CENTER FOR ECONOMIC BEHAVIOR & INEQUALITY

3. Transition Path

Adv. Macro: Heterogenous Agent Models

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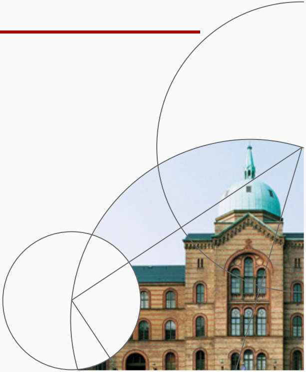

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Introduction

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  1. Based on the GEModelTools package
  2. Examples from GEModelToolsNotebooks/HANC
     (except stuff on *linearized solution* and *simulation*)

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- **Code:**
  1. Based on the GEModelTools package
  2. Examples from GEModelToolsNotebooks/HANC
     (except stuff on *linearized solution* and *simulation*)
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools
     (except stuff on *linearized solution* and *simulation*)
  3. Kirkby (2017)

Ramsey

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}$$

- **Production function:** $\Gamma_t K_t^{\alpha} L_t^{1-\alpha}$
- **Utility function:** $\frac{C_t^{1-\sigma}}{1-\sigma}$
- **Steady state:**

$$K_{ss} = \left(\frac{\left(\frac{1}{\beta} - 1 + \delta\right)}{\Gamma_{ss}\alpha} \right)^{\frac{1}{\alpha-1}}$$
$$C_{ss}^{hh} = (1-\delta)K_{ss} + \Gamma_{ss}K_{ss}^{\alpha} - K_{ss}$$

Ramsey: As an equation system

$$\begin{bmatrix} r_t^K - \alpha \Gamma_t K_t^{\alpha-1} L_t^{1-\alpha} \\ w_t - (1-\alpha)\Gamma_t K_t^{\alpha} L_t^{-\alpha} \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - \left((1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\ C_t^{hh,-\sigma} - \beta(1+r_{t+1})C_{t+1}^{hh,-\sigma} \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0,1,\dots\}, \text{given } K_{-1} \end{bmatrix} = \mathbf{0}$$

Remember: Perfect foresight

Truncated, reduced vector form

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, K_{-1}\right) = \begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{bmatrix} = \boldsymbol{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned}
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
C_t^{hh} &= \left(\beta(1+r_{t+1})\right)^{-\sigma} C_{t+1}^{hh} \text{ (backwards)} \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \text{ (forwards)} \\
L_t^{hh} &= 1
\end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduced

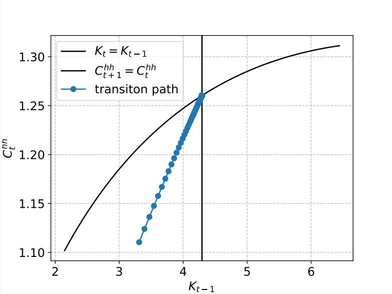
$$\boldsymbol{H}\left(\boldsymbol{K},\boldsymbol{L},\boldsymbol{\Gamma},K_{-1}\right)=\left[\begin{array}{c} A_t-A_t^{hh} \\ \forall t\in\{0,1,\ldots,T-1\} \end{array}\right]=\mathbf{0}$$

where $\boldsymbol{X}=(X_0,X_1,\ldots,X_{T-1})$, $A_{-1}^{hh}=K_{-1}$ and

$$\begin{aligned}
L_t=L_t^{hh}&=1 \\
r_t^K&=\alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} \\
w_t&=(1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\
A_t&=K_t \\
r_t&=r_t^K-\delta \\
C_t^{hh}&=\left(\beta(1+r_{t+1})\right)^{-\sigma}C_{t+1}^{hh}\,\text{(backwards)} \\
A_t^{hh}&=(1+r_t)A_{t-1}^{hh}+w_tL_t^{hh}-C_t^{hh}\,\text{(forwards)}
\end{aligned}$$

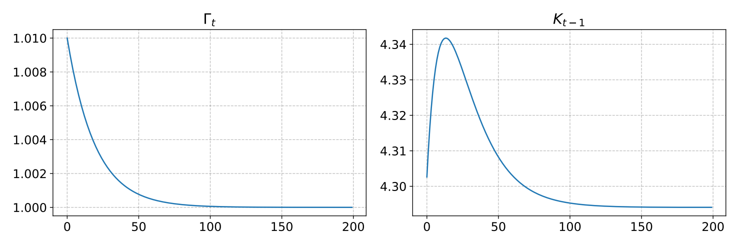
Example 1: Initially low capital

Initially away from steady state: $K_{-1} = 0.75K_{ss}$



Example 2: Technology shock

Technology shock: $\Gamma_t = 0.01\Gamma_{ss}0.95^t$ (exogenous, deterministic)



Transition path

Equation system

The model can be written as an **equation system**

$$\boldsymbol{H}\left(\{K_t, L_t; \Gamma_t\}_{t\geq 0}, \underline{\boldsymbol{D}}_0\right) = \begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \boldsymbol{D}_t - \Pi_z \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t \boldsymbol{D}_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \boldsymbol{0}$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $x_t^* = x^*\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$ and $X_t^{hh} = \boldsymbol{x}_t^{*\prime}\boldsymbol{D}_t$
2. Choice transition: $\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over β_i,z_t and a_{t-1}
4. and the policy functions $\{a_t^*\}$, $\{\ell_t^*\}$ and $\{c_t^*\}$

are such that

1. Firms maximize profits (prices) in all periods
2. Household maximize expected utility (policy functions) in all periods
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. Mutual fund balance sheet is satisfied in all periods
5. The capital market clears in all periods
6. The labor market clears in all periods
7. The goods market clears in all periods

Truncated, reduced vector form

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} r_t^k &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t \\ L_t^{hh} &= \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\ &\forall t \in \{0, 1, \ldots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduction

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0\right) = \left[\begin{array}{c} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \dots, T-1\} \end{array}\right] = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} L_t &= 1 \\ r_t^k &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t \\ &\forall t \in \{0, 1, \dots, T-1\} \end{aligned}$$

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Could we solve it with a Newton method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{K}^i, \boldsymbol{L}^i, \boldsymbol{\Gamma})$.
3. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below chosen tolerance
4. Calculate the Jacobians $\boldsymbol{H}^i_{\boldsymbol{X}} = \boldsymbol{H}_{\boldsymbol{L}}(\boldsymbol{K}^i, \boldsymbol{L}^i, \boldsymbol{\Gamma})$
5. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i + \left(\boldsymbol{H}^i_{\boldsymbol{K}}\right)^{-1} \boldsymbol{H}^i$
6. Increment i and return to step 2

Question: What is the problem?

Alternative: Use Broydens method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate the steady state Jacobian $\boldsymbol{H}_{\boldsymbol{K},ss} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}_{ss}, \boldsymbol{\Gamma}_{ss})$
3. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$.
4. Calculate Jacobian by
$$\boldsymbol{H}_{\boldsymbol{K}}^i = \begin{cases} \boldsymbol{H}_{\boldsymbol{K},ss} & \text{if } i = 0 \\ \boldsymbol{H}_{\boldsymbol{K}}^{i-1} + \frac{(\boldsymbol{H}^i - \boldsymbol{H}^{i-1}) - \boldsymbol{H}_{\boldsymbol{K}}^{i-1}(\boldsymbol{K}^i - \boldsymbol{K}^{i-1})}{\left|\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right|_2} \left(\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
5. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below tolerance
6. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i + \left(\boldsymbol{H}_{\boldsymbol{K}}^i\right)^{-1} \boldsymbol{H}^i$
7. Increment i and return to step 3

Question: What are the benefits?

Bottleneck: How do we find the Jacobian?

1. **Naive approach:** For each $s \in \{0, 1, ..., T-1\}$ do
   1.1 Set $K_t = K_{ss} + \mathbf{1}\{t = s\} \cdot \Delta, \qquad \Delta = 10^{-4}$
   1.2 Find $\boldsymbol{r}$ and $\boldsymbol{w}$
   1.3 Solve household problem backwards along transition path
   1.4 Simulate households forward along transition path
   1.5 Calculate $\frac{\partial H_t}{\partial K_s} = \frac{(A_t - A_t^{hh}) - (A_{ss} - A_{ss}^{hh})}{\Delta}$ for all t

   Bottleneck: We need T^2 solution steps and simulation steps!

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2. **Fake news algorithm:** From household Jacobian to full Jacobian

$$\boldsymbol{H_K} = \mathcal{J}^{A^{hh},r}\mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K} - \boldsymbol{I}$$

   2.1 $\mathcal{J}^{r,K}, \mathcal{J}^{w,K}$: Fast from the onset - only involve aggregates
   2.2 $\mathcal{J}^{A^{hh},r}, \mathcal{J}^{A^{hh},w}$: Only requires T solution steps and simulation steps!

What have we found?

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- **Underlying assumption:** No aggregate uncertainty
- **»Shock«, Γ:** A fully unexpected non-recurrent event $\equiv$ *MIT shock*
- **Transition path, $\boldsymbol{K}$:** Non-linear perfect foresight response to
  1. Initial distribution, $\underline{\boldsymbol{D}}_0 \neq \boldsymbol{D}_{ss}$, *or to*
  2. Shock, $\Gamma_t \neq \Gamma_{ss}$ for some t (i.e. impulse-response)

The HANC example from GEModelToolsNotebooks

- **Presentation:** I go through the code

Decomposition of GE response

- **GE transition path:** $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$
- **PE response of each:**
  1. Set $(\boldsymbol{r}, \boldsymbol{w}) \in \{(\boldsymbol{r}^*, \boldsymbol{w}_{ss}), (\boldsymbol{r}_{ss}, \boldsymbol{w}^*)\}$
  2. Solve household problem backwards along transition path
  3. Simulate households forward along transition path
  4. Calculate outcomes of interest
- **Additionally:** We can vary the initial distribution, $\underline{\boldsymbol{D}}_0$, to find the response of sub-groups

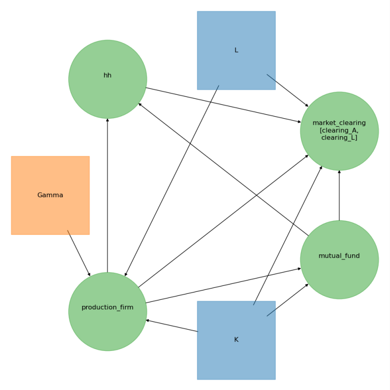
DAGs

General model class I

1. Time is discrete (index t).
2. There is a continuum of households (index i, when needed).
3. There is *perfect foresight* wrt. all aggregate variables, $\boldsymbol{X}$, indexed by $\mathcal{N}$, $\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t=0}^{\infty} = \{\boldsymbol{X}^j\}_{j\in\mathcal{N}} = \{X_t^j\}_{t=0,j\in\mathcal{N}}^{\infty}$, where $\mathcal{N} = \mathcal{Z} \cup \mathcal{U} \cup \mathcal{O}$, and $\mathcal{Z}$ are *exogenous shocks,* $\mathcal{U}$ are *unknowns*, $\mathcal{O}$ are outputs, and $\mathcal{H} \in \mathcal{O}$ are *targets.*
4. The model structure is described in terms of a set of *blocks* indexed by $\mathcal{B}$, where each block has inputs, $\mathcal{I}_b \subset \mathcal{N}$, and outputs, $\mathcal{O}_b \subset \mathcal{O}$, and there exists functions $h^o(\{\boldsymbol{X}^i\}_{i\in\mathcal{I}_b})$ for all $o \in \mathcal{O}_b$.
5. The blocks are *ordered* such that (i) each output is *unique* to a block, (ii) the first block only have shocks and unknowns as inputs, and (iii) later blocks only additionally take outputs of previous blocks as inputs. This implies the blocks can be structured as a *directed acyclical graph* (DAG).

DAG: Directed Acyclical Growth

- **Orange square:** Shocks (exogenous)
- **Purple square:** Unknowns (endogenous)
- **Green circles:** Blocks (with variables and targets inside)



General model class II

6. The number of targets are equal to the number of unknowns, and an *equilibrium* implies $\boldsymbol{X}^o = 0$ for all $o \in \mathcal{H}$. Equivalently, the model can be summarized by an *target equation system* from the unknowns and shocks to the targets,

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0},$$

and an *auxiliary model equation* to infer all variables

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z}).$$

A *steady state* satisfy

$$\boldsymbol{H}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}) = 0 \text{ and } \boldsymbol{X}_{ss} = \boldsymbol{M}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}).$$

General model class III

7. The *discretized household block* can be written recursively as

$$
\begin{aligned}
\boldsymbol{v}_t &= v(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\underline{\boldsymbol{v}}_t &= \Pi(\boldsymbol{X}_t^{hh})\boldsymbol{v}_t \\
\boldsymbol{D}_t &= \Pi(\boldsymbol{X}_t^{hh})'\underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
\boldsymbol{a}_t^* &= \boldsymbol{a}^*(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\boldsymbol{Y}_t^{hh} &= \boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
&\underline{\boldsymbol{D}}_0 \text{ is given,} \\
&\boldsymbol{X}_t^{hh} = \{\boldsymbol{X}_t^i\}_{i\in\mathcal{I}_{hh}}, \boldsymbol{Y}_t^{hh} = \{\boldsymbol{X}_t^o\}_{o\in\mathcal{O}_{hh}},
\end{aligned}
$$

   where $\boldsymbol{Y}_t$ is aggregated outputs with $\boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})$ as individual level measures.

8. Given the sequence of shocks, $\boldsymbol{Z}$, there exists a *truncation period*, T, such all variables return to steady state beforehand.

Fake News Algorithm

Fake news algorithm

Go through Section 3 of the documentation for GEModelTools

Bottlenecks

Bottlenecks

- **Small models:** Finding the stationary equilibrium
  Trick: *(Modified) policy function iteration* (Howard improvement)
  Idea: Multiple steps as once when finding the value function
  See e.g. Rendahl (2022) and eslami and Phelan (2023)
- **Bigger models:** With many unknowns and targets both computing the Jacobian and solving the equation system can be costly
  $\Rightarrow$ Auclert et. al., 2021, has some methods for speeding this up not availible in GEModelTools

Exercises

Exercises: HANCGovModel

Same model. Your choice of τ_{ss}. New questions:

1. **Define the transition path.**
2. **Plot the DAG**
3. **How does the Jacobians look like?**
4. **Find the transition path for** $G_t = G_{ss} + 0.01 G_{ss} 0.95^t$
5. **What explains household savings behavior?**
6. **What happens to consumption inequality?**

Summary

Summary and next week

- **Today:**
  1. The concept of a transition path
  2. Details of the GEModelTools package
- **Homework:** Work on completing the model extension exercise
- **Next week:** Begin working on Assignment 1