29. X has an exponential distribution with mean 1. The random variable Y is defined by the relationship $Y = \sqrt{X}$. Which of the following is the probability density function of Y?

A) $\frac{e^{-\sqrt{y}}}{2\sqrt{y}}$ B) $e^{-\sqrt{y}}$ C) $2ye^{-y^2}$ D) e^{-y^2} E) $\frac{e^{-\sqrt{y}}}{2y^2}$

30. An insurer has two lines of business that are independent of one another. The number of claims arriving per month from Line 1 of business, say N_1 has a Poisson distribution with a mean of 50 and the number of claims arriving per month from Line 2, say N_2 has a binomial distribution with $n = 100$ and $p = 0.5$. Apply the normal distribution with integer correction to determine the probability $P[\,|\,N_1 - N_2\,| > 3]$.

A) Less than 0.5 B) At least 0.5 but less than 0.55 C) At least 0.55 but less than 0.60

D) At least 0.60 but less than 0.65 E) At least 0.65

PRACTICE EXAM 12 - SOLUTIONS

1. For loss amount X, Policy 1 pays $P_1 = \begin{cases} 0 & X \leq 250 \\ X - 250 & 250 < X \leq 750 \\ 500 & 750 < X \leq 1,000 \end{cases}$

and Policy 2 pays $P_2 = \begin{cases} X & X \leq C \\ C & C < X \leq 1,000 \end{cases}$.

The pdf of X is $f_X(x) = 0.001$ for $0 \leq x \leq 1,000$

$E(P_1) = \int_{250}^{750}(x - 250) \times 0.001\,dx + 500 \times P(X > 750)$

$= 125 + 500 \times \frac{1000-750}{1000} = 250.$

$E(P_2) = \int_0^C x \times 0.001\,dx + C \times \frac{1000-C}{1000} = \frac{C^2}{2000} + \frac{1000C-C^2}{1000} = \frac{2000C-C^2}{2000}$.

We then have $\frac{2000C-C^2}{2000} = 250$. Solving the quadratic equation in C results in two roots. One root is greater than 1000, so it is rejected. The other root is $C = 292.89$. Answer: C

2. Given X, the number of patients needing cosmetic dentistry, say Y, is binomial with $n = X$ and $p = 0.25$. Then

$P(Y \geq 3) = P[(Y \geq 3) \cap (X = 2)] + P[(Y \geq 3) \cap (X = 3)] + P[(Y \geq 3) \cap (X = 4)]$

$= P(Y \geq 3|X = 2) \times P(X = 2) + P(Y \geq 3|X = 3) \times P(X = 3)$

$+ P(Y \geq 3|X = 4) \times P(X = 4)$.

$P(Y \geq 3|X = 2) = 0$ since there are only 2 patients.

$P(Y \geq 3|X = 3) = (0.25)^3 = 0.015625$ and

$P(Y \geq 3|X = 4) = P(Y = 3|X = 4) + P(Y = 4|X = 4)$

$= \binom{4}{3}(0.25)^3(0.75) + (0.25)^4 = 0.05028125.$

Then, $P(Y \geq 3) = 0.015625 \times 0.5 + 0.05028125 \times 0.3 = .023$. Answer: A:

3. If T is the time of failure, the amount paid by the warranty is $Y = \begin{cases} 100 & 0 < T \leq 1 \\ 50 & 1 < T \leq 2 \\ 150 - 50T & 2 < T \leq 3 \end{cases}$.

Y has two discrete points, and Y is continuous on the interval where $2 < T \leq 5$.

$E(Y) = 100 \times P(Y = 100) + 50 \times P(Y = 50) + \int_2^3(150 - 50t)\,f(t)\,dt.$

$P(Y = 100) = P(0 < T \leq 1) = \int_0^1 0.08t\,dt = 0.04.$

$P(Y = 50) = \int_1^2 0.08t\,dt = 0.12.$

$E(Y) = 100 \times 0.04 + 50 \times 0.12 + \int_2^3(150 - 50t) \times 0.08t\,dt = 4 + 6 + \frac{14}{3} = \frac{44}{3}$. Answer: E

4. We can solve this problem two ways. One way is to look at all combinations of 0, 1 or 2 hurricanes in total for the three months, say $X_1,\ X_2,\ X_3$.

$P(\text{total of 0 hurricanes}) = P(X_1 = 0) \times P(X_2 = 0) \times P(X_3 = 0) = 0.9^3 = 0.729$

$P(\text{total of 1 hurricane}) = P(\text{exactly one month has a hurricane})$

$= 3 \times 0.9 \times 0.9 \times (0.9 \times 0.1) = .2187$

$P(\text{total of 2 hurricanes}) = P(\text{2 months each with one hurricane and other month has 0 hurricanes})$
$+ P(\text{one month has two hurricanes and the other two have 0 hurricanes})$
$= 3 \times (0.9 \times 0.1) \times (0.9 \times 0.1) \times (0.9) + 3 \times (0.9 \times 0.1^2) \times 0.9 \times 0.9 = 0.04374.$

The probability of at most 2 hurricanes in the three months is
$0.729 + 0.2187 + 0.04374 = 0.99144.$

The probability of 3 or more hurricanes in the three months is $1 - 0.99144 = 0.00856.$

The second way to approach the problem is to recognize that X has a geometric distribution with $p = .9$. The sum of independent geometric random variables, all of which have the same p, is a negative binomial random variable with the same p and with $r =$ number of independent variables in the sum. In this case $p = .9$ and $r = 3$. The distribution of $Y = X_1 + X_2 + X_3$ is negative binomial.

$P(Y = k) = \binom{r+k-1}{k} p^r (1-p)^k = \binom{r+k-1}{r-1} p^r (1-p)^k = \binom{2+k}{2} \times .9^3 \times .1^k$
for $k = 0, 1, 2, 3, \ldots$ Then $P(Y = 0) = \binom{2}{2} \times 0.9^3 \times 0.1^0 = 0.729,$
$P(Y = 1) = \binom{3}{2} \times 0.9^3 \times 0.1^1 = 0.2187$, and $P(Y = 2) = \binom{4}{2} \times 0.9^3 \times 0.1^2 = 0.04374,$
as above. Answer: A

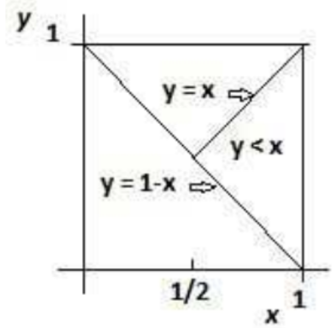
5. The median of $X + Y$ is c, which must satisfy $F_{X+Y}(c) = P[X + Y \le c] = 0.5$. Since X and Y are independent and have pdfs of $f_X(x) = e^{-x}$ and $f_Y(y) = e^{-y}$, we have
$\int_0^c \int_0^{c-x} e^{-x} e^{-y} \, dy \, dx = \int_0^c e^{-x} \left[\int_0^{c-x} e^{-y} \, dy\right] dx = 0.5$
Then $\int_0^c e^{-x} \left[1 - e^{-(c-x)}\right] dx = \int_0^c \left[e^{-x} - e^{-c}\right] dx = 1 - e^{-c} - ce^{-c} = 0.5.$
There is no algebraic solution for c, but we know that $F_{X+Y}(c)$ is an increasing function of c.
We have $F_{X+Y}(1.0) = 1 - e^{-1} - e^{-1} = 0.264$, $F(1.2) = 0.337$, $F(1.4) = 0.408$, $F(1.6) = 0.475$. Therefore we must have $c > 1.6$. Answer: E

6. Jensen's inequality states that if $h''(x) \ge 0$ at all points with non-zero density for X, then $E[h(X)] \ge h(E[X])$.
I. $h(x) = x^2 \rightarrow h''(x) = 2x \ge 0$ if $x \ge 0 \rightarrow E[X^2] \ge (E[X])^2$. True
II. $h(x) = \sqrt{x} \rightarrow h''(x) = -\frac{1}{x^{3/2}} \le 0$ if $x \ge 0 \rightarrow E[\sqrt{X}] \le \sqrt{E[X]}$. True
III. $h(x) = \ln x \rightarrow h''(x) = -\frac{1}{x^2} \le 0$ if $x \ge 0 \rightarrow E[\ln X] \le \ln E[X]$. False.
Answer: C

7. $P(K = k) = e^{-k} - e^{-k-1}$ for $k = 0, 1, 2, \ldots$
Note that $P(K = 0) = 1 - e^{-1} = p$, $P(K = 1) = e^{-1} - e^{-2} = (1 - e^{-1}) \times e^{-1} = p \times (1-p)$,
$P(K = 2) = e^{-2} - e^{-3} = (1 - e^{-1}) \times e^{-2} = p \times (1-p)^2$, and in general
$P(K = k) = e^{-k} - e^{-k-1} = (1 - e^{-1}) \times e^{-k} = p \times (1-p)^k.$
K has a geometric distribution with probability function $P(K = k) = p \times (1-p)^k$.
The mean of this distribution is $\frac{1-p}{p} = \frac{e^{-1}}{1-e^{-1}} = 0.58$ years $= 7$ months. Answer: B

8. The pgf is $P_X(t) = E[t^X]$ and the mgf is $M_X(r) = E[e^{rX}] = E[(e^r)^X] = P_X(e^r)$.
Therefore, the mgf of X is $M_X(r) = (.3 + 0.7e^r)^6$. $E[X^3] = M_X^{(3)}(0)$
(third derivative of $M_X(r)$ evaluated at $r = 0$).
$M_X'(r) = 6 \times (0.3 + 0.7e^r)^5 \times 0.7e^r$,
$M_X''(r) = 5 \times 6 \times (0.3 + 0.7e^r)^4 \times (0.7e^r)^2 + 6 \times (0.3 + 0.7e^r)^5 \times 0.7e^r$,
$$M_X^{(3)}(r) = 4 \times 5 \times 6 \times (0.3 + 0.7e^r)^3 \times (0.7e^r)^3 + 5 \times 6 \times (0.3 + 0.7e^r)^4 \times 2 \times 0.7e^r \times 0.7 + 5 \times 6 \times (0.3 + 0.7e^r)^4 \times (0.7e^r)^2 + 6 \times (0.3 + 0.7e^r)^5 \times 0.7e^r$$
$$M_X^{(3)}(0) = 4 \times 5 \times 6 \times (0.3 + 0.7)^4 \times (0.7)^3 + 5 \times 6 \times (0.3 + 0.7)^4 \times 2 \times (0.7)^2 + 5 \times 6 \times (0.3 + 0.7)^4 \times (0.7)^2 + 6 \times (0.3 + 0.7)^5 \times 0.7 = 89.46.$$ Answer: D

9. The graph below indicates the region of probability. The region is $0 < x < \frac{1}{2}$ and $1 - x < y < x$.



Then $P[Y < X] = \int_{1/2}^{1} \int_{1-x}^{x} 3x \, dy \, dx = \int_{1/2}^{1} (6x^2 - 3x) \, dx = \frac{5}{8}$. Answer: E

10. $Cov(X, Y) = E(XY) - E(X) \times E(Y)$
$E(XY) = (-1) \times 1 \times 0.1 + 1 \times 1 \times 0.3 = 0.2$
The probability function for X is
$P(X = 0) = 0.2 + 0.1 = 0.3$, $P(X = 1) = 0.1 + 0.2 + 0.3 = 0.6$, $P(X = 2) = 0.1$.
$E(X) = 0.6 + 2 \times 0.1 = 0.8$
The probability function for Y is $P(Y = -1) = 0.3$, $P(Y = 0) = 0.4$, $P(Y = 1) = 0.3$.
$E(Y) = (-1) \times 0.3 + 1 \times 0.3 = 0$. $Cov(X, Y) = 0.2 - 0.8 \times 0 = 0.2$. Answer: E

11. We identify the following events:
A - visited an acupuncture clinic , P - visited a physiotherapist , M - visited a massage therapist
We wish to find $P[A' \cap P' \cap M]$.
We use the following relationships
$P[P' \cap M] + P[P \cap M] = P[M] \to P[P' \cap M] = 0.26 - 0.18 = 0.08$,
$P[A \cap P \cap M] + P[A \cap P' \cap M] = P[A \cap M] \to P[A \cap P' \cap M] = 0.12 - 0.07 = 0.05$,
$P[A' \cap P' \cap M] + P[A \cap P' \cap M] = P[P' \cap M] \to P[A' \cap P' \cap M] = 0.08 - 0.05 = 0.03$.
Answer: D

12. We must have

$\int_0^1 f(x)\,dx \,+\, P(X=1) \,+\, \int_1^4 f(x)\,dx = \int_0^1 cx\,dx \,+\, 0.2 \,+\, \int_1^4 cx\,dx$

$= \frac{c}{2} + 0.2 + \frac{15c}{2} = 8c + 0.2 = 1$, from which we get $c = 0.1$.

$F_X(1.5) = P(X \leq 1.5) = P(0 < X < 1) + P(X = 1) + P(1 < X \leq 1.5)$

$= \int_0^1 0.1x\,dx \,+\, 0.2 \,+\, \int_1^{1.5} 0.1x\,dx = 0.3125.$ Answer: C

13. We define the following categories of policyholder

A - between ages 20 and 35, B - between 35 and 50 , C = between 50 and 65

D - over 65 and we let K denote the event that a randomly selected policyholder has high blood pressure.

We are given $P(A) = 0.1$, $P(B) = 0.2$, $P(C) = 0.3$, $P(D) = 0.4$,

$P(K|A) = 0.1$, $P(K|B) = 0.25$, $P(K|C) = 0.4$ and $P(K|D) = 0.5$.

We wish to find $P(D|K) = \frac{P(D \cap K)}{P(K)}$. We use the following rules:

$P(K) = P(A \cap K) + P(B \cap K) + P(C \cap K) + P(D \cap K)$

and $P(A \cap K) = P(K|A) \times P(A) = 0.1 \times 0.1 = 0.01$.

In a similar way, $P(B \cap K) = 0.25 \times 0.2 = 0.05$, $P(C \cap K) = 0.4 \times 0.3 = 0.12$

and $P(D \cap K) = 0.5 \times 0.4 = 0.2$.

Then $P(D|K) = \frac{0.2}{0.01+0.05+0.12+0.2} = 0.526.$ Answer: B

14. $g(t) = \Phi(t+1) - \Phi(t)$ so that $g'(t) = \phi(t+1) - \phi(t) = \frac{e^{-(t+1)^2/2}}{\sqrt{2}} - \frac{e^{-t^2/2}}{\sqrt{2}}$.

This can be factored into $g'(t) = \frac{e^{-t^2/2}}{\sqrt{2}} \times \left[e^{-(2t+1)/2} - 1\right]$.

Setting $g'(t) = 0$ results in $\frac{2t+1}{2} = 0$, or equivalently $t = -\frac{1}{2}$.

Noting that $\lim\limits_{t \to -\infty} g(t) = \lim\limits_{t \to \infty} g(t) = 0$, we see that $g(t)$ must have a maximum at $t = -\frac{1}{2}$.

The maximum value of g is $g(-\frac{1}{2}) = \Phi(\frac{1}{2}) - \Phi(-\frac{1}{2}) = 0.6915 - (1 - 0.6915) = 0.383.$

Looking at the graph of the pdf of the normal distribution, we would see that for an interval of length 1 (or any specified length), the maximum probability would be contained in the interval centred at 0. Answer: D

15. There will be one prize of \$1000. If the chosen number is a b c d (representing digits from 0 to 9), then x b c d would win \$100 if x is any of the 9 digits other than a. This is true in each of the four places for the digits. Therefore , there are $9 \times 4 = 36$ prizes of \$100.

With chosen number a b c d, a ticket would win \$10 if it is of the form

x x c d , x b x d , x b c x , a x x d , a x c x , a b x x , where x is not the appropriate digit in the chosen number. For each of those six cases, there are $9 \times 9 = 81$ choices for the two x's. Therefore, the number of \$10 prizes is $81 \times 6 = 486$.

The total paid in prizes by the city is $1000 \,+\, 100 \times 36 \,+\, 10 \times 486 = 9460$.

The city collects 10,000 from all the tickets sold and makes a profit of \$540. Answer: C

16. We must have $\int_0^{1000} cx\,dx = 1$, from which it follows that $c = \frac{1}{500{,}000}$.

The maximum reimbursement payment of 500 will occur if the loss is 700 or more.

The amount paid by the insurer when a loss occurs is $\begin{cases} 0 & 0 < x \le 200 \\ x - 200 & 200 < x \le 700 \\ 500 & 700 < x < 1000 \end{cases}$

The expected reimbursement payment is

$\int_{200}^{700} (x-200) \times \frac{x}{500{,}000}\,dx + 500 \times P(700 < X < 1000) = \frac{400}{3} + 500 \times 0.51 = 388.33.$

Answer: B

17. The mean and variance of the discrete uniform distribution on the integers $0, 1, 2, \ldots, N$ are $\frac{N}{2}$ and $\frac{(N+1)^2-1}{12}$. The variance of the Poisson with mean λ is also λ. Therefore, $\frac{(N+1)^2-1}{12} = \lambda = \frac{N}{2}$. Solving for N results in $N = 0$ or 4. Since we are told that $N > 0$, we have $N = 4$, and $\lambda = 2$.

18. The conditional pdf of Y given $X = \frac{1}{2}$ is $f(y|x = \frac{1}{2}) = \frac{f(\frac{1}{2}, y)}{f_X(\frac{1}{2})}$, where $f_X(\frac{1}{2})$ is the pdf of the marginal distribution of X at $x = \frac{1}{2}$. For $x = \frac{1}{2}$, we have $0 < y < \frac{1}{4}$, and

$f_X(\frac{1}{2}) = \int_0^{1/4} f(\frac{1}{2}, y)\,dy = \int_0^{1/4} 12 \times \frac{1}{2} \times y\,dy = \frac{3}{16}$.

The conditional distribution of Y given $X = \frac{1}{2}$ has pdf $f(y|x = \frac{1}{2}) = \frac{6y}{3/16} = 32y$ for $0 < y < \frac{1}{4}$.

The conditional expectation $E\left[Y|X = \frac{1}{2}\right] = \int_0^{1/4} y \times 32y\,dy = \frac{1}{6}$. Answer: E

19. We are given $P(X \le 400) = \int_{100}^{400} \frac{\alpha \times 100^\alpha}{x^{\alpha+1}}\,dx = 1 - \left(\frac{100}{400}\right)^\alpha = 0.6$.

Then $\alpha = \frac{ln\,0.4}{ln\,0.25} = 0.66$, Answer: C

20. The exact probability is $A = P[1000 + X = 998, 999, 1000, 1001 \text{ or } 1002]$

$= P[X = 0, \pm 1 \text{ or } \pm 2] = 0.4 + 4 \times 0.1 = 0.8.$

The mean of X is 0 and the variance of X is

$E[X^2] - 0 = (1^2 + 2^2) \times 0.2 + (3^2 + 4^2) \times 0.1 = 3.5.$

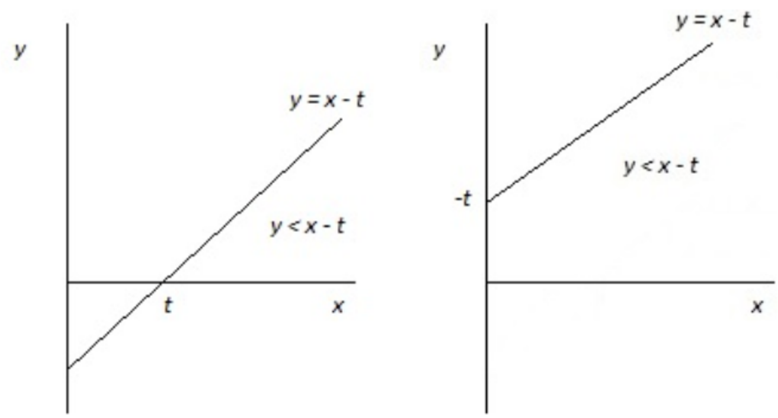
The number of jelly beans in a randomly chosen jar is in the interval $[998, 1002]$ if $-2 \le X \le 2$.

Applying the normal approximation with integer correction this probability, we have

$P[-2.5 < X \le 2.5] = P\left[\frac{-2.5}{\sqrt{3.5}} < \frac{X}{\sqrt{3.5}} \le \frac{2.5}{\sqrt{3.5}}\right] = \Phi(1.34) - \Phi(-1.34)$

$= 2 \times \Phi(1.34) - 1 = 2 \times 0.9099 - 1 = 0.8198.$ Answer: C

21. We find the cdf of T. $F_T(t) = P[T \leq t] = 1 - P[T > t]$.
$P[T > t] = P[X - Y > t] = P[Y < X - t]$.
The probability is represented by the region above the x-axis and below the line $y = x - t$ in the graphs below. The graph on the left represents the region for $t > 0$ and the graph on the right is for $t < 0$.

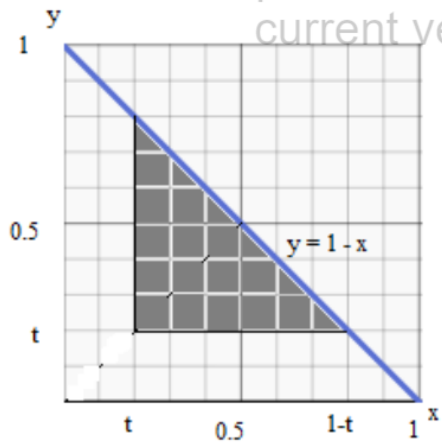


Since X and Y are independent, the joint density $f(x,y)$ is $f_X(x) \times f_Y(y) = \frac{e^{-x/2}}{2} \times e^{-y}$.
If $t > 0$, then the area of the appropriate region in the left graph is
$P[T > t] = \int_t^\infty \int_0^{x-t} \left[\frac{e^{-x/2}}{2} \times e^{-y}\right] dy\, dx = \int_t^\infty \frac{e^{-x/2}}{2} \times [1 - e^{-(x-t)}]\, dy = \frac{2e^{-t/2}}{3}$.
If $t < 0$, then the area of the appropriate region in the right graph is
$P[T > t] = \int_0^\infty \int_0^{x-t} \left[\frac{e^{-x/2}}{2} \times e^{-y}\right] dy\, dx = \int_0^\infty \frac{e^{-x/2}}{2} \times [1 - e^{-(x-t)}]\, dy = 1 - \frac{e^t}{3}$.
Answer: C

22. $P(B|A \cap C) = \frac{P(A \cap B \cap C)}{P(A \cap C)} = \frac{P(B \cap C)}{P(A \cap C)} = \frac{0.2}{0.6} = \frac{1}{3}$. Answer: D

23. We see that $F_X(0) = 1 - \sum_{n=k}^{\infty} \frac{e^{-1}}{n!} = 0$, since $\sum_{n=k}^{\infty} \frac{1}{n!} = e$.
The probability function for $k \geq 1$ is
$P(X = k) = F_X(k) - F_X(k-1) = \left(1 - \sum_{n=k}^{\infty} \frac{e^{-1}}{n!}\right) - \left(1 - \sum_{n=k-1}^{\infty} \frac{e^{-1}}{n!}\right) = \frac{e^{-1}}{(k-1)!}$
If we let $Y = X - 1$, then $P(Y = k) = P(X = k + 1) = \frac{e^{-1}}{k!}$, which is the probability function for a Poisson random variable with mean 1. Therefore, $E[X] = E[Y + 1] = 2$. Answer: C

24. The time until machine failure is $T = min\{X, Y\}$. We can find the expected time until failure as
$E[T] = \int_0^1 S_T(t)\, dt = \int_0^1 P(T > t)\, dt$.
$P(T > t) = P[(X > t) \cap (Y > t)]$. Note that for $t > \frac{1}{2}$, it is not possible to have $T > t$, since if $x > \frac{1}{2}$ then $y < 1 - x < \frac{1}{2}$. Therefore, we must have $T < \frac{1}{2}$.
Then, if $t < \frac{1}{2}$, $P[(X > t) \cap (Y > t)] = \int_t^{1-t} \int_t^{1-x} 6x\, dy\, dx = \frac{1}{2}$.
The integral is over the shaded region in the graph below.



Answer: D

25. $P(\text{Die 1}|\text{total of 8}) = \frac{P[(\text{Die 1})\cap(\text{total of 8})]}{P[\text{total of 8}]}$

$P[(\text{Die 1}) \cap (\text{total of 8})] = P[\text{total of 8}|\text{Die 1}] \times P(\text{Die 1})$

There 36 equally likely ways to toss Die 1 twice. 5 of those ways result in a total of 8; they are 2 and 6 , 3 and 5 , 4 and 4 , 5 and 3 , 6 and 2. Therefore, $P[\text{total of 8}|\text{Die 1}] = \frac{5}{36}$.

Since the die that is tossed is chosen randomly, each die has the same probability of $\frac{1}{3}$ of being the one that is tossed. Therefore, $P[(\text{Die 1}) \cap (\text{total of 8})] = P[\text{total of 8}|\text{Die 1}] \times P(\text{Die 1}) = \frac{5}{36} \times \frac{1}{3} = \frac{5}{108}$.

Since the tosses must come from one of the three dice, we have

$P[\text{total of 8}] = P[(\text{total of 8}) \cap (\text{Die 1})] + P[(\text{total of 8}) \cap (\text{Die 2})] + P[(\text{total of 8}) \cap (\text{Die 3})]$

For Die 2 and Die 3, we proceed as we did with Die 1.

$P[(\text{total of 8}) \cap (\text{Die 2})] = P[\text{total of 8}|\text{Die 2}] \times P(\text{Die 2})$.

There are 8 ways that the total could be 8 from Die 2, for a probability of $\frac{8}{36} = \frac{2}{9}$. Alternatively, Die 2 total is 8 if one toss is 3 and the other is 5. There is a $\frac{1}{3}$ probability that the first toss is a 3 and a $\frac{1}{3}$ probability that the second toss is a 5. The same is true for first toss 5 and second toss 3. The overall probability of a total of 8 from Die 2 is $\frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{2}{9}$, and

$P[(\text{total of 8}) \cap (\text{Die 2})] = P[\text{total of 8}|\text{Die 2}] \times P(\text{Die 2}) = \frac{2}{9} \times \frac{1}{3} = \frac{2}{27}$.

$P[(\text{total of 8}) \cap (\text{Die 3})] = P[\text{total of 8}|\text{Die 3}] \times P(\text{Die 3})$.

There are 9 ways that the total could be 8 from Die 3, for a probability of $\frac{9}{36} = \frac{1}{4}$. This is similar to Die 2, except one of the tosses must be 2 and one of the tosses must be 6 (8 ways in total), with the additional case that both tosses are 4, giving a total of 9 pairs of tosses that add to 8.

Then $P[(\text{total of 8}) \cap (\text{Die 3})] = P[\text{total of 8}|\text{Die 3}] \times P(\text{Die 3}) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

We then have $P[\text{total of 8}] = \frac{5}{108} + \frac{2}{27} + \frac{1}{12} = \frac{22}{108}$,

and $P(\text{Die 1}|\text{total of 8}) = \frac{P[(\text{Die 1})\cap(\text{total of 8})]}{P[\text{total of 8}]} = \frac{5/108}{22/108} = \frac{5}{22}$.

An alternative solution is based on "general reasoning" as follows. With Die 1, there are 5 ways that two tosses could add up to 8: 2+6 , 3+5 , 4+4 , 5+3 , and 6+2. With Die 2 there are ways: there are 4 ways the first toss can 3 and the second toss can be 5 (3a+5a, 3b+5a, 3a+5b, 3b+5b) and there are 4 ways the first toss can be 5 and the second toss can be 3 (reverse the previous list), so there are 8 ways the total can be 8 from two tosses of Die 2. With Die 3 there are 9 ways that the total from two tosses can be 8 (2a+6a, 2b+6a, etc, 4+4, this is similar to Die 2, but there is the 4+4 combination along with the other 8 combinations). Each of the three dice has the same chance of being chosen initially and there are $5+8+9=22$ two-toss combinations that add to 8, each being equally likely. Since 5 of the 22 combinations come from Die 1, that indicates that the is a $\frac{5}{22}$ chance that the die chosen is Die 1.

26. $P[(3 \text{ of four cars are domestic}) \cap (4 \text{ cars cross the intersection})]\backslash$
$= P[3 \text{ of } 4 \text{ cars are domestic}|4 \text{ cars cross the intersection}] \times P[4 \text{ cars cross the intersection}]$.
The Poisson with mean 4 results in $P[4 \text{ cars cross the intersection}] = \frac{4^4 e^{-4}}{4!} = 0.195367$.
Given that 4 cars cross during the minute, the number that are domestic made has a binomial distribution with $n=4$, $p=0.75$, so the probability that 3 of the 4 are domestic is
$\binom{4}{3}(.75)^3(.25) = .421875$. The overall probability is $.421875 \times .195367 = .0824$. Answer: B

27. I. $P(X=1) = P(X=1 \cap \text{Die } 1) + P(X=1 \cap \text{Die } 2)$
$= P(X=1|\text{Die } 1) \times P(\text{Die } 1) + P(X=1|\text{Die } 2) \times P(\text{Die } 2) = \frac{1}{6} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{7}{12}$.
General reasoning indicates that half the time the coin toss is a tail, so $P(X=1)$ is at least .5, but since there is a chance that 1 is tossed when the coin is head, that makes the overall probability of a 1 greater than 0.5. True.

II. $P(X=1) = \frac{7}{12}$ (from I).
In a way similar to I we get $P(X=2)$
$= P(X=2|\text{Die } 1) \times P(\text{Die } 1) + P(X=2|\text{Die } 2) \times P(\text{Die } 2) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$,
and similarly, $P(X=3) = P(X=4) = P(X=5) = P(X=6) = \frac{1}{12}$.
Then, $E[X] = \frac{7}{12} + (2+3+4+5+6) \times \frac{1}{12} = \frac{9}{4}$.
General reasoning suggests that half the time the coin toss is head and the average of the fair die is 3.5, the other half of the time the coin is a tail and the average of the die is 1. The overall average would be $\frac{1}{2} \times (3.5+1) = 2.25$. True.

III. From II, $E[X] = \frac{9}{4}$, and from I and II
$E[X^2] = 1^2 \times \frac{7}{12} + (2^2+3^2+4^2+5^2+6^2) \times \frac{1}{12} = \frac{97}{12}$.
Then $Var[X] = E[X^2] - (E[X])^2 = \frac{97}{12} - (\frac{9}{4})^2 = \frac{145}{48} = 3.02 > 3$. False Answer: C

28. Let X represent the number of pieces in a randomly chosen pack. Then $E[X] = 100]$ and let $Var[X] = \sigma^2$. $P[X \le 90] = P[X \le 90.5]$ using the integer correction. Applying the normal approximation, this becomes $P\left[\frac{X-100}{\sigma} \le \frac{-9.5}{\sigma}\right] = 0.0436$. From the normal table we see that $P[Z \le -1.71] = 0.0436$. Therefore, $\frac{-9.5}{\sigma} = -1.71$ from which we get $\sigma = 5.56$. Answer: D

29. The transformation (function) upon which Y is based is $Y = \sqrt{X} = g(X)$. Since the exponential distribution is defined only for $x > 0$, this function is increasing for all value of x for which the distribution is defined, and therefore, g has an inverse function k. The inverse function of the square root function is the squaring function, i.e., $k(y) = y^2$, so that $k(g(x)) = [g(x)]^2 = [\sqrt{x}]^2 = x$ (if $x > 0$). According to the method by which the density function of a transformed random variable is found, we have $f_Y(y) = f_X(k(y)) \times k'(y)$. In this case $f_X(x) = e^{-x}$ for the exponential random variable X with a mean of 1. Then $f_Y(y) = e^{-y^2} \times 2y$. Answer: C

30. $E[N_1] = Var[N_1] = 50$, $E[N_2] = 50$, $Var[N_2] = 25$.
$W = N_1 - N_2 \to E[W] = 50 - 50 = 0$, $Var[W] = 50 + 25 = 75$ (since N_1 and N_2 are independent).
$P[\,|\,W\,| > 3] = P[W \le -3.5] + P[W > 3.5]$ (after applying integer correction).
$P[W \le -3.5] = P\left[\frac{W}{\sqrt{75}} \le \frac{-3.5}{\sqrt{75}}\right] = \Phi(-0.40) = 1 - \Phi(0.40) = 1 - 0.655 = 0.345$.
$P[W > 3.5] = P\left[\frac{W}{\sqrt{75}} > \frac{3.5}{\sqrt{75}}\right] = 1 - \Phi(0.40) = 1 - 0.655 = 0.345$.
Combined probability is 0.69. Answer: E