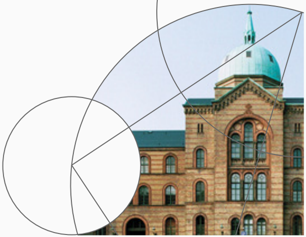
Center for Economic Behavior & Inequality

# 1. Introduction

## Adv. Macro: Heterogenous Agent Models

Jeppe Druedahl & Patrick Moran

2023



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  **Prerequisite:** *Intro. to Programming and Numerical Analysis*
  **Complicated:** *Close to the research frontier*

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  **Complicated:** *Close to the research frontier*
- **Plan for today:**
  1. More about the course
  2. Consumption-saving models
  3. Numerical dynamic programming

# Macroeconomic Models with Heterogeneous Agents

- **Model components:**
  1. Optimizing individual agents (households + firms)
  2. Idiosyncratic and aggregate risk
  3. Information flows (who knows what when $\Rightarrow$ often everything)
  4. Market clearing (Walras vs. search-and-match)

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- **HANK:** Heterogeneous Agent *New Keynesian* model
  (i.e. include price and wage setting frictions)

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  Web: sites.google.com/view/numeconcph-advmacrohet/

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- **Code:**
  1. We provide code you will build upon
  2. Based on the GEModelTools package

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- **Exam:**
  1. Hand-in 3×**assignments**
  2. **48 hour take-home:** Programming of new extension
     + analysis of model + interpretation of results

# Python

1. **Assumed knowledge:** From Introduction to Programming and Numerical Analysis you are assumed to know the basics of
   1.1 Python
   1.2 VSCode
   1.3 git
2. **Updated Python:** Install (or re-install) newest Anaconda
3. **Packages:** `pip install quantecon, EconModel, consav`
4. **GEMoodel tools:**
   4.1 Clone the `GEModelTools` repository
   4.2 Locate repository in command prompt
   4.3 Run `pip install -e .`

# Course plan

*See CoursePlan.pdf in repository*

# Knowledge

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

# Skills

1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

# Competencies

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

# Consumption-Saving

# Generations of models

1. Permanent income hypothesis (Friedman, 1957) or life-cycle model (Modigliani and Brumburg, 1954)
2. Buffer-stock consumption model (Deaton, 1991, 1992; Carroll, 1992, 1997)
3. Multiple-asset buffer-stock consumption models (e.g. Kaplan and Violante (2014))

# Consumption-saving

$$v_0(a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**
  Consumption: $c_t$
  Productivity: $z_t$
  End-of-period savings: $a_t$ (*no debt at death*)
- **Parameters:**
  Discount factor: $\beta$
  Wage: $w$
  Interest rate: $r$ (define $R \equiv 1 + r$ as interest factor)

# It is a *static* problem

$$v_0(a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **It is a *static* problem:**
  1. **Information:** $z_t$ is known for all $t$ at $t = 0$
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
  3. **Behavior:** Choose $c_0, c_1, \dots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \dots, c_{T-1}^*$

# IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$
\begin{aligned}
a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
&= R^2 a_{T-3} + Rwz_{T-2} - Rc_{T-2} + wz_{T-1} - c_{T-1} \\
&= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t}(wz_t - c_t)
\end{aligned}
$$

# IBC

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- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$R^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

# FOC and Euler-equation

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[ \sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t u'(c_t) - \lambda (1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda \left(\beta R\right)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda \left(\beta R\right)^{-t}}{-\lambda \left(\beta R\right)^{-(t+k)}} = \left(\beta R\right)^k$$

# Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned} \sum_{t=0}^{T-1} R^{-t} (\beta R)^{t/\sigma} c_0 &= s_0 + h_0 \Leftrightarrow \\ c_0^* &= \frac{1 - \frac{(\beta R)^{1/\sigma}}{R}}{1 - \left(\frac{(\beta R)^{1/\sigma}}{R}\right)^T} (s_0 + h_0) \end{aligned}$$

# Infinite horizon

- **Infinite horizon** for $\beta R < 1$: Let $T \to \infty$ to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)$$

- **Interesting properties** are e.g.:
    1. Interest rate sensitivity: $\frac{\partial c_0}{\partial r}$
    2. MPC of permanent income change: $\frac{\partial c_0}{\partial w}$
    3. MPC of future income: $\frac{\partial c_0}{\partial z_t}$
    4. MPC of windfall income: $\frac{\partial c_0}{\partial s_0}$
- **No borrowing constraints or uncertainty**
- **Other simplifications:** No age life-cycle, bequests etc.

# Liquidity/borrowing constraints

- Implied period 0 **savings** are:

$$a_0 = Ra_{-1} + wz_0 - c_0$$

- **Borrowing constraint:** $a_0 \geq -w \cdot b$
- **Maximum consumption:** $\overline{c}_0 = Ra_{-1} + wz_0 + wb$
- **Optimal consumption:** Constrained or unconstrained.

$$c_0^* = \min\left\{\overline{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)\right\}$$

- **Empirical realism.** Incl. high MPC of constrained.
- **Technical issue:** Borrowing constraints further in the future complicated the analytical solution considerably.

# Uncertainty

$$v_0(a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

s.t.

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$z_t \sim \mathcal{Z}(z_{t-1})$$

$$a_t \geq -wb$$

$$\lim_{t\to\infty}(1+r)^{-t}a_t \geq 0 \quad \text{[No-Ponzi game]}$$

- **Stochastic income** from 1st order Markov-process, $\mathcal{Z}$
- ***A true dynamic problem:***
  1. **Information:** $z_t$ is revealed period-by-period
  2. **Target:** *Expected* discounted utility, $\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$
  3. **Behavior:** Choose $c_t$ *sequentially* as information is revealed
  4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

# IBC

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \quad \Leftrightarrow \quad b_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

- **What if $T \to \infty$?** We must have $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} = 0$
    1. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
    2. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \to \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t} c_t = R a_{-1} + \sum_{t=0}^{\infty} R^{-t} w z_t$$

# Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t\left[u'(c_{t+1})\right]$:
  Increase $c_t$ by marginal $\Delta > 0$, and lower $c_{t+1}$ by $R\Delta$
  1. **Feasible:** Yes, if $a_t > -\underline{a}$
  2. **Utility change:** $u'(c_t) + \beta\,(-R)\,\mathbb{E}_t\left[u'(c_{t+1})\right] > 0$

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- **Case II:** If $u'(c_t) < \beta(1+r) \mathbb{E}_t [u'(c_{t+1})]$:
  Lower $c_t$ by marginal $\Delta > 0$, and increase $c_{t+1}$ by $R\Delta$
  1. **Feasible:** Yes (always)
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- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = -\underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t \left[u'(c_{t+1})\right]$, or
  2. **Unconstrained:** $a_t > -\underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t \left[u'(c_{t+1})\right]$

# Special case I: Quadratic utility

- **Quadratic utility:** $u(c_t) = -\frac{1}{2}(\overline{c} - c)^2$ with $\beta R = 1$
- **Euler-equation:** *Consumption = expected future consumption*

$$(\overline{c} - c_t) = \mathbb{E}_t\left[(\overline{c} - c_{t+k})\right] \Leftrightarrow c_t = \mathbb{E}_t\left[c_{t+k}\right]$$

- Use **IBC** in expectation to get **consumption function:**

$$\sum_{t=0}^{\infty} R^{-t}\mathbb{E}_0\left[c_t\right] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w\mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c_0 = ra_{-1} + \frac{r}{R}\sum_{t=0}^{T} R^{-t} w\mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c^*(a_{t-1}, z_t) = ra_{t-1} + \frac{r}{R}\sum_{k=0}^{\infty} R^{-k} w\mathbb{E}_t\left[z_{t+k}\right]$$

- **Certainty equivalence:** *Only expected income matter.*

## Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha}e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{it-1} + \psi_{it}$$
$$\psi_{it} \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function** (see proof)**:**

$$c_t^*(a_{it-1}, z_{it}) = ra_{it-1} + wz_{it} - \frac{\log\left(\beta R\right)^{\frac{1}{\alpha}} + \alpha\frac{\sigma_\psi^2}{2}}{r^2}$$

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^* \downarrow$ for given $a_{t-1}$ and $z_t$
  $\Rightarrow$ *accumulation of buffer-stock*

# Further resources

1. Lecture notes by Christopher Carroll
2. Lecture notes by Pierre-Olivier Gourinchas
3. The Economics of Consumption, Jappelli and Pistaferri (2017)
4. »Liquidity constraints and precautionary saving«
   Carroll, Holm, Kimball (JET, 2021)
5. »Theoretical Foundations of Buffer Stock Saving«
   Carroll (QE, forthcomming)

# Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*. (See Bellman, 1957, Chap. III.3.)

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- **In math:**
  1. **Value function, $v_t$:** Defined recursively from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

  with $v_T(\bullet) = 0$.

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     with $v_T(\bullet) = 0$.

  2. **Policy function, $c_t^*$:** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

# Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

1. **State variables:** $z_t$ and $a_{t-1}$
2. **Control variable:** $c_t$
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** $r$, $w$, and stuff in $u(\bullet)$

**Note:** Conceptually straightforward to extend to more goods, more assets or other states, more complex uncentainty, bounded rationality etc.

# Infinite horizon: $T \to \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

- **Contraction mapping result:** *If $\beta$ is low enough (strong enough impatience) then the value and policy function converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough $T$*
- **Maximum upper limit for $\beta$:** $\frac{1}{1+r}$
- **In practice:** Solve backwards until value and policy functions does not change anymore (given some tolerance)

# One analytical result: Natural borrowing limit

- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$

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- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$
- **Thought experiment:** Assumptions
  1. $a_{t-1} = -\frac{w\underline{z}}{r} + \Delta$
  2. $z_t = \underline{z}$, $\forall t$
  3. $c_t = 0$, $\forall t$

# One analytical result: Natural borrowing limit

- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$
- **Thought experiment:** Assumptions
  1. $a_{t-1} = -\frac{w\underline{z}}{r} + \Delta$
  2. $z_t = \underline{z}$, $\forall t$
  3. $c_t = 0$, $\forall t$
- **Implication:** For $\Delta < 0$ assets will be *decreasing without bound!*

$$
\begin{aligned}
a_t &= (1+r)\left(-\frac{w\underline{z}}{r} + \Delta\right) + w\underline{z} = -\frac{w\underline{z}}{r} + (1+r)\Delta \\
a_{t+1} &= -\frac{w\underline{z}}{r} + (1+r)^2\Delta \\
&\dots \\
a_{t+k} &= -\frac{w\underline{z}}{r} + (1+r)^k\Delta \to -\infty
\end{aligned}
$$

# One analytical result: Natural borrowing limit

- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$
- **Thought experiment:** Assumptions
  1. $a_{t-1} = -\frac{w\underline{z}}{r} + \Delta$
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- **Implication:** For $\Delta < 0$ assets will be *decreasing without bound!*

$$a_t = (1+r)\left(-\frac{w\underline{z}}{r} + \Delta\right) + w\underline{z} = -\frac{w\underline{z}}{r} + (1+r)\Delta$$

$$a_{t+1} = -\frac{w\underline{z}}{r} + (1+r)^2\Delta$$

$$\ldots$$

$$a_{t+k} = -\frac{w\underline{z}}{r} + (1+r)^k\Delta \to -\infty$$

- **Natural borrowing constraint:** $a_t > w \max\left\{-b, -\frac{\underline{z}}{r}\right\}$

# Numerical solution

# Timing of shocks

- **Realization of shocks:** First in the period before choices are made

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- **End-of-period value function** (after realization)**:**

$$\begin{aligned} v_t(z_t, a_{t-1}) &= \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(z_t, a_t) \\ &\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb \end{aligned}$$

# Discretization and linear interpolation

- **Discretization:** All state variables belong to discrete sets $\equiv$ *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#_z - 1}\}$$

$$a_t \in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#_a - 1}\}$$

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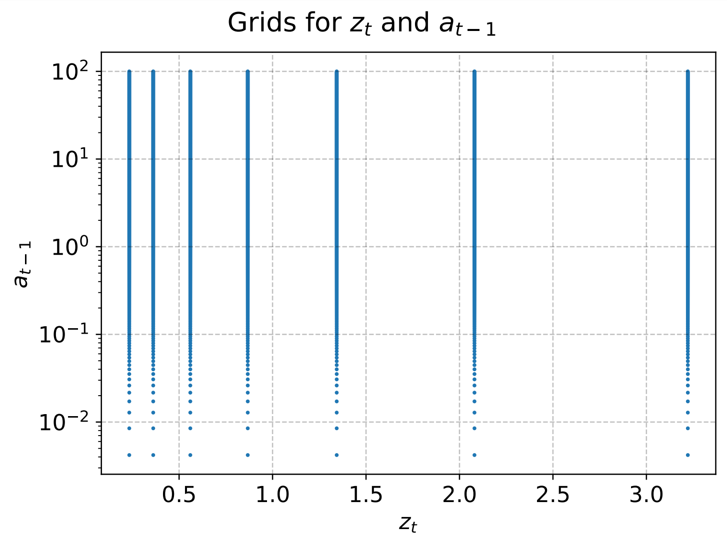
a^0 &= w \max\left\{-b, -\frac{\underline{z}}{r}\right\}
\end{aligned}
$$

- **Transition probabilities:** $\pi_{i_{z-},i_z} = \Pr[z_t = z^{i_z} \,|\, z_{t-1} = z^{i_{z-}}]$
- **Linear interpolation** (function approximation):
  1. Assume $\underline{v}_{t+1}$ is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Evaluate $\underline{v}_{t+1}(z^{i_z}, a)$ for arbitrary $a$ by

$$
\begin{aligned}
\underline{\breve{v}}_{t+1}(z^{i_z}, a) &= \underline{v}_{t+1}(z^{i_z}, a^{\iota}) + \omega_i(a - a^{\iota}) \\
\omega_i &\equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{\iota_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}} \\
\iota &\equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a
\end{aligned}
$$


Grids for $z_t$ and $a_{t-1}$
$10^2$
$10^1$
$10^0$
$10^{-1}$
$10^{-2}$
$a_{t-1}$
0.5
1.0
1.5
2.0
2.5
3.0
$z_t$

# Deriving transition probabilities

- **Specification:** Assume

$$z_t = \tilde{z}_t \xi_t, \;\; \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \;\; \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

  where $\mu_\xi$ and $\mu_\psi$ ensures $\mathbb{E}[\xi_t] = 1$, $\mathbb{E}[\tilde{z}_t] = 1$ and $\mathbb{E}[z_t] = 1$
- **Discretization of $\tilde{z}_t$:** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-}, i_{\tilde{z}}}$ given $\rho_z$ and $\sigma_\psi$ (using a method such as Tauchen (1986) or Rouwenhorst (1995))
- **Discretization of $\xi_t$:** Derive $\mathcal{G}_\xi$ and $\pi_{i_{\xi-}, i_\xi}$ given $\sigma_\xi$ (using Gauss-Hermite quadrature, see next slides)
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of $\tilde{z}_t$ and $\xi_t$ to get transition probabilities $\pi_{i_{z-}, i_z}$ (kronecker product)

# Transition probability matrix

Transition probabilities

| $z_{t-1}$ \ $z_t$ | 0.23 | 0.36 | 0.56 | 0.87 | 1.34 | 2.08 | 3.22 |
|---|---|---|---|---|---|---|---|
| 0.23 | 0.886 | 0.108 | 0.006 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.36 | 0.018 | 0.888 | 0.090 | 0.004 | 0.000 | 0.000 | 0.000 |
| 0.56 | 0.000 | 0.036 | 0.889 | 0.072 | 0.002 | 0.000 | 0.000 |
| 0.87 | 0.000 | 0.001 | 0.054 | 0.889 | 0.054 | 0.001 | 0.000 |
| 1.34 | 0.000 | 0.000 | 0.002 | 0.072 | 0.889 | 0.036 | 0.000 |
| 2.08 | 0.000 | 0.000 | 0.000 | 0.004 | 0.090 | 0.888 | 0.018 |
| 3.22 | 0.000 | 0.000 | 0.000 | 0.000 | 0.006 | 0.108 | 0.886 |

# Extra: Gauss-Hermite I

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x)g(x)dx$$

    - $f : \mathbb{R} \to \mathbb{R}$ some function
    - $g(x)$ is the probability distribution function (PDF) for $x$

- **General solution:** Turn it into a discrete sum

$$\mathbb{E}(f(x)) \approx \sum_{i=1}^{S} \omega_i f(x_i)$$

- **How to choose $S$ and the *nodes* ($x_i$) and *weights* ($\omega_i$)?**
  **Answer:** Guassian quadrature

# Extra: Gauss-Hermite II

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \sum_{i=1}^{S}\omega_i f(x_i) + \frac{S!\sqrt{\pi}}{s^S(2S)!}f^{(2S)}(\epsilon)$$

  for some $\epsilon$ and where the $(x_i, \omega_i)$'s can be easily found

- **Well behaved function:** For $S \to \infty$ we have

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx \approx \sum_{i=1}^{S}\omega_i f(x_i)$$

- **Example: Random normal variable:** $Y \sim \mathcal{N}(\mu, \sigma^2)$ so that

$$\begin{aligned}\mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} f(y)e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy \\ &\approx \frac{1}{\sqrt{\pi}}\sum_{i=1}^{S}\omega_i f(\sqrt{2}\sigma x_i + \mu)\end{aligned}$$

# Value function iteration

- **Beginning-of-period value function:**

$$\underline{v}_t(z^{i_{z-}}, a^{i_{a-}}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_{z-},i_z} v_t(z^{i_z}, a^{i_{a-}})$$

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- **End-of-period value-of-choice:**

$$v_t(z^{i_z}, a^{i_{a-}} | c_t) = u(c_t) + \beta \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} \check{v}_{t+1}(z^{i_{z+1}}, a_t)$$

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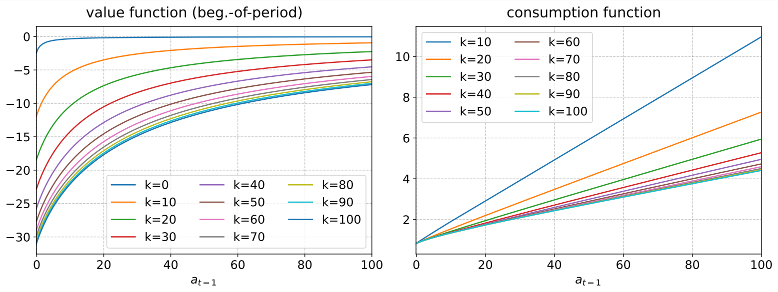
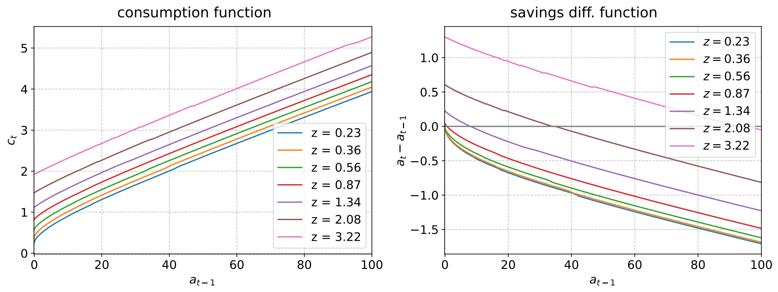
$$a_t = (1+r)a^{i_{a-}} + wz^{i_z} - c_t$$

- **Nested loops:**
  1. **Outer loop:** Backwards from $t = T-1$ (note $\underline{v}_T = 0$, or known)
  2. **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t^*(z_t, a_{t-1})$ with a *numerical optimizer*

# Convergence $(t = T - 1 - k)$



# Converged policy functions



# Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \ldots, N-1\}$

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  1. Draw $z_{it}$ given transition probabilities
  2. Use linear interpolation to evaluate

$$
\begin{aligned}
c_{it} &= \breve{c}_t^*(z_{it}, a_{it-1}) \\
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- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness

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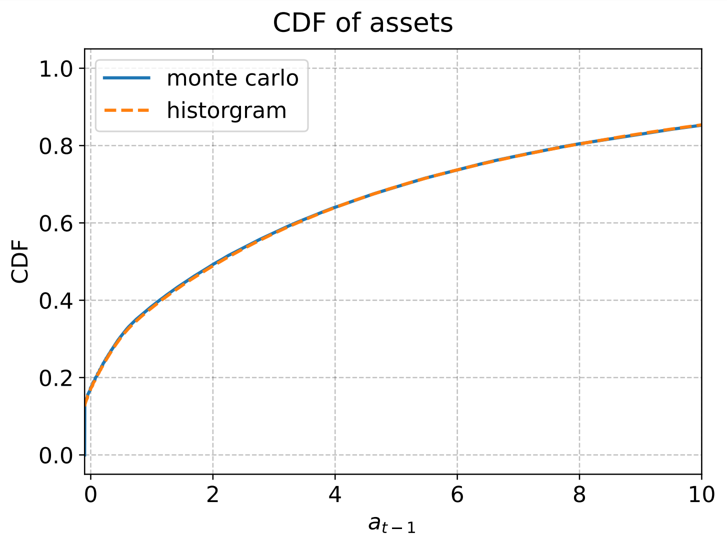
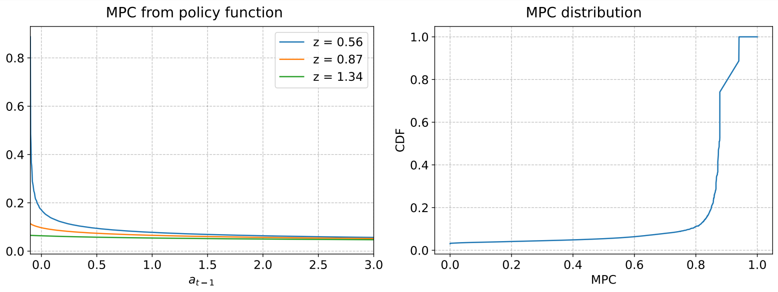
- **Review:**
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

# CDF of savings in final period



# MPCs



MPC from policy function
z = 0.56
z = 0.87
z = 1.34
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$a_{t-1}$
MPC distribution
CDF
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
MPC

# Side-note: Matrix formulation

- The histogram method can be written in **matrix form**:

$$\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$$
$$\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$$

where

$\underline{\boldsymbol{D}}_t$ is vector of length $\#_z \times \#_a$

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$\Pi_z'$ is derived from the $\pi_{i_{z-},i_z}$'s

$\Lambda_t'$ is derived from the $\iota$'s and $\omega$'s

- **Note:** Example shown in notebook
- **Further details:** Young (2010), Tan (2020), Ocampo and Robinson (2022)

# EGM

# Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

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2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

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Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} c_{+}(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand:**

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c_{+}(z^{i_z}, a^{i_a})$$

# Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand:**

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c_+(z^{i_z}, a^{i_a})$$

4. **Consumption function:** Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$
   If $m \leq m(z^{i_z}, a^0)$ con. binds: $c^*(z^{i_z}, a^{i_{a-}}) = m + w\max\left\{-b, -\frac{\underline{z}}{r}\right\}$
   Else: $c^*(z^{i_z}, a^{i_{a-}}) =$ interpolate $m(z^{i_z}, :)$ to $c(z^{i_z}, :)$ at $m$

# Practice

# In practice

- **EconModel:** Go through notebook 01. Using the EconModelClass (except part on C++)
- **ConSav:** Look at the 04. Tools folder.
- **Todays notebook:** *Consumption-Saving Model* show implementation of solution and simulation methods.

# Summary

# Summary and next week

- **Today:**
  1. Introduction to course
  2. Consumption-saving models
  3. Numerical dynamic programming
- **Next week:** Stationary equilibrium
- **Homework:**
  1. **Work on:** Familiarize your self with today's code
  2. **Read:** Aiyagari (1994),
     »Uninsured Idiosyncratic Risk and Aggregate Saving«