11. A casino is developing a gambling game. A player of the game will choose three numbers (integers) from the numbers 1,2,3,4,5,6. The casino will then randomly choose 3 numbers from 1,2,3,4,5,6. The player pays \$1 to play the game. Each time the game is played, the casino pays out according to the following schedule

Number of player's numbers chosen by casino	Amount paid by casino
3	$\$x$
2	\$0
1	\$0
0	$\$x$

Calculate x so that the casino's expected gain is \$0.05 for each play of the game.

A) 8.00 B) 8.50 C) 9.00 D) 9.50 E) 10.00

12. Suppose that X and Y have a bivariate normal distribution with $E(X) = E(Y) = 0$ and $Var(X) = Var(Y) = 1$. You are given that the 95th percentile of $W = X + Y$ is 2.8492. Calculate the conditional expectation $E[Y|X = 1]$.

A) -1.0 B) -0.50 C) 0 D) 0.50 E) 1.0

13. A loss random variable X is continuously uniformly distributed on the interval $[0, 3b]$. When a loss occurs, an insurance policy pays

i) 0 if the loss is less than b,

ii) 50% of the loss if the loss is greater then b but less than $2b$, and

iii) 100% of the loss if the loss is greater than $2b$.

Y denotes the amount paid by the insurance policy if a loss occurs.

Calculate the 80th percentile of Y?

A) $1.2b$ B) $1.6b$ C) $2.0b$ D) $2.4b$ E) $2.8b$

14. If X is a non-negative random variable with a finite, non-zero mean, then the equilibrium distribution of X has density function $g(x) = \frac{1-F_X(x)}{E(X)}$, and is defined on the same region is X.

Suppose that X has a continuous uniform distribution on $[0, 1]$. Suppose that the equilibrium distribution of X is represented by the random variable Y, and the equilibrium distribution of Y is denoted by the random variable Z. Calculate $E(Z)$.

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) 1 E) 2

15. X has an exponential distribution with mean 1 and Y has an exponential distribution with mean 2. a_X and b_X are the 5th and 95th percentiles of X, respectively, and a_Y and b_Y are the 5th and 95th percentiles of Y, respectively. Calculate $\frac{b_Y - a_Y}{b_X - a_X}$.

A) $\frac{1}{2}$ B) $ln\,2$ C) 1 D) 2 E) e

16. A loss random variable has an exponential distribution with a mean of 1. An insurer issues an insurance policy on the loss which pays the loss amount in excess of a deductible of 1. The insurer discovers that there was an underwriting error on the policy and the actual distribution of the loss random variable is exponential with a mean of 2. The insurer wishes to change the deductible of 1 to a deductible of M for the insurance policy. M is determined so that the expected insurance payout based on the original assumption of a mean of 1 is the same is the expected insurance payout based on the revised loss distribution with a mean of 2. Calculate M.

A) ln 2 B) 2 ln 2 C) 2 D) 1 + 2 ln 2 E) 2 + 2 ln 2

17. You are given an ordinary deck of 52 cards, with four suits (clubs, diamonds, hearts and spades) and cards from ace to king in each suit. You choose two cards at random. Calculate the number of two-card choices that satisfy the following criteria:

- both cards are red (diamond or heart), and
- at least one card is a face card (jack, queen or king)

A) 120 B) 135 C) 150 D) 180 E) 240

18. X has an exponential distribution with a mean of 1. Find the conditional expectation $E[X|1 < X \leq 2]$.

A) Less than 1.2 B) At least 1.2 but less than 1.4 C) At least 1.4 but less than 1.6
D) At least 1.6 but less than 1.8 E) At least 1.8

19. The waiting room of a medical clinic can hold a maximum of four people. If there are four in the clinic, an arriving patient will be turned away. An analysis of the number of people N in the waiting room at any given time results in the following probabilities:

$P(N=0)=0.1$, $P(N=1)=0.2$, $P(N=2)=0.3$, $P(N=3)=0.3$, $P(N=4)=0.1$.

The clinic has one physician and the amount of time spent with a patient has an exponential distribution with a mean of $\frac{1}{4}$ hour. Patients are seen in their order of arrival. Calculate

E[Time that a newly arriving patient will be finished seeing the physician given that there are at most 3 patients in the waiting room when the patient arrives].

A) Less than 30 minutes B) At least 30 minutes but less than 35 minutes
C) At least 35 minutes but less than 40 minutes D) At least 40 minutes but less than 45 minutes
E) At least 45 minutes

20. You are given the following cumulative distribution function: $F_X(x) = \begin{cases} 0 & x \le 1 \\ .5x - .5 & 1 < x < 2 \\ .75 & 2 \le x \le 3 \\ .25x & 3 < x \le 4 \\ 1 & x > 4 \end{cases}$

Calculate $E[X]$.

A) Less than 2 B) At least 2 but less than 2.1 C) At least 2.1 but less than 2.2

D) At least 2.2 but less than 2.3 E) At least 2.3

21. Smith is one of eight Executive Vice Presidents (EVP) of ABC Co. and Jones is one of fifteen Non-Executive Vice Presidents (VP) of ABC Co. A committee of size six is being selected and the committee must include at least 2 EVPs and at least 2 VPs. Of all the committees that can be chosen, calculate the number that include both Smith and Jones.

A) 4,949 B) 4,994 C) 5,449 D) 5,994 E) 6,449

22. A study of the cost of heating a house with natural gas versus the cost using electric heat for a typical house in Winnipeg for the month of January shows the following:

Natural Gas: average cost \$350 , standard deviation of 40 ,

Electric: average cost of \$375 , standard deviation of 30.

We make the following assumption. Two typical houses in Winnipeg are randomly chosen, one of which uses natural gas for heating and the other uses electric heat. The costs of heating the two homes are independent and each house cost is normally distributed. Calculate the probability that cost of electric heating is at least \$50 greater than the cost of gas heating for the month of January,

A) Less than 0.05 B) At least 0.05 but less than 0.10 C) At least 0.10 but less than 0.15

D) At least 0.15 but less than 0.20 E) At least 0.20

23. X and Y are random variables which have a joint distribution with joint cdf $F(x,y) = y(x^2 + xy - y^2)$ for $0 \le y \le x \le 1$.

Calculate the covariance between X and Y.

A) Less then -0.1 B) At least -0.1 but less than 0 C) At least 0 but less than 0.1

D) At least 0.1 but less than 0.2 E) At least 0.2

24. A motorcycle distributor carries motorcycles by manufactures identified as A, B, C and D. The distributor has analyzed past sales and determined that the proportion of sales by manufacturer is as follows: A: 10% B: 15% C: 35% D: 40%
The distributor is considering no longer carrying motorcycles of manufacturer A and manufacturer B. The distributor will monitor the next ten sales. The distributor will drop a manufacturer (A or B or both) if there are fewer than two sales for that manufacturer. Assuming that future sales are independent of one another and follow the historical proportions, calculate the probability that no manufacturer is dropped as a result of the next ten sales.
A) Less than 0.2 B) At least 0.2 but less than 0.4 C) At least 0.4 but less than 0.6
D) At least 0.6 but less than 0.8 E) At least 0.8

25. X has an exponential distribution with mean $k > 0$. You are given that $E[X|X > k] = 2$.
Calculate $E[X|X \leq k]$.
A) Less than 0.25 B) At least 0.25 but less than 0.50
C) At least 0.5 but less than 0.75 D) At least 0.75 but less than 1 E) Greater than 1

26. The time that it takes a technician to repair a certain malfunctioning mechanical device is uniformly distributed between 3 and 6 hours. Suppose that two devices malfunction simultaneously, and repair work on them begin simultaneously by two independent technicians. Find the expected time until the first device will be repaired.
A) 3.25 B) 3.5 C) 3.75 D) 4.0 E) 4.25

27. X is a three point discrete random variable with $P(X = 1) = p$, $P(X = 2) = q$, $P(X = 3) = r$.
You are given than $P_X'(0) = 0.25$, where $P_X(t)$ is the probability generating function of X, and $M_X'(0) = 2.125$, where $M_X(t)$ is the moment generating function of X.
Calculate r.
A) 0.075 B) 0.150 C) 0.225 D) 0.300 E) 0.375

28. The amount of time T it takes for a certain bacteria culture to double in size is a random variable with the following pdf: $f_T(t) = e^{1-t}$, $t > 1$, where t is measured in hours. Once the bacteria culture has doubled in size, the additional amount of time it takes, say U, for it to double again in size (four times the original size) has the same distribution as T and is independent of T. Calculate the probability that the bacteria culture grows to four times its original size within 4 hours.
A) $1 - e^{-2}$ B) $1 - \frac{3}{2}e^{-2}$ C) $1 - 2e^{-2}$ D) $1 - \frac{5}{2}e^{-2}$ E) $1 - 3e^{-2}$

29. A homeowner purchases an insurance policy for the risk of damage due to tornados for one tornado season.

The number of tornados X that occur in the season has the following distribution, $P(X=k) = .9 \times (.1)^k$, $k = 0, 1, 2, ...$

The amount of damage Y caused by a tornado has the following distribution

$P(Y=0) = .95$, $P(Y = 500,000) = .03$, $P(Y = 1,000,000) = .02$.

The insurance will pay 250,000 for damage that occurs as a result of the first tornado. For every tornado after the first, the insurance will pay the full amount of damage.

Calculate the expected amount that the insurer will pay on the policy for one tornado season.

A) Less than 1,000 B) At least 1,000 but less than 1,200

C) At least 1,200 but less than 1,400 D) At least 1,400 but less than 1,600

E) At least 1,600

30. X and Y are independent and both have a continuous uniform distribution on $[0,1]$.

$U = min\{X,Y\}$ and $W = max\{X,Y\}$.

Calculate the covariance between U and W.

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{1}{16}$ D) $\frac{1}{24}$ E) $\frac{1}{36}$

PRACTICE EXAM 11 - SOLUTIONS

1. $XY=1$ if either $X=1$ and $Y=1$ or $X=-1$ and $Y=-1$.
$P(X=1,Y=1)=P(X=1\,|\,Y=1)\times P(Y=1)=\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$, and
$P(X=-1,Y=-1)=P(X=-1\,|\,Y=-1)\times P(Y=-1)=\frac{1}{3}\times\frac{1}{6}=\frac{1}{18}$.
Then $P(XY=1)=P(X=1,Y=1)+P(X=-1,Y=-1)=\frac{1}{6}+\frac{1}{18}=\frac{2}{9}$.
Answer: D

2. If a loss X occurs, the insurer pays 0 if the loss is $X\leq 10{,}000$ and the insurer pays $X-10,000$ if the loss $X>10,000$. The pdf of the loss random variable given that a loss occurs is $f(x)=.00001$. Given that a loss occurs, the expected amount paid by the insurer is
$0.9\times\int_{10{,}000}^{100{,}000}(x-10,000)\times .00001\,dx=36,450$. Since there is a probability of .01 that a loss occurs, the overall expected payment by the insurer is $.01\times 36,450=364.50$. Answer: E

3. The cdf of Y_2 is $F_{Y_2}(y)=P(Y_2\leq y)=P[(\text{all three } X\text{s are } \leq y)\cup(\text{exactly two } X\text{s are } \leq y)]$
$=P(\text{all three } X\text{s are } \leq y)+P(\text{exactly two } X\text{s are } \leq y)$.
$P(\text{all three } X\text{s are } \leq y)=P[(X_1\leq y)\cap(X_2\leq y)\cap(X_3\leq y)]$.
Since the Xs are independent, this probability is $[F_X(y)]^3=y^3$.
$P(\text{exactly two } X\text{s are } \leq y)=P[(X_1\leq y)\cap(X_2\leq y)\cap(X_3>y)]$
$+P[(X_1\leq y)\cap(X_3\leq y)\cap(X_2>y)]+P[(X_2\leq y)\cap(X_3\leq y)\cap(X_2>y)]$
$=3\times[F_X(y)]^2\times[1-F_X(y)]=3y^2(1-y)$.
Then $F_{Y_2}(y)=y^3+3y^2(1-y)=3y^2-2y^3$, and the pdf of Y_2 is
$f_{Y_2}(y)=F'_{Y_2}(y)=6y-6y^2$ for y in the interval $[0,1]$.
Then $E[Y_2]=\int_0^1 y\times f_{Y_2}(y)\,dy=\int_0^1 6(y^2-y^3)\,dy=\frac{1}{2}$, and
$E[Y_2^2]=\int_0^1 y^2\times f_{Y_2}(y)\,dy=\int_0^1 6(y^3-y^4)\,dy=\frac{3}{10}$.
$Var[Y_2]=E[Y_2^2]-(E[Y_2])^2=\frac{3}{10}-\left(\frac{1}{2}\right)^2=\frac{1}{20}$. Answer: A

4. $P(S<4)$
$=P(S<4|N=0)\times P(N=0)+P(S<4|N=1)\times P(N=1)$
$+P(S<4|N=2)\times P(N=2)$.
If $N=0$ then $S=0$, so that $P(S<4|N=0)=1$.

If $N=1$ then $S=X$, so that $P(S<4|N=1)=1$.
If $N=2$ then $S=X_1+X_2$ so that
$P(S<4|N=2)=P[(X_1,X_2)=(1,1)\text{ or }(1,2)\text{ or }(2,1)]$
$=0.8\times 0.8+0.8\times 0.15+0.15\times 0.8=0.88$.
Then $P(S<4)=1\times 0.8+0.95\times .15+0.88\times .05=.9865$. Answer: C

5. Suppose that the first 6 occurs on the k-th toss. There are $k-1$ tosses that are 1,2,3,4 or 5 followed by that first 6. For each of the tosses before the first 6, each of the outcomes 1,2,3,4,5 are equally likely, each with probability 0.2, since the die is fair (and each of those tosses is not a 6). For each of the tosses before the first 6, the average, or expected, toss is

$1 \times .2 + 2 \times .2 + 3 \times .2 + 4 \times .2 + 5 \times .2 + 6 \times .2 = 3.$

Therefore, if the first 6 occurs on toss k, there are $k-1$ previous tosses, each of which is, on average 3, so the sum of all of the tosses, including the k-th one which was the first 6 is

$3(k-1)+6 = 3k+3$. And thus, $X = 3K+3$, where K is the toss number of the first 6.

The probability function for K is

$P(K=1) = \frac{1}{6}$, $P(K=2) = \frac{5}{6} \times \frac{1}{6}, \ldots, P(K=k) = \left(\frac{5}{6}\right)^{k-1} \times \frac{1}{6}$.

This is the "alternate" form of the geometric distribution reviewed in Section 6 of this book.

Using the summation relationship given, we have $E(X) = E(3K-3) = 3 \times E(K) - 3$.

$E(K) = \sum_{k=1}^{\infty}\left[k \times \left(\frac{5}{6}\right)^{k-1} \times \frac{1}{6}\right] = \frac{1}{5} \times \sum_{k=1}^{\infty}\left[k \times \left(\frac{5}{6}\right)^{k}\right] = \frac{1}{5} \times \frac{5/6}{\left(1-\frac{5}{6}\right)^2} = 6.$

Then, $E(X) = 3 \times 6 + 3 = 21$.

A way to approach this problem with general reasoning is to note that on average, the first 6 will occur on the 6th toss, since there is a $\frac{1}{6}$ chance of tossing a 6 on any given (independent) toss. Then, with an average toss of 3 (a non-6) on the previous (average) 5 tosses, the total of those 6 tosses would be $5 \times 3 + 6 = 21$.

Answer: C

6. The probability function for Y is

$P(Y=\frac{1}{2}) = P(X_1=1, X_2=2) = \frac{1}{4}$

$P(Y=1) = P(X_1=1, X_2=1 \text{ or } X_1=2, X_2=2) = \frac{1}{2}$

$P(Y=2) = P(X_1=2, X_2=1) = \frac{1}{4}$

$P_Y(t) = E[t^Y] = \frac{1}{4}t^{1/2} + \frac{1}{2}t + \frac{1}{4}t^2$, and $P_Y(0) = 0 = P(Y=0)$.

Another way to see that the answer is B is that $P_Y(0) = P(Y=0)$ for any discrete random variable Y.

Since Y cannot be 0 in this case, we have $P_Y(0) = P(Y=0) = 0$. Answer: B

7. The joint density function is $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = 2x+2y$ for $0 \le y \le x \le 1$.

For a joint distribution, the cdf of the marginal distribution of X is $F_X(x) = \lim_{y \to \infty} F(x,y)$.

For this joint distribution, the region of density has $0 \le y \le x \le 1$, so in taking that limit, y cannot be larger than x, and then $F_X(x) = F(x,x) = x^3$ for $0 \le x \le 1$.

The pdf of the marginal distribution of X is $f_X(x) = F_X'(x) = 3x^2$, and the conditional density of Y given $X=x$ is $f_{Y|X}(y|X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2x+2y}{3x^2}$ for $0 \le y \le x$.

The conditional expectation of Y given $X=x$ is $\int_0^x y \times \frac{2x+2y}{3x^2}\,dy = \frac{5x}{9}$.

If $X=0.5$. then the conditional expectation of Y given $X=0.5$ is $\frac{5}{18}$. Answer: E

8. $P(A|B) = \frac{P(A\cap B)}{P(B)} = 0.5$, $P(B|A) = \frac{P(A\cap B)}{P(A)} = 0.4 \rightarrow \frac{P(B)}{P(A)} = \frac{0.4}{0.5} = 0.8$.
$P(A\cup B) = P(A) + P(B) - P(A\cap B)$
$\rightarrow \frac{P(A\cup B)}{P(A)} = 1 + \frac{P(B)}{P(A)} - P(B|A) = 1 + 0.8 - 0.4 = 1.4$.
From $P(A\cup B) = 0.7$, we get $P(A) = \frac{P(A\cup B)}{1.4} = 0.5$.
Then $P(B) = 0.8 \times P(A) = 0.4$,
and $P(A\cap B) = P(A) + P(B) - P(A\cup B) = 0.5 + 0.4 - 0.7 = 0.2$. Answer: D

9. $A =$ male, $B =$ less than two hours per day on social media, $C =$ overweight
$n(A) = 520$, $n(B) = 420$, $n(C) = 480$, $n(A\cap B) = 100$, $n(A\cap C) = 300$,
$n(B\cap C) = 100$, $n(A\cap B\cap C) = 80$.
We wish to find $n(A'\cap B\cap C')$.
$n(A\cap B) = n(A\cap B\cap C) + n(A\cap B\cap C')$
$\rightarrow 100 = 80 + n(A\cap B\cap C') \rightarrow n(A\cap B\cap C') = 20$,
and $n(B\cap C') = n(B) - n(B\cap C) = 420 - 100 = 320$.
Finally, $n(A\cap B\cap C') + n(A'\cap B\cap C') = n(B\cap C')$
$n(A'\cap B\cap C') = 320 - 20 = 300$. Answer: B

10. $Cov(X+Y, X-Y) = Cov(X,X) + Cov,(X,Y) - Cov(X,Y) - Cov(Y,Y)$
$= Cov(X,X) - Cov(Y,Y) = Var(X) - Var(Y)$.
Since the region of joint density has $x < y$, the marginal distribution of X has pdf
$f_X(x) = \int_x^\infty 2e^{-(x+y)}\,dy = 2e^{-2x}$, $x > 0$. From this pdf we see that X has an exponential distribution with mean $\frac{1}{2}$ and variance $Var(X) = \frac{1}{4}$. The marginal distribution of Y has pdf
$f_Y(y) = \int_0^y 2e^{-(x+y)}\,dx = 2e^{-y}(1-e^{-y})$, $y > 0$.
We use the identity $\int_0^\infty t^k e^{-at}\,dt = \frac{k!}{a^{k+1}}$ (where k is an integer and $a > 0$) to get
$E(Y) = 2\int_0^\infty (ye^{-y} - ye^{-2y}) = 2\left(1 - \frac{1}{2^2}\right) = \frac{3}{2}$, and
$E(Y^2) = 2\int_0^\infty (y^2e^{-y} - y^2e^{-2y}) = 2\left(2 - \frac{2}{2^3}\right) = \frac{7}{2}$, so that
$Var(Y) = \frac{7}{2} - \left(\frac{3}{2}\right)^2 = \frac{5}{4}$. Then $Var(X) - Var(Y) = \frac{1}{4} - \frac{5}{4} = -1$. Answer: B

11. There are $\binom{6}{3} = 20$ ways that the casino can choose 3 numbers out of the set 1,2,3,4,5,6.
Of those 20 combinations,
$\binom{3}{3} \times \binom{3}{0} = 1$ would be the same all three of the numbers chosen by the player,
$\binom{3}{2} \times \binom{3}{1} = 9$ would match exactly two of the numbers chosen by the player,
$\binom{3}{1} \times \binom{3}{2} = 9$ would match exactly one of the numbers chosen by the player, and
$\binom{3}{0} \times \binom{3}{3} = 1$ would match exactly one of the numbers chosen by the player,

The probabilities of matching numbers is:

How many numbers matched	Probability
3	$\frac{1}{20} = 0.05$
2	$\frac{9}{20} = 0.45$
1	$\frac{9}{20} = 0.45$
0	$\frac{1}{20} = 0.05$

On one play of the game, the expected payout by the casino is $0.05x + 0.05x = 0.1x$ and the casino's expected gain is $1 - 0.1x$. In order for this to be 0.05 we must have $1 - 0.1x = 0.05$, so that $x = 9.50$. Answer: D

12. The mean and variance of W are $E(W) = E(W) + E(Y) = 0$ and $Var(W) = Var(X) + Var(Y) + 2Cov(X,Y) = 2 + 2Cov(X,Y)$.
The 95th percentile of W is c, where
$$P[W \leq c] = P\left[\frac{W}{\sqrt{Var(W)}} \leq \frac{c}{\sqrt{Var(W)}}\right]$$
$$= \Phi\left(\frac{c}{\sqrt{Var(W)}}\right) = \Phi\left(\frac{c}{\sqrt{2+2Cov(X,Y)}}\right) = 0.95.$$
We are given $c = 2.8942$, so that $\frac{2.8942}{\sqrt{2+2Cov(X,Y)}} = 1.645$, which gives us $Cov(X,Y) = 0.50$. For a bivariate normal distribution
$E(Y|X = x) = E(X) + \frac{Cov(X,Y)}{Var(X)} \times (x - E(X))$, so that
$E(Y|x = 1) = 0 + \frac{0.50}{1} \times (1 - 0) = 0.50$. Answer: D

13. If the loss amount is X, the amount paid by the insurance policy is $Y = \begin{cases} 0 & X < b \\ .5X & b \leq X < 2b \\ X & X \geq 2b \end{cases}$
Since X has a uniform distribution on $[0, 3b]$ we see that $P(Y < b) = P(.5X < b) = P(X < 2b) = \frac{2}{3} < .75$. Therefore, the 80th percentile of Y occurs where $X > 2b$. Since $Y = X$ for $X \geq 2b$, it follows that the 80th percentile of Y is equal to the 80th percentile of X, which is $.8 \times 3b = 2.4b$. Answer: D

14. The cdf of X is $F_X(x) = x$ for $0 \leq x \leq 1$, and $F_X(x) = 1$ for $x \geq 1$. The mean of X is $E(X) = \frac{1}{2}$.
The pdf of Y is $f_Y(y) = \frac{1-F_X(y)}{E(X)} = \frac{1-y}{1/2} = 2(1-y)$ for $0 \leq y \leq 1$ (and 0 otherwise).
The cdf of Y is $F_Y(y) = \int_0^y f_Y(t)\,dt = 2y - y^2$ for $0 \leq y \leq 1$, and the mean of Y is $E(Y) = \int_0^1 y \times 2(1-y)\,dy = \frac{1}{3}$.
The pdf of Z is $f_Z(z) = \frac{1-F_Y(z)}{E(Y)} = \frac{1-(2z-z^2)}{1/3} = 3 - 6z + 3z^2$ for $0 \leq z \leq 1$.
The mean of Z is $\int_0^1 z \times (3 - 6z + 3z^2)\,dz = \frac{1}{4}$. Answer: A

15. $F_X(x) = 1 - e^{-x}$, so that $e^{-a_X} = 0.95$ and $e^{-b_X} = 0.05$ and $a_X = -\ln 0.95$, $b_X = -\ln 0.05$.
$F_Y(y) = 1 - e^{-y/2}$, so that $e^{-a_Y/2} = 0.95$ and $e^{-b_Y/2} = 0.05$
and $a_Y = -2 \times \ln 0.95 = 2a_X$, $b_Y = -2\ln 0.05 = 2b_X$.
Then, $\frac{b_Y - a_Y}{b_X - a_X} = 2$. Answer: D

16. The expected payout based on the original assumption is $\int_1^\infty (x-1)\, e^{-x}\, dx = e^{-1}$.
The expected payout on the revised assumption is $\int_M^\infty (x-M)\, \frac{1}{2}\, e^{-x/2}\, dx = 2e^{-M/2}$.
Then setting $2e^{-M/2} = e^{-1}$ we get $M = 2 + 2\ln 2$. Answer: E

17. There are $\binom{26}{2} = \frac{26!}{24! \times 2!} = 325$ two-card choices that are both red (two cards from the 26 read cards).
There are $\binom{20}{2} = \frac{20!}{19! \times 2!} = 190$ two-card choices the are both red and both are not face cards. There are $325 - 190 = 135$ two-card choices that are both red and of which at least one is a face card.
Answer: B

18. The probability that X is between 1 and 2 is $P[1 < X \le 2] = e^{-1} - e^{-2}$.
The conditional distribution of X given that $1 < X \le 2$ has pdf
$f(x|1 < X \le 2) = \frac{f(x)}{P(1<X\le 2)} = \frac{e^{-x}}{e^{-1}-e^{-2}}$ for $1 < x \le 2$.
The conditional expectation is
$\frac{1}{e^{-1}-e^{-2}} \times \int_1^2 x\, e^{-x}\, dx = \frac{1}{e^{-1}-e^{-2}} \times \left[-xe^{-x} - e^{-x} \Big|_{x=1}^{x=2} \right] = \frac{2e^{-1}-3e^{-2}}{e^{-1}-e^{-2}} = 1.418$.
Answer: C

19. Suppose that T is the time spent by the newly arriving patient and M is the conditional distribution of the number of patients in the waiting room when the new patient arrives given that $N \le 3$. Then
$E[T|M=k] = \frac{1}{4} \times (k+1)$ (time spent with physician by the k patients in the waiting room plus time spent by newly arrived patient, in hours) for $k = 0, 1, 2, 3$.
The distribution of M is the conditional distribution of N given $N \le 3$.
$P(M=0) = P(N=0|N \le 3) = \frac{P(N=0)}{P(N\le 3)} = \frac{.1}{.9} = \frac{1}{9}$. In a similar way,
$P(M=1) = \frac{2}{9}$, $P(M=2) = \frac{1}{3}$, and $P(M=3) = \frac{1}{3}$.
Then, $E[T] = E[T|M=0] \times P(M=0) + E[T|M=1] \times P(M=1)$
$+ E[T|M=2] \times P(M=2) + E[T|M=3] \times P(M=3)$
$= \frac{1}{4} \times \frac{1}{9} + \frac{2}{4} \times \frac{2}{9} + \frac{3}{4} \times \frac{3}{9} + \frac{4}{4} \times \frac{3}{9} = \frac{26}{36}$ hours, or 43.3 minutes.
Answer: D

20. X is continuous on the interval $(1,2)$ with pdf $f_X(x) = F_X'(x) = 0.5$ and on interval $(3,4)$ with pdf $f_X(x) = F_X'(x) = 0.25$.
Since $F_X(2) = 0.75$ and $F_X(x) \to 0.5 \times 2 - 0.5 = 0.5$ as $x \to 2$, it follows that X has a discrete point of probability at $X = 2$ with $P(X = 2) = 0.75 - 0.5 = 0.25$.
Then $E[X] = \int_1^2 x \times 0.5\,dx + 2 \times 0.25 + \int_3^4 x \times 0.25\,dx = 2.125$. Answer: C

21. The committee can be made up in the following ways:
Smith + 1 EVP and Jones plus 3 VPs : number of committees is $7 \times \binom{14}{3} = 2,548$;
Smith + 2 EVPs and Jones plus 2 VPs : number of committees is $\binom{7}{2} \times \binom{14}{2} = 1,911$;
Smith + 3 EVPs and Jones plus 1 VP : number of committees is $\binom{7}{3} \times 14 = 490$.
Total number of committees is 4,949. Answer: A

22. $X =$ gas heat cost , $Y =$ electric heat cost. Since X and Y are independent, $W = Y - X$ is normal, with mean $375 - 350 = 10$, and the variance of W is $Var(W) = Var(Y) + Var(X) = 40^2 + 30^2 = 2500$, so the standard deviation of W is 50.
Then $P(W > 50) = P\left(\frac{W-10}{50} > \frac{50-10}{50}\right) = P(Z > 0.8) = 1 - \Phi(0.8) = 1 - .7881 = .2119$.
Answer: E

23. The joint density function is $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = 2x + 2y$ for $0 \le y \le x \le 1$.
For a joint distribution, the cdf of the marginal distribution of X is $F_X(x) = \lim_{y\to\infty} F(x,y)$.
For this joint distribution, the region of density has $0 \le y \le x \le 1$, so in taking that limit, y cannot be larger than x, and then $F_X(x) = F(x,x) = x^3$ for $0 \le x \le 1$.
The pdf of the marginal distribution of X is $f_X(x) = F_X'(x) = 3x^2$.
The cdf of the marginal distribution of Y is
$F_Y(y) = \lim_{x\to\infty} F(x,y) = \, = \lim_{x\to 1} F(x,y) = y + y^2 - y^3$ for $0 \le y \le 1$.
The pdf of the marginal distribution of Y is $f_Y(y) = 1 + 2y - 3y^2$ for $0 \le y \le 1$.
The covariance between X and Y is $Cov(X,Y) = E(X \times Y) - E(X) \times E(Y)$.
$E(X \times Y) = \int_0^1 \int_0^x xy\, f_{X,Y}(x,y)\,dy\,dx = \, = \int_0^1 \int_0^x xy(2x+2y)\,dy\,dx = \frac{1}{3}$.
$E(X) = \int_0^1 x\, f_X(x)\,dx = \int_0^1 x \times (3x^2)\,dx = \frac{3}{4}$.
$E(Y) = \int_0^1 y\, f_Y(y)\,dy = \int_0^1 y \times (1 + 2y - 3y^2)\,dx = \frac{23}{60}$.
$Cov(X,Y) = \frac{1}{3} - \left(\frac{3}{4} \times \frac{23}{60}\right) = \frac{11}{240} = .0458$. Answer: C

24. Let X_A be the number of sales for manufacturer A, and X_B for B, and X_{CD} for manufacturers C and D combined. X_A, X_B and X_{CD} have a multinominal distribution with

$n = 10$ and $p_A = 0.10$, $p_B = 0.15$, $p_{CD} = 0.75$

We wish to find the probability $P[(X_A \geq 2) \cap (X_B \geq 2)] = 1 - P[(X_A \leq 1) \cup (X_B \leq 1)]$.

The sales numbers that result in the event $(X_A \leq 1) \cup (X_B \leq 1)$ are as follows:

	Sales			
X_A	0	1	0	1
X_B	0	0	1	1
X_{CD}	10	9	9	8

According to the multinomial probability function,

$P[(X_A = x_A) \cap (X_B = x_B) \cap (X_{CD} = x_{CD})] = \frac{10!}{x_A! \times x_B! \times x_{CD}!} \times p_A^{x_A} \times p_B^{x_B} \times p_{CD}^{x_{CD}}$

The probabilities of the combinations above are

$P[(X_A = 0) \cap (X_B = 0) \cap (X_{CD} = 10)]$

$= \frac{10!}{0! \times 0 \times 10!} \times (0.1)^0 \times (0.15)^0 \times (0.75)^{10} = 0.0563.$

In a similar way, we get

$P[(X_A = 1) \cap (X_B = 0) \cap (X_{CD} = 9)] = 0.0751,$

$P[(X_A = 0) \cap (X_B = 1) \cap (X_{CD} = 9)] = 0.1126$, and

$P[(X_A = 1) \cap (X_B = 1) \cap (X_{CD} = 8)] = 0.1352.$

Then, $P[(X_A \leq 1) \cup (X_B \leq 1)] = 0.0563 + 0.0751 + 0.1126 + 0.1352 = 0.3792,$

and the probability that no manufacturer gets dropped is $1 - 0.3792 = 0.6208$. Answer: D

25. $f_X(x) = \frac{e^{-x/k}}{k}$ for $k > 0$. $P(X > k) = \int_k^\infty \frac{e^{-x/k}}{k}\,dx = e^{-1}$.

The density function of the conditional distribution of X given $X > k$ is

$f_X(x|X > k) = \frac{f_X(x)}{P(X>k)} = \frac{e^{-x/k}/k}{e^{-1}} = \frac{e^{-(x-k)/k}}{k}$ for $x > k$.

The conditional expectation $E[X|X > k]$ is $\int_k^\infty x \times \frac{e^{-(x-k)/k}}{k}\,dx$.

With change of variable $y = x - k$, this becomes

$\int_0^\infty (y + k)\ \frac{e^{-y/k}}{k}\,dy = k + k = 2k = 2$, so that $k = 1$.

The density of the conditional distribution of X given $X \leq 1$ is

$f_X(x|X \leq 1) = \frac{f_X(x)}{P(X\leq 1)} = \frac{e^{-x}}{1-e^{-1}}$ for $0 < x \leq 1$

The conditional expectation $E[X|X \leq 1]$ is

$\int_0^1 x \times \frac{e^{-x}}{1-e^{-1}}\,dx = \frac{1}{1-e^{-1}} \times \left[-xe^{-x} - e^{-x}\Big|_0^1 \right] = \frac{1-2e^{-1}}{1-e^{-1}} = .418.$ Answer: B

26. X_1 and X_2 are independent and each has pdf $f(x) = \frac{1}{3}$ for $3 < x < 6$.

If $Y = min\{X_1, X_2\}$, then

$P[Y > t] = P[(X_1 > t) \cap (X_2 > t)] = P[X_1 > t] \times P[X_2 > t] = \frac{6-t}{3} \times \frac{6-t}{3} = \frac{(6-t)^2}{9}$

for $3 < t < 6$. The pdf of Y is $-\frac{d}{dt} P[Y > t] = \frac{2(6-t)}{9}$ for $3 < t < 6$.

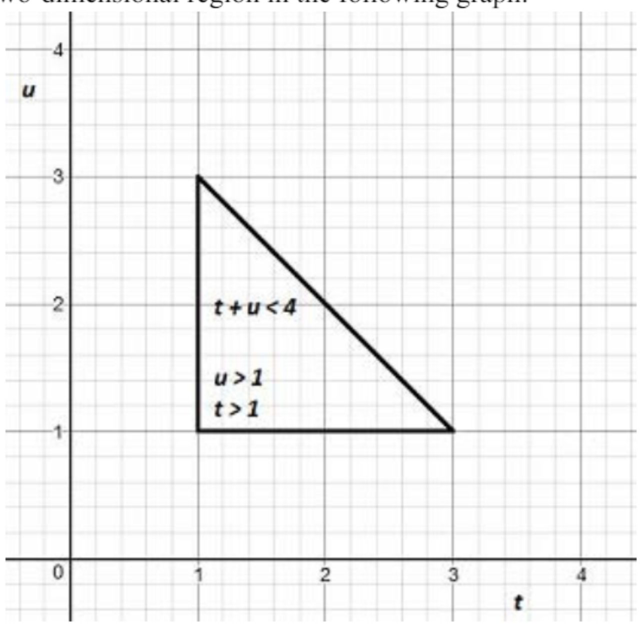
Then $E[Y] = \int_3^6 t \times \frac{2(6-t)}{9}\, dt = 4$. Answer: D

27. $p + q + r = 1$. $P_X(t) = pt + qt^2 + rt^3 \rightarrow P_X'(0) = p = 0.25$.

$M_X'(0) = E(X) = p + 2q + 3r = 2.125$.

Solving the equations results in $q = 0.375$ and $r = 0.375$. Answer: E

28. We wish to find $P(T + U \leq 2)$, where T and U are independent and $f_T(t) = e^{1-t}$, $t > 1$ and $f_U(u) = e^{1-u}$, $u > 1$. We can formulate this as the probability for the joint distribution of T and U in the two-dimensional region $1 < t + u \leq 4$. Since T and U are independent, the joint density is $f_{T,U}(t, u) = f_T(t) \times f_U(u) = e^{1-t} \times e^{1-u}$ and the probability is the double integral of the joint density over the two-dimensional region in the following graph:



The probability is

$\int_1^3 \int_1^{4-t} (e^{1-t} \times e^{1-u})\, du\, dt = \int_1^3 (e^{1-t} - e^{-2})\, dt = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$.

Answer: E

29. The expected amount paid by the insurer can be formulated as

$$\sum_{k=0}^{\infty} E[\text{payment}|k \text{ tornados}] \times P[k\,\text{tornados}]$$

$$= 0 \times P(X=0) + \sum_{k=1}^{\infty} E[\text{payment}|k \text{ tornados}] \times P[k\,\text{tornados}]$$

If there is one tornado, the expected payment is $0 \times 0.95 + 250,000 \times 0.05 = 12,500$, and the probability of one tornado is $0.9 \times 0.1 = 0.09$. If there are 2 or more tornados, the expected insurance payment is $12,500$ for the first tornado and $500,000 \times .03 + 1,000,000 \times 0.02 = 35,000$ for each tornado after the first.

If there are $k \geq 2$ tornados, then the expected payment is

$12,500 + (k-1) \times 35,000 = 35,000k - 22,500$.

The overall expected insurance payment is

$$12,500 \times 0.09 + \sum_{k=2}^{\infty} [35,000k - 22,500] \times 0.9 \times (0.1)^k.$$

We can write the summation in two parts: $\sum_{k=2}^{\infty} 35,000k \times 0.9 \times (0.1)^k$ and $-22,500 \times \sum_{k=2}^{\infty} 0.9 \times (0.1)^k$.

The second summation is

$$-22,500 \times P(X \geq 2) = -22,500 \times [1 - P(X=0,1)] = -22,500 \times [1 - 0.9 - 0.09]$$

$$= -450.$$

The first summation is $35,000 \times \sum_{k=2}^{\infty} k \times 0.9 \times (0.1)^k$

$$= 35,000 \times \left[\sum_{k=0}^{\infty} k \times 0.9 \times (0.1)^k - .09\right] = 35,000 \times (E[X] - 0.09).$$

X has a geometric distribution with $E[X] = \frac{0.1}{0.9} = \frac{1}{9}$, so the first summation is

$35,000 \times [\frac{1}{9} - 0.09] = 738.89$.

The total expected insurance payment is $12,500 \times 0.09 - 450 + 738.89 = 1{,}414$. Answer: D

30. The cdf of U is $F_U(u)$ and

$1 - F_U(u) = P[U > u] = P[(X > u) \cap (Y > u)] = P(X > u) \times P(Y > u) = (1-u)^2$.

Then $E[U] = \int_0^1 [1 - F_U(u)]\, du = \frac{1}{3}$.

The cdf of W is $F_W(w) = P[(X \leq w) \cap (Y \leq w)] = P(X \leq w) \times P(Y \leq w) = w^2$.

Then $E[W] = \int_0^1 [1 - F_W(w)]\, dw = \frac{2}{3}$.

Since one of X, Y is U and the other is W, it must be the case that $U \times W = X \times Y$. It follows that $E[U \times W] = E[X \times Y]$. Since X and Y are independent, we have

$E[X \times Y] = E[X] \times E[Y] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Then, $Cov(U, W) = E[U \times W] - E[U] \times E[W] = \frac{1}{4} - \frac{1}{3} \times \frac{2}{3} = \frac{1}{36}$. Answer: E

PRACTICE EXAM 12

1. The loss random variable X has a continuous uniform distribution on the interval $[0, 1000]$. Two insurance policies are being considered. Policy 1 pays for the loss amount in excess of a deductible amount of 250 with the additional provision that the maximum insurance payout is 500. Policy 2 has no deductible but has a maximum payout of C. Calculate the value of C for which both policies have the same expected payout when a loss occurs.

   A) Less than 250 B) At least 250 but less than 275 C) At least 275 but less than 300
   D) At least 300 but less than 325 E) At least 325

2. X denotes the number of patients that visit a dentist office in a day. The probability function of X is $P(X = 2) = 0.2$, $P(X = 3) = 0.5$, $P(X = 4) = 0.3$. 25% of the patients visiting the clinic are in need of cosmetic dentistry and the other 75% are in need of other dental services. Patients are independent of one another. Calculate the probability that the number of patients in a day are in need of cosmetic dentistry is at least 3.

   A) Less than 0.025 B) At least 0.025 but less than 0.05 C) At least 0.05 but less than 0.075
   D) At least 0.075 but less than 0.10 E) At least 0.10

3. An appliance manufacturer offers a warranty on one of its products. The price of the product is 100. If the product fails within one year the warranty will refund the full price. If the product fails in the second year, the warranty will refund one half the price. If the product fails in the third year, the warranty will refund $150 \times (1 - \frac{t}{3})$, where t represents the time until failure. The manufacturer assumes a distribution for the time from purchase until failure to have the following density, where t is measure in years: $f(t) = .08t$ for $0 < t \leq 5$, and 0 elsewhere. Calculate the expected cost of the warranty.

   A) Less than 10 B) At least 10 but less than 11 C) At least 11 but less than 12
   D) At least 12 but less than 13 E) At least 13

4. The number of hurricanes X that strike a particular island in one month has the following distribution $P(X = k) = 0.9 \times 0.1^k$, $k = 0, 1, 2, \ldots$. This is the case for each of the months of June, July and August. It is also assumed that the number of hurricanes in any given month is independent of the number in any other month. Calculate the probability of at least three hurricanes in total for the three month period of June, July and August.

   A) Less than 0.1 B) At least 0.1 but less than 0.125 C) At least 0.125 but less than 0.15
   D) At least 0.15 but less than 0.175 E) At least 0.175

5. X and Y are independent exponentially distributed random variables, each with a mean of 1. Calculate the median of the random variable $X + Y$.

A) Less than 1.0 B) At least 1.0 but less than 1.2 C) At least 1.2 but less than 1.4

D) At least 1.4 but less than 1.6 E) At least 1.6

6. Suppose that X is a non-negative random variable for which $E[\sqrt{X}]$, $E[X]$, $E[X^2]$ and $E[ln\,X]$ are all finite. Which of the following statements are always true?

I. $E[X^2] \geq (E[X])^2$ II. $\sqrt{E[X]} \geq E[\sqrt{X}]$ III. $E[ln\,X] \geq ln\,E[X]$

A) All but I B) All but II C) All but III D) All E) I only

7. A climatologist creates a probabilistic model for the time until the next major tidal wave in which X represents that time, where X has an exponential distribution with a mean of 1 year. Suppose that K represents the number of complete years until the next major tidal wave appears, with $P(K = k) = P(k < X \leq k + 1)$ for $k = 0, 1, 2, ...$ Calculate $E[K]$.

A) Less than 6 months B) At least 6 months but less than 12 months

C) At least 12 months but less than 18 months D) At least 18 months but less than 24 months

E) At least 24 months

8. The probability function of a binomial random variable is

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for integer $n \geq 1$ and $k = 0, 1, ..., n$

and the probability generating function of X is $P_X(t) = (1 - p + pt)^n = (.3 + .7t)^6$.

Calculate $E[X^3]$.

A) Less than 75 B) At least 75 but less than 80 C) At least 80 but less than 85

D) At least 85 but less than 90 E) At least 90

9. Suppose that X and Y are continuous random variables with joint density function

$$f(x,y) = \begin{cases} 3x \text{ for } 0<x<1 \text{ and } 1-x<y<1 \\ 0, \text{otherwise} \end{cases}.$$

Calculate $P[Y < X]$.

A) $\frac{1}{8}$ B) $\frac{1}{4}$ C) $\frac{3}{8}$ D) $\frac{1}{2}$ E) $\frac{5}{8}$

10. You are given the following table of joint probabilities, $P(X = x, Y = y)$

x	0	1	2
y			
-1	0.2	0.1	0
0	0.1	0.2	0.1
1	0	0.3	0

Calculate $Cov(X, Y)$.

A) -0.2 B) -0.1 C) 0 D) 0.1 E) 0.2

11. A chiropractic clinic did a survey of types of non-chiropractic treatment sought by its patients in the past year and found the following information:

(i) 29% had visited an acupuncture clinic

(ii) 52% had visited a physiotherapist

(iii) 26% had visited a massage therapist

(iv) 22% had visited both an acupuncture clinic and a physiotherapist

(v) 18% had visited a physiotherapist and a massage therapist

(vi) 12% had visited an acupuncture clinic and a massage therapist

(vii) 7% had visited acupuncture clinic and a physiotherapist and a massage therapist

Calculate the percentage of the group that has visited a massage therapist but not an acupuncture clinic nor a physiotherapist.

A) 0 B) 1% C) 2% D) 3% E) 4%

12. You are given the following information about the random variable X.
On the region $\{0 < x < 1\} \cup \{1 < x \leq 4\}$ the density function of X is $f(x) = cx$,
where c is a constant, and at the point $X = 1$, $P(X = 1) = 0.2$, and density and probability is 0 elsewhere.
Calculate $F_X(1.5)$ (F_X is the cumulative distribution function of X).

A) Less than 0.3 B) At least 0.3 but less than 0.31 C) At least 0.31 but less than 0.32
D) At least 0.32 but less than 0.33 E) At least 0.33

13. A health insurance company insures individuals over 20. The company's actuary has analyzed a large number of the company's policyholders for incidence of high blood pressure with the following result:

Age of Insured	Proportion with of High Blood Pressure	Portion of Policyholders
20 to 35	10%	10%
35 to 50	25%	20%
50 to 65	40%	30%
Over 65	50%	40%

A policyholder is selected at random and found to not have high blood pressure. Calculate the probability that the policyholder is over age 65.

A) Less than 0.45 B) At least 0.45 but less than 0.55 C) At least 0.55 but less than 0.65
D) At least 0.65 but less than 0.75 E) At least 0.75

14. X has a standard normal distribution.
The function g is defined as follows: $g(t) = P[t < X \leq t+1]$ for $-\infty < t < \infty$.
Calculate the maximum value of $g(t)$.

A) Less than 0.1 B) At least 0.1 but less than 0.2 C) At least 0.2 but less than 0.3
D) At least 0.3 but less than 0.4 E) At least 0.4

15. A mid-size city conducts a lottery in the following way. There are 10,000 lottery tickets available, with a cost of \$1 per ticket. Each ticket has a unique number, with the numbering being 0000 , 0001, ... 9999. The city will draw a number at random after all of the tickets have been sold. The prizes are as follows:

Match the number exactly - \$1000

Match exactly three digits in their correct places - \$100 (for instance if the winning number is 1234 the ticket 5234 would qualify for this prize but 2534 would not qualify since the "2" is not in the correct place)

Match exactly 2 digits in their correct places - \$10.

Calculate the city's profit from one round of the lottery.

A) Less than \$250 B) At least \$250 but less than \$500 C) At least \$500 but less than \$750
D) At least \$750 but less than \$1000 E) At least \$1000

16. An insurer determines that the size of a loss on a certain risk is the continuous random variable X with density function $f(x) = \begin{cases} cx \text{ for } 0 < x < 1000 \\ 0, \text{otherwise} \end{cases}$.

When a loss occurs, the insurer will reimburse the insurance policy holder the amount of the loss that is above a deductible amount of 200, up to a maximum reimbursement payment of 500. Calculate the insurer's expected reimbursement payment when a loss occurs.

A) Less than 300 B) At least 300 but less than 400 C) At least 400 but less than 500
D) at least 500 but less than 600 E) At least 600

17. In the course of trying to choose a discrete probability distribution as a model from some observed data, a statistician makes the following observation. If the distribution chosen, say X, is a discrete uniform distribution on the integers $0, 1, 2, \ldots, N$ (where $N > 0$), then the mean and variance of X are the same as the mean and variance of a Poisson distribution with mean λ. Calculate λ.

A) Less than .75 B) At least .75 but less than 1.25 C) At least 1.25 but less than 1.75
D) At least 1.75 but less than 2.25 E) At least 2.25

18. You are given the joint distribution of X and Y has joint pdf

$$f(x,y) = \begin{cases} 12xy \text{ for } 0 < x < 1, 0 < y < x^2 \\ 0, \text{otherwise} \end{cases}.$$

Calculate the conditional expectation $E\left[Y|X = \frac{1}{2}\right]$.

A) $\frac{1}{10}$ B) $\frac{1}{9}$ C) $\frac{1}{8}$ D) $\frac{1}{7}$ E) $\frac{1}{6}$

19. The annual loss X on an insurance policy is a random variable with density function

$$f(x) = \begin{cases} \frac{\alpha \times 100^\alpha}{x^{\alpha+1}} \text{ for } x \geq 100 \\ 0, \text{otherwise} \end{cases}.$$

You are given the 60th percentile of the distribution of X is 400. Calculate α.

A) 0.33 B) 0.50 C) 0.66 D) 1.0 E) 1.33

20. A machine on an assembly line fills successive jars with jelly beans. The jars are all the same size. When a sensor on the machine indicates that the jar is full, no more jelly beans are put into that jar. An analysis of the process shows that the number of jelly beans in a randomly filled jar is $1000 + X$, where $P(X = 0) = .4$ and $P(X = k) = .1$, where $k = \pm 1, \pm 2$ and $P(X = k) = 0.05$, where, $k = \pm 3, \pm 4$. A quality control analyst calculates A, the exact probability that the number of jelly beans in a randomly chosen jar is in the interval $[998, 1002]$. Another quality control analyst calculates B the same probability using the normal approximation to X using the integer correction. Calculate $|A - B|$.

A) 0 B) 0.01 C) 0.02 D) 0.03 E) 0.04

21. A rough model for income and expense for an operation for the coming year is that income X, is an exponential random variable with a mean of 2, and expense Y is an exponential random variable with a mean of 1. The rough model also assumes that X and Y are independent. Which of the following is the distribution function for $T = X - Y$?

A) $f_T(t) = \begin{cases} 1-\frac{e^{2t}}{3} & \text{for } t \le 0 \\ \frac{2e^{-t/2}}{3} & \text{for } t > 0 \end{cases}$ B) $f_T(t) = \begin{cases} 1-\frac{e^{t}}{3} & \text{for } t \le 0 \\ \frac{e^{-t/2}}{3} & \text{for } t > 0 \end{cases}$

C) $f_T(t) = \begin{cases} 1-\frac{e^{t}}{3} & \text{for } t \le 0 \\ \frac{2e^{-t/2}}{3} & \text{for } t > 0 \end{cases}$ D) $f_T(t) = \begin{cases} 1-\frac{e^{2t}}{3} & \text{for } t \le 0 \\ \frac{e^{-t}}{3} & \text{for } t > 0 \end{cases}$

E) $f_T(t) = \begin{cases} 1-\frac{e^{t}}{3} & \text{for } t \le 0 \\ \frac{e^{-t/2}}{2} & \text{for } t > 0 \end{cases}$

22. For three events A, B, and C, you are given:

$B \subset A$, $P(A \cap C) = 0.6$, $P(B \cap C) = 0.2$, $P(C) = 0.8$

Calculate $P(B|A \cap C)$.

A) $\frac{1}{6}$ B) $\frac{1}{5}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{1}{2}$

23. The distribution function for a discrete non-negative integer-valued random variable X is

$$F_X(k) = 1 - \sum_{n=k}^{\infty} \frac{e^{-1}}{n!}$$

Calculate the mean of X.

A) e^{-1} B) 1 C) 2 D) e E) e^2

24. A machine consists of two components, whose lifetimes have the joint density function

$$f(x, y) = \begin{cases} 6x \text{ for } 0 < x < 1, 0 < y < 1-x \\ 0, \text{otherwise} \end{cases}.$$

The machine operates until the failure of the first component. Calculate the expected time until the machine fails.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

25. A carnival game of chance is set up as follows. The game operator has three dice:

Die 1 - an ordinary die with numbers 1,2,3,4,5,6

Die 2 - a die with two faces that are 1, two faces that are 3 and two faces that are 5

Die 3 - a die with two faces that are 2, two faces that are 4 and two faces that are 6

A die is chosen at random by the game operator and it is tossed twice, with the two tosses being independent of one another. The game player is told the total of the two tosses and must choose which die was tossed. Suppose that the total is 8. Calculate the probability that it was Die 1 that was tossed.

A) $\frac{5}{22}$ B) $\frac{3}{11}$ C) $\frac{7}{22}$ D) $\frac{4}{11}$ E) $\frac{9}{22}$

26. The number of cars crossing an intersection during a one-minute period has a Poisson distribution with a mean of 4. Each car crossing is either a domestic made car or a foreign made car. The makes of successive cars are independent of one another with a probability of 0.75 that any given car crossing the intersection is domestic made. Calculate the probability that during a particular one-minute period, four cars cross the intersection and exactly three of them are domestic made.

A) Less than 0.075 B) At least 0.075 but less than 0.100 C) At least 0.100 but less than 0.125
D) At least 0.125 but less than 0.150 E) At least 0.150

27. Actuary Smith has a challenge for an actuarial student working for Smith. Smith gives the student a fair coin and two dice. One of the dice is an ordinary fair die with numbers 1 to 6. The other die has six sides that all have the number 1 on them. The student is to toss the coin. If the coin turns up "head" then the student tosses the ordinary die. If the coin toss turns of "tail", the student tosses the other die. The number that turns up is a random variable denoted X. The actuary asks the student how many of the following statements are true.

I. $P(X=1) > .5$ II. $E[X] < 3$ III. $Var[X] < 3$

A) 0 B) 1 C) 2 D) 3 E) None of A, B, C or D is correct

28. Each pack of the candy "Fizzy-O's" contains a large number of small candy pieces. The packager of the candy estimates that there are 100 pieces in an average pack. The packager does an analysis of the variation in the number of pieces in a pack and estimates that the probability of 90 or fewer pieces in a pack is 0.0436. Assuming that this probability is based on the normal approximation with integer correction applied to the number of pieces in a pack, calculate the standard deviation of the number of pieces in a pack assuming the mean is 100.

A) Less than 5.0 B) At least 5.0 but less than 5.2 C) At least 5.2 but less than 5.4
D) At least 5.4 but less than 5.6 E) At least 5.6