CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

9. A Baseline HANK Model

Adv. Macro: Heterogenous Agent Models

Jeppe Druedahl & Patrick Moran

2022

Danmarks
Grundforskningsfond
Danish National
Research Foundation

Introduction

Introduction

- **Previously:** Non-linear transition path and perfect foresight

Introduction

- **Previously:** Non-linear transition path and perfect foresight
- **Today:**
  1. Linearized IRF
  2. Linearized simulation with aggregate risk

Introduction

- **Previously:** Non-linear transition path and perfect foresight
- **Today:**
  1. Linearized IRF
  2. Linearized simulation with aggregate risk
- **Model:** Heterogeneous Agent New Keynesian (HANK) model

Introduction

- **Previously:** Non-linear transition path and perfect foresight
- **Today:**
  1. Linearized IRF
  2. Linearized simulation with aggregate risk
- **Model:** Heterogeneous Agent New Keynesian (HANK) model
- **Code:**
  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANK

Introduction

- **Previously:** Non-linear transition path and perfect foresight
- **Today:**
  1. Linearized IRF
  2. Linearized simulation with aggregate risk
- **Model:** Heterogeneous Agent New Keynesian (HANK) model
- **Code:**
  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANK
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools

HANK model

Overview

- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint

Overview

- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay dividends to households

Overview

- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay dividends to households
- **Final goods firms** (representative)
  1. Produce final good with intermediary goods
  2. Take price as given under perfect competition

Overview

- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay dividends to households
- **Final goods firms** (representative)
  1. Produce final good with intermediary goods
  2. Take price as given under perfect competition
- **Government:**
  1. Collect taxes from households
  2. Pays interest on government debt and choose public consumption

Overview

- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay dividends to households
- **Final goods firms** (representative)
  1. Produce final good with intermediary goods
  2. Take price as given under perfect competition
- **Government:**
  1. Collect taxes from households
  2. Pays interest on government debt and choose public consumption
- **Central bank**: Set nominal interest rate

Final goods firms

- Intermediary goods indexed by $j \in [0, 1]$

Final goods firms

- Intermediary goods indexed by $j \in [0, 1]$
- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

Final goods firms

- Intermediary goods indexed by $j \in [0, 1]$
- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

- **Demand curve** derived from FOC wrt. y_{jt}

$$\forall j : y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

Final goods firms

- Intermediary goods indexed by $j \in [0, 1]$
- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\,\forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

- **Demand curve** derived from FOC wrt. y_{jt}

$$\forall j : \; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- **FOC wrt.** y_{jt}

$$
\begin{aligned}
0 &= P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu} - 1} - p_{jt} \Leftrightarrow \\
\frac{p_{jt}}{P_t} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow \\
\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu - 1}} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow \\
y_{jt} &= \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t
\end{aligned}
$$

Intermediary goods firms

- ***Dynamic*** **problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt},p_{jt},n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \ \ y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** derived from FOC wrt. p_{jt} and envelope condition:

$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log\left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** derived from FOC wrt. p_{jt} and envelope condition:

$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** derived from FOC wrt. p_{jt} and envelope condition:

$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)
- **Implied dividends:** $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log(1 + \pi_t) \right]^2 Y_t$

Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$

$$- \frac{\mu}{\mu - 1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$
$$- \frac{\mu}{\mu - 1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

- **Envelope condition:** $J'_{t+1}(p_{jt}) = \frac{\mu}{\mu-1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt+1}}{p_{jt}}\right)}{p_{jt}} Y_{t+1}$

Derivation of NKPC

- **FOC wrt.** p_{jt}**:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$
$$- \frac{\mu}{\mu - 1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

- **Envelope condition:** $J'_{t+1}(p_{jt}) = \frac{\mu}{\mu-1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt+1}}{p_{jt}}\right)}{p_{jt}} Y_{t+1}$
- **FOC + Envelope + Symmetry +** $\pi_t = P_t/P_{t-1} - 1$

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\frac{Y_t}{P_t}$$
$$+ \frac{\mu}{\mu - 1}\frac{1}{\kappa}\log\left(1 + \pi_t\right)\frac{Y_t}{P_t} + \frac{\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(1 + \pi_{t+1}\right)\frac{Y_{t+1}}{P_t}}{1 + r_{t+1}}$$

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

$$\text{s.t. } a_t = (1+r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \quad, \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_t] = 1$$

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

$$\text{s.t. } a_t = (1+r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_t] = 1$$

- **Dividends:** Distributed proportional to productivity (ad hoc)

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

$$\text{s.t. } a_t = (1+r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_t] = 1$$

- **Dividends:** Distributed proportional to productivity (ad hoc)
- **Taxes:** Collected proportional to productivity (ad hoc)

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

$$\text{s.t. } a_t = (1+r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \quad, \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi),\ \mathbb{E}[z_t] = 1$$

- **Dividends:** Distributed proportional to productivity (ad hoc)
- **Taxes:** Collected proportional to productivity (ad hoc)
- **Optimality conditions:**

$$\text{FOC wrt. } c_t : 0 = c_t^{-\sigma} - \beta \mathbb{E}_t\left[v_{a,t+1}(z_{t+1}, a_t)\right]$$

$$\text{FOC wrt. } \ell_t : 0 = w_t z_t \beta \mathbb{E}_t\left[v_{a,t+1}(z_{t+1}, a_t)\right] - \varphi \ell_t^{\nu}$$

$$\text{Envelope condition: } v_{a,t}(z_t, a_{t-1}) = (1+r_t)c_t^{-\sigma}$$

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$
$$\text{s.t. } a_t = (1+r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$
$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi),\ \mathbb{E}[z_t] = 1$$

- **Dividends:** Distributed proportional to productivity (ad hoc)
- **Taxes:** Collected proportional to productivity (ad hoc)
- **Optimality conditions:**

$$\text{FOC wrt. } c_t : 0 = c_t^{-\sigma} - \beta \mathbb{E}_t\left[v_{a,t+1}(z_{t+1}, a_t)\right]$$
$$\text{FOC wrt. } \ell_t : 0 = w_t z_t \beta \mathbb{E}_t\left[v_{a,t+1}(z_{t+1}, a_t)\right] - \varphi \ell_t^{\nu}$$
$$\text{Envelope condition: } v_{a,t}(z_t, a_{t-1}) = (1+r_t)c_t^{-\sigma}$$

- **Effective labor-supply:** $n_t = z_t \ell_t$

EGM I

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t \left[v_{a,t}(z_t, a_{t-1}) \right] = \mathbb{E} \left[(1 + r_t) c_t^{-\sigma} \right]$$

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t \left[v_{a,t}(z_t, a_{t-1}) \right] = \mathbb{E}\left[(1 + r_t) c_t^{-\sigma} \right]$$

- **Endogenous grid method:** Vary z_t and a_t to find

$$\begin{aligned} c_t &= \left(\beta \underline{v}_{a,t+1}(z_t, a_t)\right)^{-\frac{1}{\sigma}} \\ \ell_t &= \left(\frac{w_t z_t}{\varphi} c_t^{-\sigma} \right)^{\frac{1}{\nu}} \\ m_t &= c_t + a_t - (w_t \ell_t - \tau_t + d_t) z_t \end{aligned}$$

EGM I

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t\left[v_{a,t}(z_t, a_{t-1})\right] = \mathbb{E}\left[(1+r_t)c_t^{-\sigma}\right]$$

- **Endogenous grid method:** Vary z_t and a_t to find

$$c_t = \left(\beta \underline{v}_{a,t+1}(z_t, a_t)\right)^{-\frac{1}{\sigma}}$$

$$\ell_t = \left(\frac{w_t z_t}{\varphi} c_t^{-\sigma}\right)^{\frac{1}{\nu}}$$

$$m_t = c_t + a_t - (w_t \ell_t - \tau_t + d_t) z_t$$

- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_t, a_{t-1}) \text{ and } \ell^*(z_t, a_{t-1}) \text{ with } m_t = (1+r_t)a_{t-1}$$

EGM I

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t \left[v_{a,t}(z_t, a_{t-1}) \right] = \mathbb{E} \left[(1 + r_t) c_t^{-\sigma} \right]$$

- **Endogenous grid method:** Vary z_t and a_t to find

$$c_t = (\beta \underline{v}_{a,t+1}(z_t, a_t))^{-\frac{1}{\sigma}}$$

$$\ell_t = \left(\frac{w_t z_t}{\varphi} c_t^{-\sigma} \right)^{\frac{1}{\nu}}$$

$$m_t = c_t + a_t - (w_t \ell_t - \tau_t + d_t) z_t$$

- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_t, a_{t-1}) \text{ and } \ell^*(z_t, a_{t-1}) \text{ with } m_t = (1 + r_t) a_{t-1}$$

- **Savings:** $a^*(z_t, a_{t-1}) = (1 + r_t) a_{t-1} - c_t^* + (w_t \ell_t^* - \tau_t + d_t) z_t$

EGM II

- **Problem:** $a^*(z_t, a_{t-1}) < 0$ violate borrowing constraint

EGM II

- **Problem:** $a^*(z_t, a_{t-1}) < 0$ violate borrowing constraint
- **Refinement if** $a^*(z_t, a_{t-1}) < 0$ **by:**
  Find ℓ^* (and c^* and n^*) with *Newton solver* assuming $a^* = 0$

EGM II

- **Problem:** $a^*(z_t, a_{t-1}) < 0$ violate borrowing constraint
- **Refinement if** $a^*(z_t, a_{t-1}) < 0$ **by:**

  Find ℓ^* (and c^* and n^*) with *Newton solver* assuming $a^* = 0$

  1. Stop if $f(\ell^*) = \ell^* - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} (c^*)^{-\frac{\sigma}{\nu}} <$ tol. where

$$\begin{aligned} c^* &= (1 + r_t)a_{t-1} + (w_t \ell^* - \tau_t + d_t)z_t \\ n^* &= z_t \ell^* \end{aligned}$$

EGM II

- **Problem:** $a^*(z_t, a_{t-1}) < 0$ violate borrowing constraint
- **Refinement if** $a^*(z_t, a_{t-1}) < 0$ **by:**

  Find ℓ^* (and c^* and n^*) with *Newton solver* assuming $a^* = 0$

  1. Stop if $f(\ell^*) = \ell^* - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} (c^*)^{-\frac{\sigma}{\nu}} <$ tol. where

  $$\begin{aligned} c^* &= (1+r_t)a_{t-1} + (w_t \ell^* - \tau_t + d_t) z_t \\ n^* &= z_t \ell^* \end{aligned}$$

  2. Set

  $$\ell^* = \frac{f(\ell^*)}{f'(\ell^*)} = \frac{f(\ell^*)}{1 - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c^*)^{-\frac{\sigma}{\nu}} w_t z_t}$$

EGM II

- **Problem:** $a^*(z_t, a_{t-1}) < 0$ violate borrowing constraint
- **Refinement if** $a^*(z_t, a_{t-1}) < 0$ **by:**

  Find ℓ^* (and c^* and n^*) with *Newton solver* assuming $a^* = 0$

  1. Stop if $f(\ell^*) = \ell^* - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} (c^*)^{-\frac{\sigma}{\nu}} <$ tol. where

     $$c^* = (1 + r_t)a_{t-1} + (w_t\ell^* - \tau_t + d_t)z_t$$
     $$n^* = z_t\ell^*$$

  2. Set

     $$\ell^* = \frac{f(\ell^*)}{f'(\ell^*)} = \frac{f(\ell^*)}{1 - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c^*)^{-\frac{\sigma}{\nu}} w_t z_t}$$

  3. Return to step 1

Government and central bank

- **Monetary policy:** Folow Taylor-rule:

$$i_t = i_t^* + \phi\pi_t$$

where i_t^* is a shock

Government and central bank

- **Monetary policy:** Folow Taylor-rule:

$$i_t = i_t^* + \phi \pi_t$$

where i_t^* is a shock

- **Fisher relationship:**

$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

Government and central bank

- **Monetary policy:** Folow Taylor-rule:

$$i_t = i_t^* + \phi\pi_t$$

where i_t^* is a shock

- **Fisher relationship:**

$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Labor: $N_t = \int n_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$ (in effective units)
2. Assets: $B_{ss} = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
3. Goods: $Y_t = \int c_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t + G_t + \frac{\mu}{\mu-1}\frac{1}{2\kappa}\left[\log\left(1+\pi_t\right)\right]^2 Y_t$

As an equation system

$$\boldsymbol{H}\left(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^{*}, \boldsymbol{Z}, \underline{\boldsymbol{D}}_0\right) = \mathbf{0}$$

$$\begin{bmatrix} \log(1+\pi_t) - \left[\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1}}{1+r_{t+1}})\right] \\ N_t - \int n_t^{*}(z_t, a_{t-1}) d\boldsymbol{D}_t \\ B_{ss} - \int a_t^{*}(z_t, a_{t-1}) d\boldsymbol{D}_t \end{bmatrix} = \mathbf{0}$$

The rest of the model is given by

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^{*}, \boldsymbol{Z})$$

As a DAG (from Auclert et al., 2021)

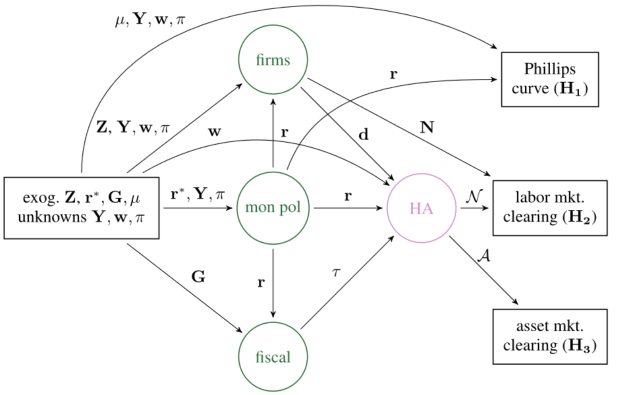

FIGURE C.1.—DAG representation of one-asset HANK economy.

Notation: $\boldsymbol{i}^* = \boldsymbol{r}^*$, $\boldsymbol{\mu}$ is a shock, $\boldsymbol{A}^{hh} = \mathcal{A}$, $\boldsymbol{N}^{hh} = \mathcal{N}$

Steady state

- **Chosen:** B_{ss}, G_{ss}, r_{ss}
- **Analytically:**
  1. **Normalization:** $Z_{ss} = N_{ss} = 1$
  2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i_{ss}^{*} = (1 + r_{ss})(1 + \pi_{ss}) - 1$
  3. **Firms:** $Y_{ss} = Z_{ss} N_{ss}$, $w_{ss} = \frac{Z_{ss}}{\mu}$ and $d_{ss} = Y_{ss} - w_{ss} N_{ss}$
  4. **Government:** $\tau_{ss} = r_{ss} B_{ss} + G_{ss}$
  5. **Assets:** $A_{ss} = B_{ss}$
- **Numerically:** Choose β and φ to get market clearing

The HANK example from GEModelToolsNotebooks I

- **Presentation:** I go through the code for finding the transition path
- **In-class exercise:**
    1. Look at the code and talk about it with the person next to you for 10 minutes
    2. Write at least one question on https://padlet.com/jeppe_druedahl/advmacrohet

Transmission mechanism to monetary policy shock

1. **Monetary policy shock:** $i_t^* \downarrow \Rightarrow i_t = i_t^* + \phi\pi_t \downarrow$
2. **Real interest rate:** $r_t = \frac{1+i_{t-1}}{1+\pi_t} \downarrow$
3. **Taxes:** $\tau_t = r_t B_{ss} \downarrow$
4. **Household consumption**, $C_t^{hh} \uparrow$, due to $r_t \downarrow$ and $\tau_t \downarrow$
5. **Firms production,** $Y_t\uparrow$, and **labor demand**, $N_t\uparrow$
6. **Inflation,** $\pi_t \uparrow$, and **wage**, $w_t\uparrow$ and **dividends**, $d_t\downarrow$
7. **Household labor supply**, $N_t^{hh}\uparrow$, due to $w_t \uparrow$ and $d_t \downarrow$, but dampened $\tau_t \downarrow$
8. **Nominal rate**, $i_t \uparrow$ due to $\pi_t \uparrow$ implying $r_t \uparrow$
9. **Household consumption**, $C_t^{hh}\uparrow$, due to $w_t \uparrow$ but dampened by $d_t \downarrow$ and $r_t \uparrow$

IRFs and simulation

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock
- **Today:** Just consider the first order solution

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock
- **Today:** Just consider the first order solution

  1. Solve for IRFs for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z})=0 \Rightarrow \boldsymbol{H}_{\boldsymbol{U}} d\boldsymbol{U}+\boldsymbol{H}_{\boldsymbol{Z}} d\boldsymbol{Z}=0 \Leftrightarrow d\boldsymbol{U}=\underbrace{-\boldsymbol{H}_{\boldsymbol{U}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}}_{\equiv \boldsymbol{G}_{\boldsymbol{U}}} d\boldsymbol{Z}$$

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock
- **Today:** Just consider the first order solution

  1. Solve for IRFs for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U^{-1}} \boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

  2. Derive all other IRFs for

$$\begin{aligned}\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U},\boldsymbol{Z}) \Rightarrow d\boldsymbol{X} &= M_{\boldsymbol{U}} d\boldsymbol{U} + M_{\boldsymbol{Z}} d\boldsymbol{Z} \\ &= \underbrace{(-M_{\boldsymbol{U}} \boldsymbol{H_U^{-1}} \boldsymbol{H_Z} + M_{\boldsymbol{Z}})}_{\equiv \boldsymbol{G}} d\boldsymbol{Z}\end{aligned}$$

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock
- **Today:** Just consider the first order solution

  1. Solve for IRFs for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

  2. Derive all other IRFs for

$$\begin{aligned}\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U},\boldsymbol{Z}) \Rightarrow d\boldsymbol{X} &= M_{\boldsymbol{U}} d\boldsymbol{U} + M_{\boldsymbol{Z}} d\boldsymbol{Z} \\ &= \underbrace{(-M_{\boldsymbol{U}}\boldsymbol{H_U}^{-1}\boldsymbol{H_Z} + M_{\boldsymbol{Z}})}_{\equiv \boldsymbol{G}} d\boldsymbol{Z}\end{aligned}$$

- **Limitations:**

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock
- **Today:** Just consider the first order solution

  1. Solve for IRFs for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z})=0 \Rightarrow \boldsymbol{H_U}d\boldsymbol{U}+\boldsymbol{H_Z}d\boldsymbol{Z}=0 \Leftrightarrow d\boldsymbol{U}=\underbrace{-\boldsymbol{H_U^{-1}}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}}d\boldsymbol{Z}$$

  2. Derive all other IRFs for

$$\begin{aligned}\boldsymbol{X}=\boldsymbol{M}(\boldsymbol{U},\boldsymbol{Z}) \Rightarrow d\boldsymbol{X} &= M_{\boldsymbol{U}}d\boldsymbol{U}+M_{\boldsymbol{Z}}d\boldsymbol{Z} \\ &= \underbrace{(-M_{\boldsymbol{U}}\boldsymbol{H_U^{-1}}\boldsymbol{H_Z}+M_{\boldsymbol{Z}})}_{\equiv \boldsymbol{G}}d\boldsymbol{Z}\end{aligned}$$

- **Limitations:**

  1. Imprecise for *large* shocks

Linearized IRFs

- **Previously:** Full non-linear transition path to an MIT-shock
- **Today:** Just consider the first order solution

  1. Solve for IRFs for unknowns

  $$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

  2. Derive all other IRFs for

  $$\begin{aligned} \boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U},\boldsymbol{Z}) \Rightarrow d\boldsymbol{X} &= M_{\boldsymbol{U}} d\boldsymbol{U} + M_{\boldsymbol{Z}} d\boldsymbol{Z} \\ &= \underbrace{(-M_{\boldsymbol{U}}\boldsymbol{H_U}^{-1}\boldsymbol{H_Z} + M_{\boldsymbol{Z}})}_{\equiv \boldsymbol{G}} d\boldsymbol{Z} \end{aligned}$$

- **Limitations:**
  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities*
     (direct in aggregate equations or through micro-behavior)

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**
    1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**
    1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
    2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
  3. Calculate the time series of other aggregate variables as

$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

  where $d\boldsymbol{X}_s$ is the IRF to a unit-shock after s periods

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
  3. Calculate the time series of other aggregate variables as

$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a unit-shock after s periods
- **Intuition:** Sum of first order effects from all previous shocks

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
  3. Calculate the time series of other aggregate variables as

$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a unit-shock after s periods
- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:**
  1. Same result if we linearize all aggregated equations and write the model in MA(∞) form
  2. The state space form can also be recovered (not needed)

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution** (advanced)**:**

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution** (advanced)**:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da^*_{i_g,s} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a^*_{i_g}}{\partial X^{hh}_{s'-s}} dX^{hh}_{s'}.$$

where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution** (advanced)**:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da^*_{i_g,s} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a^*_{i_g}}{\partial X^{hh}_{s'-s}} dX^{hh}_{s'}.$$

     where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)
  2. The policy function can there be simulated as

$$\boldsymbol{a}^*_{i_g} = \sum_{s=0}^{T} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

Advanced linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution** (advanced)**:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da^*_{i_g,s} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a^*_{i_g}}{\partial X^{hh}_{s'-s}} dX^{hh}_{s'}.$$

     where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)
  2. The policy function can there be simulated as

$$\boldsymbol{a}^*_{i_g} = \sum_{s=0}^{T} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

  3. Distribution can then be simulated forwards

The HANK example from GEModelToolsNotebooks II

- **Presentation:** I go through the code for *finding the linearized IRFS and simulating the model*
- **In-class exercise:**
  1. Look at the code and talk about it with the person next to you for 10 minutes
  2. Write at least one question on https://padlet.com/jeppe_druedahl/advmacrohet

Exercise

Exercise = Assignment II

You can start working on Assignment II: The HANK model

Summary

Summary and next week

- **Today:**
  1. A baseline HANK model
  2. Linearized IRFs and simulation
- **Next week:** Analytical Properties of HANK models
- **Homework:**
  1. Work on Assignment II
  2. Read: Auclert et al. (2018), »The Intertemporal Keynesian Cross«