Monte Carlo Methods Homework 2

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Exercise 16. *Consider the distribution on* $(0,1)$ *with*

$$\pi(x) = \frac{1}{2\sqrt{x}}.$$

Write a code that uses inversion to generate samples from π *using samples from* $\mathcal{U}(0,1)$ *and assemble a histogram of the output of your scheme. Produce a QQ plot to graphically compare your samples to* π.

Since the pdf for distribution is

$$\pi(x) = \frac{1}{2\sqrt{x}},\ x \in (0,1)$$

By integration, the cdf for distribution is

$$F(x) = \sqrt{x},\ x \in (0,1)$$

Let $U \sim \mathscr{U}(0,1)$ be the uniform distribution we sample from. The distribution we are looking for is $Y = F^{-1}(U)$.

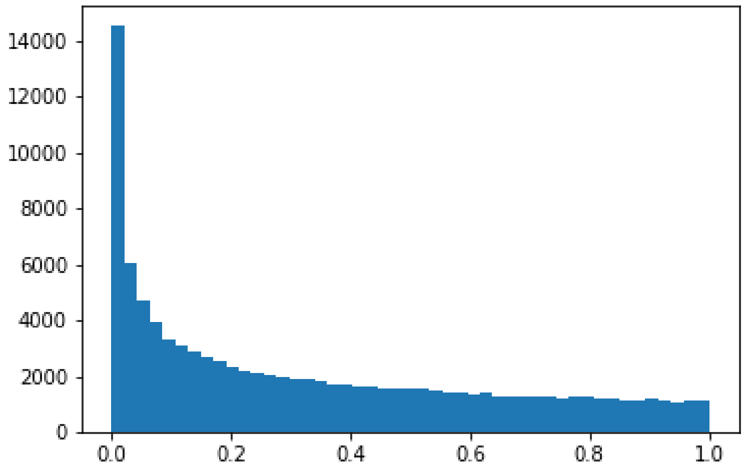
By inversion,

$$F^{-1}(u) = u^2,\ u \in (0,1).$$

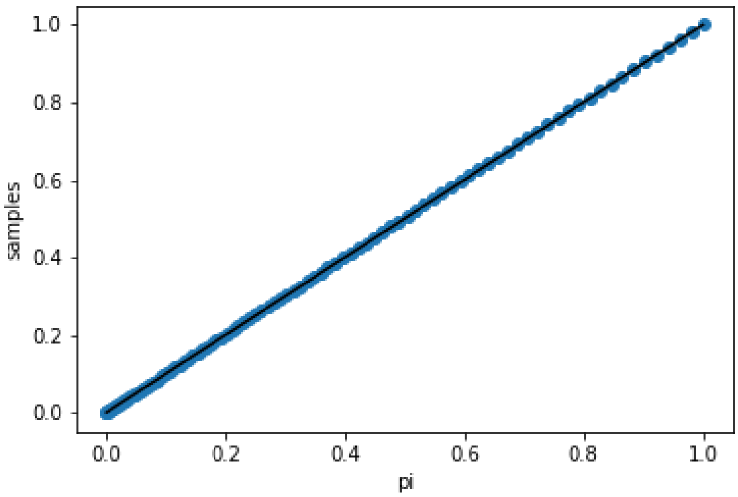
Running the code with sample size $n = 10^5$, we can plot the histogram and QQ plot.

The histogram of the scheme is shown in figure [1].

The QQ plot compared with π is shown in figure [2], indicating that sampling by inversion here is quite accurate.



Figur 1: Histogram of the scheme



Figur 2: QQ plot compared with π

Exercise 18. *Use a change of variables similar to the one you used for the Gaussian to generate a uniformly distributed sample on the unit disk given two independent samples from $\mathcal{U}(0,1)$ and produce a (2 dimensional) histogram to verify your code.*

$u1$ and $u2$ are independent $\mathscr{U}(0,1)$ random variables.

$$u_1, u_2 \sim \mathscr{U}(0,1)$$

Define the function $\varphi : [0,1]^2 \to \mathbb{R}^2$ by

$$\varphi_1(u_1,u_2) = \sqrt{u_1}\cos(2\pi u_2), \quad \varphi_2(u_1,u_2) = \sqrt{u_1}\sin(2\pi u_2)$$

The Jacobian of this transformation is the determinant of matrix

$$\begin{bmatrix} \frac{1}{2\sqrt{u_1}}\cos(2\pi u_2) & -2\pi\sqrt{u_1}\sin(2\pi u_2) \\ \frac{1}{2\sqrt{u_1}}\sin(2\pi u_2) & 2\pi\sqrt{u_1}\cos(2\pi u_2) \end{bmatrix}$$

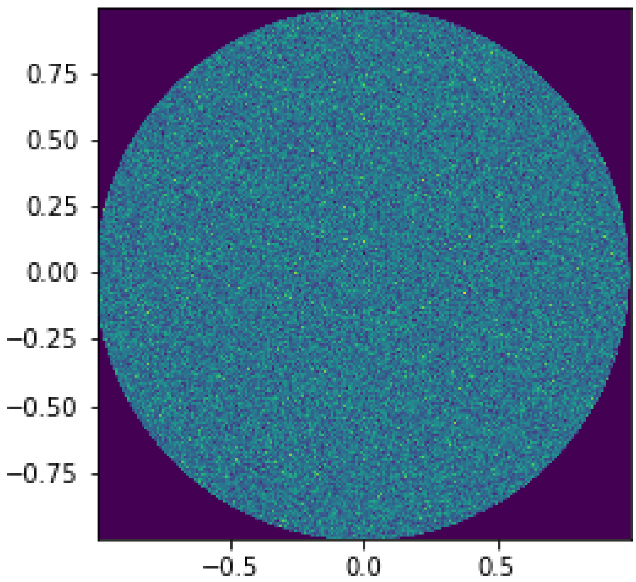
We get $|\det(D\varphi)| = \pi$.
The density function for u_1, u_2 is

$$\pi(u_1,u_2) = 1|_{0\le u_1,u_2\le 1},$$

In terms of the variables $(x,y) = \varphi(u_1,u_2)$, u_1 can be written as $u_1 = x^2 + y^2$.
Plug it in and we get the density function for x, y

$$\pi(x,y) = \frac{1}{\pi}|_{0\le x^2+y^2\le 1}.$$

The 2-dimensional histogram is shown in figure [3], which verifies my code.



Figur 3: Histogram of uniformly distributed sample on the unit disk

Exercise 19. *Write a routine to generate a single sample from the uniform measure on the unit disk from two independent samples from $\mathcal{U}(0,1)$ using rejection. Make sure you clearly identify the target density π, the reference density $\tilde{\pi}$, and your choice of K (which should be justified). There's a natural choice of K for this problem that allows you to apply the sampling algorithm without having to know the area of the unit disk in advance... what is it? Verify your code by producing a histogram. Compare (numerically) the cost of this approach with the more direct approach in Exercise 18 by comparing, e.g. the expected number of $\mathcal{U}(0,1)$ variables required per sample from the unit disk and the expected wall clock time per sample from the unit disk.*

Our target density is

$$\pi = \frac{1}{Z}|_{0\leq x^2+y^2\leq 1}$$

where Z is the normalizing constant.

The reference density is

$$\tilde{\pi} = \frac{1}{4}|_{-1\leq x,y\leq 1}$$

Because the area of the unit disk should be between 1 and 4, we have $1 \leq Z \leq 4$.

In order to satisfy $K\pi' \geq \pi$, we could set

$$K = \frac{4}{Z}$$

so that it is the smallest possible.

We do not know Z, and thus K, in advance, but it doesn't affect the rejection algorithm, since we don't have to know K in order to calculate $\frac{\pi}{K\tilde{\pi}}$. It is just the indicator function

$$\frac{\pi}{K\tilde{\pi}} = \mathbb{1}_{0\leq x^2+y^2\leq 1}$$

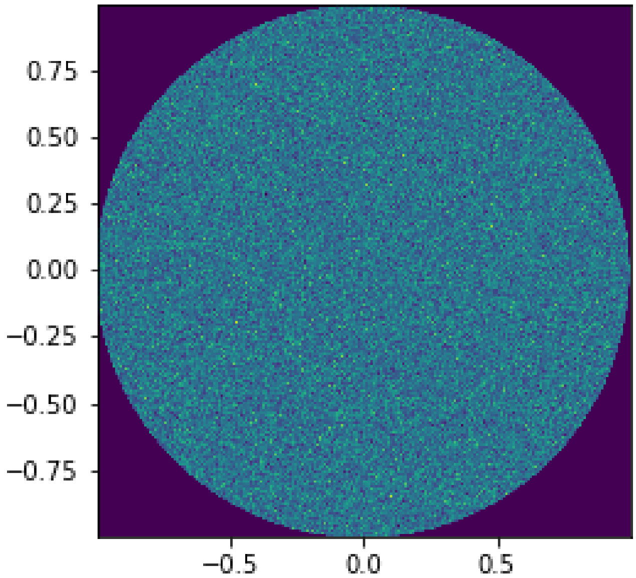
The 2-dimensional histogram is shown in figure [4], which verifies my code.

We are also interested in comparing numerically the cost of this approach with the more direct approach in Exercise 18. The results are shown in table [1].

We can see that the method change of variable in Exercise 18 perform better in terms of $\mathscr{U}(0,1)$ variables required per sample and Wall clock time per sample[U+3002]

	$\mathscr{U}(0,1)$ variables required per sample	Wall clock time per sample
Change of Variable (Ex.18)	2	8.853261470794678e-08
Rejection (Ex.19)	2.546705	6.331427693367005e-06

Tabell 1: Numerical Comparison of the cost of approaches in Exercise 18 and 19



Figur 4: Histogram of uniformly distributed sample on the unit disk

Exercise 20. *Use samples from $\mathcal{N}(m, \sigma^2)$ to estimate $\mathbf{P}[X > 2]$ for $X \sim \mathcal{N}(0,1)$ using importance sampling. By comparing the variances of the estimators for different m and σ, draw conclusions about the values of m and σ that yield the best estimators.*

From $\mathcal{N}(0,1)$

$$\pi = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Sample Y^i from $\tilde{\pi}$, which is $\mathcal{N}(m, \sigma)$: $Y^i \sim \tilde{\pi}$

$$\tilde{\pi} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-m}{\sigma})^2}$$

Therefore,

$$\frac{\pi}{\tilde{\pi}} = \sigma e^{-\frac{x^2}{2} + \frac{(x-m)^2}{2\sigma^2}}$$

The important sampling estimator is

$$\frac{1}{N}\sum_{i=1}^{N} f(Y^i)\frac{\pi(Y^i)}{\tilde{\pi}(Y^i)}, \quad \text{where } f(Y^i) = \mathbb{1}_{Y^i \geq 2}$$

The results of variances from different m and sigma are shown in figure [6]. We can see that variance = 0 is caused by no sample $Y^i \geq 2$. We are to find the minimum of the rest of variances, which is $5.98463118580311e-08$ when $m = 2$ and $\sigma = 0.5$.

m	sigma	variance
-3	0.25	0.0
-3	0.5	0.0
-3	0.75	0.0
-3	1.0	0.0
-3	1.25	0.002299567718870495
-3	1.5	0.00013101088118314336
-3	1.75	2.560668204638306e-05
-3	2.0	1.0364012201849705e-05
-3	2.25	4.577351421417621e-06
-3	2.5	3.625739910172798e-06
-3	2.75	2.741047918803314e-06
-3	3.0	2.0021073495415896e-06
-3	3.25	1.8775511884983565e-06
-3	3.5	1.6396582452610242e-06
-3	3.75	1.7873830307913963e-06
-3	4.0	1.502840218401158e-06
-3	4.25	1.2922330200486712e-06
-3	4.5	1.462102090763188e-06
-3	4.75	1.3171373261858807e-06
-3	5.0	1.434992766387211e-06
-2	0.25	0.0
-2	0.5	0.0
-2	0.75	0.0
-2	1.0	0.0024909911024851715
-2	1.25	7.272537564791693e-05
-2	1.5	1.4897728703447838e-05
-2	1.75	4.916249421814319e-06
-2	2.0	3.1352563906010494e-06
-2	2.25	2.1440376842849644e-06
-2	2.5	1.629605367575424e-06
-2	2.75	1.4856429540705991e-06
-2	3.0	1.156276030016088e-06
-2	3.25	1.2027303108175422e-06
-2	3.5	1.1676092690862465e-06
-2	3.75	1.1348351610628025e-06
-2	4.0	1.1554354461645478e-06
-2	4.25	1.0776696779854365e-06
-2	4.5	1.099963304303447e-06
-2	4.75	1.1412794829697975e-06
-2	5.0	1.1663874691411813e-06
-1	0.25	0.0
-1	0.5	0.0
-1	0.75	0.2313934841200989
-1	1.0	3.605995345572595e-05
-1	1.25	6.11965576730099e-06
-1	1.5	2.259492002557277e-06
-1	1.75	1.4385464863302848e-06
-1	2.0	1.0093890583466465e-06
-1	2.25	1.0619295795206646e-06
-1	2.5	8.596053790504692e-07
-1	2.75	7.448996575498533e-07
-1	3.0	8.795489656338138e-07
-1	3.25	8.212858157303869e-07
-1	3.5	7.574500888580535e-07
-1	3.75	8.817383818822298e-07
-1	4.0	9.591083513932206e-07
-1	4.25	9.823565893273137e-07
-1	4.5	9.323774318256147e-07
-1	4.75	1.0306801335188028e-06
-1	5.0	1.0058178598232744e-06
0	0.25	0.0
0	0.5	0.010363376202221058
0	0.75	2.1844835288632857e-05
0	1.0	2.187168759999974e-06
0	1.25	9.529494530927399e-07
0	1.5	7.038821969615281e-07
0	1.75	6.088673665896355e-07
0	2.0	5.424558055583364e-07
0	2.25	5.623634760822586e-07
0	2.5	5.841532239304844e-07
0	2.75	5.635958353979375e-07
0	3.0	5.551048858139862e-07
0	3.25	6.654191030153105e-07
0	3.5	6.137527550425296e-07
0	3.75	7.173967223483339e-07
0	4.0	5.858136251925205e-07
0	4.25	8.67108331404328e-07
0	4.5	8.11542462191197e-07
0	4.75	7.59342256271748e-07
0	5.0	8.556019043865649e-07
1	0.25	0.0019849968573028755
1	0.5	2.8457043042089263e-05
1	0.75	6.003105511580973e-07
1	1.0	3.3050502733455337e-07
1	1.25	2.967462587845565e-07
1	1.5	2.7372327261966765e-07
1	1.75	3.4445585449811583e-07
1	2.0	3.712297270103313e-07
1	2.25	3.524870391855721e-07
1	2.5	4.0645369579818477e-07
1	2.75	4.205077158145041e-07
1	3.0	4.5399465679330507e-07
1	3.25	4.99770432953995e-07
1	3.5	5.547989815007919e-07
1	3.75	5.677341478380201e-07
1	4.0	5.939142964198056e-07
1	4.25	6.225965327307193e-07
1	4.5	7.05747076375922e-07
1	4.75	7.105152004807653e-07
1	5.0	8.399626295034641e-07
2	0.25	6.125272942266353e-06
2	0.5	5.98463118580311e-08
2	0.75	9.020042182390686e-08
2	1.0	1.1152481207837525e-07
2	1.25	1.576708777736356e-07
2	1.5	1.9771840457964278e-07
2	1.75	2.494443897683303e-07
2	2.0	2.952072885712846e-07
2	2.25	3.647726504184116e-07
2	2.5	3.4604089237520224e-07
2	2.75	4.18106866862499e-07
2	3.0	4.980370004799746e-07
2	3.25	4.893844734297983e-07
2	3.5	5.507670446926358e-07
2	3.75	5.408904490407033e-07
2	4.0	6.189052887357053e-07
2	4.25	5.942544851787084e-07
2	4.5	6.835585347905221e-07
2	4.75	6.97612739622794e-07
2	5.0	7.803121630501615e-07
3	0.25	2.8368016154125153e-05
3	0.5	2.806256969013769e-07
3	0.75	1.8637266794933243e-07
3	1.0	1.828894544180269e-07
3	1.25	2.1061596561568974e-07
3	1.5	2.453609832356303e-07
3	1.75	2.5841674094843967e-07
3	2.0	3.113428085379495e-07
3	2.25	3.3972309964173497e-07
3	2.5	3.9756852584093444e-07
3	2.75	3.8329843361429066e-07
3	3.0	4.5487729185466644e-07
3	3.25	4.878221128696909e-07
3	3.5	5.09085262107557e-07
3	3.75	6.112520702635685e-07
3	4.0	6.207983231344716e-07
3	4.25	6.599378602758478e-07
3	4.5	6.802084512111757e-07
3	4.75	8.078770644202648e-07
3	5.0	8.162282466698648e-07

Figur 5: Results of variance from different m and sigma

Exercise 21. *Write a routine that uses $\mathcal{N}(0,1)$ samples to estimate the normalization constant for the density proportional to $e^{-|x|^3}$.*

The normalizing constant can be expressed as $\mathscr{Z}_p/\mathscr{Z}_q$.

We can sample Y^i from the density $\tilde{\pi}$, which is $\mathcal{N}(0,1)$

$$\tilde{\pi} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = \frac{q}{\mathcal{Z}_q} = q$$

and let

$$\pi = \frac{p}{\mathcal{Z}_p} = \frac{e^{-|x|^3}}{\mathcal{Z}_p}$$

The estimator for normalizing constant $\mathcal{Z}_p/\mathcal{Z}_q$ is

$$\frac{1}{N}\sum_{i=1}^{N}\frac{p(Y^i)}{q(Y^i)} \to \frac{\mathcal{Z}_p}{\mathcal{Z}_q}$$

From the result of running my code, the normalizing constant is 1.7854804580944723.

Exercise 22. *Repeat the last exercise for the importance sampling estimator $\widetilde{f}_N/\widetilde{1}_N$ instead of $\widetilde{f}_N$. Which of these two estimators do you prefer? Does the answer depend on m and σ?*

From $\mathcal{N}(0,1)$

$$\pi = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Sample Y^i from $\tilde{\pi}$, which is $\mathcal{N}(m,\sigma)$: $Y^i \sim \tilde{\pi}$

$$\tilde{\pi} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-m}{\sigma})^2}$$

Therefore,

$$\frac{\pi}{\tilde{\pi}} = \sigma e^{-\frac{x^2}{2}+\frac{(x-m)^2}{2\sigma^2}}$$

The important sampling estimator is

$$\sum_{i=1}^{N}\frac{f(Y^i)\frac{\pi(Y^i)}{\tilde{\pi}(Y^i)}}{\sum_{j=1}^{N}\frac{\pi(Y^j)}{\tilde{\pi}(Y^j)}}, \quad \text{where } f(Y^i) = \mathbb{1}_{Y^i \geq 2}$$

The results of variances from different m and sigma are shown in figure [6]. We can see that variance = 0 is caused by no sample $Y^i \geq 2$. We are to find the minimum of the rest of variances, which is $3.8260287681476173e-07$ when $m = 1$ and $\sigma = 1.5$.

I prefer the estimator in Exercise 20, compared with this one, because it gives smaller best variance. However, they do not reach the best variance with the same m and σ. Also, I do not see pattern of which one is better in terms of different choices of m and σ.

m	sigma	variance
-3	0.25	0.0
-3	0.5	0.0
-3	0.75	0.0
-3	1.0	0.0016956900452999926
-3	1.25	0.0013318276579447983
-3	1.5	0.0001069090994623606
-3	1.75	2.7565650258425784e-05
-3	2.0	9.780902273103804e-06
-3	2.25	5.367491225282639e-06
-3	2.5	3.864386006757438e-06
-3	2.75	2.997656512672805e-06
-3	3.0	2.496771969247076e-06
-3	3.25	2.103942043223055e-06
-3	3.5	1.7749674944906491e-06
-3	3.75	1.7429642626060763e-06
-3	4.0	1.7874412630827224e-06
-3	4.25	1.672131620282196e-06
-3	4.5	1.592357884091321e-06
-3	4.75	1.5652690666393054e-06
-3	5.0	1.5088341055708532e-06
-2	0.25	0.0
-2	0.5	0.0
-2	0.75	0.0
-2	1.0	0.001799086485350939
-2	1.25	6.285820336120053e-05
-2	1.5	1.5934847066175862e-05
-2	1.75	4.826113833501692e-06
-2	2.0	3.357055111154394e-06
-2	2.25	2.205803884217722e-06
-2	2.5	1.7183534044926906e-06
-2	2.75	1.5739710017492826e-06
-2	3.0	1.31988363856495e-06
-2	3.25	1.3935211718911355e-06
-2	3.5	1.31476452475565e-06
-2	3.75	1.4921096297595719e-06
-2	4.0	1.415840867908694e-06
-2	4.25	1.1009542972751487e-06
-2	4.5	1.4675978027535504e-06
-2	4.75	1.4162096152166777e-06
-2	5.0	1.3760275530746595e-06
-1	0.25	0.0
-1	0.5	0.0
-1	0.75	0.0017541166645592225
-1	1.0	3.752722429561137e-05
-1	1.25	5.438593543566229e-06
-1	1.5	2.582978805011084e-06
-1	1.75	1.574629488319404e-06
-1	2.0	1.1988310136266638e-06
-1	2.25	1.0671280905559911e-06
-1	2.5	1.0222437651063234e-06
-1	2.75	9.733229823523596e-07
-1	3.0	9.804798232313958e-07
-1	3.25	9.044635229826885e-07
-1	3.5	1.0130825317548232e-06
-1	3.75	1.1075974205906085e-06
-1	4.0	1.0305425376380159e-06
-1	4.25	1.0499260162492772e-06
-1	4.5	1.1941747057273323e-06
-1	4.75	1.2351284620489593e-06
-1	5.0	1.133704489928718e-06
0	0.25	0.0
0	0.5	0.0023960126448609002
0	0.75	2.1082495021341994e-05
0	1.0	2.140475039999974e-06
0	1.25	1.1427851605661698e-06
0	1.5	8.840835596783505e-07
0	1.75	6.524353312222546e-07
0	2.0	6.751739948842894e-07
0	2.25	6.51453041524807e-07
0	2.5	6.700549780745777e-07
0	2.75	7.409950108870692e-07
0	3.0	6.947434029723835e-07
0	3.25	7.684747640013332e-07
0	3.5	8.167390650134869e-07
0	3.75	8.671167669105572e-07
0	4.0	9.29995333923159e-07
0	4.25	9.764641756852357e-07
0	4.5	9.866072540342962e-07
0	4.75	8.860367550191059e-07
0	5.0	1.1351220761524125e-06
1	0.25	0.002560379173299698
1	0.5	0.00022445214715304735
1	0.75	1.8577466951621428e-06
1	1.0	4.580836460067458e-07
1	1.25	4.288869163675305e-07
1	1.5	3.8260287681476173e-07
1	1.75	4.758041550777809e-07
1	2.0	4.2319824302909983e-07
1	2.25	4.871079436815375e-07
1	2.5	5.548396465129741e-07
1	2.75	6.198594523523074e-07
1	3.0	6.676109362601696e-07
1	3.25	6.567325064248471e-07
1	3.5	7.047270959874535e-07
1	3.75	7.82661736891758e-07
1	4.0	7.90774452209309e-07
1	4.25	7.72283029572286e-07
1	4.5	9.218245074069916e-07
1	4.75	9.689738093883825e-07
1	5.0	1.0030358545529556e-06
2	0.25	0.001740051368607345
2	0.5	0.00014972500502438092
2	0.75	2.4352842008430827e-05
2	1.0	2.560257844952546e-06
2	1.25	5.560099417994329e-07
2	1.5	4.712875581020865e-07
2	1.75	3.8881913626470166e-07
2	2.0	4.934046412243392e-07
2	2.25	5.149318965987022e-07
2	2.5	5.457844168899732e-07
2	2.75	6.062658679770714e-07
2	3.0	6.117115856546346e-07
2	3.25	6.68470569857519e-07
2	3.5	7.184732265901686e-07
2	3.75	7.906245830048784e-07
2	4.0	8.371742273065544e-07
2	4.25	8.619862644985792e-07
2	4.5	7.884699928473619e-07
2	4.75	9.047897691877497e-07
2	5.0	9.482101852154156e-07
3	0.25	0.07282878185355852
3	0.5	0.0037137333538154083
3	0.75	0.0002983768327867272
3	1.0	3.7498899714424354e-05
3	1.25	3.978393652975115e-06
3	1.5	1.2141615834828376e-06
3	1.75	6.455365694882936e-07
3	2.0	6.472768214802247e-07
3	2.25	6.28844638326457e-07
3	2.5	6.494308902343787e-07
3	2.75	7.027743491500782e-07
3	3.0	7.167410315674907e-07
3	3.25	7.083776275676643e-07
3	3.5	7.370013300056286e-07
3	3.75	8.933428421328928e-07
3	4.0	8.151516946706661e-07
3	4.25	9.03569906041939e-07
3	4.5	9.013412633436977e-07
3	4.75	8.989322318234013e-07
3	5.0	1.0049408689001696e-06

Figur 6: Results of variance from different m and sigma