CENTER FOR ECONOMIC BEHAVIOR & INEQUALITY

2. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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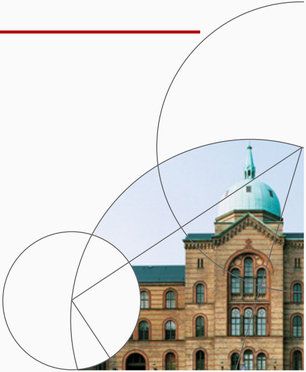

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  1. Single agent problem
  2. No interactions (only passive distribution)

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  1. What determines income and wealth inequality?
  2. What determines the real interest rate?

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- **Code:** Based on the GEModelTools package
  1. Is in active development
  2. You can help to improve interface, find bugs and features

  Documentation: See GEModelToolsNotebooks

  Original package: SSJ + course *(more complicated back-end)*

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- **Literature:** Aiyagari (1994)

Ramsey-recap

Ramsey: Firms

- **Production function:** $Y_t = F(\Gamma_t, K_{tt-1}, L_t)$ [note timing of capital] where Γ_t is technology
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1},L_t} \Pi_t$
    1. Rental rate: $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^K = F_K(\Gamma_t, K_{t-1}, L_t)$
    2. Real wage: $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = F_L(\Gamma_t, K_{t-1}, L_t)$

Zero profits: $\Pi_t = 0 \Rightarrow$

$Y_t = w_t L_t + r_t^K K_{t-1}$ [functional income distribution]

Ramsey: Zero-profit mutual fund

- **Owns all capital**
- **Capital depreciate** with rate $\delta \in (0,1)$,

$$K_t = (1-\delta)K_{t-1} + I_t$$

- **Deposits** (from households), A_{t-1}: The rate of return is

$$r_t = r_t^K - \delta$$

- **Balance sheet:**

$$A_{t-1} = K_{t-1}$$

- **Utility maximization:**

$$v_0(A_{-1}^{hh}) = \max_{\{C_t^{hh}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t^{hh})$$

s.t.

$$A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}$$

  Exogenous labor supply: $L_t^{hh} = 1$

- **Euler-equation** (implied by Lagrangian):

$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$

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- **Goods market:** $Y_t = C_t^{hh} + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears

$$\begin{aligned}
C_t^{hh} + I_t &= \left[(1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - A_t^{hh})\right] + (K_t - (1-\delta)K_{t-1}) \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t)] + (K_t - (1-\delta)K_{t-1}) \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}$$

Ramsey: Summary

- **Simplified form:**

$$
\begin{aligned}
u'(C_t^{hh}) &= \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh}) \\
K_t &= (1-\delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh}
\end{aligned}
$$

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$$u'(C_t^{hh}) = \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh}$$

- **Extended form:**

$$
\begin{aligned}
r_t^K &= F_K(\Gamma_t, K_{t-1}, L_t) \\
w_t &= F_L(\Gamma_t, K_{t-1}, L_t) \\
r_t &= r_t^K - \delta \\
A_t &= K_t \\
A_t^{hh} &= (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \\
u'(C_t^{hh}) &= \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\
A_t &= A_t^{hh} \\
L_t &= L_t^{hh}
\end{aligned}
$$

Ramsey: As an equation system

$$
\begin{bmatrix}
r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\
w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\
r_t - (r_t^K - \delta) \\
A_t - K_t \\
A_t^{hh} - \left((1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\
u'(C_t^{hh}) - \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\
A_t - A_t^{hh} \\
L_t - L_t^{hh} \\
\forall t \in \{0, 1, \dots\}, \text{given } K_{-1}
\end{bmatrix} = \mathbf{0}
$$

Note I: There is *perfect foresight*.

Note II: This is the so-called *sequence-space* formulation.

Ramsey: Steady state

- **Euler-equation** can be solved for K_{ss}:

$$u'(C_{ss}) = \beta(1 + F_K(\Gamma_{ss}, K_{ss}, 1) - \delta)u'(C_{ss}) \Leftrightarrow$$
$$F_K(K_{ss}, 1) = \frac{1}{\beta} - 1 + \delta$$

- **Accumulation equation** then implies C_{ss}:

$$K_{ss} = (1 - \delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - C_{ss} \Leftrightarrow$$
$$C_{ss} = (1 - \delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - K_{ss}$$

HANC

HANC model overview

- **Model blocks:**
    1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
    2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
    3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
    4. **Markets:** Perfect competition in labor, goods and capital markets

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  2. The Aiyagari-Bewley-Hugget-Imrohoroglu-model

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  3. The Standard Incomplete Market (SIM) model

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$\begin{aligned}
\ell_{it} &= z_{it} \\
a_{it} &= (1+r_t)a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
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  1. *Ex ante* due to different preferences, β_i

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- **Where are there heterogeneity?**
  1. *Ex ante* due to different preferences, β_i
  2. *Ex post* due to stochastic productivity, z_{it}
- **Incomplete markets due to borrowing constraint**
  (fancy words: partial self-insurrance, lack of Arrow-Debreu securities)

Recursive formulation

- **Value function** (at decision)

$$v_t(\beta_i, z_{it}, a_{it-1}) = \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(\beta_i, z_{it}, a_{it})$$

$$\text{s.t.}$$

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$$a_{it} \geq 0$$

- **Beginning-of-period value function** (before shock realization):

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E}\left[v_t(\beta_i, z_{it}, a_{it-1}) \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

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- **Envelop-condition:**

$$\underline{v}_{a,t}(\beta_i, z_{it-1}, a_{it-1}) \equiv \frac{\partial \underline{v}_t}{\partial a_{it-1}} = \mathbb{E}\left[(1 + r_t)c_{it}^{-\sigma} \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

Euler-equation

Proof: Using *variation argument* (see previous lecture)

Euler-equation:

$$\begin{aligned}
c_{it}^{-\sigma} &= \beta_i \underline{v}_{a,t+1}(\beta_i, z_{it}, a_{it}) \\
&= \beta_i \mathbb{E}_t \left[v_{a,t+1}(\beta_i, z_{it+1}, a_{it}) \right] \\
&= \beta_i (1 + r_{t+1}) \mathbb{E}_t \left[c_{it+1}^{-\sigma} \right] \\
&= \beta_i (1 + r_{t+1}) q(z_{it}, a_{it})
\end{aligned}$$

where q is the *post-decision marginal value of cash*

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

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- **Distributions** (vector of probabilities):

  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}

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  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}
  2. Productivity transition: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$ over β_i, z_{it} and a_{it-1}

Distributions and aggregates

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  3. Savings transition: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$ where again

$$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$$

Distributions and aggregates

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$$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$$

- **Aggregate consumption and savings:**

$$X_t^{hh} = \int x_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t = X^{hh}\left(\{r_\tau, w_\tau\}_{\tau \geq t}, \underline{\boldsymbol{D}}_0\right) \text{ for } x \in \{a, \ell, c\}$$

Equation system

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \boldsymbol{D}_t - \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t' \boldsymbol{D}_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \mathbf{0}$$

where $K_{-1} = \int a_{it-1} d\underline{\boldsymbol{D}}_0$

1. **Perfect forsight** wrt. aggregate variables
2. **Stationary equilibrium:** Time-constant solution.
3. **Transition path:** Time-varying solution due to e.g. initial conditions or temporary deviations of exogenous variables.

Solution method

- **Must be solved *numerically*:**
- **Household problem:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$
  1. Discretize and evaluate with interpolation
  2. Make recursion until convergence
- **Transition path:**
  1. Find the stationary equilibrium
  2. Find Jacobian around stationary equilibrium (*next time*)
  3. Solve using quasi-Newton solver (*next time*)

Solution of household problem

- **Solve:** Separately for each β_i and z_{it}
  1. Find solution from FOC for each $\tilde{a}_{it}$ in exogenous grid

$$\tilde{c}_{it}^{-\sigma} = \beta_i \underline{v}_{a,t+1}(\beta_i, z_{it}, \tilde{a}_{it}) \Leftrightarrow \tilde{c}_{it} = \left(\beta_i \underline{v}_{a,t+1}(\beta_i, z_{it}, \tilde{a}_{it})\right)^{-\frac{1}{\sigma}}$$

  2. Calculate endogenous grid $\tilde{m}_{it} = \tilde{a}_{it} + \tilde{c}_{it}$
  3. Interpolate at $m_{it} = (1+r_t)a_{it-1} + w_t z_{it} + \Pi_t$ to get optimal a_{it}
  4. Enforce constraint by $a_{it} = \max\{a_{it}, 0\}$
  5. Consumption is $c_{it} = m_{it} - a_{it}$
- **Expecation:**

$$\underline{v}_{a,t}(\beta_i, z_{it-1}, a_{it-1}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_{z-},i_z}(1+r_t)c_{it}^{-\rho}$$

Market clearing

- **Capital market:** $K_t = A_t = \int a_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t$
- **Labor market:** $L_t = \int \ell_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t = \int z_{it} d\boldsymbol{D}_t = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears

$$
\begin{aligned}
C_t^{hh} + I_t &= \int c_{it}^* d\boldsymbol{D}_t + [K_t - (1-\delta)K_{t-1}] \\
&= \int \left[(1+r_t)a_{it-1} + w_t z_{it} - a_{it}\right] d\boldsymbol{D}_t \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t] + [K_t - (1-\delta)K_{t-1}] \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$\begin{bmatrix} r_{ss}^K - F_K(\Gamma_{ss}, K_{ss}, L_{ss}) \\ w_{ss} - F_L(\Gamma_{ss}, K_{ss}, L_{ss}) \\ r_{ss} - (r_{ss}^K - \delta) \\ A_{ss} - K_{ss} \\ \boldsymbol{D}_{ss} - \Pi_z' \underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda_{ss}' \boldsymbol{D}_{ss} \\ A_{ss} - A_{ss}^{hh} \\ L_{ss} - L_{ss}^{hh} \end{bmatrix} = \mathbf{0}$$

Note I: Households still move around »inside« the distribution due to idiosyncratic shocks

Note II: Steady state for aggregates (quantities and prices) and the distribution as such

Stationary equilibrium - more verbal definition

For a given Γ_{ss}

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution $\boldsymbol{D}_{ss}$ over β_i, z_{it} and a_{it-1}
4. and the policy functions a^*_{ss}, ℓ^*_{ss} and c^*_{ss}

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Direct implementation

Technology: $F(K, L) = \Gamma K^{\alpha} L^{1-\alpha}$

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $r_{ss} = \alpha \Gamma_{ss} (K_{ss})^{\alpha-1} - \delta$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss}$

Direct implementation (alternative)

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$

Root-finding problem in r_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $K_{ss} = \left(\frac{r_{ss}+\delta}{\alpha\Gamma_{ss}}\right)^{\frac{1}{\alpha-1}}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$

Indirect implementation

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$

Consider Γ_{ss} and δ as »free« parameters:

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{K} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

Code

Calibration

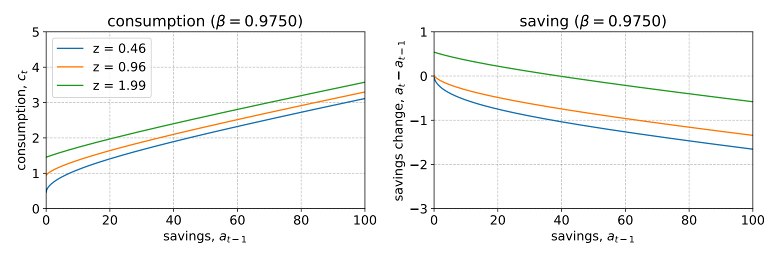
- **Preferences:** $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
  1. Discount factors: $\beta \in \{0.965, 0.975, 0.985\}$ in equal pop. shares
  2. Relative risk aversion: $\sigma = 2$
- **Income:**
  1. AR(1): $\rho_z = 0.95$
  2. Std.: $\sigma_\psi = 0.30\sqrt{(1-\rho_z^2)}$
- **Technology:** $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$
  1. Capital share: $\alpha = 0.36$
  2. TFP: $\Gamma_{ss} = 1.082$
  3. Depreciation: $\delta = 0.193$
- **Steady state:**
  1. Prices: $r_{ss} = 0.01$ and $w_{ss} = 1$
  2. Quantities: $K_{ss}/Y_{ss} = 1.776$

Consumption function

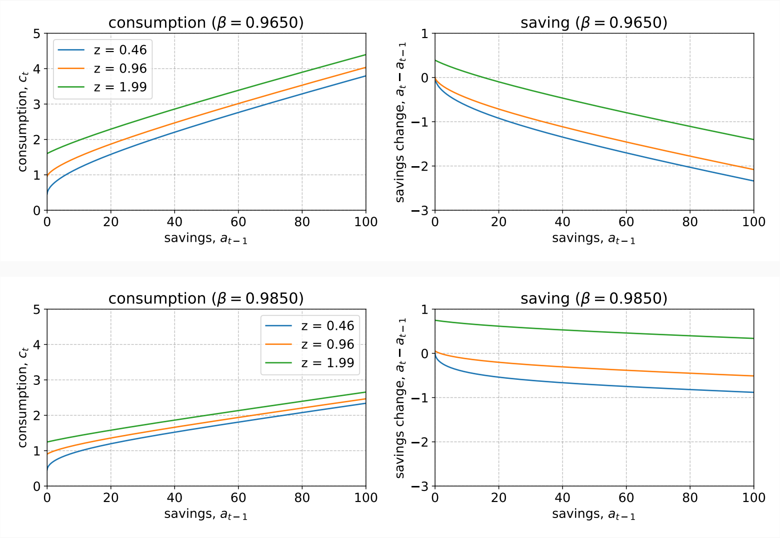
- **Euler-equation still necessary for $a_{it} > 0$:**

$$c_{it}^{-\sigma} = \beta_i (1 + r_{t+1}) \mathbb{E}_t \left[c_{it+1}^{-\sigma} \right]$$

- **Precautionary saving:**
  1. Low consumption for low cash-on-hand $\rightarrow$ *buffer-stock target*
  2. Steep slope for low cash-on-hand $\rightarrow$ *high MPC*

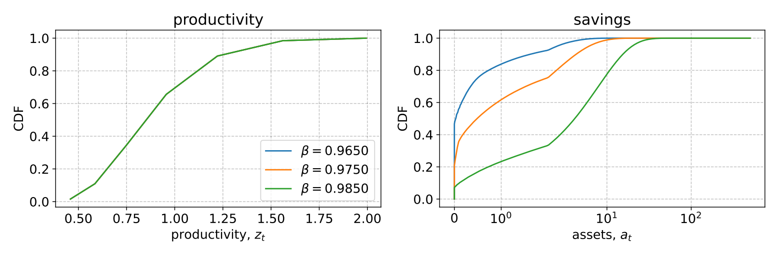


Low vs. high β_i



Distribution, D_t

- **Productivity:** Marginal distribution over only z_{it}
- **Savings:** Marginal distribution over a_{it} cond. on β_i



- **Drivers of wealth inequality:**
  1. Stochastic income
  2. Heterogeneous patience $\rightarrow$ savings behavior

Steady state interest rate

- **Representative agent / complete markets:**
  Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$

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- **Heterogeneous agents:** *No such equation exists*
  1. Euler-equation replaced by asset market clearing condition
  2. Idiosyncratic income risk affects the steady state interest rate

σ_ψ	PE ($r_{ss} = 1\%$), A^{hh}	GE, r_{ss}	GE, A^{hh}
0.09	2.78	1.00%	2.78
0.14	7.39	0.12%	2.97
0.19	13.68	-1.11%	3.30

Partial Equilibrium: Same interest rate.

General Equilibrium: Capital+labor market clearing.

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    1. **Informal:** Roughly match targets by hand
    2. **Formal:**
        - 2a. Solve root-finding problem
        - 2b. Minimize a squared loss function
    3. **Estimation:** Formal with squared loss function + standard errors

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- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

Exercise: HANCGovModel

- **No production.** No physical savings instrument
- **Households:** Get stochastic endowment z_{it} of consumption good
- **Government:**
  1. Choose government spending
  2. Collect taxes, τ_t, proportional to endowment
  3. Bonds: Pays 1 consumption good next period. Price is $p_t^B < 1$

$$p_t^B B_t = B_{t-1} + G_t - \int \tau_t z_{it} d\boldsymbol{D}_t$$

$$\tau_t = \tau_{ss} + \varphi \left(B_{t-1} - B_{ss} \right)$$

- **Market clearing:**

$$B_t = A_t^{hh}$$

Exercise: Households

Households:

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{it+1}(z_{it+1}, a_{it}) \right]$$

$$\text{s.t. } p_t^B a_{it} + c_{it} = a_{it-1} + (1-\tau_t) z_{it} \geq 0$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1} \quad , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_{it}] = 1$$

Euler-equation:

$$c_t^{-\sigma} = \beta \frac{\underline{v}_{a,t+1}(z_{it}, a_{it})}{p_t^B}$$

Envelope condition:

$$\underline{v}_{a,t}(z_{it-1}, a_{it-1}) = c_{it}^{-\sigma}$$

Exercise: Questions

1. **Define the stationary equilibrium**
2. **Solve and simulate the household problem**
   with $p^B_{ss} = 0.975$ and $\tau_{ss} = 0.12$.
3. **Find the stationary equilibrium**
   with $G_{ss} = 0.10$ and $\tau_{ss} = 0.12$.
4. **What happens for** $\tau_{ss} \in (0.11, 0.15)$?
5. **When is average household utility maximized?**

Note: Full solution in repository folder *HANCGovModel*

Summary

Summary and next week

- **Today:**
  1. The concept of a stationary equilibrium
  2. Introduction to the GEModelTools package
- **Next week:** *Transition path*
- **Homework:**
  1. Work on completing the HANCGovModel exercise (solution in repository folder HANCGovModel)
  2. Read documentation for GEModelTools (*except on linearized solution and simulation*)
  3. Skim-read at Auclert et. al. (2021) and Kirkby (2017)