PRACTICE EXAM 1 - SOLUTIONS

1. $P[E' \cup F'] = P[E'] + P[F'] - P[E' \cap F']$.
But $E' \cap F' = (E \cup F)'$, so that $P[E' \cap F'] = P[(E \cup F)'] = 1 - P[E \cup F] = 1 - 1 = 0$,
so that $P[E' \cup F'] = P[E'] + P[F']$. Answer: C

2. We define the following events:

A - the new driver has had driver education

B - the new driver has had an accident in his first year.

We are to find $P[A|\overline{B}] = \frac{P[A\cap\overline{B}]}{P[\overline{B}]}$, and we are given $P[A] = .6$, $P[B|\overline{A}] = 0.08$,
and $P[B|A] = 0.05$. Using rules of probability, $P[\overline{B}|A] = 1 - P[B|A] = 0.95$,
and hence, $P[A \cap \overline{B}] = P[\overline{B}|A] \cdot P[A] = 0.95 \times 0.6$. Also, $P[\overline{A}] = 1 - P[A] = 0.4$.

But, $P[\overline{B}|\overline{A}] = 1 - P[B|\overline{A}] = 1 - .08 = 0.92$, and hence
$P[\overline{A} \cap \overline{B}] = P[\overline{B}|\overline{A}] \cdot P[\overline{A}] = 0.92 \times 0.4$.

Thus, $P[\overline{B}] = P[A \cap \overline{B}] + P[\overline{A} \cap \overline{B}] = 0.95 \times 0.6 + .92 \times 0.4$, and then
$P[A|\overline{B}] = \frac{P[A\cap\overline{B}]}{P[\overline{B}]} = \frac{0.95\times0.6}{0.95\times0.6+0.92.4}$. Answer: E

3. $f(x) = ae^{-x} + be^{-2x} \rightarrow \int_0^\infty f(x)\,dx = a + \frac{1}{2}b = 1$.
We use the following integral rule for integer $k \geq 0$ and $c > 0$
$\int_0^\infty x^k\, e^{-cx}\,dx = \frac{k!}{c^{k+1}}$, to get $E[X] = \int_0^\infty x\, f(x)\,dx = a + \frac{1}{4}b = 1$.

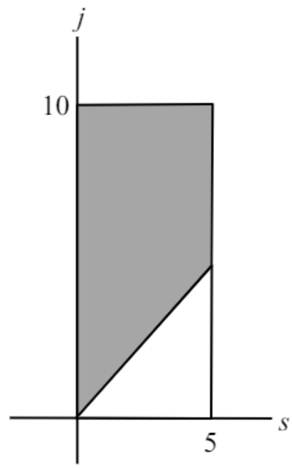
Solving the equations results in $a = 1$, $b = 0$. The probability is
$P[X < 1] = \int_0^1 e^{-x}\,dx = 1 - e^{-1} = 0.632$. Answer: B

4. The function $y = u(x) = ln\, x$ is strictly increasing (and thus, one-to-one) for all $x > 0$, with the inverse function being $x = e^y = v(y)$. Then
$f_Y(y) = f_X(v(y)) \times |v'(y)| = f_X(e^y) \times e^y = e^y \times e^{-(e^y)^2/2} \times e^y = e^{2y-\frac{1}{2}e^{2y}}$.

Alternatively, $F_Y(y) = P[Y \leq y] = P[ln\, X \leq y] = P[X \leq e^y]$, and
$F_X(x) = P[X \leq x] = \int_0^x te^{-t^2/2}dt = 1 - e^{-x^2/2}$
$\Rightarrow F_Y(y) = P[X \leq e^y] = \int_0^{e^y} te^{-t^2/2}dt = 1 - e^{-(e^y)^2/2}$
$\Rightarrow f_Y(y) = F_Y'(y) = e^{2y-\frac{1}{2}e^{2y}}$. Answer: A

5. The joint density of time until death is $f_{S,J}(s,j) = f_S(s) \times f_J(j) = \frac{1}{5} \times \frac{1}{10} = \frac{1}{50}$.
The rectangle below is the region of density for the joint distribution and the shaded region represents the event $S < J$ (Smith's time of death is less than that of Jones). The probability is $\int_0^5 \int_s^{10} \frac{1}{50}\, dj\, ds = \int_0^5 \frac{10-s}{50}\, ds = 0.75$.

Alternatively, since both S and J are uniformly distributed, and they are independent, it follows that the joint distribution is uniformly distributed on the rectangle and the probability of any region is the fraction of the region of the full rectangle on which the joint distribution is defined. The shaded region can be seen to be .75 of area of the rectangle, and therefore probability of the shaded region is .75.



Answer: E

6. Since $X \sim N(1,4)$, $Z = \frac{X-1}{2}$ has a standard normal distribution. The probability in question can be written as

$$\begin{aligned} P[X^2 - 2x \le 8] &= P[X^2 - 2X + 1 \le 9] = P[(X-1)^2 \le 9] \\ &= P[-3 \le X - 1 \le 3] \\ &= P[-1.5 \le \tfrac{X-1}{2} \le 1.5] = P[-1.5 \le Z \le 1.5] \\ &= \Phi(1.5) - [1 - \Phi(1.5)] = 0.86 \end{aligned}$$

(from the standard normal table). Answer: D

7. $E[X] = \int_0^1 x \times 2x\, dx = \frac{2}{3}$. $E[X^2] = \int_0^1 x^2 \times 2x\, dx = \frac{1}{2}$.
$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$.
$E[|X-\mu|] = E[|X - \frac{2}{3}|] = \int_0^{2/3} (\frac{2}{3} - x)(2x)\, dx + \int_{2/3}^1 (x - \frac{2}{3})(2x)\, dx = \frac{16}{81}$.

Then $\frac{E[|X-\mu|]}{Var[X]} = \frac{16/81}{1/18} = \frac{32}{9}$. Answer: C

8. Player 1 throws the dice on throws 1, 3, 5, . . . and the probability that player wins on throw $2k+1$ is $(\frac{8}{9})^{2k} \times \frac{1}{9}$ for $k = 0, 1, 2, 3, \ldots$ (there is a $\frac{1}{9}$ probability of throwing a total of 5 on any one throw of the pair of dice). The probability that player 1 wins the pot is
$\frac{1}{9} + (\frac{8}{9})^2 \times \frac{1}{9} + (\frac{8}{9})^4 \times \frac{1}{9} + \cdots = \frac{1}{9} \times \frac{1}{1-(\frac{8}{9})^2} = \frac{9}{17}$.

Player 2 throws the dice on throws $2, 4, 6, \ldots$ The probability that player 2 wins the pot on throw $2k$ is $(\frac{8}{9})^{2k-1} \times \frac{1}{9}$ for $k = 1, 2, 3, \ldots$ and the probability that player 2 wins is
$\frac{8}{9} \times \frac{1}{9} + (\frac{8}{9})^3 \times \frac{1}{9} + (\frac{8}{9})^5 \times \frac{1}{9} + \cdots = \frac{8}{9} \times \frac{1}{9} \times \frac{1}{1-(\frac{8}{9})^2} = \frac{8}{17} = 1 - \frac{9}{17}$.

If player 1 puts $1+c$ dollars into the pot, then his expected gain is $1 \times \frac{9}{17} - (1+c) \times \frac{8}{17}$.
and player 2's expected gain is $(1+c) \times \frac{8}{17} - 1 \times \frac{9}{17}$. In order for the two players to have the same expected gain, we must have $1 \times \frac{9}{17} - (1+c) \times \frac{8}{17} = 0$, so that $c = \frac{1}{8}$. Answer: C

9. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$ be the average annual income of the 100 individuals from Country A. Then
$E[S] = (\frac{1}{100})\sum_{i=1}^{100} E[X_i] = (\frac{1}{100})(100)(18,000) = 18,000$,
and $Var[S] = (\frac{1}{100})^2 \sum_{i=1}^{100} Var[X_i] = (\frac{1}{100})^2(100)(6,000)^2 = 360,000$
(being randomly chosen, the X_i's are independent, and covariances between any pair is 0). In a similar way, let $T = \frac{1}{100}\sum_{i=1}^{100} Y_i$ be the average annual income of the 100 individuals from Country B.

Then $E[T] = (\frac{1}{100})\sum_{i=1}^{100} E[Y_i] = (\frac{1}{100})(100)(31,000) = 31,000$,
and $Var[T] = (\frac{1}{100})^2 \sum_{i=1}^{100} Var[Y_i] = (\frac{1}{100})^2(100)(8,000)^2 = 640,000$.
Since the sample sizes are each 100, both S and T have distributions which are approximately normal (sample size 30 is usually the number taken in practice to use the normal approximation to the sum or mean of a random sample).

We wish to find $P[T > S + 15,000]$. $W = T - S$ has normal distribution with mean
$E[W] = E[T-S] = E[T] - E[S] = 31,000 - 18,000 = 13,000$, and with variance
$Var[W] = Var[T-S] = Var[T] + Var[S] = 640,000 + 360,000 = 1,000,000$
(T and S are independent, since the samples are drawn from two different Countries; therefore there is no covariance between T and S). Then,
$P[W > 15,000] = P[\left[\frac{W-13,000}{\sqrt{1,000,000}} > \frac{15,000-13,000}{\sqrt{1,000,000}}\right] = P[Z > 2]$
$= 1 - \Phi(2) = 1 - 0.9772 = 0.0228$ (we transform W to get $Z = \frac{W-E[W]}{\sqrt{Var[W]}}$, which has a standard normal distribution). Answer: E

10. $Y = max(X_1\,,\,X_2\,,\,X_3)$. $f_Y(y) = F_Y'(y)$, where

$$F_Y(y) = P[Y \le y] = P[max(X_1\,,\,X_2\,,\,X_3) \le y] = P[(X_1 \le y) \cap (X_2 \le y) \cap (X_3 \le y)]$$
$$= P[X_1 \le y] \times P[X_2 \le y] \times P[X_3 \le y]$$
$$= \begin{cases} 5(y-2.9)\cdot 2.5(y-2.7)\cdot 2.5(y-2.9) = 31.25(y^3 - 8.5y^2 + 24.07y - 22.707) & \text{for } 2.9 \le y \le 3.1 \\ 2.5(y-2.9) & \text{for } 3.1 \le y \le 3.3 \end{cases}$$

and $F_Y(y) = 0$ for $y \le 2.9$.

Then, $f_Y(y) = F_Y'(y) = \begin{cases} 31.25(3y^2 - 17y + 24.07) & \text{for } 2.9 \le y \le 3.1 \\ 2.5 & \text{for } 3.1 \le y \le 3.3 \end{cases}$.

Finally, $E[Y] = \int_{2.9}^{3.1} y \cdot 31.25(3y^2 - 17y + 24.07)\,dy + \int_{3.1}^{3.3} y \cdot 2.5\,dy = \frac{73}{48} + 1.6 = 3.12.$

An alternative solution uses the fact that for a non-negative random variable, $Y \ge 0$, the mean can be expressed in the form $E[Y] = \int_0^\infty [1 - F_Y(y)]\,dy$. In this case,

$$E[Y] = \int_0^{2.9} [1-0]\,dy + \int_{2.9}^{3.1} [1 - 31.25(y^3 - 8.5y^2 + 24.07y - 22.707)]\,dy$$
$$+ \int_{3.1}^{3.3} [1 - 2.5(y-2.9)]\,dy = 2.9 + \frac{41}{240} + \frac{1}{20} = 3.12.$$ Answer: C

11. The marginal density of X is $f(x) = \int_0^1 f(x,y)\,dy$.

The joint density of X and Y can be constructed from the conditional density of X given Y and the marginal density of Y, $f(x,y) = f(x|y) \times g(y)$. Then

$f(x) = \int_0^1 f(x,y)\,dy = \int_0^1 f(x|y) \times g(y)\,dy$.

But $f(x|y) = \frac{1}{\sqrt{y}}$ for $0 < x < \sqrt{y}$, or equivalently, for $x^2 < y < 1$.

Thus, $f(x) = \int_{x^2}^1 \frac{1}{\sqrt{y}} \times 1\,dy = 2y^{1/2}\Big|_{y=x^2}^{y=1} = 2(1-x)$. Answer: A

12. $Y = \text{amount paid by insurance} = \begin{cases} 0 & X < 100 \\ X - 100 & 100 \le X < 500 \\ 400 & X \ge 500 \end{cases}$

$E[Y] = \int_{100}^{500} (x-100) \times \frac{1}{1000}\,dx + 400[1 - F_X(500)] = 80 + 400(1 - \frac{1}{2}) = 280.$

$E[Y^2] = \int_{100}^{500} (x-100)^2 \times \frac{1}{1000}\,dx + 400^2[1 - F_X(500)] = \frac{64{,}000}{3} + 400^2(1 - \frac{1}{2}) = \frac{304{,}000}{3}.$

$Var[Y] = E[Y^2] - (E[Y])^2 = \frac{68{,}800}{3} = 22,933.$ Answer: D

13. $M_Z(t) = E[e^{tZ}] = E[exp(t\sum_{i=1}^n aX_i) = E[e^{atX_1} \cdot e^{atX_2} \cdots e^{atX_n}]$, and since the X_i's are independent, and since a Poisson random variable Y with mean λ_i has mgf. $M_Y(r) = e^{\lambda_i(e^r - 1)}$, this becomes

$M_Z(t) = \prod_{i=1}^n E[e^{atX_i}] = \prod_{i=1}^n M_{X_i}(at) = \prod_{i=1}^n e^{\lambda_i(e^{at}-1)} = exp[\sum_{i=1}^n \lambda_i(e^{at} - 1)].$ Answer: D

14. $Var[Y] = Var[E[Y|X]] + E[Var[Y|X]]$.
We are given $E[Y|X = x] = x$ and $Var[Y|X = x] = x^2$, so that
$E[Y|X] = X$ and $Var[Y|X] = X^2$, and then
$Var[Y] = Var[X] + E[X^2]$. We are given $E[X] = 3$, and $Var[X] = 2$,
so that $2 = Var[X] = E[X^2] - (E[X])^2 \rightarrow E[X^2] = 11$, and therefore,
$Var[Y] = 2 + 11 = 13$. Answer: E

15. Since the total probability must be 1, we have $4\theta_1 + 6\theta_2 = 1$.
The marginal distributions of X and Y have
$P[X = 1] = P[X = 5] = P[Y = 2] = P[Y = 4] = 2\theta_1 + 3\theta_2 = \frac{1}{2}$. Then, because of independence, $P[X = 1, Y = 2] = P[X = 1] \cdot P[Y = 2] = \frac{1}{4} = \theta_1 + \theta_2$.
Solving the two equations in θ_1 and θ_2 ($4\theta_1 + 6\theta_2 = 1$ and $\theta_1 + \theta_2 = \frac{1}{4}$) results in
$\theta_1 = \frac{1}{4}$, $\theta_2 = 0$. Answer: B

16. The new machine is still operating if at least one component is still working. The machine is no longer operating if both components have stopped working.

P[the new machine is still operating at time 6 months ($\frac{1}{2}$-year)]
$= 1 - P$[machine is no longer operating at $\frac{1}{2}$-year]
$= 1 - P[(X < \frac{1}{2}) \cap (Y < \frac{1}{2})] = 1 - \int_0^{1/2}\int_0^{1/2} f(x,y)\,dy\,dx$
$= 1 - \int_0^{1/2}\int_0^{1/2} (x+y)\,dy\,dx = 1 - \int_0^{1/2} (\frac{1}{2}x + \frac{1}{8})\,dx = 1 - \frac{1}{8} = \frac{7}{8}$. Answer: C

17. $P[X < 3] = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2}e^{-\lambda} = e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})$
$P[X \geq 3] = 1 - P[X < 3] = 1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})$

We are given that $P[X < 3] = 2P[X \geq 3] \rightarrow e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2}) = 2[1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})]$
$\rightarrow e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2}) = \frac{2}{3}$.
It is not possible to solve this equation algebraically, but we can substitute the 5 possible answers for λ to see which is closest. For $\lambda = 2$, we get $e^{-2}(1 + 2 + \frac{2^2}{2}) = 0.677$, which turns out to be the closest answer. Answer: A

18. $E[\frac{X}{Y}] = \sum_{x=1}^{2}\sum_{y=1}^{2} \frac{X_i}{Y_i} \times p(X_i, Y_i) = 1 \times \frac{2}{9} + \frac{1}{2} \times \frac{1}{9} + 2 \times \frac{4}{9} + 1 \times \frac{2}{9} = \frac{25}{18}$ Answer: D

19. $A_2 =$ event that second person has different birth month from the first.

$P(A_2) = \frac{11}{12} = 0.9167$.

$A_3 =$ event that third person has different birth month from first and second.

Then, the probability that all three have different birthdays is

$P[A_3 \cap A_2] = P[A_3|A_2] \times P(A_2) = \frac{10}{12} \times \frac{11}{12} = .7639$.

$A_4 =$ event that fourth person has different birth month from first three.

Then, the probability that all four have different birthdays is

$$\begin{aligned} P[A_4 \cap A_3 \cap A_2] &= P[A_4|A_3 \cap A_2] \times P[A_3 \cap A_2] \\ &= P[A_4|A_3 \cap A_2] \times P[A_3|A_2] \times P(A_2) = \frac{9}{12} \times \frac{10}{12} \times \frac{11}{12} = 0.5729. \end{aligned}$$

$A_5 =$ event that fifth person has different birth month from first four.

Then, the probability that all five have different birthdays is

$$\begin{aligned} P[A_5 \cap A_4 \cap A_3 \cap A_2] &= P[A_5|A_4 \cap A_3 \cap A_2] \times P[A_4 \cap A_3 \cap A_2] \\ &= P[A_5|A_4 \cap A_3 \cap A_2] \times P[A_4|A_3 \cap A_2] \times P[A_3|A_2] \times P(A_2) \\ &= \frac{8}{12} \times \frac{9}{12} \times \frac{10}{12} \times \frac{11}{12} = 0.3819. \end{aligned}$$

Answer: D

20. The insurer will pay $L = \begin{cases} x & \text{if } x \le 1 \text{ (million)} \\ 1 & \text{if } x > 1 \text{ (million)} \end{cases}$.

The expected payment by the insurer will be

$$\begin{aligned} E[L] = \int_0^1 x \times f(x)\,dx + \int_1^3 1 \times f(x)\,dx &= \int_0^1 x \times \frac{x(4-x)}{9}\,dx + \int_1^3 \frac{x(4-x)}{9}\,dx \\ &= \frac{13}{108} + \frac{22}{27} = \frac{101}{108} = 0.935. \end{aligned}$$

Answer: C

21. $Var[X] = E[X^2] - (E[X])^2$, and $E[X] = M'(0)$, $E[X^2] = M''(0)$.

$M'(t) = 9(\frac{2+e^t}{3})^8 \times \frac{e^t}{3}$, $M''(t) = 9 \times 8 \times (\frac{2+e^t}{3})^7 \times (\frac{e^t}{3})^2 + 9 \times (\frac{2+e^t}{3})^8 \times \frac{e^t}{3}$.

Then, $M'(0) = 3$ and $M''(0) = 8 + 3 = 11$, so that $Var[X] = 11 - 3^2 = 2$.

Alternatively, $Var[X] = \frac{d^2}{dt^2}\, ln\, M(t)\Big|_{t=0}$. In this case,

$ln\, M(t) = 9 \times ln(\frac{2+e^t}{3}) = 9 \times [ln(2+e^t) - ln\, 3]$,

so that $\frac{d}{dt}\, ln\, M(t) = \frac{9e^t}{2+e^t}$, and $\frac{d^2}{dt^2}\, ln\, M(t) = \frac{(2+e^t)(9e^t) - (9e^t)(e^t)}{(2+e^t)^2}$,

and then $\frac{d^2}{dt^2}\, ln\, M(t)\Big|_{t=0} = \frac{3 \times 9 - 9 \times 1}{3^2} = 2$.

A quicker alternative is to recognize that the given MGF is the MGF of a binomial random variable with $n = 9$ and $p = \frac{1}{3}$. In general, the MGF of a binomial random variable with parameters n and p is $M(t) = [pe^t + (1-p)]^n$. Each random variable has its own unique MGF, so $M(t) = (\frac{2+e^t}{3})^9$ must be the MGF for the binomial with $n = 9$ and $p = \frac{1}{3}$.

The variance of X is then $np(1-p) = 9 \times \frac{1}{3} \times \frac{2}{3} = 2$. Answer: A

22. The amounts won for each coin type are the component distributions:

Red coin: $X_R = \begin{cases} 100 & \text{prob. } \frac{1}{2} \\ 0 & \text{prob. } \frac{1}{2} \end{cases}$, White coin: $X_W = \begin{cases} 100 & \text{prob. } \frac{3}{4} \\ 0 & \text{prob. } \frac{1}{4} \end{cases}$,

Blue coin: $X_B = \begin{cases} 100 & \text{prob. } \frac{7}{8} \\ 0 & \text{prob. } \frac{1}{8} \end{cases}$. $P(\text{Red}) = \frac{1}{2}$, $P(\text{White}) = \frac{3}{8}$, $P(\text{Blue}) = \frac{1}{8}$.

$Y =$ amount won is a mixture of X_R, X_W and X_B , with the mixing weights above.

Then $P(Y = 100)$

$$= P(X_R = 100) \times P(\text{Red}) + P(X_W = 100) \times P(\text{White}) + P(X_B = 100) \times P(\text{Blue})$$
$$= \frac{1}{2} \times \frac{1}{2} + \frac{3}{4} \times \frac{3}{8} + \frac{7}{8} \times \frac{1}{8} = \frac{41}{64}, \text{ and } P(Y = 0) = \frac{23}{64}.$$

The expected amount won is $100 \times \frac{41}{64} = 64.0625$ on a play of the game, so the carnival should charge the player 65.0625 per play. Answer: C

23. From the given probabilities, it follows that

P[exactly two of them attend the game]

$= P$[at least two of them attend the game] $- P$[all three of them attend the game]

$= 0.80 - 0.50 = 0.30.$

We also know that P[exactly two of them attend the game]

$$= P(F \cap N \cap T') + P(F \cap N' \cap T) + P(F' \cap N \cap T) = 0.3.$$

From the symmetry of the probabilities, the three on the right hand side of the equation are equal, so that $P(F \cap N \cap T') = P(F \cap N' \cap T) = P(F' \cap N \cap T) = 0.1$.

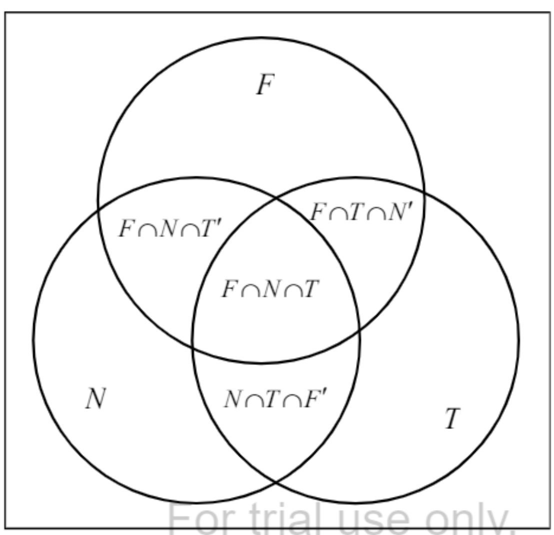
They are equal because $P(F \cap N \cap T') = P(F \cap N) - P(F \cap N \cap T)$ and

$P(F \cap T \cap N') = P(F \cap T) - P(F \cap N \cap T)$ and

$P(N \cap T \cap F') = P(N \cap T) - P(F \cap N \cap T)$, which are all equal.

The probability that Fred and Ned attend is then

$P(F \cap N) = P(F \cap N \cap T') + P(F \cap N \cap T) = 0.1 + 0.5 = 0.6.$



Answer: D

24. $f(\alpha)=\int_{\alpha}^{C}(x-\alpha)\times\frac{1}{C}\,dx=\frac{(C-\alpha)^2}{2C}\rightarrow f'(\alpha)=\frac{\alpha}{C}-1.$ Answer: D

25. Because of independence,
$P[(K=3)\cap(L=6)]=P[K=3]\times P[L=6]\ =\left[\binom{5}{3}(.3)^3(.7)^2\right]\left[\binom{10}{6}(.1)^6(.9)^4\right]$
(K and L both have binomial distributions). Answer: B

26. $f(x)=\begin{cases}x & x\le 750\\ 750 & x>750\end{cases}\rightarrow f'(x)=\begin{cases}1 & x\le 750\\ 0 & x>750\end{cases}$
This is the graph in C. Answer: C

27. The expected value of the warranty is $E[w(m)]=\int_0^{\infty}w(m)\times f(m)\,dm$, where $f(m)$ is the density function of the appliance failing at time m. We are given that the failure time has an exponential distribution with a mean of 10. The mean of an exponential distribution with parameter λ is $\frac{1}{\lambda}=10$, so that $\lambda=0.1$, and the density function is $f(m)=0.1e^{-0.1m}$. The expected value of the warranty is
$E[w(m)]=\int_0^7 v(m)\times 0.1e^{-0.1m}\,dm=\int_0^7 e^{(7-0.2m)}\times 0.1e^{-0.1m}\,dm$
$=0.1e^7\int_0^7 e^{-0.3m}dm=0.1e^7\left[\frac{1-e^{-2.1}}{.3}\right]=320.78\,.$ Answer: D

28. We define the following events: D - a person has the disease,
TP - a person tests positive for the disease. We are given $P[TP|D]=0.85$ and $P[TP|D']=0.10$ and $P[D]=0.01$. We wish to find $P[D|TP]$.

With a model population of 10,000, there would be $10,000\times P(D)=100=\#D$ people with the disease and 9,900 without the disease. The number that have the disease and test positive is $\#D\cap TP=\#D\times P[TP|D]=100\times 0.85=85$ and the number that do not have the disease and test positive is $\#D'\cap TP=\#D'\times P[TP|D']=9,900\times 0.1=990$. The total number who test positive is $\#D=\#D\cap TP\,+\,\#D'\cap TP=85+990=1075$. The probability that someone who tests positive actually has the disease is the proportion $\frac{\#D\cap TP}{TP}=\frac{85}{1075}=0.0791$.

The conditional probability approach to solving the problem is as follows.
Using the formulation for conditional probability we have $P[D|TP]=\frac{P[D\cap TP]}{P[TP]}$.
But $P[D\cap TP]=P[TP|D]\times P[D]=0.85\times 0.01=0.0085$, and
$P[D'\cap TP]=P[TP|D']\times P[D']=0.10\times 0.99=0.099$. Then,
$P[TP]=P[D\cap TP]+P[D'\cap TP]=0.1075\rightarrow P[D|TP]=\frac{0.0085}{0.1075}=0.0791.$

The following table summarizes the calculations.

$P[D] = 0.01$, given $\Rightarrow$ $P[D'] = 1 - P[D] = 0.99$

$\Downarrow$ $\Downarrow$

$P[D \cap TP]$ $P[D' \cap TP]$

$= P[TP|D] \times P[D] = 0.0085$ $= P[TP|D'] \times P[D'] = 0.099$

$\Downarrow$

$P[TP] = P[D \cap TP] + P[D' \cap TP] = 0.1075$

$\Downarrow$

$P[D|TP] = \frac{P[D \cap TP]}{P[TP]} = \frac{0.0085}{0.1075} = 0.0791.$ Answer: B

29. $Cov[X, Y] = E[XY] - E[X] \times E[Y] = E[XY] - 2.85.$

$E[XY] = \sum_{x=2}^{5} \sum_{y=0}^{2} xy \times f(x, y) = 2 \times 0 \times 0.05 + 2 \times 1 \times 0.40 + \cdots + 5 \times 1 \times 0 + 5 \times 2 \times 0$

$= 2.7 - 2.85 = -0.15.$ Answer: B

30. We identify the following events:

S - the applicant is a smoker, NS - the applicant is a non-smoker $= S'$

DS - the applicant declares to be a smoker on the application

DN - the applicant declares to be non-smoker on the application $= DS'$.

The information we are given is $P[S] = 0.3$, $P[NS] = 0.7$, $P[DN|S] = 0.4$, $P[DS|NS] = 0$. We wish to find $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]}$.

With a model population of 100 there are $30 = \#S$ smokers and $70 = \#NS$ non-smokers. The number of smokers who declare that they are non-smokers is $\#DN \cap S = \#S \times P[DN|S] = 30.4 = 12$ and since non-smokers don't lie, the number of non-smokers who declare that they are non-smokers is equal to the number of non-smokers, so $\#DN \cap NS = NS = 70$. The total number of people who declare that they are non-smokers is $\#DN \cap S + \#DN \cap NS = 12 + 70 = 82 = \#DN$.

Then, the proportion of applicants who say they are non-smokers that are actually non-smokers is $\frac{\#DN \cap NS}{DN} = \frac{70}{82} = \frac{35}{41}$.

The conditional probability approach to the solution is on the next page.

We calculate $.4 = P[DN|S] = \frac{P[DN \cap S]}{P[S]} = \frac{P[DN \cap S]}{.3} \rightarrow P[DN \cap S] = 0.12$,

and $0 = P[DS|NS] = \frac{P[DS \cap NS]}{P[NS]} = \frac{P[DS \cap NS]}{.7} \rightarrow P[DS \cap NS] = 0.$

Using the rule $P[A] = P[A \cap B] + P[A \cap B']$, and noting that $DS = DN'$ and $S = NS'$ we have $P[DS \cap S] = P[S] - P[DN \cap S] = 0.3 - 0.12 = 0.18$, and
$P[DN \cap NS] = P[NS] - P[DS \cap NS] = 0.7 - 0 = 0.7$, and
$P[DN] = P[DN \cap NS] + P[DN \cap S] = 0.7 + 0.12 = 0.82$.

Then, $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{0.7}{0.82} = \frac{35}{41}$.

These calculations can be summarized in the order indicated in the following table.

$P(S)\,, 0.3$ given	$\Rightarrow$	**1.** $P(NS) = 1 - P(S) = 0.7$ $\Downarrow$
6. DS $\Leftarrow$ $P(DS)$ $= P(DS \cap S)$ $+ P(DS \cap NS)$ $= 0.18 + 0 = 0.18$	**5.** $P(DS \cap S)$ $= P(S) - P(DN \cap S)$ $= 0.3 - 0.12 = 0.18$	**2.** $P(DS\|NS) = 0$, given $P(DS \cap NS)$ $= P(DS\|NS) \times P(NS)$ $= 0 \times 0.7 = 0$
$\Downarrow$	$\Uparrow$	
7. DN $P(DN)$ $= 1 - P(DS)$ $= 1 - 0.18$ $= 0.82$	**4.** $P(DN\|S) = .4$ given $P(DN \cap S)$ $= P(DN\|S) \times P(S)$ $= 0.4 \times 0.3 = 0.12$	**3.** $P(DN \cap NS) =$ $= P(NS) - P(DS \cap NS)$ $= 0.7 - 0 = 0.7$

Then, **8.** $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{0.7}{0.82} = \frac{35}{41}$. Answer: D

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PRACTICE EXAM 2

1. Let X have the density function $f(x) = \begin{cases} \frac{2x}{k^2} & \text{for } 0 \le x \le k \\ 0, & \text{otherwise} \end{cases}$.

For what value of k is the variance of X equal to 2?

A) 2 B) 6 C) 9 D) 18 E) 36

2. A life insurer classifies insurance applicants according to the following attributes:

M - the applicant is male

H - the applicant is a homeowner

Out of a large number of applicants the insurer has identified the following information:

40% of applicants are male, 40% of applicants are homeowners and

20% of applicants are female homeowners.

Find the percentage of applicants who are male and do not own a home.

A) 10% B) 20% C) 30% D) 40% E) 50%

3. Two components in an electrical circuit have continuous failure times X and Y. Both components will fail by time 1, but the circuit is designed so that the combined times until failure is also less than 1, so that the joint distribution of failure times satisfies the requirements $0 < x + y < 1$.

How many of the following joint density functions are consistent with an expected combined time until failure less than $\frac{1}{2}$ for the two components?

I. $f(x,y) = 2$ II. $f(x,y) = 3(x+y)$ III. $f(x,y) = 6x$ IV. $f(x,y) = 6y$

A) 0 B) 1 C) 2 D) 3 E) 4

4. In a "wheel of fortune" game, the contestant spins a dial and it ends up pointing to a number uniformly distributed between 0 and 1 (continuous). After 10,000 independent spins of the wheel find the approximate probability that the average of the 10,000 spins is less than 0.499.

A) Less than 0.34 B) At least .34 but less than 0.35 C) At least 0.35 but less than 0.36

D) At least 0.36 but less than 0.37 E) At least 0.37

5. A continuous random variable U has density function $f_U(u) = \begin{cases} 1-|u| & \text{for } -1<u<1 \\ 0, & \text{otherwise} \end{cases}$.

Which of the pairs of the following events are independent?

I. $-1 < U < 0$ II. $-\frac{1}{2} < U < \frac{1}{2}$ III. $0 < U < 1$

A) I and II only B) I and III only C) II and III only

D) I and II, II and III only E) No pairs are independent

6. An excess-of-loss insurance policy has a deductible of 1 and pays a maximum amount of 1. The loss random variable being insured by the policy has an exponential distribution with a mean of 1. Find the expected claim paid by the insurer on this policy.

A) $e^{-1} - 2e^{-2}$ B) $e^{-1} - e^{-2}$ C) $2(e^{-1} - e^{-2})$ D) e^{-1} E) $2e^{-2}$

7. If $f(x) = (k+1)x^2$ for $0 < x < 1$, find the moment generating function of X.

A) $\frac{e^t(6+6t+3t^2)}{t^3}$ B) $\frac{e^t(6-6t+3t^2)}{t^3}$ C) $\frac{e^t(6+6t+3t^2)}{t^3} - \frac{6}{t^3}$

D) $\frac{e^t(6+6t+3t^2)}{t^3} + \frac{6}{t^3}$ E) $\frac{e^t(6-6t+3t^2)}{t^3} - \frac{6}{t^3}$

8. Urn 1 contains 5 red and 5 blue balls. Urn 2 contains 4 red and 6 blue balls, and Urn 3 contains 3 red balls. A ball is chosen at random from Urn 1 and placed in Urn 2. Then a ball is chosen at random from Urn 2 and placed in Urn 3. Finally, a ball is chosen at random from Urn 3. Find the probability that the ball chosen from Urn 3 is red.

A) $\frac{15}{88}$ B) $\frac{30}{88}$ C) $\frac{45}{88}$ D) $\frac{60}{88}$ E) $\frac{75}{88}$

9. The amount of time taken by a machine repair person to repair a particular machine is a random variable with an exponential distribution with a mean of 1 hour. The repair person's employer pays the repair person a bonus of 2 whenever a repair takes less than $\frac{1}{4}$ hours, and a bonus of 1 if the repair takes between $\frac{1}{2}$ and $\frac{1}{4}$ hours. Find the average bonus received per machine repaired.

A) Less than 0.3 B) At least 0.3 but less than 0.4 C) At least 0.4 but less than 0.5

D) At least 0.5 but less than 0.6 E) At least 0.6

10. Bob and Doug are both 100-metre sprinters. Bob's sprint time is normally distributed with a mean of 10.00 seconds and Doug's sprint time is also normally distributed, but with a mean of 9.90 seconds. Both have the same standard deviation in sprint time of σ. Assuming that Bob and Doug have independent sprint times, and given that there is .95 chance that Doug beats Bob in any given race, find σ.

A) 0.040 B) 0.041 C) 0.042 D) 0.043 E) 0.044

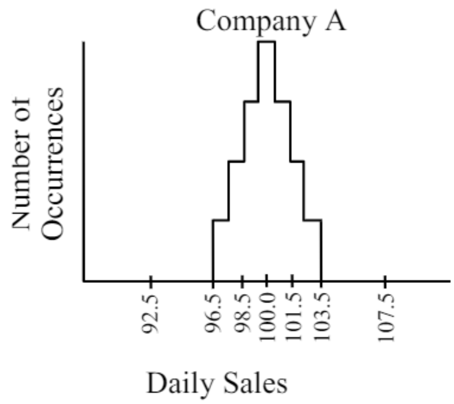
11. Let X and Y be continuous random variables with joint cumulative distribution function

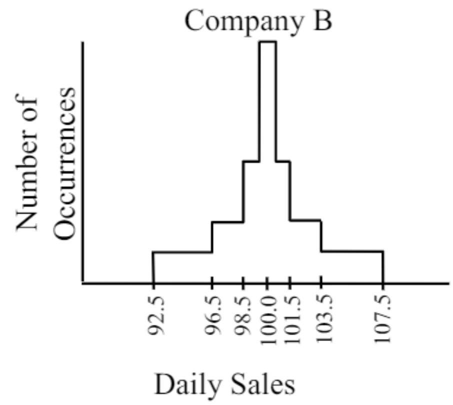
$$F(x,y) = \frac{1}{250}(20xy - x^2y - xy^2) \text{ for } 0 \le x \le 5 \text{ and } 0 \le y \le 5$$

Determine $P[X > 2]$.

A) $\frac{3}{125}$ B) $\frac{11}{50}$ C) $\frac{12}{25}$ D) $\frac{1}{250}(39y - 3y^2)$ E) $\frac{1}{250}(36y - 2y^2)$

12. A stock market analyst has recorded the daily sales revenue for two companies over the last year and displayed them in the histograms below.





The analyst noticed that a daily sales revenue above 100 for Company A was always accompanied by a daily sales revenue below 100 for Company B, and vice versa. Let X denote the daily sales revenue for Company A and let Y denote the daily sales revenue for Company B, on some future day. Assuming that daily sales revenues are independent from one day to another and all X_i's are identically distributed and all Y_i's are identically distributed, which of the following is true?

(A) Var(X) > Var(Y) and Var(X + Y) > Var(X) + Var(Y)

(B) Var(X) > Var(Y) and Var(X + Y) < Var(X) + Var(Y)

(C) Var(X) > Var(Y) and Var(X + Y) = Var(X) + Var(Y)

(D) Var(X) < Var(Y) and Var(X + Y) > Var(X) + Var(Y)

(E) Var(X) < Var(Y) and Var(X + Y) < Var(X) + Var(Y)

13. In a survey of males over the age of 30, it is found 50% are married, 40% smoke, 30% own a home and 60% own a car. It is also found that 30% are non-smoking bachelors, 40% are married car owners, 36% are non-smoking car owners, 25% own both a home and a car and 20% are married and own a home and a car. Which of the following statements is true regarding independence among the attributes of being married, being a smoker, being a car owner and being a home owner?

A) Being single and owning a car are independent

B) Being married and smoking are not independent

C) Being a smoker and owning a car are independent

D) Being a home owner and being a car owner are independent

E) Being married, being a home owner and being a car owner are mutually independent

14. A loss random variable is uniformly distributed on the integers from 0 to 11.
An insurance pays the loss in excess of a deductible of 5.5. Find the expected amount not covered by the insurance.

A) 2 B) 3 C) 4 D) 5 E) 6

15. An insurer offers an "all or nothing" policy of the following type. If the loss being insured is for an amount of D or more, then the insurance policy pays the full amount, but if the loss is less than D then the policy pays nothing. Assuming that the distribution of the loss has an exponential distribution with a mean of 2, and that $D = 2$, find the expected payout on the policy.

A) $\frac{1}{e}$ B) $\frac{2}{e}$ C) $\frac{4}{e}$ D) e E) $2e$

16. Suppose that X is a random variable with moment generating function $M(t) = \sum_{j=0}^{\infty} \frac{e^{(tj-1)}}{j!}$

Find $P[X = 2]$.

A) 0 B) $\frac{1}{2e}$ C) $\frac{e}{2}$ D) $\frac{1}{2}$ E) $\sum_{j=0}^{\infty} \frac{e^{2j-1}}{j!}$

17. Workplace accidents are categorized in three groups: minor, moderate, and severe. The probability that a given accident is minor is 0.5, that it is moderate is 0.4, and that it is severe is 0.1. Two accidents occur independently in one month. Calculate the probability that neither accident is severe and at most one is moderate.

A) 0.25 B) 0.40 C) 0.45 D) 0.56 E) 0.65

18. Customers arrive randomly and independently at a service window, and the time between arrivals has an exponential distribution with a mean of 12 minutes. Let X equal the number of arrivals per hour. What is $P[X = 10]$?

A) $\frac{10e^{-12}}{10!}$ B) $\frac{10^{12}e^{-10}}{10!}$ C) $\frac{12^{10}e^{-10}}{10!}$ D) $\frac{12^{10}e^{-12}}{10!}$ E) $\frac{5^{10}e^{-5}}{10!}$

19. A supplier of a testing device for a type of component claims that the device is highly reliable, with $P[A|B] = P[A'|B'] = 0.95$, where

$A =$ device indicates component is faulty, and $B =$ component is faulty .

You plan to use the testing device on a large batch of components of which 5% are faulty.

Find the probability that the component is faulty given that the testing device indicates that the component is faulty .

A) 0 B) 0.05 C) 0.15 D) 0.25 E) 0.50

20. Let X be a Poisson random variable with mean λ. If $P[X = 1|X \leq 1] = 0.8$, what is the value of λ?

A) 4 B) $-\ln 2$ C) 0.8 D) 0.25 E) $-\ln 0.8$

21. Let X and Y be continuous random variables with joint density function

$$f(x,y)=\begin{cases} x+y \text{ for } 0<x<1\ ,\ 0<y<1 \\ 0, \text{otherwise} \end{cases}$$

What is the marginal density function for X, where nonzero?

A) $y+\frac{1}{2}$ B) $2x$ C) x D) $\frac{x+x^2}{2}$ E) $x+\frac{1}{2}$

22. Let X_1, X_2, X_3 be uniform random variables on the interval $(0,1)$ with $Cov[X_i, X_j]=\frac{1}{24}$ for $i,j=1,2,3\,,\ i\neq j$. Calculate the variance of $X_1+2X_2-X_3$.

A) $\frac{1}{6}$ B) $\frac{1}{4}$ C) $\frac{5}{12}$ D) $\frac{1}{2}$ E) $\frac{11}{12}$

23. The joint density for liability damage X and collision damage Y when a claim occurs is $f(x,y)$, $0\leq X\leq 3\ ,\ 0\leq Y\leq 1$. Which of the following represents the probability that total loss will exceed 1?

A) $\int_0^1\int_0^{1-x} f(x,y)\,dy\,dx$ B) $1-\int_0^{1-y}\int_0^{1-x} f(x,y)\,dy\,dx$

C) $\int_0^1\int_0^{1-y} f(x,y)\,dx\,dy$ D) $1-\int_0^1\int_0^{1-y} f(x,y)\,dx\,dy$

E) $\int_0^1\int_0^{x-1} f(x,y)\,dy\,dx$

24. Medical researchers have identified three separate genes that individuals may or may not be born with. The researchers have found that 25% of the population have gene A, 20% have gene B and 10% have gene C. Furthermore, in any individual, the presence of gene A is independent of the presence of genes B or C, but no people can have both genes B and C. Find the probability that a randomly chosen individual has at least one of the three genes.

A) 0.450 B) 0.475 C) 0.500 D) 0.525 E) 0.550

25. An insurance contract reimburses a family's automobile accident losses up to a maximum of two accidents per year. The joint probability distribution for the number of accidents of a three person family (X,Y,Z) is $p(x,y,z)=k(x+2y+z)$, where

$x=0,1\ ,\ y=0,1,2\ ,\ z=0,1,2\ ,$ and

x,y,z are the number of accidents incurred by X, Y and Z, respectively.

Determine the expected number of unreimbursed accident losses given that X is not involved in any of the accidents.

A) 5/21 B) 1/3 C) 5/9 D) 4/63 E) 7/9

26. Let X be a continuous random variable with density function $f(x)=\begin{cases} \frac{x}{2} \text{ for } 0\leq x\leq 2 \\ 0, \text{otherwise} \end{cases}$.

Find $E[\,|X-E[X]|\,]$.

A) 0 B) $\frac{2}{9}$ C) $\frac{32}{81}$ D) $\frac{64}{81}$ E) $\frac{4}{3}$

27. Let X and Y be continuous random variables with joint density function

$$f(x,y)=\begin{cases}1 \text{ for } 0<y<1-|x| \text{ and } -1<x\leq 1\\ 0, \text{otherwise}\end{cases}.$$

What is $Var[X]$?

A) $\frac{1}{18}$ B) $\frac{1}{6}$ C) $\frac{2}{9}$ D) $\frac{11}{18}$ E) $\frac{2}{3}$

28. A pair of fair dice is tossed 2160 times. X is the number of times a total of 2 occurs. Find the approximate probability that X is less than 55 using the continuity correction.

A) 0.24 B) 0.26 C) 0.28 D) 0.30 E) 0.32

29. Suppose the remaining lifetimes of a husband and wife are independent and uniformly distributed on the interval [0,40]. An insurance company offers two products to married couples:

One which pays when the husband dies; and

One which pays when both the husband and wife have died.

Calculate the covariance of the two payment times.

A) 0.0 B) 44.4 C) 66.7 D) 200.0 E) 466.7

30. The mortality of a certain type of transistor is such that the probability of its breakdown in the interval $(t,$ $t+dt)$ is given by: $ce^{-ct}\,dt,\ t>0,\ c>0$. If 10 of these transistors are taken at random, then the probability that the 10th transistor that breaks down will do so during time $(v,\ v+dv)$ is

A) $10c(1-e^{-cv})^9e^{-cv}\,dv$ B) $10ce^{-10cv}\,dv$ C) $10ce^{-9cv}(1-e^{-cv})\,dv$

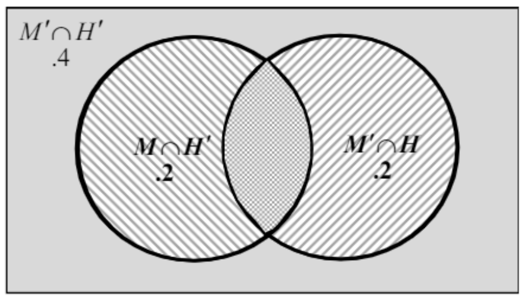
D) $10c(1-e^{-9cv})e^{-cv}\,dv$ E) $c(1-e^{-cv})^9e^{-cv}\,dv$

PRACTICE EXAM 2 - SOLUTIONS

1. $E[X]=\int_0^k x\times\frac{2x}{k^2}\,dx=\frac{2k}{3}$, $E[X^2]=\int_0^k x^2\times\frac{2x}{k^2}\,dx=\frac{k^2}{2}$
$\Rightarrow\ Var[X]=\frac{k^2}{2}-(\frac{2k}{3})^2=\frac{k^2}{18}=2\rightarrow k=6.$ Answer: B

2. $P[M]=0.4$, $P[M']=0.6$, $P[H]=0.4$, $P[H']=0.6$, $P[M'\cap H]=0.2$,
We wish to find $P[M\cap H']$. From probability rules, we have
$0.6=P[H']=P[M'\cap H']+P[M\cap H']$, and
$0.6=P[M']=P[M'\cap H]+P[M'\cap H']=0.2+P[M'\cap H']$.

Thus, $P[M'\cap H']=0.4$ and then $P[M\cap H']=0.2$. The following diagram identifies the component probabilities.



The calculations above can also be summarized in the following table. The events across the top of the table categorize individuals as male (M) or female (M'), and the events down the left side of the table categorize individuals as homeowners (H) or non-homeowners (H').

	$P(M)=0.4$, given		$P(M')=1-0.4=0.6$
$P(H)=0.4$ given	$P(M\cap H)$ $=P(H)-P(M'\cap H)=0.4-0.2=0.2$	$\Leftarrow$	$P(M'\cap H)=0.2$, given
	$\Downarrow$		
$P(H')=1-0.4=0.6$	$P(M\cap H')=P(M)-P(M\cap H)=0.4-0.2=0.2$	Answer: B	

3. $E[X+Y]=\int_0^1\int_0^{1-x}(x+y)f(x,y)\,dy\,dx$

   I. $\int_0^1\int_0^{1-x}2(x+y)\,dy\,dx=\frac{2}{3}$. Not correct.
   II. $\int_0^1\int_0^{1-x}3(x+y)(x+y)\,dy\,dx=\frac{3}{4}$. Not correct.
   III .$\int_0^1\int_0^{1-x}6x(x+y)\,dy\,dx=\frac{3}{4}$. Not correct.
   IV .$\int_0^1\int_0^{1-x}6y(x+y)\,dy\,dx=\frac{3}{4}$. Not correct.

Note that III and IV will have the same outcome by the symmetry of x and y. Answer: A

4. The outcome of a single spin is X, which has a uniform distribution on the interval $(0,1]$.

The mean and variance of X are $\frac{1}{2}$ and $\frac{1}{12}$. If $W = \sum_{i=1}^{10{,}000} X_i$, then W has an approximately normal distribution with mean $10{,}000 \times \frac{1}{2} = 5000$ and variance $10{,}000 \times \frac{1}{12} = 833.33$.

The average of the 10,000 spins is $A = \frac{W}{10{,}000}$.

$P[A < 0.499] = P[W < 4990] = P\left[\frac{W-5000}{\sqrt{833.33}} < \frac{4990-5000}{\sqrt{833.33}}\right]$

$\doteq \Phi\left(\frac{4990-5000}{\sqrt{833.33}}\right) = \Phi(-0.346) = 1 - \Phi(0.35) = 1 - 0.637 = 0.363$. Answer: D

5. $P(I) = \int_{-1}^{0} (1-|u|)\,du = \int_{-1}^{0} (1+u)\,du = \frac{1}{2}$

$P(II) = \int_{-1/2}^{1/2} (1-|u|)\,du = \int_{-1/2}^{0} (1+u)\,du + \int_{0}^{1/2} (1-u)\,du = \frac{3}{4}$

$P(III) = \int_{0}^{1} (1-|u|)\,du = \int_{0}^{1} (1-u)\,du = \frac{1}{2}$

$P(I \cap II) = P(-\frac{1}{2} < U < 0) = \int_{-1/2}^{0} (1+u)\,du = \frac{3}{8} = (\frac{1}{2})(\frac{3}{4}) = P(I) \times P(II)$

which shows that I and II are independent.

$P(I \cap III) = P(\emptyset) = 0 \neq \frac{1}{2} \times \frac{1}{2} = P(I) \times P(III)$

which shows that I and III are not independent.

$P(II \cap III) = \int_{0}^{1/2} (1-|u|)\,du = \frac{3}{8} = \frac{3}{4} \times \frac{1}{2} = P(II) \times P(III)$

which shows that II and III are independent. Answer: D

6. The loss random variable is X, which has density function $f_X(x) = e^{-x}$ for $x > 1$.

The amount paid by the insurance is $Y = \begin{cases} 0 \text{ if } X \le 1 \\ X-1 \text{ if } 1 < X \le 2 \\ 1 \text{ if } X > 2 \end{cases}$.

Then $E[Y] = \int_{1}^{2} (x-1)e^{-x}dx + 1 \times P[X > 2] = -xe^{-x}\Big|_{x=1}^{x=2} + e^{-2} = e^{-1} - e^{-2}$. Answer: B

7. Since $\int_0^1 f(x)\,dx = 1$, it follows that $(k+1) \cdot \frac{1}{3} = 1$, so that $k = 2$, and $f(x) = 3x^2$. Then,
$M_X(t) = E[e^{tX}] = \int_0^1 e^{tx} \times 3x^2 dx$.

Applying integration by parts, we have

$$\int_0^1 e^{tx} \times 3x^2 dx = \int_0^1 3x^2\, d(\tfrac{e^{tx}}{t}) = \frac{3x^2 e^{tx}}{t}\Big|_{x=0}^{x=1} - \int_0^1 \frac{6xe^{tx}}{t}\,dx$$
$$= \frac{3e^t}{t} - \int_0^1 \frac{6x}{t}\, d(\tfrac{e^{tx}}{t}) = \frac{3e^t}{t} - \left[\frac{6xe^{tx}}{t^2}\Big|_{x=0}^{x=1} - \int_0^1 \frac{6e^{tx}}{t^2}\,dx\right]$$
$$= \frac{3e^t}{t} - \frac{6e^t}{t^2} + \frac{6(e^t-1)}{t^3} = \frac{e^t(6-6t+3t^2)}{t^3} - \frac{6}{t^3}$$

Answer: E

8. $P(R_3) = P(R_3 \cap R_2 \cap R_1) + P(R_3 \cap R_2 \cap B_1) + P(R_3 \cap B_2 \cap R_1) + P(R_3 \cap B_2 \cap B_1)$.

$P(R_3 \cap R_2 \cap R_1) = P(R_3|R_2 \cap R_1) \times P(R_2|R_1) \times P(R_1) = 1 \times \frac{5}{11} \times \frac{1}{2} = \frac{5}{22}$

$P(R_3 \cap R_2 \cap B_1) = P(R_3|R_2 \cap B_1) \times P(R_2|B_1) \times P(B_1) = 1 \times \frac{4}{11} \times \frac{1}{2} = \frac{2}{11}$

$P(R_3 \cap B_2 \cap R_1) = P(R_3|B_2 \cap R_1) \times P(B_2|R_1) \times P(R_1) = \frac{3}{4} \times \frac{6}{11} \times \frac{1}{2} = \frac{9}{44}$

$P(R_3 \cap B_2 \cap B_1) = P(R_3|B_2 \cap B_1) \times P(B_2|B_1) \times P(B_1) = \frac{3}{4} \times \frac{7}{11} \times \frac{1}{2} = \frac{21}{88}$

Finally, $P(R_3) = \frac{5}{22} + \frac{2}{11} + \frac{9}{44} + \frac{21}{88} = \frac{75}{88}$. Answer: E

9. Average bonus $= 2 \cdot P[T \leq \frac{1}{4}] + 1 \times P[\frac{1}{4} < T \leq \frac{1}{2}] = 2[1 - e^{-1/4}] + [e^{-1/4} - e^{-1/2}] = 0.615$.

Answer: E

10. $B - D \sim N(.1, 2\sigma^2)$. $P[B > D] = P[B - D > 0] = P\left[\frac{B-D-0.1}{\sigma\sqrt{2}} > \frac{0-0.1}{\sigma\sqrt{2}}\right] = 0.95$

$\Rightarrow \frac{0-0.1}{\sigma\sqrt{2}} = -1.645$ (since $P[Z < 1.645] = 0.95$) $\rightarrow \sigma = 0.043$. Answer: D

11. $F_X(2) = P[X \leq 2] = \lim_{y\to\infty} F(2, y) = F(2, 5) = \frac{130}{250} = \frac{13}{25}$,

so that $P[X > 2] = 1 - P[X \leq 2] = \frac{12}{25}$. Answer: C

12. The histogram for Company B is more widely dispersed about its mean than the histogram for Company A, and therefore $Var(Y) > Var(X)$. Since daily sales revenue above 100 for Company A is always associated with daily sales revenue below 100 for Company B and vice-versa, the covariance between X and Y is negative. Therefore,

$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) < Var(X) + Var(Y)$. Answer: E

13. M - married, S - smoker, C - car owner, H - home owner

We are given, $P[M] = 0.5$, $P[S] = 0.4$, $P[C] = 0.6$, $P[H] = 0.3$, $P[M' \cap S'] = 0.3$,

$P[M \cap C] = 0.4$, $P[S' \cap C] = 0.36$, $P[H \cap C] = 0.25$, $P[M \cap C \cap H] = 0.2$.

Then, $P[M' \cap C] = P[C] - P[M \cap C] = 0.6 - 0.4 = 0.2$,

but $P[M'] \times P[C] = 0.5 \times 0.6 = .3 \rightarrow M', C$ not independent $\rightarrow$ A is false.

$P[M' \cap S'] = 0.3$, $P[M'] \times P[S'] = 0.5 \times 0.6 = 0.3 \rightarrow M', S'$ are independent

$\rightarrow M, S$ are independent $\rightarrow$ B is false.

$P[S' \cap C] = 0.36 = 0.6 \times 0.6 = P[S'] \times P[C] \rightarrow S', C$ are independent

$\rightarrow S, C$ are independent $\rightarrow$ C is true.

We can also check $P[H \cap C] = 0.25 \neq 0.3 \times 0.6 = P[H] \times P[C]$

$\rightarrow H, C$ not independent $\rightarrow$ D is false,

$P[M \cap C \cap H] = 0.2 \neq 0.5 \times 0.6 \times 0.3 = P[M] \times P[C] \times P[H]$

$\rightarrow M, C, H$ are not mutually independent. Answer: C

14. The amount not covered is

Loss	0	1	2	3	4	5	6	7	$\cdots$	11
Amt.	0	1	2	3	4	5	5.5	5.5	$\cdots$	5.5

Not Covered

Prob.	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\cdots$	$\frac{1}{12}$

The expected amount not covered by the insurance is

$\frac{1}{12} \times [0+1+2+3+4+5+5.5(6)] = 4.$ Answer: C

15. The expected payment on the policy will be

$\int_2^\infty x \times \frac{1}{2}e^{-x/2}\,dx = -xe^{-x/2} - 2e^{-x/2}\Big|_{x=2}^{x=\infty} = 4e^{-1}.$ Answer: C

16. The moment generating function for a non-negative discrete integer-valued random variable X with probability function f is defined to be $M(t) = E[e^{tX}] = \sum_{j=0}^{\infty} e^{tj} \times f(j)$. Since we are given that $M(t) = \sum_{j=0}^{\infty} \frac{e^{(tj-1)}}{j!}$, and it is known that the distribution of a random variable is uniquely determined by its moment generating function (i.e., there is precisely one probability distribution with that specified mgf), it follows that $f(j) = \frac{e^{-1}}{j!} = \frac{1}{e \cdot j!}$.

Since $f(j) = P[X=j]$, it follows that $P[X=2] = \frac{1}{2e}$. Answer: B

17. A_1 denotes the severity of accident 1 and A_2 denotes the severity of accident 2.

MI denotes the event that an accident is minor,

MO denotes the event that an accident is moderate, and

S denotes the event that an accident is severe.

The probability in question is the probability that either both are minor or exactly one is moderate (and because of independence, $P[A_1 \cap A_2] = P[A_1] \times P[A_2]$):

$P[(A_1 = MI) \cap (A_2 = MI)] + P[(A_1 = MI) \cap (A_2 = MO)]$

$+ P[(A_1 = MO) \cap (A_2 = MI)) = 0.5 \times 0.5 + 0.5 \times 0.4 + 0.4 \times .5 = 0.65$ Answer: E

18. When the time between successive arrivals has an exponential distribution with mean $\frac{1}{\alpha}$ (units of time), then the number of arrivals per unit time has a Poisson distribution with parameter (mean) α. The time between successive arrivals has an exponential distribution with mean $\frac{1}{5}$ hours (12 minutes). Thus, the number of arrivals per hour has a Poisson distribution with parameter 5, so that $P[X=10] = \frac{e^{-5}5^{10}}{10!}$.

Answer: E