

TRIVIAL SOURCE CHARACTER TABLES OF FROBENIUS GROUPS OF TYPE $(C_p \times C_p) \rtimes H$

BERNHARD BÖHMLER AND CAROLINE LASSUEUR

Dedicated to the memory of Richard Parker

ABSTRACT. Let p be a prime number. We compute the trivial source character tables of finite Frobenius groups G with an abelian Frobenius complement H and an elementary abelian Frobenius kernel of order p^2 . More precisely, we deal with infinite families of such groups which occur in the two extremal cases for the fusion of p -subgroups: the case in which there exists exactly one G -conjugacy class of non-trivial cyclic p -subgroups, and the case in which there exist exactly $p + 1$ distinct G -conjugacy classes of non-trivial cyclic p -subgroups.

1. INTRODUCTION

Let G be a finite group. Let p be a prime number dividing the order of G and let k be a large enough field of characteristic p . Permutation kG -modules and their direct summands – called *p-permutation modules* or also *trivial source modules* – are omnipresent in the modular representation theory of finite groups. They are, for example, elementary building blocks for the construction and for the understanding of different categorical equivalences between block algebras, such as splendid Rickard equivalences, p -permutation equivalences, source-algebra equivalences, or Morita equivalences with endo-permutation source. A deep understanding of the structure of these modules is therefore essential.

In this manuscript, we go back to ideas of Benson and Parker developed in [BP84]. Any trivial source kG -module can be lifted to characteristic zero and affords a well-defined ordinary character, which contains essential information about its structure. The *trivial source character table* $\text{Triv}_p(G)$ of G at the prime p collects this information in a table; it is the *species table* or *representation table* of the trivial source ring in the sense of [BP84, Ben84, Ben98]. More precisely, it provides us with information about the character values of all the indecomposable trivial source kG -modules and their Brauer quotients at all p' -conjugacy classes. See Subsection 2.3 for a precise definition.

The present article is in fact part of a program aiming at gathering information about trivial source modules of small finite groups and their associated *trivial source character tables* in a database [BFLP24]. Isolated examples – calculated by Benson, and Lux and Pahlings – can be found in [Ben84, Appendix] and [LP10, §4.10]. More recently, the first author, as part of his doctoral thesis [Böh24], developed GAP4 [GAP] and MAGMA [BCP97] algorithms, which could be used to compute the trivial source character tables of finite groups of order less than 100, as well as the trivial source character tables of various small (non-abelian) quasi-simple groups. The latter algorithms rely, in particular, on the MeatAxe algorithm, first introduced

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by R. Parker. Meanwhile, in [BFL22] and [FL23] the authors and Farrell computed *generic* trivial source character tables for the groups $\mathrm{SL}_2(q)$ and $\mathrm{PSL}_2(q)$ in cross characteristic, using the generic character tables of these groups and theoretical methods involving block theory.

Using the data produced in [BFLP24] we identified some interesting families of finite groups, of which we can calculate the trivial source character tables from a purely theoretical point of view. In this regard, the main results of this article consist in the calculation of the trivial source character tables at p of the following infinite families of Frobenius groups with an abelian Frobenius complement H and elementary abelian Frobenius kernel of rank 2:

- (I) the family of all metabelian Frobenius groups of type $(C_p \times C_p) \rtimes H$, in which there is precisely one conjugacy class of subgroups of order p ;
- (II) the family of all metabelian Frobenius groups of type $(C_p \times C_p) \rtimes H$ in which there are precisely $p + 1$ conjugacy classes of subgroups of order p .

We note that the unique group of type (I) with $|H| = p^2 - 1$ is $\mathrm{AGL}_1(p^2)$. Furthermore, if $p = 2$, then the only group of type (I) is the alternating group \mathfrak{A}_4 , while groups of type (II) only occur for odd prime numbers p . For this reason, we exclude the prime number 2 from all our calculations in this manuscript. We also emphasise that contrary to [BFL22, FL23], it is not possible to use block theoretical arguments in the present cases, because such groups possess only one p -block.

The paper is structured as follows. In Section 2 we introduce our notation and conventions. In Section 3, first we review some properties of Frobenius groups and their ordinary and Brauer characters. Then, we characterise metabelian Frobenius groups with elementary abelian Frobenius kernel. The trivial source character tables are calculated in Section 4 for groups of type (I), respectively in Section 5 for groups of type (II).

2. PRELIMINARIES

2.1. General notation. Throughout, unless otherwise stated, we adopt the notation and conventions below. We let p denote an odd prime number and G a finite group of order divisible by p . We let (K, \mathcal{O}, k) be a p -modular system, where \mathcal{O} denotes a complete discrete valuation ring of characteristic zero with field of fractions $K = \mathrm{Frac}(\mathcal{O})$ and residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p . Following [CR90, §17A], we assume that K is *sufficiently large (relative to G)*, i.e. K contains all $\exp(G)$ -th roots of unity. Then k is also sufficiently large (relative to G) and K and k are splitting fields for G and all of its subgroups. For $R \in \{\mathcal{O}, k\}$, RG -modules are assumed to be finitely generated left RG -lattices, that is, free as R -modules, and we let R denote the trivial RG -lattice.

Given a positive integer n , we let C_n denote the cyclic group of order n . We let $\mathbf{O}_p(G)$ denote the largest normal p -subgroup of G , $\mathrm{Syl}_p(G)$ denote the set of all Sylow p -subgroups of G , $\mathrm{ccls}(G)$ denote a set of representatives for the conjugacy classes of G , $[G]_{p'}$ denote a set of representatives for the p -regular conjugacy classes of G , and we let $G_{p'} := \{g \in G \mid p \nmid o(g)\}$. We recall that a group G with a normal subgroup N and a subgroup H is said to be the *internal semi-direct product of N by H* , written $G = N \rtimes H$, provided $G = NH$ and $N \cap H = \{1\}$.

Given $H \leq G$, an ordinary character ψ of H and χ an ordinary character of G , we write $\mathrm{Ind}_H^G(\psi)$ for the induction of ψ from H to G , $\mathrm{Res}_H^G(\chi)$ for the restriction of χ from G to H , $\chi^\circ := \chi|_{G_{p'}}$ for the reduction modulo p of χ , and 1_H for the trivial character of H . Given $N \trianglelefteq G$ and an ordinary character ν of G/N , we write $\mathrm{Inf}_{G/N}^G(\nu)$ for the inflation of ν from G/N to G . Similarly, we write $\mathrm{Ind}_H^G(L)$ for the induction of the kH -module L from H to G , $\mathrm{Res}_H^G(M)$ for

the restriction of the kG -module M from G to H , and $\text{Inf}_{G/N}^G(U)$ for the inflation of the $k[G/N]$ -module U from G/N to G . Moreover, if M is a kG -module, then we denote by φ_M the Brauer character afforded by M , and if $Q \leq G$ then the Brauer quotient (or Brauer construction) of M at Q is the k -vector space $M[Q] := M^Q / \sum_{R < Q} \text{tr}_R^Q(M^R)$, where M^Q denotes the fixed points of M under Q and tr_R^Q denotes the relative trace map. This vector space has a natural structure of a $kN_G(Q)$ -module, but also of a $kN_G(Q)/Q$ -module, and is equal to zero if Q is not a p -subgroup. Moreover, we use the abbreviation PIM to mean a *projective indecomposable module* and we denote by $\text{Irr}(kG)$ the set of all simple kG -modules, considered up to isomorphism. We assume that the reader is familiar with elementary notions of ordinary and modular representation theory of finite groups. We refer to [Lin18a, Web16, NT89, Hup98, CR90] for further standard notation and background results.

2.2. Character tables and decomposition matrices. We let $\text{Irr}(G)$, $\text{Lin}(G)$, and $\text{IBr}_p(G)$ denote the set of all irreducible K -characters of G , the set of all linear characters of G , and the set of all irreducible p -Brauer characters of G , respectively. We let

$$X(G) := \left(\chi(g) \right)_{\substack{\chi \in \text{Irr}(G) \\ g \in \text{ccls}(G)}} \in K^{|\text{Irr}(G)| \times |\text{ccls}(G)|}$$

denote the ordinary character table of G and we let

$$X(G, p') := \left(\chi(g) \right)_{\substack{\chi \in \text{Irr}(G) \\ g \in [G]_{p'}}} \in K^{|\text{Irr}(G)| \times |[G]_{p'}|}$$

denote the matrix obtained from $X(G)$ by removing the columns labelled by p -singular conjugacy classes. Moreover, we always assume that the first column of these matrices is labelled by the class of 1. We recall that for any $\chi \in \text{Irr}(G)$ there exist uniquely determined non-negative integers $d_{\chi\varphi}$ such that $\chi^\circ = \sum_{\varphi \in \text{IBr}_p(G)} d_{\chi\varphi} \varphi$. Then, for any $\varphi \in \text{IBr}_p(G)$, the *projective indecomposable character* associated to φ is

$$(1) \quad \Phi_\varphi := \sum_{\chi \in \text{Irr}(G)} d_{\chi\varphi} \chi.$$

The p -decomposition matrix of G is then

$$\text{Dec}_p(G) := \left(d_{\chi\varphi} \right)_{\substack{\chi \in \text{Irr}(G) \\ \varphi \in \text{IBr}_p(G)}} \in K^{|\text{Irr}(G)| \times |\text{IBr}_p(G)|}$$

and the p -projective table of G is

$$\Phi_p(G) := \left(\Phi_\varphi(x) \right)_{\substack{\varphi \in \text{IBr}_p(G) \\ x \in [G]_{p'}}} \in K^{|\text{IBr}_p(G)| \times |[G]_{p'}|},$$

which is the table of Brauer character values of the projective indecomposable kG -modules. It follows from the definitions that

$$(2) \quad \Phi_p(G) = \text{Dec}_p(G)^t \cdot X(G, p').$$

Finally, the character tables of finite cyclic groups will play an essential role in our calculations, hence in this case we fix the following labelling of the irreducible characters and conjugacy classes.

Notation 2.1. If $G := \langle x \mid x^m = 1 \rangle \cong C_m$ is a cyclic group of order $m \geq 1$, then we let $\zeta \in K$ denote a primitive m -th root of unity and we write the set of ordinary irreducible characters of G as $\text{Irr}(G) = \{\xi_1, \dots, \xi_m\}$, where

$$\xi_a(x^j) := \zeta^{(a-1)j}$$

for each $1 \leq a \leq m$ and each $0 \leq j \leq m-1$. This yields

$$X(C_m) := \left(\xi_a(x^{j-1}) \right)_{\substack{1 \leq a \leq m \\ 1 \leq j \leq m}} = \left(\zeta^{(a-1)(j-1)} \right)_{\substack{1 \leq a \leq m \\ 1 \leq j \leq m}}.$$

2.3. Trivial source character tables. Given $R \in \{\mathcal{O}, k\}$, an RG -lattice M is called a *trivial source* RG -lattice if it is isomorphic to an indecomposable¹ direct summand of an induced lattice $\text{Ind}_Q^G(R)$, where $Q \leq G$ is a p -subgroup. In addition, if Q is of minimal order subject to this property, then Q is a vertex of M . It is clear that, up to isomorphism, there are only finitely many trivial source RG -lattices.

It is well-known that any trivial source kG -module M lifts in a unique way to a trivial source $\mathcal{O}G$ -lattice \widehat{M} (see e.g. [Ben98, Corollary 3.11.4]) and we denote by $\chi_{\widehat{M}}$ the K -character afforded by \widehat{M} . Moreover, if φ_M denotes the Brauer character afforded by M , then $(\chi_{\widehat{M}})^\circ = \varphi_M$ (see e.g. [Lin18b, Proposition 5.13.6]). If M is a PIM, then $\chi_{\widehat{M}} = \Phi_\varphi$, where φ is the Brauer character afforded by the unique simple kG -module in the socle of M .

We will study trivial source modules vertex by vertex. Hence, we denote by $\text{TS}(G; Q)$ the set of isomorphism classes of trivial source kG -modules with vertex Q . We notice that $\text{TS}(G; \{1\})$ is precisely the set of isomorphism classes of PIMs of kG .

A p -subgroup $Q \leq G$ is a vertex of a trivial source kG -module M if and only if $M[Q]$ is a non-zero projective $k\overline{N}_G(Q)$ -module. Moreover, if this is the case, then the $kN_G(Q)$ -Green correspondent $f(M)$ of M is $M[Q]$ (viewed as a $kN_G(Q)$ -module). Thus, there are bijections

$$\begin{array}{ccccc} \text{TS}(G; Q) & \xrightarrow{\sim} & \text{TS}(N_G(Q); Q) & \xrightarrow{\sim} & \text{TS}(\overline{N}_G(Q); \{1\}) \\ M & \mapsto & f(M) & \mapsto & M[Q] \end{array}$$

where the inverse of the second map is given by the inflation from $\overline{N}_G(Q) := N_G(Q)/Q$ to $N_G(Q)$. These sets are also in bijection with the set of p' -conjugacy classes of $\overline{N}_G(Q)$.

Next, we let $a(kG, \text{Triv})$ denote the *trivial source ring* of kG , which is defined to be the subring of the Green ring of kG generated by the set of all isomorphism classes of trivial source kG -modules. Notice that this ring is finitely generated. By definition, the *trivial source character table of the group G at the prime p* , denoted $\text{Triv}_p(G)$, is the species table of the trivial source ring of kG . See e.g. [BP84]. However, we follow [LP10, Section 4.10] and consider $\text{Triv}_p(G)$ as the block square matrix defined according to the following convention.

Convention 2.2. First, fix a set of representatives Q_1, \dots, Q_r ($r \in \mathbb{Z}_{\geq 1}$) for the conjugacy classes of p -subgroups of G where $Q_1 := \{1\}$, $Q_r \in \text{Syl}_p(G)$ and $|Q_1| \leq \dots \leq |Q_r|$. For each $1 \leq v \leq r$ set $\overline{N}_G(Q_v) := N_G(Q_v)/Q_v$. For each pair (Q_v, s) with $1 \leq v \leq r$ and $s \in [\overline{N}_G(Q_v)]_{p'}$ there is a ring homomorphism

$$\begin{array}{ccc} \tau_{Q_v, s}^G: a(kG, \text{Triv}) & \longrightarrow & K \\ [M] & \mapsto & \varphi_{M[Q_v]}(s) \end{array}$$

mapping the class of a trivial source kG -module M to the value at s of the Brauer character $\varphi_{M[Q_v]}$ of the Brauer quotient $M[Q_v]$. (Note that the group G acts by conjugation on the

¹We emphasise here that some authors use the terminology *trivial source module* to mean a finite direct sum of indecomposable kG -modules with a trivial source. We always assume such modules to be indecomposable.

pairs (Q_v, s) and the values of $\tau_{Q_v, s}^G$ do not depend on the choice of (Q_v, s) in its G -orbit.) Then, for each $1 \leq i, v \leq r$ define a matrix

$$T_{i,v} := \left(\tau_{Q_v, s}^G([M]) \right)_{\substack{M \in \text{TS}(G; Q_i) \\ s \in [\overline{N}_G(Q_v)]_{p'}}}.$$

The *trivial source character table of G at the prime p* is then the block matrix

$$\text{Triv}_p(G) := \left[T_{i,v} \right]_{\substack{1 \leq i \leq r \\ 1 \leq v \leq r}}.$$

For convenience, we will label the columns of $\text{Triv}_p(G)$ by representatives of the p' -elements of $\overline{N}_G(Q_v)$ in $N_G(Q_v)$ ($1 \leq v \leq r$). This is possible e.g. by [Böh24, Lemma 3.1.12]. Moreover, we label the rows of $\text{Triv}_p(G)$ with the ordinary characters $\chi_{\widehat{M}}$ instead of the isomorphism classes of trivial source kG -modules M themselves.

In order to calculate the entries of $\text{Triv}_p(G)$, we use the following two well-known lemmata. The first one lets us describe certain blocks of the trivial source character table using ordinary and Brauer characters. The second lemma characterises trivial source modules with maximal vertices when a Sylow p -subgroup is normal.

Lemma 2.3. *Let $\text{Triv}_p(G) = [T_{i,v}]_{1 \leq i, v \leq r}$ be the trivial source character table of the finite group G at p . Then, the following assertions hold:*

- (a) $T_{i,v} = \mathbf{0}$ if $Q_v \not\leq_G Q_i$, so in particular $T_{i,v} = \mathbf{0}$ for every $1 \leq i < v \leq r$;
- (b) $T_{i,i} = \Phi_p(\overline{N}_G(Q_i)) = \text{Dec}_p(\overline{N}_G(Q_i))^t \cdot X(\overline{N}_G(Q_i), p')$ for every $1 \leq i \leq r$;
- (c) $T_{i,1} = (\chi_{\widehat{M}}(s))_{M \in \text{TS}(G; Q_i), s \in [G]_{p'}}$ for every $1 \leq i \leq r$.

Proof. Assertion (a) is given by [LP10, Lemma 4.10.11(b)]. The first equality in assertion (b) is given by [LP10, Lemma 4.10.11(c)] and the second equality follows from Equation (2) above. Now, if $v = 1$ and $1 \leq i \leq r$, then $M[Q_v] = M[\{1\}] = M$, so $\tau_{\{1\}, s}^G([M]) = \varphi_M(s) = \chi_{\widehat{M}}(s)$ for every $M \in \text{TS}(G; Q_i)$ and every $s \in [G]_{p'}$, proving assertion (c). \square

Lemma 2.4. *Assume G is a finite group with a normal Sylow p -subgroup $P \trianglelefteq G$ such that G/P is an abelian p' -group. Then, the following assertions hold:*

- (a) $\text{TS}(G; P) = \text{Irr}(kG) = \{\text{Inf}_{G/P}^G(S) \mid S \in \text{Irr}(k[G/P])\}$, which is the set of all 1-dimensional kG -modules (considered up to isomorphism);
- (b) $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; P)\} = \text{Inf}_{G/P}^G(\text{Irr}(G/P)) \subseteq \text{Lin}(G)$.

Proof. (a) Clearly $P = \mathbf{O}_p(G)$. Thus, it follows from Clifford's theorem that $\text{Irr}(kG) = \{\text{Inf}_{G/P}^G(S) \mid S \in \text{Irr}(k[G/P])\}$, which is precisely the set of all 1-dimensional kG -modules as G/P is an abelian p' -group (see e.g. [Web16, Corollary 6.2.2]). Now, 1-dimensional kG -modules are trivial source modules with vertex P since their restriction to P must be trivial, thus, as $\overline{N}_G(P) = G/P$ is an abelian p' -group, these already account for the whole of $\text{TS}(G; P)$. (See Subsection 2.3.)

- (b) The claim is clear from (a) and the fact that $(\chi_{\widehat{M}})^\circ$ coincides with the Brauer character of M for any $M \in \text{TS}(G; P)$. \square

We refer the reader to the survey [Las23] and to our previous paper [BFL22, §2] for further details and further properties of trivial source modules and trivial source character tables. However, we mention the following result from the thesis of the first author, which will be crucial. Inbetween, this result has also appeared in [BM25, 4.1 Theorem].

Proposition 2.5 ([Böh24, Proposition 3.1.15]). *Assume G is a finite group with a normal Sylow p -subgroup $P \trianglelefteq G$ such that G/P is abelian. Let Q be a p -subgroup of G . Then, we have $P \cap N_G(Q) = \mathbf{O}_p(N_G(Q)) \in \text{Syl}_p(N_G(Q))$ and by the Schur–Zassenhaus Theorem, we may choose a complement C of $P \cap N_G(Q)$ in $N_G(Q)$. Let S be a simple kC -module, viewed as a simple $k[QC/Q]$ -module via the canonical isomorphism $QC/Q \cong C$. Set*

$$L := \text{Ind}_{QC/Q}^{\overline{N}_G(Q)}(S) \quad \text{and} \quad U := \text{Inf}_{\overline{N}_G(Q)}^{N_G(Q)}(L).$$

Then, the following assertions hold:

- (a) *L is a projective indecomposable $k\overline{N}_G(Q)$ -module; and*
- (b) *$M := \text{Ind}_{N_G(Q)}^G(U)$ is indecomposable, hence a trivial source kG -module with vertex Q .*

In particular, any element of $\text{TS}(G; Q)$ can be obtained in this way.

3. BACKGROUND MATERIAL ON FROBENIUS GROUPS

We start by reviewing basic definitions and results about the character theory of Frobenius groups.

3.1. Frobenius groups. Recall that a finite group G admitting a non-trivial proper subgroup H such that

$$H \cap gHg^{-1} = \{1\}$$

for each $g \in G \setminus H$ is called a *Frobenius group* with *Frobenius complement* H (or a *Frobenius group with respect to H*). Frobenius proved that in such a group there exists a uniquely determined normal subgroup F such that G is the internal semi-direct product of F by H (i.e. $G = FH$ and $F \cap H = \{1\}$); concretely,

$$F = \{1\} \cup \left(G \setminus \bigcup_{g \in G} gHg^{-1} \right).$$

The normal subgroup F is called the *Frobenius kernel* of G . See e.g. [CR90, §14A]. In the sequel, we write Frobenius groups with respect to H as $F \rtimes H$. We will use the following well-known properties.

Lemma 3.1. *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel F . Then the following assertions hold.*

- (a) *If H is abelian, then H is cyclic.*
- (b) *The integer $|H|$ divides $|F| - 1$. In particular $|G : F|$ and $|F|$ are coprime integers, hence F is characteristic in G .*
- (c) *For each $f \in F \setminus \{1\}$ we have $C_G(f) \leq F$.*
- (d) *The commutator subgroup of G is $[G, G] = F \rtimes [H, H]$.*

Proof. Assertion (a) is due to Burnside and follows directly from [Hup98, 16.7 Theorem b)]. Assertion (b) is given by [Hup98, 16.6 Lemma a)]. Assertion (c) is given by [CR90, (14.4)]

Proposition (i)]. For Assertion (d), first it is clear that $[G, G] \leq F \rtimes [H, H]$ as $G/(F[H, H]) \cong H/[H, H]$ is abelian. To prove the reverse inclusion, it suffices to prove that $F = [F, H]$, since then $F[H, H] = [F, H][H, H] \leq [G, G]$. Now, as G is a Frobenius group, $H \neq 1$ and there exists $h \in H \setminus \{1\}$. Then, the $|F|$ elements of the set $\{fhf^{-1}h^{-1} \in G \mid f \in F\}$ are pairwise distinct. Else $f_1hf_1^{-1}h^{-1} = f_2hf_2^{-1}h^{-1}$ with $f_1 \neq f_2 \in F$ implies that $h \in C_G(f_2^{-1}f_1)$, contradicting (c). Finally, as $F \trianglelefteq G$, any element in this set is in F , proving that $F \leq [F, H] \leq F$. \square

3.2. Characters of Frobenius groups. The ordinary characters of Frobenius groups are well-known and given by the following theorem.

Theorem 3.2 ([CR90, (14.4) Proposition]). *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel F . Then*

$$\text{Irr}(G) = \{\text{Inf}_{G/F}^G(\psi) \mid \psi \in \text{Irr}(G/F)\} \sqcup \{\text{Ind}_F^G(\nu) \mid \nu \in T\},$$

where T is a set of representatives for the orbits of the action of G by conjugation on $\text{Irr}(F) \setminus \{1_F\}$.

Notice that the first set is precisely the set of all irreducible characters of G which contain F in their kernels.

Proposition 3.3. *Let G be a Frobenius group with cyclic Frobenius complement $H \cong C_m$ for some integer $m \geq 2$ and abelian Frobenius kernel F of order p^r for some positive integer $r \geq 1$. Then the following assertions hold:*

- (a) $\text{Irr}(G) = \{\chi_1, \dots, \chi_{m+\frac{p^r-1}{m}}\}$ where for each $1 \leq a \leq m$ we set

$$\chi_a := \text{Inf}_{G/F}^G(\xi_a)$$

with $\xi_a \in \text{Irr}(C_m)$ as defined in Notation 2.1, and

$$\{\chi_{m+1}, \dots, \chi_{m+\frac{p^r-1}{m}}\} = \{\text{Ind}_F^G(\nu) \mid \nu \in T\}$$

where T is a set of representatives for conjugation action of G on $\text{Irr}(F) \setminus \{1_F\}$;

- (b) $\text{Lin}(G) = \text{Inf}_{G/F}^G(\text{Irr}(G/F)) = \{\chi_1, \dots, \chi_m\}$ and $\text{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$ where $\varphi_a := \chi_a^\circ$ for each $1 \leq a \leq m$;
- (c) H is a set of representatives of the p -regular conjugacy classes of G ;
- (d) $\chi_a^\circ = \sum_{j=1}^m \varphi_j$ for each $1 \leq a \leq \frac{p^r-1}{m}$.

Proof. Recall from Lemma 3.1 that $\gcd(m, p) = 1$.

- (a) First, it is clear from Theorem 3.2 that G has m pairwise distinct ordinary irreducible characters which are inflated from $G/F \cong H \cong C_m$ to G . Moreover,

$$|\text{Irr}(G)| = |\text{Irr}(G/F)| + |\{\text{Ind}_F^G(\nu) \mid \nu \in T\}| = m + \frac{|F| - 1}{|H|} = m + \frac{p^r - 1}{m}$$

where the last-but-one equality holds by [Hup98, 18.7 Theorem b)].

- (b) The first claim follows from (a) and Lemma 3.1(d). Then, by Lemma 2.4, we have $|\text{IBr}_p(G)| = |\text{IBr}_p(G/F)| = |H| = m$. Since by construction $\chi_1^\circ, \dots, \chi_m^\circ$ are pairwise distinct linear Brauer characters, they already account for all the irreducible Brauer characters of G . The claim follows.

- (c) Assume that $h_1 \neq h_2 \in H \setminus \{1\}$ are conjugate in G , that is, $gh_1g^{-1} = h_2$ for some $g = fh \in FH = G$ with $f \in F$ and $h \in H$. It follows that $fhh_1h^{-1}f^{-1} = h_2$, which implies that $h_1^{-1} \cdot fh_1f^{-1} = h_1^{-1} \cdot h_2$ as H is cyclic. Since H acts fixed-point-freely on $F \setminus \{1\}$, we see that $h_1^{-1}fh_1 \in F \setminus \{f\}$. This is a contradiction, as $F \cap H = \{1\}$. The claim follows, as by (b), a set of representatives for the p -regular classes of G has size $|H|$.
- (d) It is immediate from part (b) that the first m rows of $\text{Dec}_p(G)$ are given by the identity matrix of size $m \times m$. Let now $\chi_a \in \text{Irr}(G)$ with $m+1 \leq a \leq m + \frac{p^r-1}{m}$. By (a), there exists a character $\nu \in \text{Irr}(F) \setminus \{1_F\}$ such that $\chi_a = \text{Ind}_F^G(\nu)$. By (c), we only need to prove that $\chi_a^\circ|_H = \left(\sum_{j=1}^m \varphi_j\right)|_H$. Let ρ_H be the regular character of H . Then we have

$$\chi_a^\circ|_H = (\text{Ind}_F^G(\nu))^\circ|_H = \text{Res}_H^G(\text{Ind}_F^G(\nu)) = \nu(1) \cdot \rho_H = \rho_H = \sum_{\xi \in \text{Irr}(H)} \xi = \sum_{j=1}^m \varphi_j,$$

where the third equality follows from [Hup98, 18.7 Theorem (b)] and the last equality follows from (b) and the fact that H is an abelian p' -group. \square

3.3. Frobenius groups of type $(C_p \times C_p) \rtimes H$. Recall that p denotes an odd prime number. In this article, the aim is to focus on Frobenius groups G with cyclic Frobenius complement of order m and elementary abelian Frobenius kernel of order p^2 . In particular, we will compute the trivial source character tables $\text{Triv}_p(G)$ in the following two extremal cases: first the case, in which there is precisely one G -conjugacy class of cyclic subgroups of order p (we will call this the *maximal fusion case*); second, the case in which there are precisely $p+1$ G -conjugacy classes of cyclic subgroups of order p (we call this the *minimal fusion case*). In this subsection, we characterise such groups.

Given integers $m, n > 1$, we denote by $\text{MetaFrob}(m)$ the set of isomorphism classes of metabelian Frobenius groups with Frobenius complement of order m and we set

$$\text{MetaFrob}(m, n) := \{G \in \text{MetaFrob}(m) \mid |G| = mn\}.$$

Note that Frobenius groups G with cyclic Frobenius complement of order m and elementary abelian Frobenius kernel of order p^2 comprise all elements of $\text{MetaFrob}(m, p^2)$ whose Frobenius kernels are not cyclic.

Lemma 3.4. *Let p be an odd prime number and let $m > 1$ be an integer such that $m \nmid (p-1)$. Then the following assertions hold:*

- (a) $|\text{MetaFrob}(m, p^2)| = 1$;
 (b) *if $m = p^2 - 1$ then the unique element of $\text{MetaFrob}(p^2 - 1, p^2)$ is the affine linear group $\text{AGL}_1(p^2)$ and can be identified with the subgroup*

$$\mathcal{G} := \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^\times, b \in \mathbb{F}_{p^2} \right\}$$

of $\text{GL}_2(\mathbb{F}_{p^2})$, which is a Frobenius group with Frobenius kernel and Frobenius complement

$$\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_{p^2}) \mid b \in \mathbb{F}_{p^2} \right\} \quad \text{and} \quad \mathcal{H} := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^\times \right\}$$

respectively;

- (c) if $p < m < p^2 - 1$ then the unique element of $\text{MetaFrob}(m, p^2)$ can be identified with the subgroup $\tilde{\mathcal{G}} = \mathcal{F}\tilde{\mathcal{H}} = \mathcal{F} \rtimes \tilde{\mathcal{H}}$ of \mathcal{G} , where $|\tilde{\mathcal{H}}| = m$.

Proof. (a) This follows from the formula given in [BH98, Theorem 11.7] together with [BH98, Remark 11.13.(C)].

(b) Assertion (b) is well-known and follows for example from [Jac12, Exercise 5.15.7].

(c) The group $\tilde{\mathcal{G}} = \mathcal{F} \rtimes \tilde{\mathcal{H}}$ is obviously a subgroup of \mathcal{G} . Moreover, as $\tilde{\mathcal{H}} < \mathcal{H}$ and

$$\tilde{\mathcal{G}} \setminus \tilde{\mathcal{H}} = \{f \cdot \tilde{h} \in \tilde{\mathcal{G}} \mid f \in \mathcal{F} \setminus \{1\}, \tilde{h} \in \tilde{\mathcal{H}}\} \subset \mathcal{G} \setminus \mathcal{H} = \{f \cdot h \in \mathcal{G} \mid f \in \mathcal{F} \setminus \{1\}, h \in \mathcal{H}\},$$

it follows from the definition that $\mathcal{F} \rtimes \tilde{\mathcal{H}}$ is again a Frobenius group. \square

Proposition 3.5. *Let G be a Frobenius group with Frobenius complement $H \cong C_m$ and Frobenius kernel $F \cong C_p \times C_p$ where p is an odd prime number.*

- (a) *The number of G -conjugacy classes of subgroups of G of order p equals 1 if and only if $(p+1)(p-1)_2 \mid m$ if and only if $m = (p+1) \cdot \gcd(p-1, m)$. In this case, we have $|N_H(C)| = \gcd(p-1, m)$ for any subgroup $C \leq G$ of order p .*
- (b) *The number of G -conjugacy classes of subgroups of G of order p equals $p+1$ if and only if for any $h \in H \setminus \{1\}$, there exists an integer $1 < a(h) \leq p-1$ such that $hfh^{-1} = f^{a(h)}$ for any $f \in F$. In this case, H is cyclic of order dividing $p-1$.*

Proof. Write $F = \langle x \rangle \times \langle y \rangle$ and $H = \langle h \rangle$. As F is elementary abelian of order p^2 , there are precisely $p+1$ subgroups of F of order p , namely $R_i := \langle x \cdot y^i \rangle$ ($0 \leq i \leq p-1$) and $R_p := \langle y \rangle$, and we let $X := \{R_0, \dots, R_p\}$.

- (a) The equivalent conditions and the fact that $|N_H(C)| = \gcd(p-1, m)$ for any $C \in X$ are proved in [BHHK25, Example 3.2]. It follows from the action of H on X and the induced equality $(p+1) \cdot |N_H(C)| = |H| = m$.
- (b) If the number of G -conjugacy classes of subgroups of G of order p equals $p+1$, the action of H by conjugation must map the elements of $F \setminus \{1\}$ to powers of themselves, but different from themselves by Lemma 3.1. Let $h \in H$. If the generators x and y satisfy $h x h^{-1} = x^{a_1}$ and $h y h^{-1} = y^{a_2}$ with $1 < a_2 < a_1 \leq p-1$, then $x^{a_2} y^{a_1}$ is conjugate to $(xy)^{a_2 a_1}$, which contradicts the assumption on the fusion of p -subgroups. Hence, the necessary condition holds. The sufficient condition is clear since G is a semi-direct product of F by H and F is abelian. Next, for any $C \in X$ we have that $N_G(C) = G$ acts by conjugation on C . The induced group homomorphism

$$\Theta : N_G(C) \longrightarrow \text{Aut}(C) \cong C_{p-1}, g \mapsto c_g$$

(where $c_g : C \longrightarrow C, c \mapsto gcg^{-1}$ is the automorphism of conjugation by g) is such that $\ker(\Theta) = F$ by Lemma 3.1. This yields

$$m = \frac{|G|}{|F|} \mid |\text{Aut}(C)| = p-1.$$

\square

Remark 3.6. Note that we excluded the prime number 2. In this case, however, the situation is simple: there is, up to isomorphism, only one Frobenius group of type $(C_2 \times C_2) \rtimes C_m$, namely the alternating group $\mathfrak{A}_4 = V_4 \rtimes \langle (1\ 2\ 3) \rangle$, where V_4 is the Klein-four group. Indeed, as we must

have $m \mid (p^2 - 1) = 3$, the only possibility is $m = 3$, and the only other non-abelian group of order 12 is the dihedral group of order 12, which does not have any normal subgroup isomorphic to $C_2 \times C_2$. The trivial source character table $\text{Triv}_2(\mathfrak{A}_4)$ can be found for example in [Böh24], or in [BFL22] through the isomorphism $\mathfrak{A}_4 \cong \text{PSL}_2(3)$.

4. THE MAXIMAL FUSION CASE

We now turn to the computation of the trivial source character tables of metabelian Frobenius groups with Frobenius kernel $C_p \times C_p$ and cyclic Frobenius complement C_m , where p is odd and there is precisely one conjugacy class of subgroups of order p .

Notation 4.1. Throughout this section, we assume that $G = F \rtimes H$ is a Frobenius group with Frobenius kernel $F \cong C_p \times C_p$ and cyclic Frobenius complement $H \cong C_m$, where p is odd and $m = (p + 1) \cdot \gcd(p - 1, m)$. By Lemma 3.4, up to isomorphism, there is only one group of this type: it is a subgroup of $\text{AGL}_1(p^2)$ and there is precisely one conjugacy class of subgroups of order p by Proposition 3.5(a).

We set $d(m) := \gcd(p - 1, m)$ and $e(m) := \frac{p^2 - 1}{m} = \frac{p - 1}{d(m)}$, and notice that $2 \mid d(m)$ by Proposition 3.5(a). We let $H := \langle h \rangle$ and $F := \langle x \rangle \times \langle y \rangle$ and we choose the following set of representatives for the G -conjugacy classes of p -subgroups of G :

$$\begin{aligned} Q_1 &:= \{1\}, \\ Q_2 &:= \langle x \rangle, \\ Q_3 &:= F. \end{aligned}$$

As in Proposition 3.3, we let $\text{Irr}(G) = \{\chi_1, \dots, \chi_m, \chi_{m+1}, \dots, \chi_{m+e(m)}\}$ where for each $1 \leq a \leq m$ we set $\chi_a := \text{Inf}_{G/F}^G(\xi_a)$ with $\xi_a \in \text{Irr}(H)$ as defined in Notation 2.1, and $\chi_{m+b} := \text{Ind}_F^G(\nu_b)$ for each $1 \leq b \leq e(m)$ and pairwise non-conjugate characters $\nu_b \in \text{Irr}(F) \setminus \{1_F\}$.

Lemma 4.2. *With the notation introduced in Notation 4.1, we have:*

- (a) $N_G(Q_1) = G$ and $\overline{N}_G(Q_1) \cong G$;
- (b) $N_G(Q_2) = F \rtimes N_H(Q_2)$ and $\overline{N}_G(Q_2) \cong \langle y \rangle \rtimes N_H(Q_2)$, where $|N_H(Q_2)| = d(m)$ and $N_H(Q_2) = \langle h^{p+1} \rangle$;
- (c) $N_G(Q_3) = G$ and $\overline{N}_G(Q_3) \cong H$.

Proof. Assertions (a) and (c) are straightforward from the definitions. Assertion (b) follows from the fact that G is a semi-direct product of F by H with F abelian and Proposition 3.5(a). \square

Proposition 4.3. *Let ζ be a fixed primitive m -th root of unity in K and let $\omega := \zeta^{(p+1)}$. Then the following assertions hold.*

- (a) *The ordinary character table of G restricted to the p -regular conjugacy classes is as given in Table 1.*
- (b) *The ordinary character table of $\overline{N}_G(Q_2)$ restricted to the p -regular conjugacy classes is as given in Table 2, where following Proposition 3.3 we let $\text{Irr}(\overline{N}_G(Q_2)) = \{\theta_1, \dots, \theta_{d(m)+e(m)}\}$*

	1	h^j ($1 \leq j \leq m-1$)
χ_a ($1 \leq a \leq m$)	1	$\zeta^{(a-1)j}$
χ_{m+b} ($1 \leq b \leq e(m)$)	m	0

TABLE 1. Ordinary character table of G restricted to the p -regular classes.

where for each $1 \leq b \leq e(m)$, $\theta_{b+d(m)} := \text{Ind}_{\langle y \rangle}^{\overline{N}_G(Q_2)}(\nu_b)$ for pairwise non-conjugate characters $\nu_b \in \text{Irr}(\langle y \rangle) \setminus \{1_{\langle y \rangle}\}$ and for each $1 \leq a \leq d(m)$ we let $\theta_a := \text{Inf}_{\overline{N}_G(Q_2)/\langle y \rangle}^{\overline{N}_G(Q_2)}(\xi_a)$ with $\xi_a \in \text{Irr}(N_H(Q_2))$ as defined in Notation 2.1.

	1	$h^{j(p+1)Q_2}$ ($1 \leq j \leq d(m)-1$)
θ_a ($1 \leq a \leq d(m)$)	1	$\omega^{(a-1)j}$
$\theta_{d(m)+b}$ ($1 \leq b \leq e(m)$)	$d(m)$	0

TABLE 2. Ordinary character table of $\overline{N}_G(Q_2)$ restricted to the p -regular conjugacy classes.

- (c) Setting $\varphi_a := \chi_a^\circ$ for each $1 \leq a \leq m$, then $\text{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$ and $\text{Dec}_p(G)$ is as given in Table 3.
- (d) Setting $\psi_a := \theta_a^\circ$ for each $1 \leq a \leq d(m)$, then $\text{IBr}_p(\overline{N}_G(Q_2)) = \{\psi_1, \dots, \psi_{d(m)}\}$ and $\text{Dec}_p(\overline{N}_G(Q_2))$ is as given in Table 4.

	φ_1	φ_2	\dots	φ_{m-1}	φ_m
χ_1	1	0	\dots	0	0
χ_2	0	1	\dots	\vdots	\vdots
χ_3	0	0	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	0	0
\vdots	\vdots	\vdots	\dots	1	0
χ_m	0	0	\dots	0	1
χ_{m+1}	1	1	\dots	1	1
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
$\chi_{m+e(m)}$	1	1	\dots	1	1

TABLE 3. p -decomposition matrix of G .

	ψ_1	ψ_2	\dots	$\psi_{d(m)-1}$	$\psi_{d(m)}$
θ_1	1	0	\dots	0	0
θ_2	0	1	\dots	\vdots	\vdots
θ_3	0	0	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	0	0
\vdots	\vdots	\vdots	\dots	1	0
$\theta_{d(m)}$	0	0	\dots	0	1
$\theta_{d(m)+1}$	1	1	\dots	1	1
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
$\theta_{d(m)+e(m)}$	1	1	\dots	1	1

TABLE 4. p -decomposition matrix of $\overline{N}_G(Q_2)$.

Proof. (a) For each $1 \leq a \leq m$, we have $\chi_a = \text{Inf}_{G/F}^G(\xi_a)$ with $\xi_a \in \text{Irr}(H)$. It follows from Notation 2.1 that $\chi_a(h^j) = \xi_a(h^j) = \zeta^{(a-1)j}$ for each $1 \leq j \leq m-1$. Then, as $\chi_{m+1}, \dots, \chi_{m+d(m)}$ are induced from linear characters of F to G it is immediate that their degree is m and they take value zero on the non-trivial elements of H .

(b) Analogous to (a) as $\overline{N}_G(Q_2)$ is also a Frobenius group.

(c) and (d) are immediate from Proposition 3.3(d).

□

We now come to our first main result, namely the description of the trivial source character table of G at p . In this case, it follows from the definition, that this is a 3×3 -block matrix.

Theorem 4.4. *Assume p is an odd prime number and $G = F \rtimes H$ is a Frobenius group with $F \cong C_p \times C_p$ and $H \cong C_m$ where m is an integer such that $m = (p+1) \cdot \gcd(m, p-1)$. Then, with the notation introduced in Notation 4.1 and Proposition 4.3, the trivial source character table $\text{Triv}_p(G) = [T_{i,v}]_{1 \leq i,v \leq 3}$ of G is given as described below.*

(a) *The labelling of the columns may be chosen as follows:*

- (1) *the columns of $T_{i,1}$ ($1 \leq i \leq 3$) may be labelled by the elements of H ;*
- (2) *the columns of $T_{i,2}$ ($1 \leq i \leq 3$) may be labelled by the elements of $N_H(Q_2)$;*
- (3) *the columns of $T_{i,3}$ ($1 \leq i \leq 3$) may be labelled by the elements of H .*

(b) *The ordinary characters of the trivial source modules are as follows:*

- (1) $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_1)\} = \{\chi + \sum_{\psi \in \text{Irr}(G) \setminus \text{Lin}(G)} \psi \mid \chi \in \text{Lin}(G)\};$
- (2) $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_2)\} = \left\{ \sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} \chi + \sum_{\substack{\lambda \in \text{Lin}(G), \\ \text{Res}_{N_H(Q_2)}^G(\lambda) = \theta}} \lambda \mid \theta \in \text{Irr}(N_H(Q_2)) \right\};$
- (3) $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_3)\} = \text{Lin}(G).$

As $\text{Lin}(G) = \text{Inf}_H^G(\text{Irr}(H))$ we may choose the labelling of the rows of $T_{1,v}$ and $T_{3,v}$ ($1 \leq v \leq 3$) to match that of $X(H)$. Similarly, we may choose the labelling of the rows of $T_{2,v}$ ($1 \leq v \leq 3$) to match that of $X(N_H(Q_2))$.

(c) *With the labelling of the rows and of the columns given in (a) and (b) we have:*

- (1) $T_{1,2} = T_{1,3} = T_{2,3} = \mathbf{0};$
- (2) $T_{1,1} = X(H) + \begin{pmatrix} p^2-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^2-1 & 0 & \cdots & 0 \end{pmatrix};$
- (3) $T_{2,2} = X(N_H(Q_2)) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix};$
- (4) $T_{2,1} = (A_{M,s})_{M \in \text{TS}(G; Q_2), s \in H}$ where $A_{M,s} = 0$ provided $s \in H \setminus N_H(Q_2)$ and $(A_{M,s})_{M \in \text{TS}(G; Q_2), s \in N_H(Q_2)} = (p+1) \cdot T_{2,2};$
- (5) $T_{3,2} = (B_{M,s})_{M \in \text{TS}(G; Q_3), s \in N_H(Q_2)}$ where $B_{M,s} = (T_{1,3})_{M,s}$ for each $M \in \text{TS}(G; Q_3)$ and each $s \in N_H(Q_2)$;
- (6) $T_{3,1} = T_{3,3} = X(H).$

To help visualize this result, the table version of Theorem 4.4 for $G = \text{AGL}_1(p^2)$ is given in Appendix A.

Proof. Assertion (a) is straightforward from Convention 2.2, Proposition 3.3(c) and Lemma 4.2. Next, we prove Assertion (b). To simplify, in this proof we set $N_2 := N_G(Q_2)$ and $\bar{N}_2 := \bar{N}_G(Q_2)$.

- (1) The ordinary characters of the trivial source modules in $\text{TS}(G; Q_1)$ are the ordinary characters of the PIMs of kG and can be read off from the decomposition matrix in Table 3. The claim follows.

- (2) Let $M \in \text{TS}(G; Q_2)$. By the bijections in Subsection 2.3 there exists a unique PIM P_{ψ_a} of $k\overline{N}_2$ with $1 \leq a \leq d(m)$ such that M is the Green correspondent of the inflated module $\text{Inf}_{N_2/\langle x \rangle}^{N_2}(P_{\psi_a})$. By Proposition 2.5, the induced module

$$\text{Ind}_{N_2}^G \text{Inf}_{N_2/\langle x \rangle}^{N_2}(P_{\psi_a})$$

is indecomposable, so it must be M . Next, we compute the constituents of the ordinary character of this module. First, by the above,

$$\chi_{\widehat{M}} = \text{Ind}_{N_2}^G \text{Inf}_{N_2/\langle x \rangle}^{N_2}(\Phi_{\psi_a}),$$

and by Proposition 4.3(d) the ordinary character of P_{ψ_a} is

$$\Phi_{\psi_a} = \theta_a + \sum_{b=1}^{e(m)} \theta_{d(m)+b}.$$

Now, we claim that there is a bijection

$$\text{Ind}_{N_2}^G \text{Inf}_{N_2}^{N_2} : \text{Irr}(\overline{N}_2) \setminus \text{Lin}(\overline{N}_2) \longrightarrow \text{Irr}(G) \setminus \text{Lin}(G).$$

Given $u \in \{1, \dots, e(m)\}$, as N_2 is a Frobenius group, we can write $\text{Inf}_{N_2}^{N_2}(\theta_{d(m)+u}) = \text{Ind}_F^{N_2}(\nu_u) =: I_u$ for some character $\nu_u \in \text{Irr}(F) \setminus \{1_F\}$ by Theorem 3.2. As G is also a Frobenius group, by transitivity of induction, we obtain that $\text{Ind}_{N_2}^G \text{Inf}_{N_2}^{N_2}(\theta_{d(m)+u})$ is irreducible. Hence, the map is well-defined. As both sets have the same cardinality, it remains to show that the map is injective. Assuming $\theta_{d(m)+u} \neq \theta_{d(m)+w}$ for $u, w \in \{1, \dots, e(m)\}$, then $I_u \neq I_w$. By the above, $\text{Ind}_{N_2}^G(I_u) =: \chi_u \in \text{Irr}(G) \setminus \text{Lin}(G)$. Then, by Clifford theory, the inertial subgroup of I_u is N_2 and $\text{Res}_{N_2}^G(\chi_u) = \sum_{g \in [G/N_2]} I_u^{(g)}$. Moreover, we can choose

the non-trivial coset representatives $g \in [G/N_2]$ in such a way that $g =: \ell \in H$. Now, we show that I_w is not a constituent of $\text{Res}_{N_2}^G(\chi_u)$. As $\ell \notin N_2$ there exists $q \in Q_2 \setminus \{1\}$ such that $\ell q \ell^{-1} \in F \setminus Q_2$. Since both I_u and I_w are induced from F , they do not have F in their kernels. Hence, $I_u^{(\ell)} = I_w$ is not possible for such an ℓ , as $\ker(I_u) \cap F = Q_2 = \ker(I_w) \cap F$. Hence, $\text{Ind}_{N_2}^G(I_u) \neq \text{Ind}_{N_2}^G(I_w)$ and the map is injective.

It follows that

$$\text{Ind}_{N_G(Q_2)}^G \text{Inf}_{N_2}^{N_2} \left(\sum_{b=1}^{e(m)} \theta_{d(m)+b} \right) = \sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} \chi.$$

Now we compute $\text{Ind}_{N_2}^G \text{Inf}_{N_2}^{N_2}(\theta_a)$. Clearly, the degree of this character is

$$|G : N_2| = |H : N_H(Q_2)| = \frac{m}{d(m)} = p + 1 < m,$$

so by Proposition 4.3(a), it must be a sum of $p + 1$ linear characters of G . Now, Clifford's theorem together with Gallagher's theorem (see [Hup98, 19.3 Theorem and 19.5 Theorem]) tell us that we can write

$$\text{Ind}_{N_2}^G \text{Inf}_{N_2}^{N_2}(\theta_a) = \sum_{c=1}^{p+1} \lambda_c$$

with $\lambda_1, \dots, \lambda_{p+1} \in \text{Lin}(G)$ pairwise distinct and

$$\text{Res}_{N_2}^G \left(\text{Ind}_{N_2}^G \text{Inf}_{N_2}^{N_2}(\theta_a) \right) = (p + 1) \cdot \text{Inf}_{N_2}^{N_2}(\theta_a),$$

implying that $\text{Res}_{N_2}^G(\lambda_c) = \text{Inf}_{N_2}^{N_2}(\theta_a)$ for every $1 \leq c \leq p+1$. Moreover, identifying N_2/F with $N_H(Q_2)$, then [Bou10, §1.1.3(2.e.)] yields $\text{Res}_{N_H(Q_2)}^{N_2}(\text{Inf}_{N_2}^{N_2}(\theta_a)) = \theta_a$. A counting argument shows that any linear character of G must be a constituent of an induced character $\text{Ind}_{N_2}^G \text{Inf}_{N_2}^{N_2}(\theta_{a_0})$ for some index $1 \leq a_0 \leq d(m)$. Thus, we have proved that

$$\text{Ind}_{N_2}^G \text{Inf}_{N_2}^{N_2}(\theta_a) = \sum_{c=1}^{p+1} \lambda_c = \sum_{\substack{\lambda \in \text{Lin}(G), \\ \text{Res}_{N_H(Q_2)}^G(\lambda) = \theta_a}} \lambda,$$

as required.

(3) Lemma 4.2(c) and Lemma 2.4(b) yield

$$\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_3)\} = \text{Inf}_{G/Q_3}^G(\text{Irr}(G/Q_3)) \subseteq \text{Lin}(G)$$

and the last inclusion is an equality by Lemma 3.1(d) as $|\text{Lin}(G)| = |G/[G, G]|$.

Finally, we prove Assertion (c).

(1) The claim is immediate from Lemma 2.3(a).

(2) By Lemma 2.3(b), we have

$$T_{1,1} = \Phi_p(G) = \text{Dec}_p(G)^t \cdot X(G, p').$$

Using Proposition 4.3(a) and (c) and noticing that with our choice of the labelling for the rows and the columns, $X(G, p')$ is a block matrix of type

$$\begin{pmatrix} X(H) \\ \hline m & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ m & 0 & \dots & 0 \end{pmatrix} \in K^{(m+e(m)) \times m},$$

the product is easily calculated to be as claimed, since $m \cdot e(m) = p^2 - 1$.

(3) Similarly, by Lemma 2.3(b), we have

$$T_{2,2} = \Phi_p(\overline{N}_2) = \text{Dec}_p(\overline{N}_2)^t \cdot X(\overline{N}_2, p').$$

This product is easily computed from Proposition 4.3(b) and (d), noticing that with our choice of the labelling for the rows and the columns, $X(\overline{N}_2, p')$ is a block matrix of type

$$\begin{pmatrix} X(N_H(Q_2)) \\ \hline d(m) & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ d(m) & 0 & \dots & 0 \end{pmatrix} \in K^{(d(m)+e(m)) \times d(m)}$$

and $d(m) \cdot e(m) = p - 1$.

(4) By Lemma 2.3(c), the entries of $T_{2,1}$ are obtained by evaluating the ordinary characters of the trivial source modules with vertex Q_2 obtained in (b)(2) at the elements of H . As the latter characters are induced from $N_2 = F \rtimes N_H(Q_2)$, it is clear that they take value zero on $H \setminus N_H(Q_2)$. In addition, it is also clear from Part (b)(2) (and its proof) that the values of these characters at the elements of $N_H(Q_2)$ are the values of the characters of the corresponding PIMs of \overline{N}_2 multiplied by the index $|G : N_2| = p + 1$.

- (5) By definition, we have $T_{3,2} = [\tau_{Q_2,s}^G([M])]_{M \in \text{TS}(G; Q_3), s \in [\overline{N}_G(Q_2)]_{p'}}$. So, let $M \in \text{TS}(G; Q_3)$, which is the set of all 1-dimensional kG -modules. By Lemma 4.2 and Proposition 3.3(c), we may assume that $[N_2]_{p'} = N_H(Q_2)$ and let $t \in [N_2]_{p'}$. By definition,

$$\tau_{Q_2,t}^G([M]) = \varphi_{M[Q_2]}(t).$$

Because $Q_2 \leq Q_3 = F \trianglelefteq G$, it follows from [Lin18b, Proposition 5.10.4] that $M[Q_2] = \text{Res}_{N_2}^G(M)$. Hence, $\varphi_{M[Q_2]}(t) = \chi_{\widehat{M}}(t)$, proving that the values of $T_{3,2}$ are obtained from $T_{3,1}$ by restriction to the columns labelled by the elements of $N_H(Q_2)$.

- (6) On the one hand, since $\overline{N}_G(Q_3) \cong H$, which is a p' -group, by Lemma 2.3(b), we have

$$T_{3,3} = \Phi_p(H) = \text{Dec}_p(H)^t \cdot X(H, p') = X(H),$$

as claimed. On the other hand, by Lemma 2.3(c) we have

$$T_{3,1} = (\chi_{\widehat{M}}(s))_{M \in \text{TS}(G; Q_3), s \in [G]_{p'}}.$$

Therefore, it follows from Part (a)(3) and Part (b)(3) that $T_{3,1} = X(H)$.

□

Example 4.5. Let G be the Frobenius group $(C_3 \times C_3) \rtimes C_8$ of order 72 with maximal fusion pattern, i.e. G has precisely one conjugacy class of subgroups of order 3. It follows that we have three conjugacy classes of 3-subgroups of G , namely

$$Q_1 = \{1\}, \quad Q_2 \cong C_3, \quad Q_3 \cong C_3 \times C_3.$$

Notice that G is isomorphic to the group labelled by [72, 39] in GAP's SmallGroups library, see [GAP]. The ordinary character table of G is as given in Table 5, where $\zeta_8 := \exp(\frac{2\pi i}{8})$.

	1a	8a	4a	8b	2a	8c	4b	8d	3a
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	1
χ_3	1	ζ_8^2	-1	$-\zeta_8^2$	1	ζ_8^2	-1	$-\zeta_8^2$	1
χ_4	1	ζ_8^3	$-\zeta_8^2$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$	1
χ_5	1	-1	1	-1	1	-1	1	-1	1
χ_6	1	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3	1
χ_7	1	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2	1
χ_8	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8	1
χ_9	8	0	0	0	0	0	0	0	-1

TABLE 5. Ordinary character table of $(C_3 \times C_3) \rtimes C_8$.

The trivial source character table $\text{Triv}_3(G)$ is as given in Table 6. Following our conventions we label the columns of $\text{Triv}_3(G)$ with $3'$ -elements in N_v instead of \overline{N}_v ($1 \leq v \leq 3$).

Normalisers N_i	$N_1 \cong (C_3 \times C_3) \times C_8$								$N_2 \cong (C_3 \times C_3) \times C_2$		$N_3 \cong (C_3 \times C_3) \times C_8$							
Representatives $n_j \in N_i$	1a	8a	4a	8b	2a	8c	4b	8d	1a	2a	1a	8a	4a	8b	2a	8c	4b	8d
$\chi_1 + \chi_9$	9	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
$\chi_2 + \chi_9$	9	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	0	0	0	0	0	0	0	0	0	0
$\chi_3 + \chi_9$	9	ζ_8^2	-1	$-\zeta_8^2$	1	ζ_8^2	-1	$-\zeta_8^2$	0	0	0	0	0	0	0	0	0	0
$\chi_4 + \chi_9$	9	ζ_8^3	$-\zeta_8^3$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$	0	0	0	0	0	0	0	0	0	0
$\chi_5 + \chi_9$	9	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0
$\chi_6 + \chi_9$	9	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3	0	0	0	0	0	0	0	0	0	0
$\chi_7 + \chi_9$	9	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2	0	0	0	0	0	0	0	0	0	0
$\chi_8 + \chi_9$	9	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8	0	0	0	0	0	0	0	0	0	0
$\chi_1 + \chi_3 + \chi_5 + \chi_7 + \chi_9$	12	0	0	0	4	0	0	0	3	1	0	0	0	0	0	0	0	0
$\chi_2 + \chi_4 + \chi_6 + \chi_8 + \chi_9$	12	0	0	0	-4	0	0	0	3	-1	0	0	0	0	0	0	0	0
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	1	-1	1	ζ_8	ζ_8^2	ζ_8^3	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$
χ_3	1	ζ_8^2	-1	$-\zeta_8^2$	1	ζ_8^2	-1	$-\zeta_8^2$	1	1	1	ζ_8^3	-1	$-\zeta_8^2$	1	ζ_8^3	-1	$-\zeta_8^2$
χ_4	1	ζ_8^3	$-\zeta_8^3$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$	1	-1	1	ζ_8^3	$-\zeta_8^2$	ζ_8	-1	$-\zeta_8^3$	ζ_8^2	$-\zeta_8$
χ_5	1	-1	1	-1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1
χ_6	1	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3	1	-1	1	$-\zeta_8$	ζ_8^2	$-\zeta_8^3$	-1	ζ_8	$-\zeta_8^2$	ζ_8^3
χ_7	1	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2	1	1	1	$-\zeta_8^2$	-1	ζ_8^2	1	$-\zeta_8^2$	-1	ζ_8^2
χ_8	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8	1	-1	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	ζ_8^3	ζ_8^2	ζ_8

TABLE 6. Trivial source character table of $(C_3 \times C_3) \rtimes C_8$ at $p = 3$.

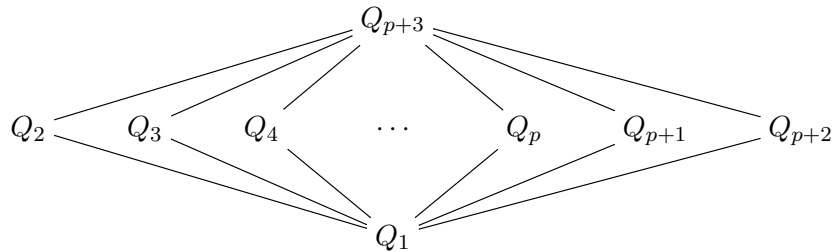
5. THE MINIMAL FUSION CASE

We now turn to the computation of the trivial source character tables of metabelian Frobenius groups with Frobenius kernel $F \cong C_p \times C_p$ and cyclic Frobenius complement $H \cong C_m$ such that no two distinct p -subgroups of G of order p are G -conjugate. Recall from Proposition 3.5(b) that $|H|$ divides $p - 1$ and that H acts on F by raising each element to a power of itself in this case. Moreover, we have $m > 1$.

Notation 5.1. Throughout this section, we adopt the following notation. We choose a generator h of C_{p-1} and let $H := \langle h^{a(m)} \rangle$, where $a(m) := \frac{p-1}{m}$. We let $\{x, y\}$ be a set of generators for F . By Proposition 3.5, we can choose the following set of representatives for the G -conjugacy classes of p -subgroups of G :

$$\begin{aligned}
Q_1 &:= \{1\}, \\
Q_i &:= \langle x \cdot y^{i-2} \rangle \quad (2 \leq i \leq p+1), \\
Q_{p+2} &:= \langle y \rangle, \\
Q_{p+3} &:= F.
\end{aligned}$$

Then, up to G -conjugation, the lattice of subgroups of G of order p is as given below.



As in Proposition 3.3, we let $\text{Irr}(G) = \{\chi_1, \dots, \chi_{m+(p+1) \cdot a(m)}\}$ where for each $1 \leq a \leq m$ we set $\chi_a := \text{Inf}_{G/F}^G(\xi_a)$ with $\xi_a \in \text{Irr}(H) = \text{Lin}(H)$ as defined in Notation 2.1, and

$$\{\chi_{m+1}, \dots, \chi_{m+(p+1) \cdot a(m)}\} = \{\text{Ind}_F^G(\nu) \mid \nu \in T\}$$

where T is a set of representatives for the conjugation action of G on $\text{Irr}(F) \setminus \{1_F\}$.

Lemma 5.2. *With the notation introduced in Notation 5.1 the following assertions hold:*

- (a) $N_G(Q_v) = G$ and $\overline{N}_G(Q_v) = G/Q_v$ for every $1 \leq v \leq p+3$; and
- (b) $\overline{N}_G(Q_v)$ is a Frobenius group with Frobenius complement $H \cong HQ_v/Q_v$ and Frobenius kernel F/Q_v for every $1 \leq v \leq p+2$.

Proof. (a) As no two distinct p -subgroups of G of order p are G -conjugate, it follows that $N_G(Q_v) = G$ for each $1 \leq v \leq p+3$. The second claim is then immediate.

- (b) For each $1 \leq v \leq p+2$, clearly $Q_v \not\trianglelefteq F$ and $Q_v \not\trianglelefteq G$ by (a). As G is a Frobenius group with respect to H , the assertion follows from the definition. \square

Notation 5.3. Following Proposition 3.3, given $v \in \{2, \dots, p+2\}$, we let $\text{Irr}(\overline{N}_G(Q_v)) = \{\theta_1, \dots, \theta_{m+\frac{p-1}{m}}\}$ where for each $1 \leq a \leq m$ we set $\theta_a := \text{Inf}_{(G/Q_v)/(F/Q_v)}^{G/Q_v}(\xi_a)$ with $\xi_a \in \text{Irr}(H)$ as defined in Notation 2.1, and

$$\{\theta_{m+1}, \dots, \theta_{m+\frac{p-1}{m}}\} = \{\text{Ind}_{F/Q_v}^{G/Q_v}(\nu) \mid \nu \in V\}$$

where V is a set of representatives for the conjugation action of G/Q_v on $\text{Irr}(F/Q_v) \setminus \{1_{F/Q_v}\}$.

Lemma 5.4. (a) *Setting $\varphi_a := \chi_a^\circ$ for each $1 \leq a \leq m$, we have $\text{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$ and $\text{Dec}_p(G)$ is as given in Table 7.*

- (b) *Setting $\psi_a := \theta_a^\circ$ for each $1 \leq a \leq m$, we have $\text{IBr}_p(\overline{N}_G(Q_v)) = \{\psi_1, \dots, \psi_m\}$ and $\text{Dec}_p(\overline{N}_G(Q_v))$ is as given in Table 8 for every $2 \leq v \leq p+2$.*

	φ_1	φ_2	\dots	φ_{m-1}	φ_m
χ_1	1	0	\dots	0	0
χ_2	0	1	\dots	\vdots	\vdots
χ_3	0	0	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	0	0
\vdots	\vdots	\vdots	\dots	1	0
χ_m	0	0	\dots	0	1
χ_{m+1}	1	1	\dots	1	1
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
$\chi_{m+(p+1) \cdot a(m)}$	1	1	\dots	1	1

TABLE 7. p -decomposition matrix of G .

	ψ_1	ψ_2	\dots	ψ_{m-1}	ψ_m
θ_1	1	0	\dots	0	0
θ_2	0	1	\dots	\vdots	\vdots
θ_3	0	0	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
\vdots	\vdots	\vdots	\dots	0	0
\vdots	\vdots	\vdots	\dots	1	0
θ_m	0	0	\dots	0	1
θ_{m+1}	1	1	\dots	1	1
\vdots	\vdots	\vdots	\dots	\vdots	\vdots
$\theta_{m+\frac{p-1}{m}}$	1	1	\dots	1	1

TABLE 8. p -decomposition matrix of $\overline{N}_G(Q_v)$ ($2 \leq v \leq p+2$).

Proof. Both (a) and (b) are immediate from Proposition 3.3(d). \square

Theorem 5.5. *Let G be a metabelian Frobenius group with cyclic complement H of order m dividing $p-1$ and Frobenius kernel $F \cong C_p \times C_p$ such that the number of G -conjugacy classes of subgroups of G of order p is precisely $p+1$. Then, the trivial source character table $\text{Triv}_p(G) = [T_{i,v}]_{1 \leq i,v \leq p+3}$ seen as a block matrix is as given in Table 9. More precisely, the labelling of the rows and columns and the entries are as described below.*

- (a) *For every $1 \leq i, v \leq p+3$ the columns of $T_{i,v}$ may be labelled by the elements of H .*

(b) *The ordinary characters of the trivial source modules are as follows:*

- (1) $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_1)\} = \{\chi + \sum_{j=m+1}^{m+(p+1) \cdot a(m)} \chi_j \mid \chi \in \text{Lin}(G)\};$
- (2) $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_i)\} = \{\lambda + \sum_{\substack{\chi \in \text{Irr}(G) \setminus \text{Lin}(G) \\ \ker(\chi) = Q_i}} \chi \mid \lambda \in \text{Lin}(G)\}$ for every $2 \leq i \leq p+2$;
- (3) $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_{p+3})\} = \text{Lin}(G).$

As $\text{Lin}(G) = \text{Inf}_H^G(\text{Irr}(H))$ we choose the labelling of the rows of $T_{i,v}$ ($1 \leq i, v \leq p+3$) to match that of $X(H)$.

(c) *With the labelling of the rows and of the columns given in (a) and (b), we have:*

- (1) $T_{i,v} = \mathbf{0}$ for every $2 \leq v < i \leq p+2$ and for every $1 \leq i < v \leq p+3$;
- (2) $T_{1,1} = X(H) + \begin{pmatrix} p^2-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^2-1 & 0 & \cdots & 0 \end{pmatrix};$
- (3) $T_{2,1} = T_{i,1} = T_{i,i} = X(H) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix}$ for every $2 \leq i \leq p+2$;
- (4) $T_{p+3,1} = T_{p+3,v} = X(H)$ for every $2 \leq v \leq p+3$.

$T_{1,1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	\cdots	$\mathbf{0}$
$T_{2,1}$	$T_{2,2} = T_{2,1}$	$\mathbf{0}$	$\mathbf{0}$	\cdots	$\mathbf{0}$
$T_{3,1} = T_{2,1}$	$\mathbf{0}$	$T_{3,3} = T_{2,1}$	$\mathbf{0}$	\cdots	$\mathbf{0}$
\vdots	\vdots	\ddots	\ddots	\ddots	\vdots
$T_{p+2,1} = T_{2,1}$	$\mathbf{0}$	\cdots	$\mathbf{0}$	$T_{p+2,p+2} = T_{2,1}$	$\mathbf{0}$
$T_{p+3,1}$	$T_{p+3,2} = T_{p+3,1}$	\cdots	\cdots	$T_{p+3,p+2} = T_{p+3,1}$	$T_{p+3,p+3} = T_{p+3,1}$

TABLE 9. Trivial source character table $\text{Triv}_p(G)$, seen as a block matrix.

Proof. (a) Let $1 \leq v \leq p+3$. Since $N_G(Q_v) = G$ by Lemma 5.2(a), we may assume that H is a set of representatives of the p -regular conjugacy classes of $N_G(Q_v)$ by Proposition 3.3(c). Thus, the claim follows from Convention 2.2.

(b) As in the proof of the previous case, because $G/Q_{p+3} \cong H$, $|\text{Lin}(G)| = |G/[G, G]|$ and $[G, G] = Q_{p+3}$ by Lemma 3.1(d), by abuse of notation we may write $\text{Lin}(G) = \text{Inf}_H^G(\text{Irr}(H))$, giving the last claim. Next, notice that by Subsection 2.3, Lemma 5.2 and Proposition 3.3, we have $|\text{TS}(G; Q_v)| = |H| = |\text{Lin}(G)|$ for every $1 \leq v \leq p+3$.

- (1) The ordinary characters of the PIMs of kG can be read off from the decomposition matrix in Table 7, that is, for each $1 \leq a \leq m$ we have

$$\Phi_{\varphi_a} = \chi_a + \sum_{j=m+1}^{m+(p+1) \cdot a(m)} \chi_j$$

where χ_a runs through $\text{Lin}(G)$.

- (2) Fix $2 \leq i \leq p+2$. We know from Lemma 5.2 that $N_G(Q_i) = G$ and $\overline{N}_G(Q_i) = G/Q_i$ is a Frobenius group with Frobenius complement $H = Q_i H/Q_i$ and Frobenius kernel F/Q_i . First, it follows from the bijections and the notation introduced in Lemma 5.4 and in Subsection 2.3 that

$$\text{TS}(G; Q_i) = \{\text{Inf}_{G/Q_i}^G(P_\psi) \mid \psi \in \text{IBr}_p(G/Q_i)\}.$$

Then, it follows from Lemma 5.4(b) that for every $1 \leq u \leq m$ the ordinary characters of the PIMs P_{ψ_u} of $k[G/Q_i]$ are given by

$$\Phi_{\psi_u} = \theta_u + \sum_{b=1}^{a(m)} \theta_{m+b},$$

where $\theta_u \in \text{Lin}(G/Q_i)$ and $\theta_{m+b} \in \text{Irr}(G/Q_i) \setminus \text{Lin}(G/Q_i)$ with $\ker(\theta_{m+b}) = Q_i$ for each $1 \leq b \leq a(m)$. Next we observe that

$$\text{Inf}_{G/Q_i}^G(\text{Irr}(G/Q_i) \setminus \text{Lin}(G/Q_i)) = \{\chi \in \text{Irr}(G) \setminus \text{Lin}(G) \mid \ker(\chi) = Q_i\}.$$

Indeed, as the kernels of the non-trivial characters of F are cyclic of order p and normal in G , it follows that the kernel of any the non-linear irreducible character of G is equal to the kernel of the character(s) it is induced from, hence also cyclic of order p . Therefore, we obtain that

$$\begin{aligned} \text{Inf}_{G/Q_i}^G(\Phi_{\psi_u}) &= \text{Inf}_{G/Q_i}^G(\theta_u + \sum_{b=1}^{a(m)} \theta_{m+b}) = \text{Inf}_{G/Q_i}^G(\theta_u) + \sum_{b=1}^{a(m)} \text{Inf}_{G/Q_i}^G(\theta_{m+b}) \\ &= \text{Inf}_{G/Q_i}^G(\theta_u) + \sum_{\substack{\chi \in \text{Irr}(G) \setminus \text{Lin}(G) \\ \ker(\chi) = Q_i}} \chi \end{aligned}$$

for each $1 \leq u \leq m$. Clearly, the characters $\text{Inf}_{G/Q_i}^G(\theta_u)$ run through $\text{Lin}(G)$ when u runs from 1 to m , and the claim follows.

- (3) The claim follows from Lemma 2.4(b), where equality holds by the argument above.

(c) We now compute the entries of $\text{Triv}_p(G)$.

- (1) The assertion is immediate from Lemma 2.3(a).
 (2) By Lemma 2.3(b), we have $T_{1,1} = \Phi_p(G)$. It is clear from Proposition 3.3 and Part (b)(1) that the degree of the characters Φ_{φ_a} ($1 \leq a \leq m$) is $1 + p^2 - 1$. Therefore, it is now only left to prove that $\chi_j(y) = 0$ for all $m+1 \leq j \leq m + (p+1) \cdot a(m)$ and for all $y \in H \setminus \{1\}$. But this is clear since for any $\nu \in \text{Irr}(F)$, we have

$$(\text{Ind}_F^G(\nu))(y) = \frac{1}{|F|} \sum_{\substack{g \in G \\ gyg^{-1} \in F}} \nu(gyg^{-1}) = 0.$$

- (3) The matrices $T_{i,i}$ ($2 \leq i \leq p+2$). Fix $i \in \{2, \dots, p+2\}$. By Lemma 2.3(b), we have $T_{i,i} = \Phi_p(\overline{N}_G(Q_i))$. By Lemma 5.2 the group $\overline{N}_G(Q_i) = G/Q_i$ is a Frobenius group with Frobenius complement H and Frobenius kernel F/Q_i . The ordinary characters of the

PIMs of $k\overline{N}_G(Q_i)$ can be read off from Table 8, namely

$$\Phi_{\psi_a} = \theta_a + \sum_{j=1}^{\frac{p-1}{m}} \theta_{m+j}$$

for each $1 \leq a \leq m$. As any element of $\text{Irr}(G/Q_i) \setminus \text{Lin}(G/Q_i)$ is induced from a linear character of F/Q_i , its degree is $[G/Q_i : F/Q_i] \cdot 1 = [G : F] = |H| = m$. Therefore, $\deg(\Phi_{\psi_a}) = 1 + \frac{p-1}{m} \cdot m = 1 + p - 1 = p$ for every $1 \leq a \leq m$. As in (c)(2), using the formula for induced characters, we see that all non-linear constituents of Φ_{ψ_a} evaluate to 0 at all $g \in H \setminus \{1\}$. The claim follows, as all the linear characters of G/Q_i are precisely the inflations of the characters in $\text{Irr}(H)$. This proves that

$$T_{i,i} = X(H) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix}.$$

The matrices $T_{i,1}$ ($2 \leq i \leq p+2$). Fix $i \in \{2, \dots, p+2\}$. By Lemma 2.3(c), we have

$$T_{i,1} = (\chi_{\widehat{M}}(s))_{M \in \text{TS}(G; Q_i), s \in [G]_{p'}}.$$

For any $M \in \text{TS}(G; Q_i)$, by the bijections in Subsection 2.3, we have

$$\chi_{\widehat{M}} = \text{Ind}_{N_G(Q_i)}^G \text{Inf}_{N_G(Q_i)/Q_i}^{N_G(Q_i)}(\Phi_{\psi_a}) = \text{Ind}_G^G \text{Inf}_{G/Q_i}^G(\Phi_{\psi_a}) = \text{Inf}_{G/Q_i}^G(\Phi_{\psi_a})$$

for a unique $a \in \{1, \dots, m\}$. Since we set in (a) that $[G]_{p'} = H$, it follows immediately that $T_{i,1} = T_{i,i}$. Moreover, we have $T_{i,i} = T_{2,2} = T_{2,1}$.

(4) Fix $v \in \{1, \dots, p+3\}$. By definition, we have

$$T_{p+3,v} = [\tau_{Q_v,s}^G([M])]_{M \in \text{TS}(G; Q_{p+3}), s \in [\overline{N}_G(Q_v)]_{p'}}.$$

So, let $M \in \text{TS}(G; Q_{p+3})$ and let $t \in [N_G(Q_v)]_{p'}$. Recall that by Lemma 5.2(a) we have $N_G(Q_v) = G$ and by Proposition 3.3(c) we may choose $[N_G(Q_v)]_{p'} = H$. By definition,

$$\tau_{Q_v,t}^G([M]) = \varphi_{M[Q_v]}(t).$$

Since $Q_v \leq Q_{p+3} = F \trianglelefteq G$, it follows from [Lin18b, Proposition 5.10.4] that $M[Q_v] = \text{Res}_{N_G(Q_v)}^G(M)$. Hence, $\varphi_{M[Q_v]}(t) = \chi_{\widehat{M}}(t)$ by Lemma 2.4(b), and moreover $T_{p+3,1} = T_{p+3,v} = X(H)$.

□

Example 5.6. Let G be the Frobenius group $(C_5 \times C_5) \rtimes C_4$ of order 100 with minimal fusion pattern, i.e. G has precisely 6 distinct conjugacy classes of subgroups of order 5. It follows that we have 8 conjugacy classes of 5-subgroups of G , namely:

$$Q_1 = \{1\}, \quad Q_2 \cong Q_3 \cong Q_4 \cong Q_5 \cong Q_6 \cong Q_7 \cong C_5, \quad Q_8 \cong C_5 \times C_5.$$

Notice that G is isomorphic to the group labelled by [100, 11] in GAP's SmallGroups library, see [GAP]. The ordinary character table of G is as given in Table 10, where $\zeta_4 := \exp(\frac{2\pi i}{4})$.

The trivial source character table $\text{Triv}_5(G)$ is as given in Table 11. Following our conventions, we label the columns of $\text{Triv}_5(G)$ with 5'-elements in $N_G(Q_v)$ instead of $\overline{N}_G(Q_v)$ ($1 \leq v \leq 8$).

	$1a$	$4a$	$2a$	$4b$	$5a$	$5b$	$5c$	$5d$	$5e$	$5f$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	ζ_4	-1	$-\zeta_4$	1	1	1	1	1	1
χ_3	1	-1	1	-1	1	1	1	1	1	1
χ_4	1	$-\zeta_4$	-1	ζ_4	1	1	1	1	1	1
χ_5	4	0	0	0	4	-1	-1	-1	-1	-1
χ_6	4	0	0	0	-1	4	-1	-1	-1	-1
χ_7	4	0	0	0	-1	-1	4	-1	-1	-1
χ_8	4	0	0	0	-1	-1	-1	4	-1	-1
χ_9	4	0	0	0	-1	-1	-1	-1	4	-1
χ_{10}	4	0	0	0	-1	-1	-1	-1	-1	4

TABLE 10. Ordinary character table of $(C_5 \times C_5) \rtimes C_4$.

Normalisers N_v		$N_1 \cong (C_5 \times C_5) \rtimes C_4$								$N_2 \cong (C_5 \times C_5) \rtimes C_4$								$N_3 \cong (C_5 \times C_5) \rtimes C_4$								$N_4 \cong (C_5 \times C_5) \rtimes C_4$								$N_5 \cong (C_5 \times C_5) \rtimes C_4$								$N_6 \cong (C_5 \times C_5) \rtimes C_4$								$N_7 \cong (C_5 \times C_5) \rtimes C_4$								$N_8 \cong (C_5 \times C_5) \rtimes C_4$							
Representatives $n_j \in N_i$		1a	4a	2a	4b	1a	4a	2a	4b	1a	4a	2a	4b	1a	4a	2a	4b	1a	4a	2a	4b	1a	4a	2a	4b	1a	4a	2a	4b	1a	4a	2a	4b	1a	4a	2a	4b																												
$\chi_1 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$		25	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_2 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$		25	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_3 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$		25	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$		25	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_1 + \chi_5$		5	1	1	1	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_2 + \chi_5$		5	ζ_4	-1	$-\zeta_4$	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																											
$\chi_3 + \chi_5$		5	-1	1	-1	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_4 + \chi_5$		5	$-\zeta_4$	-1	ζ_4	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																											
$\chi_1 + \chi_6$		5	1	1	1	0	0	0	0	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_2 + \chi_6$		5	ζ_4	-1	$-\zeta_4$	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_3 + \chi_6$		5	-1	1	-1	0	0	0	0	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_4 + \chi_6$		5	$-\zeta_4$	-1	ζ_4	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																											
$\chi_1 + \chi_7$		5	1	1	1	0	0	0	0	0	0	0	0	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_2 + \chi_7$		5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_3 + \chi_7$		5	-1	1	-1	0	0	0	0	0	0	0	0	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_4 + \chi_7$		5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																												
$\chi_1 + \chi_8$		5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_2 + \chi_8$		5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_3 + \chi_8$		5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_4 + \chi_8$		5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_1 + \chi_9$		5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_2 + \chi_9$		5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_3 + \chi_9$		5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_4 + \chi_9$		5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_1 + \chi_{10}$		5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_2 + \chi_{10}$		5	ζ_4	-1	$-\zeta_4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_3 + \chi_{10}$		5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
$\chi_4 + \chi_{10}$		5	$-\zeta_4$	-1	ζ_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0																													
χ_1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1																														
χ_2		1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1																													
χ_3		1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1																													
χ_4		1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1																													

TABLE 11. Trivial source character table of $(C_5 \times C_5) \rtimes C_4$ at $p = 5$.

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REFERENCES

- [Ben84] D. BENSON, *Modular representation theory: new trends and methods, Lecture Notes in Mathematics* **1081**, Springer-Verlag, Berlin, 1984.
- [Ben98] D. J. BENSON, *Representations and cohomology. I*, second ed., *Cambridge Studies in Advanced Mathematics* **30**, Cambridge University Press, Cambridge, 1998.
- [BP84] D. J. BENSON and R. A. PARKER, The Green ring of a finite group, *J. Algebra* **87** (1984), 290–331.
- [Böh24] B. BÖHMLER, *Trivial source character tables of small finite groups*, Dissertation, RPTU Kaiserslautern-Landau, 2024. doi:10.26204/KLUEDO/8356.
- [BFL22] B. BÖHMLER, N. FARRELL, and C. LASSUEUR, Trivial source character tables of $SL_2(q)$, *J. Algebra* **598** (2022), 308–350.
- [BFLP24] B. BÖHMLER, N. FARRELL, C. LASSUEUR, and J. PATIL, *Database of trivial source character tables, Preliminary Version*, 2024. Available at <https://representationtables.github.io/trivialsources/>.
- [BM25] R. BOLTJE and N. MONTEIRO, The ring of perfect p -permutation bimodules for blocks with cyclic defect groups, *J. Algebra* **675** (2025), 1–22.
- [BCP97] W. BOSMA, J. CANNON, and C. PLAYOUST, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.
- [Bou10] S. BOUC, *Biset functors for finite groups*, **1990**, Berlin: Springer, 2010.
- [BHHK25] T. BREUER, L. HÉTHELYI, E. HORVÁTH, and B. KÜLSHAMMER, On Frobenius graphs of diameter 3 for finite groups, *J. Algebra* **666** (2025), 507–529.
- [BH98] R. BROWN and D. K. HARRISON, Abelian Frobenius kernels and modules over number rings, *J. Pure Appl. Algebra* **126** (1998), 51–86.
- [CR90] C. W. CURTIS and I. REINER, *Methods of representation theory. Vol. I*, John Wiley & Sons, Inc., New York, 1990.
- [FL23] N. FARRELL and C. LASSUEUR, Trivial source character tables of $SL_2(q)$, Part II, *Proc. Edinburgh Math. Soc.* **66** (2023), 689–709.
- [GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.12.1*, 2022. Available at <https://www.gap-system.org>.
- [Hup98] B. HUPPERT, *Character theory of finite groups, De Gruyter Expositions in Mathematics* **25**, Walter de Gruyter & Co., Berlin, 1998.
- [Jac12] N. JACOBSON, *Basic Algebra II: Second Edition, Dover Books on Mathematics*, Dover Publications, 2012.
- [Las23] C. LASSUEUR, A tour of p -permutation modules and related classes of modules, *Jahresber. Dtsch. Math.-Ver.* **125** (2023), 137–189.
- [Lin18a] M. LINCKELMANN, *The block theory of finite group algebras. Vol. II, London Mathematical Society Student Texts* **92**, Cambridge University Press, Cambridge, 2018.
- [Lin18b] M. LINCKELMANN, *The block theory of finite group algebras. Vol. I, London Mathematical Society Student Texts* **91**, Cambridge University Press, Cambridge, 2018.
- [LP10] K. LUX and H. PAHLINGS, *Representations of groups: A computational approach*, Cambridge University Press, Cambridge, 2010.
- [NT89] H. NAGAO and Y. TSUSHIMA, *Representations of finite groups*, Academic Press, Inc., Boston, MA, 1989, Translated from the Japanese.
- [Web16] P. WEBB, *A course in finite group representation theory, Cambridge Studies in Advanced Mathematics* **161**, Cambridge University Press, Cambridge, 2016.

APPENDIX A. $\text{Triv}_p(\text{AGL}_1(p^2))$ IN TABLE FORM

To support intuition, in this appendix, we give $\text{Triv}_p(\text{AGL}_1(p^2))$ described in Theorem 4.4 in *table form*, where $\zeta := \exp(\frac{2\pi i}{p^2-1})$.

	1	h	\dots	$h^{b(p+1)-1}$ $(1 \leq b \leq p-2)$	$h^{b(p+1)}$ $(1 \leq b \leq p-2)$	$h^{b(p+1)+1}$ $(1 \leq b \leq p-2)$	\dots	h^{p^2-2}
$T_{1,1}$	$\chi_1 + \chi_{p^2}$	p^2	1	1	1	1	\dots	1
	$\chi_2 + \chi_{p^2}$	p^2	ζ	$\zeta^{b(p+1)-1}$	$\zeta^{b(p+1)}$	$\zeta^{b(p+1)+1}$	\dots	ζ^{p^2-2}
	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\dots	\vdots
	$\chi_{p^2-1} + \chi_{p^2}$	p^2	ζ^{p^2-2}	$\zeta^{(b(p+1)-1) \cdot (p^2-2)}$	$\zeta^{b(p+1) \cdot (p^2-2)}$	$\zeta^{(b(p+1)+1) \cdot (p^2-2)}$	\dots	$\zeta^{(p^2-2)^2}$
$T_{2,1}$	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+1}$	$p(p+1)$	0	0	$p+1$	0	\dots	0
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+2}$	$p(p+1)$	0	0	$\zeta^{b(p+1) \cdot (p+1)}$	0	\dots	0
	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\dots	\vdots
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+(p-1)}$	$p(p+1)$	0	0	$\zeta^{(p-2) \cdot b(p+1) \cdot (p+1)}$	0	\dots	0
$T_{3,1}$	χ_1	1	1	1	1	1	\dots	1
	χ_2	1	ζ	$\zeta^{b(p+1)-1}$	$\zeta^{b(p+1)}$	$\zeta^{b(p+1)+1}$	\dots	ζ^{p^2-2}
	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\dots	\vdots
	χ_{p^2-1}	1	ζ^{p^2-2}	$\zeta^{(b(p+1)-1) \cdot (p^2-2)}$	$\zeta^{b(p+1) \cdot (p^2-2)}$	$\zeta^{(b(p+1)+1) \cdot (p^2-2)}$	\dots	$\zeta^{(p^2-2)^2}$

TABLE 12. $T_{i,1}$ for $1 \leq i \leq 3$ (first block column of $\text{Triv}_p(\text{AGL}_1(p^2))$).

		1	h^{p+1}	$h^{2 \cdot (p+1)}$	\dots	$h^{(p-2) \cdot (p+1)}$
$T_{2,2}$	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+1}$	p	1	1	\dots	1
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+2}$	p	ζ^{p+1}	$\zeta^{2 \cdot (p+1)}$	\dots	$\zeta^{(p-2) \cdot (p+1)}$
	\vdots	\vdots	\vdots	\vdots	\dots	\vdots
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+p-1}$	p	$\zeta^{(p-2) \cdot (p+1)}$	$\zeta^{2 \cdot (p-2) \cdot (p+1)}$	\dots	$\zeta^{(p-2)^2 \cdot (p+1)}$
$T_{3,2}$	χ_1	1	1	1	\dots	1
	χ_2	1	$\zeta^{(p+1)}$	$\zeta^{2 \cdot (p+1)}$	\dots	$\zeta^{(p-2) \cdot (p+1)}$
	χ_3	1	$\zeta^{2 \cdot (p+1)}$	$\zeta^{4 \cdot (p+1)}$	\dots	$\zeta^{2 \cdot (p-2) \cdot (p+1)}$
	\vdots	\vdots	\vdots	\vdots	\dots	\vdots
	χ_{p^2-1}	1	$\zeta^{(p^2-2) \cdot (p+1)}$	$\zeta^{2 \cdot (p^2-2) \cdot (p+1)}$	\dots	$\zeta^{(p^2-2) \cdot (p-2) \cdot (p+1)}$

TABLE 13. $T_{i,2}$ for $2 \leq i \leq 3$.

		h^j ($0 \leq j \leq p^2 - 2$)
$T_{3,3}$	χ_a ($1 \leq a \leq p^2 - 1$)	$\zeta^{(a-1)j}$

TABLE 14. $T_{3,3}$.

BERNHARD BÖHMLER, LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK, WELFENGARTEN 1, 30167 HANNOVER, GERMANY
Email address: boehmler@math.uni-hannover.de

CAROLINE LASSUEUR, LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK, WELFENGARTEN 1, 30167 HANNOVER, GERMANY
Email address: lassueur@math.uni-hannover.de, lassueur@mathematik.uni-kl.de