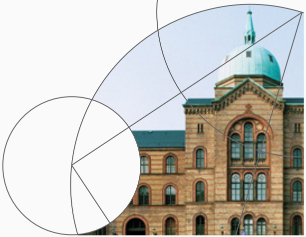
CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

2. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

Jeppe Druedahl & Patrick Moran

2023



Danmarks
Grundforskningsfond
Danish National
Research Foundation

Introduction

Introduction

- **Last time:**
  1. Single agent problem
  2. No interactions (only passive distribution)

Introduction

- **Last time:**
  1. Single agent problem
  2. No interactions (only passive distribution)
- **Today:** Interaction through Walrassian markets

Introduction

- **Last time:**
  1. Single agent problem
  2. No interactions (only passive distribution)
- **Today:** Interaction through Walrassian markets
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model

Introduction

- **Last time:**
  1. Single agent problem
  2. No interactions (only passive distribution)
- **Today:** Interaction through Walrassian markets
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Equilibrium-concept:** *Stationary equilibrium*

Introduction

- **Last time:**
  1. Single agent problem
  2. No interactions (only passive distribution)
- **Today:** Interaction through Walrassian markets
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Equilibrium-concept:** *Stationary equilibrium*
- **Code:** Based on the GEModelTools package
  1. Is in active development
  2. You can help to improve interface
  3. You can help to find bugs
  4. You can help to add features

  Documentation: See GEModelToolsNotebooks

  Original package: SSJ + course *(more complicated back-end)*

Introduction

- **Last time:**
  1. Single agent problem
  2. No interactions (only passive distribution)
- **Today:** Interaction through Walrassian markets
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Equilibrium-concept:** *Stationary equilibrium*
- **Code:** Based on the GEModelTools package
  1. Is in active development
  2. You can help to improve interface
  3. You can help to find bugs
  4. You can help to add features

  Documentation: See GEModelToolsNotebooks

  Original package: SSJ + course *(more complicated back-end)*
- **Literature:** Aiyagari (1994)

Ramsey-recap

Ramsey: Firms

- **Production function:** $Y_t = F(K_{t-1}, L_t)$ [note timing of capital]
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1}, L_t} \Pi_t$
  1. Rental rate: $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^K = F_K(K_{t-1}, L_t)$
  2. Real wage: $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = F_L(K_{t-1}, L_t)$

Zero profits: $\Pi_t = 0 \Rightarrow$

$Y_t = w_t L_t + r_t^K K_{t-1}$ [functional income distribution]

Ramsey: Zero-profit mutual fund

- **Owns all capital**
- **Capital depreciate** with rate $\delta \in (0,1)$,

$$K_t = (1-\delta)K_{t-1} + I_t$$

- **Deposits** (from households), A_{t-1}: The rate of return is

$$r_t = r_t^K - \delta$$

- **Balance sheet:**

$$A_{t-1} = K_{t-1}$$

Ramsey: Households

- **Utility maximization:**

$$v_0(A_{-1}^{hh}) = \max_{\{C_t^{hh}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t^{hh})$$

s.t.

$$A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}$$

Exogenous labor supply: $L_t^{hh} = 1$

- **Euler-equation** (implied by Lagrangian):

$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$
- **Labor market:** $L_t = L_t^{hh} = 1$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$
- **Labor market:** $L_t = L_t^{hh} = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$
- **Labor market:** $L_t = L_t^{hh} = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears

$$
\begin{aligned}
C_t^{hh} + I_t &= \left[(1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - A_t^{hh})\right] + (K_t - (1-\delta)K_{t-1}) \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t)] + (K_t - (1-\delta)K_{t-1}) \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}$$

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}$$

- **Extended form:**

$$\begin{aligned}
r_t^K &= F_K(K_{t-1}, L_t) \\
w_t &= F_L(K_{t-1}, L_t) \\
r_t &= r_t^K - \delta \\
A_t &= K_t \\
A_t^{hh} &= (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \\
u'(C_t^{hh}) &= \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\
A_t &= A_t^{hh} \\
L_t &= L_t^{hh}
\end{aligned}$$

Ramsey: As an equation system

$$
\begin{bmatrix}
r_t^K - F_K(K_{t-1}, L_t) \\
w_t - F_L(K_{t-1}, L_t) \\
r_t - (r_t^K - \delta) \\
A_t - K_t \\
A_t^{hh} - \left((1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\
u'(C_t^{hh}) - \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\
A_t - A_t^{hh} \\
L_t - L_t^{hh} \\
\forall t \in \{0, 1, \dots\}, \text{given } K_{-1}
\end{bmatrix}
= \mathbf{0}
$$

Note I: There is *perfect foresight*.

Note II: This is the so-called *sequence-space* formulation.

Ramsey: Steady state

- **Euler-equation** can be solved for K_{ss}:

$$u'(C_{ss}) = \beta(1 + F_K(K_{ss}, 1) - \delta)u'(C_{ss}) \Leftrightarrow$$
$$F_K(K_{ss}, 1) = \frac{1}{\beta} - 1 + \delta$$

- **Accumulation equation** then implies C_{ss}:

$$K_{ss} = (1 - \delta)K_{ss} + F(K_{ss}, 1) - C_{ss} \Leftrightarrow$$
$$C_{ss} = (1 - \delta)K_{ss} + F(K_{ss}, 1) - K_{ss}$$

HANC

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets

HANC model overview

- **Model blocks:**
    1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
    2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
    3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
    4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households*

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households*
- **Other names:**

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households*
- **Other names:**
  1. The Aiyagari-model

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households*
- **Other names:**
  1. The Aiyagari-model
  2. The Aiyagari-Bewley-Hugget-Imrohoroglu-model

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households*
- **Other names:**
  1. The Aiyagari-model
  2. The Aiyagari-Bewley-Hugget-Imrohoroglu-model
  3. The Standard Incomplete Market (SIM) model

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$\begin{aligned}
\ell_{it} &= z_{it} \\
a_{it} &= (1 + r_t)a_{it-1} + w_t z_{it} - c_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}$$

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$\begin{aligned}
\ell_{it} &= z_{it} \\
a_{it} &= (1 + r_t)a_{it-1} + w_t z_{it} - c_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}$$

- **Where are there heterogeneity?**

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

$$\text{s.t.}$$

$$\begin{aligned}
\ell_{it} &= z_{it} \\
a_{it} &= (1+r_t)a_{it-1} + w_t z_{it} - c_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}$$

- **Where are there heterogeneity?**
    1. *Ex ante* due to different preferences, β_i

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$\begin{aligned}
\ell_{it} &= z_{it} \\
a_{it} &= (1 + r_t) a_{it-1} + w_t z_{it} - c_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}$$

- **Where are there heterogeneity?**
  1. *Ex ante* due to different preferences, β_i
  2. *Ex post* due to stochastic productivity, z_{it}

Heterogeneous households

- **Utility maximization** for household *i*:

$$
\begin{aligned}
v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})
\end{aligned}
$$

$$
\begin{aligned}
&\text{s.t.} \\
\ell_{it} &= z_{it} \\
a_{it} &= (1 + r_t) a_{it-1} + w_t z_{it} - c_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}
$$

- **Where are there heterogeneity?**
  1. *Ex ante* due to different preferences, β_i
  2. *Ex post* due to stochastic productivity, z_{it}
- **Incomplete markets due to borrowing constraint**
  (fancy words: partial self-insurrance, lack of Arrow-Debreu securities)

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

- **Distributions** (vector of probabilities):

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

- **Distributions** (vector of probabilities):

  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

- **Distributions** (vector of probabilities):
  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}
  2. Productivity transition: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$ over β_i, z_{it} and a_{it-1}

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

- **Distributions** (vector of probabilities):

  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}
  2. Productivity transition: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$ over β_i, z_{it} and a_{it-1}
  3. Savings transition: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$ where again

$$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$$

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

- **Distributions** (vector of probabilities):

  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}
  2. Productivity transition: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$ over β_i, z_{it} and a_{it-1}
  3. Savings transition: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$ where again

$$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$$

- **Aggregate consumption and savings:**

$$X_t^{hh} = \int x_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t = X^{hh}\left(\{r_\tau, w_\tau\}_{\tau \geq t}, \underline{\boldsymbol{D}}_0\right) \text{ for } x \in \{a, \ell, c\}$$

Equation system

$$\begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \boldsymbol{D}_t - \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t' \boldsymbol{D}_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \mathbf{0}$$

where $K_{-1} = \int a_{it-1} d\underline{\boldsymbol{D}}_0$

1. **Perfect forsight** wrt. aggregate variables
2. **Stationary equilibrium:** Time-constant solution.
3. **Transition path:** Time-varying solution due to e.g. initial conditions or temporary deviations of exogenous variables.

Solution method

- **Must be solved *numerically*:**
- **Household problem:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$
  1. Discretize and evaluate with interpolation
  2. Make recursion until convergence
- **Transition path:**
  1. Find the stationary equilibrium
  2. Find Jacobian around stationary equilibrium (*next time*)
  3. Solve using quasi-Newton solver (*next time*)

Solution of household problem

- **Beginning-of-period value function:**

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E}\left[v_t(\beta_i, z_{it}, a_{it-1}) \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

Solution of household problem

- **Beginning-of-period value function:**

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E}\left[v_t(\beta_i, z_{it}, a_{it-1}) \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

- **Envelope theorem:** Differentiate with fixed a_t choice

$$\underline{v}_{a,t}(\beta_i, z_{it-1}, a_{it-1}) \equiv \frac{\partial \underline{v}_t}{\partial a_{it-1}} = \mathbb{E}\left[(1 + r_t)c_{it}^{-\sigma} \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

Solution of household problem

- **Beginning-of-period value function:**

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E}\left[v_t(\beta_i, z_{it}, a_{it-1}) \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

- **Envelope theorem:** Differentiate with fixed a_t choice

$$\underline{v}_{a,t}(\beta_i, z_{it-1}, a_{it-1}) \equiv \frac{\partial \underline{v}_t}{\partial a_{it-1}} = \mathbb{E}\left[(1+r_t)c_{it}^{-\sigma} \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

- **EGM:** Separately for each β_i and z_{it}
  1. Find solution from FOC for each $\tilde{a}_{it}$ in exogenous grid

$$\tilde{c}_{it}^{-\sigma} = \beta_i \underline{v}_{a,t+1}(\beta_i, z_{it}, \tilde{a}_{it}) \Leftrightarrow \tilde{c}_{it} = \left(\beta_i \underline{v}_{a,t+1}(\beta_i, z_{it}, \tilde{a}_{it})\right)^{-\frac{1}{\sigma}}$$

  2. Calculate endogenous grid $\tilde{m}_{it} = \tilde{a}_{it} + \tilde{c}_{it}$
  3. Interpolate at $m_{it} = (1+r_t)a_{it-1} + w_t z_{it} + \Pi_t$ to get optimal a_{it}
  4. Enforce constraint by $a_{it} = \max\{a_{it}, 0\}$
  5. Consumption is $c_{it} = m_{it} - a_{it}$

Market clearing

- **Capital market:** $K_t = A_t = \int a_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t$
- **Labor market:** $L_t = \int \ell_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t = \int z_{it} d\boldsymbol{D}_t = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears

$$
\begin{aligned}
C_t^{hh} + I_t &= \int c_{it}^* d\boldsymbol{D}_t + [K_t - (1-\delta)K_{t-1}] \\
&= \int \left[(1+r_t)a_{it-1} + w_t z_{it} - a_{it}\right] d\boldsymbol{D}_t \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t] + [K_t - (1-\delta)K_{t-1}] \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$\begin{bmatrix} r_{ss}^K - (\alpha \Gamma_{ss}(K_{ss}/L_{ss})^{\alpha-1}) \\ w_{ss} - (1-\alpha)\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha} \\ r_{ss} - (r_{ss}^K - \delta) \\ A_{ss} - K_{ss} \\ \boldsymbol{D}_{ss} - \Pi_z' \underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda_{ss}' \boldsymbol{D}_{ss} \\ A_{ss} - A_{ss}^{hh} \\ L_{ss} - L_{ss}^{hh} \end{bmatrix} = \mathbf{0}$$

Note I: Households still move around »inside« the distribution due to idiosyncratic shocks

Note II: Steady state for aggregates (quantities and prices) and the distribution as such

Stationary equilibrium - more verbal definition

For a given Γ_{ss}

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution $\boldsymbol{D}_{ss}$ over β_i, z_{it} and a_{it-1}
4. and the policy functions a^*_{ss}, ℓ^*_{ss} and c^*_{ss}

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Direct implementation

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $r_{ss} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1} - \delta$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$

Direct implementation (alternative)

Root-finding problem in r_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $K_{ss} = \left(\frac{r_{ss}+\delta}{\alpha\Gamma_{ss}}\right)^{\frac{1}{\alpha-1}}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$

Indirect implementation

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{k} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

Code

Calibration

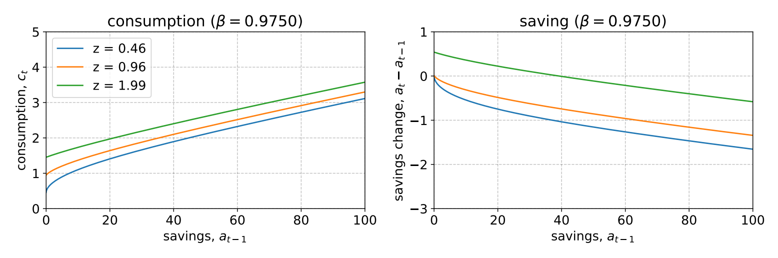
- **Preferences:** $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
    1. Discount factors: $\beta \in \{0.965, 0.975, 0.985\}$ in equal pop. shares
    2. Relative risk aversion: $\sigma = 2$
- **Income:**
    1. AR(1): $\rho_z = 0.95$
    2. Std.: $\sigma_\psi = 0.30\sqrt{(1-\rho_z^2)}$
- **Technology:** $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$
    1. Capital share: $\alpha = 0.36$
    2. TFP: $\Gamma_{ss} = 1.082$
    3. Depreciation: $\delta = 0.193$
- **Steady state:**
    1. Prices: $r_{ss} = 0.01$ and $w_{ss} = 1$
    2. Quantities: $K_{ss}/Y_{ss} = 1.776$

Consumption function

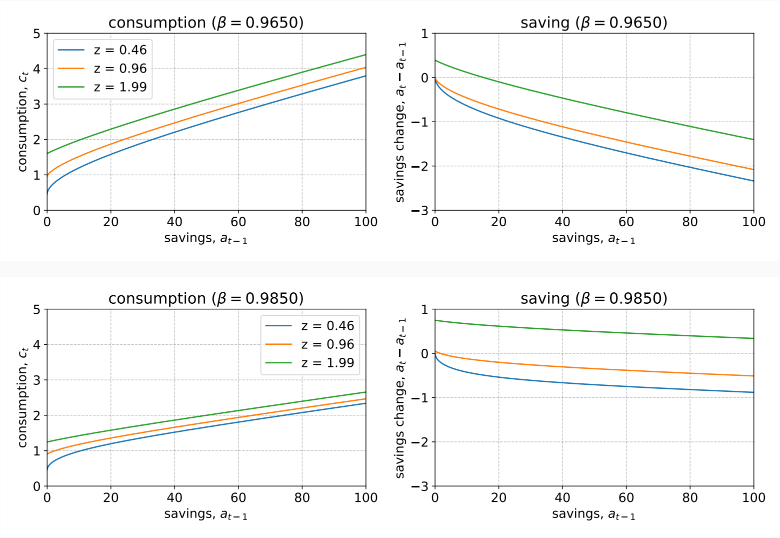
- **Euler-equation still necessary for $a_{it} > 0$:**

$$c_{it}^{-\sigma} = \beta_i (1 + r_{t+1}) \mathbb{E}_t \left[c_{it+1}^{-\sigma} \right]$$

- **Precautionary saving:**
  1. Low consumption for low cash-on-hand $\rightarrow$ *buffer-stock target*
  2. Steep slope for low cash-on-hand $\rightarrow$ *high MPC*

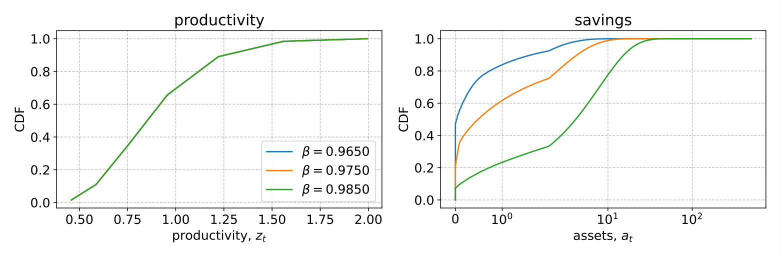


Low vs. high β_i



Distribution, D_t

- **Productivity:** Marginal distribution over only z_{it}
- **Savings:** Marginal distribution over a_{it} cond. on β_i



- **Drivers of wealth inequality:**
  1. Stochastic income
  2. Heterogeneous patience $\rightarrow$ savings behavior

Steady state interest rate

- **Representative agent / complete markets:**
  Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$

Steady state interest rate

- **Representative agent / complete markets:**
  Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$

- **Heterogeneous agents:** *No such equation exists*
  1. Euler-equation replaced by asset market clearing condition
  2. Idiosyncratic income risk affects the steady state interest rate

σ_ψ	PE ($r_{ss} = 1\%$), A^{hh}	GE, r_{ss}	GE, A^{hh}
0.09	2.78	1.00%	2.78
0.14	7.39	0.12%	2.97
0.19	13.68	-1.11%	3.30

Partial Equilibrium: Same interest rate.

General Equilibrium: Capital+labor market clearing.

Calibration

How to choose parameters?

- **External calibration:** Set subset of parameters to the *standard values in the literature* or *directly from data estimates* (e.g. income process)

How to choose parameters?

- **External calibration:** Set subset of parameters to the *standard values in the literature* or *directly from data estimates* (e.g. income process)
- **Internal calibration:** Set remaining parameters so the model fit a number of chosen *macro-level and/or micro-level targets* based on empirical estimates
    1. **Informal:** Roughly match targets by hand
    2. **Formal:**
        - 2a. Solve root-finding problem
        - 2b. Minimize a squared loss function
    3. **Estimation:** Formal with squared loss function + standard errors

How to choose parameters?

- **External calibration:** Set subset of parameters to the *standard values in the literature* or *directly from data estimates* (e.g. income process)
- **Internal calibration:** Set remaining parameters so the model fit a number of chosen *macro-level and/or micro-level targets* based on empirical estimates
    1. **Informal:** Roughly match targets by hand
    2. **Formal:**
        2a. Solve root-finding problem
        2b. Minimize a squared loss function
    3. **Estimation:** Formal with squared loss function + standard errors
- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

Exercise: HANCGovModel

- **No production.** No physical savings instrument
- **Households:** Get stochastic endowment z_{it} of consumption good
- **Government:**
  1. Choose government spending
  2. Collect taxes, τ_t, proportional to endowment
  3. Bonds: Pays 1 consumption good next period. Price is $p_t^B < 1$

$$p_t^B B_t = B_{t-1} + G_t - \int \tau_t z_{it} d\boldsymbol{D}_t$$

Exercise: Households

Households:

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{it+1}(z_{it+1}, a_{it}) \right]$$

$$\text{s.t. } p_t^B a_{it} + c_{it} = a_{it-1} + (1-\tau_t) z_{it} \geq 0$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1} \;\; , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \; \mathbb{E}[z_{it}] = 1$$

Euler-equation:

$$c_t^{-\sigma} = \beta \frac{\underline{v}_{a,t+1}(z_{it}, a_{it})}{p_t^B}$$

Envelope condition:

$$\underline{v}_{a,t}(z_{it-1}, a_{it-1}) = c_{it}^{-\sigma}$$

Exercise: Questions

1. **Define the stationary equilibrium**
2. **Solve and simulate the household problem**
   with $p^B_{ss} = 0.975$ and $\tau_{ss} = 0.12$.
3. **Find the stationary equilibrium**
   with $G_{ss} = 0.10$ and $\tau_{ss} = 0.12$.
4. **What happens for** $\tau_{ss} \in (0.11, 0.15)$?
5. **When is average household utility maximized?**

Summary

Summary and next week

- **Today:**
  1. The concept of a stationary equilibrium
  2. Introduction to the GEModelTools package
- **Next week:** *Transition path*
- **Homework:**
  1. Work on completing the HANCGovModel exercise (solution in repository folder HANCGovModel)
  2. Read documentation for GEModelTools (*except on linearized solution and simulation*)
  3. Skim-read at Auclert et. al. (2021) and Kirkby (2017)