000)^2 = 25,000,000 \rightarrow \sqrt{Var[X]} = 5,000$

Alternatively, it is also true that $Var[X] = \frac{d^2}{dt^2} ln[M(t)]\Big|_{t=0}$. In this case,

$\frac{d^2}{dt^2} ln[M(t)] = \frac{d^2}{dt^2}[-4ln(1 - 2500t)] = (-4)(-2500)^2(-1)(1 - 2500t)^{-2}$,

and when $t = 0$ this becomes $Var[X] = 25,000,000$, as before.

Another alternative solution would be to notice that $M(t) = \frac{1}{(1-ct)^{\alpha}}$ is the moment generating function for a gamma distribution with mean αc , and variance αc^2. In this case, $\alpha = 4$ and $c = 2500$, so that the variance is $4(2500)^2 = 25,000,000$. Answer: B

11. $P[\,|X-\mu|>r\cdot\sigma]\leq\frac{1}{r^2}$. In this case $\mu=0$ and $\sigma^2=4$, so that $r=4$, and $P[|X|>8]=P[|X|>4\sigma]\leq\frac{1}{16}$. Answer: A

12. $E[(X-2)^3]=E[X^3]-E[6X^2]+E[12X]-E[8] \;=M_X^{(3)}(0)-6M_X^{(2)}(0)+12M_X'(0)-8$

$M'_X(t)=-\frac{1}{(1+t)^2}\to M_X'(0)=-1$, $M_X^{(2)}(t)=\frac{2}{(1+t)^3}\to M_X^{(2)}(0)=2$,

$M_X^{(3)}(t)=-\frac{6}{(1+t)^4}\to M_X^{(3)}(0)=-6$. Then, $E[(X-2)^3]=-38$. Answer: D

13. $X=J+K+L$. Because of independence of J, K and L, we have

$$M_X(t)=M_J(t)\cdot M_K(t)\cdot M_L(t)=(1-2t)^{-3}\cdot(1-2t)^{-2.5}\cdot(1-2t)^{-4.5}=(1-2t)^{-10}.$$

We use the property of the moment generating function that states $\frac{d^n}{dt^n}M_Z(t)\Big|_{t=0}=E[Z^n]$.

Then, $\frac{d^3}{dt^3}M_X(t)=(-10)(-2)(-11)(-2)(-12)(-2)(1-2t)^{-13}$

and evaluated at $t=0$ this is $10,560$.

Alternatively, we can write $(J+K+L)^3=J^3+K^3+L^3+3J^2L+\cdots$,

and find each expectation separately (much too tedious and too much work). Answer: E

14. Since each X is either 0 or 1, it follows that

$Y=X_1X_2X_3=1$ only if $X_1=X_2=X_3=1$, and $Y=0$ otherwise.

Since X_1,X_2,X_3 form a random sample, they are mutually independent. Therefore,

$$\begin{aligned}P[Y=1] &= P[(X_1=1)\cap(X_2=1)\cap(X_3=1)]\\ &= P[(X_1=1)]\times P[(X_2=1)]\times P[(X_3=1)]=\tfrac{2}{3}\times\tfrac{2}{3}\times\tfrac{2}{3}=\tfrac{8}{27},\end{aligned}$$

and $P[Y=0]=1-P[Y=1]=\frac{19}{27}$.

The moment generating function of Y is

$M(t)=E[e^{tY}]=e^{t\cdot 0}\cdot P[Y=0]+e^{t\times 1}\cdot P[Y=1]\;=1\times\frac{19}{27}+e^t\times\frac{8}{27}=\frac{19}{27}+\frac{8}{27}e^t$.

Answer: A

15. Let A denote the event that the two balls fall into separate holes, and B denote the event that the two balls fall into the same hole. Event A is equivalent to $X = 2$ holes being left unoccupied, and event B is equivalent to $X = 3$ holes being left unoccupied.
When the balls are dropped, both balls have the same chance of dropping into any of the four holes. Therefore, each of the 16 possible outcomes (i.e., ball 1 in hole 1, 2, 3 or 4 and ball 2 in hole 1, 2, 3 or 4) is equally likely to occur. Four of these outcomes result in the two balls dropping into the same hole (event B) and the other twelve outcomes result in the balls dropping into separate holes (event A). Therefore,

$$P[A] = P[X=2] = \tfrac{12}{16} = \tfrac{3}{4} \quad \text{and} \quad P[B] = P[X=3] = \tfrac{4}{16} = \tfrac{1}{4}$$

The moment generating function of a discrete random variable X is

$$M(t) = E[e^{tX}] = \sum e^{tx} \times P[X=x].$$

The moment generating function for the random variable X described in this problem is

$$M(t) = e^{2t} \times P[X=2] + e^{3t} \times P[X=3] = e^{2t} \times \tfrac{3}{4} + e^{3t} \times \tfrac{1}{4} = \tfrac{1}{4}(3e^{2t} + e^{3t}) \ .$$

Answer: C

16. $P[\text{match only last 2}] = \frac{90}{10{,}000}$, since the 100's can be any one of 9 wrong digits and the 1000's can be any one of 10.
$P[\text{match only last 3}] = \frac{9}{10{,}000}$, since the 1000's can be any one of 9 wrong digits.
$P[\text{match all 4}] = \frac{1}{10{,}000}$.
Expected gain from ticket is
$50 \times \frac{90}{10{,}000} + 500 \times \frac{9}{10{,}000} + 5000 \times \frac{1}{10{,}000} - 2 = -0.60.$ Answer: B

17. The pdf is $c(1+x)^{-4}$ for $0 < \infty$. Therefore, $1 = \int_0^\infty c(1+x)^{-4}\,dx = \frac{c}{3}$,
from which we get $c = 3$, and $f(x) = 3(1+x)^{-4}$.
$E[X] = \int_0^\infty x \times f(x)\,dx = \int_0^\infty x \times 3(1+x)^{-4}\,dx$. This can be found using integration by parts:
$\int_0^\infty x \times 3(1+x)^{-4}\,dx = \int_0^\infty x \times d[-(1+x)^{-3}]$
$= -x(1+x)^{-3}\Big|_{x=0}^{x=\infty} - \int_0^\infty -(1+x)^{-3}\,dx = 0 + \int_0^\infty (1+x)^{-3}\,dx = \frac{1}{2}$.
Alternatively, the cumulative distribution function of X is
$F(x) = \int_0^x f(t)\,dt = \int_0^x 3(1+t)^{-4}\,dt = 1 - (1+x)^{-3}$, and we use the rule for non-negative random variables $E[X] = \int_0^\infty [1 - F(x)]\,dx = \int_0^\infty (1+x)^{-3}\,dx = \frac{1}{2}$. Answer: C

18. Since $F(x)=0$ if $x<0$ but $F(0)=1-\frac{1}{2}$, it follows that $P[X=0]=\frac{1}{2}$.
Since $F(x)$ is differentiable for $x>0$, it follows that the density function of X for $x>0$ is $f(x)=F'(x)=\frac{1}{2}e^{-x}$. The moment generating function of X is then
$M_X(t)=E[e^{tX}]=e^{t\times 0}\times P[X=0]+\int_0^\infty e^{tx}\times\frac{1}{2}e^{-x}dx=\frac{1}{2}+\frac{-1}{2(t-1)}=\frac{2-t}{2-2t}$
for $t<1$ (the improper integral converges only if $t<1$). Answer: C

19. $\frac{d}{dt}\,ln\,M(t)\Big|_{t=0}=\mu$ (mean) , $\frac{d^2}{dt^2}\,ln\,M(t)\Big|_{t=0}=\sigma^2$ (variance)
$\rightarrow\ \frac{d}{dt}\,ln\,\frac{e^{at}}{1-bt^2}=\frac{d}{dt}[at-ln(1-bt^2)]=a+\frac{2bt}{1-bt^2}$, substitute $t=0\rightarrow a=3$,
and $\frac{d^2}{dt^2}\,ln\,\frac{e^{at}}{1-bt^2}=\frac{d^2}{dt^2}[at-ln(1-bt^2)]=\frac{2b+2b^2t^2}{(1-bt^2)^2}$,
substitute $t=0\ \rightarrow\ 2b=2\ \Rightarrow\ b=1\rightarrow a+b=4$. Answer: E

20. The expected payout is $\int_0^{100}\frac{t}{5000}\,(1.1)^t\,dt=\frac{1}{5000}\left(\frac{t(1.1)^t}{ln\,1.1}-\frac{(1.1)^t}{[ln\,1.1]^2}\right)\Big|_{t=0}^{t=100}=2588$

Answer: D

21. The expected payout is $100,000(.08)+K(.02)=8000+.02K$ (since there is a .9 chance that neither dies and a .08 chance that exactly 1 dies, there must be a probability of .02 that both die). The variance is

$$\begin{aligned}74,000^2\ &=\ 100,000^2(0.08)+K^2(0.02)-(8000+0.02K)^2\\ &=\ 736,000,000-320K+0.0196.K^2\rightarrow K=500,000\end{aligned}$$

(or $-483,673$, we discard the negative root). The expected payout is
$100,000\times 0.08+500,000\times 0.02=18,000$. Answer: A

22. The expected payment is

$$0.8\times 0+0.1\times 1,000,000+0.05\times(4,000,000+9,000,000)=750,000\,.$$

Answer: C

23. From the definition of $F(x)$ we see that $F(1)=\frac{1}{2}$. This indicates that X has a point of probability at $X=1$ with $P[X=1]=\frac{1}{2}$. For $1<x<2$, the density function for X is $f(x)=F'(x)=x-1$. We formulate the variance of X as $Var[X]=E[X^2]-(E[X])^2$.
$E[X]=1\times P[X=1]+\int_1^2 x\times f(x)\,dx=1\times\frac{1}{2}+\int_1^2 x(x-1)\,dx=\frac{1}{2}+\frac{7}{3}-\frac{3}{2}=\frac{4}{3}$.
$E[X^2]=1^2\times P[X=1]+\int_1^2 x^2\times f(x)\,dx=1\times\frac{1}{2}+\int_1^2 x^2(x-1)\,dx=\frac{1}{2}+\frac{15}{4}-\frac{7}{3}=\frac{23}{12}$.
$Var[X]=\frac{23}{12}-(\frac{4}{3})^2=\frac{5}{36}$. Answer: C

24. We identify the following events:

H- toss a head ; T- toss a tail ; LH- coin is loaded toward heads

LT- coin is loaded toward tails.

We are given $P[LH] = \frac{3}{4}$, $P[LT] = \frac{1}{4}$,

$P[H|LH] = \frac{3}{4}$, $P[T|LH] = \frac{1}{4}$, $P[H|LT] = \frac{1}{4}$, $P[T|LT] = \frac{3}{4}$.

We must find the conditional probabilities $P[\text{2nd flip } H|\text{1st flip } H]$.

Then his expected gain is

$(-100)P[\text{2nd flip } H|\text{1st flip } H] + (200)(1 - P[\text{2nd flip } H|\text{1st flip } H])$.

$P[\text{2nd flip } H|\text{1st flip } H] = \frac{P[\text{2nd flip } H \cap \text{1st flip } H]}{P[\text{1st flip } H]}$.

To find both the numerator and denominator we use the rules

$P(A) = P(A \cap B) + P(A \cap \overline{B})$ and $P(A \cap B) = P(A|B) \cdot P(B)$.

The denominator is

$$\begin{aligned} P[1H] &= P[1H \cap LH] + P[1H \cap LT] \\ &= P[1H|LH] \cdot P[LH] + P[1H|LT] \cdot P[LT] = (\tfrac{3}{4})(\tfrac{3}{4}) + (\tfrac{1}{4})(\tfrac{1}{4}) = \tfrac{5}{8}. \end{aligned}$$

The numerator is

$$\begin{aligned} P[2H \cap 1H] &= P[2H \cap 1H|LH] \cdot P[LH] + P[2H \cap 1H|LT] \cdot P[LT] \\ &= (\tfrac{3}{4})^2(\tfrac{3}{4}) + (\tfrac{1}{4})^2(\tfrac{1}{4}) = \tfrac{7}{16} \end{aligned}$$

(if the coin is LH then each of the two flips has probability $\frac{3}{4}$ of being head, and if the coin is LT then each of the two flips has probability $\frac{1}{4}$ of being tail).

Then $P[\text{2nd flip } H|\text{1st flip } H] = \frac{P[\text{2nd flip } H \cap \text{1st flip } H]}{P[\text{1st flip } H]} = \frac{7/16}{5/8} = 0.7$.

Then, the expected gain is

$(-100)P[\text{2nd flip } H|\text{1st flip } H] + (200)[1 - P[\text{2nd flip } H|\text{1st flip } H]]$.

$= (-100)(0.7) + (200)(0.3) = -10$. Answer: B

25. The mode of a distribution is the point x at which the density function $f(x)$ is maximized. From the distribution function $F(x)$, we can find the density function

$f(x) = F'(x) = \frac{1}{9}(4x - x^2)$ for $0 \leq x \leq 3$. We now find where the maximum of $f(x)$ occurs on the interval $[0, 3]$. The critical points of $f(x)$ occur where $f'(x) = 0$:

$f'(x) = \frac{1}{9}(4 - 2x) = 0 \to x = 2$. To find the maximum of $f(x)$ on the interval, we calculate $f(x)$ at the critical points and at the endpoints of the interval:

$f(0) = 0$, $f(2) = \frac{4}{9}$, $f(3) = \frac{1}{3}$. The mode is at $x = 2$. Answer: D

26. $P[\text{toss head}] = P[\text{toss head} \cap \text{loaded head}] + P[\text{toss head} \cap \text{loaded tail}]$
$= P[\text{toss head}|\text{loaded head}] \times P[\text{loaded head}] + P[\text{toss head}|\text{loaded tail}] \times P[\text{loaded tail}]$
$= \frac{3}{4} \times p + \frac{1}{4} \times (1-p) = \frac{1}{4} + \frac{1}{2}p$. Then, $P[\text{toss tail}] = 1 - P[\text{toss head}] = \frac{3}{4} - \frac{1}{2}p$.

Smith's expected gain is
$(-100) \times P[\text{toss head}] + (200) \times P[\text{toss tail}]$
$= (-100) \times (\frac{1}{4} + \frac{1}{2}p) + 200 \times (\frac{3}{4} - \frac{1}{2}p) = 125 - 150p$.

In order for Smith's expected gain to be 0 we must have $125 - 150p = 0 \to p = \frac{5}{6}$.

Answer: E

27. In order for $f(x)$ to be a properly defined density function, it must satisfy $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
The integral is $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ce^{-|x|}\,dx = c \times [\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx] = c \times [1+1] = 1$,
from which it follows that $c = \frac{1}{2}$.
Then, $Var[X] = E[X^2] - (E[X])^2$.
$E[X] = \int_{-\infty}^{\infty} \frac{1}{2} xe^{-|x|}\,dx = \frac{1}{2} \times [\int_{-\infty}^{0} xe^{x}\,dx + \int_{0}^{\infty} xe^{-x}\,dx]$.

We use the integration by parts rule $\int xe^{ax}\,dx = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2}$.
Then $\int_{-\infty}^{0} xe^{x}\,dx = xe^{x} - e^{x}\Big|_{x=-\infty}^{x=0} = 0 - 1 - (0-0) = -1$,
and $\int_{0}^{\infty} xe^{-x}\,dx = -xe^{-x} - e^{-x}\Big|_{x=0}^{x=\infty} = 0 - 0 - (0-1) = 1$.
Therefore, $E[X] = -1 + 1 = 0$.

To find $E[X^2]$ we use integration by parts. $\int x^2 e^{ax}\,dx = \frac{x^2 e^{ax}}{a} - \int \frac{e^{ax}}{a} \times 2x\,dx$.
$E[X^2] = \frac{1}{2}[\int_{-\infty}^{0} x^2 e^{x}\,dx + \int_{0}^{\infty} x^2 e^{-x}\,dx]$.

Then $\int_{-\infty}^{0} x^2 e^{x}\,dx = x^2 e^{x}\Big|_{x=-\infty}^{x=0} - \int_{-\infty}^{0} e^{x} \times 2x\,dx = 0 - 0 - 2 \times (-1) = 2$,
and $\int_{0}^{\infty} x^2 e^{-x}\,dx = -x^2 e^{-x}\Big|_{x=0}^{x=\infty} + \int_{0}^{\infty} e^{-x} \cdot 2x\,dx = -0 + 0 + 2 \times 1 = 2$.
Therefore, $E[X^2] = \frac{1}{2} \times [2+2] = 2$, and $Var[X] = 2 - 0 = 2$. Answer: E

28. For each of the 400 low-risk drivers, the expected bonus payment for one month is $5.0 \times 0.9 = 4.50$, for a total expected bonus to low-risk drivers of $400 \times 4.50 = 1800$. For each of the 600 low-risk drivers, the expected bonus payment for one month is $5.0 \times 0.8 = 4.00$, for a total expected bonus to low-risk drivers of $600 \times 4.00 = 2400$. The total expected bonus payment per month for the 1000 drivers is $1800 + 2400 = 4200$. The expected bonus payment for the year is $12 \times 4200 = 50,400$. Answer: B

29. I. $\frac{d}{dt^n}\,P_X(t) = \frac{n!}{(2-t)^{n+1}} \to P[X=n] = \frac{1}{n!}\cdot\frac{n!}{2^{n+1}} = \frac{1}{2^{n+1}}$. False

II. $\frac{d}{dt}\,P_X(t) = \frac{1}{(2-t)^2} \to E[X] = P_X'(1) = 1$. True

III. $\frac{d}{dt^2}\,P_X(t) = \frac{2}{(2-t)^3} \to E[X^2] - E[X] = P_X''(1) = 2$

$\to E[X^2] = 3 \to Var[X] = 3 - 1^2 = 2$. False Answer: B

30. I. $P[X_2 = 0] = P_{X_2}(0) = a + (1-a)\times P_{X_1}(0) = a + (1-a)\times P[X_1 = 0]$. False

II. $E[X_2] = P_{X_2}'(1) = (1-a)\times P_{X_1}(1) = (1-a)\times E[X_1]$. True

III. $E[X_2^2] - E[X_2] = P_{X_2}''(1) = (1-a)\times P_{X_1}''(1) = (1-a)\times(E[X_1^2] - E[X_1])$.

From II we have $E[X_2] = (1-a)\times E[X_1]$, so it follows that $E[X_2^2] + (1-a)\times E[X_1^2]$.

$Var[X_2] = E[X_2^2] - (E[X_2])^2 = (1-a)\times E[X_1^2] - (1-a)^2\times(E[X_1])^2$

$\neq (1-a)\times\left(E[X_1^2] - (E[X_1])^2\right) = (1-a)\times Var[X_1]$. False Answer: B

SECTION 6 - FREQUENTLY USED DISCRETE DISTRIBUTIONS

Uniform distribution on N points (where $N \geq 1$ is an integer):

The probability function is $\boldsymbol{p(x) = \frac{1}{N}}$ **for** $\boldsymbol{x = 1, 2, \ldots, N}$, and $p(x) = 0$ otherwise.

Since each x has the same probability of occurring, it seems reasonable that the mean (the average) is the midpoint of the set of successive integers. The average is midway between the smallest and largest possible value of X, $\boldsymbol{E[X] = \frac{N+1}{2}}$. Another way of seeing this, is by using the rule for summing consecutive integers, $1 + 2 + \cdots + N = \frac{N(N+1)}{2}$. It then follows that

$$\begin{aligned} E[X] &= \sum_{x=1}^{N} x \times p(x) = 1 \times \frac{1}{N} + 2 \times \frac{1}{N} + \cdots + N \times \frac{1}{N} \\ &= [1 + 2 + \cdots + N] \times \frac{1}{N} = \frac{N(N+1)}{2} \times \frac{1}{N} = \frac{N+1}{2} \end{aligned}$$

The variance $\boldsymbol{Var[X] = \frac{N^2-1}{12}}$.

The probability generating function (pgf) is

$P_X(t) = \sum_{k=1}^{n} (t + t^2 + \cdots + t^n) \times \frac{1}{N} = \frac{t(t^N-1)}{N(t-1)}$ (defined for $t > 1$).

The moment generating function (mgf) is $M_X(t) = \sum_{j=1}^{N} \frac{e^{jt}}{N} = \frac{e^t(e^{Nt}-1)}{N(e^t-1)}$ for any real t.

The outcome of tossing a fair die is an example of the discrete uniform distribution with $N = 6$.

It is unlikely that the moment generating function for the discrete uniform distribution will come up in an Exam P question.

Example 6-1:

Suppose that X is a discrete random variable that is uniformly distributed on the even integers $x = 0, 2, 4, \ldots, 22$, so that the probability function of X is $p(x) = \frac{1}{12}$ for each even integer x from 0 to 22. Find $E[X]$ and $Var[X]$.

Solution:

If we consider the transformation $Y = \frac{X+2}{2}$, then the random variable Y is distributed on the points $Y = 1, 2, \ldots, 12$, with probability function $p_Y(y) = \frac{1}{12}$ for each integer y from 1 to 12. Thus, Y has the discrete uniform distribution described above with $N = 12$,

and $E[Y] = \frac{12+1}{2} = \frac{13}{2}$ and $Var[Y] = \frac{12^2-1}{12} = \frac{143}{12}$.

But we want the mean and variance of X. Since $Y = \frac{X+2}{2}$, we rewrite this as $X = 2Y - 2$ and use rules for expectation and variance to get $E[X] = 2 \times E[Y] - 2 = 11$, and

$Var[X] = 4 \times Var[Y] = \frac{143}{3}$. □

Binomial distribution with parameters n and p
(integer $n \geq 1$ and $0 \leq p \leq 1$)

Suppose that a single trial of an experiment results in either success with probability p, or failure with probability $1 - p = q$. If n independent trials of the experiment are performed, and X is the number of successes that occur, then X is an integer between 0 and n. X is said to have a binomial distribution with parameters n and p (sometimes denoted $X \sim B(n,p)$).

$\boldsymbol{p(x) = \binom{n}{x} p^x (1-p)^{n-x}}$ **for** $\boldsymbol{x = 0, 1, 2, \ldots, n}$, where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

$p(x)$ is the probability that there will be exactly x successes in the n trials of the experiment.

The average number of successes in the n trials is

$\boldsymbol{E[X] = np}$, **the mean of the binomial distribution**

and the variance is

$\boldsymbol{Var[X] = np(1-p)}$**, the variance of the binomial distribution**.

The mgf is $M_X(t) = (1-p+pe^t)^n$ and the pgf is $P_X(t) = (1-p+pt)^n$

Note that since $Var[X] = E[X^2] - (E[X])^2$, it follows that the second moment of X is $E[X^2] = np(1-p) + (np)^2$ for the binomial distribution.

In the special case of $n = 1$ (a single trial), the distribution is referred to as a **Bernoulli distribution**. If $X \sim B(n,p)$, then X is the sum of n independent Bernoulli random variables each with distribution $B(1,p)$.

For example, if $n = 3$, $p = \frac{1}{2}$, the binomial random variable has the following distribution:

X :	0	1	2	3
$p(x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This would describe the distribution of the number of heads occurring in three tosses of a fair coin. The probabilities are found as follows:

$p(0) = P[X=0] = \binom{3}{0} \frac{1}{2}^0 (1-\frac{1}{2})^{3-0} = \frac{1}{8}$, $p(1) = P[X=1] = \binom{3}{1} \frac{1}{2}^1 (1-\frac{1}{2})^{3-1} = \frac{3}{8}$, etc.

The mean is $np = 3 \times \frac{1}{2} = \frac{3}{2}$ and the variance is $np(1-p) = 3 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$.

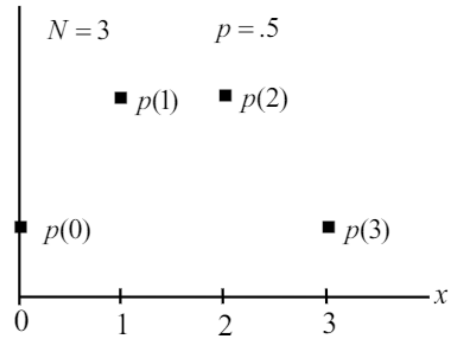
As another example, if $n = 3$, $p = .2$, the distribution is

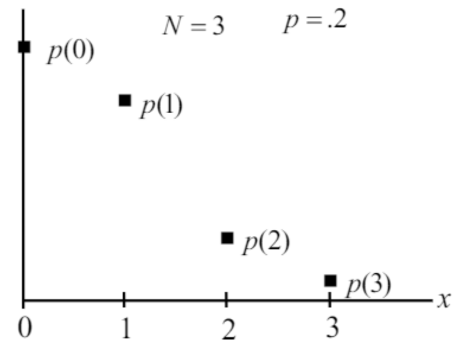
X :	0	1	2	3
$p(x)$:	0.512	0.384	0.096	0.008

As an example of a probability calculation, $P[X=2] = \binom{3}{2}(0.2)^2(1-.2)^{3-2} = 0.096$.

The mean is $np = 3(0.2) = 0.6$ and the variance is $np(1-p) = 3(0.2)(0.8) = 0.48$.

The graphs of the probability functions of these two binomial distributions are:.





For the case of $n=3$, we can illustrate fairly easily why the binomial probability function is $p(x)=\binom{3}{x}p^x(1-p)^{3-x}$ for $x=0,1,2,3$. We will use the notation S and F to denote success and failure of a particular trial of the underlying experiment. In order to have $X=0$ successes in 3 trials, the trials must be FFF. The probability of each F is $1-p$, so the probability of 3 F's in a row (because of independence of successive trials) is $(1-p)(1-p)(1-p)=(1-p)^3=\binom{3}{0}p^0(1-p)^3$. In order to have $X=1$ success, that success must occur on either the 1st, 2nd or 3rd trial. Therefore, the result of the 3 trials must be either SFF, FSF, or FFS. The probability of any one of those three sequences is $p(1-p)(1-p)=p(1-p)^2$. The combined probability of all three sequences is $3p(1-p)^2=\binom{3}{1}p^1(1-p)^2$. Similar reasoning explains the other probabilities for this binomial distribution.

Example 6-2:

Smith and Jones each take the same multiple choice test. The test has 5 questions, and each question has 5 answers (exactly one of which is right). Smith and Jones are not very well prepared for the test and they answer the questions randomly.

(i) Find the probability that they both get the same number of answers correct.

(ii) Find the probability that their papers are identical, assuming that they have answered independently of one another.

Solution:

(i) Let X be the number of answers that Smith gets correct. Then X has a binomial distribution with $n=5$, $p=.2$, and the probability function is $P[X=k]=\binom{5}{k}(.2)^k(.8)^{5-k}$.

$X:$	0	1	2	3	4	5
$p(x):$	0.32768	0.4096	0.2048	0.0512	0.0064	0.00032

The number of answers Jones gets correct, say Y, has the same distribution. Then,

$P[X=Y]=P[(X=0)\cap(Y=0)]+P[(X=1)\cap(Y=1)]+\cdots+P[(X=5)\cap(Y=5)]$

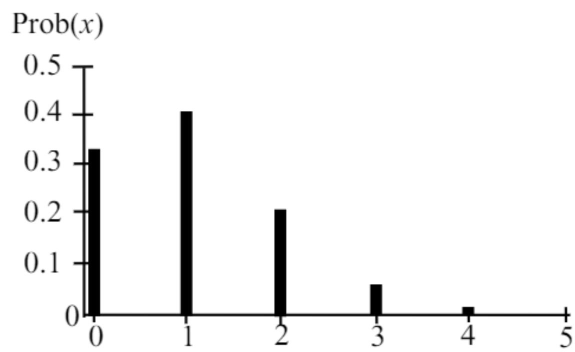
$\quad =P[X=0]\times P[Y=0]+P[X=1]\times P[Y=1]+\cdots+P[X=5]\times P[Y=5]$

(this follows from independence of X and Y), which is equal to

$(0.32768)^2+(0.4096)^2+(0.2048)^2+(0.0512)^2+(0.0064)^2+(0.00032)^2=0.3198.$

(ii) For a particular question, the probability that Jones picks (at random) the same answer as Smith is 0.2. Since all 5 questions are answered independently, the probability of both papers being identical is $(0.2)^5 = .00032$. □

The following is a probability plot for the binomial distribution in Example 6-2.

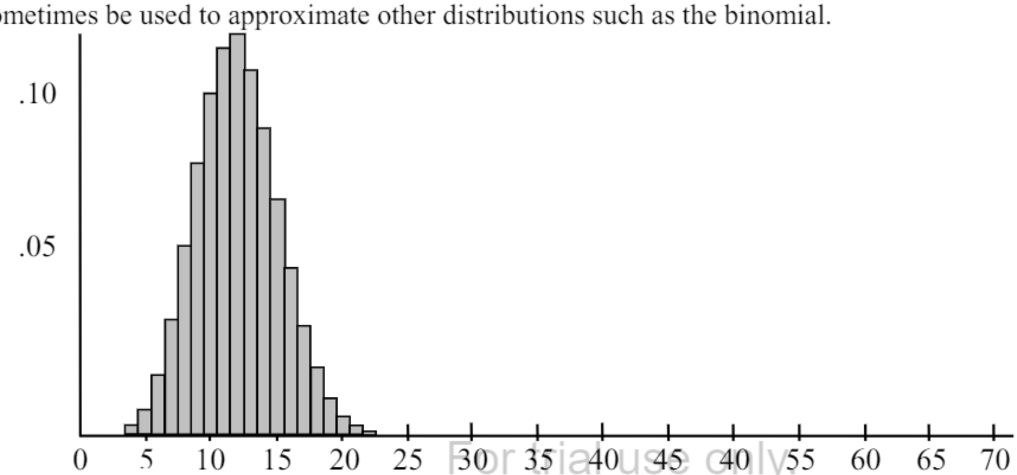


Example 6-3:

If X is the number of "6"'s that turn up when 72 ordinary dice are independently thrown, find the expected value of X^2.

Solution: X has a binomial distribution with $n = 72$ and $p = \frac{1}{6}$. Then
$E[X] = np = 12$, and $Var[X] = np(1-p) = 10$. But $Var[X] = E[X^2] - (E[X])^2$,
so that $E[X^2] = Var[X] + (E[X])^2 = 10 + 12^2 = 154$.
Note that the probability function of X is $P[X = k] = \binom{72}{k}(\frac{1}{6})^k(\frac{5}{6})^{72-k}$ for $k = 0, 1, 2, ..., 72$.

Below is a histogram for the distribution of X. The probabilities are very small for X-values above 20 or so. For instance, $P[X = 25] = 0.00010195$, $P[X = 30] = 3.5 \times 10^{-7}$ and $P[X = 2] = 0.0002035$. The mode of the distribution occurs at $X = 12$, with a probability of $P[X = 12] = 0.12525$. $X = 12$ is also the mean of the distribution. As n gets larger in a binomial distribution, the histogram takes on more of a bell shape. Later on we will see the normal approximation applied to a distribution. The normal distribution is a continuous random variable with a bell-shaped density that can sometimes be used to approximate other distributions such as the binomial.



□

Poisson distribution with parameter λ ($\lambda > 0$)

This distribution is defined for all integers $0, 1, 2, \ldots$

The probability function is $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, 2, 3, \ldots,$

λ may be referred to as the Poisson parameter for the distribution.

The mean and variance are equal to the Poisson parameter, $E[X] = Var[X] = \lambda$,

$M_X(t) = e^{\lambda(e^t - 1)}$ and $P_X(t) = e^{\lambda(t-1)}$.

The Poisson distribution is often used as a model for counting the number of events of a certain type that occur in a certain period of time. Suppose that X represents the number of customers arriving for service at a bank in a one hour period, and that a model for X is the Poisson distribution with parameter λ.

Under some reasonable assumptions (such as independence of the numbers arriving in different time intervals) it is possible to show that the number arriving in any time period also has a Poisson distribution with the appropriate parameter that is "scaled" from λ.

Suppose that $\lambda = 40$, meaning that X, the number of bank customers arriving in one hour, has a mean of 40. If Y represents the number of customers arriving in 2 hours, then Y has a Poisson distribution with a parameter of 80. For any time interval of length t, the number of customers arriving in that time interval has a Poisson distribution with parameter (mean) $\lambda t = 40t$. For instance, the number of customers arriving during a 15-minute period ($t = \frac{1}{4}$ hour) will have a Poisson distribution with parameter (mean) $40 \times \frac{1}{4} = 10$.

As an example, for $\lambda = .5$, the following is a partial description of the Poisson distribution:

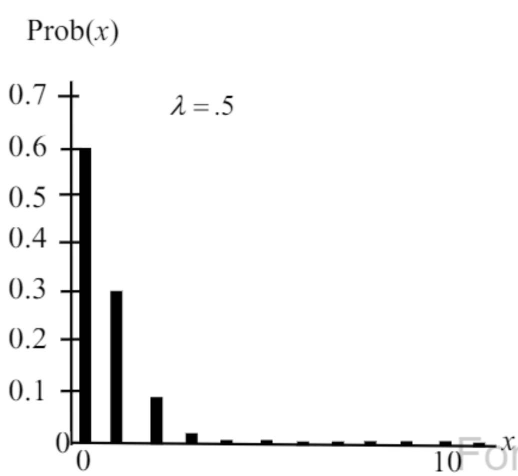
X :	0	1	2	3	4	5	...
$p(x)$:	0.6065	0.3033	0.0758	0.0126	0.0016	0.0002	...

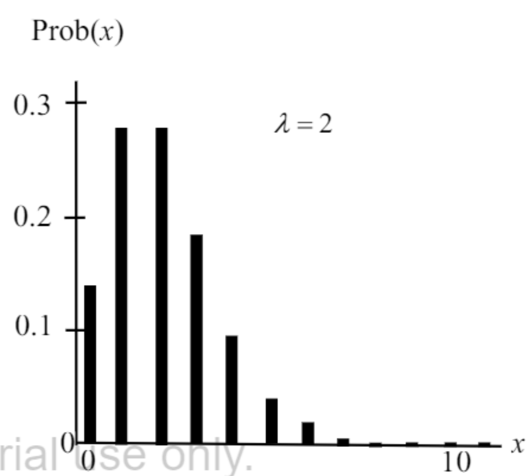
An example of the calculation of these values is $P[X = 3] = p(3) = \frac{e^{-.5}(.5)^3}{3!} = 0.012636$.

For $\lambda = 2$, the following is a partial description of the Poisson distribution:

X :	0	1	2	3	4	5 ...
$p(x)$:	0.1353	0.2707	0.2707	0.1804	0.0902	0.0361 . .

Histograms of these two Poisson distributions are as follows:





Example 6-4:

The number of home runs in a baseball game is assumed to have a Poisson distribution with a mean of 3. As a promotion, a company pledges to donate \$10,000 to charity for each home run hit up to a maximum of 3. Find the expected amount that the company will donate. Another company will donate \$C for each home run over 3 hit during the game, and C is chosen so that the second company's expected donation is the same as the first. Find C.

Solution:

Let X be the number of home runs hit in the game and let Y be the first company's donation, and let Z be the second company's donation. The probability function for X is $P[X=n]=\frac{e^{-3}3^n}{n!}$. The distribution of X is

$X:$	0	1	2	3	4	5 ...
$p(x):$	0.0498	0.1494	0.2240	0.2240	0.1680	0.1008
$Y:$	0	$10,000$	$20,000$	$30,000$	$30,000$	$30,000$
$Z:$	0	0	0	0	C	$2C$

$$E[Y] = 0\times 0.0498 + 10,000\times 0.1494 + 20,000\times 0.2240 + 30,000\times(0.2240+0.1680+\cdots)$$
$$= 10,000\times 0.1494 + 20,000\times 0.2240 + 30,000\times(1-p(0)-p(1)-p(2))$$
$$= 10,000\times 0.1494) + 20,000\times 0.2240 + (30,000)\times(1-0.0498-0.1494-0.2240) = 23,278.$$

To find $E[Z]$ we look at X as the sum of two new random variables U and W..

$X:$	0	1	2	3	4	5	6	...	x	...
$U:$	0	1	2	3	3	3	3	...	3	...
$W:$	0	0	0	0	1	2	3	...	$x-3$	...

We see that $X=U+W$, and therefore, $3=E[X]=E[U]+E[W]$. We can find $E[U]$ in the same way we found $E[Y]$ above,

$E[U]=0\times p(0)+1\times p(1)+2\times p(2)+3\times[1-p(0)-p(1)-p(2)]$

$=0.1494+2\times 0.2240+3\times(1-.0498-.1494-.2240)=2.3278$ (reduced by a factor of 10,000 from Y).

It follows that $E[W]=E[X]-E[U]=3-2.3278=0.6722$. Then, since $Z=CW$, we get $E[Z]=0.6722C$.

In order for this to be 23,278 we require $C=\frac{23{,}278}{0.6722}=34,630$. □

Example 6-5:

Assume that the number of hits, X, per baseball game, has a Poisson distribution. If the probability of a no-hit game is $\frac{1}{10{,}000}$, find the probability of having 4 or more hits in a particular game.**Solution:**

$P[X=0]=\frac{e^{-\lambda}\times\lambda^0}{0!}=e^{-\lambda}=\frac{1}{10{,}000}\rightarrow \lambda=ln\,10,000$.

$P[X\geq 4]=1-(P[X=0]+P[X=1]+P[X=2]+P[X=3])$

$=1-(\frac{e^{-\lambda}\times\lambda^0}{0!}+\frac{e^{-\lambda}\times\lambda^1}{1!}+\frac{e^{-\lambda}\times\lambda^2}{2!}+\frac{e^{-\lambda}\times\lambda^3}{3!})$

$=1-(\frac{1}{10{,}000}+\frac{ln 10{,}000}{10{,}000}+\frac{(ln 10{,}000)^2}{2\times 10{,}000}+\frac{(ln 10{,}000)^3}{6\times 10{,}000})=0.9817.$ □