HOW DO WE MEASURE AND MANAGE INTEREST RATE RISK OF FIXED INCOME SECURITIES?

DV01/PVBP/PV01

DV01 = "dollar value of an 01"

PV01 = "present value of an 01"

PVBP = "present value of a basis point"

All of the above refer to the change in price of a fixed income security when a relevant interest rate (usually the yield to maturity) changes by 1 basis point = 0.01%

EXAMPLE

We have a bond with the following characteristics:

- $100 face value
- Exactly 1 year to maturity
- Semi-annual coupon of 8%/year
- Yield to maturity of 6%/year

BOND PRICE/YIELD FORMULA

$$P = \frac{\$4}{1 + y/2} + \frac{\$4}{(1 + y/2)^2} + \frac{\$100}{(1 + y/2)^2}$$

$$= \frac{\$4}{1 + y/2} + \frac{\$104}{(1 + y/2)^2}$$

$$P(0.06) = \frac{\$4}{1 + 0.06/2} + \frac{\$104}{(1 + 0.06/2)^2} = \$101.91347$$

WHAT IF YIELD TO MATURITY INCREASES BY 0.01% ???

$$P(y) = \frac{\$4}{1 + y/2} + \frac{\$104}{(1 + y/2)^2}$$

$$P(0.0601) = \frac{\$4}{1 + 0.0601/2} + \frac{\$104}{(1 + 0.0601/2)^2} = \$101.903764$$

Note that yield went up and price went down.

EXAMPLE (cont)

So, if yield to maturity changes by 1 basis point (0.01%), price of \$100 face value of bond has declined from P = \$101.9134697 to P' = \$101.9037644. The PV01 for \$100 face value is therefore

$$P - P' = \$0.00970529$$

CAN'T WE USE CALCULUS FOR THIS?

$$P(y) = \frac{\$4}{1 + y/2} + \frac{\$104}{(1 + y/2)^2}$$

By definition,

$$\frac{dP}{dy} = \lim_{h \to 0} \frac{P(y+h) - P(y)}{h},$$

$$\text{So } \left(\frac{dP}{dy}\right) h \approx P(y+h) - P(y),$$

provided that h is small

CAN'T WE USE CALCULUS FOR THIS? (cont)

$$P(y) = \frac{\$4}{1 + y/2} + \frac{\$104}{(1 + y/2)^2}$$

$$= \$4z + \$104z^2, \; z = (1 + y/2)^{-1}$$

$$\frac{dP}{dy} = \frac{dP(z)}{dz}\frac{dz}{dy}$$

$$= (\$4 + \$208z)\left[\tfrac{1}{2}(1 + y/2)^{-2}\right]$$

$$= -\frac{\$2}{(1 + y/2)^2} - \frac{\$104}{(1 + y/2)^3}$$

CAN'T WE USE CALCULUS FOR THIS? (cont)

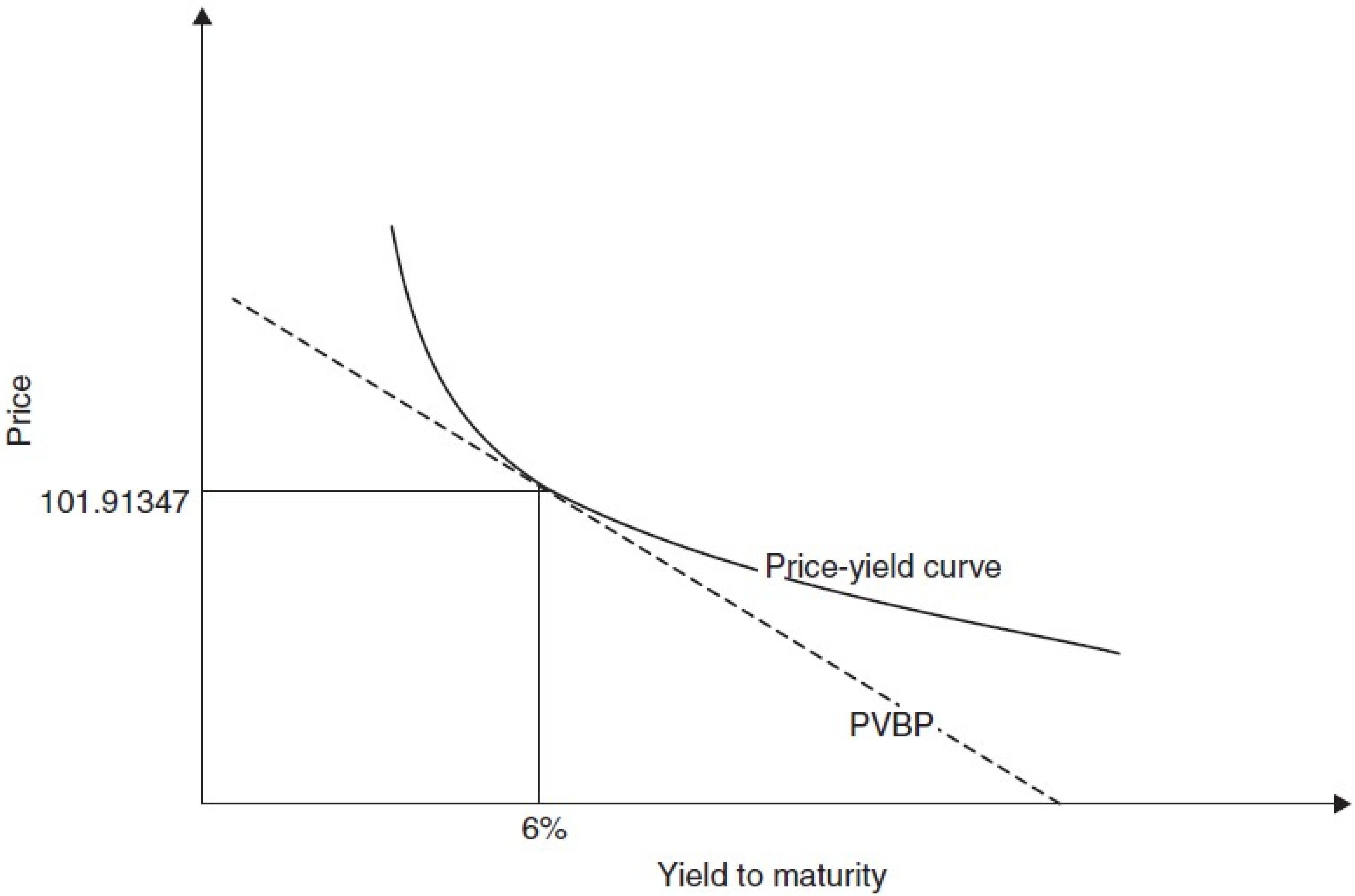
$$\left.\frac{dP}{dy}\right|_{y=6\%} = -\frac{\$2}{(1+0.06/2)^2} - \frac{\$104}{(1+0.06/2)^3}$$

$$= -\$97.05992439$$

$$\text{Hence}\left(\left.\frac{dP}{dy}\right|_{y=6\%}\right)h = (-\$97.05992439)(-0.0001)$$

$$\text{or } \$0.009705992439$$

Note that this is quite close to the previous value of $0.00970529


Price
101.91347
Price-yield curve
PVBP
6%
Yield to maturity

BETTER ESTIMATE OF PV01

Rather than compute PV01 at 6% by incrementing by a full basis point, better to compute P(0.05995) – P(0.06005). Since

$$P(0.05995) = \frac{\$4}{1+0.05995/2} + \frac{\$104}{(1+0.05995/2)^2} = \$101.9183229$$

$$P(0.06005) = \frac{\$4}{1+0.06005/2} + \frac{\$104}{(1+0.06005/2)^2} = \$101.9086169$$

$$P(0.05995) - P(0.06005) = 0.00970599245$$

Note that this is quite close to the estimate using the derivative, $0.009705992439

VARIOUS DISPLAY CONVENTIONS

Result is more meaningful if it is expressed <u>per million dollars of face value</u>. In this case, the PV01 for $1,000,000 face value would then be

$$P - P' = \$97.05$$

Or maybe it should be expressed as

$$P' - P = -\$97.05,$$

Many other variations are reasonable.

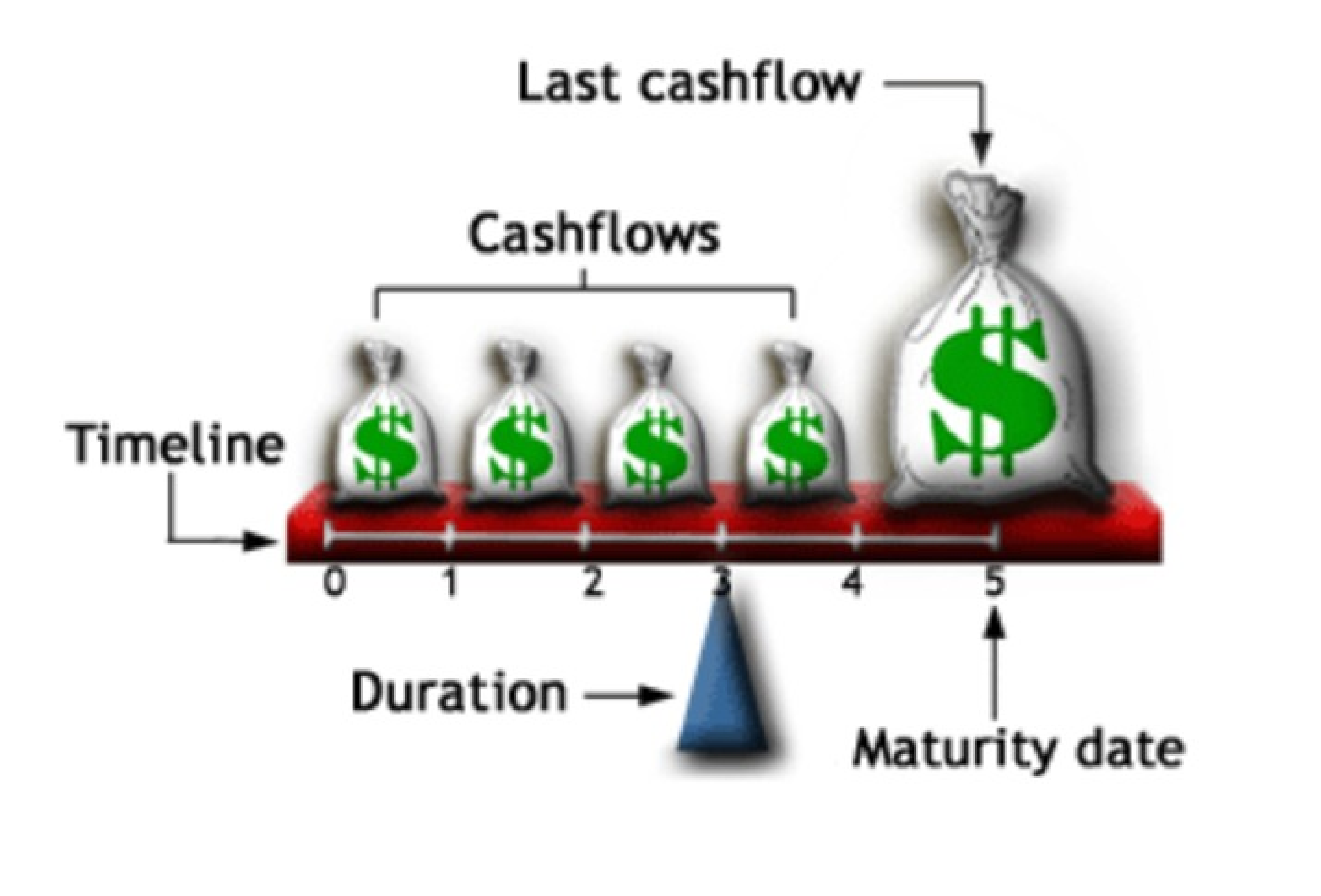
DURATION

MACAULAY DURATION

Average time to payment of all cashflows associated with a bond, weighted by present value, e.g.

$$\text{Macaulay Duration} = \frac{\sum_{i=1}^{N} t_i \mathrm{PV}(CF_i)}{\sum_{i=1}^{N} \mathrm{PV}(CF_i)}$$

$$= \frac{1}{P} \sum_{i=1}^{N} t_i \mathrm{PV}(CF_i)$$


Last cashflow
Cashflows
Timeline
0
1
2
3
4
5
Duration
Maturity date

MACAULAY DURATION, STRAIGHT BOND

- **Assumptions:**
  - Annual compounding
  - N periods
  - Yield to maturity, y
  - At a coupon period
- For $i < N$, $Cf_i = c/(1+y)^i$, $t_i = i$
- For $i = N$, $Cf_i = (1 + c)/(1+y)^N$

MACAULAY DURATION, STRAIGHT BOND (cont)

$$\text{Macaulay Duration} = \frac{\sum_{i=1}^{N} t_i \, \text{PV}\left(CF_i\right)}{\sum_{i=1}^{N} \text{PV}\left(CF_i\right)}$$

EXAMPLE: Straight bond with 3 coupon periods

- Assumptions:
  - Annual compounding
  - N = 3
  - Yield to maturity = coupon rate = 5%/year
  - At a coupon period
- Bond Price = P = 100%
- Duration (see duration spreadsheet) calculated as 2.859 years (Avg. waiting time to receive $1 is 2.859 years)

EXAMPLE: 30 year “long” treasury bond

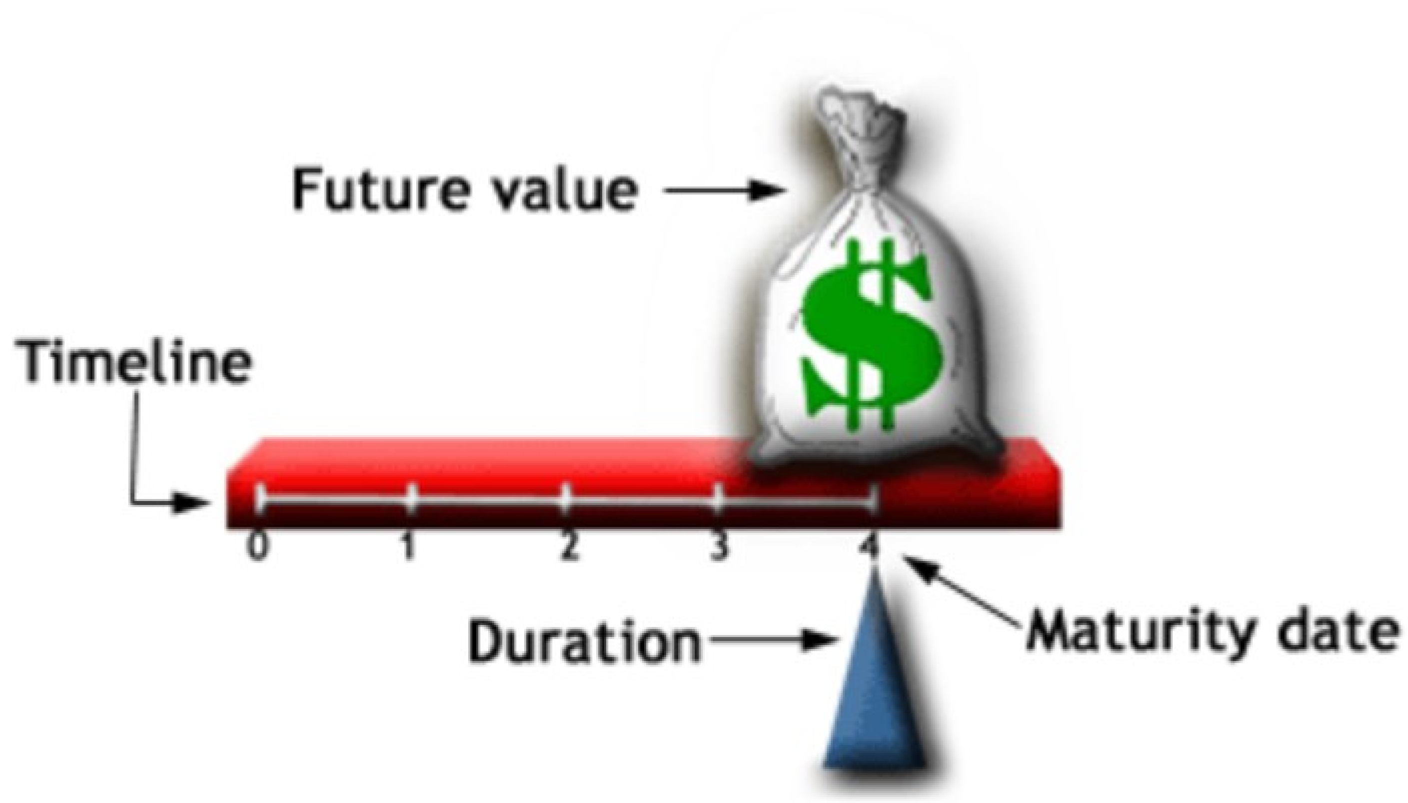
- **Assumptions:**
  - **Semi-annual compounding**
  - **N = 60**
  - **Yield to maturity = coupon rate = 5%/year**
  - **At a coupon period**
- **Bond Price = P = 100%**
- **Duration (see duration spreadsheet) calculated as 15.84 years (Avg. waiting time to receive $1 is 15.84 years)**

SPECIAL CASE: Duration of a single cashflow

Assume unit cashflow C at time T.

$$D = \frac{T\,\mathrm{PV}(T)}{\mathrm{PV}(T)} = T$$

Duration of a Zero-Coupon Bond



CONCLUSIONS AND OBSERVATIONS

- **Duration of a coupon bond is shorter than its maturity**
- **Duration of a zero coupon bond (i.e. only payment at maturity) is the same as its maturity**
- **As coupons go up, duration gets shorter**
- **As coupons go down, duration gets longer**

MORE CONCLUSIONS AND OBSERVATIONS

- The duration of a 30 year treasury is about 15 years (depending on details) because present value of cashflows far in the future is small.
- Strictly speaking, the duration calculation given here only makes sense if we know for sure that the cashflows are definitely going to be paid.
- Mcaulay Duration assumes constant yield to maturity

MODIFIED DURATION

Defined as the <u>percentage change in price for an infinitesimal change in yield,</u> *i.e.*

$$\text{Mod. Dur.} = -\frac{1}{P}\frac{dP}{dy}$$

$$= -\frac{d\ln(P)}{dy}$$

MACAULAY AND MODIFIED DURATION, STRAIGHT BOND

$$P\left(y\right)=\sum_{i=1}^{N} c\left(1+y\right)^{-i}+\left(1+y\right)^{-N}$$

MACAULAY AND MODIFIED DURATION, STRAIGHT BOND

RESULT:

$$-\frac{1}{P}\frac{dP}{dy}=\frac{1}{1+y}\left\{\frac{1}{P}\left[\sum_{i=1}^{N} ic\left(1+y\right)^{-i}+N\left(1+y\right)^{-N}\right]\right\}$$

Since y << 1,

Mod. Dur ≈ D

MACAULAY AND MODIFIED DURATION, ONE CASHFLOW

$$P(y) = e^{-yT}$$

$$\frac{dP}{dy} = -Te^{-yT}, \text{ so}$$

$$-\frac{1}{P}\frac{dP}{dy} = T$$

Macaulay and modified duration are the same for a single cashflow

IMPLICATION OF Mod Dur ≈ D

Modified duration = measure of sensitivity to a unit change in interest rates

Macaulay duration = average time to the cashflows, weighted by present value.

So "duration", whether modified or Macaulay is a measure of interest rate risk

Mod. Dur vs PVBP

$$\text{Mod. Dur.} = -\frac{1}{P}\frac{dP}{dy}$$

$$\approx -\frac{1}{P}\frac{\text{PVBP}}{\delta y}$$

$$= -\frac{1}{P}\text{PVBP}\ (\text{if } \delta y = 1)$$

Mod. Dur vs PVBP (cont)

$$\text{Mod. Dur.} \approx -\frac{1}{P}\frac{\text{PVBP}}{\delta y},$$

So

$$-(\text{Mod. Dur.})P\,\delta y \approx \text{PVBP}$$

CONVEXITY

Defined as the <u>second derivative of price with respect to yield per unit price,</u> *i.e.*

$$\mathrm{Convexity} = \frac{1}{P}\frac{d2P}{dy2}$$