

# TRIVIAL SOURCE CHARACTER TABLES OF FROBENIUS GROUPS OF TYPE $(C_p \times C_p) \rtimes H$

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*Dedicated to the memory of Richard Parker*

**ABSTRACT.** Let  $p$  be a prime number. We compute the trivial source character tables of finite Frobenius groups  $G$  with an abelian Frobenius complement  $H$  and an elementary abelian Frobenius kernel of order  $p^2$ . More precisely, we deal with infinite families of such groups which occur in the two extremal cases for the fusion of  $p$ -subgroups: the case in which there exists exactly one  $G$ -conjugacy class of non-trivial cyclic  $p$ -subgroups, and the case in which there exist exactly  $p+1$  distinct  $G$ -conjugacy classes of non-trivial cyclic  $p$ -subgroups.

## 1. INTRODUCTION

Let  $G$  be a finite group. Let  $p$  be a prime number dividing the order of  $G$  and let  $k$  be a large enough field of characteristic  $p$ . Permutation  $kG$ -modules and their direct summands – called  *$p$ -permutation modules* or also *trivial source modules* – are omnipresent in the modular representation theory of finite groups. They are, for example, elementary building blocks for the construction and for the understanding of different categorical equivalences between block algebras, such as splendid Rickard equivalences,  $p$ -permutation equivalences, source-algebra equivalences, or Morita equivalences with endo-permutation source. A deep understanding of the structure of these modules is therefore essential.

In this manuscript, we go back to ideas of Benson and Parker developed in [BP84]. Any trivial source  $kG$ -module can be lifted to characteristic zero and affords a well-defined ordinary character, which contains essential information about its structure. The *trivial source character table*  $\mathrm{Triv}_p(G)$  of  $G$  at the prime  $p$  collects this information in a table; it is the *species table* or *representation table* of the trivial source ring in the sense of [BP84, Ben84, Ben98]. More precisely, it provides us with information about the character values of all the indecomposable trivial source  $kG$ -modules and their Brauer quotients at all  $p'$ -conjugacy classes. See Subsection 2.3 for a precise definition.

The present article is in fact part of a program aiming at gathering information about trivial source modules of small finite groups and their associated *trivial source character tables* in a database [BFLP24]. Isolated examples – calculated by Benson, and Lux and Pahlings – can be found in [Ben84, Appendix] and [LP10, §4.10]. More recently, the first author, as part of his doctoral thesis [Böh24], developed GAP4 [GAP] and MAGMA [BCP97] algorithms, which could be used to compute the trivial source character tables of finite groups of order less than 100, as well as the trivial source character tables of various small (non-abelian) quasi-simple groups. The latter algorithms rely, in particular, on the MeatAxe algorithm, first introduced

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by R. Parker. Meanwhile, in [BFL22] and [FL23] the authors and Farrell computed *generic* trivial source character tables for the groups  $\mathrm{SL}_2(q)$  and  $\mathrm{PSL}_2(q)$  in cross characteristic, using the generic character tables of these groups and theoretical methods involving block theory.

Using the data produced in [BFLP24] we identified some interesting families of finite groups, of which we can calculate the trivial source character tables from a purely theoretical point of view. In this regard, the main results of this article consist in the calculation of the trivial source character tables at  $p$  of the following infinite families of Frobenius groups with an abelian Frobenius complement  $H$  and elementary abelian Frobenius kernel of rank 2:

- (I) the family of all metabelian Frobenius groups of type  $(C_p \times C_p) \rtimes H$ , in which there is precisely one conjugacy class of subgroups of order  $p$ ;
- (II) the family of all metabelian Frobenius groups of type  $(C_p \times C_p) \rtimes H$  in which there are precisely  $p+1$  conjugacy classes of subgroups of order  $p$ .

We note that the unique group of type (I) with  $|H| = p^2 - 1$  is  $\mathrm{AGL}_1(p^2)$ . Furthermore, if  $p = 2$ , then the only group of type (I) is the alternating group  $\mathfrak{A}_4$ , while groups of type (II) only occur for odd prime numbers  $p$ . For this reason, we exclude the prime number 2 from all our calculations in this manuscript. We also emphasise that contrary to [BFL22, FL23], it is not possible to use block theoretical arguments in the present cases, because such groups possess only one  $p$ -block.

The paper is structured as follows. In Section 2 we introduce our notation and conventions. In Section 3, first we review some properties of Frobenius groups and their ordinary and Brauer characters. Then, we characterise metabelian Frobenius groups with elementary abelian Frobenius kernel. The trivial source character tables are calculated in Section 4 for groups of type (I), respectively in Section 5 for groups of type (II).

## 2. PRELIMINARIES

**2.1. General notation.** Throughout, unless otherwise stated, we adopt the notation and conventions below. We let  $p$  denote an odd prime number and  $G$  a finite group of order divisible by  $p$ . We let  $(K, \mathcal{O}, k)$  be a  $p$ -modular system, where  $\mathcal{O}$  denotes a complete discrete valuation ring of characteristic zero with field of fractions  $K = \mathrm{Frac}(\mathcal{O})$  and residue field  $k = \mathcal{O}/J(\mathcal{O})$  of characteristic  $p$ . Following [CR90, §17A], we assume that  $K$  is *sufficiently large* (relative to  $G$ ), i.e.  $K$  contains all  $\exp(G)$ -th roots of unity. Then  $k$  is also sufficiently large (relative to  $G$ ) and  $K$  and  $k$  are splitting fields for  $G$  and all of its subgroups. For  $R \in \{\mathcal{O}, k\}$ ,  $RG$ -modules are assumed to be finitely generated left  $RG$ -lattices, that is, free as  $R$ -modules, and we let  $R$  denote the trivial  $RG$ -lattice.

Given a positive integer  $n$ , we let  $C_n$  denote the cyclic group of order  $n$ . We let  $\mathbf{O}_p(G)$  denote the largest normal  $p$ -subgroup of  $G$ ,  $\mathrm{Syl}_p(G)$  denote the set of all Sylow  $p$ -subgroups of  $G$ ,  $ccls(G)$  denote a set of representatives for the conjugacy classes of  $G$ ,  $[G]_{p'}$  denote a set of representatives for the  $p$ -regular conjugacy classes of  $G$ , and we let  $G_{p'} := \{g \in G \mid p \nmid o(g)\}$ . We recall that a group  $G$  with a normal subgroup  $N$  and a subgroup  $H$  is said to be the *internal semi-direct product of  $N$  by  $H$* , written  $G = N \rtimes H$ , provided  $G = NH$  and  $N \cap H = \{1\}$ .

Given  $H \leq G$ , an ordinary character  $\psi$  of  $H$  and  $\chi$  an ordinary character of  $G$ , we write  $\mathrm{Ind}_H^G(\psi)$  for the induction of  $\psi$  from  $H$  to  $G$ ,  $\mathrm{Res}_H^G(\chi)$  for the restriction of  $\chi$  from  $G$  to  $H$ ,  $\chi^\circ := \chi|_{G_{p'}}$  for the reduction modulo  $p$  of  $\chi$ , and  $1_H$  for the trivial character of  $H$ . Given  $N \trianglelefteq G$  and an ordinary character  $\nu$  of  $G/N$ , we write  $\mathrm{Inf}_{G/N}^G(\nu)$  for the inflation of  $\nu$  from  $G/N$  to  $G$ . Similarly, we write  $\mathrm{Ind}_H^G(L)$  for the induction of the  $kH$ -module  $L$  from  $H$  to  $G$ ,  $\mathrm{Res}_H^G(M)$  for

the restriction of the  $kG$ -module  $M$  from  $G$  to  $H$ , and  $\text{Inf}_{G/N}^G(U)$  for the inflation of the  $k[G/N]$ -module  $U$  from  $G/N$  to  $G$ . Moreover, if  $M$  is a  $kG$ -module, then we denote by  $\varphi_M$  the Brauer character afforded by  $M$ , and if  $Q \leq G$  then the Brauer quotient (or Brauer construction) of  $M$  at  $Q$  is the  $k$ -vector space  $M[Q] := M^Q / \sum_{R < Q} \text{tr}_R^Q(M^R)$ , where  $M^Q$  denotes the fixed points of  $M$  under  $Q$  and  $\text{tr}_R^Q$  denotes the relative trace map. This vector space has a natural structure of a  $kN_G(Q)$ -module, but also of a  $kN_G(Q)/Q$ -module, and is equal to zero if  $Q$  is not a  $p$ -subgroup. Moreover, we use the abbreviation PIM to mean a *projective indecomposable module* and we denote by  $\text{Irr}(kG)$  the set of all simple  $kG$ -modules, considered up to isomorphism. We assume that the reader is familiar with elementary notions of ordinary and modular representation theory of finite groups. We refer to [Lin18a, Web16, NT89, Hup98, CR90] for further standard notation and background results.

**2.2. Character tables and decomposition matrices.** We let  $\text{Irr}(G)$ ,  $\text{Lin}(G)$ , and  $\text{IBr}_p(G)$  denote the set of all irreducible  $K$ -characters of  $G$ , the set of all linear characters of  $G$ , and the set of all irreducible  $p$ -Brauer characters of  $G$ , respectively. We let

$$X(G) := \left( \chi(g) \right)_{\substack{\chi \in \text{Irr}(G) \\ g \in \text{ccls}(G)}} \in K^{|\text{Irr}(G)| \times |\text{ccls}(G)|}$$

denote the ordinary character table of  $G$  and we let

$$X(G, p') := \left( \chi(g) \right)_{\substack{\chi \in \text{Irr}(G) \\ g \in [G]_{p'}}} \in K^{|\text{Irr}(G)| \times |[G]_{p'}|}$$

denote the matrix obtained from  $X(G)$  by removing the columns labelled by  $p$ -singular conjugacy classes. Moreover, we always assume that the first column of these matrices is labelled by the class of 1. We recall that for any  $\chi \in \text{Irr}(G)$  there exist uniquely determined non-negative integers  $d_{\chi\varphi}$  such that  $\chi^\circ = \sum_{\varphi \in \text{IBr}_p(G)} d_{\chi\varphi} \varphi$ . Then, for any  $\varphi \in \text{IBr}_p(G)$ , the *projective indecomposable character* associated to  $\varphi$  is

$$(1) \quad \Phi_\varphi := \sum_{\chi \in \text{Irr}(G)} d_{\chi\varphi} \chi.$$

The  $p$ -decomposition matrix of  $G$  is then

$$\text{Dec}_p(G) := \left( d_{\chi\varphi} \right)_{\substack{\chi \in \text{Irr}(G) \\ \varphi \in \text{IBr}_p(G)}} \in K^{|\text{Irr}(G)| \times |\text{IBr}_p(G)|}$$

and the  $p$ -projective table of  $G$  is

$$\Phi_p(G) := \left( \Phi_\varphi(x) \right)_{\substack{\varphi \in \text{IBr}_p(G) \\ x \in [G]_{p'}}} \in K^{|\text{IBr}_p(G)| \times |[G]_{p'}|},$$

which is the table of Brauer character values of the projective indecomposable  $kG$ -modules. It follows from the definitions that

$$(2) \quad \Phi_p(G) = \text{Dec}_p(G)^t \cdot X(G, p').$$

Finally, the character tables of finite cyclic groups will play an essential role in our calculations, hence in this case we fix the following labelling of the irreducible characters and conjugacy classes.

**Notation 2.1.** If  $G := \langle x \mid x^m = 1 \rangle \cong C_m$  is a cyclic group of order  $m \geq 1$ , then we let  $\zeta \in K$  denote a primitive  $m$ -th root of unity and we write the set of ordinary irreducible characters of  $G$  as  $\text{Irr}(G) = \{\xi_1, \dots, \xi_m\}$ , where

$$\xi_a(x^j) := \zeta^{(a-1)j}$$

for each  $1 \leq a \leq m$  and each  $0 \leq j \leq m-1$ . This yields

$$X(C_m) := \left( \xi_a(x^{j-1}) \right)_{\substack{1 \leq a \leq m \\ 1 \leq j \leq m}} = \left( \zeta^{(a-1)(j-1)} \right)_{\substack{1 \leq a \leq m \\ 1 \leq j \leq m}}.$$

**2.3. Trivial source character tables.** Given  $R \in \{\mathcal{O}, k\}$ , an  $RG$ -lattice  $M$  is called a *trivial source RG-lattice* if it is isomorphic to an indecomposable<sup>1</sup> direct summand of an induced lattice  $\text{Ind}_Q^G(R)$ , where  $Q \leq G$  is a  $p$ -subgroup. In addition, if  $Q$  is of minimal order subject to this property, then  $Q$  is a vertex of  $M$ . It is clear that, up to isomorphism, there are only finitely many trivial source  $RG$ -lattices.

It is well-known that any trivial source  $kG$ -module  $M$  lifts in a unique way to a trivial source  $\mathcal{O}G$ -lattice  $\widehat{M}$  (see e.g. [Ben98, Corollary 3.11.4]) and we denote by  $\chi_{\widehat{M}}$  the  $K$ -character afforded by  $\widehat{M}$ . Moreover, if  $\varphi_M$  denotes the Brauer character afforded by  $M$ , then  $(\chi_{\widehat{M}})^\circ = \varphi_M$  (see e.g. [Lin18b, Proposition 5.13.6]). If  $M$  is a PIM, then  $\chi_{\widehat{M}} = \Phi_\varphi$ , where  $\varphi$  is the Brauer character afforded by the unique simple  $kG$ -module in the socle of  $M$ .

We will study trivial source modules vertex by vertex. Hence, we denote by  $\text{TS}(G; Q)$  the set of isomorphism classes of trivial source  $kG$ -modules with vertex  $Q$ . We notice that  $\text{TS}(G; \{1\})$  is precisely the set of isomorphism classes of PIMs of  $kG$ .

A  $p$ -subgroup  $Q \leq G$  is a vertex of a trivial source  $kG$ -module  $M$  if and only if  $M[Q]$  is a non-zero projective  $k\overline{N}_G(Q)$ -module. Moreover, if this is the case, then the  $kN_G(Q)$ -Green correspondent  $f(M)$  of  $M$  is  $M[Q]$  (viewed as a  $kN_G(Q)$ -module). Thus, there are bijections

$$\begin{array}{ccccc} \text{TS}(G; Q) & \xrightarrow{\sim} & \text{TS}(N_G(Q); Q) & \xrightarrow{\sim} & \text{TS}(\overline{N}_G(Q); \{1\}) \\ M & \mapsto & f(M) & \mapsto & M[Q] \end{array}$$

where the inverse of the second map is given by the inflation from  $\overline{N}_G(Q) := N_G(Q)/Q$  to  $N_G(Q)$ . These sets are also in bijection with the set of  $p'$ -conjugacy classes of  $\overline{N}_G(Q)$ .

Next, we let  $a(kG, \text{Triv})$  denote the *trivial source ring* of  $kG$ , which is defined to be the subring of the Green ring of  $kG$  generated by the set of all isomorphism classes of trivial source  $kG$ -modules. Notice that this ring is finitely generated. By definition, the *trivial source character table of the group  $G$  at the prime  $p$* , denoted  $\text{Triv}_p(G)$ , is the species table of the trivial source ring of  $kG$ . See e.g. [BP84]. However, we follow [LP10, Section 4.10] and consider  $\text{Triv}_p(G)$  as the block square matrix defined according to the following convention.

**Convention 2.2.** First, fix a set of representatives  $Q_1, \dots, Q_r$  ( $r \in \mathbb{Z}_{\geq 1}$ ) for the conjugacy classes of  $p$ -subgroups of  $G$  where  $Q_1 := \{1\}$ ,  $Q_r \in \text{Syl}_p(G)$  and  $|Q_1| \leq \dots \leq |Q_r|$ . For each  $1 \leq v \leq r$  set  $\overline{N}_G(Q_v) := N_G(Q_v)/Q_v$ . For each pair  $(Q_v, s)$  with  $1 \leq v \leq r$  and  $s \in [\overline{N}_G(Q_v)]_{p'}$  there is a ring homomorphism

$$\begin{aligned} \tau_{Q_v, s}^G : a(kG, \text{Triv}) &\longrightarrow K \\ [M] &\mapsto \varphi_{M[Q_v]}(s) \end{aligned}$$

mapping the class of a trivial source  $kG$ -module  $M$  to the value at  $s$  of the Brauer character  $\varphi_{M[Q_v]}$  of the Brauer quotient  $M[Q_v]$ . (Note that the group  $G$  acts by conjugation on the

<sup>1</sup>We emphasise here that some authors use the terminology *trivial source module* to mean a finite direct sum of indecomposable  $kG$ -modules with a trivial source. We always assume such modules to be indecomposable.

pairs  $(Q_v, s)$  and the values of  $\tau_{Q_v, s}^G$  do not depend on the choice of  $(Q_v, s)$  in its  $G$ -orbit.) Then, for each  $1 \leq i, v \leq r$  define a matrix

$$T_{i,v} := \left( \tau_{Q_v, s}^G([M]) \right)_{\substack{M \in \text{TS}(G; Q_i) \\ s \in [\overline{N}_G(Q_v)]_{p'}}}.$$

The *trivial source character table of  $G$  at the prime  $p$*  is then the block matrix

$$\text{Triv}_p(G) := \begin{bmatrix} T_{i,v} \end{bmatrix}_{\substack{1 \leq i \leq r \\ 1 \leq v \leq r}}.$$

For convenience, we will label the columns of  $\text{Triv}_p(G)$  by representatives of the  $p'$ -elements of  $\overline{N}_G(Q_v)$  in  $N_G(Q_v)$  ( $1 \leq v \leq r$ ). This is possible e.g. by [Böh24, Lemma 3.1.12]. Moreover, we label the rows of  $\text{Triv}_p(G)$  with the ordinary characters  $\chi_{\widehat{M}}$  instead of the isomorphism classes of trivial source  $kG$ -modules  $M$  themselves.

In order to calculate the entries of  $\text{Triv}_p(G)$ , we use the following two well-known lemmata. The first one lets us describe certain blocks of the trivial source character table using ordinary and Brauer characters. The second lemma characterises trivial source modules with maximal vertices when a Sylow  $p$ -subgroup is normal.

**Lemma 2.3.** *Let  $\text{Triv}_p(G) = [T_{i,v}]_{1 \leq i, v \leq r}$  be the trivial source character table of the finite group  $G$  at  $p$ . Then, the following assertions hold:*

- (a)  $T_{i,v} = \mathbf{0}$  if  $Q_v \not\leq_G Q_i$ , so in particular  $T_{i,v} = \mathbf{0}$  for every  $1 \leq i < v \leq r$ ;
- (b)  $T_{i,i} = \Phi_p(\overline{N}_G(Q_i)) = \text{Dec}_p(\overline{N}_G(Q_i))^t \cdot X(\overline{N}_G(Q_i), p')$  for every  $1 \leq i \leq r$ ;
- (c)  $T_{i,1} = (\chi_{\widehat{M}}(s))_{M \in \text{TS}(G; Q_i), s \in [G]_{p'}}$  for every  $1 \leq i \leq r$ .

*Proof.* Assertion (a) is given by [LP10, Lemma 4.10.11(b)]. The first equality in assertion (b) is given by [LP10, Lemma 4.10.11(c)] and the second equality follows from Equation (2) above. Now, if  $v = 1$  and  $1 \leq i \leq r$ , then  $M[Q_v] = M[\{1\}] = M$ , so  $\tau_{\{1\}, s}^G([M]) = \varphi_M(s) = \chi_{\widehat{M}}(s)$  for every  $M \in \text{TS}(G; Q_i)$  and every  $s \in [G]_{p'}$ , proving assertion (c).  $\square$

**Lemma 2.4.** *Assume  $G$  is a finite group with a normal Sylow  $p$ -subgroup  $P \trianglelefteq G$  such that  $G/P$  is an abelian  $p'$ -group. Then, the following assertions hold:*

- (a)  $\text{TS}(G; P) = \text{Irr}(kG) = \{\text{Inf}_{G/P}^G(S) \mid S \in \text{Irr}(k[G/P])\}$ , which is the set of all 1-dimensional  $kG$ -modules (considered up to isomorphism);
- (b)  $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; P)\} = \text{Inf}_{G/P}^G(\text{Irr}(G/P)) \subseteq \text{Lin}(G)$ .

*Proof.* (a) Clearly  $P = \mathbf{O}_p(G)$ . Thus, it follows from Clifford's theorem that  $\text{Irr}(kG) = \{\text{Inf}_{G/P}^G(S) \mid S \in \text{Irr}(k[G/P])\}$ , which is precisely the set of all 1-dimensional  $kG$ -modules as  $G/P$  is an abelian  $p'$ -group (see e.g. [Web16, Proposition 6.2.2]). Now, 1-dimensional  $kG$ -modules are trivial source modules with vertex  $P$  since their restriction to  $P$  must be trivial, thus, as  $\overline{N}_G(P) = G/P$  is an abelian  $p'$ -group, these already account for the whole of  $\text{TS}(G; P)$ . (See Subsection 2.3.)

- (b) The claim is clear from (a) and the fact that  $(\chi_{\widehat{M}})^\circ$  coincides with the Brauer character of  $M$  for any  $M \in \text{TS}(G; P)$ .  $\square$

We refer the reader to the survey [Las23] and to our previous paper [BFL22, §2] for further details and further properties of trivial source modules and trivial source character tables. However, we mention the following result from the thesis of the first author, which will be crucial. Inbetween, this result has also appeared in [BM25, 4.1 Theorem].

**Proposition 2.5** ([Böh24, Proposition 3.1.15]). *Assume  $G$  is a finite group with a normal Sylow  $p$ -subgroup  $P \trianglelefteq G$  such that  $G/P$  is abelian. Let  $Q$  be a  $p$ -subgroup of  $G$ . Then, we have  $P \cap N_G(Q) = \mathbf{O}_p(N_G(Q)) \in \text{Syl}_p(N_G(Q))$  and by the Schur–Zassenhaus Theorem, we may choose a complement  $C$  of  $P \cap N_G(Q)$  in  $N_G(Q)$ . Let  $S$  be a simple  $kC$ -module, viewed as a simple  $k[QC/Q]$ -module via the canonical isomorphism  $QC/Q \cong C$ . Set*

$$L := \text{Ind}_{QC/Q}^{\overline{N}_G(Q)}(S) \quad \text{and} \quad U := \text{Inf}_{\overline{N}_G(Q)}^{N_G(Q)}(L).$$

Then, the following assertions hold:

- (a)  $L$  is a projective indecomposable  $k\overline{N}_G(Q)$ -module; and
- (b)  $M := \text{Ind}_{N_G(Q)}^G(U)$  is indecomposable, hence a trivial source  $kG$ -module with vertex  $Q$ .

In particular, any element of  $\text{TS}(G; Q)$  can be obtained in this way.

### 3. BACKGROUND MATERIAL ON FROBENIUS GROUPS

We start by reviewing basic definitions and results about the character theory of Frobenius groups.

**3.1. Frobenius groups.** Recall that a finite group  $G$  admitting a non-trivial proper subgroup  $H$  such that

$$H \cap gHg^{-1} = \{1\}$$

for each  $g \in G \setminus H$  is called a *Frobenius group* with *Frobenius complement*  $H$  (or a *Frobenius group with respect to  $H$* ). Frobenius proved that in such a group there exists a uniquely determined normal subgroup  $F$  such that  $G$  is the internal semi-direct product of  $F$  by  $H$  (i.e.  $G = FH$  and  $F \cap H = \{1\}$ ); concretely,

$$F = \{1\} \cup \left( G \setminus \bigcup_{g \in G} gHg^{-1} \right).$$

The normal subgroup  $F$  is called the *Frobenius kernel* of  $G$ . See e.g. [CR90, §14A]. In the sequel, we write Frobenius groups with respect to  $H$  as  $F \rtimes H$ . We will use the following well-known properties.

**Lemma 3.1.** *Let  $G$  be a Frobenius group with Frobenius complement  $H$  and Frobenius kernel  $F$ . Then the following assertions hold.*

- (a) *If  $H$  is abelian, then  $H$  is cyclic.*
- (b) *The integer  $|H|$  divides  $|F| - 1$ . In particular  $|G : F|$  and  $|F|$  are coprime integers, hence  $F$  is characteristic in  $G$ .*
- (c) *For each  $f \in F \setminus \{1\}$  we have  $C_G(f) \leq F$ .*
- (d) *The commutator subgroup of  $G$  is  $[G, G] = F \rtimes [H, H]$ .*

*Proof.* Assertion (a) is due to Burnside and follows directly from [Hup98, 16.7 Theorem b)]. Assertion (b) is given by [Hup98, 16.6 Lemma a)]. Assertion (c) is given by [CR90, (14.4)

Proposition (i)]. For Assertion (d), first it is clear that  $[G, G] \leq F \rtimes [H, H]$  as  $G/(F[H, H]) \cong H/[H, H]$  is abelian. To prove the reverse inclusion, it suffices to prove that  $F = [F, H]$ , since then  $F[H, H] = [F, H][H, H] \leq [G, G]$ . Now, as  $G$  is a Frobenius group,  $H \neq 1$  and there exists  $h \in H \setminus \{1\}$ . Then, the  $|F|$  elements of the set  $\{fhf^{-1}h^{-1} \in G \mid f \in F\}$  are pairwise distinct. Else  $f_1hf_1^{-1}h^{-1} = f_2hf_2^{-1}h^{-1}$  with  $f_1 \neq f_2 \in F$  implies that  $h \in C_G(f_2^{-1}f_1)$ , contradicting (c). Finally, as  $F \trianglelefteq G$ , any element in this set is in  $F$ , proving that  $F \leq [F, H] \leq F$ .  $\square$

**3.2. Characters of Frobenius groups.** The ordinary characters of Frobenius groups are well-known and given by the following theorem.

**Theorem 3.2** ([CR90, (14.4) Proposition]). *Let  $G$  be a Frobenius group with Frobenius complement  $H$  and Frobenius kernel  $F$ . Then*

$$\text{Irr}(G) = \{\text{Inf}_{G/F}^G(\psi) \mid \psi \in \text{Irr}(G/F)\} \sqcup \{\text{Ind}_F^G(\nu) \mid \nu \in T\},$$

where  $T$  is a set of representatives for the orbits of the action of  $G$  by conjugation on  $\text{Irr}(F) \setminus \{1_F\}$ .

Notice that the first set is precisely the set of all irreducible characters of  $G$  which contain  $F$  in their kernels.

**Proposition 3.3.** *Let  $G$  be a Frobenius group with cyclic Frobenius complement  $H \cong C_m$  for some integer  $m \geq 2$  and abelian Frobenius kernel  $F$  of order  $p^r$  for some positive integer  $r \geq 1$ . Then the following assertions hold:*

- (a)  $\text{Irr}(G) = \{\chi_1, \dots, \chi_{m+\frac{p^r-1}{m}}\}$  where for each  $1 \leq a \leq m$  we set

$$\chi_a := \text{Inf}_{G/F}^G(\xi_a)$$

with  $\xi_a \in \text{Irr}(C_m)$  as defined in Notation 2.1, and

$$\{\chi_{m+1}, \dots, \chi_{m+\frac{p^r-1}{m}}\} = \{\text{Ind}_F^G(\nu) \mid \nu \in T\}$$

where  $T$  is a set of representatives for conjugation action of  $G$  on  $\text{Irr}(F) \setminus \{1_F\}$ ;

- (b)  $\text{Lin}(G) = \text{Inf}_{G/F}^G(\text{Irr}(G/F)) = \{\chi_1, \dots, \chi_m\}$  and  $\text{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$  where  $\varphi_a := \chi_a^\circ$  for each  $1 \leq a \leq m$ ;
- (c)  $H$  is a set of representatives of the  $p$ -regular conjugacy classes of  $G$ ;
- (d)  $\chi_a^\circ = \sum_{j=1}^m \varphi_j$  for each  $1 \leq a \leq \frac{p^r-1}{m}$ .

*Proof.* Recall from Lemma 3.1 that  $\gcd(m, p) = 1$ .

- (a) First, it is clear from Theorem 3.2 that  $G$  has  $m$  pairwise distinct ordinary irreducible characters which are inflated from  $G/F \cong H \cong C_m$  to  $G$ . Moreover,

$$|\text{Irr}(G)| = |\text{Irr}(G/F)| + |\{\text{Ind}_F^G(\nu) \mid \nu \in T\}| = m + \frac{|F| - 1}{|H|} = m + \frac{p^r - 1}{m}$$

where the last-but-one equality holds by [Hup98, 18.7 Theorem b)].

- (b) The first claim follows from (a) and Lemma 3.1(d). Then, by Lemma 2.4, we have  $|\text{IBr}_p(G)| = |\text{IBr}_p(G/F)| = |H| = m$ . Since by construction  $\chi_1^\circ, \dots, \chi_m^\circ$  are pairwise distinct linear Brauer characters, they already account for all the irreducible Brauer characters of  $G$ . The claim follows.

- (c) Assume that  $h_1 \neq h_2 \in H \setminus \{1\}$  are conjugate in  $G$ , that is,  $gh_1g^{-1} = h_2$  for some  $g = fh \in FH = G$  with  $f \in F$  and  $h \in H$ . It follows that  $fhh_1h^{-1}f^{-1} = h_2$ , which implies that  $h_1^{-1} \cdot fh_1f^{-1} = h_1^{-1} \cdot h_2$  as  $H$  is cyclic. Since  $H$  acts fixed-point-freely on  $F \setminus \{1\}$ , we see that  $h_1^{-1}fh_1 \in F \setminus \{f\}$ . This is a contradiction, as  $F \cap H = \{1\}$ . The claim follows, as by (b), a set of representatives for the  $p$ -regular classes of  $G$  has size  $|H|$ .
- (d) It is immediate from part (b) that the first  $m$  rows of  $\text{Dec}_p(G)$  are given by the identity matrix of size  $m \times m$ . Let now  $\chi_a \in \text{Irr}(G)$  with  $m+1 \leq a \leq m + \frac{p^r-1}{m}$ . By (a), there exists a character  $\nu \in \text{Irr}(F) \setminus \{1_F\}$  such that  $\chi_a = \text{Ind}_F^G(\nu)$ . By (c), we only need to prove that  $\chi_a^\circ|_H = (\sum_{i=j}^m \varphi_j)|_H$ . Let  $\rho_H$  be the regular character of  $H$ . Then we have

$$\chi_a^\circ|_H = (\text{Ind}_F^G(\nu))^\circ|_H = \text{Res}_H^G(\text{Ind}_F^G(\nu)) = \nu(1) \cdot \rho_H = \rho_H = \sum_{\xi \in \text{Irr}(H)} \xi = \sum_{j=1}^m \varphi_j,$$

where the third equality follows from [Hup98, 18.7 Theorem (b)] and the last equality follows from (b) and the fact that  $H$  is an abelian  $p'$ -group.

□

**3.3. Frobenius groups of type  $(C_p \times C_p) \rtimes H$ .** Recall that  $p$  denotes an odd prime number. In this article, the aim is to focus on Frobenius groups  $G$  with cyclic Frobenius complement of order  $m$  and elementary abelian Frobenius kernel of order  $p^2$ . In particular, we will compute the trivial source character tables  $\text{Triv}_p(G)$  in the following two extremal cases: first the case, in which there is precisely one  $G$ -conjugacy class of cyclic subgroups of order  $p$  (we will call this the *maximal fusion case*); second, the case in which there are precisely  $p+1$   $G$ -conjugacy classes of cyclic subgroups of order  $p$  (we call this the *minimal fusion case*). In this subsection, we characterise such groups.

Given integers  $m, n > 1$ , we denote by  $\text{MetaFrob}(m)$  the set of isomorphism classes of metabelian Frobenius groups with Frobenius complement of order  $m$  and we set

$$\text{MetaFrob}(m, n) := \{G \in \text{MetaFrob}(m) \mid |G| = mn\}.$$

Note that Frobenius groups  $G$  with cyclic Frobenius complement of order  $m$  and elementary abelian Frobenius kernel of order  $p^2$  comprise all elements of  $\text{MetaFrob}(m, p^2)$  whose Frobenius kernels are not cyclic.

**Lemma 3.4.** *Let  $p$  be an odd prime number and let  $m > 1$  be an integer such that  $m \nmid (p-1)$ . Then the following assertions hold:*

- (a)  $|\text{MetaFrob}(m, p^2)| = 1$ ;
- (b) if  $m = p^2 - 1$  then the unique element of  $\text{MetaFrob}(p^2 - 1, p^2)$  is the affine linear group  $\text{AGL}_1(p^2)$  and can be identified with the subgroup

$$\mathcal{G} := \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^\times, b \in \mathbb{F}_{p^2} \right\}$$

of  $\text{GL}_2(\mathbb{F}_{p^2})$ , which is a Frobenius group with Frobenius kernel and Frobenius complement  $\mathcal{F} := \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_{p^2}) \mid b \in \mathbb{F}_{p^2} \right\}$  and  $\mathcal{H} := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_{p^2}) \mid a \in \mathbb{F}_{p^2}^\times \right\}$  respectively;

- (c) if  $p < m < p^2 - 1$  then the unique element of  $\text{MetaFrob}(m, p^2)$  can be identified with the subgroup  $\tilde{\mathcal{G}} = \mathcal{F}\tilde{\mathcal{H}} = \mathcal{F} \rtimes \tilde{\mathcal{H}}$  of  $\mathcal{G}$ , where  $|\tilde{\mathcal{H}}| = m$ .

*Proof.* (a) This follows from the formula given in [BH98, Theorem 11.7.] together with [BH98, Remark 11.13.(C)].

- (b) Assertion (b) is well-known and follows for example from [Jac12, Exercise 5.15.7].

- (c) The group  $\tilde{\mathcal{G}} = \mathcal{F} \rtimes \tilde{\mathcal{H}}$  is obviously a subgroup of  $\mathcal{G}$ . Moreover, as  $\tilde{\mathcal{H}} < \mathcal{H}$  and

$$\tilde{\mathcal{G}} \setminus \tilde{\mathcal{H}} = \{f \cdot \tilde{h} \in \tilde{\mathcal{G}} \mid f \in \mathcal{F} \setminus \{1\}, \tilde{h} \in \tilde{\mathcal{H}}\} \subset \mathcal{G} \setminus \mathcal{H} = \{f \cdot h \in \mathcal{G} \mid f \in \mathcal{F} \setminus \{1\}, h \in \mathcal{H}\},$$

it follows from the definition that  $\mathcal{F} \rtimes \tilde{\mathcal{H}}$  is again a Frobenius group.  $\square$

**Proposition 3.5.** Let  $G$  be a Frobenius group with Frobenius complement  $H \cong C_m$  and Frobenius kernel  $F \cong C_p \times C_p$  where  $p$  is an odd prime number.

- (a) The number of  $G$ -conjugacy classes of subgroups of  $G$  of order  $p$  equals 1 if and only if  $(p+1)(p-1)_2 \mid m$  if and only if  $m = (p+1) \cdot \gcd(p-1, m)$ . In this case, we have  $|N_H(C)| = \gcd(p-1, m)$  for any subgroup  $C \leq G$  of order  $p$ .
- (b) The number of  $G$ -conjugacy classes of subgroups of  $G$  of order  $p$  equals  $p+1$  if and only if for any  $h \in H \setminus \{1\}$ , there exists an integer  $1 < a(h) \leq p-1$  such that  $hfh^{-1} = f^{a(h)}$  for any  $f \in F$ . In this case,  $H$  is cyclic of order dividing  $p-1$ .

*Proof.* Write  $F = \langle x \rangle \times \langle y \rangle$  and  $H = \langle h \rangle$ . As  $F$  is elementary abelian of order  $p^2$ , there are precisely  $p+1$  subgroups of  $F$  of order  $p$ , namely  $R_i := \langle x \cdot y^i \rangle$  ( $0 \leq i \leq p-1$ ) and  $R_p := \langle y \rangle$ , and we let  $X := \{R_0, \dots, R_p\}$ .

- (a) The equivalent conditions and the fact that  $|N_H(C)| = \gcd(p-1, m)$  for any  $C \in X$  are proved in [BHHK25, Example 3.2]. It follows from the action of  $H$  on  $X$  and the induced equality  $(p+1) \cdot |N_H(C)| = |H| = m$ .
- (b) If the number of  $G$ -conjugacy classes of subgroups of  $G$  of order  $p$  equals  $p+1$ , the action of  $H$  by conjugation must map the elements of  $F \setminus \{1\}$  to powers of themselves, but different from themselves by Lemma 3.1. Let  $h \in H$ . If the generators  $x$  and  $y$  satisfy  $hxh^{-1} = x^{a_1}$  and  $hyh^{-1} = y^{a_2}$  with  $1 < a_2 < a_1 \leq p-1$ , then  $x^{a_2}y^{a_1}$  is conjugate to  $(xy)^{a_2a_1}$ , which contradicts the assumption on the fusion of  $p$ -subgroups. Hence, the necessary condition holds. The sufficient condition is clear since  $G$  is a semi-direct product of  $F$  by  $H$  and  $F$  is abelian. Next, for any  $C \in X$  we have that  $N_G(C) = G$  acts by conjugation on  $C$ . The induced group homomorphism

$$\Theta : N_G(C) \longrightarrow \text{Aut}(C) \cong C_{p-1}, g \mapsto c_g$$

(where  $c_g : C \longrightarrow C, c \mapsto gcg^{-1}$  is the automorphism of conjugation by  $g$ ) is such that  $\ker(\Theta) = F$  by Lemma 3.1. This yields

$$m = \frac{|G|}{|F|} \left| |\text{Aut}(C)| \right| = p-1.$$

$\square$

**Remark 3.6.** Note that we excluded the prime number 2. In this case, however, the situation is simple: there is, up to isomorphism, only one Frobenius group of type  $(C_2 \times C_2) \rtimes C_m$ , namely the alternating group  $\mathfrak{A}_4 = V_4 \rtimes \langle (1 \ 2 \ 3) \rangle$ , where  $V_4$  is the Klein-four group. Indeed, as we must

have  $m \mid (p^2 - 1) = 3$ , the only possibility is  $m = 3$ , and the only other non-abelian group of order 12 is the dihedral group of order 12, which does not have any normal subgroup isomorphic to  $C_2 \times C_2$ . The trivial source character table  $\text{Triv}_2(\mathfrak{A}_4)$  can be found for example in [Böh24], or in [BFL22] through the isomorphism  $\mathfrak{A}_4 \cong \text{PSL}_2(3)$ .

#### 4. THE MAXIMAL FUSION CASE

We now turn to the computation of the trivial source character tables of metabelian Frobenius groups with Frobenius kernel  $C_p \times C_p$  and cyclic Frobenius complement  $C_m$ , where  $p$  is odd and there is precisely one conjugacy class of subgroups of order  $p$ .

**Notation 4.1.** Throughout this section, we assume that  $G = F \rtimes H$  is a Frobenius group with Frobenius kernel  $F \cong C_p \times C_p$  and cyclic Frobenius complement  $H \cong C_m$ , where  $p$  is odd and  $m = (p+1) \cdot \gcd(p-1, m)$ . By Lemma 3.4, up to isomorphism, there is only one group of this type: it is a subgroup of  $\text{AGL}_1(p^2)$  and there is precisely one conjugacy class of subgroups of order  $p$  by Proposition 3.5(a).

We set  $d(m) := \gcd(p-1, m)$  and  $e(m) := \frac{p^2-1}{m} = \frac{p-1}{d(m)}$ , and notice that  $2 \mid d(m)$  by Proposition 3.5(a). We let  $H := \langle h \rangle$  and  $F := \langle x \rangle \times \langle y \rangle$  and we choose the following set of representatives for the  $G$ -conjugacy classes of  $p$ -subgroups of  $G$ :

$$\begin{aligned} Q_1 &:= \{1\}, \\ Q_2 &:= \langle x \rangle, \\ Q_3 &:= F. \end{aligned}$$

As in Proposition 3.3, we let  $\text{Irr}(G) = \{\chi_1, \dots, \chi_m, \chi_{m+1}, \dots, \chi_{m+e(m)}\}$  where for each  $1 \leq a \leq m$  we set  $\chi_a := \text{Inf}_{G/F}^G(\xi_a)$  with  $\xi_a \in \text{Irr}(H)$  as defined in Notation 2.1, and  $\chi_{m+b} := \text{Ind}_F^G(\nu_b)$  for each  $1 \leq b \leq e(m)$  and pairwise non-conjugate characters  $\nu_b \in \text{Irr}(F) \setminus \{1_F\}$ .

**Lemma 4.2.** *With the notation introduced in Notation 4.1, we have:*

- (a)  $N_G(Q_1) = G$  and  $\overline{N}_G(Q_1) \cong G$ ;
- (b)  $N_G(Q_2) = F \rtimes N_H(Q_2)$  and  $\overline{N}_G(Q_2) \cong \langle y \rangle \rtimes N_H(Q_2)$ , where  $|N_H(Q_2)| = d(m)$  and  $N_H(Q_2) = \langle h^{p+1} \rangle$ ;
- (c)  $N_G(Q_3) = G$  and  $\overline{N}_G(Q_3) \cong H$ .

*Proof.* Assertions (a) and (c) are straightforward from the definitions. Assertion (b) follows from the fact that  $G$  is a semi-direct product of  $F$  by  $H$  with  $F$  abelian and Proposition 3.5(a).  $\square$

**Proposition 4.3.** *Let  $\zeta$  be a fixed primitive  $m$ -th root of unity in  $K$  and let  $\omega := \zeta^{(p+1)}$ . Then the following assertions hold.*

- (a) *The ordinary character table of  $G$  restricted to the  $p$ -regular conjugacy classes is as given in Table 1.*
- (b) *The ordinary character table of  $\overline{N}_G(Q_2)$  restricted to the  $p$ -regular conjugacy classes is as given in Table 2, where following Proposition 3.3 we let  $\text{Irr}(\overline{N}_G(Q_2)) = \{\theta_1, \dots, \theta_{d(m)+e(m)}\}$*

	1	$h^j$ ( $1 \leq j \leq m-1$ )
$\chi_a$ ( $1 \leq a \leq m$ )	1	$\zeta^{(a-1)j}$
$\chi_{m+b}$ ( $1 \leq b \leq e(m)$ )	$m$	0

TABLE 1. Ordinary character table of  $G$  restricted to the  $p$ -regular classes.

where for each  $1 \leq b \leq e(m)$ ,  $\theta_{b+d(m)} := \text{Ind}_{\langle y \rangle}^{\overline{N}_G(Q_2)}(\nu_b)$  for pairwise non-conjugate characters  $\nu_b \in \text{Irr}(\langle y \rangle) \setminus \{1_{\langle y \rangle}\}$  and for each  $1 \leq a \leq d(m)$  we let  $\theta_a := \text{Ind}_{\overline{N}_G(Q_2)/\langle y \rangle}^{\overline{N}_G(Q_2)}(\xi_a)$  with  $\xi_a \in \text{Irr}(N_H(Q_2))$  as defined in Notation 2.1.

	1	$h^{j(p+1)}Q_2$ ( $1 \leq j \leq d(m)-1$ )
$\theta_a$ ( $1 \leq a \leq d(m)$ )	1	$\omega^{(a-1)j}$
$\theta_{d(m)+b}$ ( $1 \leq b \leq e(m)$ )	$d(m)$	0

TABLE 2. Ordinary character table of  $\overline{N}_G(Q_2)$  restricted to the  $p$ -regular conjugacy classes.

- (c) Setting  $\varphi_a := \chi_a^\circ$  for each  $1 \leq a \leq m$ , then  $\text{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$  and  $\text{Dec}_p(G)$  is as given in Table 3.
- (d) Setting  $\psi_a := \theta_a^\circ$  for each  $1 \leq a \leq d(m)$ , then  $\text{IBr}_p(\overline{N}_G(Q_2)) = \{\psi_1, \dots, \psi_{d(m)}\}$  and  $\text{Dec}_p(\overline{N}_G(Q_2))$  is as given in Table 4.

	$\varphi_1$	$\varphi_2$	.....	$\varphi_{m-1}$	$\varphi_m$
$\chi_1$	1	0	.....	0	0
$\chi_2$	0	1	.....	.....	.....
$\chi_3$	0	0	.....	.....	.....
$\vdots$	.....	.....	.....	.....	.....
$\vdots$	.....	.....	.....	0	0
$\vdots$	.....	.....	.....	0	0
$\chi_m$	0	0	.....	0	1
$\chi_{m+1}$	1	1	.....	1	1
$\vdots$	.....	.....	.....	.....	.....
$\chi_{m+e(m)}$	1	1	.....	1	1

TABLE 3.  $p$ -decomposition matrix of  $G$ 

	$\psi_1$	$\psi_2$	.....	$\psi_{d(m)-1}$	$\psi_{d(m)}$
$\theta_1$	1	0	.....	0	0
$\theta_2$	0	1	.....	.....	.....
$\theta_3$	0	0	.....	.....	.....
$\vdots$	.....	.....	.....	.....	.....
$\vdots$	.....	.....	.....	0	0
$\vdots$	.....	.....	.....	1	0
$\theta_{d(m)}$	0	0	.....	0	1
$\theta_{d(m)+1}$	1	1	.....	1	1
$\vdots$	.....	.....	.....	.....	.....
$\theta_{d(m)+e(m)}$	1	1	.....	1	1

TABLE 4.  $p$ -decomposition matrix of  $\overline{N}_G(Q_2)$ 

*Proof.* (a) For each  $1 \leq a \leq m$ , we have  $\chi_a = \text{Ind}_{G/F}^G(\xi_a)$  with  $\xi_a \in \text{Irr}(H)$ . It follows from Notation 2.1 that  $\chi_a(h^j) = \xi_a(h^j) = \zeta^{(a-1)j}$  for each  $1 \leq j \leq m-1$ . Then, as  $\chi_{m+1}, \dots, \chi_{m+d(m)}$  are induced from linear characters of  $F$  to  $G$  it is immediate that their degree is  $m$  and they take value zero on the non-trivial elements of  $H$ .

- (b) Analogous to (a) as  $\overline{N}_G(Q_2)$  is also a Frobenius group.
- (c) and (d) are immediate from Proposition 3.3(d).

□

We now come to our first main result, namely the description of the trivial source character table of  $G$  at  $p$ . In this case, it follows from the definition, that this is a  $3 \times 3$ -block matrix.

**Theorem 4.4.** *Assume  $p$  is an odd prime number and  $G = F \rtimes H$  is a Frobenius group with  $F \cong C_p \times C_p$  and  $H \cong C_m$  where  $m$  is an integer such that  $m = (p+1) \cdot \gcd(m, p-1)$ . Then, with the notation introduced in Notation 4.1 and Proposition 4.3, the trivial source character table  $\text{Triv}_p(G) = [T_{i,v}]_{1 \leq i,v \leq 3}$  of  $G$  is given as described below.*

(a) *The labelling of the columns may be chosen as follows:*

- (1) *the columns of  $T_{i,1}$  ( $1 \leq i \leq 3$ ) may be labelled by the elements of  $H$ ;*
- (2) *the columns of  $T_{i,2}$  ( $1 \leq i \leq 3$ ) may be labelled by the elements of  $N_H(Q_2)$ ;*
- (3) *the columns of  $T_{i,3}$  ( $1 \leq i \leq 3$ ) may be labelled by the elements of  $H$ .*

(b) *The ordinary characters of the trivial source modules are as follows:*

- (1)  $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_1)\} = \{\chi + \sum_{\psi \in \text{Irr}(G) \setminus \text{Lin}(G)} \psi \mid \chi \in \text{Lin}(G)\};$
- (2)  $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_2)\} = \{\sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} \chi + \sum_{\substack{\lambda \in \text{Lin}(G), \\ \text{Res}_{N_H(Q_2)}^G(\lambda) = \theta}} \lambda \mid \theta \in \text{Irr}(N_H(Q_2))\};$
- (3)  $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_3)\} = \text{Lin}(G).$

As  $\text{Lin}(G) = \text{Inf}_H^G(\text{Irr}(H))$  we may choose the labelling of the rows of  $T_{1,v}$  and  $T_{3,v}$  ( $1 \leq v \leq 3$ ) to match that of  $X(H)$ . Similarly, we may choose the labelling of the rows of  $T_{2,v}$  ( $1 \leq v \leq 3$ ) to match that of  $X(N_H(Q_2))$ .

(c) *With the labelling of the rows and of the columns given in (a) and (b) we have:*

- (1)  $T_{1,2} = T_{1,3} = T_{2,3} = \mathbf{0};$
- (2)  $T_{1,1} = X(H) + \begin{pmatrix} p^2-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^2-1 & 0 & \cdots & 0 \end{pmatrix};$
- (3)  $T_{2,2} = X(N_H(Q_2)) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix};$
- (4)  $T_{2,1} = (A_{M,s})_{M \in \text{TS}(G; Q_2), s \in H}$  where  $A_{M,s} = 0$  provided  $s \in H \setminus N_H(Q_2)$  and  

$$(A_{M,s})_{M \in \text{TS}(G; Q_2), s \in N_H(Q_2)} = (p+1) \cdot T_{2,2};$$
- (5)  $T_{3,2} = (B_{M,s})_{M \in \text{TS}(G; Q_3), s \in N_H(Q_2)}$  where  $B_{M,s} = (T_{1,3})_{M,s}$  for each  $M \in \text{TS}(G; Q_3)$  and each  $s \in N_H(Q_2);$
- (6)  $T_{3,1} = T_{3,3} = X(H).$

To help visualize this result, the table version of Theorem 4.4 for  $G = \text{AGL}_1(p^2)$  is given in Appendix A.

*Proof.* Assertion (a) is straightforward from Convention 2.2, Proposition 3.3(c) and Lemma 4.2. Next, we prove Assertion (b). To simplify, in this proof we set  $N_2 := N_G(Q_2)$  and  $\overline{N}_2 := \overline{N}_2(Q_2)$ .

- (1) The ordinary characters of the trivial source modules in  $\text{TS}(G; Q_1)$  are the ordinary characters of the PIMs of  $kG$  and can be read off from the decomposition matrix in Table 3. The claim follows.

- (2) Let  $M \in \text{TS}(G; Q_2)$ . By the bijections in Subsection 2.3 there exists a unique PIM  $P_{\psi_a}$  of  $k\bar{N}_2$  with  $1 \leq a \leq d(m)$  such that  $M$  is the Green correspondent of the inflated module  $\text{Inf}_{N_2/\langle x \rangle}^{N_2}(P_{\psi_a})$ . By Proposition 2.5, the induced module

$$\text{Ind}_{N_2}^G \text{Inf}_{N_2/\langle x \rangle}^{N_2}(P_{\psi_a})$$

is indecomposable, so it must be  $M$ . Next, we compute the constituents of the ordinary character of this module. First, by the above,

$$\chi_{\widehat{M}} = \text{Ind}_{N_2}^G \text{Inf}_{N_2/\langle x \rangle}^{N_2}(\Phi_{\psi_a}),$$

and by Proposition 4.3(d) the ordinary character of  $P_{\psi_a}$  is

$$\Phi_{\psi_a} = \theta_a + \sum_{b=1}^{e(m)} \theta_{d(m)+b}.$$

Now, we claim that there is a bijection

$$\text{Ind}_{N_2}^G \text{Inf}_{\bar{N}_2}^{N_2} : \text{Irr}(\bar{N}_2) \setminus \text{Lin}(\bar{N}_2) \longrightarrow \text{Irr}(G) \setminus \text{Lin}(G).$$

Given  $u \in \{1, \dots, e(m)\}$ , as  $N_2$  is a Frobenius group, we can write  $\text{Inf}_{\bar{N}_2}^{N_2}(\theta_{d(m)+u}) = \text{Ind}_F^{\bar{N}_2}(\nu_u) =: I_u$  for some character  $\nu_u \in \text{Irr}(F) \setminus \{1_F\}$  by Theorem 3.2. As  $G$  is also a Frobenius group, by transitivity of induction, we obtain that  $\text{Ind}_{N_2}^G \text{Inf}_{\bar{N}_2}^{N_2}(\theta_{d(m)+u})$  is irreducible. Hence, the map is well-defined. As both sets have the same cardinality, it remains to show that the map is injective. Assuming  $\theta_{d(m)+u} \neq \theta_{d(m)+w}$  for  $u, w \in \{1, \dots, e(m)\}$ , then  $I_u \neq I_w$ . By the above,  $\text{Ind}_{N_2}^G(I_u) =: \chi_u \in \text{Irr}(G) \setminus \text{Lin}(G)$ . Then, by Clifford theory, the inertial subgroup of  $I_u$  is  $N_2$  and  $\text{Res}_{N_2}^G(\chi_u) = \sum_{g \in [G/N_2]} I_u^{(g)}$ . Moreover, we can choose

the non-trivial coset representatives  $g \in [G/N_2]$  in such a way that  $g =: \ell \in H$ . Now, we show that  $I_w$  is not a constituent of  $\text{Res}_{N_2}^G(\chi_u)$ . As  $\ell \notin N_2$  there exists  $q \in Q_2 \setminus \{1\}$  such that  $\ell q \ell^{-1} \in F \setminus Q_2$ . Since both  $I_u$  and  $I_w$  are induced from  $F$ , they do not have  $F$  in their kernels. Hence,  $I_u^{(\ell)} = I_w$  is not possible for such an  $\ell$ , as  $\ker(I_u) \cap F = Q_2 = \ker(I_w) \cap F$ . Hence,  $\text{Ind}_{N_2}^G(I_u) \neq \text{Ind}_{N_2}^G(I_w)$  and the map is injective.

It follows that

$$\text{Ind}_{N_G(Q_2)}^G \text{Inf}_{\bar{N}_2}^{N_2} \left( \sum_{b=1}^{e(m)} \theta_{d(m)+b} \right) = \sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} \chi.$$

Now we compute  $\text{Ind}_{N_2}^G \text{Inf}_{\bar{N}_2}^{N_2}(\theta_a)$ . Clearly, the degree of this character is

$$|G : N_2| = |H : N_H(Q_2)| = \frac{m}{d(m)} = p + 1 < m,$$

so by Proposition 4.3(a), it must be a sum of  $p + 1$  linear characters of  $G$ . Now, Clifford's theorem together with Gallagher's theorem (see [Hup98, 19.3 Theorem and 19.5 Theorem]) tell us that we can write

$$\text{Ind}_{N_2}^G \text{Inf}_{\bar{N}_2}^{N_2}(\theta_a) = \sum_{c=1}^{p+1} \lambda_c$$

with  $\lambda_1, \dots, \lambda_{p+1} \in \text{Lin}(G)$  pairwise distinct and

$$\text{Res}_{N_2}^G \left( \text{Ind}_{N_2}^G \text{Inf}_{\bar{N}_2}^{N_2}(\theta_a) \right) = (p + 1) \cdot \text{Inf}_{\bar{N}_2}^{N_2}(\theta_a),$$

implying that  $\text{Res}_{N_2}^G(\lambda_c) = \text{Inf}_{\overline{N}_2}^{N_2}(\theta_a)$  for every  $1 \leq c \leq p+1$ . Moreover, identifying  $N_2/F$  with  $N_H(Q_2)$ , then [Bou10, §1.1.3(2.e.)] yields  $\text{Res}_{N_H(Q_2)}^{N_2}\left(\text{Inf}_{\overline{N}_2}^{N_2}(\theta_a)\right) = \theta_a$ . A counting argument shows that any linear character of  $G$  must be a constituent of an induced character  $\text{Ind}_{N_2}^G \text{Inf}_{\overline{N}_2}^{N_2}(\theta_{a_0})$  for some index  $1 \leq a_0 \leq d(m)$ . Thus, we have proved that

$$\text{Ind}_{N_2}^G \text{Inf}_{\overline{N}_2}^{N_2}(\theta_a) = \sum_{c=1}^{p+1} \lambda_c = \sum_{\substack{\lambda \in \text{Lin}(G), \\ \text{Res}_{N_H(Q_2)}^G(\lambda) = \theta_a}} \lambda,$$

as required.

(3) Lemma 4.2(c) and Lemma 2.4(b) yield

$$\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_3)\} = \text{Inf}_{G/Q_3}^G(\text{Irr}(G/Q_3)) \subseteq \text{Lin}(G)$$

and the last inclusion is an equality by Lemma 3.1(d) as  $|\text{Lin}(G)| = |G/[G, G]|$ .

Finally, we prove Assertion (c).

- (1) The claim is immediate from Lemma 2.3(a).
- (2) By Lemma 2.3(b), we have

$$T_{1,1} = \Phi_p(G) = \text{Dec}_p(G)^t \cdot X(G, p').$$

Using Proposition 4.3(a) and (c) and noticing that with our choice of the labelling for the rows and the columns,  $X(G, p')$  is a block matrix of type

$$\begin{pmatrix} X(H) \\ \hline m & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ m & 0 & \cdots & 0 \end{pmatrix} \in K^{(m+e(m)) \times m},$$

the product is easily calculated to be as claimed, since  $m \cdot e(m) = p^2 - 1$ .

(3) Similarly, by Lemma 2.3(b), we have

$$T_{2,2} = \Phi_p(\overline{N}_2) = \text{Dec}_p(\overline{N}_2)^t \cdot X(\overline{N}_2, p').$$

This product is easily computed from Proposition 4.3(b) and (d), noticing that with our choice of the labelling for the rows and the columns,  $X(\overline{N}_2, p')$  is a block matrix of type

$$\begin{pmatrix} X(N_H(Q_2)) \\ \hline d(m) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ d(m) & 0 & \cdots & 0 \end{pmatrix} \in K^{(d(m)+e(m)) \times d(m)}$$

and  $d(m) \cdot e(m) = p - 1$ .

- (4) By Lemma 2.3(c), the entries of  $T_{2,1}$  are obtained by evaluating the ordinary characters of the trivial source modules with vertex  $Q_2$  obtained in (b)(2) at the elements of  $H$ . As the latter characters are induced from  $N_2 = F \rtimes N_H(Q_2)$ , it is clear that they take value zero on  $H \setminus N_H(Q_2)$ . In addition, it is also clear from Part (b)(2) (and its proof) that the values of these characters at the elements of  $N_H(Q_2)$  are the values of the characters of the corresponding PIMs of  $\overline{N}_2$  multiplied by the index  $|G : N_2| = p + 1$ .

- (5) By definition, we have  $T_{3,2} = [\tau_{Q_2,s}^G([M])]_{M \in \text{TS}(G; Q_3), s \in [\bar{N}_G(Q_2)]_{p'}}$ . So, let  $M \in \text{TS}(G; Q_3)$ , which is the set of all 1-dimensional  $kG$ -modules. By Lemma 4.2 and Proposition 3.3(c), we may assume that  $[N_2]_{p'} = N_H(Q_2)$  and let  $t \in [N_2]_{p'}$ . By definition,

$$\tau_{Q_2,t}^G([M]) = \varphi_{M[Q_2]}(t).$$

Because  $Q_2 \leq Q_3 = F \trianglelefteq G$ , it follows from [Lin18b, Proposition 5.10.4] that  $M[Q_2] = \text{Res}_{N_2}^G(M)$ . Hence,  $\varphi_{M[Q_2]}(t) = \chi_M(t)$ , proving that the values of  $T_{3,2}$  are obtained from  $T_{3,1}$  by restriction to the columns labelled by the elements of  $N_H(Q_2)$ .

- (6) On the one hand, since  $\bar{N}_G(Q_3) \cong H$ , which is a  $p'$ -group, by Lemma 2.3(b), we have

$$T_{3,3} = \Phi_p(H) = \text{Dec}_p(H)^t \cdot X(H, p') = X(H),$$

as claimed. On the other hand, by Lemma 2.3(c) we have

$$T_{3,1} = (\chi_M(s))_{M \in \text{TS}(G; Q_3), s \in [G]_{p'}}.$$

Therefore, it follows from Part (a)(3) and Part (b)(3) that  $T_{3,1} = X(H)$ . □

**Example 4.5.** Let  $G$  be the Frobenius group  $(C_3 \times C_3) \rtimes C_8$  of order 72 with maximal fusion pattern, i.e.  $G$  has precisely one conjugacy class of subgroups of order 3. It follows that we have three conjugacy classes of 3-subgroups of  $G$ , namely

$$Q_1 = \{1\}, \quad Q_2 \cong C_3, \quad Q_3 \cong C_3 \times C_3.$$

Notice that  $G$  is isomorphic to the group labelled by [ 72, 39 ] in GAP's SmallGroups library, see [GAP]. The ordinary character table of  $G$  is as given in Table 5, where  $\zeta_8$  denotes a primitive 8-th root of unity.

	1a	8a	4a	8b	2a	8c	4b	8d	3a
$\chi_1$	1	1	1	1	1	1	1	1	1
$\chi_2$	1	$\zeta_8$	$\zeta_8^2$	$\zeta_8^3$	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	1
$\chi_3$	1	$\zeta_8^2$	-1	$-\zeta_8^2$	1	$\zeta_8^2$	-1	$-\zeta_8^2$	1
$\chi_4$	1	$\zeta_8^3$	$-\zeta_8^2$	$\zeta_8$	-1	$-\zeta_8^3$	$\zeta_8^2$	$-\zeta_8$	1
$\chi_5$	1	-1	1	-1	1	-1	1	-1	1
$\chi_6$	1	$-\zeta_8$	$\zeta_8^2$	$-\zeta_8^3$	-1	$\zeta_8$	$-\zeta_8^2$	$\zeta_8^3$	1
$\chi_7$	1	$-\zeta_8^2$	-1	$\zeta_8^2$	1	$-\zeta_8^2$	-1	$\zeta_8^2$	1
$\chi_8$	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	$\zeta_8^3$	$\zeta_8^2$	$\zeta_8$	1
$\chi_9$	8	0	0	0	0	0	0	0	-1

TABLE 5. Ordinary character table of  $(C_3 \times C_3) \rtimes C_8$

The trivial source character table  $\text{Triv}_3(G)$  is as given in Table 6. Following our conventions we label the columns of  $\text{Triv}_3(G)$  with  $3'$ -elements in  $N_v$  instead of  $\bar{N}_v$  ( $1 \leq v \leq 3$ ).

Normalisers $N_i$ Representatives $n_j \in N_i$	$N_1 \cong (C_3 \times C_3) \rtimes C_8$								$N_2 \cong (C_3 \times C_3) \rtimes C_2$		$N_3 \cong (C_3 \times C_3) \rtimes C_8$								
	1a	8a	4a	8b	2a	8c	4b	8d	1a	2a	1a	8a	4a	8b	2a	8c	4b	8d	
$\chi_1 + \chi_9$	9	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
$\chi_2 + \chi_9$	9	$\zeta_8$	$\zeta_8^2$	$\zeta_8^3$	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	0	0	0	0	0	0	0	0	0	0	0
$\chi_3 + \chi_9$	9	$\zeta_8^2$	-1	$-\zeta_8^2$	1	$\zeta_8^2$	-1	$-\zeta_8^2$	0	0	0	0	0	0	0	0	0	0	0
$\chi_4 + \chi_9$	9	$\zeta_8^3$	$-\zeta_8^2$	$\zeta_8$	-1	$-\zeta_8^3$	$\zeta_8^2$	$-\zeta_8$	0	0	0	0	0	0	0	0	0	0	0
$\chi_5 + \chi_9$	9	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0
$\chi_6 + \chi_9$	9	$-\zeta_8$	$\zeta_8^2$	$-\zeta_8^3$	-1	$\zeta_8$	$-\zeta_8^2$	$\zeta_8^3$	0	0	0	0	0	0	0	0	0	0	0
$\chi_7 + \chi_9$	9	$-\zeta_8^2$	-1	$\zeta_8^2$	1	$-\zeta_8^2$	-1	$\zeta_8^2$	0	0	0	0	0	0	0	0	0	0	0
$\chi_8 + \chi_9$	9	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	$\zeta_8^3$	$\zeta_8^2$	$\zeta_8$	0	0	0	0	0	0	0	0	0	0	0
$\chi_1 + \chi_3 + \chi_5 + \chi_7 + \chi_9$	12	0	0	0	4	0	0	0	3	1	0	0	0	0	0	0	0	0	0
$\chi_2 + \chi_4 + \chi_6 + \chi_8 + \chi_9$	12	0	0	0	-4	0	0	0	3	-1	0	0	0	0	0	0	0	0	0
$\chi_1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_2$	1	$\zeta_8$	$\zeta_8^2$	$\zeta_8^3$	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	1	-1	1	$\zeta_8$	$\zeta_8^2$	$\zeta_8^3$	-1	$-\zeta_8$	$-\zeta_8^2$	$-\zeta_8^3$	
$\chi_3$	1	$\zeta_8^2$	-1	$-\zeta_8^2$	1	$\zeta_8^2$	-1	$-\zeta_8^2$	1	1	1	$\zeta_8^2$	-1	$-\zeta_8^2$	1	$\zeta_8^2$	-1	$-\zeta_8^2$	
$\chi_4$	1	$\zeta_8^3$	$-\zeta_8^2$	$\zeta_8$	-1	$-\zeta_8^3$	$\zeta_8^2$	$-\zeta_8$	1	-1	1	$\zeta_8^3$	$-\zeta_8^2$	$\zeta_8$	-1	$-\zeta_8^3$	$\zeta_8^2$	$-\zeta_8$	
$\chi_5$	1	-1	1	-1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1	
$\chi_6$	1	$-\zeta_8$	$\zeta_8^2$	$-\zeta_8^3$	-1	$\zeta_8$	$-\zeta_8^2$	$\zeta_8^3$	1	-1	1	$-\zeta_8$	$\zeta_8^2$	$-\zeta_8^3$	-1	$\zeta_8$	$-\zeta_8^2$	$\zeta_8^3$	
$\chi_7$	1	$-\zeta_8^2$	-1	$\zeta_8^2$	1	$-\zeta_8^2$	-1	$\zeta_8^2$	1	1	1	$-\zeta_8^2$	-1	$\zeta_8^2$	1	$-\zeta_8^2$	-1	$\zeta_8^2$	
$\chi_8$	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	$\zeta_8^3$	$\zeta_8^2$	$\zeta_8$	1	-1	1	$-\zeta_8^3$	$-\zeta_8^2$	$-\zeta_8$	-1	$\zeta_8^3$	$\zeta_8^2$	$\zeta_8$	

TABLE 6. Trivial source character table of  $(C_3 \times C_3) \rtimes C_8$  at  $p = 3$ 

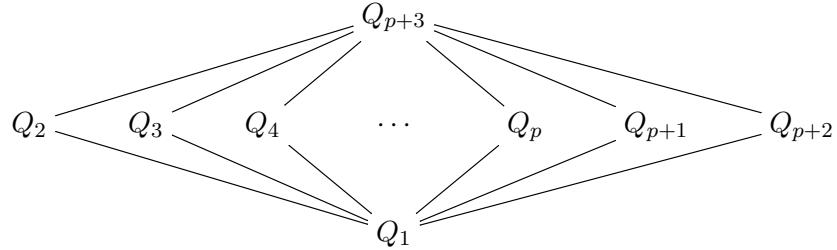
## 5. THE MINIMAL FUSION CASE

We now turn to the computation of the trivial source character tables of metabelian Frobenius groups with Frobenius kernel  $F \cong C_p \times C_p$  and cyclic Frobenius complement  $H \cong C_m$  such that no two distinct  $p$ -subgroups of  $G$  of order  $p$  are  $G$ -conjugate. Recall from Proposition 3.5(b) that  $|H|$  divides  $p - 1$  and that  $H$  acts on  $F$  by raising each element to a power of itself in this case. Moreover, we have  $m > 1$ .

**Notation 5.1.** Throughout this section, we adopt the following notation. We choose a generator  $h$  of  $C_{p-1}$  and let  $H := \langle h^{a(m)} \rangle$ , where  $a(m) := \frac{p-1}{m}$ . We let  $\{x, y\}$  be a set of generators for  $F$ . By Proposition 3.5, we can choose the following set of representatives for the  $G$ -conjugacy classes of  $p$ -subgroups of  $G$ :

$$\begin{aligned} Q_1 &:= \{1\}, \\ Q_i &:= \langle x \cdot y^{i-2} \rangle \quad (2 \leq i \leq p+1), \\ Q_{p+2} &:= \langle y \rangle, \\ Q_{p+3} &:= F. \end{aligned}$$

Then, up to  $G$ -conjugation, the lattice of subgroups of  $G$  of order  $p$  is as given below.



As in Proposition 3.3, we let  $\text{Irr}(G) = \{\chi_1, \dots, \chi_{m+(p+1)\cdot a(m)}\}$  where for each  $1 \leq a \leq m$  we set  $\chi_a := \text{Inf}_{G/F}^G(\xi_a)$  with  $\xi_a \in \text{Irr}(H) = \text{Lin}(H)$  as defined in Notation 2.1, and

$$\{\chi_{m+1}, \dots, \chi_{m+(p+1)\cdot a(m)}\} = \{\text{Ind}_F^G(\nu) \mid \nu \in T\}$$

where  $T$  is a set of representatives for the conjugation action of  $G$  on  $\text{Irr}(F) \setminus \{1_F\}$ .

**Lemma 5.2.** *With the notation introduced in Notation 5.1 the following assertions hold:*

- (a)  $N_G(Q_v) = G$  and  $\overline{N}_G(Q_v) = G/Q_v$  for every  $1 \leq v \leq p+3$ ; and
- (b)  $\overline{N}_G(Q_v)$  is a Frobenius group with Frobenius complement  $H \cong HQ_v/Q_v$  and Frobenius kernel  $F/Q_v$  for every  $1 \leq v \leq p+2$ .

*Proof.* (a) As no two distinct  $p$ -subgroups of  $G$  of order  $p$  are  $G$ -conjugate, it follows that  $N_G(Q_v) = G$  for each  $1 \leq v \leq p+3$ . The second claim is then immediate.

- (b) For each  $1 \leq v \leq p+2$ , clearly  $Q_v \trianglelefteq F$  and  $Q_v \trianglelefteq G$  by (a). As  $G$  is a Frobenius group with respect to  $H$ , the assertion follows from the definition.  $\square$

**Notation 5.3.** Following Proposition 3.3, given  $v \in \{2, \dots, p+2\}$ , we let  $\text{Irr}(\overline{N}_G(Q_v)) = \{\theta_1, \dots, \theta_{m+\frac{p-1}{m}}\}$  where for each  $1 \leq a \leq m$  we set  $\theta_a := \text{Ind}_{(G/Q_v)/(F/Q_v)}^{G/Q_v}(\xi_a)$  with  $\xi_a \in \text{Irr}(H)$  as defined in Notation 2.1, and

$$\{\theta_{m+1}, \dots, \theta_{m+\frac{p-1}{m}}\} = \{\text{Ind}_{F/Q_v}^{G/Q_v}(\nu) \mid \nu \in V\}$$

where  $V$  is a set of representatives for the conjugation action of  $G/Q_v$  on  $\text{Irr}(F/Q_v) \setminus \{1_{F/Q_v}\}$ .

**Lemma 5.4.** (a) Setting  $\varphi_a := \chi_a^\circ$  for each  $1 \leq a \leq m$ , we have  $\text{IBr}_p(G) = \{\varphi_1, \dots, \varphi_m\}$  and  $\text{Dec}_p(G)$  is as given in Table 7.

- (b) Setting  $\psi_a := \theta_a^\circ$  for each  $1 \leq a \leq m$ , we have  $\text{IBr}_p(\overline{N}_G(Q_v)) = \{\psi_1, \dots, \psi_m\}$  and  $\text{Dec}_p(\overline{N}_G(Q_v))$  is as given in Table 8 for every  $2 \leq v \leq p+2$ .

	$\varphi_1$	$\varphi_2$	$\dots$	$\varphi_{m-1}$	$\varphi_m$
$\chi_1$	1	0	$\dots$	0	0
$\chi_2$	0	1	$\dots$	$\vdots$	$\vdots$
$\chi_3$	0	0	$\dots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	1	0
$\chi_m$	0	0	$\dots$	0	1
$\chi_{m+1}$	1	1	$\dots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\chi_{m+(p+1)\cdot a(m)}$	1	1	$\dots$	1	1

TABLE 7.  $p$ -decomposition matrix of  $G$

	$\psi_1$	$\psi_2$	$\dots$	$\psi_{m-1}$	$\psi_m$
$\theta_1$	1	0	$\dots$	0	0
$\theta_2$	0	1	$\dots$	$\vdots$	$\vdots$
$\theta_3$	0	0	$\dots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	1	0
$\theta_m$	0	0	$\dots$	0	1
$\theta_{m+1}$	1	1	$\dots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\theta_{m+\frac{p-1}{m}}$	1	1	$\dots$	1	1

TABLE 8.  $p$ -decomposition matrix of  $\overline{N}_G(Q_v)$   
( $2 \leq v \leq p+2$ )

*Proof.* Both (a) and (b) are immediate from Proposition 3.3(d).  $\square$

**Theorem 5.5.** Let  $G$  be a metableian Frobenius group with cyclic complement  $H$  of order  $m$  dividing  $p-1$  and Frobenius kernel  $F \cong C_p \times C_p$  such that the number of  $G$ -conjugacy classes of subgroups of  $G$  of order  $p$  is precisely  $p+1$ . Then, the trivial source character table  $\text{Triv}_p(G) = [T_{i,v}]_{1 \leq i,v \leq p+3}$  seen as a block matrix is as given in Table 9. More precisely, the labelling of the rows and columns and the entries are as described below.

- (a) For every  $1 \leq i, v \leq p+3$  the columns of  $T_{i,v}$  may be labelled by the elements of  $H$ .

(b) *The ordinary characters of the trivial source modules are as follows:*

- (1)  $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_1)\} = \{\chi + \sum_{j=m+1}^{m+(p+1)\cdot a(m)} \chi_j \mid \chi \in \text{Lin}(G)\};$
- (2)  $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_i)\} = \{\lambda + \sum_{\substack{\chi \in \text{Irr}(G) \setminus \text{Lin}(G) \\ \ker(\chi) = Q_i}} \chi \mid \lambda \in \text{Lin}(G)\}$  for every  $2 \leq i \leq p+2$ ;
- (3)  $\{\chi_{\widehat{M}} \mid M \in \text{TS}(G; Q_{p+3})\} = \text{Lin}(G).$

As  $\text{Lin}(G) = \text{Inf}_H^G(\text{Irr}(H))$  we choose the labelling of the rows of  $T_{i,v}$  ( $1 \leq i, v \leq p+3$ ) to match that of  $X(H)$ .

(c) *With the labelling of the rows and of the columns given in (a) and (b), we have:*

- (1)  $T_{i,v} = \mathbf{0}$  for every  $2 \leq v < i \leq p+2$  and for every  $1 \leq i < v \leq p+3$ ;
- (2)  $T_{1,1} = X(H) + \begin{pmatrix} p^2-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^2-1 & 0 & \cdots & 0 \end{pmatrix};$
- (3)  $T_{2,1} = T_{i,1} = T_{i,i} = X(H) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix}$  for every  $2 \leq i \leq p+2$ ;
- (4)  $T_{p+3,1} = T_{p+3,v} = X(H)$  for every  $2 \leq v \leq p+3$ .

$T_{1,1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$T_{2,1}$	$T_{2,2} = T_{2,1}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$T_{3,1} = T_{2,1}$	$\mathbf{0}$	$T_{3,3} = T_{2,1}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$T_{p+2,1} = T_{2,1}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$T_{p+2,p+2} = T_{2,1}$	$\mathbf{0}$
$T_{p+3,1}$	$T_{p+3,2} = T_{p+3,1}$	$\cdots$	$\cdots$	$T_{p+3,p+2} = T_{p+3,1}$	$T_{p+3,p+3} = T_{p+3,1}$

TABLE 9. Trivial source character table  $\text{Triv}_p(G)$ , seen as a block matrix

*Proof.* (a) Let  $1 \leq v \leq p+3$ . Since  $N_G(Q_v) = G$  by Lemma 5.2(a), we may assume that  $H$  is a set of representatives of the  $p$ -regular conjugacy classes of  $N_G(Q_v)$  by Proposition 3.3(c). Thus, the claim follows from Convention 2.2.

(b) As in the proof of the previous case, because  $G/Q_{p+3} \cong H$ ,  $|\text{Lin}(G)| = |G/[G, G]|$  and  $[G, G] = Q_{p+3}$  by Lemma 3.1(d), by abuse of notation we may write  $\text{Lin}(G) = \text{Inf}_H^G(\text{Irr}(H))$ , giving the last claim. Next, notice that by Subsection 2.3, Lemma 5.2 and Proposition 3.3, we have  $|\text{TS}(G; Q_v)| = |H| = |\text{Lin}(G)|$  for every  $1 \leq v \leq p+3$ .

- (1) The ordinary characters of the PIMs of  $kG$  can be read off from the decomposition matrix in Table 7, that is, for each  $1 \leq a \leq m$  we have

$$\Phi_{\varphi_a} = \chi_a + \sum_{j=m+1}^{m+(p+1)\cdot a(m)} \chi_j$$

where  $\chi_a$  runs through  $\text{Lin}(G)$ .

- (2) Fix  $2 \leq i \leq p+2$ . We know from Lemma 5.2 that  $N_G(Q_i) = G$  and  $\overline{N}_G(Q_i) = G/Q_i$  is a Frobenius group with Frobenius complement  $H = Q_iH/Q_i$  and Frobenius kernel  $F/Q_i$ . First, it follows from the bijections and the notation introduced in Lemma 5.4 and in Subsection 2.3 that

$$\text{TS}(G; Q_i) = \{\text{Inf}_{G/Q_i}^G(P_\psi) \mid \psi \in \text{IBr}_p(G/Q_i)\}.$$

Then, it follows from Lemma 5.4(b) that for every  $1 \leq u \leq m$  the ordinary characters of the PIMs  $P_{\psi_u}$  of  $k[G/Q_i]$  are given by

$$\Phi_{\psi_u} = \theta_u + \sum_{b=1}^{a(m)} \theta_{m+b},$$

where  $\theta_u \in \text{Lin}(G/Q_i)$  and  $\theta_{m+b} \in \text{Irr}(G/Q_i) \setminus \text{Lin}(G/Q_i)$  with  $\ker(\theta_{m+b}) = Q_i$  for each  $1 \leq b \leq a(m)$ . Next we observe that

$$\text{Inf}_{G/Q_i}^G(\text{Irr}(G/Q_i) \setminus \text{Lin}(G/Q_i)) = \{\chi \in \text{Irr}(G) \setminus \text{Lin}(G) \mid \ker(\chi) = Q_i\}.$$

Indeed, as the kernels of the non-trivial characters of  $F$  are cyclic of order  $p$  and normal in  $G$ , it follows that the kernel of any the non-linear irreducible character of  $G$  is equal to the kernel of the character(s) it is induced from, hence also cyclic of order  $p$ . Therefore, we obtain that

$$\begin{aligned} \text{Inf}_{G/Q_i}^G(\Phi_{\psi_u}) &= \text{Inf}_{G/Q_i}^G(\theta_u + \sum_{b=1}^{a(m)} \theta_{m+b}) = \text{Inf}_{G/Q_i}^G(\theta_u) + \sum_{b=1}^{a(m)} \text{Inf}_{G/Q_i}^G(\theta_{m+b}) \\ &= \text{Inf}_{G/Q_i}^G(\theta_u) + \sum_{\substack{\chi \in \text{Irr}(G) \setminus \text{Lin}(G) \\ \ker(\chi) = Q_i}} \chi \end{aligned}$$

for each  $1 \leq u \leq m$ . Clearly, the characters  $\text{Inf}_{G/Q_i}^G(\theta_u)$  run through  $\text{Lin}(G)$  when  $u$  runs from 1 to  $m$ , and the claim follows.

- (3) The claim follows from Lemma 2.4(b), where equality holds by the argument above.

(c) We now compute the entries of  $\text{Triv}_p(G)$ .

- (1) The assertion is immediate from Lemma 2.3(a).  
(2) By Lemma 2.3(b), we have  $T_{1,1} = \Phi_p(G)$ . It is clear from Proposition 3.3 and Part (b)(1) that the degree of the characters  $\Phi_{\varphi_a}$  ( $1 \leq a \leq m$ ) is  $1 + p^2 - 1$ . Therefore, it is now only left to prove that  $\chi_j(y) = 0$  for all  $m+1 \leq j \leq m + (p+1) \cdot a(m)$  and for all  $y \in H \setminus \{1\}$ . But this is clear since for any  $\nu \in \text{Irr}(F)$ , we have

$$(\text{Ind}_F^G(\nu))(y) = \frac{1}{|F|} \sum_{\substack{g \in G \\ gyg^{-1} \in F}} \nu(gyg^{-1}) = 0.$$

- (3) The matrices  $T_{i,i}$  ( $2 \leq i \leq p+2$ ). Fix  $i \in \{2, \dots, p+2\}$ . By Lemma 2.3(b), we have  $\overline{T}_{i,i} = \Phi_p(\overline{N}_G(Q_i))$ . By Lemma 5.2 the group  $\overline{N}_G(Q_i) = G/Q_i$  is a Frobenius group with Frobenius complement  $H$  and Frobenius kernel  $F/Q_i$ . The ordinary characters of the

PIMs of  $k\overline{N}_G(Q_i)$  can be read off from Table 8, namely

$$\Phi_{\psi_a} = \theta_a + \sum_{j=1}^{\frac{p-1}{m}} \theta_{m+j}$$

for each  $1 \leq a \leq m$ . As any element of  $\text{Irr}(G/Q_i) \setminus \text{Lin}(G/Q_i)$  is induced from a linear character of  $F/Q_i$ , its degree is  $[G/Q_i : F/Q_i] \cdot 1 = [G : F] = |H| = m$ . Therefore,  $\deg(\Phi_{\psi_a}) = 1 + \frac{p-1}{m} \cdot m = 1 + p - 1 = p$  for every  $1 \leq a \leq m$ . As in (c)(2), using the formula for induced characters, we see that all non-linear constituents of  $\Phi_{\psi_a}$  evaluate to 0 at all  $g \in H \setminus \{1\}$ . The claim follows, as all the linear characters of  $G/Q_i$  are precisely the inflations of the characters in  $\text{Irr}(H)$ . This proves that

$$T_{i,i} = X(H) + \begin{pmatrix} p-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & \cdots & 0 \end{pmatrix}.$$

The matrices  $T_{i,1}$  ( $2 \leq i \leq p+2$ ). Fix  $i \in \{2, \dots, p+2\}$ . By Lemma 2.3(c), we have

$$T_{i,1} = (\chi_{\widehat{M}}(s))_{M \in TS(G; Q_i), s \in [G]_{p'}}.$$

For any  $M \in TS(G; Q_i)$ , by the bijections in Subsection 2.3, we have

$$\chi_{\widehat{M}} = \text{Ind}_{N_G(Q_i)}^G \text{Inf}_{N_G(Q_i)/Q_i}^{N_G(Q_i)}(\Phi_{\psi_a}) = \text{Ind}_G^G \text{Inf}_{G/Q_i}^G(\Phi_{\psi_a}) = \text{Inf}_{G/Q_i}^G(\Phi_{\psi_a})$$

for a unique  $a \in \{1, \dots, m\}$ . Since we set in (a) that  $[G]_{p'} = H$ , it follows immediately that  $T_{i,1} = T_{i,i}$ . Moreover, we have  $T_{i,i} = T_{2,2} = T_{2,1}$ .

(4) Fix  $v \in \{1, \dots, p+3\}$ . By definition, we have

$$T_{p+3,v} = [\tau_{Q_v,s}^G([M])]_{M \in TS(G; Q_{p+3}), s \in [\overline{N}_G(Q_v)]_{p'}}.$$

So, let  $M \in TS(G; Q_{p+3})$  and let  $t \in [N_G(Q_v)]_{p'}$ . Recall that by Lemma 5.2(a) we have  $N_G(Q_v) = G$  and by Proposition 3.3(c) we may choose  $[N_G(Q_v)]_{p'} = H$ . By definition,

$$\tau_{Q_v,t}^G([M]) = \varphi_{M[Q_v]}(t).$$

Since  $Q_v \leq Q_{p+3} = F \trianglelefteq G$ , it follows from [Lin18b, Proposition 5.10.4] that  $M[Q_v] = \text{Res}_{N_G(Q_v)}^G(M)$ . Hence,  $\varphi_{M[Q_v]}(t) = \chi_{\widehat{M}}(t)$  by Lemma 2.4(b), and moreover  $T_{p+3,v} = T_{p+3,v} = X(H)$ .

□

**Example 5.6.** Let  $G$  be the Frobenius group  $(C_5 \times C_5) \rtimes C_4$  of order 100 with minimal fusion pattern, i.e.  $G$  has precisely 6 distinct conjugacy classes of subgroups of order 5. It follows that we have 8 conjugacy classes of 5-subgroups of  $G$ , namely:

$$Q_1 = \{1\}, Q_2 \cong Q_3 \cong Q_4 \cong Q_5 \cong Q_6 \cong Q_7 \cong C_5, Q_8 \cong C_5 \times C_5.$$

Notice that  $G$  is isomorphic to the group labelled by [ 100, 11 ] in GAP's SmallGroups library, see [GAP]. The ordinary character table of  $G$  is as given in Table 10, where  $\zeta_4$  denotes a primitive 4-th root of unity.

The trivial source character table  $\text{Triv}_5(G)$  is as given in Table 11. Following our conventions, we label the columns of  $\text{Triv}_5(G)$  with 5'-elements in  $N_G(Q_v)$  instead of  $\overline{N}_G(Q_v)$  ( $1 \leq v \leq 8$ ).

	$1a$	$4a$	$2a$	$4b$	$5a$	$5b$	$5c$	$5d$	$5e$	$5f$
$\chi_1$	1	1	1	1	1	1	1	1	1	1
$\chi_2$	1	$\zeta_4$	-1	$-\zeta_4$	1	1	1	1	1	1
$\chi_3$	1	-1	1	-1	1	1	1	1	1	1
$\chi_4$	1	$-\zeta_4$	-1	$\zeta_4$	1	1	1	1	1	1
$\chi_5$	4	0	0	0	4	-1	-1	-1	-1	-1
$\chi_6$	4	0	0	0	-1	4	-1	-1	-1	-1
$\chi_7$	4	0	0	0	-1	-1	4	-1	-1	-1
$\chi_8$	4	0	0	0	-1	-1	-1	4	-1	-1
$\chi_9$	4	0	0	0	-1	-1	-1	-1	4	-1
$\chi_{10}$	4	0	0	0	-1	-1	-1	-1	-1	4

TABLE 10. Ordinary character table of  $(C_5 \times C_5) \rtimes C_4$

Normalisers $N_w$	$N_1 \cong (C_5 \times C_5) \rtimes C_4$	$N_2 \cong (C_5 \times C_5) \rtimes C_4$	$N_3 \cong (C_5 \times C_5) \rtimes C_4$	$N_4 \cong (C_5 \times C_5) \rtimes C_4$	$N_5 \cong (C_5 \times C_5) \rtimes C_4$	$N_6 \cong (C_5 \times C_5) \rtimes C_4$	$N_7 \cong (C_5 \times C_5) \rtimes C_4$	$N_8 \cong (C_5 \times C_5) \rtimes C_4$
Representatives $n_j \in N_i$	1a 4a 2a 4b							
$\chi_1 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$	25 1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_2 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$	25 $\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_3 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$	25 -1	-1 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 + \chi_{10}$	25 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_1 + \chi_5$	5 1 1 1	5 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_2 + \chi_5$	5 $\zeta_4$ -1	- $\zeta_4$ 5	$\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_3 + \chi_5$	5 -1 1 -1	5 -1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_4 + \chi_5$	5 - $\zeta_4$ -1	$\zeta_4$ 5	- $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_1 + \chi_6$	5 1 1 1	0 0 0 0	5 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_2 + \chi_6$	5 $\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	5 -1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_3 + \chi_6$	5 -1 1 -1	0 0 0 0	5 -1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_4 + \chi_6$	5 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	5 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_1 + \chi_7$	5 1 1 1	0 0 0 0	0 0 0 0	5 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_2 + \chi_7$	5 $\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	0 0 0 0	5 $\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	0 0 0 0
$\chi_3 + \chi_7$	5 -1 1 -1	0 0 0 0	0 0 0 0	5 -1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_4 + \chi_7$	5 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	5 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_1 + \chi_8$	5 1 1 1	0 0 0 0	0 0 0 0	5 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_2 + \chi_8$	5 $\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_3 + \chi_8$	5 -1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_4 + \chi_8$	5 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
$\chi_1 + \chi_9$	5 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	5 1 1 1	0 0 0 0
$\chi_2 + \chi_9$	5 $\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	- $\zeta_4$ 0	0 0 0 0
$\chi_3 + \chi_9$	5 -1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	5 -1 1 -1	0 0 0 0
$\chi_4 + \chi_9$	5 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	5 - $\zeta_4$ -1	$\zeta_4$ 0 0 0 0
$\chi_1 + \chi_{10}$	5 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	5 - $\zeta_4$ -1	$\zeta_4$ 0 0 0 0
$\chi_2 + \chi_{10}$	5 $\zeta_4$ -1	- $\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	5 -1 1 -1	0 0 0 0
$\chi_3 + \chi_{10}$	5 -1 1 -1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	5 -1 1 -1	0 0 0 0
$\chi_4 + \chi_{10}$	5 - $\zeta_4$ -1	$\zeta_4$ 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	5 - $\zeta_4$ -1	$\zeta_4$ 0 0 0 0
$\chi_1$	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
$\chi_2$	1 $\zeta_4$ -1	- $\zeta_4$ 1	$\zeta_4$ -1	- $\zeta_4$ 1	$\zeta_4$ -1	- $\zeta_4$ 1	$\zeta_4$ -1	- $\zeta_4$ 1
$\chi_3$	1 -1 1 -1	-1 1 -1 1	-1 1 -1 1	-1 1 -1 1	-1 1 -1 1	-1 1 -1 1	-1 1 -1 1	-1 1 -1 1
$\chi_4$	1 - $\zeta_4$ -1	$\zeta_4$ 1						

TABLE 11. Trivial source character table of  $(C_5 \times C_5) \rtimes C_4$  at  $p = 5$

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APPENDIX A.  $\mathrm{Triv}_p(\mathrm{AGL}_1(p^2))$  IN TABLE FORM

To support intuition, in this appendix, we give  $\mathrm{Triv}_p(\mathrm{AGL}_1(p^2))$  described in Theorem 4.4 in *table form*, where we let  $\zeta$  denote a primitive  $(p^2 - 1)$ -th root of unity in  $K$ .

		1	$h$	...	$h^{b(p+1)-1}$	$h^{b(p+1)}$	$h^{b(p+1)+1}$	$h^{b(p+1)+2}$	...	$h^{p^2-2}$
$T_{1,1}$	$\chi_1 + \chi_{p^2}$	$p^2$	1	...	1	1	1	1	...	1
	$\chi_2 + \chi_{p^2}$	$p^2$	$\zeta$	...	$\zeta^{b(p+1)-1}$	$\zeta^{b(p+1)}$	$\zeta^{b(p+1)+1}$	$\zeta^{b(p+1)+2}$	...	$\zeta^{p^2-2}$
	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
	$\chi_{p^2-1} + \chi_{p^2}$	$p^2$	$\zeta^{p^2-2}$	...	$\zeta^{(b(p+1)-1) \cdot (p^2-2)}$	$\zeta^{b(p+1) \cdot (p^2-2)}$	$\zeta^{(b(p+1)+1) \cdot (p^2-2)}$	$\zeta^{(b(p+1)+2) \cdot (p^2-2)}$	...	$\zeta^{(p^2-2)^2}$
$T_{2,1}$	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+1}$	$p(p+1)$	0	...	0	$p+1$	0	0	...	0
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+2}$	$p(p+1)$	0	...	0	$\zeta^{b(p+1) \cdot (p+1)}$	0	0	...	0
	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+(p-1)}$	$p(p+1)$	0	...	0	$\zeta^{(p-2) \cdot b(p+1)} \cdot (p+1)$	0	0	...	0
$T_{3,1}$	$\chi_1$	1	1	...	1	1	1	1	...	1
	$\chi_2$	1	$\zeta$	...	$\zeta^{b(p+1)-1}$	$\zeta^{b(p+1)}$	$\zeta^{b(p+1)+1}$	$\zeta^{b(p+1)+2}$	...	$\zeta^{p^2-2}$
	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
	$\chi_{p^2-1}$	1	$\zeta^{p^2-2}$	...	$\zeta^{(b(p+1)-1) \cdot (p^2-2)}$	$\zeta^{b(p+1) \cdot (p^2-2)}$	$\zeta^{(b(p+1)+1) \cdot (p^2-2)}$	$\zeta^{(b(p+1)+2) \cdot (p^2-2)}$	...	$\zeta^{(p^2-2)^2}$

TABLE 12.  $T_{i,1}$  for  $1 \leq i \leq 3$  (first block column of  $\text{Triv}_p(\text{AGL}_1(p^2))$ ).

	1	$h^{p+1}$	$h^{2 \cdot (p+1)}$	...	$h^{(p-2) \cdot (p+1)}$	
$T_{2,2}$	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+1}$	$p$	1	1	...	1
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+2}$	$p$	$\zeta^{p+1}$	$\zeta^{2 \cdot (p+1)}$	...	$\zeta^{(p-2) \cdot (p+1)}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
	$\chi_{p^2} + \sum_{a=0}^p \chi_{a(p-1)+p-1}$	$p$	$\zeta^{(p-2) \cdot (p+1)}$	$\zeta^{2 \cdot (p-2) \cdot (p+1)}$	...	$\zeta^{(p-2)^2 \cdot (p+1)}$
$T_{3,2}$	$\chi_1$	1	1	1	...	1
	$\chi_2$	1	$\zeta^{(p+1)}$	$\zeta^{2 \cdot (p+1)}$	...	$\zeta^{(p-2) \cdot (p+1)}$
	$\chi_3$	1	$\zeta^{2 \cdot (p+1)}$	$\zeta^{4 \cdot (p+1)}$	...	$\zeta^{2 \cdot (p-2) \cdot (p+1)}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
	$\chi_{p^2-1}$	1	$\zeta^{(p^2-2) \cdot (p+1)}$	$\zeta^{2 \cdot (p^2-2) \cdot (p+1)}$	...	$\zeta^{(p^2-2) \cdot (p-2) \cdot (p+1)}$

TABLE 13.  $T_{i,2}$  for  $2 \leq i \leq 3$ .

		$h^j \ (0 \leq j \leq p^2 - 2)$
$T_{3,3}$	$\chi_a \ (1 \leq a \leq p^2 - 1)$	$\zeta^{(a-1)j}$

TABLE 14.  $T_{3,3}$ .

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