ASSIGNMENT II

Vision: This project teaches you to solve for the *stationary equilibrium* and *transition path* in a heterogeneous agent model and analyze the economic results.

- **Problem:** The problem consists of

  1. A number of questions (page 2)
  2. A model (page 3-4)

- **Code:** The problem is designed to be solved with the *GEModelTools* package.

- **Structure:** Your project should consist of

  1. A single self-contained pdf-file with all results
  2. A single Jupyter notebook showing how the results are produced
  3. Well-documented *.py* files

- **Hand-in:** Upload a single zip-file on Absalon (and nothing else)

- **Deadline:** 24th of November 2023

- **Exam:** Your Assignment II will be a part of your exam portfolio.
  You can incorporate feedback before handing in the final version.

HANC with a Welfare State

a) **Find the stationary equilibrium without a government** ($G_t = L_t^G = \chi_t = 0$)**.** Report the expected discounted utility.
Code is provided as a starting point.

b) **Find optimal welfare policies I.** Choose G_t and L_t^G to maximize expected discounted utility in the stationary equilibrium. Keep $\chi_t = 0$. Report G_t/Y_t.
Hint: You can use that $G_t = \Gamma^G L_t^G$ is always optimal cf. 8

c) **Find optimal welfare policies II.** Repeat b) allowing for $\chi_t \neq 0$. Discuss whether positive or negative transfer are optimal.

d) **Increased TFP.** Repeat question c) with $\Gamma^Y = 1.1$. Comment on the differences.

e) **Transition path**. Compute the transition path from the stationary equilibrium in c) to the one in d). Argue for you choice of policies path of G_t, L_t^G and χ_t.

1. Model

Households. The model has a continuum of infinitely lived households indexed by $i \in [0,1]$. Households are *ex ante* homogeneous. Households choose consumption and how much labor to supply. Savings is in terms of capital, which is rented out to firms at the rental rate, r_t^K. There are no possibilities to borrow. Households are *ex post* heterogeneous in terms of their stochastic labor productivity, z_{it}, and their (end-of-period) savings, a_{it-1}. The distribution of households over idiosyncratic states is denoted $\underline{\boldsymbol{D}}_t$ before shocks are realized and $\boldsymbol{D}_t$ afterwards. The real wage is w_t, and real-profits are Π_t. The government imposes a proportional tax rate, τ_t, and provides real lump-sum transfers is χ_t and a flow of services S_t. Households choose consumption, c_{it}, and labor supply, ℓ_{it}.

The utility function is

$$u(c_{it}, S_t, \ell_{it}) = \frac{c_{it}^{1-\sigma}}{1-\sigma} + \frac{(S_t + \underline{S})^{1-\omega}}{1-\omega} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu}. \tag{1}$$

The household problem is

$$\begin{aligned} v_t(z_{it}, a_{it-1}) &= \max_{c_{it}, \ell_{it}} u(c_{it}, S_t, \ell_{it}) + \beta \mathbb{E}_t \left[v_{t+1}(z_{it+1}, a_{it}) \right] \\ \text{s.t. } a_{it} + c_{it} &= (1 + r_t) a_{it-1} + (1 - \tau_t) w_t z_{it} \ell_{it} + \chi_t + \Pi_t \\ \log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_{it}] = 1 \\ a_{it} &\geq 0. \end{aligned} \tag{2}$$

where $r_t \equiv r_t^K - \delta$. The expected discounted utility is

$$\overline{v}_t = \sum_{k=0}^{\infty} \beta^t \int u(c_{it}, S_t, \ell_{it}) d\boldsymbol{D}_{t+k}. \tag{3}$$

The aggregate quantities of central interest are

$$C_t^{hh} = \int c_{it} d\boldsymbol{D}_t \tag{4}$$

$$L_t^{hh} = \int \ell_{it} z_{it} d\boldsymbol{D}_t \tag{5}$$

$$A_t^{hh} = \int a_{it} d\boldsymbol{D}_t \tag{6}$$

From here on the sub-script i is left out if not strictly necessary.

Firms. A representative firm rents capital, K_{t-1}, and hires labor L_t^Y to produce goods, with the production function

$$Y_t = \Gamma^Y K_{t-1}^{\alpha} (L_t^Y)^{1-\alpha} \tag{7}$$

where Γ^Y is TFP and α is the Cobb-Douglas weight parameter on capital. Capital depreciates with the rate $\delta \in (0,1)$. The real rental price of capital is r_t^K and the real wage is w_t. Profits are $\Pi_t = Y_t - w_t L_t^Y - r_t^K K_{t-1}$. The households own the representative firm in equal shares.

The law-of-motion for capital is $K_t = (1-\delta)K_{t-1} + I_t$.

Government.

The government purchases goods, G_t, and hire labor L_t^G, to produce government services according to

$$S_t = \min\{G_t, \Gamma^G L_t^G\} \tag{8}$$

The government runs a balanced budget each period such that

$$G_t + w_t L_t^G + \chi_t = \int \tau_t w_t \ell d\boldsymbol{D}_t = \tau_t w_t L_t^{hh}$$

Market clearing. Market clearing implies

1. Asset market: $K_t = A_t^{hh}$
2. Labor market: $L_t^Y + L_t^G = L_t^{hh}$
3. Goods market: $Y_t = C_t^{hh} + G_t + I_t$

2. Calibration

1. **Preferences:** $\sigma = \omega = 2$, $\underline{S} = 10^{-8}$, $\varphi = 1.0$, $\nu = 1.0$
2. **Income process:** $\rho_z = 0.96$, $\sigma_\psi = 0.15$,
3. **Production:** $\Gamma^Y = \Gamma^G = 1$, $\alpha_{ss} = 0.30$, $\delta = 0.10$

3. Solving the household problem

The envelope condition implies

$$\underline{v}_t(z_{it}, a_{t-1}) = (1 + r_t^K - \delta)c_{it}^{-\sigma} \tag{9}$$

The first order conditions imply

$$c_{it} = (\beta \mathbb{E}(\underline{v}_{t+1}(z_{it+1}, a_{it}))^{-\frac{1}{\sigma}} \tag{10}$$

$$\ell_{it} = \left(\frac{(1-\tau_{it})w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} c_{it}^{-\sigma/\nu} \tag{11}$$

The household problem can be solved with an extended EGM:

1. Calculate c_{it} and ℓ_{it} over end-of-period states from FOCs
2. Construct endogenous grid $m_{it} = c_{it} + a_{it} - (1-\tau_t)w_t \ell_{it} z_{it}$
3. Use linear interpolation to find consumption $c^*(z_{it}, a_{it-1})$ and labor supply $\ell^*(z_{it}, a_{it-1})$ with $m_{it} = (1+r_t)a_{it-1}$
4. Calculate savings $a^*(z_{it}, a_{it-1}) = (1+r_t)a_{it-1} + (1-\tau_t)w_t \ell_{it}^* z_{it} - c_{it}^*$
5. If $a^*(z_{it}, a_{it-1}) < 0$ set $a^*(z_{it}, a_{it-1}) = 0$ and search for ℓ_{it} such that $f(\ell_{it}) \equiv \ell_{it} - \left(\frac{(1-\tau_t)w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} c_{it}^{-\sigma/\nu} = 0$ holds and

$$c_{it} = (1+r_t)a_{it-1} + (1-\tau_t)w_t \ell_{it} z_{it}$$

. This can be done with a Newton solver with an update from step j to step $j+1$ by

$$\begin{aligned}
\ell_{it}^{j+1} &= \ell_{it}^j - \frac{f(\ell_{it})}{f'(\ell_{it})} \\
&= \ell_{it}^j - \frac{\ell_{it} - \left(\frac{(1-\tau_t)w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} c_{it}^{-\sigma/\nu}}{1 - \left(\frac{(1-\tau_y)w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} (-\sigma/\nu)\, c_{it}^{-\sigma/\nu-1} \frac{\partial c_{it}}{\partial \ell_{it}}} \\
&= \ell_t^j - \frac{\ell_t - \left(\frac{(1-\tau)w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} c_{it}^{-\sigma/\nu}}{1 - \left(\frac{(1-\tau_t)w_{it} z_{it}}{\varphi}\right)^{\frac{1}{\nu}} (-\sigma/\nu)\, c_{it}^{-\sigma/\nu-1}(1-\tau)w_t z_{it}}
\end{aligned}$$

Implementation hints

One way to think about the model is as follows:

1. Let K_t and L_t^Y be the variables we guess on (i.e. unknowns)
2. Let G_t and χ_t to be fully exogenous policy choices
3. Let $L_t^G = \Gamma^G / G_t$ be an implied policy choice (why can we use this?)
4. Let $\tau_t = \frac{G_t + w_t L_t^G}{w_t (L_t^Y + L_t^G)}$ be another implied policy choice (why can we use this?)
5. Ensure the asset market and the labor market both clears

This can be used both for finding the *steady state* and for solving for the *transition path*. For the latter see the code in Listing 1 below. In question b)-c), you should expect to find

b) $\overline{v}_{ss} = -138.97$

c) $\overline{v}_{ss} = -136.89$

d) $\overline{v}_{ss} = -128.84$

```
@nb.njit
def government(par,ini,ss,G,L_Y,w,chi,
    L_G,L,tau,wt,B,S):

    # a. total employment
    L_G[:] = ?
    L[:] = ?

    # b. implied taxation
    tau[:] = ?
    wt[:] = ?

    # c. debt
    B[:] = ?

    # d. service flow
    S[:] = ?

```

Listing 1: Inspiration for blocks.py