CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

11. Introducing HANK

Adv. Macro: Heterogenous Agent Models

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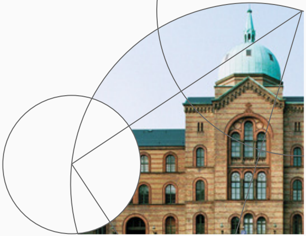
Danmarks
Grundforskningsfond
Danish National
Research Foundation



Introduction

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- **Today:**
  1. Linearized Impulse Response Function (IRF)
  2. Linearized simulation with aggregate risk
- **Relevance:** Business cycle analysis
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Boppart et. al. (2018), »Exploiting MIT shocks in heterogeneous-agent economies: The impulse response as a numerical derivative«
  3. Documentation for GEModelTools

IRFs and simulation

Reminder of model class

- **Unknowns:** $\boldsymbol{U}$
- **Shock:** $\boldsymbol{Z}$
- **Additional variables:** $\boldsymbol{X}$
- **Target equation system:**

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0}$$

- **Auxiliary model equations:**

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z})$$

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  2. Derive all other IRFs for

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- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**
  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities* (direct in aggregate equations or through micro-behavior)

Aggregate risk (dynamic equilibrium)

- **Aggregate stochastic variables:** $\boldsymbol{Z}$ follow some known process with innovations ϵ. *State space form*: RHS is what is known today

$$\begin{bmatrix} \underline{\boldsymbol{D}}_{t+1} \\ \boldsymbol{X}_t \\ \boldsymbol{Z}_t \end{bmatrix} = \mathcal{M}\left(\begin{bmatrix} \underline{\boldsymbol{D}}_t \\ \boldsymbol{X}_{t-1} \\ \boldsymbol{Z}_{t-1} \end{bmatrix}, \boldsymbol{\epsilon}_t \right)$$

$\neq$ perfect foresight wrt. future agg. variables in *sequence-space*

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- **Observation:** Linearization of aggregate variables imply *certainty equivalence* with respect to these

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- **Insight:** *The IRF from an MIT shock is <u>equivalent</u> to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)

Comparisons

- **State-space approach with linearization**: Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)
  Con:
  1. Harder to implement in my view
  2. Valuable to be able to interpret Jacobians

  Pro**:**
  1. More similar to standard approaches for RBC and NK models
  2. Easier path to 2nd and higher order approximations
- **Global solution:** The distribution of households is a state variable for each household $\Rightarrow$ *explosion in complexity*
  1. Original: Krusell and Smith (1997, 1998); Algan et al. (2014);
  2. Deep learning: Fernández-Villaverde et al. (2021); Maliar et al. (2021); Han et al. (2021); Kase et al. (2022); Azinovic et al. (2022); Gu et al. (2023); Chen et al. (2023)
- **Discrete aggregate risk:** Lin and Peruffo (2023)

Example: Global HANC (Krusell-Smith)

- **Recursive formulation of household problem:**

$$v(\boldsymbol{D}_t, \Gamma_t, z_{it}, a_{it-1}) = \max_{a_{it}, c_{it}} u(c_{it}) + \beta \mathbb{E}_t \left[v(\boldsymbol{D}_{t+1}, \Gamma_{t+1}, z_{it+1}, a_{it}) \right]$$

$$\text{s.t.}$$

$$\begin{aligned}
K_{t-1} &= \int a_{it-1} d\boldsymbol{D}_t \\
r_t &= \alpha \Gamma_t K_{t-1}^{\alpha-1} - \delta \\
w_t &= (1-\alpha)\Gamma_t K_{t-1}^{\alpha} \\
a_{it} + c_{it} &= (1+r_t)a_{it-1} + w_t z_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0,
\end{aligned}$$

- **Problem:** How to discretize $\boldsymbol{D}_t$?
  Note: $\boldsymbol{D}_t$ needed directly for K_{t-1} and indirectly for $K_t, K_{t+1} \ldots$

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.

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$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T-1} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

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- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:** Same result if we linearize all aggregated equations and write the model in MA(∞) form

Generalized linearized simulation [advanced]

- **Generality:** Add auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations

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- **Full distribution:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da_{i_g,s}^{*} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a_{i_g}^{*}}{\partial X_{s'-s}^{hh}} dX_{s'}^{hh}.$$

  where $\partial \boldsymbol{a}_{i_g}^{*} / \partial X_k^{hh}$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

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  2. The policy function can there be simulated as

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  2. The policy function can there be simulated as

$$\boldsymbol{a}^*_{i_g,t} = \sum_{s=0}^{T-1} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

  3. Distribution can then be simulated forwards

Calculating moments - variance

- **Identical and independent distributed innovations:**

$$\mathbb{E}\left[\epsilon_t^i \epsilon_{t'}^j\right] = \begin{cases} \sigma_i & \text{if } t = t' \text{ and } i = j \\ 0 & \text{el} \end{cases}$$

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- **Calculating moments such as var**(dC_t) from the IRFs:

$$\begin{aligned} \text{var}(dC_t) &= \mathbb{E}\left[\left(\sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} dC_s \epsilon_{t-s}^i\right)^2\right] \\ &= \sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} \mathbb{E}\left[\epsilon_{t-s}^i \epsilon_{t-s}^i\right] \left(dC_s^i\right)^2 \\ &= \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1} \left(dC_s^i\right)^2 \end{aligned}$$

where dC_s^i is the IRF to a unit-shock to i after s periods and σ_i is the standard deviation of shock i

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

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- **Covariance decomposition:**

$$\frac{\text{contribution from one shock}}{\text{contributions from all shocks}} = \frac{\sigma_j^2 \sum_{s=0}^{T-1-k} dC_s^j dY_{s+k}^j}{\sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i}$$

Estimation

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- **Also possible:** *Bayesian likelihood estimation* (see SSJ)
- **Speed:** For a new set of parameters?
  1. Only shock processes change $\Rightarrow$ *same Jacobians* ($\boldsymbol{G_U}$, $\boldsymbol{G}$)
  2. Only need to re-compute Jacobian of aggregate variables? (only single block?)
  3. Also need to re-compute Jacobian of household problem?
  4. Also need to find stationary equilibrium again?

Sticky prices

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  2. Supply labor and choose consumption
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- **Central bank**: Set nominal interest rate

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- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

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- **Demand curve** derived from FOC wrt. y_{jt}

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- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- **FOC wrt.** y_{jt}

$$
\begin{aligned}
0 &= P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu} - 1} - p_{jt} \Leftrightarrow \\
\frac{p_{jt}}{P_t} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow \\
\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu - 1}} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow \\
y_{jt} &= \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t
\end{aligned}
$$

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

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$$\text{s.t. } y_{jt} = \Gamma_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

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Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right) \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1} \frac{w_t}{\Gamma_t} \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$

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$$+ \frac{\mu}{\mu - 1}\frac{1}{\kappa}\log\left(1 + \pi_t\right)\frac{Y_t}{P_t} + \frac{\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(1 + \pi_{t+1}\right)\frac{Y_{t+1}}{P_t}}{1 + r_{t+1}}$$

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$$\log(1+\pi_t) = \kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log\left(1+\pi_{t+1}\right)}{1+r_{t+1}}$$

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Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_{it} and a_{it-1}

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{it+1}, a_{it}) \right]$$

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EGM I

- **Beginning-of-period value function:**

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$$
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  3. Return to step 1

Government and central bank

- **Monetary policy:** Follow Taylor-rule:

$$i_t = i_t^* + \phi\pi_t + \phi^Y(Y_t - Y_{ss})$$

where i_t^* is a shock

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- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Assets: $B_{ss} = \int a_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t$
2. Labor: $N_t = \int n_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t$ (in effective units)
3. Goods: $Y_t = \int c_t^*(z_{it}, a_{it-1}) d\boldsymbol{D}_t + G_t + \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log(1+\pi_t)\right]^2 Y_t$

As an equation system

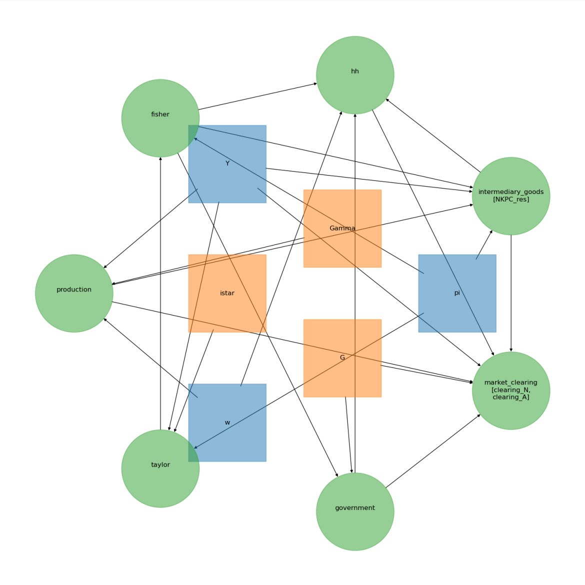
$$\boldsymbol{H}\left(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^{*}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0\right) = \mathbf{0}$$

$$\begin{bmatrix} \log(1+\pi_t) - \left[\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1}}{1+r_{t+1}})\right] \\ N_t - \int n_t^{*}(z_{it}, a_{it-1}) d\boldsymbol{D}_t \\ B_{ss} - \int a_t^{*}(z_{it}, a_{it-1}) d\boldsymbol{D}_t \end{bmatrix} = \mathbf{0}$$

The rest of the model is given by

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^{*}, \boldsymbol{\Gamma})$$

As a DAG



hh
fisher
Y
intermediary_goods
[NKPC_res]
Gamma
production
istar
pi
G
market_clearing
[clearing_N,
clearing_A]
w
taylor
government

Steady state

- **Chosen:** B_{ss}, G_{ss}, r_{ss}
- **Analytically:**
  1. **Normalization:** $Z_{ss} = N_{ss} = 1$
  2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i^*_{ss} = (1 + r_{ss})(1 + \pi_{ss}) - 1$
  3. **Firms:** $Y_{ss} = Z_{ss} N_{ss}$, $w_{ss} = \frac{Z_{ss}}{\mu}$ and $d_{ss} = Y_{ss} - w_{ss} N_{ss}$
  4. **Government:** $\tau_{ss} = r_{ss} B_{ss} + G_{ss}$
  5. **Assets:** $A_{ss} = B_{ss}$
- **Numerically:** Choose β and φ to get market clearing

Transmission mechanism to monetary policy shock

1. **Monetary policy shock:** $i_t^* \downarrow \Rightarrow i_t = i_t^* + \phi\pi_t \downarrow$
2. **Real interest rate:** $r_t = \frac{1+i_{t-1}}{1+\pi_t} \downarrow$
3. **Taxes:** $\tau_t = r_t B_{ss} \downarrow$
4. **Household consumption**, $C_t^{hh} \uparrow$, due to $r_t \downarrow$ and $\tau_t \downarrow$
5. **Firms production,** $Y_t \uparrow$, and **labor demand**, $N_t \uparrow$
6. **Inflation,** $\pi_t \uparrow$, and **wage**, $w_t \uparrow$ and **dividends**, $d_t \downarrow$
7. **Household labor supply**, $N_t^{hh} \uparrow$, due to $w_t \uparrow$ and $d_t \downarrow$, but dampened $\tau_t \downarrow$
8. **Nominal rate**, $i_t \uparrow$ due to $\pi_t \uparrow$ implying $r_t \uparrow$
9. **Household consumption**, $C_t^{hh} \uparrow$, due to $w_t \uparrow$ but dampened by $d_t \downarrow$ and $r_t \uparrow$

Representative agent

- **Replace market clearing conditions with FOCs:**

$$C_t^{-\sigma} = \beta(1 + r_{t+1}) C_{t+1}^{-\sigma}$$
$$\varphi N_t^{\nu} = w_t C_t^{-\sigma}$$

- From resource constraint: $C_t = Y_t - G_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log\left(1 + \pi_t\right)\right]^2 Y_t$
- Ensure same steady state: $\beta^{RA} = \frac{1}{1+r_{ss}}, \ \varphi^{RA} = \frac{w_{ss}\left(C_{ss}^{hh}\right)^{-\sigma}}{\left(N_{ss}\right)^{\nu}}$
- **Intertemporal budget constraint:**

$$C_0 + \frac{C_1}{1 + r_1} + \ldots = (1 + r_0)\, A_{-1} + Y_0^{RA} + \frac{Y_1^{RA}}{1 + r_1} \ldots$$

where $Y_t^{RA} = w_t N_t + d_t - \tau_t$ is household income

Summary

- **Today:**
  1. Aggregate risk and linearized dynamics (IRF and simulation)
  2. Calculating aggregate moments (for calibration or estimation)
  3. HANK with sticky prices

Exercise

Exercise

1. Compute the non-linear response to a temporary increase in government spending
2. Compute the linearized IRF to the same shock and compare
3. Sketch the transmission mechanism of government spending
4. Analyze how the aggressiveness of monetary policy affects the effectiveness of fiscal policy
5. Compare you previous results with the effects of a public transfer

Summary

Summary and next week

- **Today:**
  1. Aggregate risk and linearized dynamics (IRF and simulation)
  2. Calculating aggregate moments (for calibration or estimation)
  3. A baseline HANK (sticky prices)
- **Next week:** More on HANK models
- **Homework:**
  1. Work on exercise
  2. Skim-read Auclert et al. (2023),
     »The Intertemporal Keynesian Cross«