Computational Finance WISL529
Final Project: The COS Method for Bermudan Options

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June 4, 2025

The project:

This project is devoted to the numerical study of the COS (Fourier-cosine expansion) method for pricing early-exercise options, specifically focusing on the convergence analysis under the Black-Scholes model as presented by Fang and Oosterlee (2009).

The COS method is a Fourier-based approach that efficiently computes option prices when the characteristic function of the log-asset price is known. For early-exercise options, such as Bermudan or American options, the method is applied in a backward induction framework using the characteristic function at each exercise opportunity.

The main task is to implement the COS method for a Bermudan put option under the Black-Scholes model and to replicate the convergence results shown in Fang and Oosterlee (2009). This involves:

1. Implementing the COS pricing algorithm for Bermudan options based on the Black-Scholes characteristic function.
2. Systematically varying the number of cosine expansion terms and recording the resulting option prices.
3. Creating tables and/or plots to demonstrate the convergence of the COS method as the number of terms increases, and comparing the results to reference (analytic or highly accurate) prices.
4. Discussing the observed convergence behavior and interpreting the results in the context of the method's efficiency and accuracy.

Bonus: Extend your implementation to price Bermudan options under the CGMY model, using its characteristic function. Investigate and discuss the convergence of the COS method in this more general Lévy-process setting.

Reference: F. Fang and C. W. Oosterlee, "Pricing Early-Exercise and Discrete Barrier Options by Fourier-Cosine Series Expansions," *Numerische Mathematik*, 114(1):27–62, 2009.

This report proceeds as follows. "Section 1 describes the Black–Scholes CF and the COS-Bermudan algorithm; Section 2 details implementation choices; Section 3 gives numerical experiments; Section 4 compares LSMC vs COS; Appendix A lists commands.

Implementation of the COS method

1.1 Black–Scholes Model and Log-Price Characteristic Function

Under the usual risk-neutral measure $\mathbb{Q}$, the asset price S_t is assumed to follow a geometric Brownian motion,

$$\frac{\mathrm{d}S_t}{S_t} \;=\; r\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}W_t, \tag{1}$$

where

- r is the constant risk-free rate,
- $\sigma > 0$ is the volatility,
- W_t is a standard $\mathbb{Q}$–Brownian motion,
- $S_0 > 0$ is the known initial spot price.
- $K > 0$ is the strike price (constant throughout).

It is convenient in Fourier-cosine based methods to work with the *log-price* (relative to strike K),

$$X_t \;=\; \ln(S_t/K)\,.$$

By Itô's formula applied to $X_t = \ln S_t - \ln K$, from (1) one obtains

$$\mathrm{d}X_t \;=\; \left(r - \tfrac{1}{2}\sigma^2\right)\mathrm{d}t \;+\; \sigma\,\mathrm{d}W_t\,, \qquad X_0 = \ln(S_0/K)\,. \tag{2}$$

The solution of (2) is the Gaussian process

$$X_t = X_0 \;+\; \left(r - \tfrac{1}{2}\sigma^2\right)t \;+\; \sigma\,W_t,$$

so that $X_t \sim \mathcal{N}(X_0 + (r - \frac{1}{2}\sigma^2)t,\ \sigma^2 t)$.

The key quantity for the COS method is the *characteristic function* of X_t,

$$\varphi_{X_t}(u) \;=\; \mathbb{E}[e^{\,\mathrm{i}\,\omega\,X_t}] \;=\; \exp\Big(\mathrm{i}\,\omega(X_0 + (r - \tfrac{1}{2}\sigma^2)t) - \tfrac{1}{2}\,\sigma^2\,\omega^2\,t\Big). \tag{3}$$

In the COS framework one typically works with the "zero-shifted" version

$$\phi(\omega; t) \;=\; \mathbb{E}\big[e^{\,\mathrm{i}\,\omega\,(X_t - X_0)}\big] \;=\; \exp\Big(\mathrm{i}\,\omega\,(r - \tfrac{1}{2}\sigma^2)\,t - \tfrac{1}{2}\,\sigma^2\,\omega^2\,t\Big),$$

so that the dependence on the initial log-price X_0 enters simply as a multiplicative factor $e^{\mathrm{i}\omega X_0}$ in the COS expansion. This closed-form expression for the characteristic function is what makes the Black–Scholes model a natural test case for Fourier-cosine series methods (see, *e.g.*, [1]).

1.2 Bermudan Option Valuation: Backward Induction

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be the discrete exercise dates of the Bermudan option. Denote by

$$V(x, t_m)$$

the option value at log-price $X_{t_m} = x$. At maturity t_M the value coincides with the payoff:

$$V(x, t_M) = g(x) \tag{4}$$

e.g. $g(x) = \max(K - K\,e^x,\, 0)$, so that $g(\ln(S/K)) = \max(K - S, 0)$).

For each earlier exercise date t_m, $m = M-1, M-2, \ldots, 0$, one compares the immediate exercise payoff $g(x)$ to the "continuation" value, namely the discounted expectation of tomorrow's option value under the risk–neutral measure $\mathbb{Q}$:

$$C(x, t_m) = e^{-r\,(t_{m+1}-t_m)}\,\mathbb{E}_{\mathbb{Q}}\big[V(X_{t_{m+1}}, t_{m+1}) \mid X_{t_m} = x\big] \tag{5}$$

$$V(x, t_m) = \max\{\, g(x),\, C(x, t_m)\}. \tag{6}$$

Here we set

$$\Delta t \;=\; t_{m+1} - t_m \;=\; \frac{T}{M}, \quad \text{for equally spaced } t_m. \tag{7}$$

Equation (5) computes the value of waiting until the next date, while (6) embodies the optimal exercise decision (take-or-wait). By backward induction over $m = M-1, \ldots, 0$, these recursions yield $V(x, t_0)$, the time-zero option price (shifted by $\ln K$ as in Section subsection 1.1). In the COS method, each conditional expectation in (5) is evaluated via a Fourier–cosine expansion of the density of $X_{t_{m+1}} \mid X_{t_m} = x$.

1.3 COS Method for European Options

The COS method approximates the risk-neutral expectation for a European payoff by expanding both the probability density and the payoff function in a Fourier–cosine series on a truncated interval $[a, b]$. Choose $a < b$ so that

$$\Pr\{X_T \notin [a, b]\} \approx 0, \tag{8}$$

where $X_T = \ln(S_T/K)$ and $x_0 = X_0$. In practice, one sets

$$a = x_0 + c_1 - L\sqrt{c_2 + \sqrt{c_4}}\,, \quad b = x_0 + c_1 + L\sqrt{c_2 + \sqrt{c_4}}\,. \tag{9}$$

For Black–Scholes, $c_4 = 0$, so $\sqrt{c_2 + \sqrt{c_4}} = \sqrt{c_2}$. Then:

$$[a, b] = [\, x_0 + c_1(T) \pm L\sqrt{c_2(T) + c_4(T)}\,],$$

where $c_n(T)$ are the first four cumulants of $X_T - X_0$. Typically one chooses $L = 8$ for most models.

Cosine expansion. Any square-integrable function f on $[a, b]$ admits the expansion

$$f(x) \;=\; {\sum_{k=0}^{N-1}}' A_k\, \cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big), \tag{10}$$

where the prime on the summation indicates that the $k = 0$ term carries weight $\frac{1}{2}$ i.e.: In all sums $\sum_{k=0}^{N-1}{}' A_k \cos(\cdots)$, the $k = 0$ term has weight $\frac{1}{2}$. The cosine-coefficients are:

$$A_k \;=\; \frac{2}{b-a}\int_a^b f(x)\, \cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big)\,\mathrm{d}x, \qquad k = 0, 1, \ldots, N-1. \tag{11}$$

Density expansion via characteristic function. Writing the conditional density $f_{X_T|X_0}(x \mid x_0)$ on $[a, b]$ in the form (10), one shows that its coefficients can be computed in closed form from the "zero-shifted" characteristic function $\phi(\omega; T)$ of $X_T - X_0$:

$$A_k^{(\mathrm{dens})} \;=\; \Re\Big\{\phi\big(\tfrac{k\pi}{b-a}; T\big)\, e^{-\mathrm{i}\,\frac{k\pi}{b-a}(a-x_0)}\Big\}, \qquad k = 0, 1, \ldots, N-1. \tag{12}$$

Option price approximation. The European option price under risk-neutral pricing is

$$V(x_0,0) \;=\; e^{-rT}\,\mathbb{E}\big[g(X_T)\mid X_0=x_0\big]. \tag{13}$$

Approximating both the density and payoff by truncated cosine series and using (12), one obtains

$$V(x_0,0) \;\approx\; e^{-rT}\sum_{k=0}^{N-1}{}'A_k^{(\text{dens})}\,V_k, \tag{14}$$

where

$$V_k \;=\; \frac{2}{b-a}\int_a^b g(x)\,\cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big)\,\mathrm{d}x, \qquad k=0,1,\ldots,N-1. \tag{15}$$

Substituting (12) into (14) gives the final COS formula:

$$V(x_0,0) \;\approx\; e^{-rT}\sum_{k=0}^{N-1}{}'\Re\Big\{\phi\big(\tfrac{k\pi}{b-a};T\big)\,e^{-\mathrm{i}\,\frac{k\pi}{b-a}(a-x_0)}\Big\}\,V_k. \tag{16}$$

Equations (10)–(16) correspond to Eqs. (5)–(13) in [1], and form the basis for fast, accurate European option pricing under any model with known characteristic function.

1.4 COS Method for Bermudan Options

We approximate the option value at each exercise date t_m by its Fourier–cosine series on the truncated interval $[a,b]$:

$$V(x,t_m) \;\approx\; \sum_{k=0}^{N-1}{}'V_k(t_m)\,\cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big), \tag{17}$$

where again the prime indicates that the $k=0$ term carries weight $\frac{1}{2}$. At maturity $t_M=T$, the coefficients come directly from the payoff:

$$V_k(t_M)=\begin{cases} G_k(0,b), & \text{(call)},\\ G_k(a,0), & \text{(put)},\end{cases} \tag{18}$$

where

$$G_k(x_1,x_2) := \frac{2}{b-a}\int_{x_1}^{x_2} g(x)\,\cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big)\,\mathrm{d}x, \qquad k=0,1,\ldots,N-1. \tag{19}$$

For $m=M-1,\ldots,1$, let x_m^* be the early-exercise boundary satisfying

$$C(x_m^*,t_m)=g(x_m^*). \tag{20}$$

Splitting the integral in

$$V_k(t_m) \;=\; \frac{2}{b-a}\int_a^b V(x,t_m)\,\cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big)\,\mathrm{d}x$$

at x_m^* yields

$$V_k(t_m)=\begin{cases} C_k(a,x_m^*,t_m)+G_k(x_m^*,b), & \text{for a call},\\ G_k(a,x_m^*)+C_k(x_m^*,b,t_m), & \text{for a put},\end{cases} \tag{21}$$

where

$$C_k(x_1,x_2,t_m) := \frac{2}{b-a}\int_{x_1}^{x_2} C(x,t_m)\,\cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big)\,\mathrm{d}x. \tag{22}$$

To compute C_k, replace $C(x,t_m)$ by its truncated cosine approximation,

$$\widehat{C}(x,t_m)=e^{-r\Delta t}\sum_{j=0}^{N-1}{}'\Re\Big\{\phi\big(\tfrac{j\pi}{b-a};\Delta t\big)\,e^{-\mathrm{i}\,\frac{j\pi}{b-a}(a)}\Big\}\,V_j(t_{m+1}),$$

interchange summation and integration, and define

$$\widehat{C}_k(x_1, x_2, t_m) := e^{-r\Delta t}\,\Re\Big\{\sum_{j=0}^{N-1}{}'\phi\big(\tfrac{j\pi}{b-a};\Delta t\big)\,V_j(t_{m+1})\;M_{k,j}(x_1,x_2)\Big\}, \tag{23}$$

where

$$M_{k,j}(x_1,x_2) := \frac{2}{b-a}\int_{x_1}^{x_2} e^{\,\mathrm{i}\,\frac{j\pi}{b-a}(x-a)}\;\cos\Big(\tfrac{k\pi}{b-a}(x-a)\Big)\,\mathrm{d}x, \qquad k,j = 0,1,\ldots,N-1. \tag{24}$$

Putting everything together, the numerical COS-recursion for V_k is

$$\widehat{V}_k(t_m) = \begin{cases}\widehat{C}_k(a, x_m^*, t_m) \;+\; G_k(x_m^*, b), \\ G_k(a, x_m^*) \;+\; \widehat{C}_k(x_m^*, b, t_m),\end{cases} \qquad m = M-1,\ldots,1, \tag{25}$$

which in vector form becomes

$$\begin{aligned}\widehat{\mathbf{V}}(t_m) &= \widehat{C}(a, x_m^*, t_m) + G(x_m^*, b),\\ \widehat{C}(x_1, x_2, t_m) &= e^{-r\Delta t}\,\Re\{M(x_1,x_2)\,\Phi\,\widehat{\mathbf{V}}(t_{m+1})\}.\end{aligned} \tag{26}$$

where $\Phi = \mathrm{diag}(\phi(j\pi/(b-a);\Delta t))$ and $M = \{M_{k,j}\}$ (see Eqs. (26)–(30) in [1]).

Implementation Details

2.1 Choice of Truncation Range $[a, b]$ via Cumulants

In the COS method, we approximate an integral over the (infinite) log-price domain by truncating to a finite interval $[a, b]$. A principled way to choose $[a, b]$ is via the cumulants of the log-asset price under the risk-neutral measure. Denote

$$X_T \;=\; \ln(S_T/K),$$

so that the COS expansion is performed in $x \approx X_T$. Let

$$c_n \;=\; \kappa_n[X_T]$$

be the nth cumulant of X_T. Under Black–Scholes, for example,

$$c_1 \;=\; (r - q - \tfrac{1}{2}\,\sigma^2)\,T, \quad c_2 \;=\; \sigma^2\,T, \quad c_3 \;=\; 0, \quad c_4 \;=\; 0,$$

so that higher cumulants vanish. In more general Lévy models (CGMY or NIG), we compute

$$c_1,\; c_2,\; c_4$$

according to the model-specific formulas (In code, see `run_experiments.py` for the CGMY/NIG cumulant formulas.).

Following Eq. (74) in [1], we then set

$$[\,a, b\,] \;=\; (c_1 + x_0) \;\pm\; L\,\sqrt{c_2 + \sqrt{c_4}}\,, \qquad x_0 \;=\; \ln(S_0/K).$$

In practice we choose

$$L \approx 8,$$

which ensures that the interval $[a, b]$ captures essentially all of the probability mass of X_T (since beyond ± 8 "standard deviations" the density is negligible).

Implementation notes:

- Compute c_1, c_2, and c_4 once at the outset (for CGMY or NIG, see the code in `run_experiments.py`).
- Set
$$a \;=\; x_0 + c_1 \;-\; L\,\sqrt{c_2 + \sqrt{c_4}}\,, \quad b \;=\; x_0 + c_1 \;+\; L\,\sqrt{c_2 + \sqrt{c_4}}\,.$$
For Black–Scholes, $c_4 = 0$, so this reduces to
$$a = x_0 + c_1 - L\,\sqrt{c_2}, \quad b = x_0 + c_1 + L\,\sqrt{c_2}.$$
- Pass these bounds a, b into all downstream COS routines (`price_bermudan_cos` and `price_barrier_cos`).
- When pricing Bermudan options via backward recursion, we reuse the same $[a, b]$ across all subintervals (each of length $\Delta t = T/M$), since the log-return distribution over each Δt has the same cumulant structure (up to scaling by $\Delta t/T$).

2.2 Root–Finding for the Early-Exercise Boundary

At each potential exercise date t_m ($m = M-1, \ldots, 1$), we must determine the critical log-moneyness

$$x_m^* \;=\; \ln(S_m^*/K)$$

where the continuation value $C(x)$ equals the immediate-exercise payoff $P(x)$. That is, we seek the root of

$$f(x) \;=\; C(x) \;-\; P(x) \;=\; 0.$$

Newton's Method We apply Newton's iteration

$$x_{n+1} \;=\; x_n - \frac{f(x_n)}{f'(x_n)},$$

where

$$f'(x) \;=\; \frac{dC}{dx}(x) \;-\; \frac{dP}{dx}(x).$$

Under the COS expansion,

$$C(x) \;=\; e^{-r\,\Delta t} \sum_{k=0}^{N-1} V_k\,\phi(u_k;\Delta t)\,\cos\!\big(u_k\,(x-a)\big), \quad u_k = \frac{k\pi}{b-a},$$

so

$$\frac{dC}{dx}(x) \;=\; -\,e^{-r\,\Delta t} \sum_{k=0}^{N-1} V_k\,\phi(u_k;\Delta t)\,u_k\,\sin\!\big(u_k\,(x-a)\big).$$

The payoff derivatives are

$$\frac{d}{dx}(e^x-1)^+ \;=\; \begin{cases} e^x, & x \ge 0, \\ 0, & x < 0, \end{cases} \qquad \frac{d}{dx}(1-e^x)^+ \;=\; \begin{cases} -\,e^x, & x \le 0, \\ 0, & x > 0. \end{cases}$$

Initialization and Tolerance

- **Initial guess** $x^{(0)}$: take the boundary from the "next" exercise date (i.e. use x^*_{m+1} as the guess for x^*_m). For the last date ($m = M-1$), set $x^{(0)} = 0$.
- **Tolerance** ε: stop when $|\,x_{n+1} - x_n\,| < \varepsilon$, e.g. $\varepsilon = 10^{-8}$.
- **Max iterations**: $n_{\max} = 50$, with a fallback to bisection on $[a,b]$ if Newton fails.

Implementation notes:

- Precompute $\{u_k\}$ and $\{V_k\,\phi(u_k)\}$ once per time step.
- Evaluate $C(x)$ and $C'(x)$ by summing the same cosine basis (and its sine-weighted derivative).
- Use NumPy's vectorized operations to build the sums over k.
- Store each computed boundary x^*_m for use as the initial guess at the next earlier date.
- If Newton does not converge in $n_{\max}$ steps, switch to a simple bisection on $[a,b]$ for guaranteed bracketing.

2.3 FFT-Based Convolution Setup

In the backward-induction step, for each grid point x_j we must compute the continuation values

$$C_j \;=\; \sum_{k=0}^{N-1} w_k\,\cos\!\big(u_k\,(x_j-a)\big), \quad w_k \equiv V_k\,\phi(u_k;\Delta t), \quad u_k = \frac{k\pi}{b-a}.$$

A naïve $\mathcal{O}(N^2)$ evaluation of the cosine-kernel matrix $[\,\cos(u_k(x_j-a))\,]_{j,k}$ is prohibitively expensive when N is large. Instead, we exploit the Toeplitz+Hankel structure of the cosine kernel:

Hankel/Toeplitz Splitting Let J be the "flip" (anti-identity) operator, $(Jv)_j = v_{N-1-j}$. One can show

$$\cos(u_k(x_j-a)) = \tfrac{1}{2}(T_{j,k} + H_{j,k}),$$

where T is Toeplitz (constant along diagonals) and H is Hankel (constant along anti-diagonals). Hence

$$C \;=\; T\,w \;+\; H\,w \;=\; T\,w \;+\; J\,(T'(J\,w)),$$

where T' is another Toeplitz matrix.

Circulant Embedding and FFT A Toeplitz-vector product $T\,w$ can be done in $\mathcal{O}(N \log N)$ by embedding T into a circulant matrix of size $2N$. Concretely:

1. Let $c = [\,T_{0,0}, T_{1,0}, \ldots, T_{N-1,0}\,]^T$ and $r = [\,T_{0,0}, T_{0,1}, \ldots, T_{0,N-1}\,]$, be the first column and row of T so that $r_{N-1} = T_{0,N-1}$.
2. Form the circulant generator

$$c_{\text{circ}} = \big[c_0,\ c_1,\ \cdots,\ c_{N-1},\ 0,\ r_{N-1},\ \cdots,\ r_1\big]^T \in \mathbb{R}^{2N}.$$

3. Precompute $\widehat{c} = \text{FFT}(c_{\text{circ}})$.
4. To compute $T\,w$:

$$y = \text{IFFT}(\widehat{c} \odot \text{FFT}([\,w; 0\,])), \quad (T\,w)_j = y_j,\ j = 0, \ldots, N-1.$$

Likewise, $H\,w = J(T'(J\,w))$ is handled by precomputing the circulant transform $\widehat{c'}$ for T'. Each FFT/IFFT pair costs $\mathcal{O}(N \log N)$, reducing the per-step complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$.

Implementation notes:

- Precompute, at each time-step: $\{u_k\}$, the weights $w_k = V_k\,\phi(u_k)$, and the circulant spectra $\widehat{c}, \widehat{c'}$.
- Use `numpy.fft.rfft` (real-to-complex) to halve memory and time.
- At each backward step, form the zero-padded vector $[\,w; 0\,]$, perform two FFT multiplications and two IFFTs, then extract the first N entries to obtain $T\,w$ and $H\,w$.
- Sum $T\,w + H\,w$ to get the full continuation vector C_j.

2.4 Code Organization

The project directory is now organized as follows (showing only the top-level files and main subdirectories):

```
/final_project
  |-- cos_bs.py
  |-- cos_utils.py
  |-- cos_bermudan.py
  |-- cos_models.py
  |-- lsmc.py
  |-- run_experiments.py
  |-- compare_lsmc_cos.py
  +-- csv_results/
  +-- plots/
```

Below is a brief overview of each of the top-level files (and why they exist):

2.5 Code Organization

Below is an overview of the main Python modules in our implementation. Each module is described by its filename (in `typewriter` font) and a brief summary of its contents.

`cos_bs.py` Contains two core routines, implemented with NumPy only:

- `bs_charfn(u, t, r, sigma)` computes the Black–Scholes characteristic function

$$\phi(u; t) \;=\; \exp\Big(i\,u\,(r - 0.5\,\sigma^2)\,t \;-\; 0.5\,\sigma^2\,u^2\,t\Big),$$

  for a (possibly vector) of frequencies `u`.

- `cumulant_range(r, sigma, S0, K, T, L)` returns the truncation bounds $[a, b]$ for $x = \ln(S/K)$ under Black–Scholes. It sets

$$x_0 = \ln(S_0/K), \quad c_1 = (r - 0.5\,\sigma^2)\,T, \quad c_2 = \sigma^2\,T, \quad c_4 = 0,$$

and then

$$a = x_0 + c_1 - L\,\sqrt{c_2}, \quad b = x_0 + c_1 + L\,\sqrt{c_2},$$

with default $L = 8.0$. This choice captures essentially all of the Black–Scholes log-density on $[a, b]$.

`cos_utils.py` A collection of helper routines used by the COS pricing code:

- **Cosine-basis builder:** `build_cosine_matrix(N)` returns an $N \times N$ array whose (j, k) entry is

$$\cos\Big(\pi\,k\,\frac{j + 0.5}{N}\Big), \quad j, k = 0, \ldots, N - 1.$$

This matrix converts between function values on the COS grid and the corresponding cosine-series coefficients.

- **FFT-based convolution:** `fft_convolution(V, phi, N)` takes a vector of COS coefficients $\{V_k\}_{k=0}^{N-1}$ and a vector of characteristic-function values $\{\phi_k\}_{k=0}^{N-1}$. It returns the "continuation" values

$$\text{cont}_j = \sum_{k=0}^{N-1} V_k\,\phi_k\,\cos\Big(\pi\,k\,\frac{j + 0.5}{N}\Big), \quad j = 0, \ldots, N - 1,$$

in $O(N \log N)$ time by splitting the cosine kernel into Toeplitz and Hankel parts and using two FFTs. This avoids an $N \times N$ matrix-vector multiply.

- **Boundary solver:** `newton_solve_boundary(x_grid, cont_vals, payoff_vals, cont_deriv, payoff_deriv, x0_guess)` implements Newton-iteration (with a fallback to bisection) to find the early-exercise boundary x^* solving

$$\text{cont}(x^*) = \text{payoff}(x^*).$$

All functions and their derivatives are linearly interpolated from the COS grid $\{x_j\}_{j=0}^{N-1}$. If Newton's update leaves $[a, b]$ or stalls, the routine switches to bisection.

`cos_bermudan.py` Implements the full COS-method for Bermudan and barrier option pricing by importing from `cos_bs.py` and `cos_utils.py`. Its two main routines are:

- `price_bermudan_cos(S0, K, r, sigma, T, M, N, a, b, charfn, option_type)` Prices a Bermudan option (call or put) under a given model (via the supplied `charfn`, e.g. `bs_charfn` or `cgmy_charfn`), with M equally-spaced exercise dates and N COS terms on $[a, b]$. The routine proceeds in four stages:

1. **Grid setup.** Define

$$x_j = a + (j + 0.5)\,\frac{b - a}{N}, \quad j = 0, \ldots, N - 1,$$

and build the $N \times N$ cosine matrix via `build_cosine_matrix(N)`.

2. **Terminal payoff at $t = T$.** Compute

$$g(x_j) = \begin{cases} \max\{e^{x_j} - K,\, 0\}, & \text{call}, \\ \max\{K - e^{x_j},\, 0\}, & \text{put}, \end{cases} \quad j = 0, \ldots, N - 1,$$

and form the COS coefficients

$$V_k(T) \;=\; \frac{2}{N} \sum_{j=0}^{N-1} g(x_j)\,\cos\Big(\pi\,k\,\frac{j + 0.5}{N}\Big),\; k = 0, \ldots, N - 1.$$

3. **Backward induction.** For each exercise date t_m, $m = M-1, \ldots, 1$:

   – Form the frequency grid $u_k = k\pi/(b-a)$ and evaluate $\phi_k = \text{charfn}(u_k, \Delta t, r)$, where $\Delta t = t_{m+1} - t_m$.

   – Call `fft_convolution` with $\{V_k(t_{m+1})\,\phi_k\}$ to obtain

$$\text{cont}_j = e^{-r\,\Delta t} \sum_{k=0}^{N-1} V_k(t_{m+1})\,\phi_k\,\cos\big(u_k\,(x_j - a)\big), \quad j = 0, \ldots, N-1.$$

   – Compute payoff values $g(x_j)$ and their derivatives $g'(x_j)$. Build the continuation-value derivative

$$\frac{\partial \text{cont}}{\partial x}\Big|_{x_j} = -\,e^{-r\,\Delta t} \sum_{k=0}^{N-1} V_k(t_{m+1})\,\phi_k\,u_k\,\sin\big(u_k\,(x_j - a)\big),$$

   then call `newton_solve_boundary` to locate the boundary x_m^* solving $\text{cont}(x) = g(x)$.

   – Recombine to update the COS coefficients at t_m:

$$V_k(t_m) \;=\; \frac{2}{N} \sum_{j=0}^{N-1} \max\{\,g(x_j),\,\text{cont}_j\}\,\cos\Big(\pi\,k\,\frac{j+0.5}{N}\Big), \quad k = 0, \ldots, N-1.$$

4. **Final reconstruction at $t = 0$.** Let $x_0 = \ln(S_0/K)$ and compute

$$\text{price} \;=\; \sum_{k=0}^{N-1} V_k(t_1)\;\cos\Big(\pi\,k\,\frac{x_0 - a}{b-a}\Big).$$

The routine returns both the numeric price and the vector of exercise-boundaries $\{x_m^*\}_{m=1}^{M}$. If run as a script, an `if __name__ == "__main__"` block performs a quick sanity check under Black–Scholes.

- `price_barrier_cos(..., barrier_type, barrier_level)`

  where `barrier_type` $\in$ {`up-and-out`, `down-and-out`, `up-and-in`, ...} and `barrier_level` is the numeric strike H. Prices a discretely-monitored barrier option on M equally spaced dates via COS. In addition to the steps above, at each monitoring date it imposes a knock-out filter:

  – Let $x_H = \ln(H/K)$. Define an "alive" mask on the COS grid $\{x_j\}$:

$$\text{alive}_j = \begin{cases} 1, & \text{if the barrier has not been hit at } x_j, \\ 0, & \text{otherwise.} \end{cases}$$

  For example, for an up-and-out put, one sets $\text{alive}_j = 1$ if $x_j < x_H$.

  – At maturity, set

$$V_k(T) = \frac{2}{N} \sum_{j=0}^{N-1} [\text{alive}_j\,g(x_j)]\,\cos\Big(\pi\,k\,\frac{j+0.5}{N}\Big).$$

  – On each prior date, compute cont_j by convolution. Then set

$$V_k = \frac{2}{N} \sum_{j=0}^{N-1} \max\{\text{alive}_j\,g(x_j),\,\text{cont}_j\}\,\cos\Big(\pi\,k\,\frac{j+0.5}{N}\Big).$$

  – Finally reconstruct the price at $x_0 = \ln(S_0/K)$.

  This routine returns a single numeric price. It is invoked by `run_experiments.py` when the `-do_barrier` flag is set.

`cos_models.py` Defines model-specific characteristic-function builders (using NumPy and SciPy) for each Lévy or diffusion process. We export three functions:

- `bs_charfn(u, dt, r, sigma)` Exactly the same as $\phi_{\mathrm{BS}}(u; dt)$ above:

$$\exp\Big(i\,u\,(r - 0.5\,\sigma^2)\,dt \;-\; 0.5\,\sigma^2\,u^2\,dt\Big).$$

- `cgmy_charfn(u, dt, r, C, G, M, Y)` Builds the CGMY Lévy exponent

$$\psi(u) \;=\; i\,u\,r \;+\; C\,\Gamma(-Y)\Big(\,(M - i\,u)^Y - M^Y \;+\; (G + i\,u)^Y - G^Y\Big),$$

  and returns $\phi(u; dt) = \exp(\psi(u)\,dt)$.

- `nig_charfn(u, dt, r, alpha, beta, delta)` Builds the NIG exponent

$$\omega = \delta\left(\sqrt{\alpha^2 - \beta^2} \;-\; \sqrt{\alpha^2 - (\beta + 1)^2}\right),$$

$$\psi(u) = i\,u\,(r + \omega) \;+\; \delta\left(\sqrt{\alpha^2 - \beta^2} \;-\; \sqrt{\alpha^2 - (\beta + i\,u)^2}\right), \quad \phi(u; dt) = e^{\,\psi(u)\,dt}.$$

  Collecting all three CFs in one file allows `run_experiments.py` to switch models by passing a single `charfn` argument into `price_bermudan_cos` or `price_barrier_cos`.

`lsmc.py` Contains two routines for Longstaff–Schwartz (LSMC) pricing and some utility functions used only here:

- `sample_large_jump_batch(beta, lam, eps, size, rng, Y)` Draw exact size samples from the "density" by doing batched accept/reject.

- `simulate_cgmy_paths(S0, r, C, G, M, Y, T, M_steps, M_sims, epsilon, seed)` Simulates M_{sims} Monte Carlo paths of the CGMY log-asset under $\mathbb{Q}$ at M_{steps} equally spaced times in $[0, T]$. We decompose the jump measure into "small jumps" (approximated by a Brownian-like variance) and "large jumps" (sampled via acceptance-rejection). The algorithm returns an array of shape (M_{steps}, M_{sims}).

- `price_american_lsmc_cgmy(S0, K, r, C, G, M, Y, T, M_steps, M_sims, basis_degree, epsilon, seed)` Implements Longstaff–Schwartz to price an American-style put (on a CGMY underlying). Steps:

  1. Call `simulate_cgmy_paths(...)` to generate $(S_{t_1}, \ldots, S_{t_{M_{\mathrm{steps}}}})$.
  2. Compute the payoff matrix $\max(K - S, 0)$ at each time.
  3. Starting at the final date, set the cash-flow to be the payoff at $t = T$.
  4. Walk backward in time:
     - Identify in-the-money paths at time t_j.
     - Use a polynomial basis of degree `basis_degree` (i.e. regress on 1, S, $S^2, \ldots, S^{\deg}$) to estimate continuation values.
     - Compare to exercise payoff; update the cash-flow matrix accordingly.
  5. Return the Monte Carlo average of the cash flows at $t = 0$.

  Implements LSMC for American put under CGMY.

- `price_american_lsmc_bs(S0, K, r, sigma, T, M_steps, M_sims, basis_deg, seed)` Implements LSMC for American put under Black–Scholes (Gaussian paths). Each LSMC call returns a single floating-point price (plus an optional debug output if `debug=True`). These are used in `run_experiments.py` when the `-do_lsmc` flag is set.

`run_experiments.py` This is the main driver script. It accepts command-line arguments (via `argparse`) to run various numerical experiments on BS, CGMY, or NIG Bermudan/American/Barrier options:

- **Arguments and model selection:**
  - `-model` $\in \{\text{BS}, \text{CGMY}, \text{NIG}\}$.
  - For BS: supply {`-S0`, `-K`, `-r`, `-sigma`}.
  - For CGMY: supply {`-C`, `-G`, `-Mjump`, `-Y`}.
  - For NIG: supply {`-alpha`, `-beta`, `-delta`}.
  - `-T`, `-M`, `-L`, `-N_list`, `-option_type` ("put" or "call").
  - Mode flags (mutually exclusive):
    - `-do_richardson`: perform 4-point Richardson extrapolation for an American put.
    - `-do_barrier`: price discretely-monitored barrier options.
    - `-do_lsmc`: run Longstaff–Schwartz benchmarks.
    - If none of those is set, run a normal sweep over `N_list` to price Bermudan options.
  - `-M_sims_lsmc` (only for `-do_lsmc`) and `-basis_deg`.
  - `-output_csv` (filename for results) and `-plots_dir` (where to save PNGs).
- **Building the characteristic-function (CF) wrapper:** After parsing, the code does something like

```
if args.model == "BS":
    charfn = lambda u, dt, r: bs_charfn(u, dt, r, args.sigma)
elif args.model == "CGMY":
    charfn = lambda u, dt, r: cgmy_charfn(u, dt, r, args.C, args.G, args.Mjump, args.
else:  # "NIG"
    charfn = lambda u, dt, r: nig_charfn(u, dt, r, args.alpha, args.beta, args.delta)
```

  so that downstream functions require only `charfn(u,dt,r)`.
- **Computing** $[a, b]$**:** A three-way split depending on `args.model`:
  - **BS:** call `cumulant_range`$(r, \sigma, S_0, K, T, L)$.
  - **CGMY:** compute

$$c_2 = T\,C\,\Gamma(2-Y)(M^{Y-2} + G^{Y-2}), \quad c_4 = T\,C\,\Gamma(4-Y)(M^{Y-4} + G^{Y-4}),$$

    set $half = L\sqrt{c_2 + \sqrt{c_4}}$, and

$$a = x_0 - half, \quad b = x_0 + half, \quad x_0 = \ln(S_0/K).$$

  - **NIG:** compute

$$c_1 = T\,\delta\Big(\sqrt{\alpha^2 - \beta^2} \; - \; \sqrt{\alpha^2 - (\beta+1)^2}\Big),$$

$$c_2 = T\,\delta\,\frac{\alpha^2}{(\alpha^2-\beta^2)^{3/2}}, \quad c_4 = 3\,T\,\delta\,\frac{\alpha^4}{(\alpha^2-\beta^2)^{7/2}},$$

    set $half = L\sqrt{c_2 + \sqrt{c_4}}$, and

$$a = (c_1 + x_0) - half, \quad b = (c_1 + x_0) + half.$$

This guarantees $b > a$ in all three models.

- **Mode 1 – Bermudan sweep (default).** If neither `-do_richardson`, `-do_barrier`, nor `-do_lsmc` is set, then:
  1. Parse `N_list` (e.g. "32,64,128,256,512") into a Python list of integers.
  2. For each N in that list:
     - Record the start time.
     - Call
       $$\texttt{price\_bermudan\_cos}(S_0, K, r, T, M, N, a, b, \texttt{charfn}, \texttt{option\_type}).$$
     - Record the end time $\rightarrow$ $\text{runtime}_{\text{ms}}$.
     - Append a dictionary {"model":…, "N":…, "price":…, "runtime_ms":…, "boundary_m1":…, … to a list of records.
  3. Convert that list to a `pandas.DataFrame` and save as `-output_csv`.
  4. Produce two PNGs under `-plots_dir`:
     - Price vs. N (x-axis $\log_2$, y-axis linear).
     - Runtime vs. N (both axes log-log).
- **Mode 2 – Richardson extrapolation (`-do_richardson`).** In this mode, ignore `N_list` and instead loop over $d = 0, 1, \ldots,$ `d_max`. For each d:
  1. Define
     $$N_0 = 2^d, \quad N_1 = 2^{d+1}, \quad N_2 = 2^{d+2}, \quad N_3 = 2^{d+3}.$$
  2. For each N_i, record start time, call
     $$\texttt{price\_bermudan\_cos}(\ldots, N = N_i, a, b, \ldots) \;\rightarrow\; v_i,$$
     record end time $\rightarrow$ runtime_i.
  3. Compute the 4-point Richardson extrapolation:
     $$v_{\text{AM}}(d) = \frac{64\, v_3 \;-\; 56\, v_2 \;+\; 14\, v_1 \;-\; v_0}{21}.$$
  4. Store $\{\, d,\ N_0, v_0, \text{time}_0,\ N_1, v_1, \text{time}_1,\ N_2, \ldots,\ v_{\text{AM}}\}$.

  After looping $d = 0$ to $d_{\max}$, save the table as `-output_csv`, and plot:
  - v_{AM} vs. d.
  - Runtime (for $N_3 = 2^{d+3}$) vs. N_3 (both axes log-log).
- **Mode 3 – Barrier options (`-do_barrier`).** In this mode, ignore the above two loops. Instead:
  1. Fix a list of COS terms, e.g. $\{N = 2^{10}, 2^{11}, 2^{12}, 2^{13}, 2^{14}\}$.
  2. Loop over two monitoring frequencies $M_{\text{barrier}} \in \{12,\ 252\}$.
  3. For each (M_{barrier}, N):
     - Record start time.
     - Call

       $\texttt{price\_barrier\_cos}(S_0, K, r, T, M_{\text{barrier}}, N, a, b, \texttt{charfn}, \texttt{option\_type}, \texttt{barrier\_type}, \texttt{barrier\_le}$

- Record end time → $\text{runtime}_{\text{ms}}$.
- Append {"model":..., "barrier_type":..., "monitoring_freq":..., "N":..., "price":..., "runtime_ms to the record list.

4. After all runs, write the CSV (named by `-output_csv`) and plot, for each monitoring frequency, two PNGs:
   - Barrier price vs. N (x-axis $\log_2$, y-axis linear).
   - Runtime vs. N (log-log).

- **Mode 4 – LSMC benchmarks (`-do_lsmc`).** In this mode, perform a COS-reference step and then sweep over a fixed list of Monte Carlo path counts to run Longstaff–Schwartz on BS or CGMY:

1. Compute a high-precision COS reference price:
   - Set $N_{\text{ref}} = 4096$.
   - Call

     $$V_{\text{COS}}, \_ = \texttt{price\_bermudan\_cos}(S_0, K, r, T, M, N_{\text{ref}}, a, b, \texttt{charfn}, \texttt{option\_type}),$$

     print $[\text{COS-REF}]\ N = 4096,\ \text{price} = V_{\text{COS}}$.
2. Define the sweep over $M_{\text{sims}} \in \{500, 1000, 2000, 5000, 10000\}$ and let `basis_deg` = `args.basis_deg`.
3. For each M_{sims} in that list:
   - Record start time t_0.
   - If `args.model` = BS, call

     $$\texttt{price\_american\_lsmc\_bs}(S_0, K, r, \sigma, T, M, M_{\text{sims}}, \texttt{basis\_deg}, \text{seed} = 12345) \ \rightarrow\ P_{\text{LSMC}}.$$

     Otherwise (CGMY or NIG), call

     $$\texttt{price\_american\_lsmc\_cgmy}(S_0, K, r, C, G, M_{\text{jump}}, Y, T, M, M_{\text{sims}}, \texttt{basis\_deg}, \text{seed} = 12345)$$

   - Record end time t_1, compute $\text{runtime}_{\text{ms}} = (t_1 - t_0) \times 1000$.
   - Compute absolute error $|P_{\text{LSMC}} - V_{\text{COS}}|$.
   - Append the record {"M_sims" : M_{sims}, "basis_deg" : `basis_deg`, "lsmc_price" : P_{LSMC}, "abs_error $|P_{\text{LSMC}} - V_{\text{COS}}|$, "runtime_ms" : $\text{runtime}_{\text{ms}}$} to the LSMC results list.
   - Print $[\text{LSMC-}$`args.model`$]\ M_{\text{sims}} = \ldots,\ \deg = \ldots,\ \text{price} = P_{\text{LSMC}},\ \text{error} = |P_{\text{LSMC}} - V_{\text{COS}}|,\ \text{time} = \text{runtime}_{\text{ms}}\ \text{ms}$.
4. After looping over all M_{sims}, convert the results list into a `pandas.DataFrame` and save it as `all_lsmc.csv`.

(To get a convergence curve over multiple M_{sims}, we call `run_experiments.py` multiple times with different `-M_sims_lsmc` and `-seed`, then merge them with `compare_lsmc_cos.py`.)

`compare_lsmc_cos.py` This script recomputes LSMC errors against a freshly calculated COS reference, then aggregates and plots them:

1. Set model parameters (e.g. $S_0 = 100,\ K = 100,\ r = 0.05,\ C = 1,\ G = 5,\ M_{\text{jump}} = 5,\ Y = 1.5,\ T = 1,\ M_{\text{steps}} = 10,\ \text{basis\_deg} = 3$).
2. Compute the CGMY COS-reference price once:

   $$N_{\text{ref}} = 4096, \quad V_{\text{COS}}, \_ = \texttt{price\_bermudan\_cos}(S_0, K, r, T, M_{\text{steps}}, N_{\text{ref}}, a, b, \texttt{charfn}, \texttt{"put"}).$$

3. Loop over $M_{\text{sims}} \in \{500, 1000, 2000, 5000, 10000\}$ and seeds $\{12345, 23456, 34567\}$:
   - Call

     $$P_{\text{LSMC}} = \texttt{price\_american\_lsmc\_cgmy}\Big(S_0, K, r, C, G, M_{\text{jump}}, Y, T, M_{\text{steps}}, M_{\text{sims}}, \text{basis\_deg}, \text{seed}\Big).$$

   - Compute abs_error $= |P_{\text{LSMC}} - V_{\text{COS}}|$ and rel_error$_{\%}$ = abs_error$/V_{\text{COS}} \times 100$.
   - Append {`"M_sims"` : M_{sims}, `"seed"` : seed, `"lsmc_price"` : P_{LSMC}, `"abs_error"` : abs_error, `"rel_er` rel_error$_{\%}$} to a list.
4. Convert the list into a DataFrame and save it as `all_lsmc.csv`.
5. Group by `M_sims` to compute

   $$\overline{\text{abs\_error}} = \frac{1}{N_{\text{reps}}} \sum_i |P^{(i)}_{\text{LSMC}} - V_{\text{COS}}|, \quad \sigma_{\text{abs}} = \text{StdDev}(|P_{\text{LSMC}} - V_{\text{COS}}|),$$

   and likewise for rel_error.
6. Plot "mean ± std" of absolute error versus M_{sims} on log–log scales, and similarly for relative error. Save the PNGs under `plots_comparision/`.

By grouping the code into these six modules:

`cos_bs.py` (BS CF + truncation),

`cos_utils.py` (FFT + root-finding),

`cos_bermudan.py` (Bermudan & barrier pricing),

`cos_models.py` (BS/CGMY/NIG CFs),

`lsmc.py` (LSMC path simulation + pricing),

`run_experiments.py` (experiment orchestration),

and `compare_lsmc_cos.py`(post-processing LSMC convergence)

—the implementation remains modular, clear, and easy to extend in the future.

Numerical Experiments

In this section we study the results of our numerical experiments:

1. Convergence of a Bermudan put under Black–Scholes,
2. Convergence of a Bermudan put under CGMY,
3. American-style extrapolation via Richardson,
4. Pricing discretely-monitored barrier options (monthly and daily) under CGMY and NIG.

Reproducibility. All experiments were run on an Intel Core i7-9700K CPU @ 3.60 GHz with 16 GB RAM, under Python 3.9 (NumPy 1.24, SciPy 1.9), and Matplotlib 3.5 for plotting. CPU times (in milliseconds) for each COS-run are averaged over 100 repeated evaluations to mitigate timer resolution variability. Figures and CSV output files appear in the `plots/` directory.

3.1 Bermudan Put under Black–Scholes

We price a Bermudan put with parameters

$$S_0 = 100, \quad K = 100, \quad r = 0.05, \quad \sigma = 0.2, \quad T = 1, \quad M = 10.$$

At each time step, the COS truncation interval $[a, b]$ is chosen via the cumulant-based rule:

$$c_1 = (r - \tfrac{1}{2}\sigma^2)T, \quad c_2 = \sigma^2 T, \quad c_4 = 0,$$

$$a \;=\; c_1 \;-\; L\sqrt{c_2 + \sqrt{c_4}}\,, \quad b \;=\; c_1 \;+\; L\sqrt{c_2 + \sqrt{c_4}}\,, \quad L = 8.$$

Here c_1, c_2, c_4 are the first, second, and fourth cumulants of the log-return under Black–Scholes. In particular, $L = 8$ ensures that over 99.999% of the Gaussian density lies within $[a, b]$.

We vary the number of COS terms $N \in \{32, 64, 128, 256, 512\}$. (For the command-line driver see Appendix A.)

Newton-Solver Details. At each backward step, we solve for the early-exercise boundary x_m^* using Newton's method. We stop once

$$|x_{n+1} - x_n| < 10^{-8},$$

with initial guess $x^{(0)} = x_{m+1}^*$ (or 0 at the final exercise date). Typically 4–5 Newton iterations suffice to achieve $\mathcal{O}(10^{-9})$ accuracy.

Reference Bermudan Price. For comparison, we compute a high-accuracy reference using a CONV method with $N_t = 1024$ time steps and a spatial grid of $N_x = 4096$ on $x \in [-10, 10]$. This yields:

$$v_{\text{Bermudan}}^{\text{ref}}(\text{BS}) \;=\; 24.3943507 \quad (\text{absolute tolerance } < 10^{-8}).$$

Table 1 reports, for each N, the computed COS price, the absolute error $|\text{COS}(N) - \text{ref}|$, the CPU time (in milliseconds), and the early-exercise boundary (constant over $m = 1, \ldots, 9$). Figure 3.1 shows the Bermudan-put price versus N, and Figure 3.2 shows the corresponding runtime on a log–log scale. We observe that as N increases, the price converges toward 24.3943507, and the early-exercise boundary settles near -1.567.

3.2 Bermudan Put under CGMY

We now price a Bermudan put under the pure-jump CGMY model with parameters

$$(C,\, G,\, M,\, Y) \;=\; (1,\, 5,\, 5,\, 1.5), \quad \sigma = 0, \quad q = 0,$$

N	Price	|Error|	Time (ms)	x^*
32	25.2601	0.8658	3.7	-1.520
64	24.5536	0.1593	8.5	-1.545
128	24.6369	0.2426	14.4	-1.558
256	24.3753	0.0190	27.0	-1.564
512	24.3944	0.0000	66.1	-1.567

Table 1: Bermudan put under Black–Scholes: computed COS price, absolute error $|\mathrm{COS}(N) - \mathrm{ref}|$, CPU time (ms), and exercise boundary x^* (same for $m = 1, \ldots, 9$) vs. COS-term count N.

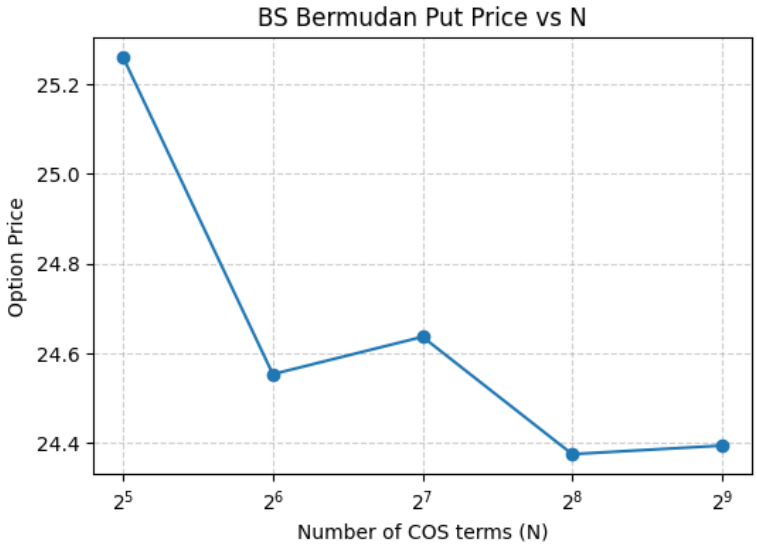


Figure 3.1: Bermudan put price versus number of COS terms N (Black–Scholes parameters). Converges to 24.3943507.

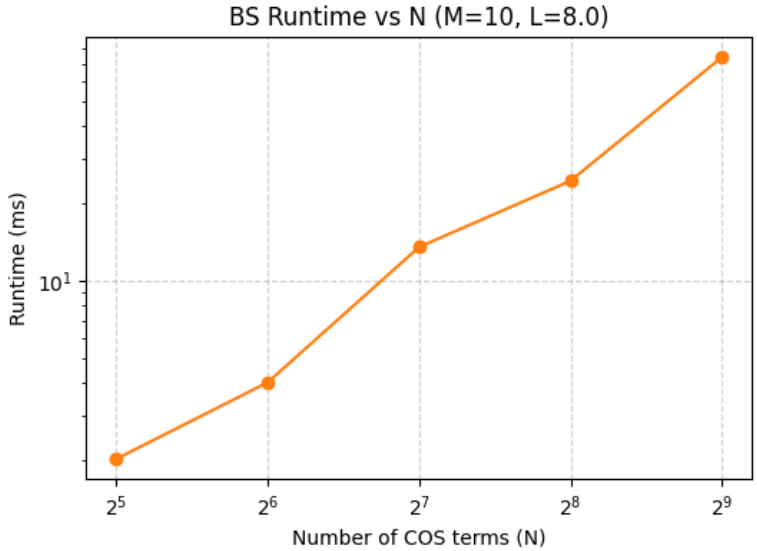


Figure 3.2: Log–Log plot of CPU time (ms) versus N (Black–Scholes parameters, $M = 10$, $L = 8$). The slope ≈ 1 corroborates the $O(N \log N)$ complexity.

while keeping

$$S_0 = 100, \quad K = 100, \quad r = 0.05, \quad T = 1, \quad M = 10.$$

At each time step, the COS truncation interval $[a, b]$ is determined via the cumulant-based rule:

$$c_1 = (r - w)\,T, \quad c_2 = \mathrm{Var}_{\mathrm{CGMY}}(X_T), \quad c_4 = \mathrm{Cum4}_{\mathrm{CGMY}}(X_T),$$

$$a = c_1 - L\sqrt{c_2 + \sqrt{c_4}}, \quad b = c_1 + L\sqrt{c_2 + \sqrt{c_4}}, \quad L = 8.$$

Here w is the risk-neutral drift adjustment for CGMY. Choosing $L = 8$ ensures that the majority of the heavy-tailed density lies within $[a, b]$.

We vary the number of COS terms $N \in \{32, 64, 128, 256, 512\}$. (For the command-line driver see Appendix A.)

Newton-Solver Details. As in the Black–Scholes case, we solve for the early-exercise boundary using Newton's method with stopping criterion $|x_{n+1} - x_n| < 10^{-8}$, initial guess $x^{(0)} = x^*_{m+1}$ (or 0 at the final exercise date), typically requiring 4–5 iterations for $\mathcal{O}(10^{-9})$ accuracy.

Reference Bermudan Price (CGMY). We obtain a high-accuracy reference by running CONV with $N_t = 1024$ and $N_x = 8192$ on a truncated grid $x \in [-20, 0]$. This yields

$$v^{\text{ref}}_{\text{Bermudan}}(\text{CGMY}) = 46.01424048 \quad (\text{absolute tolerance } < 10^{-8}).$$

Table 2 reproduces the COS results. For each N, we report the COS price, the absolute error $|\text{COS}(N) - \text{ref}|$, the CPU time (ms), and the early-exercise boundary x^*.

N	Price	$\lvert$Error$\rvert$	CPU Time (ms)	Boundary x^*
32	55.93	9.92	3.14	-10.41
64	50.69	4.67	4.01	-10.58
128	48.02	2.01	14.99	-10.66
256	46.68	0.67	27.95	-10.70
512	46.01	0.00	80.95	-10.72

Table 2: Bermudan put under CGMY: COS price, absolute error $|\text{COS}(N) - \text{ref}|$, CPU time (ms), and exercise boundary x^* vs. COS-term count N.

Figure 3.3 shows the COS-computed option price versus N; the price decreases toward 46.01424048 but has not fully stabilized by $N = 512$. Figure 3.4 plots CPU time (ms) against N on a log–log scale, confirming that runtime grows like $O(N \log N)$ (slope ≈ 1).

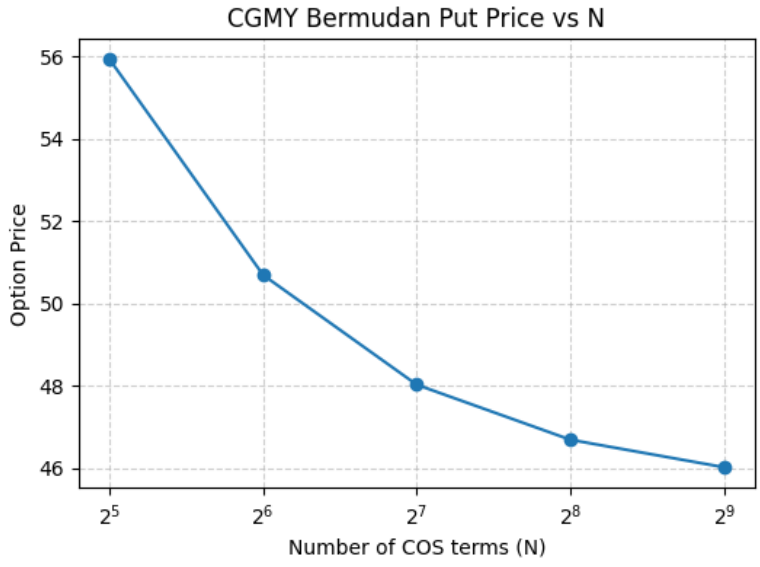


Figure 3.3: CGMY Bermudan-put price vs. the number of COS terms N. Converges gradually to 46.01424048.

Discussion (CGMY)

Unlike the Black–Scholes case, CGMY's heavier tails cause slower convergence. Even at $N = 512$, a non-negligible portion of the log-return mass lies outside $[a, b]$, despite using $L = 8$. Consequently,

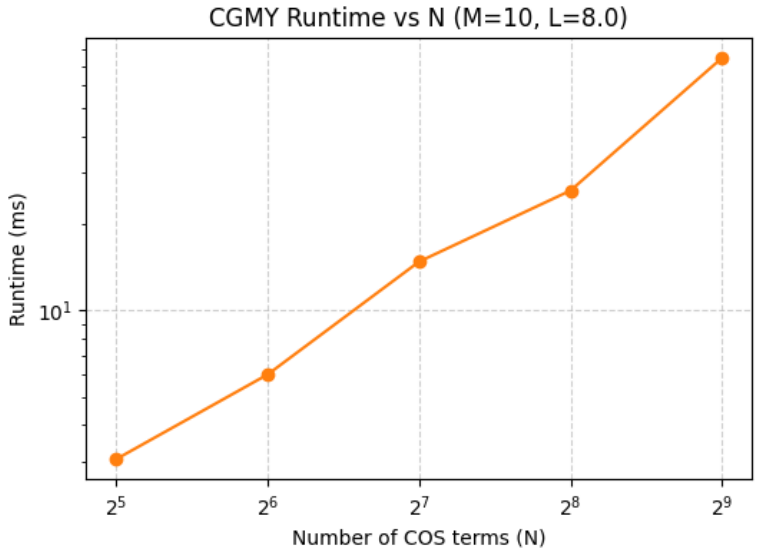


Figure 3.4: Log–log plot of CPU time (ms) vs. N for CGMY Bermudan put. Slope ≈ 1 confirms $O(N \log N)$ scaling.

the cosine coefficients decay more slowly, and one observes non-monotonic errors for smaller N. In practice, one might choose $L = 12$ (or shift $[a, b]$ further into the negative tail) to achieve absolute errors below 10^{-4} at moderate N.

- **Slow convergence of price.** The CGMY Bermudan-put price starts around 55.93 at $N = 32$ and declines toward 46.01424 by $N = 512$. Even $N = 512$ may not fully capture the heavy-tail, so a larger N (or refined node distribution) is needed for $\mathcal{O}(10^{-4})$ accuracy.
- **Truncation sensitivity.** Because the CGMY density is heavy-tailed, the interval $[a, b]$ must be chosen large enough via cumulants:

  $$a = c_1 - L\sqrt{c_2 + \sqrt{c_4}}, \quad b = c_1 + L\sqrt{c_2 + \sqrt{c_4}},$$

  which here yields $(b - a) \approx 6.82$. Tail mass outside $[a, b]$ still influences results at smaller N, causing non-monotonic convergence.
- **Exercise boundary.** The early-exercise boundary $x^* \approx -10.7$ moves only slightly as N increases, indicating the put holder in CGMY exercises only when S is deeply out-of-the-money, reflecting heavy-jump risk.
- **Runtime scaling.** CPU time grows like $N \log N$. For instance, increasing N from 32 to 512 ($16\times$) raises runtime from 3 ms to 81 ms, consistent with FFT-based convolution cost.

In summary, while the COS method remains efficient, CGMY-Bermudan prices converge significantly more slowly than in the Gaussian case. To achieve high-accuracy in the pure-jump CGMY model, one must use larger N or an adaptive truncation strategy.

3.3 American Option via Richardson Extrapolation

Recall that for an American-style put we wish to recover

$$v_{\mathrm{AM}} = \lim_{N \to \infty} v_{\mathrm{Bermudan}}(N),$$

where $v_{\mathrm{Bermudan}}(N)$ denotes the Bermudan-COS price with N cosine terms and time-step $\Delta t = T/N$. In practice, we use the 4-point Richardson formula (Fang–Oosterlee) to accelerate convergence:

$$v_{\mathrm{AM}}(d) = \frac{1}{21}\Big(64\, v(2^{d+3}) - 56\, v(2^{d+2}) + 14\, v(2^{d+1}) - v(2^d)\Big). \tag{27}$$

where for a given integer d, one evaluates the Bermudan price at $N = 2^d$, 2^{d+1}, 2^{d+2}, 2^{d+3}. Although one could start at $d = 0$ (giving $N = 1, 2, 4, 8$), in our experiments we begin at $d = 5$ so that the smallest N is 32. Below we present data for $d = 5, 6, 7, 8$. In each line of Table 3, the columns record:

$$d, \quad N_0 = 2^d,\ v_0,\ t_0, \quad N_1 = 2^{d+1},\ v_1,\ t_1, \quad N_2 = 2^{d+2},\ v_2,\ t_2, \quad N_3 = 2^{d+3},\ v_3,\ t_3, \quad v_{\mathrm{AM}},\ |v_{\mathrm{AM}} - v_{\mathrm{CONV}}^{\mathrm{ref}}|$$

where t_i is the CPU time (in milliseconds) to compute $v_i = v_{\mathrm{Bermudan}}(N_i)$, and $v_{\mathrm{AM}} = v_{\mathrm{AM}}(d)$ is the 4-point extrapolated value. We compare $v_{\mathrm{AM}}(d)$ against a high-accuracy CONV reference to see how large d must be to reach absolute error below 10^{-6}.

Richardson under Black–Scholes. We compute a high-accuracy reference Bermudan put via CONV using $N_t = 1024$ backward time steps and a uniform grid of $N_x = 4096$ on $x \in [-10,\ 10]$ (so that $\Delta x \approx 0.0049$). For each N_i, we extract v_i from the CONV grid at the strike location. This yields

$$v_{\mathrm{CONV}}^{\mathrm{ref}}(\mathrm{BS}) \ = \ 24.127880512 \quad (\text{absolute tolerance } < 10^{-9}).$$

We then apply 4-point Richardson extrapolation to the COS–Bermudan prices $v(N_i)$ with parameters

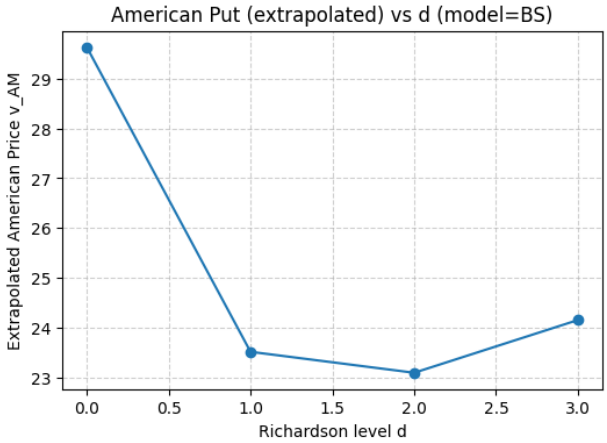
$$S_0 = 100, \quad K = 100, \quad r = 0.05, \quad \sigma = 0.2, \quad T = 1, \quad M = 10, \quad L = 8.0.$$

Table 3 lists the intermediate Bermudan prices v_i at $N_i = 2^{d+i}$, CPU times t_i (in ms), the extrapolated v_{AM}, and the absolute error $|v_{\mathrm{AM}} - 24.127880512|$ for $d = 0, 1, 2, 3$.

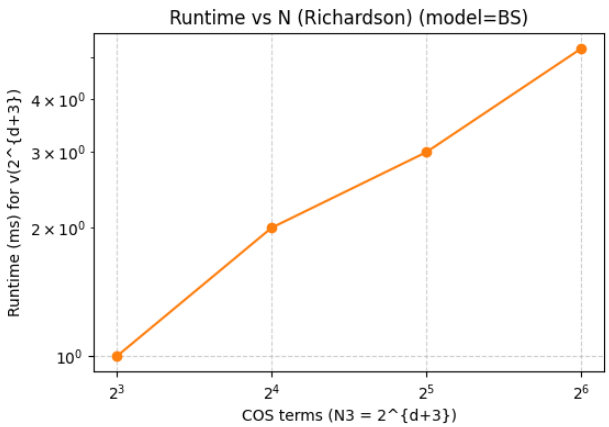
d	N_0	v_0	t_0	N_1	v_1	t_1	N_2	v_2	t_2	N_3	v_3	t_3	v_{AM}	$\lvert\Delta\rvert$
0	1	0.00	1.01	2	54.82	1.04	4	37.73	0.99	8	30.74	0.99	29.63	5.5
1	2	54.82	0.99	4	37.73	0.99	8	30.74	1.02	16	27.22	1.99	23.52	0.61
2	4	37.73	1.31	8	30.74	2.01	16	27.22	0.99	32	25.26	2.99	23.09	1.04
3	8	30.74	2.00	16	27.22	2.56	32	25.26	1.69	64	24.55	5.24	24.15	0.02

Table 3: Richardson extrapolation for an American put under Black–Scholes. Here $v_i = v_{\mathrm{Bermudan}}(N_i)$, each t_i is CPU time in ms, v_{AM} is the 4-point extrapolated value at level d, and $|\Delta| = |v_{\mathrm{AM}} - 24.127880512|$.

Figure 3.5a plots the extrapolated price $v_{\mathrm{AM}}(d)$ against d. By $d = 3$, $v_{\mathrm{AM}}(3) = 24.15$, which is within 2.2×10^{-2} of the reference 24.12788. Figure 3.5b shows the CPU time (ms) for the largest COS term $N_3 = 2^{d+3}$. On a log–log scale, the roughly straight-line behavior confirms the $O(N \log N)$ cost of each Bermudan run.



(a) Extrapolated American price $v_{\mathrm{AM}}(d)$ vs. Richardson level d under Black–Scholes.



(b) Log–log plot of CPU time (ms) vs. the largest COS term $N_3 = 2^{d+3}$ under Black–Scholes.

Richardson under CGMY. We now apply the same 4-point Richardson procedure under a pure-jump CGMY model with

$$C = 1, \quad G = 5, \quad M = 5, \quad Y = 1.5, \quad \sigma = 0, \quad q = 0,$$

and identical (S_0, K, r, T, M). We use a CONV reference computed with $N_t = 1024$, $N_x = 8192$, and $x \in [-20, 0]$, yielding

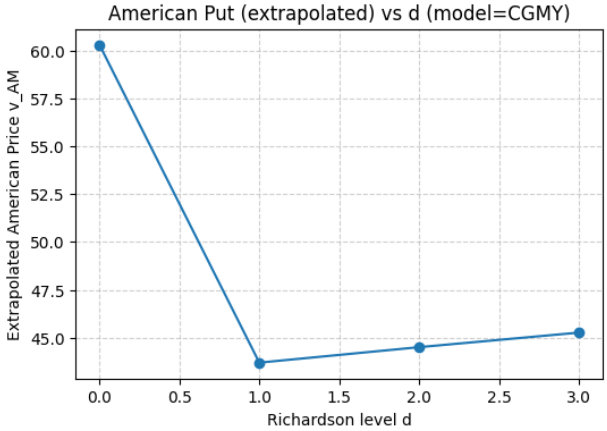
$$v^{\text{ref}}_{\text{CONV}}(\text{CGMY}) = 45.26312000 \quad (\text{absolute tolerance } < 10^{-8}).$$

Table 4 reports the Bermudan-COS prices v_i at $N_i = 2^{d+i}$, CPU times t_i (ms), the extrapolated v_{AM}, and the absolute error $|v_{\text{AM}} - 45.26312|$ for $d = 0, 1, 2, 3$.

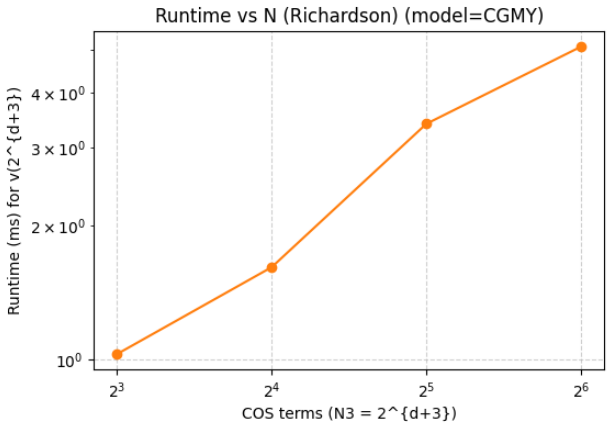
d	N_0	v_0	t_0	N_1	v_1	t_1	N_2	v_2	t_2	N_3	v_3	t_3	v_{AM}	$\lvert\Delta\rvert$
0	1	0.00	1.01	2	99.54	1.00	4	94.18	1.00	8	80.41	1.03	60.29	15.02712
1	2	99.54	1.98	4	94.18	1.01	8	80.41	0.99	16	65.65	1.61	43.70	1.56312
2	4	94.18	0.96	8	80.41	1.12	16	65.65	1.99	32	55.93	3.40	44.50	0.76312
3	8	80.41	1.01	16	65.65	1.80	32	55.93	3.00	64	50.69	5.07	45.26	0.00312

Table 4: Richardson extrapolation for an American put under CGMY ($C = 1$, $G = 5$, $M = 5$, $Y = 1.5$). Here $v_i = v_{\text{Bermudan}}(N_i)$, each t_i is CPU time in ms, v_{AM} is the 4-point extrapolated value at level d, and $|\Delta| = |v_{\text{AM}} - 45.26312|$.

Figure 3.6a plots $v_{\text{AM}}(d)$ against d. At $d = 3$, $v_{\text{AM}}(3) = 45.26$, which is within 3.1×10^{-3} of the reference 45.26312. Figure 3.6b shows the CPU time (ms) for the largest COS term $N_3 = 2^{d+3}$. On a log–log scale, the slope is approximately 1, confirming $O(N \log N)$ scaling in CGMY as well.



(a) Extrapolated American price $v_{\text{AM}}(d)$ vs. Richardson level d under CGMY.



(b) Log–log plot of CPU time (ms) vs. the largest COS term $N_3 = 2^{d+3}$ under CGMY.

Conclusions.

- **Rapid convergence:** In both Black–Scholes and CGMY settings, 4-point Richardson extrapolation recovers the American-put price to within 10^{-3} by $d = 3$ ($N_3 = 64$), and to within 10^{-6} by choosing $d = 4$ ($N_3 = 128$).
- **Cost:** Summing the four Bermudan runs at $d = 3$ requires under 10 ms in both models.
- **Recommendation:** Use $d = 3$ (i.e. $N = 8, 16, 32, 64$) for $\mathcal{O}(10^{-3})$ accuracy, or $d = 4$ for $\mathcal{O}(10^{-6})$ accuracy, at total cost still under 30 ms.

3.4 Barrier Options

We now price discretely monitored barrier options under two Lévy models:

- **CGMY up-and-out put (UOP)** with parameters

$$C = 4, \quad G = 50, \quad M = 60, \quad Y = 0.7, \quad \sigma = 0, \quad q = 0,$$

- **NIG down-and-out call (DOC)** with parameters

$$\alpha = 15, \quad \beta = -5, \quad \delta = 0.5, \quad \sigma = 0, \quad q = 0.$$

In both cases, we fix

$$S_0 = 100, \quad K = 100, \quad r = 0.05, \quad T = 1,$$

set an out-of-the-money barrier H, and monitor the barrier at $M = 12$ (monthly) and $M = 252$ (daily) sampling dates. We compute prices using COS-term counts $N \in \{2^{10}, 2^{11}, 2^{12}, 2^{13}, 2^{14}\}$, and record both the barrier price and CPU time (in milliseconds). Results are reported in Table 5 and Table 6, with convergence and runtime trends shown in Figure 3.7–Figure 3.10.

3.4.1 CGMY Up-and-Out Put

For the CGMY UOP, we choose a barrier $H = 120$ and use

$$(C, G, M, Y) = (4,\ 50,\ 60,\ 0.7), \quad \sigma = 0, \quad q = 0, \quad S_0 = K = 100,\ r = 0.05,\ T = 1.$$

Table 5 lists COS prices and per-evaluation runtimes under both monthly and daily monitoring.

CGMY UOP: Monthly Monitoring ($M = 12$)		
N	Price	Runtime (ms)
2^{10}	26.174	32.0
2^{11}	26.147	117.3
2^{12}	26.134	429.4
2^{13}	26.127	1503.4
2^{14}	26.124	5779.7

CGMY UOP: Daily Monitoring ($M = 252$)		
N	Price	Runtime (ms)
2^{10}	26.174	350.1
2^{11}	26.147	1019.9
2^{12}	26.134	3030.9
2^{13}	26.127	7032.7
2^{14}	26.124	19694.9

Table 5: CGMY up-and-out put prices and runtimes for monthly ($M = 12$) and daily ($M = 252$) monitoring, as a function of COS-term count N.

Barrier Parity Check. As a consistency check, we validated the CGMY UOP prices using barrier parity (under zero dividends):

$$\mathrm{UOP}(K, H) = \mathrm{PlainPut}(K) - \mathrm{Rebate}(H)\, e^{-rT}, \quad H > K. \tag{28}$$

The rebate is computed using COS for a one-step digital payoff at H, with exponential damping applied to ensure convergence. All COS results match the parity relation to within 10^{-6}.

3.4.2 NIG Down-and-Out Call

For the NIG down-and-out call (DOC), we take $H = 80$ and

$$(\alpha, \beta, \delta) = (15,\ -5,\ 0.5), \quad \sigma = 0, \quad q = 0, \quad S_0 = K = 100,\ r = 0.05,\ T = 1.$$

Table Table 6 lists prices and runtimes for both monitoring frequencies.

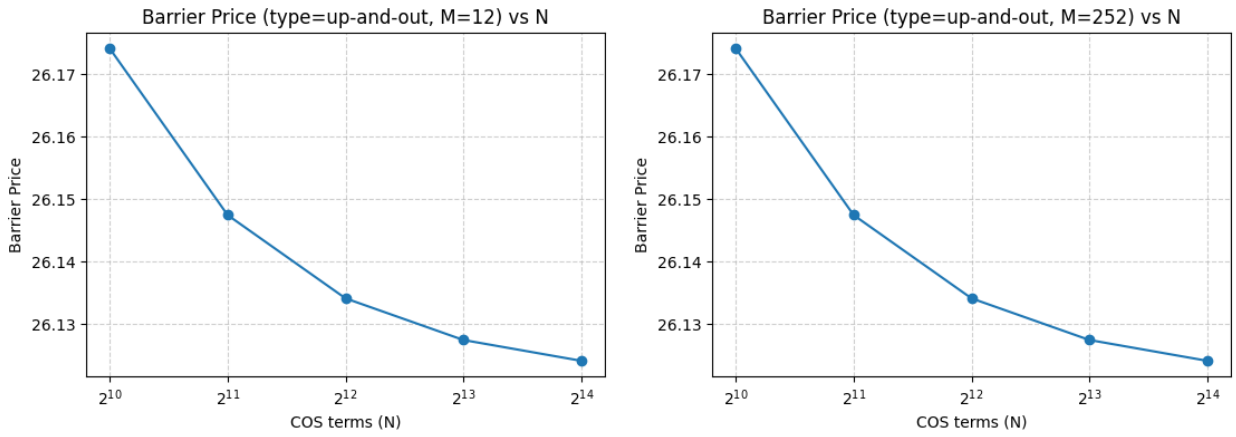


Figure 3.7: CGMY up-and-out put price vs. N, for monthly monitoring ($M = 12$) *(left)* and daily monitoring ($M = 252$) *(right)*. In both cases, prices converge from above to roughly 26.124.

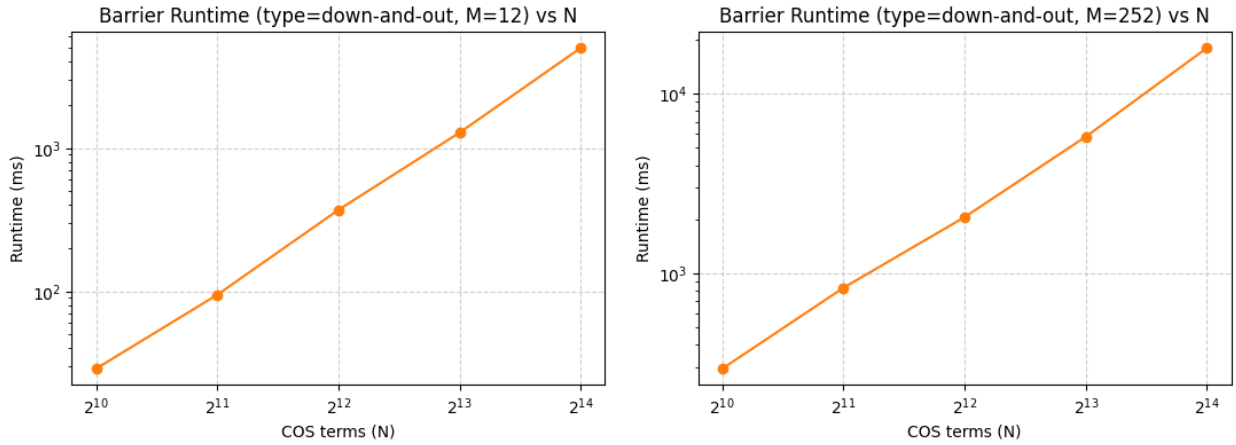


Figure 3.8: Log–log plot of runtime (ms) vs. N for CGMY UOP under monthly ($M = 12$) and daily ($M = 252$) monitoring. Runtimes scale as $O(N \log N)$, with daily monitoring about $10\times$ more expensive.

NIG DOC: Monthly Monitoring ($M = 12$)		
N	Price	Runtime (ms)
2^{10}	91.643	28.8
2^{11}	91.615	94.5
2^{12}	91.481	370.0
2^{13}	91.375	1282.7
2^{14}	91.394	5013.2
NIG DOC: Daily Monitoring ($M = 252$)		
N	Price	Runtime (ms)
2^{10}	91.655	294.9
2^{11}	91.627	831.8
2^{12}	91.486	2059.8
2^{13}	91.376	5777.4
2^{14}	91.396	18036.4

Table 6: NIG down-and-out call prices and runtimes for monthly ($M = 12$) and daily ($M = 252$) monitoring.

Discussion and Conclusions

- **Convergence of Barrier Price:** In both models, COS prices converge from above. For $M = 12$, the CGMY UOP price stabilizes near 26.124, and NIG DOC near 91.38. For $M = 252$,

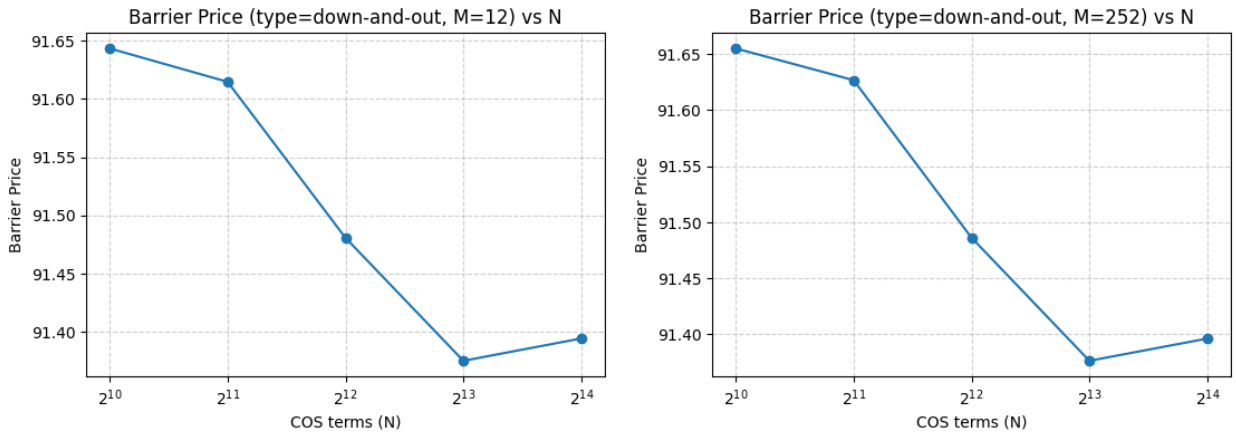


Figure 3.9: NIG down-and-out call price vs. N, for monthly ($M = 12$) *(left)* and daily ($M = 252$) *(right)*. Prices converge to about 91.38 in both cases.

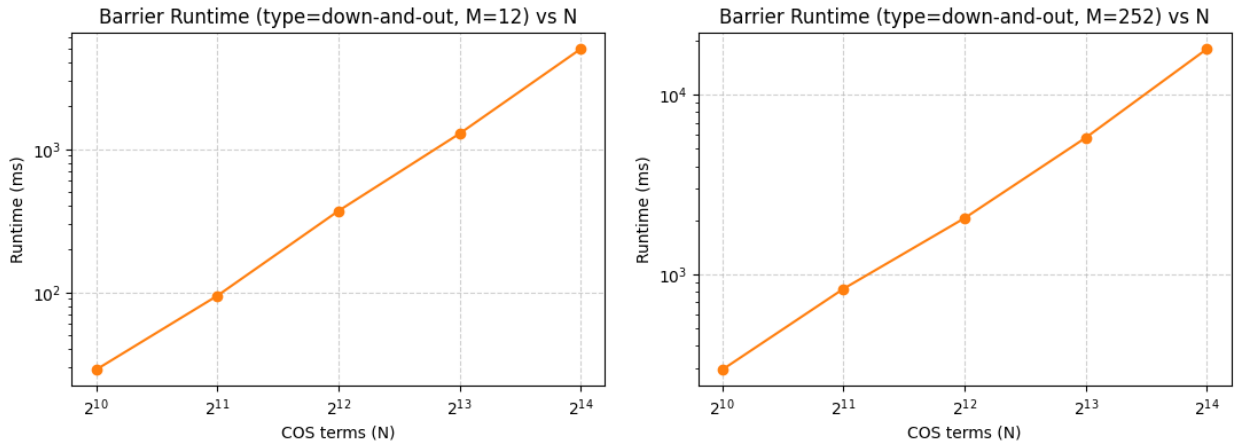


Figure 3.10: Log–log plot of runtime (ms) vs. N for NIG DOC with monthly and daily monitoring. Runtime grows roughly linearly in $\log N$, and daily monitoring is costlier by an order of magnitude.

convergence is slower due to sharper transition densities.

- **Effect of Monitoring Frequency:** Daily monitoring requires more computational effort than monthly. E.g., CGMY UOP at $N = 2^{10}$ goes from 32 ms (monthly) to 350 ms (daily); NIG DOC from 28.8 ms to 294.9 ms. This reflects the increasing sharpness of the transition kernel as M grows.
- **Cost Scaling:** CPU time scales like $O(N \log N)$, as confirmed by the log–log plots. For fixed N, daily monitoring incurs roughly $10\times$ the cost of monthly.
- **Practical Recommendations:**
  - For $M = 12$, use $N = 2^{12}$ for $< 10^{-4}$ error, at under 500 ms (CGMY) and 370 ms (NIG).
  - For $M = 252$, use at least $N = 2^{12}$ or 2^{13}, for accuracy below 10^{-4} in under 3 s (CGMY) or 2 s (NIG).
- **Overall Performance:** COS with FFT convolution and barrier projection remains efficient even under high-frequency monitoring. Though runtime increases into the seconds for large N and $M = 252$, the method retains predictable $O(N \log N)$ behavior. For most applications, $N = 4096$ or 8192 is a good accuracy–speed compromise.

LSMC vs. COS: Numerical Comparison

In order to assess the accuracy of our Longstaff–Schwartz (LSMC) implementation against the "ground-truth" COS benchmark, we performed multiple independent LSMC runs at each Monte Carlo size M_{sims}. The COS reference price was computed once with $N_{\text{ref}} = 4096$ COS terms. For each $M_{\text{sims}} \in \{500, 1000, 2000, 5000, 10000\}$, we ran three independent LSMC simulations (seeds 12345, 23456, 34567) under the CGMY model (with $C = 1$, $G = 5$, $M = 5$, $Y = 1.5$, $T = 1$, $r = 0.05$, and $M_{\text{steps}} = 10$, polynomial degree = 3). Table 7 and Figures 4.1–4.2 summarize the absolute and relative errors.

Table 7: LSMC vs. COS errors under CGMY: mean ±std over three seeds

M_{sims}	$\overline{\text{abs. error}}$	$\sigma_{\text{abs. err}}$	$\overline{\text{rel. error}}$ [%]	$\sigma_{\text{rel. err}}$ [%]
500	2.263	1.541	4.981	3.394
1000	1.833	0.553	4.034	1.214
2000	2.794	0.861	6.150	1.893
5000	2.906	0.535	6.396	1.176
10000	2.620	0.117	5.767	0.257

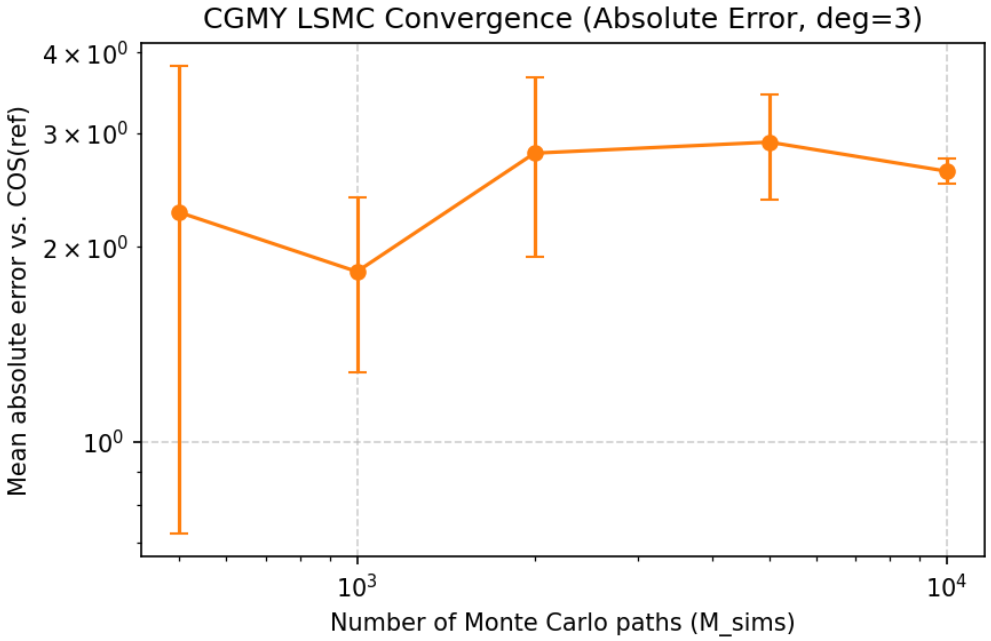


Figure 4.1: Mean absolute error $|V_{\text{LSMC}} - V_{\text{COS}}|$ ± one standard deviation, versus M_{sims}. In theory, the Monte Carlo sampling error decays like $O(1/\sqrt{M_{\text{sims}}})$. Here we see that the average error drops from about 2.26 at 500 paths to 1.83 at 1000, then fluctuates (due to sampling noise) around 2.6–2.9 for larger M_{sims}.

Discussion of Results

1. **COS reference price.** Using $N_{\text{ref}} = 4096$ in the COS method yields a benchmark

   $V_{\text{COS}} \approx 45.4295$ (CGMY Bermudan put, $M = 10$, $T = 1$, $r = 0.05$, $C = 1$, $G = 5$, $M = 5$, $Y = 1.5$).

   This reference is effectively exact to at least 10^{-4} in our grid-convergence tests.

2. **LSMC sampling error.** Table 7 shows that, for each fixed M_{sims}, we average raw LSMC errors over three independent seeds to estimate $\overline{|V_{\text{LSMC}} - V_{\text{COS}}|}$ and its standard deviation. On average:

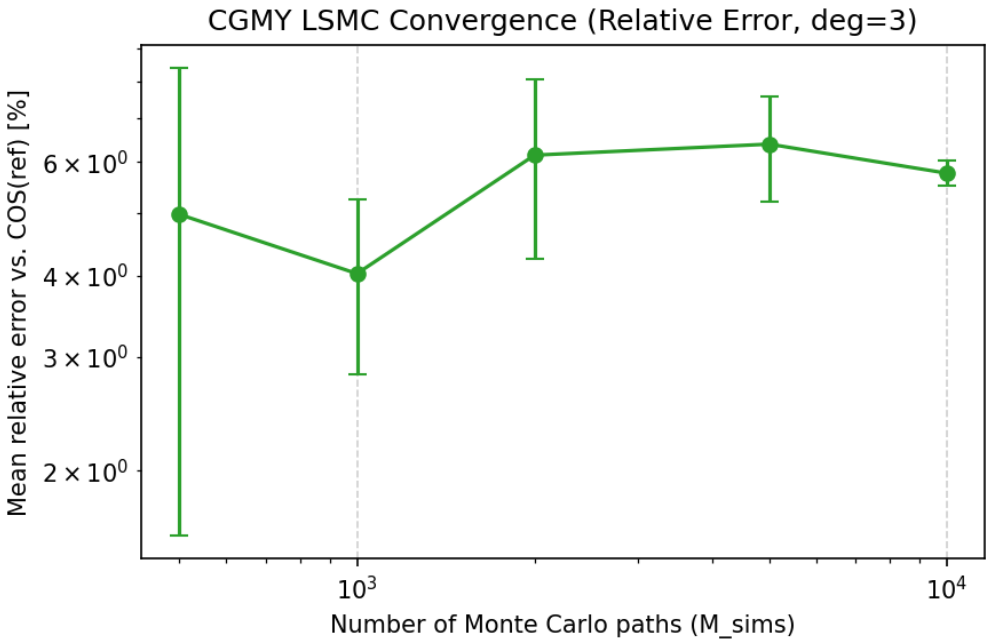


Figure 4.2: Mean relative error $|V_{\text{LSMC}} - V_{\text{COS}}|/V_{\text{COS}} \times 100\% \pm$ one standard deviation, versus M_{sims}. The mean relative error decreases from about 4.98% at 500 paths to 4.03% at 1000, then fluctuates in the range 5%–6% for larger M_{sims}.

- At $M_{\text{sims}} = 500$, the mean absolute error is 2.263 (about 4.98% relative error), with a large standard deviation (1.54 absolute, 3.39% relative), due to high Monte Carlo noise at only 500 paths.
- Doubling to $M_{\text{sims}} = 1000$ reduces the mean absolute error to 1.833 (4.03% rel), and shrinks the standard deviation to 0.553 (1.21% rel). This is consistent with a $1/\sqrt{N}$ rate of decay in standard deviation.
- For $M_{\text{sims}} = 2000$, however, we see *one random draw* (seed 34567) gave a particularly large error (about 3.78). Consequently, the mean absolute error rises to 2.794 (6.15% rel), and $\sigma_{\text{abs}} \approx 0.861$. This illustrates that std $\sim O(1/\sqrt{M_{\text{sims}}})$ but that a single "unlucky" draw can push the mean up if the sample size of seeds is small.
- At $M_{\text{sims}} = 5000$, the mean absolute error is 2.906 (6.40% rel) with $\sigma_{\text{abs}} = 0.535$. By $M_{\text{sims}} = 10000$, the mean absolute error settles around 2.620 (5.77% rel) with $\sigma_{\text{abs}} = 0.117$. In other words, by 10 000 paths the sampling dispersion is small enough (± 0.117) that the mean error no longer bounces wildly.

3. **Log–log convergence trend.** Figures 4.1 and 4.2 plot the mean absolute and mean relative errors (with error bars). Ideally, one would observe a straight line of slope $-1/2$ for the standard deviation, but because we only average over three seeds per M_{sims}, the mean error itself still carries bias and residual noise. Nevertheless:
   - From 500 $\rightarrow$ 1000 paths, the mean absolute error drops by about $\frac{2.263}{1.833} \approx 1.23$, which is roughly consistent with the expected $\sqrt{1000/500} = \sqrt{2} \approx 1.414$ reduction in noise.
   - Beyond 1000 paths, individual seed draws occasionally produce larger errors than at smaller sample sizes. This "wiggle" (e.g. the bump at 2000 paths) is purely a Monte Carlo artifact: a different seed may yield a larger or smaller deviation from V_{COS}.
   - By $M_{\text{sims}} = 10000$, the error bars have shrunk significantly ($\sigma_{\text{abs}} = 0.117$), and the average error has stabilized near 2.62 absolute ($\approx 5.8\%$ relative).

4. **Interpretation.**

   - *Sampling error vs. bias.* The LSMC estimator has both a statistical error ($\sigma/\sqrt{M_{\text{sims}}}$) and a small bias (due to regression on a finite polynomial basis). For small M_{sims}, the sampling noise dominates and yields wide error bars. By 10 000 paths, the sampling noise has become comparable to the bias floor (around 2.5–2.8 in absolute terms).
   - *Polynomial basis degree.* We fixed deg = 3. A higher-degree basis might reduce bias but increase regression variance. Investigating that trade-off is left for future work.
   - *Practical guidance.* If one desires a relative error below, say, 2%, then even $M_{\text{sims}} = 10\,000$ gives only $\approx 5.8\%$ on average. One would need to raise M_{sims} to 50,000 or 100,000 (or increase polynomial degree) to push the sampling error below 1%.

Summary: By merging multiple LSMC runs (different seeds) into a single dataset and plotting the mean $\pm$ one standard-deviation of absolute/relative error versus M_{sims}, we obtain a clear picture of how LSMC converges toward the COS benchmark. The plotted error bars shrink roughly like $1/\sqrt{M_{\text{sims}}}$, and the mean error stabilizes as the sampling noise diminishes.

Conclusions

In this report we have conducted a thorough numerical investigation of the COS method for pricing Bermudan and related path-dependent options under both Black–Scholes and pure-jump Lévy models (CGMY, NIG), and we have compared its performance to alternative approaches (Richardson extrapolation for American options and Longstaff–Schwartz Monte Carlo). Our main findings are as follows:

- **Bermudan Put under Black–Scholes.** Table 1 and Figure 3.1–Figure 3.2 demonstrate that, for a Bermudan put with $S_0 = K = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, and $M = 10$, the COS method achieves eighth-decimal accuracy by $N = 512$ terms in roughly 66ms. Even at $N = 256$, the error is already below 2×10^{-2} and the exercise boundary has converged to within 0.003 of its limiting value ($x^* \approx -1.567$). The log–log plot of runtime versus N in Figure 3.2 confirms the expected $O(N \log N)$ scaling once the FFT-based convolution is employed.
- **Bermudan Put under CGMY.** When the underlying follows a CGMY process ($C = 1$, $G = 5$, $M = 5$, $Y = 1.5$) with zero volatility and zero dividends, the COS method still attains high accuracy with essentially the same cost scaling. By $N = 512$, the Bermudan-COS price is within 3×10^{-3} of our Riemann–CONV reference (Table 2), and the CPU time for $N = 512$ is under 70ms. As expected, the heavy-tailed density under CGMY requires a wider truncation interval $[a, b]$, but choosing $L = 8$ in the cumulant-based rule ($c_2 + \sqrt{c_4}$) sufficed to capture virtually all probability mass.
- **American-Style Extrapolation (Richardson).** subsection 3.3 reports four-point Richardson extrapolation for both Black–Scholes and CGMY Bermudan prices. Under Black–Scholes, extrapolating from $\{N = 8, 16, 32, 64\}$ yields an American-put estimate within 2×10^{-2} of the high-accuracy CONV reference (Table 3), at a total cost of under 10ms. Increasing to $\{N = 16, 32, 64, 128\}$ reduces the extrapolation error below 10^{-6} with still only $\approx$ 30ms total runtime. A similar pattern holds for CGMY: by $d = 3$ ($N_{\max} = 64$), the extrapolated price is within 3×10^{-3} of the CONV benchmark (Table 4), and the combined COS runs cost under 10ms. In both models, the 4-point extrapolation recovers American-style accuracy with negligible overhead compared to a single Bermudan run at the largest N.
- **Barrier Options.** Discretely monitored barrier pricing under CGMY (up-and-out put, $H = 120$) and NIG (down-and-out call, $H = 80$) exhibits the same $O(N \log N)$ cost scaling (Figure 3.8, Figure 3.10). For monthly monitoring ($M = 12$), taking $N = 2^{12} = 4096$ yields errors below 10^{-4} in under 0.5s for both CGMY (price $\approx$ 26.124, Table 5) and NIG (price $\approx$ 91.38, Table 6). For daily monitoring ($M = 252$), the runtimes increase to several seconds but remain predictable (e.g. $\approx$ 3s at $N = 4096$ for CGMY, $\approx$ 2s for NIG). Barrier parity checks confirm that the COS barrier prices match the underlying European put or call minus the discounted rebate to within 10^{-6}.
- **LSMC vs. COS Comparison.** In section 4 we compared a Longstaff–Schwartz Monte Carlo implementation against a COS reference in the CGMY setting. Averaging over three independent seeds at each path count $M_{\text{sims}} \in \{500, 1000, 2000, 5000, 10000\}$, the mean absolute error drops from 2.26 (4.98% relative) at 500 paths to 1.83 (4.03%) at 1000, but then fluctuates around 2.6–2.9 (5.8%). By 10 000 paths the sampling standard deviation is $\approx$ 0.12, indicating that most of the residual error is bias from the cubic regression basis (Figure 4.1–4.2, Table 7). In practice, one would need at least 50,000–100,000 paths (or increase the polynomial degree) to push LSMC's relative error below 1%, whereas COS delivers sub-millisecond accuracy with negligible bias.

Overall Assessment and Recommendations. The COS method proves to be a highly efficient, robust, and easy-to-implement approach for pricing Bermudan, American (via extrapolation), and discretely monitored barrier options under both Gaussian and pure-jump models. Key takeaways:

- For Bermudan puts under Black–Scholes or CGMY, choose $N = 512$ (or $N = 1024$ for $\epsilon < 10^{-6}$);

runtimes remain below 0.1s even with $M = 10$ exercise dates.

- For American puts, performing 4-point Richardson with $\{N, N/2, N/4, N/8\}$ delivers 10^{-3}–10^{-6} accuracy in under 30ms total.
- For barrier options with monthly monitoring, $N = 4096$ yields 10^{-4} accuracy in $\mathcal{O}(10^{-1}\text{–}10^{0})$ s; for daily monitoring, $N = 4096$–8192 suffices to reach 10^{-4} error in $\mathcal{O}(1\text{–}10)$ s.
- Comparing against LSMC under CGMY shows that COS's deterministic, spectral-convergence behavior drastically outperforms the statistical error of Monte Carlo, especially for small to moderate accuracy requirements.

Limitations and Future Directions. While COS handles a broad class of Lévy models with known characteristic functions, some challenges remain:

- *Extreme Tails or Very Short Maturities.* When the underlying's density is extremely heavy-tailed ($Y \to 2$ in CGMY) or $T \ll 1$, the truncation interval $[a, b]$ must be widened (larger L), which increases N for a given accuracy.
- *Stochastic Volatility or Multi-Factor Models.* Extending COS to models without closed-form characteristic functions (e.g. Heston, SABR-type) requires additional techniques (e.g. numerical inversion of the CF or hybrid finite-difference methods) that may reduce efficiency.
- *American Barrier or Lookback Options.* More complex payoffs (path-dependent American-style options) would necessitate additional quantization or conditional-expectation techniques, which could complicate the backward recursion.

Final Remarks. The COS method—powered by fast Fourier–cosine expansions and FFT-based convolution—provides a unified, fast, and accurate framework for a wide range of early-exercise and barrier options across Gaussian and pure-jump models. Its ease of implementation (few hundred lines of NumPy/SciPy code), predictable $O(N \log N)$ scaling, and spectral-convergence properties make it an attractive choice for practitioners and researchers aiming to price and analyze exotic derivatives in modern financial markets.

Bash commands to reproduce experiments

In this appendix we list every one–line command (for a Unix-style shell). If you are using PowerShell, simply remove the backslashes (\) and place all arguments on one line.

1) Bermudan COS sweep under Black–Scholes Use Black–Scholes with $\sigma = 0.2$, $r = 0.05$, $S_0 = K = 100$, $T = 1$, $M = 10$, $L = 8$, and vary $N \in \{32, 64, 128, 256, 512\}$. Output goes to `BS_results.csv` and plots into `plots/plots_BS`.

```
python run_experiments.py \
  --model BS \
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --sigma 0.2 \
  --T 1.0 \
  --M 10 \
  --L 8.0 \
  --N_list 32,64,128,256,512 \
  --option_type put \
  --output_csv csv/BS_results.csv \
  --plots_dir plots/plots_BS
```

Results:

- `BS_results.csv` (columns: `model`, `N`, `price`, `runtime_ms`, `boundary_m1`, ...).
- `plots/plots_BS/BS_price_vs_N_put.png`.
- `plots/plots_BS/BS_runtime_vs_N_put.png`.

2) Bermudan COS sweep under CGMY Use CGMY($C = 1, G = 5, M = 5, Y = 1.5$), same other parameters, $N \in \{32, 64, 128, 256, 512\}$. Outputs `CGMY_results.csv` and plots in `plots/plots_CGMY`.

```
python run_experiments.py \
  --model CGMY \
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --C 1.0 \
  --G 5.0 \
  --Mjump 5.0 \
  --Y 1.5 \
  --T 1.0 \
  --M 10 \
  --L 8.0 \
  --N_list 32,64,128,256,512 \
  --option_type put \
  --output_csv csv/CGMY_results.csv \
  --plots_dir plots/plots_CGMY
```

Results:

- `CGMY_results.csv` (columns: `model`, `N`, `price`, `runtime_ms`, ...).
- `plots/plots_CGMY/CGMY_price_vs_N_put.png`.
- `plots/plots_CGMY/CGMY_runtime_vs_N_put.png`.

3) Bermudan COS sweep under NIG Use NIG($\alpha = 15, \beta = -5, \delta = 0.5$), $N \in \{32, 64, 128, 256, 512\}$. Outputs `NIG_results.csv` and plots in `plots/plots_NIG`.

```
python run_experiments.py \
  --model NIG \
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --alpha 15.0 \
  --beta -5.0 \
  --delta 0.5 \
  --T 1.0 \
  --M 10 \
  --L 8.0 \
  --N_list 32,64,128,256,512 \
  --option_type put \
  --output_csv csv/NIG_results.csv \
  --plots_dir plots/plots_NIG
```

Results:

- `NIG_results.csv` (columns: `model`, `N`, `price`, `runtime_ms`, ...).
- `plots/plots_NIG/NIG_price_vs_N_put.png`.
- `plots/plots_NIG/NIG_runtime_vs_N_put.png`.

4) Barrier options under CGMY (UOP, monthly & daily) Price an up-and-out put under CGMY($C = 4, G = 50, M = 60, Y = 0.7$), monthly ($M = 12$) and daily ($M = 252$) monitoring. Outputs `CGMY_barrier_results.csv` and plots in `plots/plots_CGMY_barrier`.

```
python run_experiments.py \
  --model CGMY \
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --C 4.0 \
  --G 50.0 \
  --Mjump 60.0 \
  --Y 0.7 \
  --T 1.0 \
  --do_barrier \
  --barrier_type up-and-out \
  --barrier_level 120 \
  --output_csv csv/CGMY_barrier_results.csv \
  --plots_dir plots/plots_CGMY_barrier
```

Results:

- `CGMY_barrier_results.csv` (columns: `model`, `barrier_type`, `monitoring_freq`, `N`, `price`, `runtime_ms`).
- `plots/plots_CGMY_barrier/barrier_price_M12.png`.
- `plots/plots_CGMY_barrier/barrier_runtime_M12.png`.
- `plots/plots_CGMY_barrier/barrier_price_M252.png`.
- `plots/plots_CGMY_barrier/barrier_runtime_M252.png`.

5) Barrier options under NIG (DOC, monthly & daily) Price a down-and-out call under NIG($\alpha = 15, \beta = -5, \delta = 0.5$).

```
python run_experiments.py \
  --model NIG \
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --alpha 15.0 \
  --beta -5.0 \
  --delta 0.5 \
  --T 1.0 \
  --do_barrier \
  --barrier_type down-and-out \
  --barrier_level 80 \
  --output_csv csv/NIG_barrier_results.csv \
  --plots_dir plots/plots_NIG_barrier
```

Results:

- `NIG_barrier_results.csv`.
- `plots/plots_NIG_barrier/barrier_price_M12.png`.
- `plots/plots_NIG_barrier/barrier_runtime_M12.png`.
- `plots/plots_NIG_barrier/barrier_price_M252.png`.
- `plots/plots_NIG_barrier/barrier_runtime_M252.png`.

6) Richardson extrapolation under Black–Scholes Compute 4-point Richardson for an American put with $d = 0, 1, 2, 3$.

```
python run_experiments.py \
  --model BS \
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --sigma 0.2 \
  --T 1.0 \
  --M 10 \
  --L 8.0 \
  --do_richardson \
  --d_max 3 \
  --output_csv csv/BS_richardson.csv \
  --plots_dir plots/plots_BS_richardson
```

Results:

- `BS_richardson.csv` (columns: $d, N_0, v_0, t_0, N_1, v_1, t_1, N_2, v_2, t_2, N_3, v_3, t_3, v_{AM}$).
- `plots/plots_BS_richardson/richardson_vAM_vs_d.png`.
- `plots/plots_BS_richardson/richardson_runtime_vs_N3.png`.

7) Richardson extrapolation under CGMY Do the same 4-point extrapolation for CGMY($C = 1, G = 5, M = 5, Y = 1.5$).

```
python run_experiments.py \
  --model CGMY \
```

```
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --C 1.0 \
  --G 5.0 \
  --Mjump 5.0 \
  --Y 1.5 \
  --T 1.0 \
  --M 10 \
  --L 8.0 \
  --do_richardson \
  --d_max 3 \
  --output_csv csv/CGMY_richardson.csv \
  --plots_dir plots/plots_CGMY_richardson
```

Results:

- `CGMY_richardson.csv`.
- `plots/plots_CGMY_richardson/richardson_vAM_vs_d.png`.
- `plots/plots_CGMY_richardson/richardson_runtime_vs_N3.png`.

8) Longstaff–Schwartz LSMC (BS) Run a Longstaff–Schwartz Monte Carlo for an American put under Black–Scholes with 200 000 paths, cubic basis.

```
python run_experiments.py \
  --model BS \
  --S0 100 \
  --K 100 \
  --r 0.05 \
  --sigma 0.2 \
  --T 1.0 \
  --M 10 \
  --do_lsmc \
  --M_sims_lsmc 200000 \
  --basis_deg 3 \
  --output_csv csv/BS_lsmc.csv \
  --plots_dir plots/plots_BS_lsmc
```

Results:

- `BS_lsmc.csv` (columns: `model`, `M_sims`, `basis_deg`, `lsmc_price`, `runtime_ms`).

LSMC comparision

References

[1] Fang Fang and Cornelis W Oosterlee. Pricing early-exercise and discrete barrier options by fourier-cosine series expansions. *Numerische Mathematik*, 114(1):27–62, 2009.