For trial use only.
Content may not
represent the most
current version

26. If "failure" refers to a hurricane that results in no damage and "success" refers a hurricane that causes damage, then the distribution of X the number of failures until the 2nd success has a negative binomial distribution with probability function
$p(X=x) = \binom{r+x-1}{r-1} p^r (1-p)^x$ for $x = 0, 1, 2, 3, ...,$ where $r = 2$ and $p = .4$

We can also describe this in terms of total number of hurricanes $n = r + x = 2 + x$,
so that $P(N = n) = P(X = n-2) = \binom{n-1}{2-1} p^2 (1-p)^{n-2} = \binom{n-1}{2-1} (.4)^2 (.6)^{n-2}$
for $n = 2 + x = 2, 3, \ldots$. We wish find n that maximizes this probability.
$P(N=2) = 0.16$, $P(N=3) = 2 \times 0.16 \times 0.6 = 0.192$, $P(N=4) = 0.1728$.

The probabilities continue to decrease, because for $n \geq 3$, we have $\frac{n}{n-1} \times 0.6 < 1$.
Answer: B

27. $P(2) = P(1) = P(0)$, $P(3) = \frac{1}{2} \times P(2) = \frac{1}{2!} \times P(0)$, . . .
$P(k) = \frac{1}{(k-1)!} \times P(0)$. The probability function must satisfy the requirement
$\sum_{i=0}^{\infty} P(i) = 1$ so that $P(0) + \sum_{i=1}^{\infty} \frac{1}{(i-1)!} \times P(0) = P(0)(1+e) = 1$
(this uses the series expansion for e^x at $x = 1$). Then, $P(0) = \frac{1}{e+1}$. Answer: C

28. Suppose the mean of Y is λ_Y. Then this is also the variance, since Y has a Poisson distribution.
The same is true for X, with mean and variance of λ_X. The second moment of a Poisson random variable with mean λ is $\lambda + \lambda^2$, because $Var[W] = E[W^2] - (E[W])^2$
so $E[W^2] = Var[W] + (E[W])^2$.
We are given $\lambda_X = \lambda_Y - 8$, and $E[X^2] = \lambda_X + \lambda_X^2 = 0.6 \times E[Y^2] = 0.6 \times (\lambda_Y + \lambda_Y^2)$.
This can be written as $(\lambda_Y - 8) + (\lambda_Y - 8)^2 = 0.6 \times (\lambda_Y + \lambda_Y^2)$. This results in a quadratic equation in λ_Y, with roots $\lambda_Y = 4$ or 35. We reject the root of 4, since this would result in $\lambda_X = -4$, but a negative mean is not allowed for a Poisson random variable. Therefore, $\lambda_Y = 35$, which is both the mean and variance of Y. Answer: E

For trial use only.
Content may not
represent the most
current version

For trial use only.
Content may not
represent the most
current version
For trial use only.
Content may not
represent the most
current version

For trial use only. Content may not represent the most current version

SECTION 7 - FREQUENTLY USED CONTINUOUS DISTRIBUTIONS

Note that for a continuous random variable X, the following probabilities are the same: $P[a < X < b]$, $P[a < X \leq b]$, $P[a \leq X < b]$, $P[a \leq X \leq b]$.

Uniform distribution on the interval (a, b) (where $-\infty < a < b < \infty$):

The density function is constant, $\boldsymbol{f(x) = \frac{1}{b-a}}$ **for** $\boldsymbol{a < x < b}$, and $f(x) = 0$ otherwise.

The distribution function is $F(x) = \int_a^x f(x)\,dx = \frac{x-a}{b-a}$, so $\boldsymbol{F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}}$.

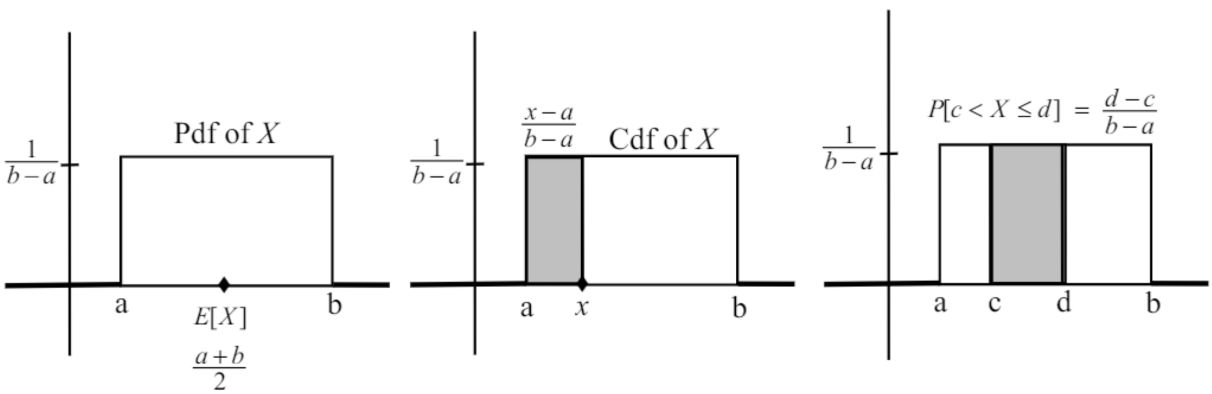
The mean and variance are $\boldsymbol{E[X] = \frac{a+b}{2}}$ and $\boldsymbol{Var[X] = \frac{(b-a)^2}{12}}$.

The moment generating function is $M_X(t) = \frac{e^{bt} - e^{at}}{(b-a) \times t}$ for any real t.

The n-th moment of X is $E[X^n] = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$. The median is $\frac{a+b}{2}$, the same as the mean.

This is a symmetric distribution about the mean; the mean is the midpoint of the interval (a, b).

The probability of the subinterval (c, d) of (a, b) is $P[c < X \leq d] = \frac{d-c}{b-a}$.



Example 7-1:

Suppose that X has a uniform distribution on the interval $(0, a)$, where $a > 0$. Find $P[X > X^2]$.

Solution:

If $a \leq 1$, then $X > X^2$ is always true for $0 < X < a$, so that $P[X > X^2] = 1$.

If $a > 1$, then $X > X^2$ only if $X < 1$, which has probability

$P[X < 1] = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{a}\,dx = \frac{1}{a}$. Thus, $P[X > X^2] = min[1, \frac{1}{a}]$. □

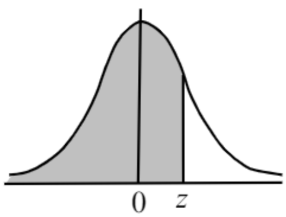
For trial use only. Content may not represent the most current version

For trial use only.
Content may not
represent the most
current version

The Normal Distribution

The **standard normal** distribution, $Z \sim N(0,1)$, has a mean of 0 and variance of 1. A table of probabilities for the standard normal distribution is provided on the exam. The density function is $\phi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}$ for $-\infty < z < \infty$. $E[Z] = 0$, $Var[Z] = 1$.
The moment generating function is $M_Z(t) = exp\left[\frac{t^2}{2}\right]$.

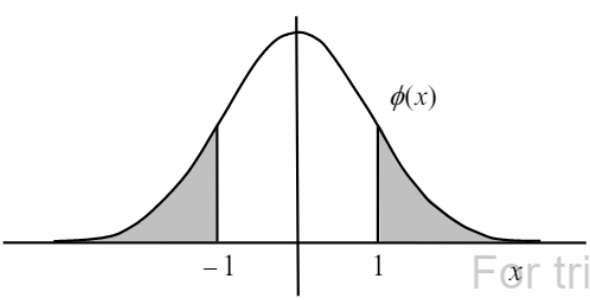
The density function has the following bell-shaped graph. The shaded area is the distribution function $P[Z \le z]$, which is denoted $\Phi(z)$.The graph and an excerpt from the standard normal distribution table are given below.

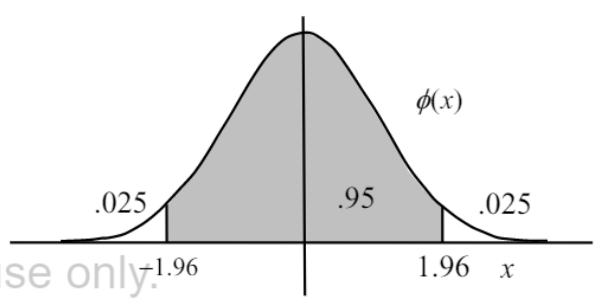


	0.00	0.01	0.02	0.03
0.0	0.5000	0.5040	0.5080	0.5120
0.1	0.5398	0.5438	0.5478	0.5517
0.2	0.5793	0.5832	0.5871	0.5910
0.3	0.6179	0.6217	0.6255	0.6293
0.4	0.6554	0.6591	0.6628	0.6664
0.5	0.6915	0.6950	0.6985	0.7019
0.6	0.7257	0.7291	0.7324	0.7357
0.7	0.7580	0.7611	0.7642	0.7673
0.8	0.7881	0.7910	0.7939	0.7967
0.9	0.8159	0.8186	0.8212	0.8238

A normal distribution table is provided at the exam. The full table can be found just before the practice exam section later in this study guide. The entries in the table are probabilities of the form $\Phi(z) = P[Z \le z]$. The 95-th percentile of Z is 1.645 (sometimes denoted $z_{.05}$) since $\Phi(1.645) = 0.950$ (the shaded region to the left of $z = 1.645$ in the graph above).

We use the symmetry of the standard normal distribution to find $\Phi(z)$ for negative values of z.
For instance, $\Phi(-1) = P[Z \le -1] = P[Z \ge 1] = 1 - \Phi(1)$ since the two regions have the same area (probability). This is illustrated in the left graph below. The two outside areas are equal, the left area is $\Phi(-1)$ and the right area is $1 - \Phi(1)$. Notice also in the right graph below that $P[-1.96 \le Z \le 1.96] = 0.95$, since $P[Z > 1.96] = 1 - \Phi(1.96) = 0.025$, and this area is deleted from both ends of the curve.





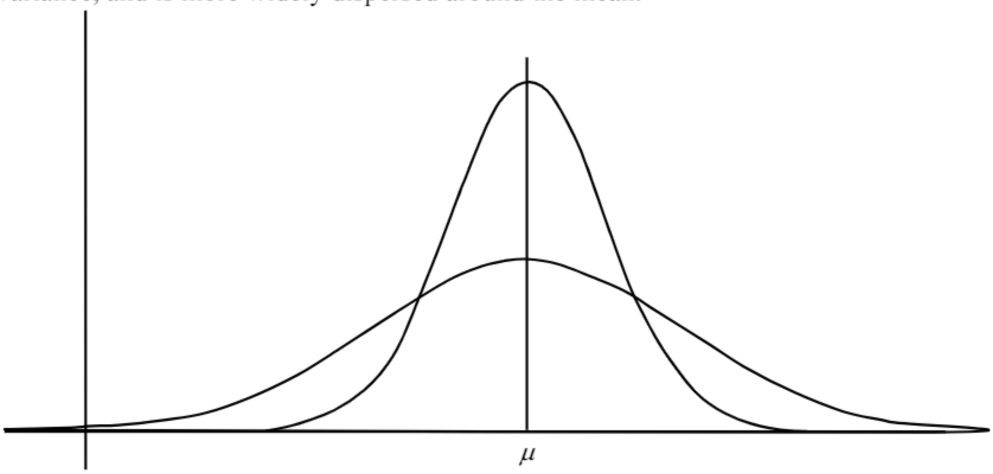
For trial use only.
Content may not
represent the most
current version

For trial use only.
Content may not
represent the most
current version

The general form of the normal distribution has mean μ and variance σ^2. This is a continuous distribution with a "bell-shaped" density function similar to that of the standard normal, but symmetric around the mean μ. The pdf is $f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-(x-\mu)^2/2\sigma^2}$ for $-\infty < x < \infty$. and the mean, variance and moment generating function of X are

$$\boldsymbol{E[X] = \mu\,,\ Var[X] = \sigma^2\,,\ M_X(t) = exp\Big[\mu t + \tfrac{\sigma^2 t^2}{2}\Big].}$$

Note also that for the normal distribution, **mean = median = mode =** $\boldsymbol{\mu}$. μ is the "center" of the distribution, and the variance σ^2 is a measure of how widely dispersed the distribution is. The graph shows the density functions of two normal distributions with a common mean μ. The distribution with the "flatter" graph has the larger variance, and is more widely dispersed around the mean.



Given any normal random variable $X \sim N(\mu, \sigma^2)$, it is possible to find $P[r < X < s]$ by first "standardizing" . This means that we define the random variable Z as follows: $Z = \frac{X-\mu}{\sigma}$.

This may be referred to as the "Z-score for X", or simply the Z-score.

Then

$$P[r < X < s] = P\Big[\tfrac{r-\mu}{\sigma} < \tfrac{X-\mu}{\sigma} < \tfrac{s-\mu}{\sigma}\Big] = P\Big[\tfrac{r-\mu}{\sigma} < Z < \tfrac{s-\mu}{\sigma}\Big] = \Phi\Big(\tfrac{s-\mu}{\sigma}\Big) - \Phi\Big(\tfrac{r-\mu}{\sigma}\Big)$$

For example, suppose that X has a normal distribution with mean 1 and variance 4. Then

$$P[X \le 2.5] = P\Big[\tfrac{X-1}{\sqrt{4}} \le \tfrac{2.5-1}{\sqrt{4}}\Big] = P\big[Z \le .75\big] = \Phi(.75) = .7734.$$

We have found $\Phi(.75)$ from the standard normal table.

The 95-th percentile of X can be found as follows. Let us denote the 95-th percentile of X by c. Then $P[X \le c] = 0.95 \ \Rightarrow\ P\Big[\frac{X-1}{\sqrt{4}} \le \frac{c-1}{\sqrt{4}}\Big] = \Phi\Big(\frac{c-1}{\sqrt{4}}\Big) = 0.95 \ \Rightarrow\ \frac{c-1}{\sqrt{4}} = 1.645 \ \Rightarrow\ c = 4.29$. We have used the value 1.645, which is the 95-th percentile of the standard normal.

For trial use only.
Content may not
represent the most
current version

For trial use only. Content may not represent the most current version

Example 7-2:

If for a certain normal random variable X, $P[X < 500] = 0.5$ and $P[X > 650] = 0.0227$, find the standard deviation of X.

Solution:

The normal distribution is symmetric about its mean, with $P[X < \mu] = 0.5$ for any normal random variable. Thus, for this normal X we have $\mu = 500$. Then, $P[X > 650] = .0227 = P\left[\frac{X-500}{\sigma} > \frac{150}{\sigma}\right]$. From the standard normal table, we see that $\Phi(2.00) = .9773$. Since $\frac{X-500}{\sigma}$ has a standard normal distribution, it follows from the table for the standard normal distribution that $\frac{150}{\sigma} = 2.00$ and $\sigma = 75$. $\square$

Approximating a distribution using a normal distribution

Given a random variable X with mean μ and variance σ^2, probabilities related to the distribution of X are sometimes approximated by assuming the distribution of X is approximately $N(\mu, \sigma^2)$. The SOA/CAS probability exam regularly has questions involving the normal approximation. It has sometimes been the case that a question asks for the approximate probability for some interval. This will almost always mean that the normal approximation should be applied, even if it is not specifically mentioned. Later in the study guide we will see the justification for using the normal approximation for a sum of random variables. It is in this context that approximate probabilities have come up on the exam.

Integer correction for the normal approximation to an integer-valued random variable:

The normal distribution is continuous, but it can be used to approximate a discrete integer- valued distribution. In such a case (if instructed to do so) we can apply the following procedure.

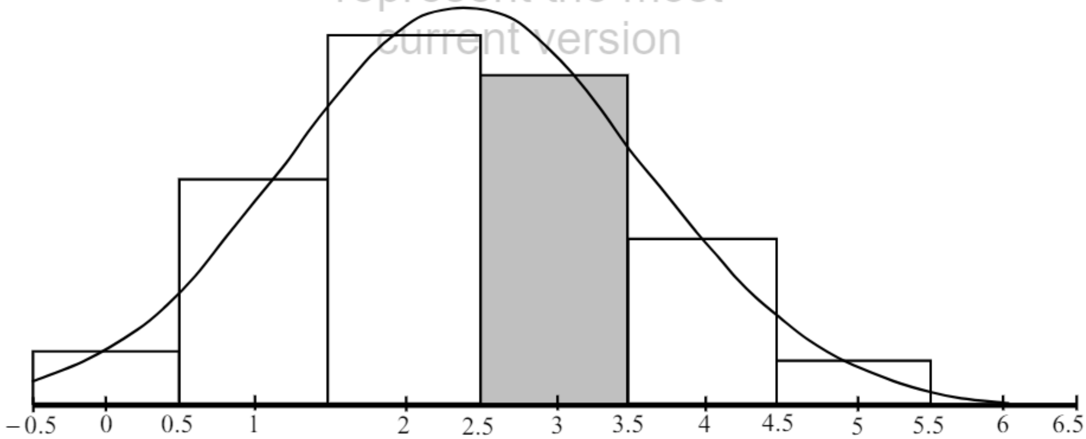
If X is discrete and integer-valued then an **"integer or continuity correction"** may be applied in the following way. If n and m are integers, the probability $P[n \le X \le m]$ is approximated by using a normal random variable Y with the same mean and variance as X, and then finding the probability $P[n - \frac{1}{2} \le Y \le m + \frac{1}{2}]$. We extend the interval $[n, m]$ to $[n - \frac{1}{2}, m + \frac{1}{2}]$. The reasoning behind this can be seen from the following graphs, in which a normal density function is superimposed over the histogram of an integer-valued random variable.

The integer-valued random variable X in the following graphs happens to be a binomial with $N = 6$ and $p = .4$, so the mean and variance are $6 \times 0.4 = 2.4$ and $6 \times 0.4 \times 0.6 = 1.44$. The way in which the normal approximation is applied, is to use the normal distribution with the same mean ($\mu = 2.4$) and variance ($\sigma^2 = 1.44$) as the original distribution. The density function in the graphs is of that normal distribution. In the first graph, the shaded region is the actual binomial probability that the outcome of the binomial distribution is 3.

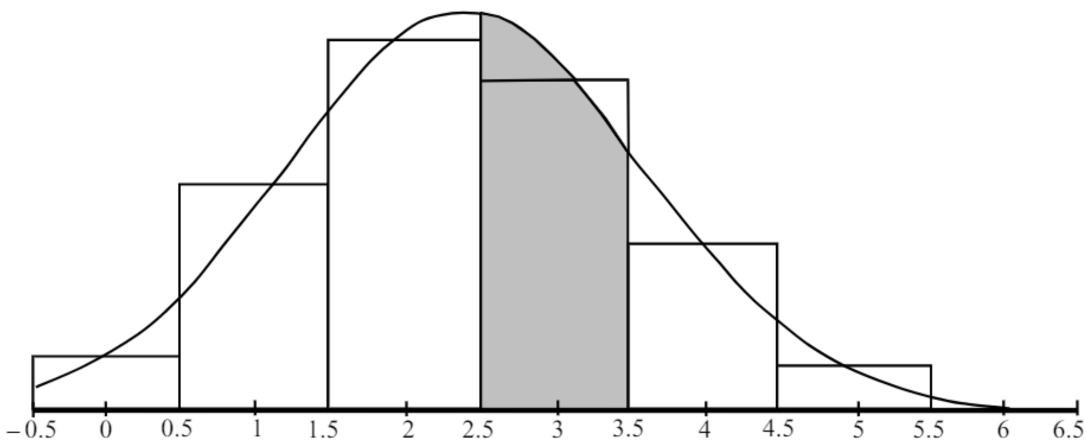
This actual probability is $\binom{6}{3}(0.4)^3(0.6)^{6-3} = 0.27648$.

For trial use only. Content may not represent the most current version

For trial use only.
Content may not
represent the most
current version



Since the normal distribution Y is a continuous distribution, in order to calculate a probability using the normal distribution, we must integrate the density over an interval. In order to use the normal distribution to approximate the probability that the binomial outcome is 3, we integrate over an interval of length 1 centered at $x = 3$. This is the integral from 2.5 to 3.5 of the normal density, so the normal approximation probability is $P[2.5 < Y < 3.5]$. This is the shaded region in the next graph.



For a normal distribution with mean 2.4 and variance 1.44, this probability is:

$P[2.5 < Y < 3.5] = P[\frac{2.5-2.4}{\sqrt{1.44}} < \frac{Y-2.4}{\sqrt{1.44}} < \frac{3.5-2.4}{\sqrt{1.44}}] = P[0.0833 < Z < 0.9167]$

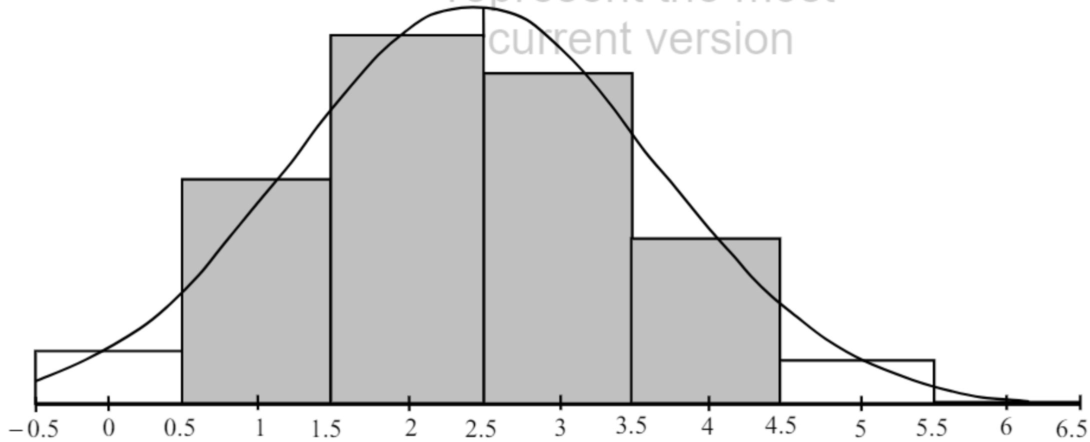
$= \Phi(0.92) - \Phi(0.08) = 0.8212 - 0.5319 = 0.29$. The exact binomial probability is 0.27648.

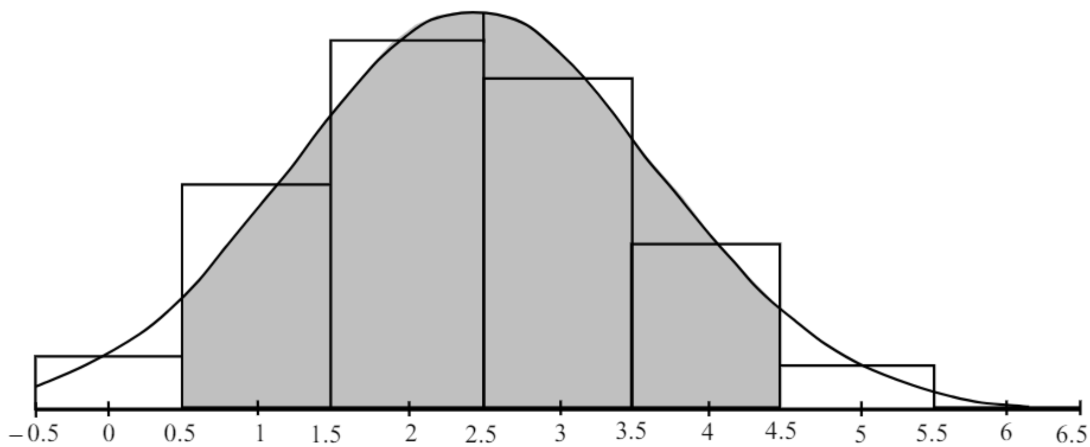
In general, the normal approximation for an integer value $X = k$ is the normal distribution (Y) probability on the interval from $k - \frac{1}{2}$ to $k + \frac{1}{2}$ ($P[k - \frac{1}{2} < Y < k + \frac{1}{2}]$). For the probability of several successive integer values, we have a series of intervals. For instance, to find the probability that $1 \leq X \leq 4$, we would approximate the probability at $X = 1$, $X = 2$, $X = 3$ and $X = 4$ and add them up. This corresponds to finding the normal probability for Y on the intervals from 0.5 to 1.5, from 1.5 to 2.5, from 2.5 to 3.5 and from 3.5 to 4.5. When we combine these, we get the probability from 0.5 to 4.5, $P[0.5 < Y < 4.5]$. This is illustrated in the following graphs. The shaded region of the first graph is the actual binomial probability $P[1 \leq X \leq 4]$.

The shaded region of the second graph is the normal approximation probability $P[0.5 < Y < 4.5]$.

For trial use only.
Content may not
represent the most
current version

For trial use only. Content may not represent the most current version





Note that if we were asked to approximate the probability $P[1 < X < 4]$, then this would be $P[X = 2 \text{ or } 3]$, which we would approximate it as $P[1.5 < Y < 3.5]$. If we were asked to approximate $P[X \leq 5]$, we would approximate it as $P[Y < 5.5]$.

There is a little bit of a vague area regarding the use of the integer correction on the exam and it may be worthwhile to calculate probabilities both with and without the integer correction. If the probability corresponding to the integer correction is one of the possible answers, it should be the correct answer (unless there is an indication that the integer correction should not be used). Note also that the phrase "integer correction" is sometimes referred to as "continuity correction".

We will consider sums of independent random variables in more detail later in this study guide, but one rule to make a note of now is the following. If X_1 and X_2 are independent normal random variables with means μ_1 and μ_2, and variances σ_1^2 and σ_2^2, respectively, then $W = X_1 + X_2$ is also a normal random variable, and has mean $\mu_1 + \mu_2$, and variance $\sigma_1^2 + \sigma_2^2$.

For trial use only. Content may not represent the most current version

For trial use only. Content may not represent the most current version

Example 7-3:

Suppose that a multiple choice exam has 40 questions, each with 5 possible answers. A well prepared student feels that he has a probability of 0.5 of getting any particular question correct, with independence from one question to another. Apply the normal approximation to X, the number of correct answers out of 40, to determine the probability of getting at least 25 correct. Find the probability with the integer correction, and then without the correction.

Solution:

The number of questions answered correctly, say X, has a binomial distribution with mean $(40)(.5) = 20$ and variance $40 \times 0.5 \times 0.5 = 10$. Applying the normal approximation Y to X, with integer correction to find the probability of answering at least 25 correct, we get

$$P[X \geq 25] = P[Y \geq 24.5] = P\left[\frac{Y-20}{\sqrt{10}} \geq \frac{24.5-20}{\sqrt{10}}\right] = P[Z \geq 1.42] = 1 - \Phi(1.42) = 0.078.$$

Without the integer correction, the probability is

$$P[Y \geq 25] = P\left[\frac{Y-20}{\sqrt{10}} \geq \frac{25-20}{\sqrt{10}}\right] = P[Z \geq 1.58] = 1 - \Phi(1.58) = 0.057.$$

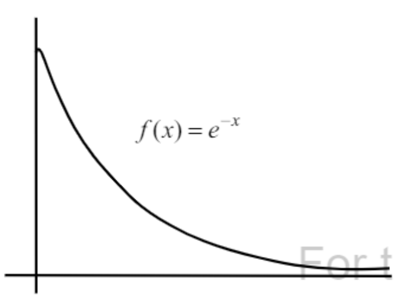
There is a noticeable difference between the two approaches. If X has a much larger standard deviation, then the difference is not so noticeable. □

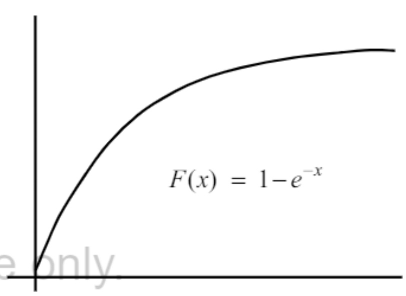
Exponential distribution with mean $\theta > 0$

The density function is $\boldsymbol{f(x) = \frac{1}{\theta} e^{-x/\theta}}$ **for** $\boldsymbol{x > 0}$, and $f(x) = 0$ otherwise.
The distribution function is $\boldsymbol{F(x) = 1 - e^{-x/\theta}}$ **for** $\boldsymbol{x \geq 0}$ and
the survival function is $\boldsymbol{S(x) = 1 - F(x) = P[X > x] = e^{-x/\theta}}$.
The mean is $\boldsymbol{E[X] = \theta}$, the variance is $\boldsymbol{Var[X] = \theta^2}$ and
the moment generating function is $M_X(t) = \frac{1}{1-t\theta}$ for $t < \frac{1}{\theta}$
The k-th moment is $E[X^k] = \int_0^\infty x^k \times \frac{1}{\theta} e^{-x/\theta} dx = k! \times \theta^k$, $k = 1, 2, 3, \ldots$
A general and useful identity from which this follows is $\int_0^\infty y^n \times e^{-cy}\, dy = \frac{n!}{c^{n+1}}$ (by mathematical induction).

The exponential distribution is often used as a model for the time until some specific event occurs, say the time until the next earthquake at a certain location.

The graphs of the pdf and cdf for the exponential distribution with mean 1 are





For trial use only. Content may not represent the most current version

For trial use only.
Content may not
represent the most
current version

Example 7-4:

The random variable T has an exponential distribution such that $P[T \leq 2] = 2 \times P[T > 4]$.

Find $Var[T]$.

Solution:

Suppose that T has mean θ. Then $P[T \leq 2] = 1 - e^{-2/\theta} = 2\,P[T > 4] = 2e^{-4/\theta}$

$\Rightarrow\ 2x^2 + x - 1 = 0$, where $x = e^{-2/\theta}$. Solving the quadratic equation results in $x = \frac{1}{2},\ -1$. We ignore the negative root, so that $e^{-2/\theta} = \frac{1}{2}$, and $\theta = \frac{2}{ln\,2}$.

Then, $Var[T] = \theta^2 = \frac{4}{(ln\,2)^2}$. □

Example 7-5:

The initial cost of a machine is 3. The lifetime of the machine has an exponential distribution with a mean of 3 years. The manufacturer is considering offering a warranty and considers two types of warranties. Warranty 1 pays 3 if the machine fails in the first year, 2 if the machine fails in the second year, and 1 if the machine fails in the third year, with no payment if the machine fails after 3 years. Warranty 2 pays $3e^{-t}$ if the machine fails at time t years (with no limit on the time of failure). Find the expected warranty payment under each of the two warranties.

Solution:

The pdf of T is $f(t) = \frac{1}{3}e^{-t/3}$, and the cdf is $F(t) = 1 - e^{-t/3}$.
Let X be the amount paid by warranty 1. The distribution of X is

$X:$	3	2	1	0
$p(x):$	$P[0 < T \leq 1]$	$P[1 < T \leq 2]$	$P[2 < T \leq 3]$	$P[T > 3]$
	$1 - e^{-\frac{1}{3}} = 0.2835$	$e^{-\frac{1}{3}} - e^{-\frac{2}{3}} = 0.2031$	$e^{-\frac{2}{3}} - e^{-1} = 0.1455$	$e^{-1} = 0.3679$

$E[X] = 3 \times 0.2835 + 2 \times 0.2031 + 1 \times 0.1455 = 1.40$.

Let Y be the amount paid by warranty 2. Then $Y = 3e^{-T}$.

$E[Y] = \int_0^\infty 3e^{-t} \times \frac{1}{3}e^{-t/3}\,dt = \int_0^\infty e^{-4t/3}\,dt = \frac{3}{4}$. □

There are a few additional properties satisfied by the exponential distribution that are worth noting.

(i) Lack of memory property: for $x, y > 0$,

$P[X > x + y | X > x] = \frac{P\{X > x+y \cap X > x]}{P[X > x]} = \frac{P\{X > x+y]}{P[X > x]} = \frac{e^{-(x+y)/\theta}}{e^{-x/\theta}} = e^{-y/\theta} = P[X > y]$.

We can interpret this as follows. Suppose that X represents the time, measured from now, in weeks until the next insurance claim filed by a company, and suppose also that X has an exponential distribution with mean θ. Suppose that 5 weeks have passed without an insurance claim, and we want to know the distribution of the time until the next insurance claim as measured from our new time origin, which is 5 weeks after the previous time origin. According to the lack of memory property, the fact that there have been no claims in the past 5 weeks is irrelevant, and measuring time starting from our new time origin, the

For trial use only.
Content may not
represent the most
current version

For trial use only. Content may not represent the most current version

time until the next claim is exponential with the same mean θ. In fact, no matter how many claims have occurred in the past 5 weeks, as measured from now, the time until the next claim has an exponential distribution with mean θ; the distribution has "forgotten" what has happened prior to now and the "clock" measuring time until the next claim is restarted now.

(ii) Link between the exponential distribution and Poisson distribution:
Suppose that X has an exponential distribution with mean θ and we regard X as the time between successive occurrences of some type of event (say the event is the arrival of a new insurance claim at an insurance office), where time is measured in some appropriate units (seconds, minutes, hours or days, etc.). Now, we imagine that we choose some starting time (say labeled as $t=0$), and from now we start recording times between successive events. For instance, the first claim may arrive in 2 days, then the next claim arrives 3 days after that, etc.

Let N represent the number of events (claims) that have occurred when one unit of time has elapsed. Then N will be a random variable related to the times of the occurring events. It can be shown that the distribution of N is Poisson with a mean of $\frac{1}{\theta}$.

(iii) The minimum of a collection of independent exponential random variables:
Suppose that independent random variables $Y_1, Y_2, ..., Y_n$ each have exponential distributions with means $\theta_1, \theta_2, \ldots, \theta_n$, respectively. Let $Y = min\{Y_1, Y_2, \ldots, Y_n\}$. Then Y has an exponential distribution with mean $\frac{1}{\frac{1}{\theta_1}+\frac{1}{\theta_2}+\cdots+\frac{1}{\theta_n}}$. An interpretation of this relationship is as follows. Suppose that an insurer has two types of insurance policies, basic coverage and extended coverage. Suppose insurance policies are independent of one another and that the time X_B until a claim from a basic policy is exponential with a mean of 4 weeks, and the time X_E until a claim from an extended policy is exponential with a mean of 2 weeks. The time until the next claim of any type is $X = min\{X_B, X_E\}$. X will be exponential with mean $\frac{1}{\frac{1}{4}+\frac{1}{2}} = \frac{4}{3}$ weeks. Another way of interpreting this is that the average number of claims per week for basic policies is $\frac{1}{4}$ (one every 4 weeks) and the average number of claims per week for extended policies is $\frac{1}{2}$, so the average number of claims per week for the two policy types combined is $\frac{3}{4}$. We can descibe the average of $\frac{3}{4}$ claims per weeks an average of $\frac{4}{3}$ weeks between claims.

Example 7-6:
Verify algebraically the validity of properties (i) and (iii) of the exponential distribution described above.

Solution:

(i) Suppose that X has an exponential distribution with parameter λ. Then
$P[X > x+y|X > x] = \frac{P[(X>x+y)\cap(X>x)]}{P[X>x]} = \frac{P[X>x+y]}{P[X>x]} = \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} = e^{-\lambda y}$,
and $P[X > y] = e^{-\lambda y}$.

For trial use only. Content may not represent the most current version

For trial use only. Content may not represent the most current version

(iii) Suppose that independent random variables $Y_1, Y_2, ..., Y_n$ have exponential distributions with means $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$ respectively. Let $Y = min\{Y_1, Y_2, \ldots, Y_n\}$. Then,

$$\begin{aligned} P[Y > y] &= P[Y_i > y \text{ for all } i = 1, 2, ..., n] = P[(Y_1 > y) \cap (Y_2 > y) \cap \cdots \cap (Y_n > y)] \\ &= P[Y_1 > y] \times P[Y_2 > y] \times \cdots \times P[Y_n > y] \text{ (because of independence of the } Y_i\text{'s)} \\ &= e^{-\lambda_1 y} \times e^{-\lambda_2 y} \times \cdots \times e^{-\lambda_n y} = e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)y}. \end{aligned}$$

The cdf of Y is then
$F_Y(y) = P[Y \leq y] = 1 - P[Y > y] = 1 - e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)y}$ and the pdf of Y is
$f_Y(y) = F_Y'(y) = (\lambda_1 + \lambda_2 + \cdots + \lambda_n)e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)y}$, which is the pdf of an exponential distribution with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$. □

Gamma distribution with parameters $\alpha > 0$ and $\theta > 0$

The pdf is $f(x) = \frac{x^{\alpha-1} \times e^{-x/\theta}}{\theta^\alpha \times \Gamma(\alpha)}$ for $x > 0$, and $f(x) = 0$ otherwise.
$\Gamma(\alpha)$ is the **gamma function**, which is defined for $\alpha > 0$ to be $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \times e^{-y} dy$.
If n is a positive integer it can be shown by induction that $\Gamma(n) = (n-1)!$ (factorial).
The mean, variance and moment generating function of X are

$$E[X] = \alpha\theta, \quad Var[X] = \alpha\theta^2, \quad \text{and} \quad M_X(t) = \left(\frac{1}{1-\theta t}\right)^\alpha \text{ for } t < \frac{1}{\theta}$$

If this distribution was to show up on Exam P, it would likely have the parameter α as an integer n, in which case the density would be $f(x) = \frac{x^{n-1} \times e^{-x/\theta}}{\theta^\alpha \times (n-1)!}$.

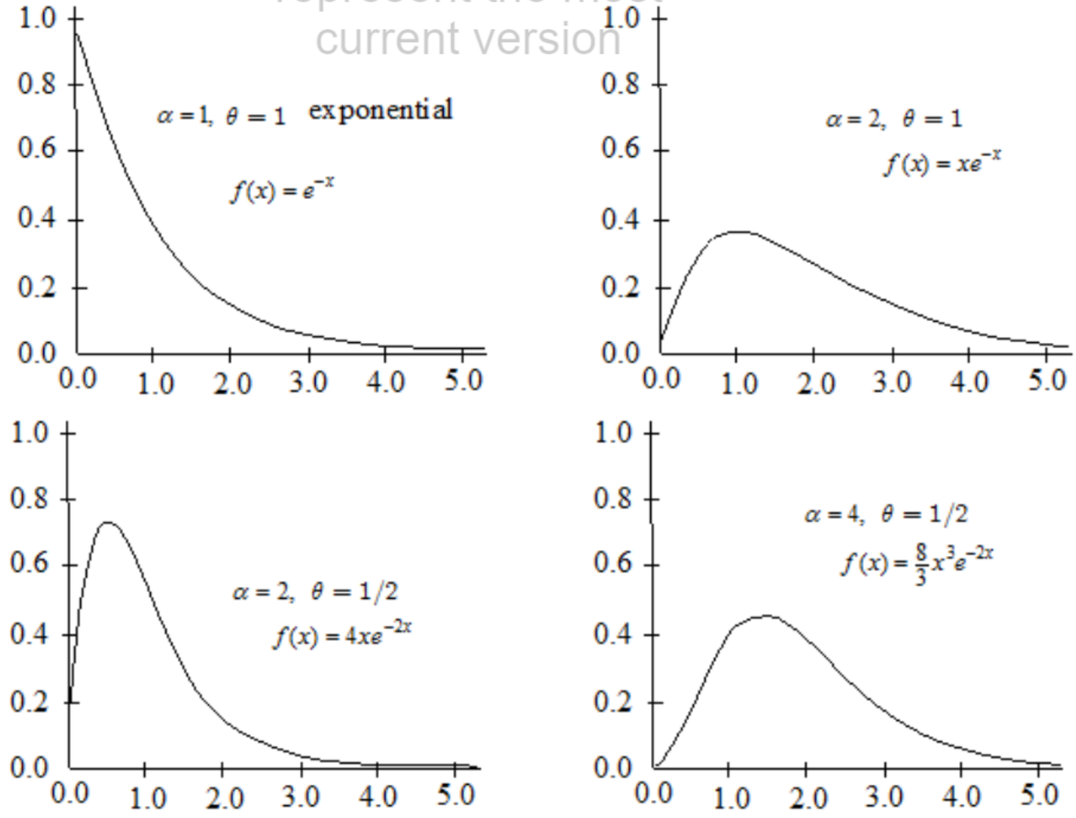
Note that the exponential distribution with mean θ is a special case of the gamma distribution with $\alpha = 1$ and θ. The graphs below illustrate the density functions of a few gamma distributions with various combinations of parameters α and θ.

The cdf of the gamma distribution can be complicated. If α is an integer then it is possible to find the cdf by integration by parts. But this would tend to be quite tedious unless α is 1 or 2.

As α gets larger, the pdf becomes more weighted to the right and is more spread out. As θ gets smaller, the pdf is more weighted to the left and the graph of the pdf becomes more peaked.

For trial use only. Content may not represent the most current version

For trial use only. Content may not represent the most current version



Gamma distribution density functions

For Exam P it is very important to be familiar with the uniform, normal and exponential distributions. The current sample question file on the SOA website has two questions out of about 330 that refer to the gamma distribution.

Example 7-7:

An insurer will pay 80% of the loss incurred on a loss of amount X. The loss random variable X has pdf $f(x)=\frac{3{,}000{,}000}{x^4}$ for $x>100$, and $f(x)=0$ for $x\le 100$.

Find the standard deviation of the amount paid by the insurer.

Solution:

If Y is the amount paid by the insurer, then $Y=0.8X$, so that

$Var[Y]=(0.8)^2\times Var[X]$, and $\sqrt{Var[Y]}=0.8\times\sqrt{Var[X]}$.

$Var[X]=E[X^2]-(E[X])^2$, where $E[X]=\int_{100}^{\infty}x\times\frac{3{,}000{,}000}{x^4}\,dx=150$ and

$E[X^2]=\int_{100}^{\infty}x^2\times\frac{3{,}000{,}000}{x^4}\,dx=30,000$. Then $Var[X]=30,000-(150)^2=7,500$,

and $\sqrt{Var[Y]}=0.8\times\sqrt{7,500}=69.3$. □

For trial use only. Content may not represent the most current version

For trial use only. Content may not represent the most current version

SUMMARY OF CONTINUOUS DISTRIBUTIONS

Distribution	Parameters	PDF, $p(x)$	Mean, $E[X]$	Variance, $Var[X]$	MGF, $M_X(t)$
Uniform	$a < b$	$\frac{1}{b-a}$, $a < x < b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)\times t}$
Normal	μ (any number), $\sigma^2 > 0$	$\frac{1}{\sigma\cdot\sqrt{2\pi}} \times e^{-(x-\mu)^2/2\sigma^2}$ $-\infty < x < \infty$	μ	σ^2	$exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$
Exponential	$\frac{1}{\lambda} = \theta > 0$	$\lambda e^{-\lambda x}$, $x > 0$ $F(x) = 1 - e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$
Gamma	$\alpha > 0\,, \beta > 0$	$\frac{\beta^\alpha \times x^{\alpha-1} \times e^{-\beta x}}{\Gamma(\alpha)}$, $x > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$

For trial use only. Content may not represent the most current version

For trial use only.
Content may not
represent the most
current version

PROBLEM SET 7

Frequently Used Continuous Distributions

1. Let X be a random variable with a continuous uniform distribution on the interval $(1, a)$ where $a > 1$. If $E[X] = 6 \times Var[X]$, then $a =$
A) 2 B) 3 C) $3\sqrt{2}$ D) 7 E) 8

2. A large wooden floor is laid with strips 2 inches wide and with negligible space between strips. A uniform circular disk of diameter 2.25 inches is dropped at random on the floor. What is the probability that the disk touches three of the wooden strips?
A) $\frac{1}{\sqrt{\pi}}$ B) $\frac{1}{\pi}$ C) $\frac{1}{4}$ D) $\frac{1}{8}$ E) $\frac{1}{\pi^2}$

3. If X has a continuous uniform distribution on the interval from 0 to 10, then what is $P[X + \frac{10}{X} > 7]$?
A) $\frac{3}{10}$ B) $\frac{31}{70}$ C) $\frac{1}{2}$ D) $\frac{39}{70}$ E) $\frac{7}{10}$

Problems 4 and 5 relate to the following information.

Three individuals are running a one kilometer race. The completion time for each individual is a random variable. X_i is the completion time, in minutes, for person i.

X_1 : uniform distribution on the interval $[2.9\,,\,3.1]$

X_2 : uniform distribution on the interval $[2.7\,,\,3.1]$

X_3 : uniform distribution on the interval $[2.9\,,\,3.3]$

The three completion times are independent of one another.

4. Find the probability that the earliest completion time is less than 3 minutes.
A) 0.89 B) 0.91 C) 0.94 D) 0.96 E) 0.98

5. Find the probability that the latest completion time is less than 3 minutes (nearest .01).
A) 0.03 B) 0.06 C) 0.09 D) 0.12 E) 0.15

6. A student received a grade of 80 in a math final where the mean grade was 72 and the standard deviation was s. In the statistics final, he received a 90, where the mean grade was 80 and the standard deviation was 15. If the standardized scores (i.e., the scores adjusted to a mean of 0 and standard deviation of 1) were the same in each case, then $s =$
A) 10 B) 12 C) 16 D) 18 E) 20

For trial use only.
Content may not
represent the most
current version

For trial use only. Content may not represent the most current version

7. If X has a standard normal distribution and $Y = e^X$, what is the k-th moment of Y?

A) 0 B) 1 C) $e^{k/2}$ D) $e^{k^2/2}$

E) 1 if $k = 2m - 1$ and $e^{(2m-1)(2m-3)\cdots 3\cdot 1}$ if $k = 2m$

8. (SOA) For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000. For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000. Assume that the total claim amounts of the two companies are independent. What is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

A) 0.180 B) 0.185 C) 0.217 D) 0.223 E) 0.240

9. (SOA) The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively. What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?

A) $\frac{1}{18}\left(1-e^{-2/3}-e^{-1/2}+e^{-7/6}\right)$ B) $\frac{1}{18}e^{-7/6}$ C) $1-e^{-2/3}-e^{-1/2}+e^{-7/6}$

D) $1-e^{-2/3}-e^{-1/2}+e^{-1/3}$ E) $1-\frac{1}{3}e^{-2/3}-\frac{1}{6}e^{-1/2}+\frac{1}{18}e^{-7/6}$

10. Let X be a continuous random variable with density function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $-\infty < x < \infty$. Calculate $E[X|X \geq 0]$.

A) 0 B) $\frac{1}{\sqrt{2\pi}}$ C) $\frac{1}{2}$ D) $\sqrt{\frac{2}{\pi}}$ E) 1

11. (SOA) Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation 0.0056h. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation 0.0044h. Assuming the two measurements are independent random variables, what is the probability that their average value is within 0.005h of the height of the tower?

A) 0.38 B) 0.47 C) 0.68 D) 0.84 E) 0.90

12. A new car battery is sold for \$100 with a 3-year limited warranty. If the battery fails at time t $(0 < t < 3)$, the battery manufacturer will refund \$100$(1 - \frac{t}{3})$. After analyzing battery performance, the battery manufacturer uses the (continuous) uniform distribution on the interval $(0, n)$ as the model for time until failure for the battery (n in years). The battery manufacturer determines that the expected cost of the warranty is \$10. Find n.

A) 3 B) 5 C) 10 D) 15 E) 30

For trial use only. Content may not represent the most current version

For trial use only. Content may not represent the most current version

13. (SOA) The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

A) 6,321 B) 7,358 C) 7,869 D) 10,256 E) 12,642

14. (SOA) A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $0.5x$ if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made. At what level must x be set if the expected payment made under this insurance is to be 1000 ?

A) 3858 B) 4449 C) 5382 D) 5644 E) 7235

15. (SOA) The time to failure of a component in an electronic device has an exponential distribution with a median of four hours. Calculate the probability that the component will work without failing for at least five hours.

A) 0.07 B) 0.29 C) 0.38 D) 0.42 E) 0.57

16. An insurer uses the exponential distribution with mean μ as the model for the total annual claim occurring from a particular insurance policy in the current one year period. The insurer assumes an inflation factor of 10% for the one year period following the current one year period. Using the insurer's assumption, find the coefficient of variation ($\frac{standard\ deviation}{expected\ value}$) for the annual claim paid on the policy for the one year period following the current one year period.

A) 1.21 B) 1.1 C) 1 D) $\frac{1}{1.1}$ E) $\frac{1}{1.21}$

17. Average loss size per policy on a portfolio of policies is 100. Actuary 1 assumes that the distribution of loss size has an exponential distribution with a mean of 100, and Actuary 2 assumes that the distribution of loss size has a pdf of $f_2(x) = \frac{2\theta^2}{(x+\theta)^3}$, $x > 0$. If m_1 and m_2 represent the median loss sizes for the two distributions, find $\frac{m_1}{m_2}$.

A) .6 B) 1.0 C) 1.3 D) 1.7 E) 2.0

18. The time until the occurrence of a major hurricane is exponentially distributed. It is found that it is 1.5 times as likely that a major hurricane will occur in the next ten years as it is that the next major hurricane will occur in the next five years. Find the expected time until the next major hurricane.

A) 5 B) $5\,ln\,2$ C) $\frac{5}{ln\,2}$ D) $10\,ln\,2$ E) $\frac{10}{ln\,2}$

For trial use only. Content may not represent the most current version

For trial use only. Content may not represent the most current version

19. (SOA) A driver and a passenger are in a car accident. Each of the independently has a probability of 0.3 of being hospitalized. When a hospitalization occurs, the loss is uniformly distributed on $[0, 1]$. When two hospitalizations occur, the losses are independent. Calculate the expected number of people in the car who are hospitalized, given that the total loss due to hospitalizations from the accident is less than 1.

A) 0.510 B) 0.534 C) 0.600 D) 0.628 E) 0.800

20. X has a gamma distribution with pdf is $f(x) = \frac{x^{\alpha-1} \times e^{-x/\theta}}{\theta^{\alpha} \times \Gamma(\alpha)}$ for $x > 0$,

and $f(x) = 0$ otherwise. You are given

(i) $E[X] = 2 \times Var[X]$ and (ii) $M_X(1) = 16$ (moment generating function of X).

Calculate $E[X]$.

A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) 1 D) 2 E) 4

For trial use only. Content may not represent the most current version

For trial use only. Content may not represent the most current version

PROBLEM SET 7 SOLUTIONS

1. $E[X] = \frac{1+a}{2}$ and $Var[X] = \frac{(a-1)^2}{12}$, so that
$\frac{a+1}{2} = 6 \times \frac{(a-1)^2}{12} \Rightarrow a^2 - 3a = 0 \Rightarrow a = 0, 3 \Rightarrow a = 3$ (since $a > 0$). Answer: B

2. Let us focus on the left-most point p on the disk. Consider two adjacent strips on the floor. Let the interval $[0, 2]$ represent the distance as we move across the left strip from left to right. If p is between 0 and 1.75, then the disk lies within the two strips.

If p is between 1.75 and 2, the disk will lie on 3 strips (the first two and the next one to the right). Since any point between 0 and 2 is equally likely as the left most point p on the disk (i.e. uniformly distributed between 0 and 2) it follows that the probability that the disk will touch three strips is $\frac{0.25}{2} = \frac{1}{8}$.
Answer: D

3. Since the density function for X is $f(x) = \frac{1}{10}$ for $0 < x < 10$, we can regard X as being positive. Then
$P[X + \frac{10}{X} > 7] = P[X^2 - 7X + 10 > 0] = P[(X-5)(X-2) > 0]$
$= P[X > 5] + P[X < 2]$ (since $(t-5)(t-2) > 0$ if either both $t-5, t-2 > 0$ or both $t-5, t-2 < 0$) $= \frac{5}{10} + \frac{2}{10} = \frac{7}{10}$. Answer: E

4. $f_{X_1}(t) = \frac{1}{0.2} = 5$ for $2.9 \le t \le 3.1$, $F_{X_1}(t) = P[X_1 \le t] = 5(t - 2.9)$ for $2.9 \le t \le 3.1$
$f_{X_2}(t) = \frac{1}{0.4} = 2.5$ for $2.7 \le t \le 3.1$, $F_{X_2}(t) = P[X_2 \le t] = 2.5(t - 2.7)$ for $2.7 \le t \le 3.1$
$f_{X_3}(t) = \frac{1}{0.4} = 2.5$ for $2.9 \le t \le 3.3$, $F_{X_3}(t) = P[X_3 \le t] = 2.5(t - 2.9)$ for $2.9 \le t \le 3.3$
$P[min(X_1, X_2, X_3) < 3] = 1 - P[min(X_1, X_2, X_3) \ge 3]$
$= 1 - P[(X_1 \ge 3) \cap (X_2 \ge 3) \cap (X_3 \ge 3)]$
$= 1 - [1 - F_{X_1}(3)] \times [1 - F_{X_2}(3)] \times [1 - F_{X_3}(3)]$
$= 1 - [1 - 5(3 - 2.9)] \times [1 - 2.5(3 - 2.7)] \times [1 - 2.5(3 - 2.9)] = 0.90625$. Answer: B

5. $P[max(X_1, X_2, X_3) < 3] = P[(X_1 < 3) \cap (X_2 < 3) \cap (X_3 < 3)]$
$= F_{X_1}(3) \times F_{X_2}(3) \times F_{X_3}(3) = [5(3 - 2.9)] \times [2.5(3 - 2.7)] \times [2.5(3 - 2.9)] = 0.09375$
Answer: C

6. The standardized statistics score is $\frac{90-80}{15} = \frac{2}{3}$. The standardized math score is
$\frac{80-72}{s} = \frac{8}{s} = \frac{2}{3} \rightarrow s = 12$. Answer: B

7. The k-th moment of Y is $E[Y^k] = E[e^{kX}] = M_X(k) = e^{k^2/2}$ (since $\mu = 0$ and $\sigma^2 = 1$). Answer: D

For trial use only. Content may not represent the most current version

For trial use only.
Content may not
represent the most
current version

8. We denote by X_A and X_B the total claim amount for the coming year for Company A and B, respectively. We are asked to find $P[X_B > X_A]$. X_A is a mixture of two parts.

There is a discrete part,

P[Company A has no claims in the coming year] $= P[X_A = 0] = 0.6$

and a continuous part

P[Company A has some claims in the coming year], which is equal to

$P[X_A$ has a normal distribution with mean 10,000 and standard deviation 2,000$] = 0.4$

X_B is similar. There is a discrete part

P[Company B has no claims in the coming year] $= P[X_B = 0] = 0.7$

and a continuous part

P[Company B has some claims in the coming year], which is equal to

$P[X_B$ has a normal distribution with mean 9,000 and standard deviation 2,000$] = 0.3$

Therefore, $X_A = \begin{cases} 0 & \text{prob. } 0.6 \\ Y_A\text{, normal, mean 10,000, std. dev. 2000} & \text{prob. } 0.4 \end{cases}$ and

$X_B = \begin{cases} 0 & \text{prob. } 0.7 \\ Y_B\text{, normal, mean 9,000, std. dev. 2000} & \text{prob. } 0.3 \end{cases}$

We use the following probability rule:

$$P[C] = P[C|D_1] \times P[D_1] + P[C|D_2] \times P[D_2] + \cdots + P[C|D_n] \times P[D_n],$$

for any event C and any partition of events $D_1, D_2, ..., D_n$

In this case, the event C is $X_B > X_A$, and the partition has 4 events:

D_1 : Company A has no claims and Company B has no claims
D_2 : Company A has no claims and Company B has some claims
D_3 : Company A has some claims and Company B has no claims
D_4 : Company A has some claims and Company B has some claims

The companies have independent claim amounts, so we can use the following rule for independent events: $P[U \cap V] = P[U] \times P[V]$.

The probabilities of the partition events are

$P[D_1] = P$[(no claims for company A) $\cap$ (no claims for company B)]
$= P$[no claims for company A] $\times P$[no claims for company B] $= 0.6 \times 0.7 = 0.42$,
$P[D_2] = 0.6 \times 0.3 = 0.18$, $P[D_3] = 0.4 \times 0.7 = 0.28$, $P[D_4] = 0.4 \times 0.3 = 0.12$.

Using the partition rule above, we have

$$P[X_B > X_A] = P[X_B > X_A|D_1] \times P[D_1] + P[X_B > X_A|D_2] \times P[D_2]$$
$$+ P[X_B > X_A|D_3] \times P[D_3] + P[X_B > X_A|D_4] \times P[D_4].$$

$P[X_B > X_A|D_1] = 0$, since in this case $X_A = X_B = 0$.

$$P[X_B > X_A|D_2] = P[Y_B > 0] = P\left[\frac{Y_B - 9{,}000}{2{,}000} > \frac{0 - 9{,}000}{2{,}000}\right] = P[Z > -4.5] = \Phi(4.5) = 1.$$

In this case $X_A = 0$ and $X_B = Y_B$ has a normal distribution, and we can standardize the probability; Z has a standard normal distribution (mean 0, standard deviation 1).

$$P[X_B > X_A|D_3 = P[Y_A < 0] = P\left[\frac{Y_A - 10{,}000}{2{,}000} < \frac{0 - 10{,}000}{2{,}000}\right]$$
$$= P[Z < -5] = \Phi(-5) = 1 - \Phi(5) = 0.$$

For trial use only.
Content may not
represent the most
current version