CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

7. Transition Path

Adv. Macro: Heterogenous Agent Models

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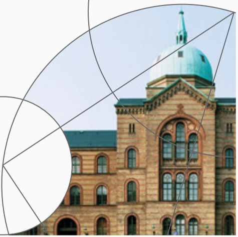

Introduction

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- **Code:**
  1. Based on the GEModelTools package
  2. Examples from GEModelToolsNotebooks/HANC
     (except stuff on *linearized solution* and *simulation*)

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- **Code:**
  1. Based on the GEModelTools package
  2. Examples from GEModelToolsNotebooks/HANC
     (except stuff on *linearized solution* and *simulation*)
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools
     (except stuff on *linearized solution* and *simulation*)
  3. Kirkby (2017)

Transition path

Equation system

The model can be written as an **equation system**

$$\boldsymbol{H}\left(\{K_t, L_t; \Gamma_t\}_{t\geq 0}, \underline{\boldsymbol{D}}_0\right) = \begin{bmatrix} r_t - (\alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} - \delta) \\ w_t - (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t - \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t' \boldsymbol{D}_t \\ K_t - \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t - 1 \\ \forall t \in \{0, 1, \dots\} \end{bmatrix} = \boldsymbol{0}$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $a_t^* = a^*\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$
2. Choice transition: $\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over z_t and a_{t-1}
4. and the policy functions $\{a_t^*(z_t, a_{t-1})\}$ and $\{c_t^*(z_t, a_{t-1})\}$

are such that

1. Firms maximize profits (prices) in all periods
2. Household maximize expected utility (policy functions) in all periods
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. The capital market clears in all periods
5. The labor market clears in all periods
6. The goods market clears in all periods

Truncated, reduced vector form:

$$\boldsymbol{H}(\boldsymbol{K},\boldsymbol{\Gamma},\underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t^{hh} - K_t \\ \forall t \in \{0,1,\ldots,T-1\} \end{bmatrix} = \boldsymbol{0}$$

where $\boldsymbol{K} = (K_0, K_1, \ldots, K_{T-1})$ and $\boldsymbol{\Gamma} = (\Gamma_0, \Gamma_1, \ldots, \Gamma_{T-1})$ and

$$\begin{aligned} L_t &= 1 \\ r_t &= \alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} - \delta \\ w_t &= (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ &\forall t \in \{0,1,\ldots,T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Could we solve it with a Newton method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$.
3. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below chosen tolerance
4. Calculate the Jacobian $\boldsymbol{H}_{\boldsymbol{K}}^i = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$
5. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i + \left(\boldsymbol{H}_{\boldsymbol{K}}^i\right)^{-1} \boldsymbol{H}^i$
6. Increment i and return to step 2

Question: What is the problem?

Alternative: Use Broydens method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate the steady state Jacobian $\boldsymbol{H}_{\boldsymbol{K},ss} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}_{ss}, \boldsymbol{\Gamma}_{ss})$
3. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$.
4. Calculate Jacobian by
$$\boldsymbol{H}_{\boldsymbol{K}}^i = \begin{cases} \boldsymbol{H}_{\boldsymbol{K},ss} & \text{if } i = 0 \\ \boldsymbol{H}_{\boldsymbol{K}}^{i-1} + \frac{(\boldsymbol{H}^i - \boldsymbol{H}^{i-1}) - \boldsymbol{H}_{\boldsymbol{K}}^{i-1}(\boldsymbol{K}^i - \boldsymbol{K}^{i-1})}{\left|\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right|_2} \left(\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
5. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below tolerance
6. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i + \left(\boldsymbol{H}_{\boldsymbol{K}}^i\right)^{-1} \boldsymbol{H}^i$
7. Increment i and return to step 3

Question: What are the benefits?

Bottleneck: How do we find the Jacobian?

1. **Naive approach:** For each $s \in \{0, 1, ..., T-1\}$ do
   1.1 Set $K_t = K_{ss} + \mathbf{1}\{t = s\} \cdot \Delta, \qquad \Delta = 10^{-4}$
   1.2 Find $\boldsymbol{r}$ and $\boldsymbol{w}$
   1.3 Solve household problem backwards along transition path
   1.4 Simulate households forward along transition path
   1.5 Calculate $\frac{\partial H_t}{\partial K_s} = \frac{(A_t^{hh} - K_t) - (A_{ss}^{hh} - K_{ss})}{\Delta}$ for all t

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2. **Fake news algorithm:** From household Jacobian to full Jacobian

$$\boldsymbol{H_K} = \mathcal{J}^{A^{hh},r}\mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K} - \boldsymbol{I}$$

   2.1 $\mathcal{J}^{r,K}, \mathcal{J}^{w,K}$: Fast from the onset - only involve aggregates
   2.2 $\mathcal{J}^{A^{hh},r}, \mathcal{J}^{A^{hh},w}$: Only requires T solution steps and simulation steps!

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- **Underlying assumption:** No aggregate uncertainty
- **»Shock«, Γ:** A fully unexpected non-recurrent event $\equiv$ *MIT shock*
- **Transition path, $\boldsymbol{K}$:** Non-linear perfect foresight response to
  1. Initial distribution, $\underline{\boldsymbol{D}}_0 \neq \boldsymbol{D}_{ss}$, *or to*
  2. Shock, $\Gamma_t \neq \Gamma_{ss}$ for some t (i.e. impulse-response)

The HANC example from GEModelToolsNotebooks

- **Presentation:** I go through the code
- **In-class exercise:**
  1. Look at the code and talk about it with the person next to you for 10 minutes
  2. Write at least one question on https://padlet.com/jeppe_druedahl/advmacrohet

Decomposition of GE response

- **GE transition path:** $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$
- **PE response of each:**
  1. Set $(\boldsymbol{r}, \boldsymbol{w}) \in \{(\boldsymbol{r}^*, \boldsymbol{w}_{ss}), (\boldsymbol{r}_{ss}, \boldsymbol{w}^*)\}$
  2. Solve household problem backwards along transition path
  3. Simulate households forward along transition path
  4. Calculate outcomes of interest
- **Additionally:** We can vary the initial distribution, $\underline{\boldsymbol{D}}_0$, to find the response of sub-groups

DAGs

General model class I

1. Time is discrete (index t).
2. There is a continuum of households (index i, when needed).
3. There is *perfect foresight* wrt. all aggregate variables, $\boldsymbol{X}$, indexed by $\mathcal{N}$, $\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t=0}^{\infty} = \{\boldsymbol{X}^j\}_{j\in\mathcal{N}} = \{X_t^j\}_{t=0,j\in\mathcal{N}}^{\infty}$, where $\mathcal{N} = \mathcal{Z} \cup \mathcal{U} \cup \mathcal{O}$, and $\mathcal{Z}$ are *exogenous shocks,* $\mathcal{U}$ are *unknowns*, $\mathcal{O}$ are outputs, and $\mathcal{H} \in \mathcal{O}$ are *targets.*
4. The model structure is described in terms of a set of *blocks* indexed by $\mathcal{B}$, where each block has inputs, $\mathcal{I}_b \subset \mathcal{N}$, and outputs, $\mathcal{O}_b \subset \mathcal{O}$, and there exists functions $h^o(\{\boldsymbol{X}^i\}_{i\in\mathcal{I}_b})$ for all $o \in \mathcal{O}_b$.
5. The blocks are *ordered* such that (i) each output is *unique* to a block, (ii) the first block only have shocks and unknowns as inputs, and (iii) later blocks only additionally take outputs of previous blocks as inputs. This implies the blocks can be structured as a *directed acyclical graph* (DAG).

DAG: Directed Acyclical Growth

From Auclert et al. (2021):

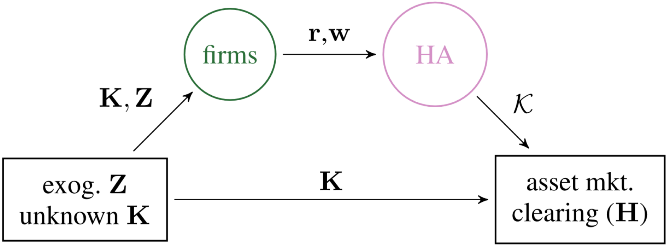

FIGURE 3.—DAG representation of Krusell–Smith economy.

General model class II

6. The number of targets are equal to the number of unknowns, and an *equilibrium* implies $\boldsymbol{X}^o = 0$ for all $o \in \mathcal{H}$. Equivalently, the model can be summarized by an *target equation system* from the unknowns and shocks to the targets,

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0},$$

and an *auxiliary model equation* to infer all variables

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z}).$$

A *steady state* satisfy

$$\boldsymbol{H}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}) = 0 \text{ and } \boldsymbol{X}_{ss} = \boldsymbol{M}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}).$$

General model class III

7. The *discretized household block* can be written recursively as

$$
\begin{aligned}
\boldsymbol{v}_t &= v(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\underline{\boldsymbol{v}}_t &= \Pi(\boldsymbol{X}_t^{hh})\boldsymbol{v}_t \\
\boldsymbol{D}_t &= \Pi(\boldsymbol{X}_t^{hh})'\underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
\boldsymbol{a}_t^* &= \boldsymbol{a}^*(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\boldsymbol{Y}_t^{hh} &= \boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
&\underline{\boldsymbol{D}}_0 \text{ is given,} \\
&\boldsymbol{X}_t^{hh} = \{\boldsymbol{X}_t^i\}_{i\in\mathcal{I}_{hh}}, \boldsymbol{Y}_t^{hh} = \{\boldsymbol{X}_t^o\}_{o\in\mathcal{O}_{hh}},
\end{aligned}
$$

   where $\boldsymbol{Y}_t$ is aggregated outputs with $\boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})$ as individual level measures.

8. Given the sequence of shocks, $\boldsymbol{Z}$, there exists a *truncation period*, T, such all variables return to steady state beforehand.

Fake News Algorithm

Fake news algorithm

Go through Section 3 of the documentation for GEModelTools

Exercises

Exercises: Model extensions

1. **Firms:** Unchanged
2. **Households:** New budget constraint with labor taxes, τ_t

$$a_t + c_t = (1 + r_t)a_{t-1} + (1 - \tau_t)w_t z_t \geq 0$$

3. **Government:** Set taxes and government consumption, and government bonds follows the law-of-motion

$$B_{t+1} = (1 + r_t)B_t + G_t - \int \tau_t z_t d\boldsymbol{D}_t$$

4. **Asset market clearing:** $K_t + B_t = \int a_t^*(z_t, a_{t-1})d\boldsymbol{D}_t$
5. **Define and find the stationary equilibrium and transition path**
6. **What is the response to a persistent shock to G_t when the budget is balanced in all periods?**

Summary

Summary and next week

- **Today:**
  1. The concept of a transition path
  2. Details of the GEModelTools package
- **Next week:** More on interesting heterogeneous agent models
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read: Auclert et. al. (2021), »Demographics, Wealth, and Global Imbalances in the Twenty-First Century«