Action Potential Propagation in Soma-Axon Model

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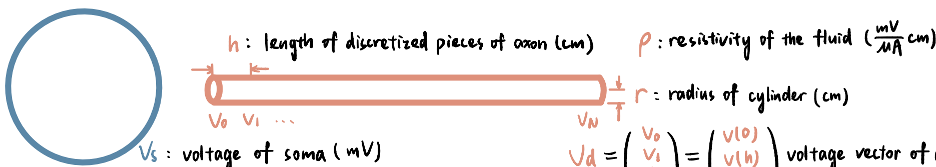
1. Introduction

This project investigates how action potential propagates through the soma and axion.
The soma is the cell body and the axon is the long threadlike part of the nerve cell.
We assume both of the structure satisfy Hodgkin-Huxley equations.
Here's a graph of the soma and axon, as well as how some parameters are introduced.
The soma is in blue while axon in pink.



$$V_d = \begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} v(0) \\ v(h) \\ \vdots \\ v(L) \end{pmatrix}$$ voltage vector of axon (mV)

C_d: membrane capacitance of axon $\left(\frac{\mu F}{cm^2}\right)$

2. Equations

The Hodgkin-Huxley equations for the soma is an ordinary differential equation of voltage with respect to time, while the Hodgkin-Huxley equations for the axon is a partial differential equation with respect to time and space.

$$\begin{cases} C_s \dfrac{dV_s}{dt} + g_s(V_s - E_s) = I_{apply} - I_0 \\ C_d \dfrac{\partial V_d}{\partial t} + g_d(V_d - E_d) = \dfrac{r}{2\rho} \dfrac{\partial^2 V_d}{\partial x^2} \end{cases}$$

Here g_s and g_d are membrane conductance. We assume they are the same, but they can actually be different.

$$g_s = g_d = g_{Na} + g_K + g_L = \bar{g}_{Na} m^3 h + \bar{g}_K n^4 + \bar{g}_L$$

m, n, h are gating variables following

$$\begin{cases} m' = \alpha_m(v)(1-m) - \beta_m(v)m \\ h' = \alpha_h(v)(1-h) - \beta_h(v)h \\ n' = \alpha_n(v)(1-n) - \beta_n(v)n \end{cases}$$

The functions of V are the opening and closing rate constants for the gates

The weighted average of potentials by conductance Es and Ed are given by

$$E_s = E_d = \frac{g_{Na} E_{Na} + g_K E_K + g_L E_L}{g_{Na} + g_K + g_L}$$

The I_{apply} and I_0 are current per unit area applied to the soma and from soma to the axon

The current from soma to axon is I_0 multiplied by the area of the soma A_s:

$$A_s I_0 = -\frac{\pi r^2}{\rho} \frac{V_1 - V_0}{h}$$

The boundary condition for the axon PDE is

$$V_0 = V_s$$

meaning continuous voltage from soma to axon at $x=0$

3. Numerical Method

We want to transform the equations into a system of linear equations so that we can update the voltages at each time step.

We define matrix L as the second-order derivative matrix of voltage by space

$$\frac{d^2 V_d}{dx^2} = L V_d = L \begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ V_N \end{pmatrix} = \frac{1}{h^2} \begin{pmatrix} V_0 - 2V_1 + V_2 \\ V_1 - 2V_2 + V_3 \\ \vdots \\ V_{N-2} - 2V_{N-1} + V_N \end{pmatrix}$$

the equations at each time step are

$$\begin{cases} \text{update } V_s : C_s \dfrac{V_0^{n+1} - V_0^n}{\Delta t} + g(V_0^{n+1} - E_s) = I_{apply} + \dfrac{\pi r^2}{\rho} \dfrac{V_1^{n+1} - V_0^{n+1}}{h} \\ \text{update } V_d : C_d \dfrac{V_d^{n+1} - V_d^n}{\Delta t} + g(V_d^{n+1} - E_d) = \dfrac{r}{2\rho} L V^{n+1} + \dfrac{r}{2\rho} \begin{pmatrix} \frac{1}{h^2} V_0^{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{cases}$$

We rearrange the equation into unknown on the left and known on the right

$$V_0^{n+1} \left(\frac{C_s}{\Delta t} + g_s + \frac{\pi r^2}{\rho h} \right) + V_1^{n+1} \left(-\frac{\pi r^2}{\rho h} \right) = C_s \frac{V_0^n}{\Delta t} + g_s E_s + I_{apply}$$

$$V_j^{n+1} \left(\frac{C_d}{\Delta t} + g_d + \frac{r}{\rho h^2} \right) + V_{j-1}^{n+1} \left(-\frac{r}{2\rho h^2} \right) + V_{j+1}^{n+1} \left(-\frac{r}{2\rho h^2} \right) = g_d E_d + C_d \frac{V_j^n}{\Delta t}, \quad 2 \le j \le N$$

We change them into matrix form

$$
\underbrace{\begin{pmatrix}
\frac{C_s}{\Delta t}+g_s+\frac{\pi r^2}{\rho h} & -\frac{\pi r^2}{\rho h} & 0 & \cdots & \cdots \\
-\frac{r}{2\rho h^2} & \frac{C_d}{\Delta t}+g_d+\frac{r}{\rho h^2} & -\frac{r}{2\rho h^2} & 0 & \cdots \\
0 & -\frac{r}{2\rho h^2} & \frac{C_d}{\Delta t}+g_d+\frac{r}{\rho h^2} & -\frac{r}{2\rho h^2} & \cdots \\
\vdots & 0 & \ddots & & 0 \\
 & & \ddots & & \cdots \\
0 & \cdots & & -\frac{r}{\rho h^2} & \frac{C_d}{\Delta t}+g_d+\frac{r}{\rho h^2}
\end{pmatrix}}_{X}
\underbrace{\begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ \\ V_N \end{pmatrix}^{n+1}}_{Y}
=
\underbrace{\begin{pmatrix}
C_s\frac{V_0^n}{\Delta t}+g_s E_s+I_{apply} \\
g_d E_d + C_d\frac{V_1^n}{\Delta t} \\
g_d E_d + C_d\frac{V_j^n}{\Delta t} \\
\vdots \\
g_d E_d + C_d\frac{V_N^n}{\Delta t}
\end{pmatrix}}_{Z}
$$

The matrix X is a sparse matrix with most of the values 0

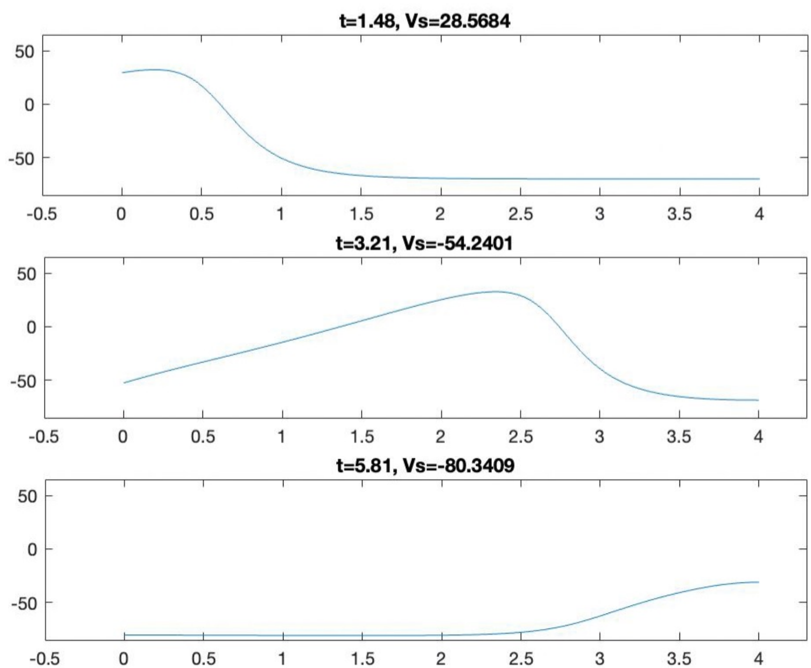
We do $Y = X \backslash Z$ in Matlab

4. Results and Discussions

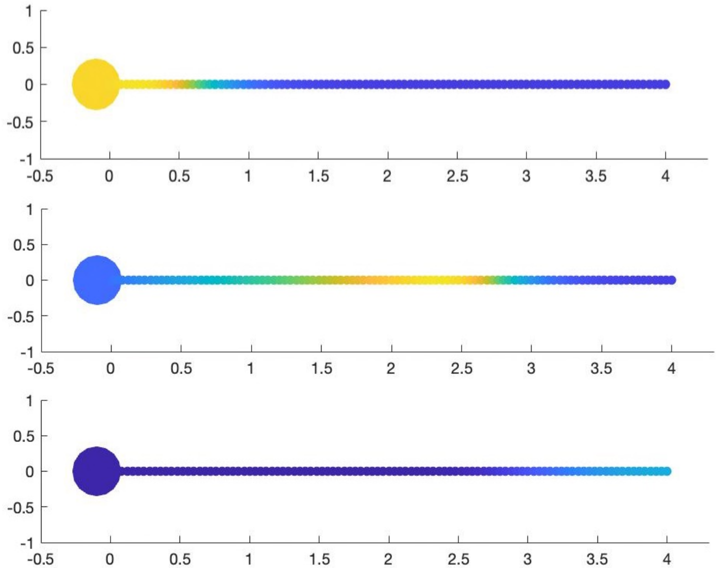
4.1 Animation

We apply starting voltage of -55 to the soma

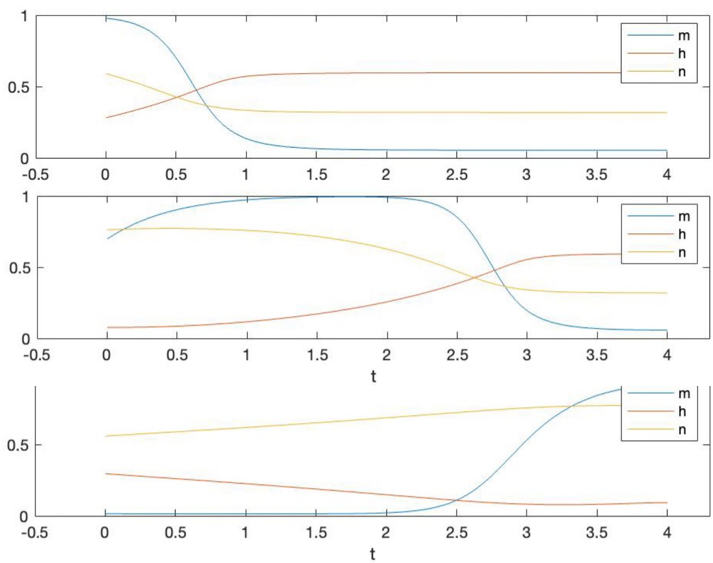
I plotted the voltages along the axon at different time: x-axis is the position on axon, y-axis is the voltage, and the plot is drawn again at every time step.



I also plotted the animation of soma and axon, with warmer color indicating higher voltage and cooler color indicating lower voltage: x-axis is the position on axon, y-axis is the voltage, and the plot is drawn again at every time step. (the same as previous plots)

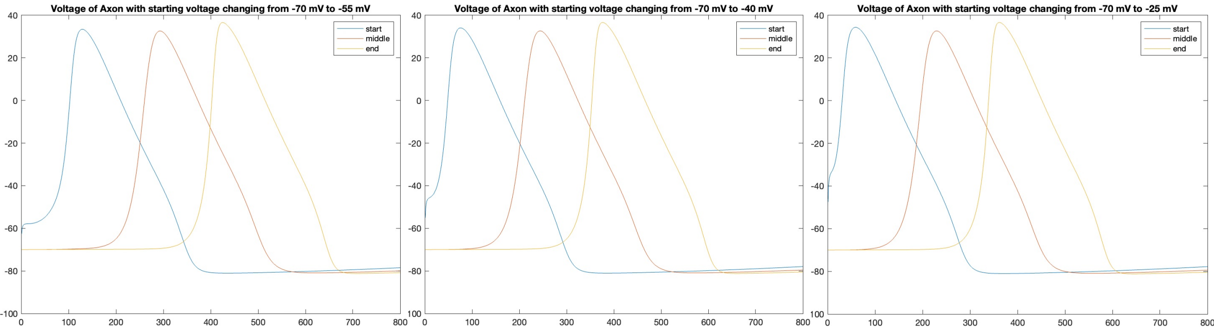


These plots are how the m, h, n gates are at different time (the same as previous plot)



4.2 Different Starting Voltage

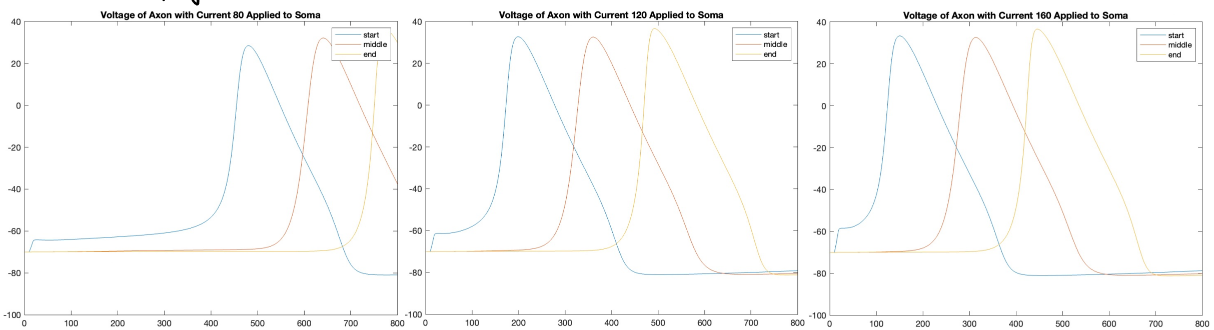
We apply different starting voltage to the soma, and see how the voltages along the axon with time. We observe the voltage at the start, middle, and end of axon.



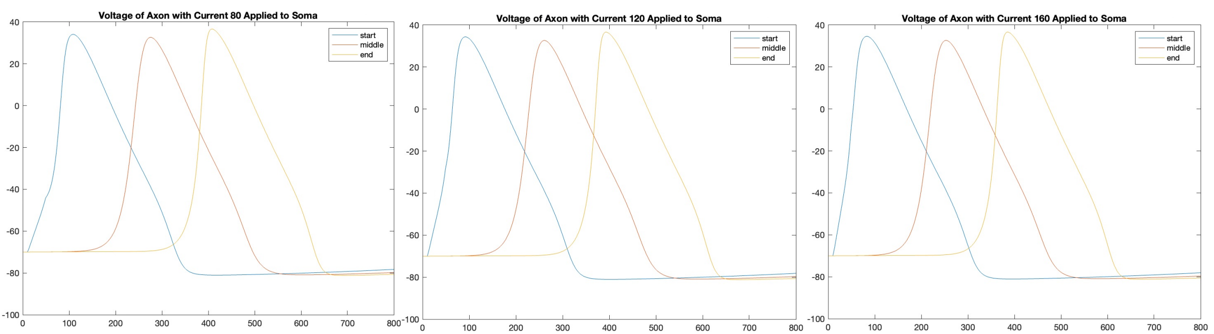
We find that the larger the voltage decreases at the beginning, the quicker the voltage changes in response to starting voltage change in the soma.

4.3 Different Current Applied to Soma

We apply different current applied to the soma from time 0.1s to 0.2s.



We apply different current applied to the soma from time 0.1s to 0.5s.



We find that the larger the current applied to soma and the longer time current applied, the quicker the voltage changes in response to starting voltage change in the soma.

5. Summary and Conclusion

I implemented a model of soma and axon, and observe how the action potential propagates through the system. I changed different starting voltage and current applied to soma as variables. We find that the larger the voltage decreases at the beginning, the current applied to soma and the longer time current applied, the quicker the voltage changes in response to starting voltage change in the soma.

6. References

Frank Hoppensteadt, Charles Peskin : Modeling and Simulation in Medicine and Life Sciences

lecture notes from Charles Peskin : Propagation of the Action Potential

lecture notes from Magnus Richardson : Introduction to the Theoretical Neuroscience Week 3 Lecture Notes Cable Theory

```
% Here is the main program, the other function programs are downloaded from
professor Charles Peskin's website with minor modifications
clear
clc
close all
global check;
global t1p t2p ip; %parameters for the function izero(t)
in_HH
in_mhnv
r = 0.0238; % cm
rho = 0.0354; % mV/uA
Cs=1.0;     %(muF/cm^2)
Cd=1.0;     %(muF/cm^2)
gNabar=120; %((muA/mV)/cm^2) max possible Na+ conductance per unit area
gKbar=36;  %((muA/mV)/cm^2) max possible K+ conductance per unit area
gLbar=0.3;  %((muA/mV)/cm^2) leakage conductance per unit area
ENa = 45;   %(mV)
EK = -82;   %(mV)
EL = -59;   %(mV)
% Define the x grid and initial conditions
L = 4; % cm
A = 100; % cm^2
N = 101;
h = L/(N-1);
x = (0:N-1)'*h;
% Matrix for solving the linear system
M = sparse(N+1,N+1);
% Second derivative matrix (from last time)
L = zeros(N,N);
vd = -70*ones(N,1);
m = alpham(vd)./(alpham(vd)+betam(vd));
n = alphan(vd)./(alphan(vd)+betan(vd));
jj = alphah(vd)./(alphah(vd)+betah(vd)); % CALL H GATES J GATES - avoids
conflict with dx
dt = 1e-2;
tf = 8;
clockmax=tf/dt;
allt = zeros(clockmax);
allv = zeros(clockmax,N);
allm = allv;
alln = allv;
allh = allv;
Z = zeros(N+1,1);
for clock=1:clockmax
   t=clock*dt;                        %note time
   ms=snew(ms,alpham(vs),betam(vs),dt); %update m
   hs=snew(hs,alphah(vs),betah(vs),dt); %update h
   ns=snew(ns,alphan(vs),betan(vs),dt); %update n
   m=snew(m,alpham(vd),betam(vd),dt); %update m
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```
    jj=snew(jj,alphah(vd),betah(vd),dt); %update h
    n=snew(n,alphan(vd),betan(vd),dt); %update n
    gNas=gNabar*(ms^3)*hs;    %sodium conductance
    gKs =gKbar*(ns^4);    %potassium conductance
    gs=gNas+gKs+gLbar;         %total conductance
    gEs=gNas*ENa+gKs*EK+gLbar*EL;          %gE=g*E
    gNa=gNabar.*(m.^3).*jj;    %sodium conductance
    gK =gKbar.*(n.^4);    %potassium conductance
    g=gNa+gK+gLbar;         %total conductance
    gE=gNa*ENa+gK*EK+gLbar*EL;         %gE=g*E
    vvector = [vs; vd];
    vold = vvector;
    M(1,1) = Cs/dt + gs + pi*r^2/(rho*h*A);
    M(1,2) = -pi*r^2/(rho*h*A);
    Z(1) = Cs/dt*vvector(1) + gEs + izero(t);
    for j=2:N
        M(j,j) = Cd/dt + g(j-1) + r/(rho*h^2);
        M(j,j+1) = -r/(2*rho*h^2);
        M(j,j-1) = -r/(2*rho*h^2);
    end
    M(N+1,N) = -r/(rho*h^2);
    M(N+1,N+1) = Cd/dt + g(N) + r/(rho*h^2);
    for i=2:N+1
        Z(i) = Cd/dt*vvector(i) + gE(i-1);
    end
    vvector = M \ Z;
    vs = vvector(1);
    vd = vvector(2:N+1);

%     Check
      if(check)
        I0=-pi*r^2/rho*(vvector(2)-vvector(1))/h;
        RHS1=izero(t)-I0;
        LHS1=Cs*(vvector(1)-vold(1))/dt+gs*vvector(1)-gEs;
        LHS1-RHS1;
        L=zeros(N,N+1);
        for iRow=1:N-1
        L(iRow,iRow+1)=-2/h^2;
        L(iRow,iRow)=1/h^2;
        L(iRow,iRow+2)=1/h^2;
        end
        L(N,N+1)=-2/h^2;
        L(N,N)=2/h^2;
        LHS2=Cd*(vvector(2:end)-vold(2:end))/dt+g.*(vvector(2:end))-gE;
        RHS2=r/(2*rho)*(L*vvector);
        max(abs(RHS2-LHS2));
      end
    % Plot results
    allt(clock)=clock;
```

```
   allv(clock,:)=vd;
   allh(clock,:)=j;
   allm(clock,:)=m;
   alln(clock,:)=n;
%     figure(1)
%     subplot(3,1,1),plot(x,vd)
%     ylim([-85 65])
%     xlim([-0.5 4.3])
%     title(strcat('t=',num2str(t),', Vs=',num2str(vs)))
%     subplot(3,1,2)
%     rsoma = 0.1;
%     xx=[-rsoma 0:h:4]';
%     scatter(xx,0*xx,[1000; 36*ones(N,1)],vvector,'filled');
%     caxis([-80, 40])
%     xlim([-0.5 4.3])
%     subplot(3,1,3),plot(x,m,x,jj,x,n)
%     ylim([0 1])
%     legend('m','h','n')
%     xlabel('t')
%     xlim([-0.5 4.3])
%     drawnow
end
figure(2)
plot(allv(:,1));
hold on
plot(allv(:,(N+1)/2))
plot(allv(:,N))
hold off
legend('start','middle','end')
title("Voltage of Axon with Area 10")
% title("Voltage of Axon with Current 80 Applied to Soma")
drawnow
```