Chapter 4
Risk management

4.1 Introduction

In Chapter 2 we study the basic formulation of decision-making problems using stochastic programming. In these problems we consider that the objective of the decision-making agents is either maximizing profit (e.g., the financial profit of an electricity producer) or minimizing cost (e.g., the electricity procurement cost of an industrial consumer). In stochastic programming, where uncertain data are modeled as stochastic processes, the profit or cost is a random variable that can be characterized by a probability distribution. In an optimization problem involving a random objective function it is necessary to optimize a function characterizing the distribution of this random variable, for instance, its expected value. This is the criterion that is generally used in stochastic programming problems. Therefore, the problem consisting in maximizing "the profit" obtained by a decision-making agent results in maximizing the expected profit achieved by this agent.

Despite the numerous advantages of representing a random variable by its expected value, its main drawback is that the remaining parameters characterizing the distribution associated with the random variable are neglected. For instance, a random variable representing a profit with an expected value acceptable for the decision maker could also present a non-negligible probability of experiencing negative profits or losses.

In order to control the risk of experiencing profit distributions with non-desirable properties, e.g., with a high probability of low profit, risk control constitutes an important issue when formulating stochastic programming models. The most usual way of managing risk is to include in the formulation of the problem a term measuring the risk associated with a profit distribution. This term is usually referred to as risk functional or risk measure. Examples of risk measures are the variance of the profit, the probability of falling behind a target value, or the expected value of the profit being inferior to a specified value.

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In this chapter we define and formulate some of the most usual risk measures regarding stochastic programming problems in electricity markets and financial optimization. For the sake of simplicity, all measures considered in this chapter are formulated using a two-stage stochastic programming problem whose objective function consists in maximizing the expected profit attained by a particular decision maker. Specifically, we consider the following risk measures:

1. Variance
2. Shortfall probability
3. Expected shortage
4. Value-at-Risk (VaR)
5. Conditional Value-at-Risk (CVaR).

We also deal with another risk management strategy based on imposing stochastic dominance constraints in the formulation of stochastic programming problems.

The rest of the chapter is organized as follows. Section 4.2 defines concepts related to risk-neutral and risk-averse decision-making problems. In Section 4.3 the risk measures enumerated above are defined, formulated, and tested by means of illustrative examples. Lastly, both a summary of the chapter and some relevant conclusions are provided in Section 4.4.

4.2 Risk Control in Stochastic Programming Problems

4.2.1 Risk-Neutral Decision Making

In Chapter 2 we present the general formulation of a two-stage stochastic programming problem,

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)}$$

$$\boldsymbol{c}^\top \boldsymbol{x} + \sum_{\omega \in \Omega} \pi(\omega) \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega) \tag{4.1}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.2}$$

$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega), \ \forall \omega \in \Omega \tag{4.3}$$

$$\boldsymbol{x} \in X, \ \boldsymbol{y}(\omega) \in Y, \ \forall \omega \in \Omega, \tag{4.4}$$

where $\boldsymbol{x}$ and $\boldsymbol{y} = \{\boldsymbol{y}(\omega)\,; \forall\omega \in \Omega\}$ are the first- and second-stage decision variable vectors, respectively, and c, $\boldsymbol{q}(\omega)$, $\boldsymbol{b}$, $\boldsymbol{h}(\omega)$, $\boldsymbol{A}$, $\boldsymbol{T}(\omega)$, and $\boldsymbol{W}(\omega)$ are known vectors and matrices of appropriate size.

Defining

$$f(\boldsymbol{x}, \omega) = \tag{4.5}$$
$$\boldsymbol{c}^\top \boldsymbol{x} + \max_{\boldsymbol{y}(\omega)} \{\boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega) : \boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega),\ \boldsymbol{y}(\omega) \in Y\},$$

problem (4.1)-(4.4) can be equivalently expressed in compact form as

$$\text{Maximize}_{\boldsymbol{x}}$$
$$\mathcal{E}_\omega\{f(\boldsymbol{x}, \omega)\} \tag{4.6}$$

subject to

$$\boldsymbol{x} \in X,\ \forall\omega \in \Omega. \tag{4.7}$$

The objective of problem (4.6)-(4.7) is to maximize the expected value of the function $f(\boldsymbol{x}, \omega)$, which can represent, for instance, the profit achieved by a power producer within a given planning horizon.

From a more abstract perspective, it is embedded in the definition of $f(\boldsymbol{x}, \omega)$ that, after the decision on $\boldsymbol{x}$ and the observation of ω, the second-stage decision $\boldsymbol{y}(\omega)$ has to be an optimal solution to the remaining optimization problem

$$\text{Maximize}_{\boldsymbol{y}(\omega)}$$
$$\boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega) \tag{4.8}$$

subject to

$$\boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega) - \boldsymbol{T}(\omega)\boldsymbol{x} \tag{4.9}$$
$$\boldsymbol{y}(\omega) \in Y. \tag{4.10}$$

Conceptually, it is convenient to view $f(\boldsymbol{x}, \omega)$ as the value of the random value $f(\boldsymbol{x}, \cdot)$ at the argument ω. Then, for $\boldsymbol{x}$ varying in X, a family $\{f(\boldsymbol{x}, \cdot) : \boldsymbol{x} \in X\}$ of random variables is induced by the two-stage decision problem under uncertainty. Finding a best $\boldsymbol{x}$ now corresponds to finding a best random variable in this family. In (4.6)-(4.7) this is accomplished by ranking the random variables according to their expectations and picking the biggest. Certainly, other modes of ranking are possible too. They will lead to other types of stochastic problems which we will study later on.

Since vectors of variables $\boldsymbol{x}$ and $\boldsymbol{y}(\omega)$ are obtained by maximizing the expected profit without modeling the risk, we denote (4.1)-(4.4) and the equivalent problem (4.6)-(4.7) as *risk-neutral problems*.

The following example illustrates the concept of risk-neutral problem.

Illustrative Example 4.1 (Risk-neutral problem).

Consider the problem faced by an electricity retailer that seeks to determine the purchases in the futures market in order to maximize the profit resulting from selling energy to a group of clients. In doing so, the retailer buys energy in both the pool and a futures market. The price of the energy in the pool in each period t is assumed to be unknown and is characterized as a random variable, which is modeled using a set of scenarios, $\lambda^{\mathrm{P}}_{t\omega}$. The retailer participates in the futures market by buying energy through three different contracts, $f = 1, 2, 3$, defined by a purchasing price, λ^{F}_{f}, and a maximum quantity of power that can be purchased, $X^{\max}_{f}$. The power purchased in contract f, x_f, is delivered in each period of the planning horizon. The demand of the consumers in each period t, P^{C}_{t}, is assumed to be known (deterministic), while the selling price of the energy to the clients, λ^{C}, is fixed to \$35/MWh. For this illustrative example, the considered planning horizon comprises 3 hourly periods.

The client demands for the three periods are 150, 225, and 175 MW. The data concerning the forward contracts available in the futures market are given in Table 4.1. The price of the electricity in the pool in every period is represented by a set of 10 equiprobable scenarios. Pool price data are provided in Table 4.2.

Table 4.1 Illustrative Example 4.1: forward contract data

Contract #	Price (\$/MWh)	Maximum quantity (MW)
1	34	50
2	35	30
3	36	25

Table 4.2 Illustrative Example 4.1: pool price data (\$/MWh)

Scenario #	Period #			Scenario #	Period #		
	1	2	3		1	2	3
1	28.5	36.3	31.4	6	29.2	34.8	31.2
2	27.3	37.5	29.6	7	34.1	36.9	35.4
3	29.4	35.7	31.3	8	33.4	35.4	34.9
4	33.9	35.4	35.1	9	28.4	36.3	32.9
5	34.5	38.9	37.5	10	27.6	38.9	32.1

The formulation of the risk-neutral profit maximization problem is

$$\text{Maximize}_{x_f, y_{t\omega}}$$
$$\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3} \sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{\omega=1}^{10} \pi_\omega \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \tag{4.11}$$

subject to

$$0 \leq x_f \leq X_f^{\max}, \quad f = 1, 2, 3 \tag{4.12}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1, 2, 3;\ \omega = 1, \ldots, 10 \tag{4.13}$$

$$y_{t\omega} \geq 0, \quad t = 1, 2, 3;\ \omega = 1, \ldots, 10, \tag{4.14}$$

where x_f is the power purchased from forward contract f and $y_{t\omega}$ is the power purchased from the pool in period t and scenario ω. Note that x_f is a here-and-now decision variable, whereas $y_{t\omega}$ is wait-and-see.

The objective function (4.11) represents the expected profit achieved by the retailer. This expected profit is equal to the revenue obtained from selling energy to clients minus the cost of purchasing in both the futures and the pool markets. Observe that the two first terms concerning the revenue and the cost of purchasing in the futures market are deterministic, whereas the third term corresponding to the expected cost of purchasing in the pool is evaluated over all pool price scenarios. Parameter π_ω in this last term refers to the probability of scenario ω. Note that the term modeling the revenue of the retailer is constant in this problem and thus it can be removed from the objective fucntion.

Constraints (4.12) establish the lower and upper bounds for the power purchased from the futures market in each contract f. The power balance in each period and scenario is stated in (4.13). Finally, constraints (4.14) impose the non-negativity of variables $y_{t\omega}$.

The optimal solution to problem (4.11)-(4.14) in terms of the optimal purchases in the futures market is $\{x_1^*, x_2^*, x_3^*\} = \{0, 0, 0\}$. That is, the retailer prefers to supply the demand of its clients only buying from the pool. This result is reasonable since the mean value of the pool prices provided in Table 4.2 is \$33.46/MWh, which is smaller than the prices of the three available forward contracts (Table 4.1).

The resulting expected profit is \$618.75. Table 4.3 provides the value of the profit for each pool price scenario. Observe that the profit exhibits high volatility varying from minus \$1240 to plus \$1580.

Fig. 4.1 represents the cumulative distribution function (cdf) of the profit. Since risk-neutral problem (4.11)-(4.14) only focuses on maximizing the expected profit, the risk of experiencing low profits in some scenarios is not avoided. Observe that there exists a probability equal to 0.2 of experiencing losses.

□

Table 4.3 Illustrative Example 4.1: profit per scenario

Scenario #	Profit ($)	Scenario #	Profit ($)
1	1312.50	6	1580.00
2	1537.50	7	-362.50
3	1330.00	8	167.50
4	57.50	9	1065.00
5	-1240.00	10	740.00

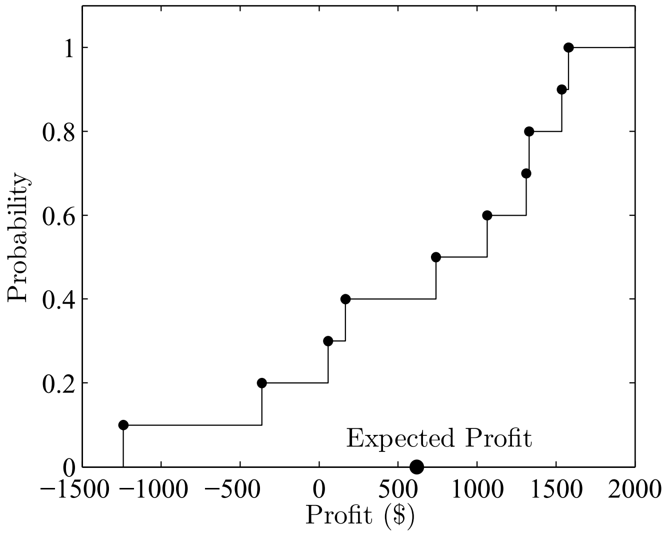

Fig. 4.1 Illustrative Example 4.1: profit cdf for the risk-neutral case

4.2.2 Risk-Averse Decision Making

The main disadvantage of ignoring risk in problem (4.1)-(4.4) is that the optimal values of variables $\boldsymbol{x}$ and $\boldsymbol{y}(\omega)$ may lead to the maximum expected profit at the expense of experiencing very low profits in some unfavorable scenarios. In order to avoid such situations, it is advisable to include in the formulation of the problem a term modeling the risk of variability associated with the profit $f(\boldsymbol{x},\omega)$. Thus, we introduce the function $r_\omega\{f(\boldsymbol{x},\omega)\}$ that assigns to a given random variable representing profit, $f(\boldsymbol{x},\omega)$, $\forall\omega \in \Omega$, a real number characterizing the risk associated with that profit. The function $r_\omega\{f(\boldsymbol{x},\omega)\}$ is referred to as *risk measure*.

Risk measures can be incorporated either into the objective function of the problem, as the mean-risk approach proposed in [91], or as an additional set of constraints in the problem formulation.

Consider the two-stage stochastic programming problem

$$\text{Maximize}_{\boldsymbol{x}}$$
$$\mathcal{E}_\omega\big\{f(\boldsymbol{x},\omega)\big\} - \beta r_\omega\big\{f(\boldsymbol{x},\omega)\big\} \tag{4.15}$$

subject to

$$\boldsymbol{x} \in X, \tag{4.16}$$

where $\beta \in [0,\infty)$ is a weighting parameter used to materialize the tradeoff between expected profit and risk aversion. If $\beta = 0$, the risk term in the objective function is neglected and the resulting problem becomes the risk-neutral one. As β increases, the expected profit term becomes less significant with respect to the risk term.

Observe that the risk faced by the decision maker can be also controlled by including the risk measure as an additional constraint, i.e.,

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)}$$
$$\mathcal{E}_\omega\{f(\boldsymbol{x},\omega)\} \tag{4.17}$$

subject to

$$\boldsymbol{x} \in X \tag{4.18}$$
$$r_\omega\big\{f(\boldsymbol{x},\omega)\big\} \le \delta, \tag{4.19}$$

where δ represents the maximum risk that the decision maker is willing to take.

The optimal solution obtained from solving problem (4.15)-(4.16) or (4.17)-(4.19) depends on the value of parameters β/δ. This optimal solution, in terms of expected profit and risk for a given value of parameters β/δ, defines an efficient point. In other words, an efficient point is a pair expected profit/risk in such a way that it is impossible to find a set of decision variables yielding simultaneously greater expected profit and lower risk. This way, a solution with greater expected profit than that of an efficient point can only be obtained at the cost of experiencing a higher risk, and viceversa. A collection of efficient points (obtained for different values of β/δ) defines an efficient frontier [91]. Fig. 4.2 illustrates an example of efficient frontier for problem (4.15)-(4.16). Observe that a small value of β yields a solution with high expected profit and also high risk. On the contrary, a large value of β achieves a solution with smaller expected profit and smaller risk. Thus, efficient frontiers are relevant instruments used by decision makers to resolve the tradeoff between expected profit and risk. Finally, note that efficient frontiers are composed of a finite set of efficient points (thus, they are not continuous) and the line resulting from joining efficient points is not necessarily either convex or concave.

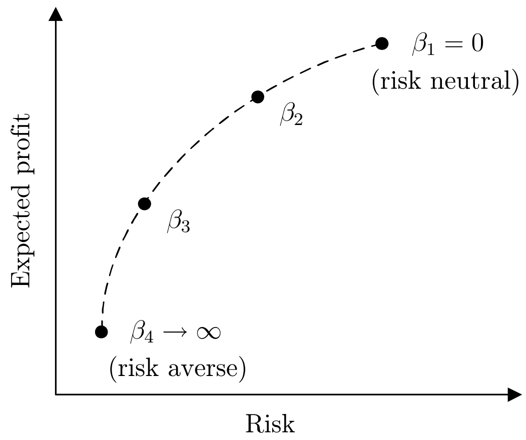

Fig. 4.2 Example of efficient frontier

4.3 Risk Measures

Risk measures are needed for characterizing the risk associated with a given decision. This way, risk measures enable us to compare two different decisions in terms of the risk involved.

In the technical literature it is possible to find a wide set of risk measures used for different applications. Artzner et al. present in [7] a set of desirable properties that risk measures should fulfill. These properties are the following:

1. Translation invariance
2. Subadditivity
3. Positive homogeneity
4. Monotonicity.

Measures satisfying these four properties are defined as *coherent risk measures.*

Let us consider two possible random outcomes (e.g., profits) $f_1(\omega)$ and $f_2(\omega) \in F$ and a risk measure $r_\omega\{f(\omega)\}$, that is a function of the random outcome $f(\omega) \in F$, where F is the set of all possible random profit outcomes. If the monetary units of random outcomes $f(\omega)$ are identical to those of the risk measure $r_\omega\{f(\omega)\}$, the properties of coherent risk measures are formulated as follows:

1. Translation invariance. For all $f_1(\omega) \in F$ and all real numbers a, it holds that $r_\omega\{f_1(\omega) + a\} = r_\omega\{f_1(\omega)\} + a$.
2. Subadditivity. For all $f_1(\omega)$, $f_2(\omega) \in F$, it is satisfied that $r_\omega\{f_1(\omega) + f_2(\omega)\} \leq r_\omega\{f_1(\omega)\} + r_\omega\{f_2(\omega)\}$.

3. Positive homogeneity. For all $f_1(\omega) \in F$ and all real numbers a, it is verified that $r_\omega\{a \times f_1(\omega)\} = a \times r_\omega\{f_1(\omega)\}$.
4. Monotonicity. For all $f_1(\omega), f_2(\omega) \in F$, if $f_1(\omega) \geq f_2(\omega)$, then $r_\omega\{f_1(\omega)\} \leq r_\omega\{f_2(\omega)\}$.

Next, we introduce and analyze several risk measures used in stochastic programming. For illustrative purposes, we consider a two-stage problem where the stochastic processes involved are represented by a discrete set of scenarios.

4.3.1 Variance

The use of the variance of a profit/cost distribution as a risk measure was first proposed by Nobel laureate H. M. Markowitz. Due to the important mean-variance model first proposed in [91], Markowitz is known as the father of the modern portfolio theory and is considered to be responsible for the upgrade of financial theory to a scientific discipline.

Basically, the mean-variance model considers that a decision (or position) can be characterized by two parameters: the expected return (expected profit/cost) and the variance of this return, which, as a dispersion measure, is used to model the risk faced by the decision maker. Therefore, a large variance indicates that there exists a high risk of experiencing a profit different from the expected one.

Considering the profit $f(\boldsymbol{x}, \omega)$, the variance can be formulated as

$$\mathrm{V}(\boldsymbol{x}) = \mathcal{E}_\omega \left\{ \Big(f(\boldsymbol{x}, \omega) - \mathcal{E}_\omega\big\{ f(\boldsymbol{x}, \omega) \big\} \Big)^2 \right\}. \tag{4.20}$$

Observe that the decision maker desires to obtain a variance as low as possible in order to avoid the variability of the profit.

The variance can be incorporated into the risk-neutral problem (4.1) as shown below:

$$\begin{gathered} \text{Maximize}_{\boldsymbol{x}, \boldsymbol{y}(\omega)} \\ (1-\beta) \left(\boldsymbol{c}^\top \boldsymbol{x} + \sum_{\omega \in \Omega} \pi(\omega) \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega) \right) - \\ \beta \sum_{\omega \in \Omega} \pi(\omega) \left(f(\boldsymbol{x}, \omega) - \sum_{\omega' \in \Omega} \pi(\omega') f(\boldsymbol{x}, \omega') \right)^2 \end{gathered} \tag{4.21}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.22}$$

$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega), \ \forall \omega \in \Omega \tag{4.23}$$

$$\boldsymbol{x} \in X, \ \boldsymbol{y}(\omega) \in Y, \ \forall \omega \in \Omega. \tag{4.24}$$

Note that problem (4.21)-(4.24) is a quadratic problem, which can turn into a quadratic mixed-integer problem if there exists any integer requirement in sets X or Y. Observe that the variance of the profit is multiplied by the parameter β, whereas the expected profit is multiplied by $(1-\beta)$. Here, β lies in the interval [0,1]. Thus, if $\beta = 0$, the variance is neglected, while the expected profit is disregarded if $\beta = 1$. Observe that the formulation with $\beta \in [0,1]$ is equivalent to (4.15)-(4.16), where $\beta \in [0, \infty)$. It should be noted that both formulations obtain the same efficient points. However, the advantage of using the formulation with $\beta \in [0,1]$ is that the value of this parameter is limited to a finite interval.

The usage of the variance of the profit as a risk-measure is illustrated in the following example.

Illustrative Example 4.2 (Variance).

Let us consider the risk-neutral problem (4.11)-(4.14) presented in Illustrative Example 4.1. If the variance of the profit is included in this problem to hedge against profit variability, the resulting formulation is

$$\text{Maximize}_{x_f, y_{t\omega}}$$

$$(1-\beta)\left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{\omega=1}^{10} \pi_\omega \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega}\right) -$$
$$\beta \sum_{\omega=1}^{10} \pi_\omega \left(\lambda_{t\omega}^{\mathrm{P}} y_{t\omega} - \sum_{\omega'=1}^{10} \pi_{\omega'} \sum_{t=1}^{3} \lambda_{t\omega'}^{\mathrm{P}} y_{t\omega'}\right)^2 \tag{4.25}$$

subject to

$$0 \le x_f \le X_f^{\max}, \quad f = 1,2,3 \tag{4.26}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1,2,3; \ \omega = 1, \ldots, 10 \tag{4.27}$$

$$y_{t\omega} \ge 0, \quad t = 1,2,3; \ \omega = 1, \ldots, 10. \tag{4.28}$$

Observe that since the pool price is the only stochastic process, the variance of the profit coincides with the variance of the cost of purchasing in the pool.

The resulting cumulative distribution functions for the profit in cases $\beta = \{0,1\}$ are represented in Fig. 4.3. The optimal purchases in the futures market for $\beta = 1$ are $\{x_1^*, x_2^*, x_3^*\} = \{50, 30, 25\}$. That is, the minimum variance is obtained if the retailer purchases as much as possible in the futures market.

For this situation, the expected profit is equal to \$208.65, that is, a value much smaller than that obtained for the case $\beta = 0$ (\$618.75). However, this reduction of 66.27% in the expected profit leads to a decrease of 56% in the profit of the worst scenario, from minus \$1240 to minus \$545.50. This result indicates that the futures market is an effective tool to reduce the risk associated with profit variability.

Note that one of the worst features of using the variance as risk measure is that in addition to penalizing the "worst" zone of the profit distribution (values smaller than the expected profit), it also penalizes those scenarios with profit higher than the expected profit. For this reason, a high reduction of the profit in the best scenarios can be also observed in Fig. 4.3. In other words, altering the variance affects both tails of the profit distribution.

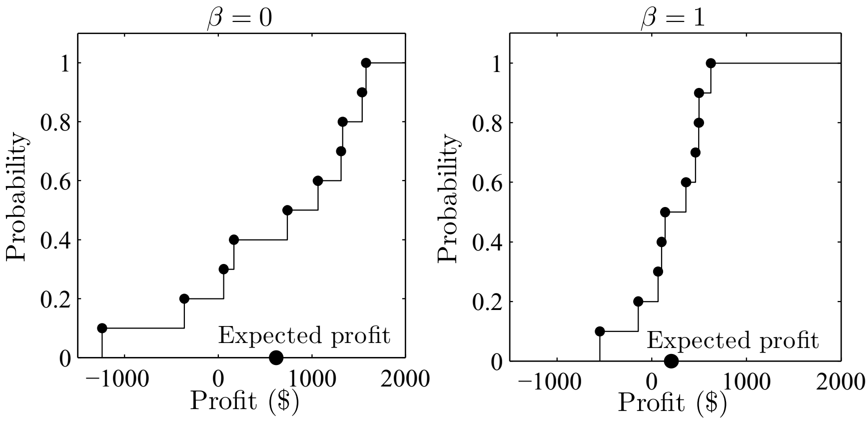

Fig. 4.3 Illustrative Example 4.2: cdfs for $\beta = 0$ and $\beta = 1$

The profit per scenario for $\beta = 1$ is provided in Table 4.4. As previously stated, the expected profit is equal to \$208.65. Given this value and the profit per scenario provided in Table 4.4, the variance of the profit is computed as

$$\begin{gathered}0.1 \times (463.50 - 208.65)^2 + 0.1 \times (499.50 - 208.65)^2 + \\ 0.1 \times (502.00 - 208.65)^2 + 0.1 \times (69.50 - 208.65)^2 + \\ 0.1 \times (-545.50 - 208.65)^2 + 0.1 \times (626.00 - 208.65)^2 + \\ 0.1 \times (-140.50 - 208.65)^2 + 0.1 \times (106.00 - 208.65)^2 + \\ 0.1 \times (363.00 - 208.65)^2 + 0.1 \times (143.00 - 208.65)^2 = \\ (\$^2)128{,}720.\end{gathered} \tag{4.29}$$

Table 4.4 Illustrative Example 4.2: profit per scenario for $\beta = 1$

Scenario #	Profit ($)	Scenario #	Profit ($)
1	463.50	6	626.00
2	499.50	7	-140.50
3	502.00	8	106.00
4	69.50	9	363.00
5	-545.50	10	143.00

Finally, Fig. 4.4 provides the efficient frontier of problem (4.25)-(4.28). As expected, the variance of the profit decreases as the value of β increases. As a consequence of the decrease in the variance, the expected profit also diminishes.

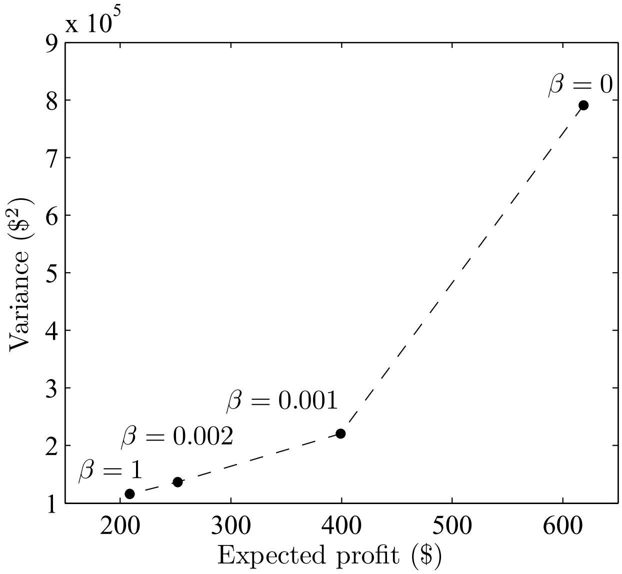

Fig. 4.4 Illustrative Example 4.2: efficient frontier

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4.3.2 Shortfall Probability

The shortfall probability, $\mathrm{SP}(\eta, \boldsymbol{x})$, is equal to the probability of the profit being less than a pre-fixed value η. Mathematically the shortfall probability is defined as

$$\mathrm{SP}(\eta, \boldsymbol{x}) = P\Big(\omega | f(\boldsymbol{x}, \omega) < \eta\Big), \ \forall \eta \in \mathbb{R}. \tag{4.30}$$

To reduce the risk faced by the decision maker, it is desirable that the shortfall probability of the profit is as low as possible.

The shortfall probability is equal to the sum of the probabilities of those profit scenarios on the left of η. Regarding the cumulative distribution function, the shortfall probability is equal to the value of that distribution for a profit less than η.

The shortfall probability can be incorporated into the risk-neutral problem (4.1)-(4.4) as shown below:

$$\text{Maximize}_{\boldsymbol{x}, \boldsymbol{y}(\omega), \theta(\omega)}$$
$$(1-\beta)\left(\boldsymbol{c}^\top \boldsymbol{x} + \sum_{\omega \in \Omega} \pi(\omega) \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\right) - \beta \sum_{\omega \in \Omega} \pi(\omega)\theta(\omega) \tag{4.31}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.32}$$
$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega), \ \forall \omega \in \Omega \tag{4.33}$$
$$\eta - \Big(\boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\Big) \le M\theta(\omega), \ \forall \omega \in \Omega \tag{4.34}$$
$$\theta(\omega) \in \{0, 1\}, \ \forall \omega \in \Omega \tag{4.35}$$
$$\boldsymbol{x} \in X, \ \boldsymbol{y}(\omega) \in Y, \ \forall \omega \in \Omega, \tag{4.36}$$

where $\theta(\omega)$ is a binary variable that is equal to 1 if the profit in scenario ω is smaller than η (and equal to 0 otherwise) and M is a sufficiently large constant. Thus, the term $\sum_{\omega \in \Omega} \pi(\omega)\theta(\omega)$ represents the cumulative probability of all scenarios whose profit is less than η, i.e., it is equal to the shortfall probability.

A drawback of the shortfall probability is that it gives no information about the profit distribution beyond the parameter η. Thus, a "fat tail" that may appear in profit distributions is not detected by this risk measure. Moreover, the use of the fixed target η in addition to the fact that this measure is not expressed in profit units causes the shortfall probability not to satisfy the properties of coherent risk measures.

The following example illustrates the formulation of the shortfall probability.

Illustrative Example 4.3 (Shortfall probability).

Consider the risk-neutral problem (4.11)-(4.14) presented in Illustrative Example 4.1. The formulation of this problem including the shortfall probability in its objective function is

$$\text{Maximize}_{x_f, y_{t\omega}, \theta_\omega}$$
$$(1-\beta)\left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{\omega=1}^{10} \pi_\omega \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \right) - \beta \sum_{\omega=1}^{10} \pi_\omega \theta_\omega \tag{4.37}$$

subject to

$$0 \le x_f \le X_f^{\max}, \quad f = 1,2,3 \tag{4.38}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1,2,3;\ \omega = 1,\ldots,10 \tag{4.39}$$

$$y_{t\omega} \ge 0, \quad t = 1,2,3;\ \omega = 1,\ldots,10 \tag{4.40}$$

$$\eta - \left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \right) \le M\theta_\omega, \quad \forall \omega = 1,\ldots,10 \tag{4.41}$$

$$\theta_\omega \in \{0,1\}, \quad \forall \omega = 1,\ldots,10. \tag{4.42}$$

For this example, we use $\eta = \$175$ and M=10,000.

Fig. 4.5 represents the cumulative distribution functions of the profit for $\beta = \{0,1\}$. In the case $\beta = 0$, the shortfall probability is equal to 0.4. That is, there is a probability of 0.4 of experiencing a profit less than $\eta = \$175$. For $\beta = 1$, the shortfall probability is reduced up to 0.2. In this case, the expected profit is \$539.44, i.e, a 13% smaller than that corresponding to the risk-neutral case (\$618.75). The optimal purchases in the futures market for this case are $\{x_1^*, x_2^*, x_3^*\} = \{49, 0, 0\}$.

The profit and the optimal value of θ_ω per scenario for $\beta = 1$ are provided in Table 4.5. Observe that variable θ_ω is equal to 1 in all scenarios with a profit smaller than $\eta = \$175$, being 0 otherwise. Variable θ_ω is 1 for scenarios 5 and 7, with profits equal to minus \$804.27 and minus \$147.08, respectively. The shortfall probability is calculated as

$$\sum_{\omega=1}^{10} \pi_\omega \theta_\omega^* = 0.1 + 0.1 = 0.2. \tag{4.43}$$

The efficient frontier of the shortfall probability versus the expected profit is depicted in Fig. 4.6. In this figure we can observe that the shortfall probability decreases as the weighting parameter β increases. Note that the difference in magnitude between the shortfall probability and the expected profit causes that large values of β are needed to reduce effectively the shortfall probability.

□

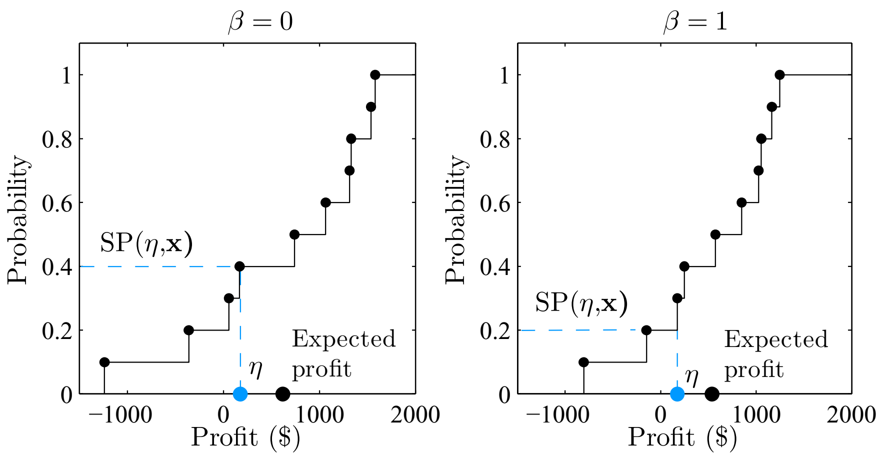

Fig. 4.5 Illustrative Example 4.3: cdfs for $\beta = 0$ and $\beta = 1$

Table 4.5 Illustrative Example 4.3: profit and variable θ_ω per scenario for $\beta = 1$

Scenario #	Profit ($)	θ_ω	Scenario #	Profit ($)	θ_ω
1	1028.54	0	6	1247.08	0
2	1165.42	0	7	-147.08	1
3	1055.83	0	8	250.73	0
4	175.00	0	9	849.58	0
5	-804.27	1	10	573.54	0

4.3.3 Expected Shortage

The expected shortage, $\mathrm{ES}(\eta, \boldsymbol{x})$, is the expectation of the profit in those scenarios with a profit smaller than a pre-fixed value η. The expected shortage is defined as

$$\mathrm{ES}(\eta, \boldsymbol{x}) = \eta - \frac{1}{\mathrm{SP}(\eta, \boldsymbol{x})} \mathcal{E}_\omega \left\{ \max\Big\{ \eta - f(\boldsymbol{x}, \omega), 0 \Big\} \right\}, \ \forall \eta \in \mathbb{R}. \tag{4.44}$$

Expression $\max\{\eta - f(\boldsymbol{x}, \omega), 0\}$ is different from zero in all scenarios in which the profit is smaller than η, being zero otherwise. In order to properly calculate the expected value of the profit over such scenarios, it is necessary not to take into account the probability of those scenarios with a profit greater than η from $\mathcal{E}_\omega\big\{\max\{\eta - f(\boldsymbol{x}, \omega), 0\}\big\}$. For this reason the expectation expression above must be divided by the sum of the probabilities of all scenarios with a profit smaller than η. The sum of these probabilities is equal to the shortfall probability $\mathrm{SP}(\eta, \boldsymbol{x})$, defined in Subsection 4.3.2.

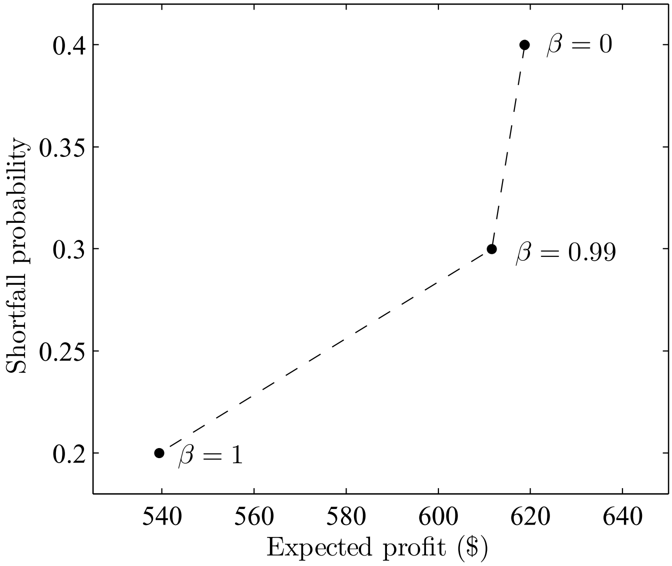

Fig. 4.6 Illustrative Example 4.3: efficient frontier

It should be noted that some authors define the expected shortage as $\mathcal{E}_\omega\{\max\{\eta - f(\boldsymbol{x},\omega), 0\}\}$, [105]. Observe that if the expected shortage is defined as in (4.44), it is equivalent to the conditional expectation of $f(\boldsymbol{x},\omega)$, i.e.,

$$\mathrm{ES}(\eta, \boldsymbol{x}) = \mathcal{E}_\omega\{f(\boldsymbol{x},\omega) | f(\boldsymbol{x},\omega) < \eta\}. \tag{4.45}$$

Profit distributions with a high expected shortage are desirable to reduce the risk of experiencing low profits in the worst scenarios. Observe that while the shortage probability represents a probability, the expected shortage is measured in profit units.

The expected shortage can be incorporated into the risk-neutral problem (4.1)-(4.4) as

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega),s(\omega)}$$

$$(1-\beta)\left(\boldsymbol{c}^\top \boldsymbol{x} + \sum_{\omega\in\Omega} \pi(\omega)\boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\right) - \beta \sum_{\omega\in\Omega} \pi(\omega)s(\omega) \tag{4.46}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.47}$$

$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega),\ \forall \omega \in \Omega \tag{4.48}$$

$$\eta - \left(\boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\right) \le s(\omega),\ \forall \omega \in \Omega \tag{4.49}$$

$$s(\omega) \ge 0,\ \forall \omega \in \Omega \tag{4.50}$$

$$\boldsymbol{x} \in X,\ \boldsymbol{y}(\omega) \in Y,\ \forall \omega \in \Omega, \tag{4.51}$$

where $s(\omega)$ is a continuous and non-negative variable equal to the maximum value between $\eta - \left(\boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\right)$ and 0.

Once problem (4.46)-(4.51) is solved and the optimal values of variables $s(\omega)$ are obtained, the expected shortage is equal to

$$\eta - \frac{1}{\sum_{\omega \in \Omega | s(w) \ge 0} \pi(\omega)} \sum_{\omega \in \Omega} \pi(\omega) s(\omega).$$

Note that the use of the fixed target η causes the expected shortage not to satisfy the properties of coherence as described in Section 4.3.

The formulation of the expected shortage is illustrated in the following example.

Illustrative Example 4.4 (Expected shortage).

The formulation of problem (4.11)-(4.14) in Illustrative Example 4.1 if the expected shortage is included in its objective function is

$$\text{Maximize}_{x_f, y_{t\omega}, s_\omega}$$

$$(1-\beta)\left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{\omega=1}^{10} \pi_\omega \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega}\right) - \beta \sum_{\omega \in \Omega} \pi_\omega s_\omega \tag{4.52}$$

subject to

$$0 \le x_f \le X_f^{\max}, \quad f = 1, 2, 3 \tag{4.53}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1, 2, 3;\ \omega = 1, \ldots, 10 \tag{4.54}$$

$$y_{t\omega} \ge 0, \quad t = 1, 2, 3;\ \omega = 1, \ldots, 10 \tag{4.55}$$

$$\eta - \left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega}\right) \le s_\omega, \quad \forall \omega = 1, \ldots, 10 \tag{4.56}$$

$$s_\omega \ge 0, \quad \forall \omega = 1, \ldots, 10. \tag{4.57}$$

For this example we consider $\eta = \$0$.

In the case $\beta = 0$, the expected shortage is equal to minus \$801.25. That is, the expected value of the profit below $\eta = \$0$ is equal to minus \$801.25. For $\beta = 1$, the expected shortage increases, being equal to minus \$342.75. For this case, the expected profit is equal to \$208.65, and the optimal decision vector is $\{x_1^*, x_2^*, x_3^*\} = \{50, 30, 25\}$.

The cumulative distribution functions of the profit for $\beta = \{0, 1\}$ are represented in Fig. 4.7. The efficient frontier is provided in Fig. 4.8. Observe that the increase of the expected shortage is obtained by means of a significant reduction of the expected profit.

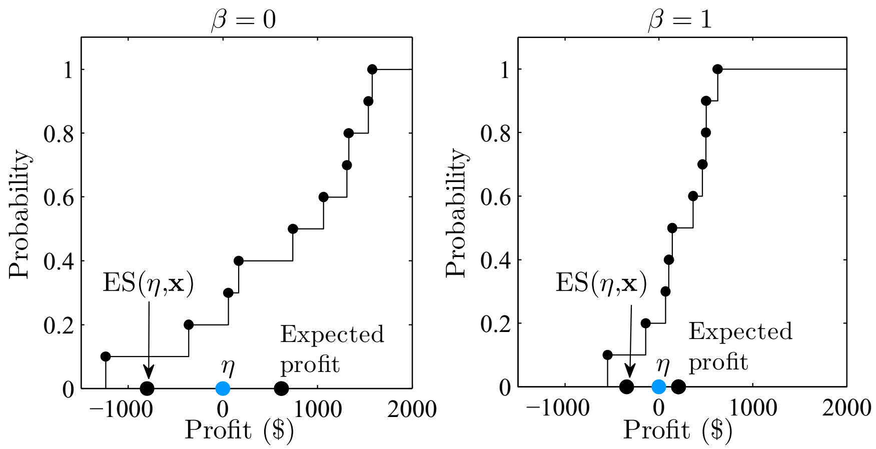

Fig. 4.7 Illustrative Example 4.4: cdfs for $\beta = 0$ and $\beta = 1$

The profit and the optimal value of s_ω per scenario for $\beta = 1$ are provided in Table 4.6. Variable s_ω is equal to the difference between $\eta = \$0$ and the profit in scenario ω if this difference is positive, being 0 otherwise. In Table 4.6 we can observe that s_ω is different from zero in scenarios 5 and 7. Given these values of variable s_ω the expected shortage is computed as

$$\eta - \frac{1}{\sum_{\omega=1|s_\omega^* \geq 0}^{10} \pi_\omega} \sum_{\omega=1}^{10} \pi_\omega s_\omega^* =$$

$$0 - \frac{1}{0.1 + 0.1}(0.1 \times 545.50 + 0.1 \times 140.50) =$$

$$= -\$342.75.$$

□

Table 4.6 Illustrative Example 4.4: profit and variable s_ω per scenario for $\beta = 1$

Scenario #	Profit ($)	s_ω	Scenario #	Profit ($)	s_ω
1	463.50	0	6	626.00	0
2	499.50	0	7	-140.50	140.50
3	502.00	0	8	106.00	0
4	69.50	0	9	363.00	0
5	-545.50	545.50	10	143.00	0

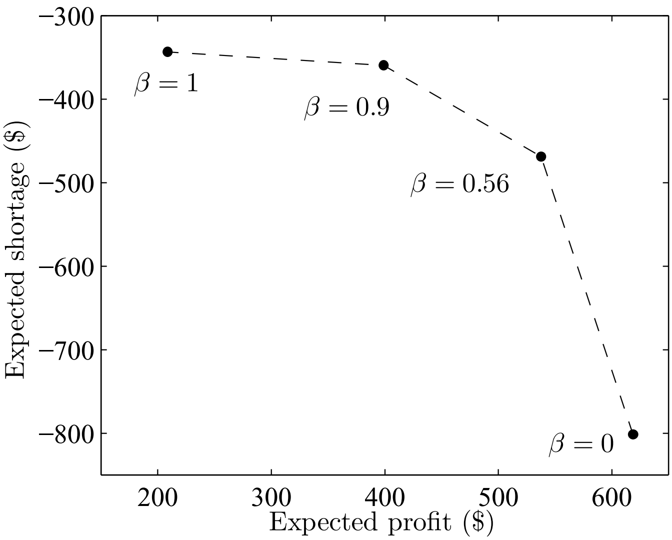

Fig. 4.8 Illustrative Example 4.4: efficient frontier

4.3.4 *Value-at-Risk*

For a given $\alpha \in (0,1)$, the value-at-risk, VaR, is equal to the largest value η ensuring that the probability of obtaining a profit less than η is lower than $1-\alpha$. In other words, the $\mathrm{VaR}(\alpha, \boldsymbol{x})$ is the $(1-\alpha)$-quantile of the profit distribution. Mathematically, the $\mathrm{VaR}(\alpha, \boldsymbol{x})$ is defined as

$$\mathrm{VaR}(\alpha, \boldsymbol{x}) = \max\Big\{\eta : P\big(\omega | f(\boldsymbol{x}, \omega) < \eta\big) \leq 1-\alpha\Big\}, \; \forall \alpha \in (0,1). \qquad (4.58)$$

Note that η is not a given parameter, but the risk measure associated with the random variable representing the profit.

The $\mathrm{VaR}(\alpha, \boldsymbol{x})$ can be incorporated into the risk-neutral problem (4.1)-(4.4) as

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega),\eta,\theta(\omega)}$$
$$(1-\beta)\left(\boldsymbol{c}^\top \boldsymbol{x} + \sum_{\omega\in\Omega} \pi(\omega)\boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\right) + \beta\eta \tag{4.59}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.60}$$
$$\boldsymbol{T}(\omega)x + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega),\ \forall\omega\in\Omega \tag{4.61}$$
$$\sum_{\omega\in\Omega} \pi(\omega)\theta(\omega) \le 1-\alpha \tag{4.62}$$
$$\eta - \left(\boldsymbol{c}^\top\boldsymbol{x} + \boldsymbol{q}(\omega)^\top\boldsymbol{y}(\omega)\right) \le M\theta(\omega),\ \forall\omega\in\Omega \tag{4.63}$$
$$\theta(\omega)\in\{0,1\},\ \forall\omega\in\Omega \tag{4.64}$$
$$\boldsymbol{x}\in X,\ \boldsymbol{y}(\omega)\in Y,\ \forall\omega\in\Omega, \tag{4.65}$$

where η is a variable whose optimal value is equal to the $\text{VaR}(\alpha,\boldsymbol{x})$, $\theta(\omega)$ is a binary variable which is equal to 1 if the profit in scenario ω is less than η (and equal to 0 otherwise) and M is a large enough constant.

A serious shortcoming of the VaR is that it gives no information about the profit distribution beyond its value. Thus, a “fat tail” appearing in profit distributions is not detected by the VaR. On the other hand, the VaR satisfies all coherence properties except subadditivity.

The usage of the VaR as a risk measure is illustrated by means of the following example.

Illustrative Example 4.5 (VaR). The formulation of problem (4.11)-(4.14) in Illustrative Example 4.1 incorporating the VaR as a risk-control mechanism is

$$\text{Maximize}_{x_f,y_{t\omega},\theta_\omega,\eta}$$
$$(1-\beta)\left(\sum_{t=1}^{3}\lambda^{\text{C}}P_t^{\text{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3}\lambda_f^{\text{F}}x_f - \sum_{\omega=1}^{10}\pi_\omega\sum_{t=1}^{3}\lambda_{t\omega}^{\text{P}}y_{t\omega}\right) + \beta\eta \tag{4.66}$$

subject to

$$0 \leq x_f \leq X_f^{\max}, \quad f = 1, 2, 3 \tag{4.67}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1, 2, 3; \ \omega = 1, \ldots, 10 \tag{4.68}$$

$$y_{t\omega} \geq 0, \quad t = 1, 2, 3; \ \omega = 1, \ldots, 10 \tag{4.69}$$

$$\eta - \left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3} \sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \right) \leq M\theta_\omega, \quad \forall \omega = 1, \ldots, 10 \tag{4.70}$$

$$\sum_{\omega=1}^{10} \pi_\omega \theta_\omega \leq 1 - \alpha \tag{4.71}$$

$$\theta_\omega \in \{0, 1\}, \quad \forall \omega = 1, \ldots, 10, \tag{4.72}$$

where $\alpha = 0.8$ and M=10,000.

Fig. 4.9 provides the cumulative distribution functions of the profit for $\beta = \{0, 1\}$. The VaR for $\beta = 0$ is \$57.50, which is equivalent to say that the 0.2-quantil of the profit distribution is equal to \$57.50. Observe that the VaR coincides with the value of the third scenario with smallest profit. Thus, the probability of the profit being less than the profit under this scenario is equal to 0.2, which is the value of $1 - \alpha = 1 - 0.8 = 0.2$.

The VaR and the expected profit for $\beta = 1$ are \$177.50 and \$535.75, respectively, which correspond to the decision vector $\{x_1^*, x_2^*, x_3^*\} = \{50, 0, 0\}$. This result indicates that the highest VaR is not obtained by means of the maximum participation in the futures market, $\{x_1, x_2, x_3\} = \{50, 30, 25\}$. In fact, for the maximum participation in the futures market the resulting VaR is equal to \$69.50 and the expected profit is equal to \$208.65. Observe that this pair of VaR and expected profit is outperformed by the solution for $\beta = 1$. For this reason, the solution attained with $\{x_1, x_2, x_3\} = \{50, 30, 25\}$ (maximum participation in the futures market) is not an efficient point.

The profit and the optimal value of θ_ω per scenario for $\beta = 1$ are provided in Table 4.7. The variable θ_ω is equal to 1 if the profit in scenario ω is smaller than the optimal value of η. For $\beta = 1$, the optimal value of η is \$177.50, which is precisely the value of the VaR. In Table 4.7, we observe that θ_ω is different from zero in scenarios 5 and 7. That is, the variable θ_ω is equal to 1 in all scenarios with a profit smaller than \$177.50.

The efficient frontier is depicted in Fig. 4.10. In this case, only two efficient points are obtained, which correspond to the extreme values of the parameter β (0 and 1).

□

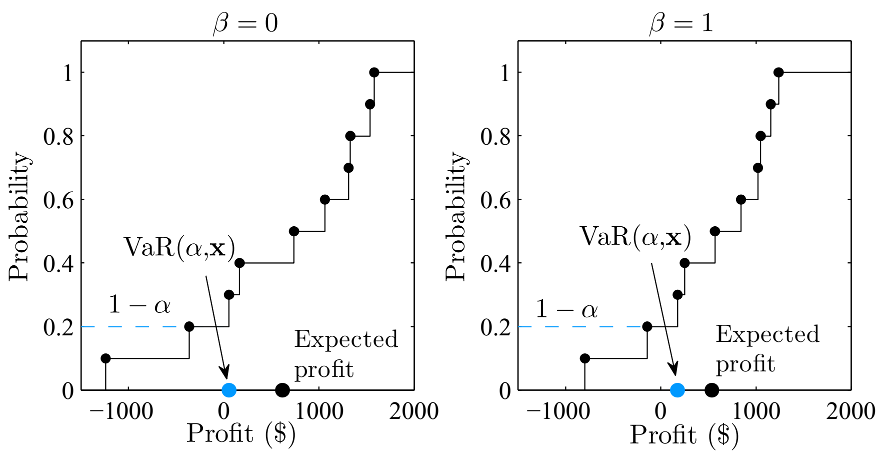

Fig. 4.9 Illustrative Example 4.5: cdfs for $\beta = 0$ and $\beta = 1$

Table 4.7 Illustrative Example 4.5: profit and variable θ_ω per scenario for $\beta = 1$

Scenario #	Profit ($)	θ_ω	Scenario #	Profit ($)	θ_ω
1	1022.50	0	6	1240.00	0
2	1157.50	0	7	-142.50	1
3	1050.00	0	8	252.50	0
4	177.50	0	9	845.00	0
5	-795.00	1	10	570.00	0

4.3.5 Conditional Value-at-Risk

For a given $\alpha \in (0,1)$, the conditional value-at-risk, CVaR, is defined as the expected value of the profit smaller than the $(1-\alpha)$-quantile of the profit distribution. If all profit scenarios are equiprobable, CVaR$(\alpha, \boldsymbol{x})$ is computed as the expected profit in the $(1-\alpha) \times 100\%$ worst scenarios. The CVaR is also known as mean excess loss or average value-at-risk.

Mathematically, the CVaR$(\alpha, \boldsymbol{x})$ for a discrete distribution is defined as, [124, 125],

$$\text{CVaR}(\alpha, \boldsymbol{x}) = \max\left\{\eta - \frac{1}{1-\alpha}\mathcal{E}_\omega\Big\{\max\{\eta - f(\boldsymbol{x},\omega), 0\}\Big\}\right\}, \ \forall \alpha \in (0,1). \tag{4.73}$$

The CVaR$(\alpha, \boldsymbol{x})$ can be incorporated into the risk-neutral problem (4.1) as

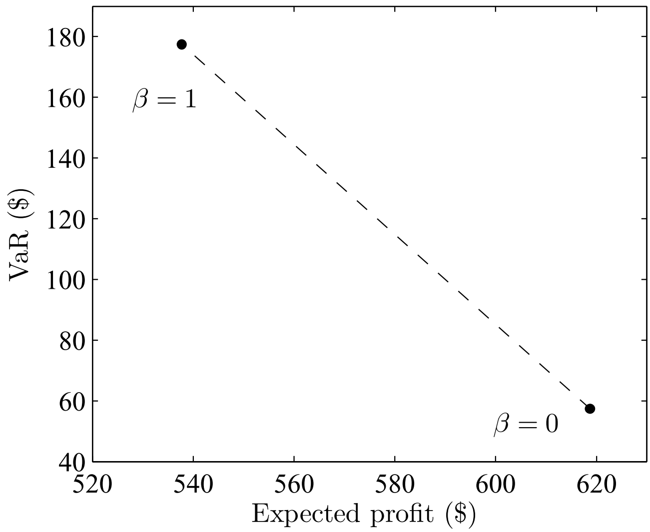

Fig. 4.10 Illustrative Example 4.5: efficient frontier

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega),\eta,s(\omega)}$$

$$(1-\beta)\left(\boldsymbol{c}^\top\boldsymbol{x} + \sum_{\omega\in\Omega}\pi(\omega)\boldsymbol{q}(\omega)^\top\boldsymbol{y}(\omega)\right) + \beta\left(\eta - \frac{1}{1-\alpha}\sum_{\omega\in\Omega}\pi(\omega)s(\omega)\right) \tag{4.74}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.75}$$

$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega),\ \forall\omega\in\Omega \tag{4.76}$$

$$\eta - (\boldsymbol{c}^\top\boldsymbol{x} + \boldsymbol{q}(\omega)^\top\boldsymbol{y}(\omega)) \le s(\omega),\ \forall\omega\in\Omega \tag{4.77}$$

$$s(\omega) \ge 0,\ \forall\omega\in\Omega \tag{4.78}$$

$$\boldsymbol{x}\in X,\ \boldsymbol{y}(\omega)\in Y,\ \forall\omega\in\Omega, \tag{4.79}$$

where η is an auxiliary variable and $s(\omega)$ is a continuous non-negative variable equal to the maximum of $\eta - (\boldsymbol{c}^\top\boldsymbol{x} + \boldsymbol{q}(\omega)^\top\mathbf{y}(\omega))$ and 0.

One of the most important advantages of the CVaR is its ability to quantify fat tails beyond the VaR, apart from being a coherent risk measure as established in Section 4.3, [7]. Moreover no binary variables are needed for its calculation.

The following example illustrates the formulation of the CVaR.

Illustrative Example 4.6 (CVaR).

The formulation of problem (4.11)-(4.14) in Illustrative Example 4.1 including the CVaR in the objective function is

$$\text{Maximize}_{x_f, y_{t\omega}, s_\omega, \eta}$$
$$(1-\beta)\left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{\omega=1}^{10} \pi_\omega \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega}\right) +$$
$$\beta\left(\eta - \frac{1}{1-\alpha}\sum_{\omega=1}^{10} \pi_\omega s_\omega\right) \tag{4.80}$$

subject to

$$0 \le x_f \le X_f^{\max}, \quad f = 1,2,3 \tag{4.81}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1,2,3;\ \omega = 1,\ldots,10 \tag{4.82}$$

$$y_{t\omega} \ge 0, \quad t = 1,2,3;\ \omega = 1,\ldots,10 \tag{4.83}$$

$$\eta - \left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3}\sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega}\right) \le s_\omega, \quad \forall \omega = 1,\ldots,10 \tag{4.84}$$

$$s_\omega \ge 0, \quad \forall \omega = 1,\ldots,10. \tag{4.85}$$

Fig. 4.11 represents the cumulative distribution functions for $\beta = \{0,1\}$. For $\beta = 0$, the CVaR is equal to minus \$801.25, whereas it is equal to minus \$343.00 for $\beta = 1$. In this last case, the expected profit is equal to \$208.65. The optimal decision vector is $\{x_1^*, x_2^*, x_3^*\} = \{0,0,0\}$ for $\beta = 0$, and $\{x_1^*, x_2^*, x_3^*\} = \{50,30,25\}$ for $\beta = 1$.

The profit and the optimal value of s_ω per scenario for $\beta = 1$ are provided in Table 4.8. The variable s_ω is equal to the difference between η and the profit in scenario ω if this difference is positive, being 0 otherwise. In Table 4.8, we observe that s_ω is different from zero only in scenario 5. For the values of variable s_ω listed in this table and the optimal value of $\eta = -\$140.50$, the CVaR is computed as

$$\eta - \frac{1}{1-\alpha}\sum_{\omega=1}^{10} \pi_\omega s_\omega =$$
$$-140.50 - \frac{1}{1-0.8}(0.1 \times 405.00) =$$
$$= -\$343.$$

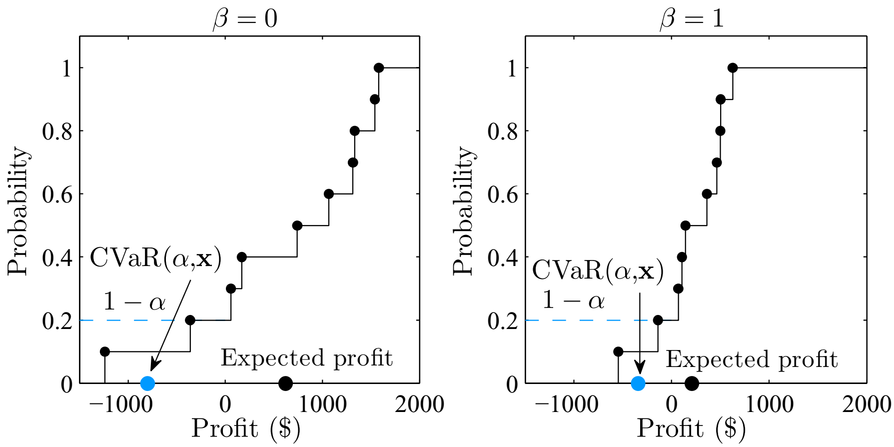

Fig. 4.11 Illustrative Example 4.6: cdfs for $\beta = 0$ and $\beta = 1$

Table 4.8 Illustrative Example 4.6: profit and variable s_ω per scenario for $\beta = 1$

Scenario #	Profit ($)	s_ω	Scenario #	Profit ($)	s_ω
1	463.50	0	6	626.00	0
2	499.50	0	7	-140.50	0
3	502.00	0	8	106.00	0
4	69.50	0	9	363.00	0
5	-545.50	405.00	10	143.00	0

The efficient frontier expressed in terms of the CVaR is provided in Fig. 4.12.

□

4.3.6 Stochastic Dominance

Stochastic dominance allows managing risk from a different point of view than that pertaining to the previously analyzed risk measures. Rather than searching for the "best" member of the profit distribution $f(\boldsymbol{x}, \omega)$, we seek "acceptable" members, and optimize over them. This leads to a recently introduced class of stochastic programming problems [39, 40, 98].

The stochastic dominance is a well-established concept in decision theory, which enables the comparison of two random variables in terms of the desired acceptability. In this section we deal with first- and second-order stochastic dominance.

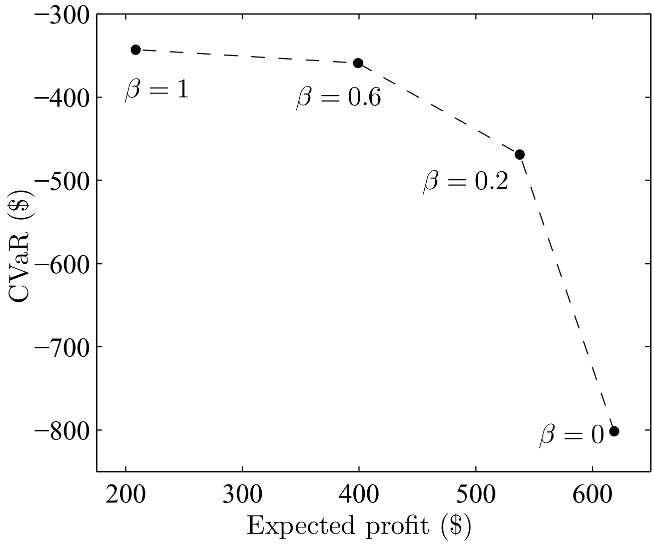

Fig. 4.12 Illustrative Example 4.6: efficient frontier

4.3.6.1 First-Order Stochastic Dominance Constraints (FOSDCs)

When preferring large realizations of random variables to small ones, a random variable $f_1(\omega)$ is said to dominate a random variable $f_2(\omega)$ to first order $\Big(f_1(\omega) \succeq_{(1)} f_2(\omega)\Big)$ if and only if [66]

$$P\Big(\omega | f_1(\omega) > \eta\Big) \geq P\Big(\omega | f_2(\omega) > \eta\Big). \tag{4.86}$$

Expression (4.86) means that, for all possible values of η, the probability of $f_1(\omega)$ being greater than η is higher than that of $f_2(\omega)$. For convenience, expression (4.86) can be also written as

$$P\Big(\omega | f_1(\omega) \leq \eta\Big) \leq P\Big(\omega | f_2(\omega) \leq \eta\Big). \tag{4.87}$$

Note that (4.87) can be equivalently expressed in terms of the cumulative distribution functions of variables $f_1(\omega)$ and $f_2(\omega)$,

$$F_1(\eta) \leq F_2(\eta), \ \forall \eta \in \mathbb{R}, \tag{4.88}$$

where $F(\eta) = P\Big(\omega | f(\omega) \leq \eta\Big)$ is the cumulative distribution function of random variable $f(\omega)$.

For example, consider a stochastic programming problem whose objective function consists in maximizing the expected profit. In this case, first-order stochastic dominance constraints can be easily incorporated to impose that the resulting profit dominates a pre-specified benchmark profit profile. Thus, we ensure that the resulting profit is "better" than a given benchmark that is acceptable for the decision maker. Clearly, caution should be exercised when selecting an appropriate benchmark in order to avoid unfeasible instances in which the resulting profit cannot outperform the pre-fixed benchmark. The formulation of this problem is

$$\text{Maximize}_{\boldsymbol{x}, \boldsymbol{y}(\omega), \theta(\omega, \upsilon)}$$
$$\boldsymbol{c}^\top \boldsymbol{x} + \sum_{\omega \in \Omega} \pi(\omega) \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega) \tag{4.89}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.90}$$
$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega), \; \forall \omega \in \Omega \tag{4.91}$$
$$k(\upsilon) - \left(\boldsymbol{c}^\top \boldsymbol{x} - \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\right) + \epsilon \leq M\theta(\omega, \upsilon), \; \forall \omega \in \Omega, \forall \upsilon \in \Upsilon \tag{4.92}$$
$$\sum_{\omega \in \Omega} \pi(\omega)\theta(\omega, \upsilon) \leq \sum_{\upsilon' \in \Upsilon | k(\upsilon') \leq k(\upsilon)} \tau(\upsilon'), \; \forall \upsilon \in \Upsilon \tag{4.93}$$
$$\theta(\omega, \upsilon) \in \{0, 1\}, \; \forall \omega \in \Omega, \forall \upsilon \in \Upsilon \tag{4.94}$$
$$\boldsymbol{x} \in X, \; \boldsymbol{y}(\omega) \in Y, \; \forall \omega \in \Omega, \tag{4.95}$$

where the random variable $\{k(\upsilon), \forall \upsilon \in \Upsilon\}$ provides the pre-fixed profit benchmark that is acceptable for the decision maker, $\theta(\omega, \upsilon)$ is a binary variable that is equal to 1 if the benchmark scenario $k(\upsilon)$ is greater than the profit in scenario ω (and 0 otherwise), $\tau(\upsilon)$ represents the probability associated with scenario υ of the benchmark random variable, M is a large enough constant, and ϵ is a positive, small constant, which is used to ensure that if $k(\upsilon)$ is equal to the profit in scenario ω the binary variable $\theta(\omega, \upsilon)$ is equal to 1.

Observe that imposing benchmark k in problem (4.89)-(4.95) ensures the acceptability of the resulting profit distribution in spite of maximizing the expected profit. For this reason, it is also possible to replace the expected profit in the objective function by another function $g(\boldsymbol{x})$ that may characterize the preferences of the decision maker for the resulting decision vector $\boldsymbol{x}$.

The following illustrative example characterizes the usage of first-order stochastic dominance constraints in a stochastic programming problem.

Illustrative Example 4.7 (First-order stochastic dominance constraints).

In order to examine the influence of using first-order stochastic dominance constraints on problem (4.11)-(4.14) in Illustrative Example 4.1, two

5-scenario benchmark profiles are used. The parameters of these benchmarks are provided in Table 4.9.

Table 4.9 Illustrative Example 4.7: benchmark data

Scenario #	Probability	Benchmark 1 ($)	Benchmark 2 ($)
1	0.05	-1000	-750
2	0.25	-500	100
3	0.30	0	200
4	0.25	500	400
5	0.15	1000	500

The formulation of problem (4.11)-(4.14) including first-order stochastic dominance constraints is

$$\text{Maximize}_{x_f, y_{t\omega}, \theta_{\omega\upsilon}}$$

$$\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3} \sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{\omega=1}^{10} \pi_\omega \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \tag{4.96}$$

subject to

$$0 \leq x_f \leq X_f^{\max}, \quad f = 1,2,3 \tag{4.97}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1,2,3; \ \omega = 1,\ldots,10 \tag{4.98}$$

$$y_{t\omega} \geq 0, \quad t = 1,2,3; \ \omega = 1,\ldots,10 \tag{4.99}$$

$$k_\upsilon - \left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3} \sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \right) + \epsilon \leq M \theta_{\omega\upsilon},$$
$$\omega = 1,\ldots,10; \ \upsilon = 1,\ldots,5 \tag{4.100}$$

$$\sum_{\omega=1}^{10} \pi_\omega \theta_{\omega\upsilon} \leq \sum_{\upsilon'=1,\ldots,5 | k_{\upsilon'} \leq k_\upsilon} \tau_{\upsilon'}, \quad \upsilon = 1,\ldots,5 \tag{4.101}$$

$$\theta_{\omega\upsilon} \in \{0,1\}, \quad \forall \omega = 1,\ldots,10; \ \upsilon = 1,\ldots,5. \tag{4.102}$$

In this case we select M =10,000 and $\epsilon = 0.01$.

The cumulative distribution functions of the profit obtained by imposing the two benchmarks are provided in Fig. 4.13. Observe that benchmark 2 is comparatively more restrictive and allows reducing the worst scenario from minus \$1000, in the case considering benchmark 1, to minus \$750. The optimal decision vector and expected profit for the case with benchmark 1 are $\{x_1^*, x_2^*, x_3^*\} = \{26.97, 0, 0\}$ and \$575.06, respectively. For benchmark 2, the result is $\{x_1^*, x_2^*, x_3^*\} = \{50, 7.63, 0\}$ with an expected profit of \$502.51. Note that in order to meet the requirements enforced by benchmark 2, the retailer must purchase in the futures market a higher quantity of energy than in the case of benchmark 1.

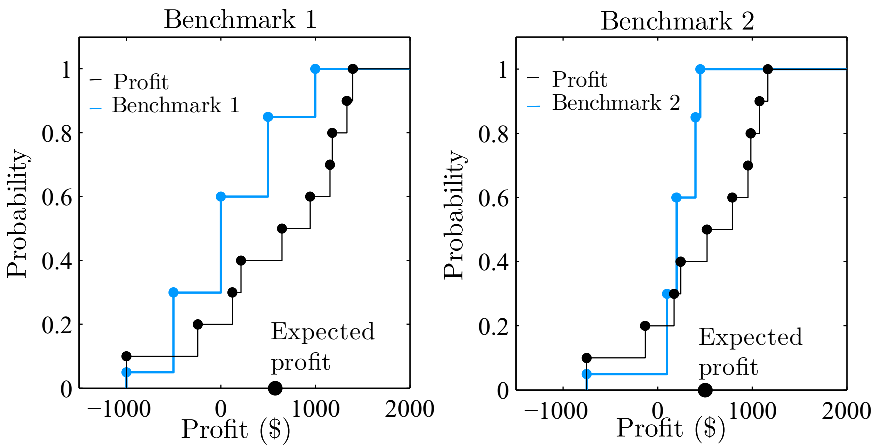

Fig. 4.13 Illustrative Example 4.7: cdfs imposing benchmarks 1 and 2

The profit per price scenario and the optimal value of $\theta_{\omega\upsilon}$ per each price scenario and benchmark scenario are provided in Table 4.10. The variable $\theta_{\omega\upsilon}$ is equal to 1 if the profit in scenario ω is less than the benchmark scenario υ. For instance, $\theta_{52} = 1$ because the profit in scenario 5, minus \$999.99, is smaller than that in scenario 2 of the benchmark, minus \$500. Observe that constraint (4.101) is satisfied for all benchmark scenarios.

□

4.3.6.2 Second-Order Stochastic Dominance Constraints (SOSDCs)

When preferring large realizations of random variables to small ones, a random variable $f_1(\omega)$ is said to dominate a random variable $f_2(\omega)$ to second order $\left(f_1(\omega) \succeq_{(2)} f_2(\omega)\right)$ if and only if, [65],

$$F_1^{(2)}(\eta) \leq F_2^{(2)}(\eta), \ \forall \eta \in \mathbb{R}, \tag{4.103}$$

with

Table 4.10 Illustrative Example 4.7: solution for benchmark 1

Scenario #	Profit ($)	$\theta_{\omega 1}$	$\theta_{\omega 2}$	$\theta_{\omega 3}$	$\theta_{\omega 4}$	$\theta_{\omega 5}$
1	1156.09	0	0	0	0	1
2	1332.55	0	0	0	0	1
3	1178.98	0	0	0	0	1
4	122.22	0	0	0	1	1
5	-999.99	0	1	1	1	1
6	1396.62	0	0	0	0	1
7	-243.84	0	1	1	1	1
8	213.34	0	0	0	1	1
9	946.34	0	0	0	0	1
10	648.31	0	0	0	0	1
	$\sum_{\omega=1}^{10} \pi_\omega \theta_{\omega\upsilon}$	0.00	0.20	0.20	0.40	1.00
	$\sum_{\upsilon'=1,\ldots,5 \mid k_{\upsilon'} \le k_\upsilon} \tau'_\upsilon$	0.05	0.30	0.60	0.85	1.00

$$F_1^{(2)}(\eta) = \int_{-\infty}^{\eta} F_1(t)\, dt, \; \forall \eta \in \mathbb{R}, \tag{4.104}$$

where $F_1(\eta)$ represents the cumulative distribution function of the random variable $f_1(\omega)$, i.e., $F_1(\eta) = P\Big(\omega | f_1(\omega) \le \eta\Big)$. Therefore, $F_1^{(2)}(\eta)$ represents the area below $F_1(\eta)$ within the interval $(-\infty, \eta]$. Hence it can be said that $f_1(\omega) \succeq_{(2)} f_2(\omega)$ holds if and only if the area below $F_1(\eta)$ is less than or equal to the area below $F_2(\eta)$ in all intervals $(-\infty, \eta]$, $\forall \eta \in \mathbb{R}$.

A useful equivalent definition of second-order stochastic dominance of $f_1(\omega)$ above $f_2(\omega)$, when preferring greater returns, is

$$\mathcal{E}_\omega \Big\{ \big(\eta - f_1(\omega)\big)_+ \Big\} \le \mathcal{E}_\omega \Big\{ \big(\eta - f_2(\omega)\big)_+ \Big\}, \; \forall \eta \in \mathbb{R}, \tag{4.105}$$

where operator $(a)_+$ denotes the greatest value between 0 and a.

Therefore, a random variable $f_1(\omega)$ stochastically dominates a random variable $f_2(\omega)$ to the second order if and only if, for any value η, the expected shortfall of $f_1(\omega)$ below η is less than or equal to the expected shortfall of $f_2(\omega)$ below η.

Second-order stochastic dominance can be incorporated into the risk-neutral problem (4.1)-(4.4) in order to assess the risk aversion with respect to a given benchmark profile k, which is selected by the decision maker. Hence, decision $\boldsymbol{x}$ is considered acceptable if $f(\boldsymbol{x}, \omega) \succeq_{(2)} k$.

Note that there might be many decisions $x \in X$ that are considered acceptable. Thus, over all "acceptable" $x \in X$, we select the decision achieving the maximum expected profit. This leads to the following stochastic programming problem with dominance constraints

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega),s(\omega,\upsilon)}$$
$$\boldsymbol{c}^\top \boldsymbol{x} + \sum_{\omega \in \Omega} \pi(\omega) \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega) \tag{4.106}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4.107}$$
$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega),\ \forall \omega \in \Omega \tag{4.108}$$
$$k(\upsilon) - (\boldsymbol{c}^\top \boldsymbol{x} - \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)) \leq s(\omega, \upsilon),\ \forall \omega \in \Omega,\ \forall \upsilon \in \Upsilon \tag{4.109}$$
$$\sum_{\omega \in \Omega} \pi(\omega) s(\omega, \upsilon) \leq \sum_{\upsilon' \in \Upsilon} \tau(\upsilon') \max\Big(k(\upsilon) - k(\upsilon'), 0\Big),\ \forall \upsilon \in \Upsilon \tag{4.110}$$
$$s(\omega, \upsilon) \geq 0,\ \forall \omega \in \Omega, \forall \upsilon \in \Upsilon \tag{4.111}$$
$$\boldsymbol{x} \in X,\ \boldsymbol{y}(\omega) \in Y,\ \forall \omega \in \Omega, \tag{4.112}$$

where the random variable $\{k(\upsilon), \forall \upsilon \in \Upsilon\}$ provides the pre-fixed profit benchmark that is acceptable for the decision maker and $s(\omega, \upsilon)$ is a variable that measures the shortfall of the profit in scenario ω below the benchmark scenario υ.

The formulation of the second-order stochastic dominance constrains is illustrated by means of the following example.

Illustrative Example 4.8 (Second-order stochastic dominance constraints).

In order to study the effect of including second-order stochastic dominance constraints in problem (4.11)-(4.14) of Illustrative Example 4.1, two 5-scenario benchmark profiles are used. The parameters of these benchmarks are provided in Table 4.11.

Table 4.11 Illustrative Example 4.8: benchmark data

Scenario #	Probability	Benchmark 1 ($)	Benchmark 2 ($)
1	0.05	-1000	-750
2	0.10	-750	-500
3	0.20	50	50
4	0.35	500	500
5	0.30	1000	700

The formulation of problem (4.11)-(4.14) including second-order stochastic dominance constraints is

$$\text{Maximize}_{x_f, y_{t\omega}, s_{\omega\upsilon}}$$
$$\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3} \sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{\omega=1}^{10} \pi_\omega \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \tag{4.113}$$

subject to

$$0 \le x_f \le X_f^{\max}, \quad f = 1, 2, 3 \tag{4.114}$$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^{\mathrm{C}}, \quad t = 1, 2, 3;\ \omega = 1, \ldots, 10 \tag{4.115}$$

$$y_{t\omega} \ge 0, \quad t = 1, 2, 3;\ \omega = 1, \ldots, 10 \tag{4.116}$$

$$k_\upsilon - \left(\sum_{t=1}^{3} \lambda^{\mathrm{C}} P_t^{\mathrm{C}} - \sum_{f=1}^{3} \sum_{t=1}^{3} \lambda_f^{\mathrm{F}} x_f - \sum_{t=1}^{3} \lambda_{t\omega}^{\mathrm{P}} y_{t\omega} \right) \le s_{\omega\upsilon},$$
$$\omega = 1, \ldots, 10;\ \upsilon = 1, \ldots, 5 \tag{4.117}$$

$$\sum_{\omega=1}^{10} \pi_\omega s_{\omega\upsilon} \le \sum_{\upsilon'=1}^{5} \tau'_\upsilon \max\left((k_\upsilon - k_{\upsilon'}, 0) \right), \quad \upsilon = 1, \ldots, 5 \tag{4.118}$$

$$s_{\omega\upsilon} \ge 0, \quad \forall \omega = 1, \ldots, 10;\ \upsilon = 1, \ldots, 5. \tag{4.119}$$

Fig. 4.14 provides the cumulative distribution functions of the profit obtained by imposing the two benchmarks in Table 4.11. The expected profit for the case with benchmark 1 is \$552.31, which corresponds to the optimal decision vector $\{x_1^*, x_2^*, x_3^*\} = \{41, 0, 0\}$. In this case, the worst scenario involves a profit of minus \$875. On the other hand, for benchmark 2, the optimal decision vector is $\{x_1^*, x_2^*, x_3^*\} = \{50, 28.8, 0\}$, with an associated expected profit of \$404.63, being the profit of the worst scenario minus \$625.

The profit per price scenario, and the optimal value of $s_{\omega\upsilon}$ per both price scenario and benchmark scenario are provided in Table 4.12. The variable $s_{\omega\upsilon}$ is equal to the difference between the k_υ scenario of the benchmark and the profit in scenario ω if this difference is positive, being 0 otherwise. For instance, $s_{53} =$ \$925.0 because the value of the scenario 3 of the benchmark, \$50, minus the profit in scenario 5, minus \$875.00, is equal to \$925.00. Observe that constraint (4.118) is satisfied for all benchmark scenarios and is active for benchmark scenarios 1 and 2.

□

4.4 Summary and Conclusions

In this chapter we discuss the need of risk control in stochastic programming problems concerning electricity markets. We observe that accounting for risk

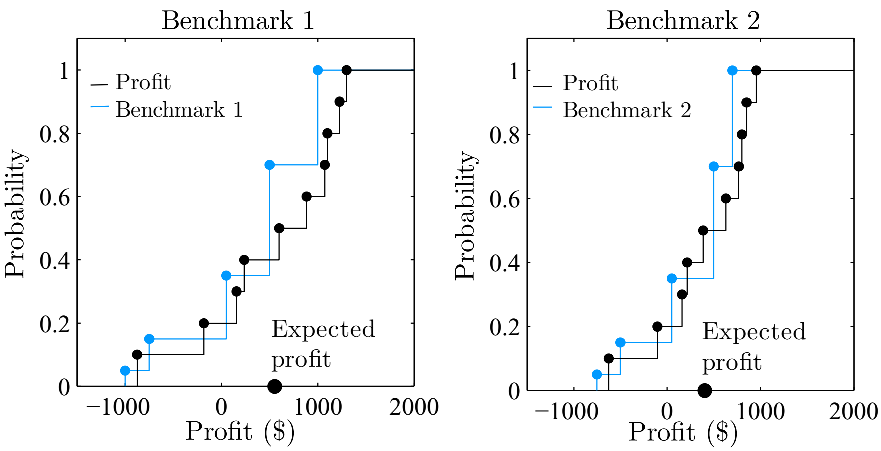

Fig. 4.14 Illustrative Example 4.8: cdfs imposing benchmarks 1 and 2

Table 4.12 Illustrative Example 4.8: solution for benchmark 1

Scenario #	Profit ($)	$s_{\omega 1}$	$s_{\omega 2}$	$s_{\omega 3}$	$s_{\omega 4}$	$s_{\omega 5}$
1	1074.63	0	0	0	0	0
2	1225.81	0	0	0	0	0
3	1100.34	0	0	0	0	0
4	155.93	0	0	0	344.1	844.1
5	-875.00	0	125.0	925.0	1375.0	1875.0
6	1301.12	0	0	0	0	0
7	-182.05	0	0	232.0	682.0	1182.0
8	237.22	0	0	0	262.8	762.8
9	884.55	0	0	0	0	115.4
10	600.56	0	0	0	0	399.4
	$\sum_{\omega=1}^{10} \pi_\omega s_{\omega\upsilon}$	0.00	12.50	115.71	266.39	517.88
	$\sum_{\upsilon'=1}^{5} \tau'_\upsilon \max(k_\upsilon - k_{\upsilon'}, 0)$	0.00	12.50	132.50	290.00	640.00

in the formulation of these problems enables to make informed decisions avoiding undesirable outcomes in the "worst" scenarios.

Usually, risk management is performed by means of the so-called risk measures. A risk measure is a function that associates a given random variable (profit or cost) with a real number characterizing the risk. This real number serves us to compare different decisions in terms of risk.

In this chapter, we analyze different risk measures characterizing the risk of a profit objective function. The first one is the profit variance, first proposed by Nobel laureate H. M. Markowitz in his pioneering work [91]. The variance is an intuitive dispersion measure of the profit that, if included in a stochastic programming problem, reduces the probability of experiencing a profit different from the expected one. However, its drawback is that in addition of penalizing low-profit scenarios, it also penalizes those scenarios with profits higher than the expected profit.

The shortfall probability and the expected shortfall are linear measures that characterize the risk as the probability of the profit being smaller of a given target value for profit, and the expected value of the profit being below that target, respectively [129]. These two measures are easy to implement, but require to specify an *arbitrary* target value for profit. Moreover, they are not coherent risk measures as defined in [7].

Other risk measures extensively used for characterizing risk are the value-at-risk (VaR) and the conditional value-at-risk (CVaR). These measures require the use of a probability, $(1-\alpha)$, that makes reference to the part of the profit distribution that is desired to be *managed* in order to hedge the risk. The VaR is the $(1-\alpha)$-quantile of the profit distribution, whereas the CVaR is the expected value of the profit distribution below that $(1-\alpha)$-quantile. Since these measures do not use a profit target value, they are more suitable for modeling risk. In addition, the CVaR is a coherent risk measure.

A different point of view for risk control consists in using stochastic dominance constraints in the formulation of the considered problem. Using a profit benchmark that is acceptable for the decision maker enables to impose that the resulting profit is "better" than this benchmark. We study first- and second-order stochastic dominance constraints. First-order constraints are enforced by means of a set of auxiliary binary variables, whereas second-order constraints are formulated with continuous variables. The main drawback of using stochastic dominance constraints are the need of specifying a benchmark profile and a significantly higher computational cost than that required by other risk measures.

Currently, the CVaR is widely used because, besides being a coherent risk measure, it can be expressed using a linear formulation. For these two reasons, the CVaR is the most used risk measure in problems pertaining to electricity markets.

4.5 Exercises

Exercise 4.1. Consider the problem faced by a producer owning a 100-MW power plant. This producer seeks to determine the on/off status of its unit and the selling of power in the futures market for a planning horizon of 4 hours. This producer can sell power in the futures market by means of three contracts with prices 40, 42, and \$45/MWh, each contract involving an amount of power up to 25 MW throughout the 4-hour market horizon. This