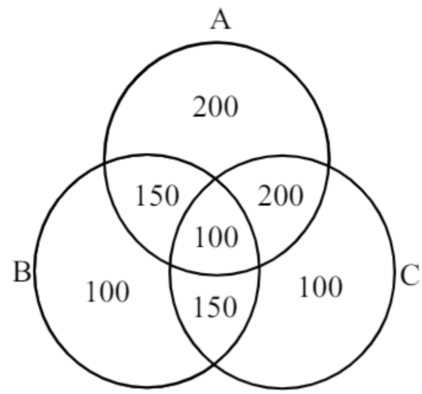
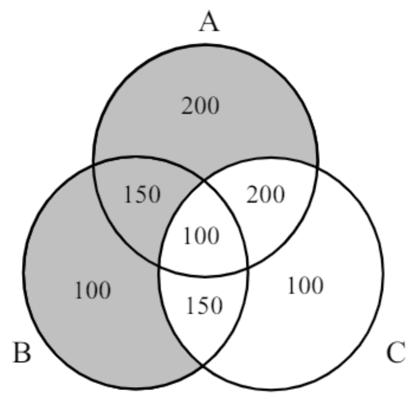
We can express the numbers in the component sets in the following Venn diagram.



(ii) The number who had either symptom A or B (or both) but not C is $n[(A \cup B) \cap C'] = 200 + 150 + 100 = 450$. This represented in the following Venn diagram.



(iii) The number who had all three symptoms is $n(A \cap B \cap C)$, which we have already identified as 100.

4. $A =$ auto insurance purchased

$H =$ homeowner insurance purchased

$R =$ renter insurance purchased

$P(\cdot)$ denotes the percent of the agent's clients

We wish to find $P(A \cap R)$.

We are given $P(H \cap R) = 0$ (homeowner and renter insurance are mutually exclusive), and

(i) $1 - P(A \cup H \cup R) = 0.17$

(ii) $P(A) = 0.64$

(iii) $P(H) = 2 \times P(R)$

(iv) $P(A \cap H) + P(A \cap R) + P(H \cap R) - 2 \times P(A \cap H \cap R) = 0.35$

(v) $P(H \cap A') = 0.11$.

From (i) and (iv) we get $P(A \cap H) + P(A \cap R) = 0.35$.

From (ii) we get

$P(A \cup H \cup R)$

$= P(A) + P(H) + P(R) - P(A \cap H) - P(A \cap R) - P(H \cap R) + P(A \cap H \cap R) = 0.83,$

which becomes $0.64 + P(H) + \frac{1}{2} \times P(H) - 0.35 = 0.83$, so that $P(H) = 0.36$ and $P(R) = 0.18$. From (v) we get $0.36 = P(H) = P(H \cap A) + P(H \cap A') = P(H \cap A) + 0.11$,

so that $P(H \cap A) = 0.25$. Then from $P(A \cap H) + P(A \cap R) = 0.35$, we get $P(A \cap R) = 0.10$.

Answer: B

5. We are given $n(C) = 34$ (number who watched CBS) , $n(N) = 15$, $n(A) = 10$, $n(C \cap N) = 7$, $n(C \cap A) = 6$, $n(N \cap A) = 5$, $n(C \cap N \cap A) = 4$, $n(H) = 18$, and $H \cap (C \cup N \cup A) = 0$.

The number who watched at least one channel is

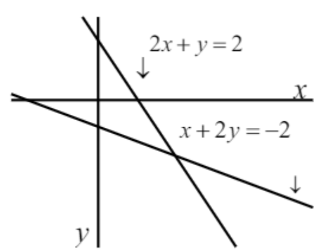
$n(H \cup C \cup N \cup A) = n(H) + n(C \cup N \cup A)$, and

$n(C \cup N \cup A) = 34 + 15 + 10 - 7 - 6 - 5 + 4 = 45.$

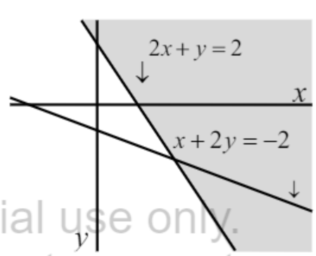
The number who watched at least one channel is $18 + 45 = 63$. Answer: B

6. $\lim_{N \to \infty} \frac{5^N}{N!} = \frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdots}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots} \leq \frac{5^5}{5!} \cdot \left(\frac{5}{6}\right)^{\infty} = 0.$ Answer: A

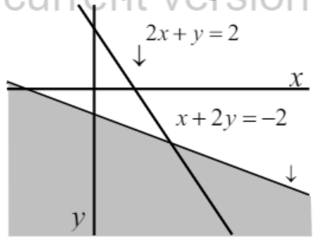
7. The graphs of the lines are illustrated in the following graph.



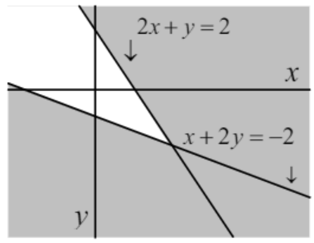
The inequality $2x + y > 2$ is represented by the region "above" the line $2x + y = 2$:



The inequality $x+2y<-2$ is represented by the region "below" the line $x+2y=-2$:



The union of the two regions is:



Answer: B

8. The quadratic equation ax^2+bx+c has no real roots if $b^2-4ac<0$.
Thus, $9+8k<0 \rightarrow k<-\frac{9}{8}$. Answer: A

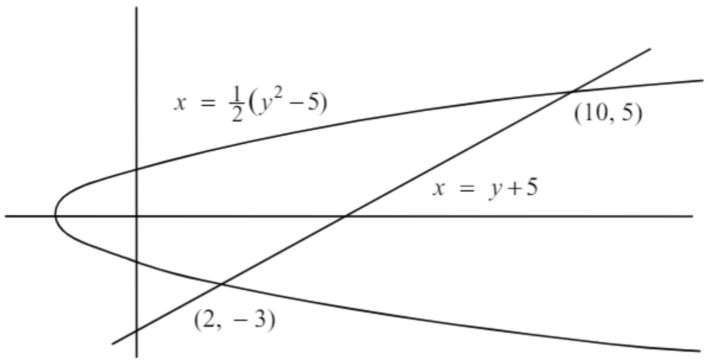
9. $1/(1/x)=x$. Answer: D

10. 1st derivative - $e^x(x^4+4x^3)$; 2nd derivative - $e^x(x^4+8x^3+12x^2)$
3rd derivative - $e^x(x^4+12x^3+36x^2+24x)$
4th derivative - $e^x(x^4+16x^3+72x^2+96x+24)$
It might be possible to determine a general expression for the nth derivative in terms of n then substitute $x=0$. However, note that the 1st, 2nd and 3rd derivatives ($n=1,2,3$) must be 0 if $x=0$, but the 4th derivative is not 0 at $x=0$. This eliminates answers A, B, C and E.

11. We define $F(t)$ to be the population at time t. Then $F(0)=6$, $\lim_{t\rightarrow\infty}F(t)=30$,
and $F'(t)=\frac{Ae^t}{(.02A+e^t)^2}$. Then (using the substitution $u=.02A+e^t$), we have
$F(s)-F(0)=\int_0^s F'(t)\,dt=\int_0^s \frac{Ae^t}{(.02A+e^t)^2}\,dt=-\frac{A}{.02A+e^t}\Big|_{t=0}^{t=s}=\frac{A}{.02A+1}-\frac{A}{.02A+e^s}$,
so that $F(s)=6+\frac{A}{.02A+1}-\frac{A}{.02A+e^s}$.

Then $\lim_{s\rightarrow\infty}F(s)=6+\frac{A}{.02A+1}=30\rightarrow \frac{A}{.02A+1}=24\rightarrow A=46.15$.
Therefore, $F(s)=30-\frac{46.15}{.923+e^s}$. In order to have $F(t)=10$, we have
$30-\frac{46.15}{.923+e^s}=10\rightarrow s=0.325$. Answer: A

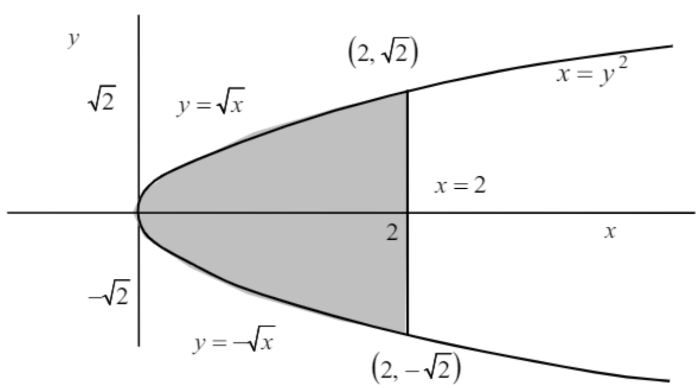
12. The line and the parabola intersect at y-values that are the solutions of $y+5=\frac{1}{2}(y^2-5)$, so that $y=-3\ (x=2)\ ,\ 5\ (x=10)$.
The graph below indicates the closed region bounded by the line and the parabola.
The area of the region is $\int_{-3}^{5}[(y+5)-\frac{1}{2}(y^2-5)]\,dy=\frac{128}{3}$.



Answer: E

13. $F(x)=f(g(x))$, where $g(x)=x^{1/3}$ and $f(z)=\int_0^z\sqrt{1+t^4}\,dt$. Applying the Chain Rule results in $F'(x)=f'(g(x))\cdot g'(x)=f'(x^{1/3})\times\frac{1}{3}x^{-2/3}=\sqrt{1+(x^{1/3})^4}\times\frac{1}{3}x^{-2/3}$.
At $x=0$, this becomes $\frac{1}{0^+}$. Answer: E

14. The region of integration is illustrated in the graph below. For each x between 0 and 2, we have $-\sqrt{x}\le y\le\sqrt{x}$, or equivalently, for $-\sqrt{2}\le y\le\sqrt{2}$, we have $y^2\le x\le 2$. The integral becomes $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{y^2}^{2}f(x,y)\,dx\,dy$.



Answer: B

15. $\int_0^1 \int_x^1 \frac{1}{1+y^2}\,dy\,dx = \int_0^1 \int_0^y \frac{1}{1+y^2}\,dx\,dy = \int_0^1 \frac{y}{1+y^2}\,dy = \frac{1}{2}\,ln(1+y^2)\Big|_0^1 = \frac{1}{2}\,ln\,2$

Note that if we try to solve the integral directly as written, we get

$\int_0^1 \int_x^1 \frac{1}{1+y^2}\,dy\,dx = \int_0^1 \left[arctan(y)\Big|_x^1\right] dx = \int_0^1 [\frac{\pi}{4} - arctan(x)]\,dx$

which is a more difficult integral to determine. Answer: B

16. Total salary $= 100,000[1 + (1.05) + (1.05)^2 + \cdots + (1.05)^{34}]$

$= 100,000 \cdot \frac{1.05^{35}-1}{1.05-1} = 9,032,031.$

Note that salary in 35th year has grown for 34 years since the first year of employment. Answer: E

17. Value at end of 35 years of deposits is

$$100,000(.03)(1.04)^{34} + 100,000(.03)(1.05)(1.04)^{33} + 100,000(.03)(1.05)^2(1.04)^{32}$$
$$+ \cdots + 100,000(.03)(1.05)^{33}(1.04) + 100,000(.03)(1.05)^{34}$$
$$= 100,000(.03)[(1.04)^{34} + (1.05)(1.04)^{33} + (1.05)^2(1.04)^{32}$$
$$+ \cdots + (1.05)^{33}(1.04) + (1.05)^{34}]$$

(this represents accumulation of 1st yr. deposit, 2nd year deposit, 3rd year deposit, . . . , 34th year deposit, and 35th year deposit) . If we factor out $(1.04)^{34}$ the sum becomes

$$100,000(.03)(1.04)^{34} \times [1 + \frac{1.05}{1.04} + (\frac{1.05}{1.04})^2 + \cdots + (\frac{1.05}{1.04})^{33} + (\frac{1.05}{1.04})^{34}]$$
$$= 100,000(.03)(1.04)^{34} \times [\frac{(\frac{1.05}{1.04})^{35}-1}{\frac{1.05}{1.04}-1}] = 470,978.$$

Answer: C

SECTION 1 - BASIC PROBABILITY CONCEPTS

PROBABILITY SPACES AND EVENTS

Sample point and sample space: A sample point is the simple outcome of a random experiment. The probability space (also called sample space) is the collection of all possible sample points related to a specified experiment. When the experiment is performed, one of the sample points will be the outcome. The probability space is the "full set" of possible outcomes of the experiment.

Mutually exclusive outcomes: Outcomes are mutually exclusive if they cannot occur simultaneously. They are also referred to as **disjoint** outcomes.

Exhaustive outcomes: Outcomes are exhaustive if they combine to be the entire probability space, or equivalently, if at least one of the outcomes must occur whenever the experiment is performed.

Event: Any collection of sample points, or any subset of the probability space is referred to as an event. We say that "event A has occurred" if the experimental outcome was one of the sample points in A.

Union of events $\boldsymbol{A}$ and $\boldsymbol{B}$: $A \cup B$ denotes the union of events A and B, and consists of all sample points that are in either A or B.

Union of events $\boldsymbol{A_1, A_2, ..., A_n}$: $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$ denotes the union of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are in at least one of the A_i's. This definition can be extended to the union of infinitely many events.

Intersection of events $\boldsymbol{A_1, A_2, ..., A_n}$: $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$ denotes the intersection of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are simultaneously in all of the A_i's. ($A \cap B$ is also denoted $A \cdot B$ or AB).

Mutually exclusive events $\boldsymbol{A_1, A_2, ..., A_n}$: Two events are mutually exclusive if they have no sample points in common, or equivalently, if they have **empty intersection**. Events $A_1, A_2, ..., A_n$ are mutually exclusive if $A_i \cap A_j = \emptyset$ for all $i \neq j$, where $\emptyset$ denotes the empty set with no sample points. Mutually exclusive events cannot occur simultaneously.

Exhaustive events $\boldsymbol{B_1, B_2, ..., B_n}$: If $B_1 \cup B_2 \cup \cdots \cup B_n = S$, the entire probability space, then the events $B_1, B_2, ..., B_n$ are referred to as exhaustive events.

Complement of event A: The complement of event A consists of all sample points in the probability space that are **not in A**. The complement is denoted $\overline{A}$, $\sim A$, A' or A^c and is equal to $\{x : x \notin A\}$. When the underlying random experiment is performed, to say that the complement of A has occurred is the same as saying that A has not occurred.

Subevent (or subset) A of event B: If event B contains all the sample points in event A, then A is a subevent of B, denoted $A \subset B$. The occurrence of event A implies that event B has occurred.

Partition of event A: Events $C_1, C_2, ..., C_n$ form a partition of event A if $A = \bigcup_{i=1}^{n} C_i$ and the C_i's are mutually exclusive.

DeMorgan's Laws:

(i) $(A \cup B)' = A' \cap B'$, to say that $A \cup B$ has not occurred is to say that A has not occurred and B has not occurred; this rule generalizes to any number of events;
$\left(\bigcup_{i=1}^{n} A_i\right)' = (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n' = \bigcap_{i=1}^{n} A_i'$

(ii) $(A \cap B)' = A' \cup B'$, to say that $A \cap B$ has not occurred is to say that either A has not occurred or B has not occurred (or both have not occurred); this rule generalizes to any number of events, $\left(\bigcap_{i=1}^{n} A_i\right)' = (A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n' = \bigcup_{i=1}^{n} A_i'$

Indicator function for event A: The function $I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is the indicator function for event A, where x denotes a sample point. $I_A(x)$ is 1 if event A has occurred.

Basic set theory was reviewed in Section 0 of these notes. The Venn diagrams presented there apply in the sample space and event context presented here.

Example 1-1:

Suppose that an "experiment" consists of tossing a six-faced die. The **probability space** of outcomes consists of the set $\{1, 2, 3, 4, 5, 6\}$, each number being a **sample point** representing the number of spots that can turn up when the die is tossed. The outcomes 1 and 2 (or more formally, $\{1\}$ and $\{2\}$) are an example of **mutually exclusive** outcomes, since they cannot occur simultaneously on one toss of the die. The collection of all the outcomes (sample points) 1 to 6 are **exhaustive** for the experiment of tossing a die since one of those outcomes must occur. The collection $\{2, 4, 6\}$ represents the **event** of tossing an even number when tossing a die. We define the following events.

$A = \{1, 2, 3\} =$ "a number less than 4 is tossed",

$B = \{2, 4, 6\} =$ "an even number is tossed",

$C = \{4\} =$ "a 4 is tossed",

$D = \{2\} =$ "a 2 is tossed".

Then (i) $A \cup B = \{1, 2, 3, 4, 6\}$,

(ii) $A \cap B = \{2\}$,

(iii) A and C are mutually exclusive since $A \cap C = \emptyset$ (when the die is tossed it is not possible for both A and C to occur),

(iv) $D \subset B$,

(v) $A' = \{4, 5, 6\}$ (complement of A),

(vi) $B' = \{1, 3, 5\}$,

(vii) $A \cup B = \{1, 2, 3, 4, 6\}$, so that $(A \cup B)' = \{5\} = A' \cap B'$ (this illustrates one of DeMorgan's Laws). □

Some rules concerning operations on events:

(i) $A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$ and
$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$ for any events
$A, B_1, B_2, ..., B_n$

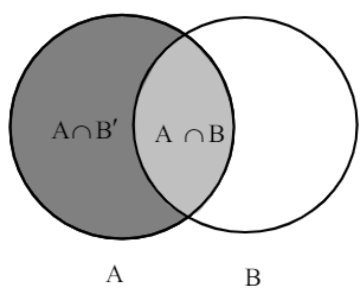
(ii) If $B_1, B_2, ..., B_n$ are exhaustive events $\left(\bigcup\limits_{i=1}^{n} B_i = S, \text{the entire probability space} \right)$,
then for any event A,

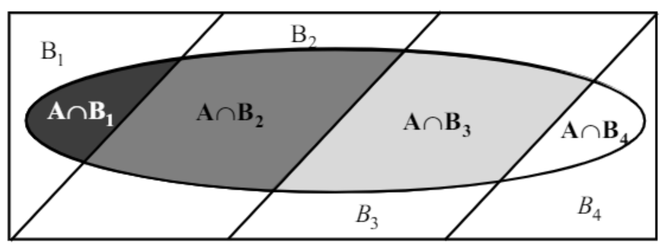
$$A = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$

If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive events, then they form a **partition of the probability space**. For example, the events $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$ and $B_3 = \{5, 6\}$ form a partition of the probability space for the outcomes of tossing a single die.

The general idea of a partition is illustrated in the diagram below. As a special case of a partition, if B is any event, then B and B' form a partition of the probability space. We then get the following identity for any two events A and B:

$\boldsymbol{A = A \cap (B \cup B') = (A \cap B) \cup (A \cap B')}$; note also that $A \cap B$ and $A \cap B'$ form a partition of event A.





(iii) For any event A, $A \cup A' = S$, the entire probability space, and $A \cap A' = \emptyset$

(iv) $A \cap B' = \{x : x \in A \text{ and } x \notin B\}$ is sometimes denoted $A - B$, and consists of all sample points that are in event A but not in event B

(v) If $A \subset B$ then $A \cup B = B$ and $A \cap B = A$

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PROBABILITY

Probability function for a discrete probability space: A discrete probability space (or sample space) is a set of a finite or countable infinite number of sample points. $P[a_i]$ or p_i denotes the probability that sample point (or outcome) a_i occurs. There is some underlying "random experiment" whose outcome will be one of the a_i's in the probability space. Each time the experiment is performed, one of the a_i's will occur. The probability function P must satisfy the following two conditions:

(i) $0 \leq P[a_i] \leq 1$ for each a_i in the sample space, and
(ii) $P[a_1] + P[a_2] + \cdots = \sum_{\text{all } i} P[a_i] = 1$ (total probability for a probability space is always 1)

This definition applies to both finite and infinite probability spaces.

Tossing an ordinary die is an example of an experiment with a finite probability space $\{1, 2, 3, 4, 5, 6\}$. An example of an experiment with an infinite probability space is the tossing of a coin until the first head appears. The toss number of the first head could be any positive integer, 1, or 2, or 3, ... The probability space for this experiment is the infinite set of positive integers $\{1, 2, 3, ...\}$ since the first head could occur on any toss starting with the first toss. The notion of discrete random variable covered later is closely related to the notion of discrete probability space and probability function.

Uniform probability function: If a probability space has a finite number of sample points, say k points, $a_1, a_2, ..., a_k$, then the probability function is said to be **uniform** if each sample point has the same probability of occurring ; $P[a_i] = \frac{1}{k}$ for each $i = 1, 2, ..., k$. Tossing a fair die would be an example of this, with $k = 6$.

Probability of event A: An event A consists of a subset of sample points in the probability space. In the case of a discrete probability space, the probability of A is
$P[A] = \sum_{a_i \in A} P[a_i]$, the sum of $P[a_i]$ over all sample points in event A.

Example 1-2: In tossing a "fair" die, it is assumed that each of the six faces has the same chance of $\frac{1}{6}$ of turning up. f this is true, then the probability function $P(j) = \frac{1}{6}$ for $j = 1, 2, 3, 4, 5, 6$ is a uniform probability function on the sample space $\{1, 2, 3, 4, 5, 6\}$.
The event "an even number is tossed" is $A = \{2, 4, 6\}$, and has probability
$P[A] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$. □

Continuous probability space: An experiment can result in an outcome which can be any real number in some interval. For example, imagine a simple game in which a pointer is spun randomly and ends up pointing at some spot on a circle. The angle from the vertical (measured clockwise) is between 0 and 2π. The probability space is the interval $(0, 2\pi]$, the set of possible angles that can occur. We regard this as a continuous probability space. In the case of a continuous probability space (an interval), we describe probability by assigning probability to subintervals rather than individual points. If the spin is "fair", so that all points on the circle are equally likely to occur, then intuition suggests that the probability assigned to an interval would be the fraction that the interval is of the full circle. For instance, the probability that the pointer ends up between "3 O'clock" and "9 O'clock" (between $\pi/2$ or 90° and $3\pi/2$ or 270° from the vertical) would be 0.5, since that subinterval is one-half of the full circle. The notion of a continuous random variable, covered later in this study guide, is related to a continuous probability space.

Some rules concerning probability:

(i) $\boldsymbol{P[S]=1}$ **if S is the entire probability space** (when the underlying experiment is performed, some outcome must occur with probability 1; for instance $S=\{1,2,3,4,5,6\}$ for the die toss example).

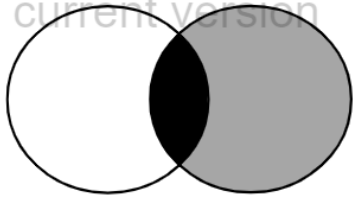
(ii) $\boldsymbol{P[\emptyset]=0}$ (the probability of no face turning up when we toss a die is 0).

(iii) **If events $A_1, A_2, \ldots, A_n$ are mutually exclusive (also called disjoint) then**
$$\boldsymbol{P[\bigcup_{i=1}^{n} A_i] = P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = \sum_{i=1}^{n} P[A_i]}.$$
This extends to infinitely many mutually exclusive events. This rule is similar to the rule discussed in Section 0 of this study guide, where it was noted that the number of elements in the union of mutually disjoint sets is the sum of the numbers of elements in each set. When we have mutually exclusive events and we add the event probabilities, there is no double counting.

(iv) For any event A, $\boldsymbol{0 \le P[A] \le 1}$.

(v) If $A \subset B$ then $P[A] \le P[B]$.

(vi) For any events A, B and C, $\boldsymbol{P[A \cup B] = P[A] + P[B] - P[A \cap B]}$.
This relationship can be explained as follows. We can formulate $A \cup B$ as the union of three mutually exclusive events as follows: $A \cup B = (A \cap B') \cup (A \cap B) \cup (B \cap A')$.
This is expressed in the following Venn diagram.



$A \cap B'$ $A \cap B$ $B \cap A'$

Since these are mutually exclusive events, it follows that
$P[A \cup B] = P[A \cap B'] + P[A \cap B] + P[B \cap A']$.
From the Venn diagram we see that $A = (A \cap B') \cup (A \cap B)$, so that
$\boldsymbol{P[A] = P[A \cap B'] + P[A \cap B]}$, and we also see that $P[B \cap A'] = P[B] - P[A \cap B]$.
It then follows that
$P[A \cup B] = (P[A \cap B'] + P[A \cap B]) + P[B \cap A'] = P[A] + P[B] - P[A \cap B]$.

We subtract $P[A \cap B]$ because $P[A] + P[B]$ counts $P[A \cap B]$ twice. $P[A \cup B]$ is the probability that at least one of the two events A, B occurs. This was reviewed in Section 0, where a similar rule was described for counting the number of elements in $A \cup B$.

For the union of three sets we have
$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

(vii) For any event A, $\boldsymbol{P[A'] = 1 - P[A]}$.

(viii) For any events A and B, $\boldsymbol{P[A] = P[A \cap B] + P[A \cap B']}$
(this was mentioned in (vi), it is illustrated in the Venn diagram above).

(ix) For exhaustive events $B_1, B_2, ..., B_n$, $P[\bigcup_{i=1}^{n} B_i] = 1$.
If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive, they form a **partition** of the entire probability space, and for any event A,
$$\boldsymbol{P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n] = \sum_{i=1}^{n} P[A \cap B_i].}$$

(x) If P is a uniform probability function on a probability space with k points, and if event A consists of m of those points, then $P[A] = \frac{m}{k}$.

(xi) **The words "percentage" and "proportion" are used as alternatives to "probability".**
As an example, if we are told that the percentage or proportion of a group of people that are of a certain type is 20%, this is generally interpreted to mean that a randomly chosen person from the group has a 20% probability of being of that type. This is the "long-run frequency" interpretation of probability. As another example, suppose that we are tossing a fair die. In the long-run frequency interpretation of probability, to say that the probability of tossing a 1 is $\frac{1}{6}$ is the same as saying that if we repeatedly toss the die, the proportion of tosses that are 1's will approach $\frac{1}{6}$.

(xii) for any events $A_1, A_2, ..., A_n$, $P[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} P[A_i]$, with equality holding if and only if the events are mutually exclusive

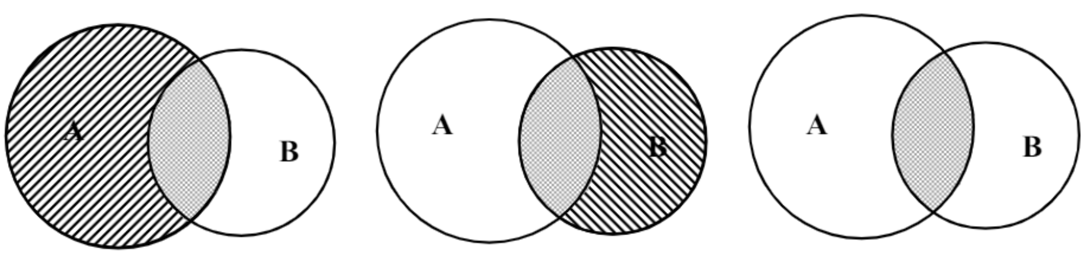
Example 1-3: Suppose that $P[A \cap B] = 0.2$, $P[A] = 0.6$, and $P[B] = 0.5$.
Find $P[A' \cup B']$, $P[A' \cap B']$, $P[A' \cap B]$ and $P[A' \cup B]$.
Solution: Using probability rules we get the following.
$P[A' \cup B'] = P[(A \cap B)'] = 1 - P[A \cap B] = 0.8.$

$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 0.6 + 0.5 - 0.2 = 0.9$
$\rightarrow P[A' \cap B'] = P[(A \cup B)'] = 1 - P[A \cup B] = 1 - 0.9 = 0.1.$

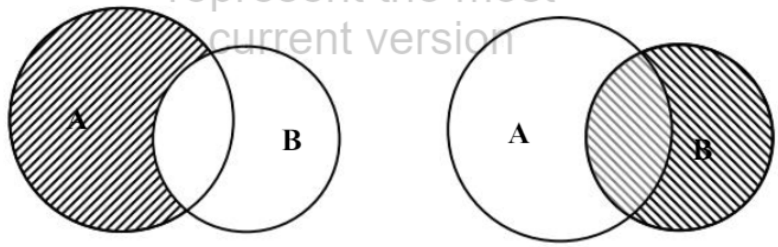
$P[B] = P[B \cap A] + P[B \cap A'] \rightarrow P[A' \cap B] = 0.5 - 0.2 = 0.3.$

$P[A' \cup B] = P[A'] + P[B] - P[A' \cap B] = 0.4 + 0.5 - 0.3 = 0.6.$

The following Venn diagrams illustrate the various combinations of intersections, unions and complements of the events A and B.

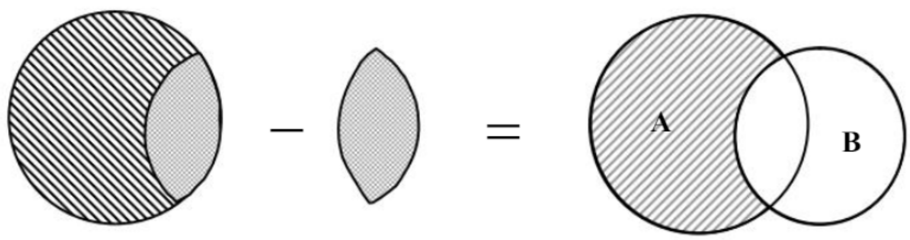


$P[A] = .6$ $P[B] = .5$ $P[A \cap B] = .2$



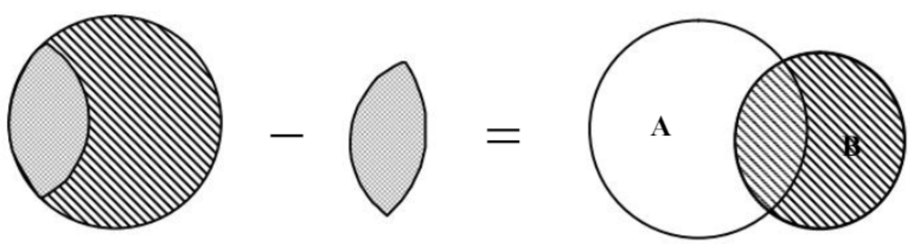
$P[A \cap B'] = .4$ $\qquad$ $P[A' \cap B] = .3$

From the following Venn diagrams we see that $P[A \cap B'] = P[A] - P[A \cap B] = 0.6 - 0.2 = 0.4$ and $P[A' \cap B] = P[B] - P[A \cap B] = 0.5 - 0.2 = 0.3$.



$P[A]$ $-$ $P[A \cap B]$ $=$ $P[A \cap B']$

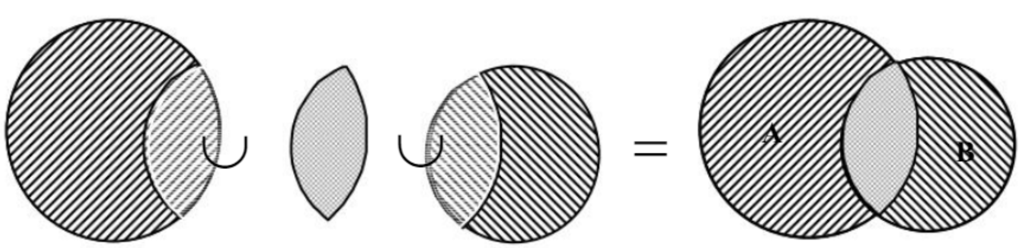
$.6$ $-$ $.2$ $=$ $.4$



$P[B]$ $-$ $P[A \cap B]$ $=$ $P[A' \cap B]$

$.5$ $-$ $.2$ $=$ $.3$

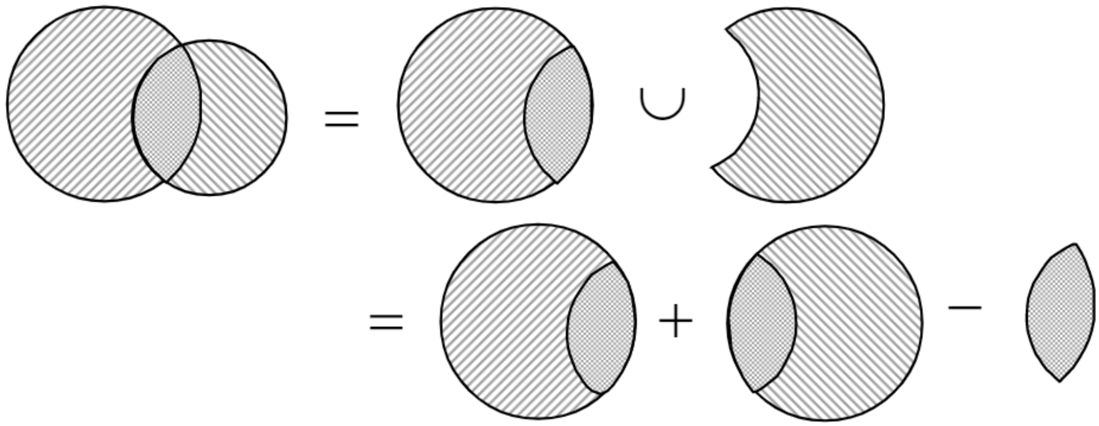
The following Venn diagrams shows how to find $P[A \cup B]$.



$P[A \cap B']$ $+$ $P[A \cap B]$ $+$ $P[A' \cap B]$ $=$ $P[A \cup B]$

$.4$ $+$ $.2$ $+$ $.3$ $=$ $.9$

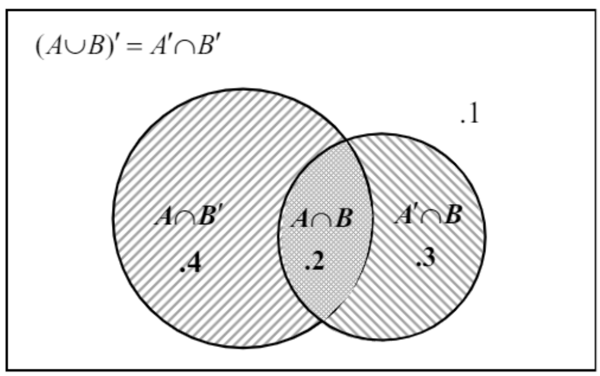
The relationship $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ is explained in the following Venn



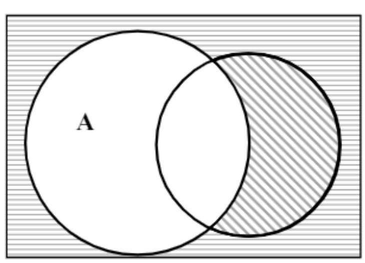
$$P[A \cup B] \quad = \quad P[A] \quad + \quad P[B] \quad - \quad P[A \cap B]$$

$$= .6 + .5 - .2 = .9$$

The components of the events and their probabilities are summarized in the following diagram.



We can represent a variety of events in Venn diagram form and find their probabilities from the component events described in the previous diagram. For instance, the complement of A is the combined shaded region in the following Venn diagram, and the probability is $P[A'] = 0.3 + 0.1 = 0.4$. We can get this probability also from $P[A'] = 1 - P[A] = 1 - 0.6 = 0.4$.



Another efficient way of summarizing the probability information for events A and B is in the form of a table.

	$P[A] = .6$ (given)	$P[A']$
$P[B] = 0.5$ (given)	$P[A \cap B] = 0.2$ (given)	$P[A' \cap B]$
$P[B']$	$P[A \cap B']$	$P[A' \cap B']$

Complementary event probabilities can be found from $P[A'] = 1 - P[A] = 0.4$ and $P[B'] = 1 - P[B] = 0.5$. Also, across each row or along each column, the "intersection probabilities" add up to the single event probability at the end or top:

$$
\begin{aligned}
&P[B] = P[A \cap B] + P[A' \cap B] \rightarrow 0.5 = 0.2 + P[A' \cap B] \rightarrow P[A' \cap B] = 0.3,\\
&P[A] = P[A \cap B] + P[A \cap B'] \rightarrow 0.6 = 0.2 + P[A \cap B'] \rightarrow P[A \cap B'] = 0.4, \text{ and}\\
&P[A'] = P[A' \cap B] + P[A' \cap B'] \rightarrow 0.4 = 0.3 + P[A' \cap B'] \rightarrow P[A' \cap B'] = 0.1 \text{ or}\\
&P[B'] = P[A \cap B'] + P[A' \cap B'] \rightarrow 0.5 = 0.4 + P[A' \cap B'] \rightarrow P[A' \cap B'] = 0.1.
\end{aligned}
$$

These calculations can be summarized in the next table.

		$P[A] = 0.6$ (given)	$\Rightarrow$	$P[A'] = 1 - P[A] = 0.4$
		$\Uparrow$		$\Uparrow$
$P[B] = 0.5$ (given)	$\Leftarrow$	$P[A \cap B] = 0.2$ (given)	$+$	$P[A' \cap B] = \mathbf{0.3}$
		$+$		$+$
$P[B'] = \mathbf{0.5}$	$\Leftarrow$	$P[A \cap B'] = \mathbf{0.4}$	$+$	$P[A' \cap B'] = \mathbf{0.1}$

$\square$

Example 1-4:

A survey is made to determine the number of households having electric appliances in a certain city. It is found that 75% have radios (R), 65% have electric irons (I), 55% have electric toasters (T), 50% have (IR), 40% have (RT), 30% have (IT), and 20% have all three. Find the probability that a household has at least one of these appliances.

Solution:

$$
\begin{aligned}
P[R \cup I \cup T] &= P[R] + P[I] + P[T]\\
&\quad - P[R \cap I] - P[R \cap T] - P[I \cap T] + P[R \cap I \cap T]\\
&= 0.75 + 0.65 + 0.55 - 0.5 - 0.4 - 0.3 + 0.2 = 0.95.
\end{aligned}
$$

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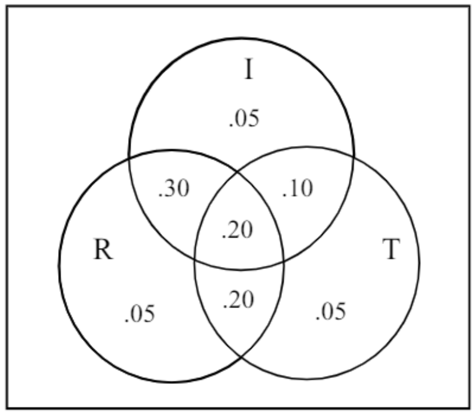
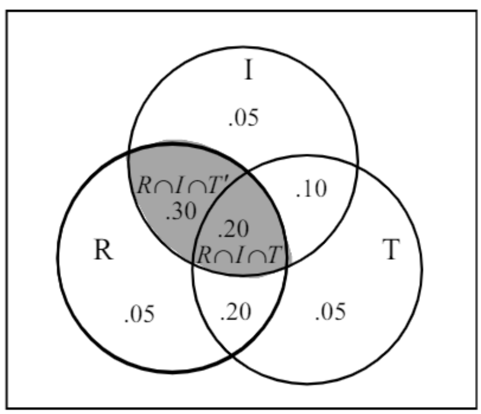
The following diagram deconstructs the three events.



The entries in this diagram were calculated from the "inside out". For instance, since $P(R \cap I) = 0.5$ (given), and since $P(R \cap I \cap T) = 0.2$ (also given), it follows that
$P(R \cap I \cap T') = 0.3$, since
$0.5 = P(R \cap I) = P(R \cap I \cap T) + P(R \cap I \cap T') = 0.2 + P(R \cap I \cap T')$
(this uses the rule $P(A) = P(A \cap B) + P(A \cap B')$, where $A = R \cap I$ and $B = T$).

This is illustrated in the following diagram.



The value ".05" that is inside the diagram for event R refers to $P(R \cap I' \cap T')$ (the proportion who have radios but neither irons nor toasters). This can be found in the following way.

First we find $P(R \cap I')$:
$0.75 = P(R) = P(R \cap I) + P(R \cap I') = 0.5 + P(R \cap I') \rightarrow P(R \cap I') = 0.25$.
$P(R \cap I')$ is the proportion with radios but not irons; this is the "0.05" inside R combined with the ".2" in the lower triangle inside $R \cap T$.

Then we find $P(R \cap I' \cap T)$:

$$0.4 = P(R \cap T) = P(R \cap I \cap T) + P(R \cap I' \cap T)$$
$$= 0.2 + P(R \cap I' \cap T) \rightarrow P(R \cap I' \cap T) = 0.2.$$

Finally we find $P(R \cap I' \cap T')$:

$$0.25 = P(R \cap I') = P(R \cap I' \cap T) + P(R \cap I' \cap T')$$
$$= 0.2 + P(R \cap I' \cap T') \rightarrow P(R \cap I' \cap T') = 0.05.$$

The other probabilities in the diagram can be found in a similar way. Notice that $P(R \cup I \cup T)$ is the sum of the probabilities of all the disjoint pieces inside the three events,
$P(R \cup I \cup T) = 0.05 + 0.05 + 0.05 + 0.1 + 0.2 + 0.3 + 0.2 = 0.95$.
We can also use the rule
$P(R \cup I \cup T) = P(R) + P(I) + P(T) - P(R \cap I) - P(R \cap T) - P(I \cap T) + P(R \cap I \cap T)$
$= 0.75 + 0.65 + 0.55 - 0.5 - 0.4 - 0.3 + 0.2 = 0.95$.
Either way, this implies that 5% of the households have none of the three appliances. □

It is possible that information is given in terms of numbers of units in each category rather than proportion of probability of each category that was given in Example 1-4.

Example 1-5:

In a survey of 120 students, the following data was obtained.
60 took English, 56 took Math, 42 took Chemistry, 34 took English and Math, 20 took Math and Chemistry, 16 took English and Chemistry, 6 took all three subjects.
Find the number of students who took

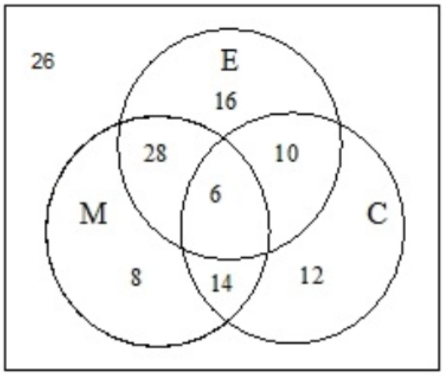
(i) none of the subjects,
(ii) Math, but not English or Chemistry,
(iii) English and Math but not Chemistry.

Solution:

Since $E \cap M$ has 34 and $E \cap M \cap C$ has 6, it follows that $E \cap M \cap C'$ has 28.
The other entries are calculated in the same way (very much like the previous example).

(i) The total number of students taking any of the three subjects is $E \cup M \cup C$, and is $16 + 28 + 6 + 10 + 8 + 14 + 12 = 94$. The remaining 26 (out of 120) students are not taking any of the three subjects (this could be described as the set $E' \cap M' \cap C'$).
(ii) $M \cap E' \cap C'$ has 8 students.
(iii) $E \cap M \cap C'$ has 28 students.

The following diagram illustrates how the numbers of students can be deconstructed.



□

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