6. We first find the conditional distribution of Y given $X = 1$.
$P[Y=0|X=1] = \frac{P[X=1,Y=0]}{P[X=1]} = \frac{0.05}{P[X=1]}$, $P[Y=1|X=1] = \frac{0.125}{P[X=1]}$;
this requires $P[X=1] = P[X=1,Y=0] + P[X=1,Y=1] = 0.05 + 0.125 = 0.175.$

The conditional distribution of Y given $X = 1$ is
$P[Y=0|X=1] = \frac{P[X=1,Y=0]}{P[X=1]} = \frac{0.05}{0.175} = \frac{2}{7}$, $P[Y=1|X=1] = \frac{0.125}{0.175} = \frac{5}{7}$.

The conditional variance is $Var[Y|X=1] = E[Y^2|X=1] - (E[Y|X=1])^2$, where
$E[Y^2|X=1] = 0^2 \times \frac{2}{7} + 1^2 \times \frac{5}{7} = \frac{5}{7}$, $E[Y|X=1] = 0 \times \frac{2}{7} + 1 \times \frac{5}{7} = \frac{5}{7}$.

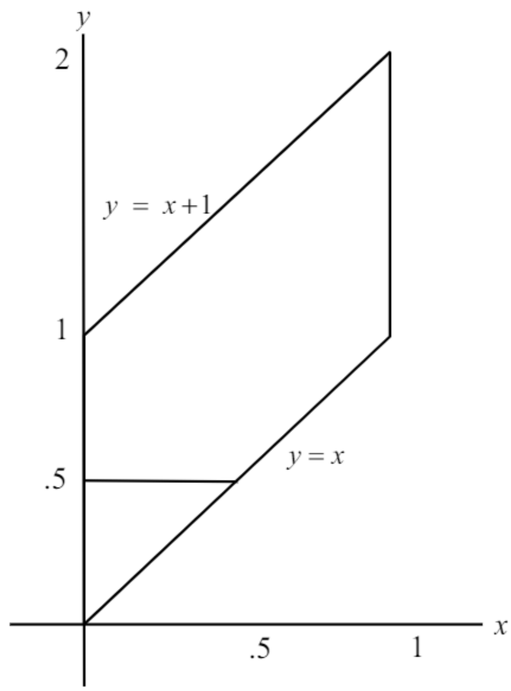
Then $Var[Y|X=1] = \frac{5}{7} - (\frac{5}{7})^2 = 0.204.$ Answer: C

7. The marginal distribution of X is found by summing probabilities over the other variable Y.
$P[X=0] = \sum_{y=0}^{2} P[X=0, Y=y] = \frac{1}{6} + 0 + 0 = \frac{1}{6}$
$P[X=1] = \sum_{y=0}^{2} P[X=1, Y=y] = \frac{1}{12} + \frac{1}{6} + 0 = \frac{1}{4}$
$P[X=2] = \sum_{y=0}^{2} P[X=2, Y=y] = \frac{1}{12} + \frac{1}{3} + \frac{1}{6} = \frac{7}{12}$
$E[X] = \sum_{x=0}^{2} x \times P[X=x] = 0 \times \frac{1}{6} + 1 \times \frac{1}{4} + 2 \times \frac{7}{12} = \frac{17}{12}$
$E[X^2] = \sum_{x=0}^{2} x^2 \times P[X=x] = 0^2 \times \frac{1}{6} + 1^2 \times \frac{1}{4} + 2^2 \times \frac{7}{12} = \frac{31}{12}$
$Var[X] = E[X^2] - (E[X])^2 = \frac{31}{12} - (\frac{17}{12})^2 = 0.576$ Answer: B

8. We wish to find $P[1 < Y < 3|X=2] = \int_1^3 f(y|X=2)\,dy = \int_1^3 \frac{f(2,y)}{f_X(2)}\,dy$
$f(2,y) = \frac{2}{2^2(2-1)} \times y^{-[(2)(2)-1]/(2-1)} = \frac{1}{2} y^{-3}$
$f_X(2) = \int_1^\infty f(2,y)\,dy = \int_1^\infty \frac{1}{2} y^{-3}\,dy = \frac{1}{4} \Rightarrow \frac{f(2,y)}{f_X(2)} = 2y^{-3}$
$P[1 < Y < 3|X=2] = \int_1^3 2y^{-3}\,dy = -y^{-2}\Big|_{y=1}^{y=3} = -\frac{1}{9} - (-1) = \frac{8}{9}$ Answer: E

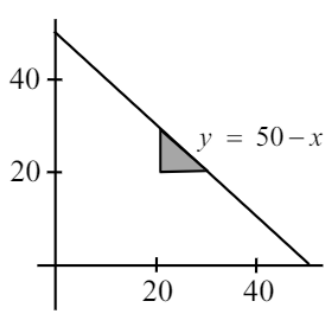
9. We are given that the marginal distribution of X is uniform on the interval $(0,1)$, so that $f_X(x) = 1$ for $0 < x < 1$. We are also given that the conditional distribution of Y given $X = x$ is uniform on the interval $x < y < x+1$, so that $f_{Y|X}(y|X=x) = 1$ for $x < y < x+1$. The density function for the joint distribution of X and Y is $f_{X,Y}(x,y) = f_{Y|X}(y|X=x) \times f_X(x) = 1$ on the region $0 < x < y < x+1 < 2$; this region of probability is the parallelogram in the graph below.
The event "damage to the other driver's car will be greater than 0.5" when an accident occurs is the event "$Y > 0.5$". This is the upper region of the parallelogram below. The probability is the double integral of the

joint distribution density on that region. We can also find the probability by first finding the probability of the lower triangular region and then taking the complement.



$P[Y > 0.5] = 1 - P[Y \leq 0.5]$.
$P[Y \leq 0.5] = \int_0^{0.5} \int_x^{0.5} 1\, dy\, dx = \int_0^{0.5} (0.5 - x)\, dx = \frac{1}{8} \Rightarrow P[Y > 0.5] = \frac{7}{8}$ Answer: D

10. The region of joint density is the region in the first quadrant below the line $y = 50 - x$ (with horizontal intercept 50 and vertical intercept 50). If X and Y denote the failure time of the two components, then the event that both components are still functioning 20 months from now has probability $P[(X > 20) \cap (Y > 20)]$. This is the shaded region in the graph at the right. From the graph it can be seen that the region of probability for this event is the triangular region bounded on the left by $x = 20$ and on the right by $x = 30$, and bounded below by $y = 20$, and bounded above by $y = 50 - x$.
The probability of the event is $\int_{20}^{30} \int_{20}^{50-x} f(x,y)\, dy\, dx$.



Answer: B

11. $Cov(X,Y) = E[XY] - E[X] \times E[Y]$.

We use the expression $E[X] = \int\int x \times f(x,y)\,dy\,dx$ to find $E[X]$. Since the region of probability is defined with $x \le y \le 2x$, we apply double integration in the $dy\,dx$ order. It would be possible to reverse the order, but that would not make the solution any more efficient. $E[Y]$ and $E[XY]$ are found in a similar way.

$$E[X] = \int_0^1 \int_x^{2x} x \times \tfrac{8}{3} \times xy\,dy\,dx = \tfrac{8}{3} \times \int_0^1 \int_x^{2x} x^2 y\,dy\,dx$$
$$= \tfrac{8}{3} \times \int_0^1 [\tfrac{x^2y^2}{2}\Big|_{y=x}^{y=2x}]\,dx = \tfrac{8}{3} \times \int_0^1 [\tfrac{3x^4}{2}]\,dx = \tfrac{8}{3} \times \tfrac{3}{2} \times \tfrac{1}{5} = \tfrac{4}{5}$$

$$E[Y] = \int_0^1 \int_x^{2x} y \times \tfrac{8}{3} \times xy\,dy\,dx = \tfrac{8}{3} \times \int_0^1 \int_x^{2x} xy^2\,dy\,dx$$
$$= \tfrac{8}{3} \times \int_0^1 [\tfrac{xy^3}{3}\Big|_{y=x}^{y=2x}]\,dx = \tfrac{8}{3} \times \int_0^1 [\tfrac{7x^4}{3}]\,dx = \tfrac{8}{3} \times \tfrac{7}{3} \times \tfrac{1}{5} = \tfrac{56}{45}$$

$$E[XY] = \int_0^1 \int_x^{2x} xy \times \tfrac{8}{3} \times xy\,dy\,dx = \tfrac{8}{3} \times \int_0^1 \int_x^{2x} x^2y^2\,dy\,dx$$
$$= \tfrac{8}{3} \times \int_0^1 [\tfrac{x^2y^3}{3}\Big|_{y=x}^{y=2x}]\,dx = \tfrac{8}{3} \times \int_0^1 [\tfrac{7x^5}{3}]\,dx = \tfrac{8}{3} \times \tfrac{7}{3} \times \tfrac{1}{6} = \tfrac{28}{27}$$

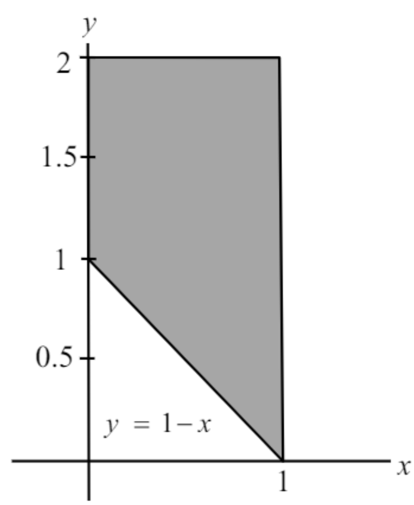
Then $Cov(X,Y) = \frac{28}{27} - \frac{4}{5} \times \frac{56}{45} = 0.041$. Answer: A

12. The range $x^2 \le y \le x$ is only valid for $0 \le x \le 1$. This is true since $x^2 > x$ for $x > 1$ and $x^2 > 0 > x$ for $x < 0$. Therefore, the range for y is $0 \le x^2 \le y \le x \le 1$, so that $0 \le y \le 1$. Also, the inequality $x^2 \le y$ is equivalent to $x \le \sqrt{y}$, so that $x^2 \le y \le x$ is equivalent to $y \le x \le \sqrt{y}$. The marginal density function of Y is found by integrating the joint density over the range for the other variable x;

$g(y) = \int_y^{\sqrt{y}} 15y\,dx = 15yx\Big|_{x=y}^{x=\sqrt{y}} = 15y(\sqrt{y} - y) = 15(y^{3/2} - y^2)$ for $0 \le y \le 1$

Note that it is true that for any particular x we have $x^2 \le y \le x$. However, since x can be any number from 0 to 1, y can also be any number from 0 to 1. Answer: E

13. We are asked to find $P[X + Y \ge 1]$. The joint distribution of X and Y is defined on the rectangle $0 < X < 1,\ 0 < Y < 2$. The region representing the probability in question is the region on and above the line $x + y = 1$ that is inside the rectangle. The probability is the double integral of the joint density function, integrated over the region of probability. This can be expressed as $\int_0^1 \int_{1-x}^2 f(x,y)\,dy\,dx = \int_0^1 \int_{1-x}^2 \frac{2x+2-y}{4}\,dy\,dx$ which becomes $\int_0^1 \frac{5x^2+6x+1}{8}\,dx = \frac{17}{24} = 0.71$.



Answer: D

14. $f_{Y|X}(y|X=x) = \frac{1}{x}$ for $0 < y < x$, $f_X(x) = \frac{1}{12}$ for $0 < x < 12$
$f_{X,Y}(x,y) = f_{Y|X}(y|X=x) \times f_X(x) = \frac{1}{12x}$ for $0 < y < x < 12$
Then $E[XY] = \int_0^{12}\int_0^x xy \times \frac{1}{12x}\,dy\,dx = \int_0^{12}\int_0^x \frac{y}{12}\,dy\,dx = 24.$
We are given that X has a uniform distribution on $(0,12)$, and therefore, $E[X] = 6$.
Also, $E[Y] = \int_0^{12}\int_0^x y \times \frac{1}{12x}\,dy\,dx = \int_0^{12}\frac{x}{24}dx = 3$ (note that if we had used the reverse order of integration then we would have $E[Y] = \int_0^{12}\int_y^{12} y \times \frac{1}{12x}\,dx\,dy = \int_0^{12}\frac{y}{12}\,ln(\frac{12}{y})\,dy$, which would require integration by parts). Finally, $Cov(X,Y) = 24 - 6 \times 3 = 6$. Answer: C

15. If we find the conditional density function $f_{Y|X}(y|X=\frac{1}{3})$, then
$P[Y < X|X = \frac{1}{3}] = P[Y < \frac{1}{3}|X = \frac{1}{3}] = \int_0^{1/3} f_{Y|X}(y|X=\frac{1}{3})\,dy.$
The conditional density is $f_{Y|X}(y|X=\frac{1}{3}) = \frac{f(\frac{1}{3},y)}{f_X(\frac{1}{3})}$.
The joint density is $f(\frac{1}{3},y) = 24(\frac{1}{3})y = 8y$, $0 < y < 1 - \frac{1}{3}$
and the marginal density of X at $X = \frac{1}{3}$ is $f_X(\frac{1}{3}) = \int_0^{2/3} 24 \times \frac{1}{3} \times y\,dy = \frac{16}{9}$.
The conditional density is $f_{Y|X}(y|X=\frac{1}{3}) = \frac{8y}{\frac{16}{9}} = \frac{9y}{2}$.
The conditional probability is $P[Y < X|X = \frac{1}{3}] = \int_0^{1/3}\frac{9y}{2}\,dy = \frac{1}{4}$. Answer: C

16. We first find the value of the constant k that makes $f(x,y)$ a properly defined density function for the joint distribution. The requirement that must be met is that the double integral of $f(x,y)$ over the x-y region of density must be 1. The x-y region of density is the square $0 < x < 1$, $0 < y < 1$. Thus,
$\int_0^1\int_0^1 kx\,dy\,dx = k\int_0^1 x\,dx = k \times \frac{1}{2} = 1$, from which we get $k = 2$ and $f(x,y) = 2x$.
We use following definition of $Cov(X,Y)$: $Cov(X,Y) = E[XY] - E[X] \times E[Y]$.
$E[XY] = \int_0^1\int_0^1 (xy)f(x,y)\,dy\,dx = \int_0^1\int_0^1 (xy)(2x)\,dy\,dx = \frac{1}{3}$,
$E[X] = \int_0^1\int_0^1 (x)f(x,y)\,dy\,dx = \int_0^1\int_0^1 (x)(2x)\,dy\,dx = \frac{2}{3}$, and
$E[Y] = \int_0^1\int_0^1 (y)f(x,y)\,dy\,dx = \int_0^1\int_0^1 (y)(2x)\,dy\,dx = \frac{1}{2}$.
Then, $Cov(X,Y) = \frac{1}{3} - \frac{2}{3} \times \frac{1}{2} = 0$.
There is another way that this covariance of 0 could have been found.
The density function of the marginal distribution of X is $f_X(x) = \int_0^1 2x\,dy = 2x$ for $0 < x < 1$, and the density function of the marginal distribution of Y is $f_Y(y) = \int_0^1 2x\,dx = 1$ for $0 < y < 1$.
We can then see that $f(x,y) = 2x = 2x \times 1 = f_X(x) \times f_Y(y)$, which indicates that X and Y are independent. If two random variables are independent, then they have covariance of 0. Answer: B

17. The distribution of the number of tornadoes in county Q given there are none in county P is

$P[Q=n|P=0]=\frac{P[(Q=n)\cap(P=0)]}{P[P=0]}$, for $n=0,1,2,3$.

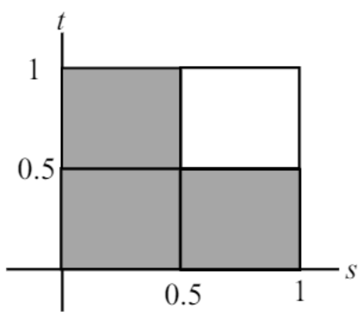
The denominator is

$$P[P=0]=P[(Q=0)\cap(P=0)]+P[(Q=1)\cap(P=0)]+P[(Q=2)\cap(P=0)]$$
$$+P[(Q=3)\cap(P=0)]=0.12+0.06+0.05+0.02=0.25.$$

Then, $P[Q=0|P=0]=\frac{P[(Q=0)\cap(P=0)]}{P[P=0]}=\frac{0.12}{0.25}=0.48$, $P[Q=1|P=0]=0.24$,
$P[Q=2|P=0]=0.20$, $P[Q=3|P=0]=0.08$.

Then, $E[Q|P=0]=0\times 0.48+1\times 0.24+2\times 0.2+3\times 0.08=0.88$
$E[Q^2|P=0]=0^2\times 0.48+1^2\times 0.24+2^2\times 0.2+3^2\times 0.08=1.76$, and
$Var[Q|P=0]=E[Q^2|P=0]-(E[Q|P=0])^2=1.76-(0.88)^2=0.9856$. Answer: D

18. Suppose that the times of failure of the two devices are X and Y.
We wish to find the probability that at least one failure occurs by time 0.5.
This is $P[(X<0.5)\cup(Y<0.5)]$. The region of probability is the shaded in the graph below. The probability is the double integral over region E (let t be the horizontal, s vertical).
The first integral corresponds to the square $0.5<s<1$, $0<t<0.5$, and the second integral corresponds to the rectangle $0<s<0.5$, $0<t<1$.



Answer: E

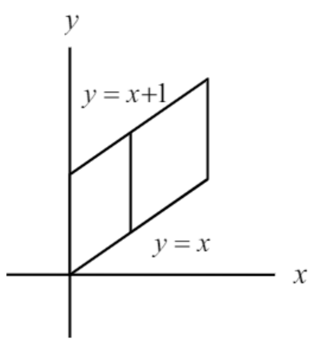
19. We must have $Y\le X$ since no more than proportion X buy the supplementary policy. The region of positive joint density is the triangular region $0\le y\le x\le 1$ (below the line $y=x$, inside the unit square). We wish to find the conditional probability $P[Y<.05|X=.1]$. The conditional density for Y given $X=0.1$ is $f_{Y|X}(y|X=0.1)=\frac{f(0.1,y)}{f_X(0.1)}$, where $f_X(0.1)$ is the density function of the marginal distribution of X at 0.1. In general, $f_X(x)=\int_{-\infty}^{\infty}f(x,y)\,dy$ gives the density of the marginal distribution of X from a joint distribution. In this case, based on where the joint density is non-zero, we have
$f_X(0.1)=\int_0^{0.1}f(0.1,y)\,dy=\int_0^{0.1}2(0.1+y)\,dy=0.03.$
Then, $f_{Y|X}(y|X=0.1)=\frac{f(0.1,y)}{f_X(0.1)}=\frac{2(0.1+y)}{0.03}$, $y<0.1$ (since $Y<X$), and
$P[Y<0.05|X=0.1]=\int_0^{0.05}\frac{2(0.1+y)}{0.03}\,dy=0.4167$. Answer: D

20. To find the conditional variance of Y given that $X = x$, we must find the density function of the conditional distribution of Y given $X = x$. This is $f_{Y|X}(y|X = x) = \frac{f(x,y)}{f_X(x)}$. We must find the density function of the marginal distribution of X. This is found by integrating the joint distribution density with respect to y over the appropriate region:

$$f_X(x) = \int_x^{x+1} 2x\, dy = 2x \text{ for } 0 < x < 1$$

Then, $f_{Y|X}(y|X = x) = \frac{2x}{2x} = 1$ and this conditional density is valid for $x < y < x + 1$.

Therefore, the conditional distribution of Y given $X = x$ is uniform on the interval $x < y < x + 1$ (a uniform distribution has a constant density). The variance of the continuous uniform distribution on an interval of length 1 is $\frac{1}{12}$.



Answer: A

21. $F_X(t) = P[X \le t] = P\left[\frac{X-\mu_X}{\sigma} \le \frac{t-\mu_X}{\sigma}\right] = P\left[W \le \frac{t-\mu_X}{\sigma}\right]$, where $W \sim N(0,1)$
$F_Y(t) = P[Y \le t] = P\left[\frac{Y-\mu_Y}{\sigma} \le \frac{t-\mu_Y}{\sigma}\right] = P\left[V \le \frac{t-\mu_Y}{\sigma}\right]$, where $V \sim N(0,1)$
$F_X(t) \ge F_Y(t)$ is equivalent to $\frac{t-\mu_X}{\sigma} \ge \frac{t-\mu_Y}{\sigma}$, which is equivalent to $\mu_X \le \mu_Y$

Note that the fact the X and Y have a bivariate distribution with correlation coefficient ρ_{XY} is irrelevant - we are comparing probabilities of the marginal distributions of X and Y (however, we do use the fact that X and Y have common variance σ^2). Answer: B

22. The moment generating function of X_1 and X_2 is
$M(t_1, t_2) = E[e^{t_1X_1+t_2X_2}] = \int_0^1\int_0^1 e^{t_1x_1+t_2x_2}dx_1\, dx_2 = \frac{(e^{t_1}-1)(e^{t_2}-1)}{t_1t_2}$. Answer: B

23. The moment generating function for X is $M_X(t_1) = M(t_1, 0) = \frac{1}{3} + \frac{2}{3}e^{t_1}$.
Then,

$$E[X] = M'_X(0) = \frac{2}{3} \text{, and } E[X^2] = M''_X(0) = \frac{2}{3}$$

so that

$$Var[X] = \frac{2}{3} - (\frac{2}{3})^2 = \frac{2}{9}$$

Answer: D

24. We wish to find $P[X<0.2]$. The region of density for the joint distribution is below the line $x+y=1$. The region of probability for the event in question is shaded below. he probability is found by integrating the joint density over that region.

$$P[X<0.2]=\int_0^{0.2}\int_0^{1-x}6[1-(x+y)]\,dy\,dx=\int_0^{0.2}3(1-x)^2\,dx=0.488.$$

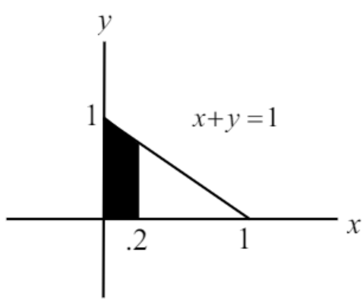
It would also be possible to solve this problem by first finding the marginal distribution of X, and then find $P[X<0.2]$. The density function of the marginal distribution of X is found by integrating the joint density in the "y-direction" over the appropriate range.

Since $x+y<1$ is equivalent to $y<1-x$, the appropriate range for integration over y is from $y=0$ to $y=1-x$. Therefore, $f_X(x)=\int_0^{1-x}f(x,y)\,dy=\int_0^{1-x}6[1-(x+y)]\,dy=3(1-x)^2$

This is exactly the "inside integral" in the double integration above.

Then, $P[X<0.2]=\int_0^{0.2}3(1-x)^2\,dx=0.488$, as before.

This second approach is essentially identical to the first approach.



Answer: C

25. $P[Y<X]=\int_0^{\infty}\int_y^{\infty}f_X(x)f_Y(y)\,dx\,dy$ (since X and Y are independent, the joint density function of X and Y is the product of the two separate density functions). The density function of X is $\frac{1}{\alpha}e^{-x/\alpha}$, and of Y is $\frac{1}{\beta}e^{-x/\beta}$, so that

$P[Y<X]=\int_0^{\infty}\int_y^{\infty}\frac{1}{\alpha}e^{-x/\alpha}\frac{1}{\beta}e^{-y/\beta}dx\,dy=\int_0^{\infty}\frac{1}{\beta}e^{-y/\beta}e^{-y/\alpha}\,dy=\frac{\frac{1}{\beta}}{\frac{1}{\alpha}+\frac{1}{\beta}}=\frac{\alpha}{\alpha+\beta}$ Answer: A

26. The marginal density function of X is $f_X(x)=\int_0^{2-x}\frac{3}{4}(2-x-y)\,dy=\frac{3}{8}(2-x)^2$, and then $E[X]=\int_0^2 x\times\frac{3}{8}(2-x)^2dx=.5$. In a similar way, the marginal density functio n of Y is $f_Y(y)=\int_0^{2-y}\frac{3}{4}(2-x-y)\,dx=\frac{3}{8}(2-y)^2$, and then $E[Y]=\int_0^2 y\times\frac{3}{8}(2-y)^2dy=0.5$ (this could be anticipated from the symmetry of x and y in the joint density function). Then, $E[X+Y]=1$.

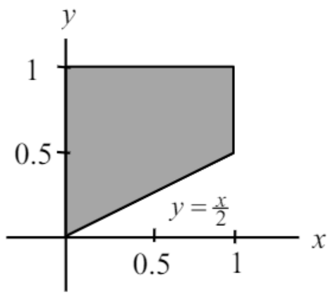
Alternatively, we can find

$$E[X+Y]=\int_0^2\int_0^{2-x}(x+y)\times\frac{3}{4}(2-x-y)\,dy\,dx=1$$ Answer: C

27. The marginal density function of Y is $f_Y(y)=\int_0^y(6xy+3x^2)dx=4y^3$, and then $P[Y<0.5]=\int_0^{0.5}4y^3\,dy=(0.5)^4$. Answer: C

28. The shaded region in the graph below corresponds to the event that $X < 2Y$. The probability is

$$P[X<2Y] = P[\tfrac{X}{2}<Y] = \int_0^1\int_{x/2}^1 (x+y)\,dy\,dx = \int_0^1 [x(1-\tfrac{x}{2})+\tfrac{1}{2}(1-\tfrac{x^2}{4})]dx = \frac{19}{24}$$



Answer: D

29. The device fails when the second circuit fails, which is at time Y. We wish to find $E[Y]$.

This is $E[Y] = \int_0^\infty\int_x^\infty y\times 6e^{-x}e^{-2y}\,dy\,dx = \int_0^\infty 6e^{-x}(\int_x^\infty y\times e^{-2y}\,dy)dx$.

We find $\int_x^\infty y\times e^{-2y}\,dy$ by integration by parts :

$$\int_x^\infty y\times e^{-2y}\,dy = \int_x^\infty y\,d(-\tfrac{1}{2}e^{-2y}) = -\tfrac{1}{2}ye^{-2y}\Big|_{y=x}^{y=\infty} - \int_x^\infty(-\tfrac{1}{2}e^{-2y})dy$$
$$= -0+\tfrac{1}{2}xe^{-2x}+\tfrac{1}{4}e^{-2x}$$

Then, $E[Y] = \int_0^\infty 6e^{-x}(\frac{1}{2}xe^{-2x}+\frac{1}{4}e^{-2x})\,dx = \int_0^\infty(3xe^{-3x}+\frac{3}{2}e^{-3x})\,dx$.

From integration by parts, we get $\int_0^\infty 3xe^{-3x}\,dx = \int_0^\infty x\,d(-e^{-3x}) = -\frac{1}{3}xe^{-3x}\Big|_{x=0}^{x=\infty} = \frac{1}{3}$.

Also, $\int_0^\infty \frac{3}{2}e^{-3x}\,dx = \frac{1}{2}$, so that $E[Y] = \frac{1}{3}+\frac{1}{2} = \frac{5}{6}$.

Alternatively, we can find the pdf of the marginal distribution of Y first:

$f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 6e^{-x}e^{-2y}\,dx = 6(1-e^{-y})e^{-2y} = 6(e^{-2y}-e^{-3y})$, for $0<y<\infty$.

Then, $E[Y] = \int_0^\infty y\cdot 6(e^{-2y}-e^{-3y})\,dy$. After integration by parts, this becomes $\frac{5}{6}$.
Answer: D

30. $E[X] = \int_2^{10}\int_0^1 x \times \frac{1}{64}(10 - xy^2)\,dy\,dx$

The "inside" integral is

$\int_0^1 x \times \frac{1}{64}(10 - xy^2)\,dy = \frac{1}{64} \times \int_0^1 (10x - x^2y^2)\,dy = \frac{1}{64} \times 10x - \frac{x^2}{3}]$

The complete integral is $\int_2^{10} \frac{1}{64} \times [10x - \frac{x^2}{3}]\,dx = \frac{1}{64} \times [(5x^2 - \frac{x^3}{9})\Big|_{x=2}^{x=10}] = 5.8.$

Note that we could have found $f_X(x)$, the marginal density function of X first and the have found $E[X]$.

This would be done as follows:

$f_X(x) = \int_0^1 \frac{1}{64}(10 - xy^2)\,dy = \frac{1}{64}(10 - \frac{x}{3})$, and then

$$E[X] = \int_2^{10} x \times f_X(x)\,dx = \int_2^{10} x \times \frac{1}{64}(10 - \frac{x}{2})\,dx = \frac{1}{64} \times \int_2^{10}(10x - \frac{x^2}{3})\,dx$$
$$= \frac{1}{64} \times (5x^2 - \frac{x^3}{9})\Big|_{x=2}^{x=10}] = 5.8 \text{ (as in the first approach).}$$

Answer: C

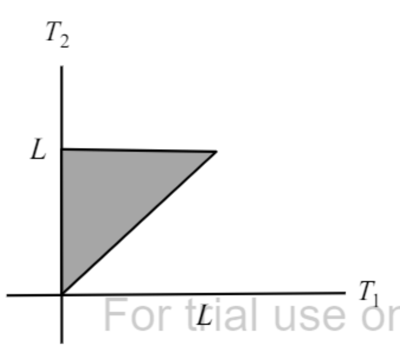
31. $P[X + Y + Z = 2|X = 0] = \frac{P[(Y+Z=2)\cap(X=0)]}{P[X=0]}$.

$P[X = 0] = \sum_{y=0}^{2}\sum_{z=0}^{2} f(0, y, z) = \frac{1}{81}[(6-0) + (6-1) + (6-2)$
$+ (6-1) + (6-2) + (6-3) + (6-2) + (6-3) + (6-4)] = \frac{36}{81}.$

$P[(Y + Z = 2) \cap (X = 0)] = f(0,0,2) + f(0,1,1) + f(0,2,0) = \frac{4}{81} + \frac{4}{81} + \frac{4}{81}.$

$P[X + Y + Z = 2|X = 0] = \frac{12/81}{36/81} = \frac{1}{3}.$ Answer: B

32. The graph at the right indicates the region of non-zero density for the joint distribution of T_1 and T_2. The expected value is $\int_0^L \int_0^{t_2} (t_1^2 + t_2^2)\, f(t_1, t_2)\,dt_1\,dt_2$. We are told that the joint distribution is uniform over the triangular region, and therefore the joint density function $f(t_1, t_2)$ is constant over the region and numerically equal to $\frac{1}{\text{area of region}} = \frac{2}{L^2}$ (half of the $L \times L$ square). The expected value is $\int_0^L \int_0^{t_2} (t_1^2 + t_2^2) \times \frac{2}{L^2}\,dt_1\,dt_2 = \frac{2}{L^2} \times \int_0^L \frac{4t_2^3}{3}\,dt_2 = \frac{2L^2}{3}.$



Answer: C

33. The company considers the two bids further if the two bids are within 20 of one another. If we let X be the amount of the first bid and Y the amount of the second bid, then the (x, y) region for which the company will consider the bids further satisfies $x - 20 < y < x + 20$. This is the complement of the union of the two regions $(y \le x - 20) \cup (y \ge x + 20)$. We are told that both X and Y are uniformly distributed between 2000 and 2200, so that

$$f_X(x) = \frac{1}{200} = 0.005 \ \text{ for } \ 2000 \le x \le 2200$$

and $$f_Y(y) = \frac{1}{200} = 0.005 \ \text{ for } \ 2000 \le y \le 2200$$

Since X and Y are independent, the density function of the joint distribution is

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) = (0.005)^2$$

for

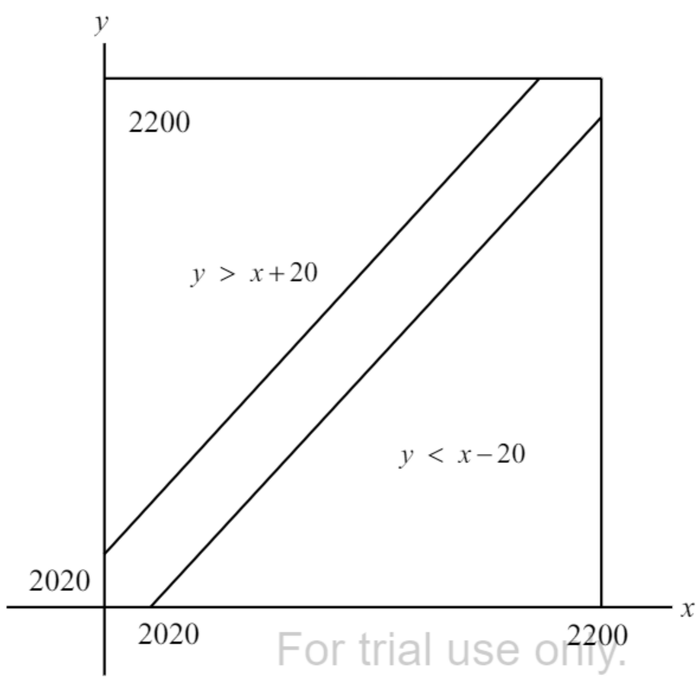
$$2000 \le x \le 2200 \ \text{ and } \ 2000 \le y \le 2200$$

The probability of a two-dimensional region is the double integral of the joint density over the region. Since the joint density is constant, the probability is the region of the area multiplied by that constant.

The region $y \le x - 20$ (and inside the square) has probability $\frac{1}{2} \times 180 \times 180 \times (.005)^2 = 0.405$ (since the triangular region below the lower line has base 180 and height 180).

The region $y \ge x + 20$ also has probability $\frac{1}{2} \times 180 \times 180 \times (0.005)^2 = 0.405$ (same size triangle). Therefore,

P[the two bids are within 20 of one another]
$= P[X - 20 < Y < X + 20] \ = \ 1 - 0.405 - 0.405 \ = \ 0.19.$



Answer: B

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34. Conditional expectations are usually found as follows.

$E[W|W>a] = \frac{\int_a^\infty w\, f_W(w)\, dw}{P[W>a]}$ (with summation used in the discrete case), with a similar formulation for $E[W|W<a]$ or $E[W|a<W<b]$. In the specific case that W is a non-negative random variable ($W \geq 0$), we have $E[W|W>0] = \frac{\Sigma w\, f(w)}{P[W>0]}$ (in the discrete case), and notice that the numerator is $E[W]$ (unconditional expectation), so that $E[W|W>0] = \frac{E[W]}{P[W>0]}$. In this case,

$E[L] = 0 \times 0.9 + 500 \times 0.06 + \cdots + 100,000 \times 0.001 = 290$, and $P[L>0] = 0.1$, so that the conditional expectation becomes $E[L|L>0] = \frac{290}{0.1} = 2900$.

The problem can be solved in an alternative way. We first determine the conditional distribution of L given that $L>0$ (where L denotes that amount of the loss). L has a discrete distribution, and the probability function of L given that $L>0$ is found from the following relationship.

For $x>0$, $P[L=x|L>0] = \frac{P[L=x]}{P[L>0]} = \frac{P[L=x]}{0.1} = 10P[L=x]$.

The conditional distribution of L given $L>0$ is

x	$P[L=x\|L>0]$
500	0.600
$1,000$	0.300
$10,000$	0.080
$50,000$	0.010
$100,000$	0.010

The expected amount of the loss given that the loss is greater than 0 is the expectation of this conditional distribution of L given that $L>0$. This expectation is

$$E[L|L>0] = 500 \times .6 + 1,000 \times .3 + 10,000 \times .08$$
$$+ 50,000 \times .01 + 100,000 \times .01 = 2,900$$

Answer: D

35. Let X and Y denote the two loss amounts (<u>not</u> payment amounts). We consider the following combinations of X and Y that result in the total benefit <u>payment</u> not exceeding 5.

Case 1: $0<X\leq 1$ (so loss X results in no payment) and $0<Y\leq 7$ (so that loss Y results in a maximum payment of 5 after applying the deductible of 2).

Case 2: $1<X\leq 6$ (so loss X results in a maximum payment of 5 after the deductible of 1 is applied) and $0<Y\leq 2$ (so loss Y results in no payment).

Case 3: $1<X\leq 6$ and $2\leq Y\leq 7$ and $(X-1)+(Y-2)\leq 5$ ($X-1$ is paid for loss X and $Y-2$ is paid for loss Y). The last condition is equivalent to $X+Y\leq 8$. The probability that the total benefit paid does not exceed 5 is the sum of the probabilities for Cases 1, 2 and 3.

$P[\text{Case 1}] = P[(0 < X \le 1) \cap (0 < Y \le 7)] = P[0 \le X \le 1] \times P[0 < Y \le 7] = \frac{1}{10} \times \frac{7}{10} = 0.07$

(we have used the independence of X and Y to find the probability of the intersection)

$$\begin{aligned} P[\text{Case 2}] &= P[(1 < X \le 6) \cap (0 < Y \le 2)] \\ &= P[1 < X \le 6] \times P[0 < Y \le 2] = \tfrac{5}{10} \times \tfrac{2}{10} = 0.10\,. \end{aligned}$$

$$\begin{aligned} P[\text{Case 3}] &= \int_1^6 \int_2^{8-x} f(x,y)\,dy\,dx = \int_1^6 \int_2^{8-x} f_X(x) \times f_Y(y)\,dy\,dx \\ &= \int_1^6 \int_2^{8-x} \tfrac{1}{10} \times \tfrac{1}{10}\,dy\,dx = \int_1^6 \int_2^{8-x} (0.01)\,dy\,dx \\ &= 0.01 \times \int_1^6 [8 - x - 2]\,dx = .01 \times (6x - \tfrac{x^2}{2}\Big|_{x=1}^{x=6}) = 0.125 \end{aligned}$$

(note that $f(x,y) = f_X(x) \times f_Y(y)$ because X and Y are independent).

The total probability is $0.07 + 0.10 + 0.125 = 0.295$.

Once we have identified Cases 1, 2 and 3, this problem could be approached from a graphical point of view. Since X and Y are independent and uniform, the joint distribution of X and Y is uniform on the square $0 < x < 10$, $0 < y < 10$, with joint density $0.1 \times 0.1 = 0.01$.

Since the joint distribution is uniform, the probability of any event involving X and Y is equal to the constant density (0.01 in this case) multiplied by the area of the region representing the event.

The three regions for Cases 1, 2 and 3 are indicated in the graph below.

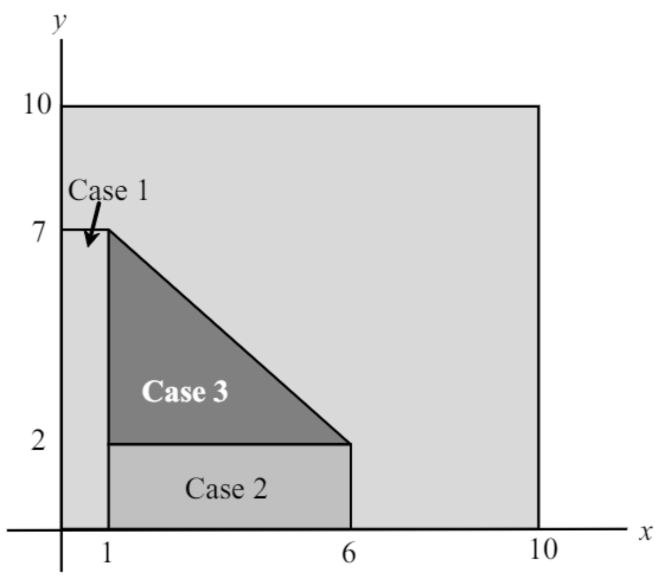
The 10×10 square is the full region for the joint distribution.

The rectangular area for Case 1 is $1 \times 7 = 7$ for a probability of $7 \times 0.01 = 0.07$.

The rectangular area for Case 2 is $5 \times 2 = 10$ for a probability of $10 \times 0.01 = 0.10$.

The triangular are for Case 3 is $\frac{1}{2} \times 5 \times 5 = 12.5$ for a probability of $12.5 \times 0.01 = 0.125$.

The total probability for Cases 1, 2 and 3 combined is again 0.295.



Answer: C

36. $p_X(x) = \sum_y p(x,y)$. $p_X(0) = p(0,1) + p(0,2) = \frac{1}{12} + \frac{2}{12} = \frac{1}{4}$,
$p_X(1) = p(1,2) + p(1,3) = \frac{4}{12} + \frac{5}{12} = \frac{3}{4}$, and $p_X(x) = 0$, otherwise. Answer: B

37. The new amount paid to the surgeon is $X' = X + 100$, and the new amount of hospital charges is $Y' = 1.1Y$. We wish to find
$Var[X' + Y'] = Var[X'] + Var[Y'] + 2Cov(X', Y')$.
$Var[X'] = Var[X + 100] = Var[X] = 5,000$, and
$Var[Y'] = Var[1.1Y] = 1.1^2 \times Var[Y] = 1.21 \times 10,000 = 12,100$.
$Cov(X', Y') = Cov(X + 100, 1.1Y) = 1.1 \times Cov(X, Y)$.
We have used the covariance rule $Cov(aU + b, cW + d) = acCov(U, W)$.
We still must know $Cov(X, Y)$ to complete the problem.
We are given $Var[X + Y] = 17,000$, and we use the relationship
$$17,000 = Var[X+Y] = Var[X] + Var[Y] + 2Cov(X,Y)$$
$$= 5,000 + 10,000 + 2Cov(X,Y) \rightarrow Cov(X,Y) = 1,000.$$

Then $Cov(X', Y') = 1.1Cov(X, Y) = 1,100$.
Finally, $Var[X' + Y'] = Var[X'] + Var[Y'] + 2Cov(X', Y')$
$= 5,000 + 12,100 + 2(1,100) = 19,300$. Answer: C

38. $C_1 = X + Y$, $C_2 = X + 1.2Y$. We use the following rules:
$Cov(U, U) = Var(U)$
$$Cov(aU + bV + c, dS + eT + f)$$
$$= adCov(U,S) + aeCov(U,T) + bdCov(V,S) + beCov(V,T)$$
and $Cov(U, V) = Cov(V, U)$

Then, $Cov(C_1, C_2) = Cov(X + Y, X + 1.2Y)$
$$= Cov(X,X) + 1.2Cov(X,Y) + Cov(Y,X) + 1.2Cov(Y,Y)$$
$$= Var(X) + 2.2Cov(X,Y) + 1.2Var(Y)$$

From the given information, we have $Var(X) = E(X^2) - [E(X)]^2 = 2.4$
$Var(Y) = E(Y^2) - [E(Y)]^2 = 2.4$. Also,
$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
$\rightarrow 8 = 2.4 + 2.4 + 2Cov(X, Y) \rightarrow 1.6$
Then, $Cov(C_1, C_2) = 2.4 + 2.2(1.6) + 1.2(2.4) = 8.8$. Answer: A

39. $Cov(X, X + cY) = Cov(X, X) + cCov(X, Y) = Var[X] + cCov(X, Y) = 25 - 10c$
This is set equal to $Cov(X, Z) = 2.5$, so that $25 - 10c = 2.5 \rightarrow c = 2.25$. Answer: B

40. Distribution of T given claim amount $X = x$ is uniform on interval $(x, 2x)$ and has pdf $f_{T|X}(t|X = x) = \frac{1}{x}$ for $x < t < 2x$. The pdf of X is $f_X(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$.

The density function of the joint distribution between T and X is
$f_{X,T}(x,t) = f_{T|X}(t|X = x) \times f_X(x) = \frac{1}{x} \times \frac{3}{8}x^2 = \frac{3}{8}x$ for $0 \leq x < t < 2x \leq 4$
(since $x \leq 2$, it follows that $2x \leq 4$).

The event $T > 3$ is illustrated in the graph below. In order to have $T > 3$, it must be true that $x \geq 1.5$, since if $x < 1.5$ then $t < 2x < 3$. Thus, the region of probability for the event $T > 3$ is $1.5 \leq x \leq 2$, and $3 < t < 2x$. The probability is

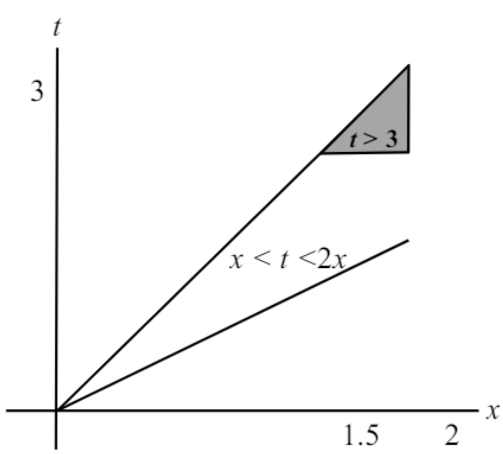
$$P[T > 3] = \int_{1.5}^{2}\int_{3}^{2x} f_{X,T}(x,t)\,dt\,dx = \int_{1.5}^{2}\int_{3}^{2x} \frac{3}{8}x\,dt\,dx = \int_{1.5}^{2} \frac{3}{8}x(2x-3)\,dx = \frac{11}{64}.$$

Alternatively, we can express the conditional probability $P[T > 3|X = x]$ as

$$P[T > 3|X = x] = \begin{cases} 0 & \text{if } x \leq 1.5 \text{ (since then } 2x \leq 3) \\ \frac{2x-3}{x} & \text{if } 1.5 < x \leq 2 \end{cases}$$

Then, $P[T > 3] = \int_{1.5}^{2} P[T > 3|X = x] \cdot f_X(x)\,dx = \int_{1.5}^{2} \frac{2x-3}{x} \cdot \frac{3x^2}{8}\,dx = \frac{11}{64}$.

The graph of the probability region is below.



Answer: A

41. $T_B \sim$ exponential mean 2, pdf $f_B(s) = \frac{1}{2}e^{-s/2}$.
$T_D \sim$ exponential mean 3, $f_D(t) = \frac{1}{3}e^{-t/3}$.
Since T_B and T_D are independent, the joint density is
$f_{B,D}(s,t) = f_B(s) \times f_D(t) = \frac{1}{2}e^{-s/2} \times \frac{1}{3}e^{-t/3}$, and
$P[T_D < T_B] = \int_0^\infty \int_t^\infty (\frac{1}{2}e^{-s/2})(\frac{1}{3}e^{-t/3})\, ds\, dt \ = \int_0^\infty \frac{1}{3}e^{-5t/6}\, dt = 0.4.$

A more general reasoning approach to the solution is the following.
In the next 6 days we expect 3 Basic claims (one every 2 days) and 2 Deluxe claims (one every 3 days). Of the next 5 claims, there is a $\frac{2}{5} = 0.4$ chance that it is from a Deluxe policy on average.
Answer: C

42. We first note that X and Y are independent. This is true because the joint density can be factored in a function of x alone multiplied by a function of y alone. Another way to verify independence is to note that the marginal density of X is

$$f_X(x) = \int_0^\infty f(x,y)\, dy = \int_0^\infty 2e^{-(x+2y)}\, dy = e^{-x} \text{ for } x > 0,$$

and the marginal density of Y is

$$f_Y(y) = \int_0^\infty f(x,y)\, dx = \int_0^\infty 2e^{-(x+2y)}\, dx = 2e^{-2y} \text{ for } y > 0\,.$$

Then, since $f(x,y) = f_X(x) \times f_Y(y)$ and since the region of density is rectangular (the entire first quadrant), it follows that X and Y are independent. Since they are independent, the variance of Y does not depend on X, so the conditional variance of Y given X is the same as the variance of Y, and the conditional variance of Y given that $X > 3$ and $Y > 3$ is the same as the conditional variance of Y given that $Y > 3$. The conditional density of Y given that $Y > 3$ is

$$f(y|Y > 3) = \frac{f_Y(y)}{P(Y>3)} \text{ for } y > 3$$
$$P(Y > 3) = \int_3^\infty 2e^{-2y}\, dy = e^{-6}$$

so

$$f(y|Y > 3) = \frac{2e^{-2y}}{e^{-6}} \text{ for } y > 3\,.$$

If we make the change of variable $z = y - 3$
then this becomes $f(z|Z > 0) = \frac{2e^{-2(z+3)}}{e^{-6}} = 2e^{-2z}$ for $z > 0$.
This is an exponential density with a mean of $\frac{1}{2}$. The variance of an exponential distribution is the square of the mean, which is $\frac{1}{2^2} = \frac{1}{4}$.

Another point to note, once we have determined that the marginal distribution of Y is exponential with mean 0.5, because of the "lack-of-memory" property of the exponential distribution, the conditional distribution of Y given $Y > a$ is still exponential with mean 0.5, for any $a > 0$.

Answer: A.

43. The conditional density of Y given X is $f_{Y|X}(y|x) = \frac{1}{x}$ on the interval $[0, x]$ (uniform).

The region of non-zero joint density of X and Y is on the region $0 < y < x < 1$, since Y has non-zero density only on the interval $[0, x]$. The joint density of X and Y on that region is

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) \times f_X(x) = \frac{1}{x} \times 2x = 2 \text{ for } 0 < x < y < 1.$$

We can summarize the joint density function as $f_{X,Y}(x, y) = \begin{cases} 2 & \text{if } 0 < y < x < 1 \\ 0 & \text{if } 0 < x < y < 1 \end{cases}$

(note that we can ignore the region $y = x$, since it has area 0 in the two-dimensional region of probability). This is a (joint) uniform distribution on the triangular region $0 < x < y < 1$. The marginal density of Y is $f_Y(y) = \int_0^1 f_{X,Y}(x, y)\, dx = \int_0^y 0\, dx + \int_y^1 2\, dx = 2(1 - y)$, and this is defined on the region $0 < y < 1$. This is true because $f(x, y) = 0$ for $0 < x < y < 1$ and $f(x, y) = 2$ for $0 < y < x < 1$.

Then the conditional density of X given $Y = y$ is

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$ for $y < x < 1$, and 0 otherwise.

It is also true in general that i the joint distribution of X and Y is uniform (has constant density) on a region, then the conditional density of Y given X, or of X given Y will be uniform on the appropriate region of non-zero density. Once we have determined that the joint density of X and Y is 2 (constant, and therefore uniform) on the region $0 < y < x < 1$, we know that the conditional density of X given Y will be constant on the interval of definition. The interval of definition for X given Y is $y < x < 1$, which has length $1 - y$. Therefore, the conditional density of X given Y is the constant $\frac{1}{1-y}$ (the density of a uniform is $\frac{1}{\text{interval length}}$).

Answer: E

44. The key point to note in this problem is that we are given the cdf of survival for someone born in the same year as the insured. This is not the cdf of survival for a 40-year old, it is the cdf of survival for a newborn. If we define X to be the time until death for the 40-year old, then the distribution of X is the conditional distribution of T (time from birth until death) given that $T > 40$ (given survival from birth to age 40). The expected payment is

$$5000 \times P(X < 10) = 5000 \times P(T < 50|T > 40) = 5000 \times \frac{P(40<T<50)}{P(T>40)}$$

$$= 5000 \times \frac{F_T(50) - F_T(40)}{1 - F_T(40)} = 5000 \times 0.0696 = 348$$

Answer: B

45. We use the conditioning formula for variance.

$$Var[N] = E[\,Var[N|\lambda]\,] + Var[\,E[N|\lambda]\,]$$

Since the distribution of workplace accidents is Poisson with mean λ, we have

$$E[N|\lambda] = \lambda \text{ and } Var[N|\lambda] = \lambda$$

Then, since the distribution of λ is uniform on the interval $[0, 3]$, we have $E[\,Var[N|\lambda]\,] = E[\lambda] = 1.5$, and $Var[\,E[N|\lambda]\,] = Var[\lambda] = \frac{3^2}{12} = 0.75$ (the variance of a uniform random variable is the square of the interval length divided by 12).

Then $Var[N] = 1.5 + 0.75 = 2.25$. Answer: E

46. Since $Y = 2$, a 5 can be rolled on the first roll or the third or later roll, and also there is no 6 on the first roll. Given that there is no 6 on the first roll, the probability of a 5 on the first roll is $\frac{1}{5}$. The number of rolls until a 5 appears has a geometric distribution with probability function $P(X = n) = \frac{1}{6} \times (\frac{5}{6})^{n-1}$, with mean $\frac{1}{1/6} = 6$. If the first 5 does not appear on the first roll, then it will appear on the third or later roll, so the expected number of rolls needed to roll a 5 given that a 5 did not occur on the first two rolls is $2 + 6 = 8$. The overall expected number of rolls until a 5 given that the first 6 is on the second roll is $1 \times 0.2 + 8 \times 0.8 = 6.6$.
Answer: D

47. If W is a discrete random variable with probability function $P(W = k) = p_k$, then the moment generating function of W is $M_W(t) = \Sigma p_k e^{-kt}$. Also, the moment generating function of the sum of independent random variables is the product of the separate moment generating functions. Then $M_{X+Y}(t) = M_X(t) \times M_Y(t)$. Since X and Y are identically distributed, we have $M_X(t) = M_Y(t)$, so $0.09e^{-2t} + 0.24e^{-t} + 0.34 + 0.24e^{t} + 0.09e^{2t} = [M_X(t)]^2$.

Algebraically, we see that $M_X(t) = 0.3e^{-t} + 0.4 + 0.3e^{t}$, so that the distribution of X (and Y) is $P(X = -1) + 0.3$, $P(X = 0) = 0.4$, $P(X = 1) = 0.3$. Then $P(X \leq 0) = 0.7$.
Answer: E

48. The marginal density for X at 0,75 is

$$f_X(0.75) = \int_0^1 f(0.75, y)\,dy = \int_0^{0.5} 1.5\,dy \;+\; \int_{0.5}^1 0.75\,dy = 1.125\,.$$

The conditional density of Y given $X = 0.75$ is then

$$f_{Y|X}(y|x = 0.75) = \frac{f(0.75,y)}{f_X(0.75)} = \begin{cases} \frac{1.5}{1.125} = \frac{4}{3} & \text{for } 0 < y < 0.5 \\ \frac{.75}{1.125} = \frac{2}{3} & \text{for } 0.5 < y < 1 \end{cases}$$

Then, $E(Y|X = 0.75) = \int_0^{0.5} \frac{4}{3} y\,dy + \int_{0.5}^1 \frac{2}{3} y\,dy = \; = 0.417$ and

$E(Y^2|X = .75) = \int_0^{0.5} \frac{4}{3} y^2\,dy + \int_{0.5}^1 \frac{2}{3} y^2\,dy = 0.25$

$Var(Y|X = 0.75) = 0.25 - 0.417^2 = 0.076$

Answer: C

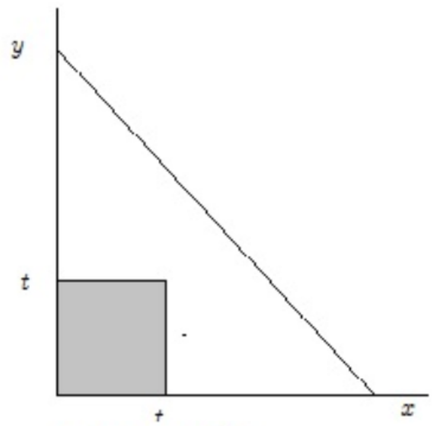
49. The joint density can be written as $f(x,y) = e^{-x} \times \frac{1}{2} sin\, y = g(x) \times h(y)$.
Since the joint density is defined on a rectangular region and since it factors into the form $g(x) \times h(y)$, it follows that the marginal distributions of X and Y are independent. Therefore

$$P[(X < 1) \cap (Y < \tfrac{\pi}{2})] = P[X < 1] \times P[Y < \tfrac{\pi}{2}] = (\int_0^1 e^{-x}\,dx)(\int_0^{\pi/2} \tfrac{1}{2} sin\, y\,dy)$$
$$= (1 - e^{-1}) \times \tfrac{1}{2}$$

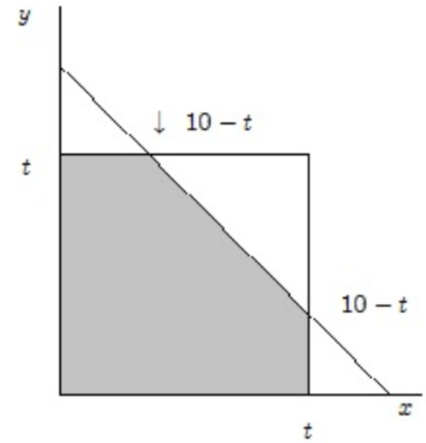
Answer: A

50. We wish to find $E[N_2|N_1 = 2]$. If we know the conditional distribution of N_2 given $N_1 = 1$, then $E[N_2|N_1 = 2] = \sum_{i=1}^{\infty} i \times P(N_2 = i|N_1 = 2)$.
From the given joint distribution, we can find the unconditional probability that $N_1 = 1$:
$P(N_1 = 2) = \sum_{j=1}^{\infty} P(2,j) = \sum_{j=1}^{\infty} \frac{3}{4} \times \frac{1}{4} e^{-2}(1 - e^{-2})^{n_2 - 1} = \frac{3}{16} e^{-2} \times \sum_{k=0}^{\infty} a^k = \frac{3}{4} e^{-2} \times \frac{1}{1-a}$,
where $a = 1 - e^{-1}$, so that $P(N_1 = 2) = \frac{3}{16}$.
Then $P(N_2 = i|N_1 = 2) = \frac{P(N_2 = i \cap N_1 = 2)}{P(N_1 = 2)} = e^{-2}(1 - e^{-2})^{i-1}$, and
$E[N_2|N_1 = 2] = \sum_{i=1}^{\infty} i \times e^{-2}(1 - e^{-2})^{i-1} = \frac{e^{-2}}{1 - e^{-2}} \times \sum_{i=1}^{\infty} i(1 - e^{-2})^i$.
We use the rule $\sum_{i=1}^{\infty} i \times a^i = \frac{a}{(1-a)^2}$, so get $E[N_2|N_1 = 2] = \frac{e^{-2}}{1 - e^{-2}} \times \frac{1 - e^{-2}}{(e^{-2})^2} = e^2$.
Answer: E

51. We denote by T the time until the machine fails. Then $t = max\{X, Y\}$. The cdf of T will be $F_T(t) = P[T \le t] = P[X \le t \cap Y \le t]$. The region of non-zero probability for X and Y can also be described as $0 < x < 10$, $0 < y < 10 - x$. Then, $F_T(t) = \int_0^t \int_0^t f(x,y)\,dy\,dx$.
If $t < 5$, then $F_T(t) = \int_0^t \int_0^t \frac{1}{50}\,dy\,dx = \frac{t^2}{50}$, and if $t > 5$ (first shaded region below), then $F_T(t) = \int_0^{10-t}\int_0^t \frac{1}{50}\,dy\,dx + \int_{10-t}^t \int_0^{10-x} \frac{1}{50}\,dy\,dx = \frac{10t-t^2}{50} + \frac{10t-50}{50} = \frac{20t-t^2-50}{50}$.
(second shaded region below).
$E[T] = \int_0^{10} [1 - F_T(t)]\,dt = \int_0^5 (1 - \frac{t^2}{50})\,dt + \int_5^{10} (1 - \frac{20t-t^2-50}{50}) = \frac{15}{6} + \frac{5}{6} = 5$.



Shaded area is t^2



Shaded area is $t^2 - \frac{(2t-10)^2}{2} = 20t - t^2 - 50$

As an alternative, we can condition the expectation of T over whether or not $X < Y$ or $X > Y$.
$E[T] = E[T|X < Y] \times P(X < Y) + E[T|X > Y] \times P(X > Y)$. From the symmetry of the region, we see that $P(X < Y) = P(X > Y) = 0.5$. Furthermore, if $X < Y$, then $T = max\{X, Y\} = Y$, and Y is uniformly distributed between X and $10 - X$, and will have a mean of 5 (midpoint of X and $10 - X$), so $E[T|X < Y] = 5$. By the symmetry of the situation, it is also true that $E[T|X > Y] = 5$. It then follows that $E[T] = 5$. Answer: D

52. Total time is 60, so $x < 60$ and whatever the value of x, we must have $y < 60 - x$.
Answer: E

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