CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

14a. HANK-SAM

Adv. Macro: Heterogenous Agent Models

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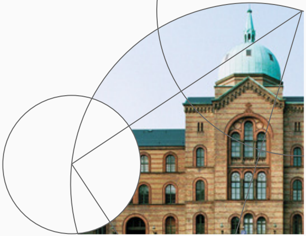
2023



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Introduction

Introduction

- **Further advanced topics:**
  1. Policy analysis
  2. Life-cycle
  3. Endogenous idiosyncratic risk (time-varying)
  4. Discrete choices (with *taste shocks*)
  5. Bounded rationality (non-FIRE)

Introduction

- **Further advanced topics:**
  1. Policy analysis
  2. Life-cycle
  3. Endogenous idiosyncratic risk (time-varying)
  4. Discrete choices (with *taste shocks*)
  5. Bounded rationality (non-FIRE)
- **Example:** *HANK-SAM*

  Broer et. al. (2023),

  »Fiscal stimulus policies according to HANK-SAM«

Policies

Policies

1. **Policies as shocks:** *Choose ARMA(p,q) process and study IRFs* (also for utility, inequality and social welfare)
   Non-linear transition: Interaction with initial state and other shocks
2. **Policies from targets:** Make different policies comparable by achieving the *exact* same path of outcomes
3. **Policy rules:** Parameterize and study effects on IRFs to shocks
4. **Optimal policy with quadratic loss function** (McKay and Wolf, 2023)
   4.1 Easy to solve numerically with *ad hoc* loss function
   4.2 Harder to derive loss function from social welfare (Ramsey problem)
5. **Optimal policy:** Discretion, commitment, timeless perspective (Dávila and Schaab, 2023)

Life-cycle

Households with life-cycle

- **Age:** $h_{it} \in \{0, 1, \ldots, \#_h - 1\}$
- **Mortality:** $\delta\left(h_{it}\right) \in [0, 1]$, $\delta(\#_h - 1) = 1$, $\zeta(j) = \int \mathbf{1}\{h_{it} = j\} d\boldsymbol{D}_{ss}$
- **Income profile:** $\mathcal{Z}(z_{it}, h_{it+1})$
- **Household problem**, $\{r_t, w_t, q_t\} \rightarrow \{a_{it}, b_{it}, c_{it}, \ell_{it}\}$:

$$v_0(\beta_i, h_{it}, z_{it}, a_{it-1}) = \max_{c_{it}} u(c_t) + \beta_i (1 - \delta(h_{it})) \underline{v}_{t+1}\left(\beta_i, h_{it+1}, z_{it+1}, a_{it}\right)$$

$$\begin{aligned}
\text{s.t. } \ell_{it} &= z_{it} \\
a_{it} &= \begin{cases} (1 + r_t) q_t + w_t \ell_{it} - c_{it} + \Pi_t & \text{if } h_{it} = 0 \\ (1 + r_t) a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t & \text{else} \end{cases} \\
b_{it} &= \delta(h_{it}) a_{it} \\
h_{it+1} &= \begin{cases} 0 & \text{with prob. } \delta(h_{it}) \\ h_{it} + 1 & \text{else} \end{cases} \\
z_{it+1} &\sim \mathcal{Z}(z_{it}, h_{it+1}), \; \mathbb{E}\left[z_{it}\right] = 1 \\
a_{it} &\geq 0
\end{aligned}$$

Equation system

$$
\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{q}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t - A_t^{hh} \\ \zeta(0) q_t - B_{t-1}^{hh} \\ \forall t \in \{0, 1, \dots, T-1\} \end{bmatrix} = \mathbf{0}
$$

where $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$
\begin{aligned}
L_t &= 1 \\
A_t &= K_t \\
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha) \Gamma_t (K_{t-1}/L_t)^{\alpha} \\
\boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\
A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\
&\forall t \in \{0, 1, \dots, T-1\}
\end{aligned}
$$

Endogenous idiosyncratic risk

Consumption problem

- **Recursive household problem:**

$$v_t(u_{it}, a_{it-1}) = \max_{c_{it}} u(c_{it}) + \beta \underline{v}_t\left(u_{it}, a_{it}\right)$$

s.t.

$$a_{it} = (1 + r_t)a_{it-1} + y_{it} - c_{it}$$

$$y_{it} = (1 - \tau_t)w_t \cdot \begin{cases} 1 & \text{if } u_{it} = 0 \\ \phi \in (0,1) & \text{if } u_{it} = 1 \end{cases}$$

$$a_{it} \geq 0$$

- Working if $u_{it} = 0$, unemployed if $u_{it} = 1$
- **Solution method:** *Standard EGM*

External endogenous risk

- **Expectation step:**

$$\underline{v}_t(u_{it-1}, a_{it-1}) = \mathbb{E}\left[v_t(u_{it}, a_{it-1}) \,|\, u_{it-1}, a_{it-1}, \delta_t, \lambda_t^u\right]$$

s.t.

$$\pi_t(u_{it} \,|\, u_{it-1}) = \begin{cases} \delta_t & \text{if } u_{it} = 1 \text{ and } u_{it-1} = 0 \\ 1 - \delta_t & \text{if } u_{it} = 0 \text{ and } u_{it-1} = 0 \\ \lambda_t^u & \text{if } u_{it} = 0 \text{ and } u_{it-1} = 1 \\ 1 - \lambda_t^u & \text{if } u_{it} = 1 \text{ and } u_{it-1} = 1 \end{cases}$$

- **Stochastic transition matrix:** $\Pi_{t,z} = \Pi_z(\delta_t, \lambda_t^u)$
- **Envelope condition:** *Nothing changed*
- **Transition steps:**

$$\boldsymbol{D}_t = \Pi'_{t,z}\underline{\boldsymbol{D}}_t$$

$$\underline{\boldsymbol{D}}_{t+1} = \Lambda'_t \boldsymbol{D}_t$$

Internal endogenous risk

- **Expectation step:**

$$\underline{v}_t(u_{it-1}, a_{it-1} \mid s_{it}) = \mathbb{E}\left[v_t(u_{it}, a_{it-1}) \mid u_{it-1}, a_{it-1}, s_{it}, \delta_t, \lambda_t^{u,s}\right]$$

s.t.

$$\pi_t(u_{it} \mid u_{it-1}, s_{it}) = \begin{cases} \delta_t & \text{if } u_{it} = 1 \text{ and } u_{it-1} = 0 \\ 1 - \delta_t & \text{if } u_{it} = 0 \text{ and } u_{it-1} = 0 \\ \lambda_t^{u,s} s_{it} & \text{if } u_{it} = 0 \text{ and } u_{it-1} = 1 \\ 1 - \lambda_t^{u,s} s_{it} & \text{if } u_{it} = 1 \text{ and } u_{it-1} = 1 \end{cases}$$

- **Search decision:**
  1. Discrete search choice: $s_{it} \in \{0, 1\}$
  2. Search cost: λ if $s_{it} = 1$
  3. Taste shocks: $\varepsilon(s_{it}) \sim$ Extreme value (Iskhakov et. al., 2017)
- **See also:** Bardóczy (2021)

Discrete search decision

- **Standard logit formula:**

$$\underline{v}_t(u_{it-1}, a_{it-1}) = \max_{s_{it}\in\{0,1\}} \left\{\underline{v}_t(u_{it-1}, a_{it-1} \mid s_{it}) - \lambda \mathbf{1}_{s_{it}=1} + \sigma_\varepsilon \varepsilon\left(s_{it}\right)\right\}$$

$$= \sigma_\varepsilon \log\left(\exp \frac{\underline{v}_t(u_{it-1}, a_{it-1} \mid 0)}{\sigma_\varepsilon} + \exp \frac{\underline{v}_t(u_{it-1}, a_{it-1} \mid 1)}{\sigma_\varepsilon}\right)$$

- **Transition matrix:**

$$\Pi_{t,z} = \Pi_z\left(\left\{r_\tau, w_\tau, \tau_\tau, \delta_\tau, \lambda_\tau^{u,s}\right\}_{\tau\geq t}\right)$$

Envelope condition

- **Choice probabilities:**

$$P_t(s \mid u_{it-1}, a_{it-1}) = \frac{\exp \frac{\underline{v}_t(u_{it-1}, a_{it-1} \mid s)}{\sigma_\xi}}{\sum_{s' \in \{0,1\}} \exp \frac{\underline{v}_t(u_{it-1}, a_{it-1} \mid s')}{\sigma_\xi}}$$

- **Envelope condition:**

$$\begin{aligned} \underline{v}_{a,t}(u_{t-1}, a_{t-1}) &= \sum_{s \in \{0,1\}} P_t(s \mid u_{it-1}, a_{it-1}) \pi_t(u_{it} \mid u_{it-1}, s) v_{a,t}(u_{it}, a_{it-1}) \\ &= \sum_{s \in \{0,1\}} P_t(s \mid u_{it-1}, a_{it-1}) \pi_t(u_{it} \mid u_{it-1}, s) c_t^*(u_{it}, a_{it-1})^{-\sigma} \end{aligned}$$

- **Break of *monotonicity*** $\Rightarrow$ FOC still *necessary*, but not *sufficient*
  1. **Normally:** Savings $\uparrow$ $\Rightarrow$ future consumption $\uparrow$ $\Rightarrow$ marginal utility $\downarrow$
  2. **Now also:** Future search jump $\downarrow$ $\Rightarrow$ future income $\downarrow$
     $\Rightarrow$ future consumption $\downarrow$ $\Rightarrow$ marginal utility $\uparrow$

Upper envelope for given z^{i_z}

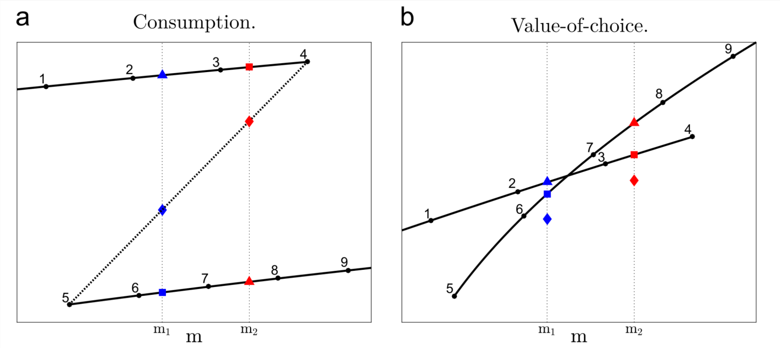
1. **Generate candidate points:** $\forall i_a \in \{0, 1, \ldots, \#_a - 1\}$

$$
\begin{aligned}
w^{i_a} &= \beta \underline{v}_{t+1}(z^{i_z}, a^{i_a}) \\
c^{i_a} &= u'^{-1}\left(\beta \underline{v}_{a,t+1}(z^{i_z}, a^{i_a})\right) \\
m^{i_a} &= a^{i_a} + c^{i_a} \\
v^{i_a} &= u\left(c^{i_a}\right) + w^{i_a}
\end{aligned}
$$

2. **Apply upper-envelope:** $\forall i_{a-} \in \{0, 1, \ldots, \#_a - 1\}$

$$
\begin{aligned}
c^*(a^{i_{a-}}) = \max_{j \in \{0,1,\ldots \#_a - 2\}} & \; u\left(c^{i_{a-}}\right) + w^{i_{a-}} \text{ s.t.} \\
m^{i_{a-}} &= (1 + r_t) a^{i_{a-}} + w_t z^{i_z} \in \left[m^j, m^{j+1}\right] \\
c^{i_{a-}} &= \min\left\{\text{interp } \left\{m^{i_a}\right\} \to \left\{c^{i_a}\right\} \text{ at } m^{i_{a-}}, m^{i_{a-}}\right\} \\
a^{i_{a-}} &= m^{i_{a-}} - c^{i_{a-}} \\
w^{i_{a-}} &= \text{interp } \left\{a^{i_a}\right\} \to \left\{w^{i_a}\right\} \text{ at } a^{i_{a-}}
\end{aligned}
$$

Illustration



1. **Numbering:** Different levels of end-of-period assets, a^{i_a}
2. **Problem:** Find the consumption function at m_1 and m_2
3. **Largest value-of-choice:** Denoted by the *triangles*

Source: Druedahl and Jørgensen (2017), $G^2 EGM$

Example

- **Beg.-of-period value function:**

$$\underline{v}_{t+1}(a_t) = \sqrt{m_{t+1}} + \eta \max\{m_{t+1} - \underline{m}, 0\}$$
$$\text{where } m_{t+1} = (1+r)a_t + 1$$

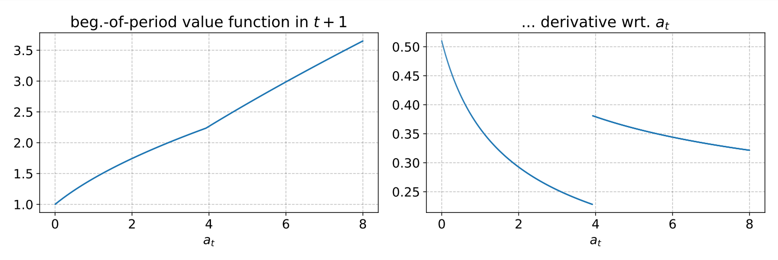
- **Derivative:**

$$\underline{v}_{a,t+1}(a_t) = \frac{1}{2}(1+r)m_{t+1}^{-\frac{1}{2}} + (1+r)\eta\mathbf{1}\{m_{t+1} > \underline{m}\}$$

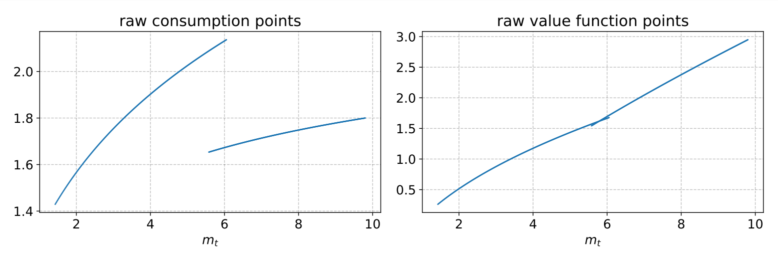
- **Budget constraint:**

$$a_t + c_t = (1+r)a_{t-1} + 1$$

Next-period values

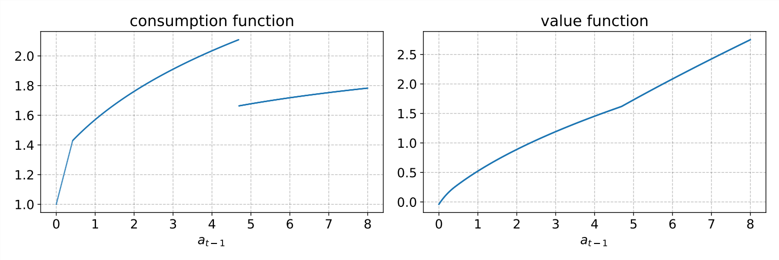


Raw values of c^{i_a} and v^{i_a}



Problem: Overlaps $\Rightarrow$ not a function m_t!

Result after upper envelope



General problem structure

- **General problem structure with *nesting*:**

$$
\begin{aligned}
\overline{v}_t\left(\overline{x}_t, d_t, e_t, m_t\right) &= \max_{c_t \in [0, m_t]} u(c_t, d_t, e_t) + \beta \underline{v}_{t+1}\left(\underline{\Gamma}_t\left(\overline{x}_t, d_t, e_t, a_t\right)\right) \\
&\text{with } a_t = m_t - c_t \\
v(x_t) &= \max_{d_t \in \Omega^d(x_t)} \overline{v}_t\left(\overline{\Gamma}_t\left(x_t, d_t\right)\right) \\
\underline{v}_t\left(\underline{x}_t\right) &= \max_{e_t \in \Omega^e(\underline{x}_t)} \mathbb{E}\left[v\left(\Gamma\left(\underline{x}_t, e_t\right)\right) \mid \underline{x}_t, e_t\right]
\end{aligned}
$$

- **Finding c_t:** *EGM with upper envelope can (typically) still be used*
- **Finding d_t and e_t:**
  1. Combination of discrete and continuous choices
  2. Typically requires use of numerical optimizer or root-finder
- **Druedahl (2021),** »A Guide on Solving Non-Convex Consumption-Saving Models« (costly with extra states in $\overline{v}$)

Non-FIRE

Motivating example

- **FIRE:** *Full International Rational Expectations*
- **IKC:** $d\boldsymbol{Y} = \boldsymbol{M}^r d\boldsymbol{r} + \boldsymbol{M} d\boldsymbol{Y}$ where $M_{t,s} = \frac{\partial C_t}{\partial Y_s}$ and similar for $\boldsymbol{M}^r$
- **Myopic behavior:**
  1. Agents *never* thinks about the future
  2. Agents gradually observe current aggregate variables

$$\boldsymbol{M}^{\text{myopic}} = \begin{bmatrix} M_{0,0} & 0 & 0 & \cdots \\ M_{1,0} & M_{0,0} & 0 & \cdots \\ M_{2,0} & M_{1,0} & M_{0,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  Consider $t = 1$:

  - $M_{1,0} dY_0$: Effect from past shock observed
  - $M_{0,0} dY_1$: Effect of unexpected change in period 1

Sticky expectations

- **Sticky expectations:** A fraction $1-\theta$ updates expectations each period (from Carroll et. al., 2020)

$$\boldsymbol{M}^{\text{sticky}} = \begin{bmatrix} M_{0,0} & (1-\theta)M_{0,1} & (1-\theta)M_{0,2} & \cdots \\ M_{1,0} & (1-\theta)M_{1,1} + \theta M_{0,0} & (1-\theta)M_{1,2} + \theta(1-\theta)M_{0,1} & \cdots \\ M_{2,0} & (1-\theta)M_{2,1} + \theta M_{1,0} & \vdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Consider $t=0$:

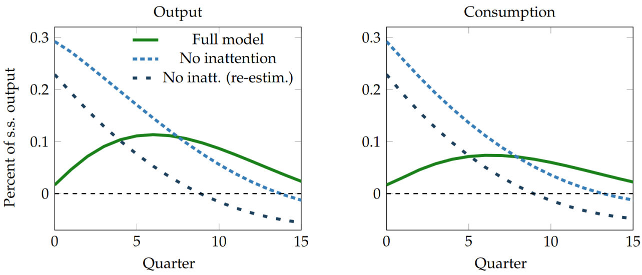
1. Non-updaters: $\theta M_{0,0} dY_0$
2. Updaters: $(1-\theta)\sum_{s\geq 0} M_{0,s} dY_s$

Consider $t=1$:

1. Ingoing updaters: $(1-\theta)\sum_{s\geq 0} M_{1,s} dY_s$
2. Ingoing non-updaters: $\theta\left(M_{1,0} dY_0 + M_{0,0} dY_1\right)$
3. New updaters: $\theta(1-\theta)\sum_{s\geq 1} M_{0,s} dY_{s+1}$

Hump-shaped response to monetary policy

Figure 4: Impulse responses with and without inattention

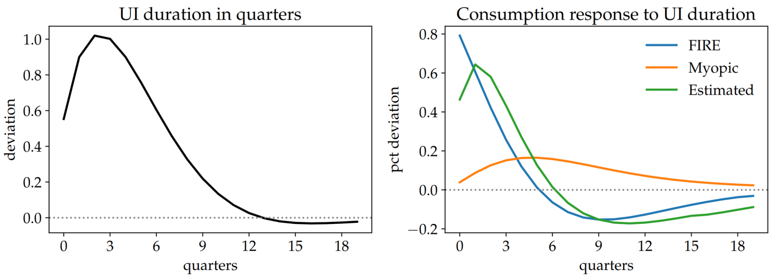


Note. This figure shows the general equilibrium paths of output and consumption in our estimated HA model with different assumptions on inattention. The solid green uses our baseline estimates of household inattention. The dashed blue line is the impulse response when the inattention parameter is set to $\theta = 0$, holding all other parameters fixed at their estimated value in table 2. The dotted dark blue line reestimates the model parameters without inattention.

Source: Auclert et. al. (2020), »Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model«

UI extensions might be less powerful

Figure 4: Partial-equilibrium Consumption Responses to an UI Extension



Source: Bardóczy and Guerreiro (2020), »Unemployment Insurance in Macroeconomic Stabilization with Imperfect Expectations«

HANK-SAM

My ongoing work

- **Zero liqduity:**
  Broer, Druedahl, Harmenberg and Öberg (2023),
  »The Unemployment-Risk Channel in Business-Cycle Fluctuations«
- **Positive liqudity:**
  Broer, Druedahl, Harmenberg and Öberg (2023),
  »Fiscal stimulus policies according to HANK-SAM«

Fiscal stimulus policies according to HANK-SAM

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Code: *HANK-SAM* in *GEModelToolsNotebooks*

Household problem

$$v_t(\beta_i, u_{it}, a_{it-1}) = \max_{c_{it}, a_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta_i \mathbb{E}_t \left[v_{t+1}\left(\beta_i, u_{it+1}, a_{it}\right)\right]$$

$$\text{s.t. } a_{it} + c_{it} = (1+r_t)a_{it-1} + (1-\tau_t)y_t(u_{it}) + \text{div}_t + \text{transfer}_t$$

$$a_{it} \geq 0$$

1. **Dividends and government transfers:** div_t and transfer_t
2. **Real wage**: w_t
3. **Income tax:** τ_t
4. **Separation rate** for employed: δ_t
5. **Job-finding rate** for unemployed: $\lambda_t^{u,s} s(u_{it-1})$
   (where $s(u_{it-1})$ is exogenous search effectiveness)
6. US-style duration-dependent **UI system:**
   a) High replacement rate $\overline{\phi}$, first $\overline{u}$ months
   b) Low replacement rate $\underline{\phi}$, after $\overline{u}$ months

Income process

- **Income** is

$$y_{it}(u_{it}) = w_{ss} \cdot \begin{cases} 1 & \text{if } u_{it} = 0 \\ \overline{\phi}\mathrm{UI}_{it} + (1 - \mathrm{UI}_{it})\underline{\phi} & \text{else} \end{cases}$$

where share of the month with UI is

$$\mathrm{UI}_{it} = \begin{cases} 0 & \text{if } u_{it} = 0 \\ 1 & \text{else if } u_{it} < \overline{u}_t \\ 0 & \text{else if } u_{it} > \overline{u}_t + 1 \\ \overline{u} - (u_{it} - 1) & \text{else} \end{cases}$$

- **Note:** *Hereby $\overline{u}$ becomes a continuous variables*

Transition probabilities

- **Beginning-of-period value function:**

$$\underline{v}_t\left(\beta_i, u_{it-1}, a_{it-1}\right) = \mathbb{E}\left[v_t(\beta_i, u_{it}, a_{it-1}) \,\middle|\, u_{it-1}, a_{it-1}\right]$$

- **Grids:** $u_{it} \in \{0, 1, \ldots, \#_u - 1\}$ for $\#_u - 1$
- **Workers** with $u_{it-1} = 0$:

$$u_{it} = \begin{cases} 0 & \text{with } 1 - \delta_t \\ 1 & \text{with } \delta_t \end{cases}, \; \mathrm{UI}_{it} = \begin{cases} 1 & \text{with } \pi^{\mathrm{UI}} \\ 0 & \text{with } 1 - \pi^{\mathrm{UI}} \end{cases}$$

- **Unemployed** with $u_{it-1} = 1$:

$$u_{it} = \begin{cases} 0 & \text{with } \lambda_t^{u,s} \mathrm{s}(u_{it-1}) \\ \min\left\{u_{it-1} + 1, \#_u - 1\right\} & \text{with } 1 - \lambda_t^{u,s} s(u_{it-1}) \end{cases}, \; \mathrm{UI}_{it} = \mathrm{UI}_{it-1}$$

Hiring and firing

- **Job value:**

$$V_t^j = p_t^x Z_t - w_{ss} + \beta^{\text{firm}} \mathbb{E}_t \left[(1 - \delta_{ss}) V_{t+1}^j \right]$$

- **Vacancy value:**

$$V_t^v = -\kappa + \lambda_t^v V_t^j + (1 - \lambda_t^v)(1 - \delta_{ss}) \beta^{\text{firm}} \mathbb{E}_t \left[V_{t+1}^v \right]$$

- **Free entry** implies

$$V_t^v = 0$$

Labor market dynamics

- **Labor market tightness** is given by

$$\theta_t = \frac{v_t}{S_t}$$

- **Cobb-Douglas matching function** implies:

$$\lambda_t^v = A\theta_t^{-\alpha}$$
$$\lambda_t^{u,s} = A\theta_t^{1-\alpha}$$

- **Law of motion for unemployment:**

$$u_t = u_{t-1} + \delta_t(1 - u_{t-1}) - \lambda_t^{u,s} S_t$$

Standard New Keynesian block

- **Intermediate goods price:** p_t^X
- Dixit-Stiglitz **demand curve** $\Rightarrow$ **Phillips curve** relating marginal cost, $MC_t = p_t^x$, and **final goods price inflation**, $\Pi_t = P_t / P_{t-1}$,

$$1 - \epsilon + \epsilon p_t^x = \phi \pi_t (1 + \pi_t) - \phi \beta^{\text{firm}} \mathbb{E}_t \left[\pi_{t+1} (1 + \pi_{t+1}) \frac{Y_{t+1}}{Y_t} \right]$$

  with output $Y_t = Z_t (1 - u_t)$
- **Flexible price limit:** $\phi \to 0$
- **Taylor rule:**

$$1 + i_t \quad = \quad (1 + i_{ss}) \left(\frac{1 + \pi_t}{1 + \pi_{ss}} \right)^{\delta_\pi}$$

Government

- **Unemployment insurance:** $\Phi_t = w_{ss}\left(\overline{\phi}_t \mathrm{UI}_t^{hh} + \underline{\phi}\left(u_t - \mathrm{UI}_t^{hh}\right)\right)$
- **Total expenses:** $X_t = \Phi_t + G_t + \mathrm{transfer}_t$
- **Total taxes:** $\mathrm{taxes}_t = \tau_t\left(\Phi_t + w_{ss}(1-u_t)\right)$
- **Government budget** is

$$q_t B_t = (1 + q_t\delta_q)B_{t-1} + X_t - \mathrm{taxes}_t$$

- **Tax rule:**

$$\tilde{\tau}_t = \frac{(1 + q_t\delta_q)\,B_{t-1} + X_t - q_{ss}B_{ss}}{\Phi_t + w_{ss}(1-u_t)}$$

$$\tau_t = \omega\tilde{\tau}_t + (1-\omega)\tau_{ss}$$

Equilibrium

1. **Financial markets:**

$$\frac{1+\delta_q q_{t+1}}{q_t} = \frac{1+i_t}{1+\pi_{t+1}}$$

$$1+r_t = \begin{cases} \frac{(1+\delta_q q_0)B_{-1}}{A_{-1}^{hh}} & \text{if } t=0 \\ \frac{1+i_{t-1}}{1+\pi_t} & \text{else} \end{cases}$$

2. **Market clearing:**

$$A_t^{hh} = q_t B_t$$

$$Y_t = C_t^{hh} + G_t$$

Summary

Summary

- **Mixed advanced:**
  1. Policies (shocks, targets, rule, optimal)
  2. Life-cycle (age, mortality, income profile)
  3. Endogenous idiosyncratic risk (external/internal)
  4. Discrete choices with taste shocks (upper envelope, non-convex)
  5. Bounded rationality (manipulation of Jacobian, myopic, sticky)
- **HANK-SAM** (labor market dynamics, fiscal policy)