ASSIGNMENT I: THE AIYGARI MODEL

Vision: This project teaches you to solve for the *stationary equilibrium* in a neoclassical-style heterogeneous agent model and analyze the economic results.

- **Problem:** The problem consists of
  1. A number of questions (page 2)
  2. A model (page 3 onward, incl. solution tricks)
- **Code:** The problem is designed to be solved with the *GEModelTools* package.
- **Structure:** Your project should consist of
  1. A single self-contained pdf-file with all results
  2. A single Jupyter notebook showing how the results are produced
  3. Well-documented *.py* files
- **Hand-in:** Upload a single zip-file on Absalon (and nothing else)
- **Deadline:** 14th of October 2022
- **Exam:** Your Aiygari-project will be a part of your exam portfolio.
  You can incorporate feedback before handing in the final version.

Questions

1. **Define the stationary equilibrium for the model on the next page**
2. **Solve for the stationary equilibrium**

   Show aggregate quantities and prices

   Illustrate household behavior

   Note: *You can restrict attention to equilibria with a positive real interest rate*
3. **Illustrate how changes in the tax rates affect the stationary equilibrium**
4. **Discuss the social optimal level of taxation**

   Begin with average household utility as a social welfare criterion

   Other aspects of social welfare can also be introduced

   Note: *Searching over a fixed grid of tax rates is fine*
5. **Suggest and implement an extension which improves the tax system**

   The definition of »improves« is up to you

1. Model

Households. The model has a continuum of infinitely lived households indexed by $i \in [0,1]$. Households are *ex ante* heterogeneous in terms of their dis-utility of labor, φ_i, and their time-invariant productivity, ζ_i. Households are *ex post* heterogeneous in terms of their time-varying stochastic productivity, z_t, and their (end-of-period) savings, a_{t-1}. The distribution of households over idiosyncratic states is denoted $\underline{\boldsymbol{D}}_t$ before shocks are realized and $\boldsymbol{D}_t$ afterwards. Households choose to supply labor, ℓ_t, and consumption, c_t. Households are not allowed to borrow. The real interest rate is r_t, the real wage is w_t, and real-profits are Π_t. Interest-rate income is taxed with the rate $\tau_t^a \in [0,1]$ and labor income is taxed with the rate $\tau_t^\ell \in [0,1]$.

The household problem is

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t,\ell_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi_i \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \mid z_t, a_t\right] \qquad (1)\\
\text{s.t. } a_t + c_t &= (1+\tilde{r}_t)a_{t-1} + \tilde{w}_t \ell_t \zeta_i z_t + \Pi_t \\
\log z_{t+1} &= \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi),\ \mathbb{E}[z_t] = 1 \\
a_t &\geq 0
\end{aligned}
$$

where $\tilde{r}_t = (1-\tau_t^a) r_t$ and $\tilde{w}_t = (1-\tau_t^\ell) w_t$. Aggregate quantities are

$$L_t^{hh} = \int \ell_t \zeta_i z_t d\boldsymbol{D}_t \qquad (2)$$

$$C_t^{hh} = \int c_t d\boldsymbol{D}_t \qquad (3)$$

$$A_t^{hh} = \int a_t d\boldsymbol{D}_t \qquad (4)$$

Firms. A representative firm rents capital, K_{t-1}, and hire labor, L_t, to produce goods, with the production function

$$Y_t = \Gamma K_{t-1}^{\alpha} L_t^{1-\alpha} \qquad (5)$$

where Γ is technology. Capital depreciates with the rate $\delta \in (0,1)$. The real rental price of capital is r_t^K and the real wage is w_t. Profits are

$$\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1} \qquad (6)$$

The law-of-motion for capital is

$$K_t = (1-\delta)K_{t-1} + I_t \tag{7}$$

The households own the representative firm in equal shares.

Government. The budget constraint for the government is

$$\begin{aligned} B_t &= (1+r_t^B)B_{t-1} + G_t - \int \left[\tau_t^a r_t a_{t-1} + \tau_t^\ell w_t \ell_t \zeta_i z_t\right] d\boldsymbol{D}_t \\ &= (1+r_t^B)B_{t-1} + G_t - \tau_t^a r_t A_t^{hh} - \tau_t^\ell w_t L_t^{hh} \end{aligned} \tag{8}$$

where G_t is exogenous government spending not entering household utility, B_t is (end-of-period) government bonds, and r_t^B is the real interest rate on government bonds.

Market clearing. Arbitrage implies that all assets must give the same rate of return

$$r_t = r_t^B = r_t^K - \delta \tag{9}$$

Market clearing implies

1. Labor market: $L_t = L_t^{hh}$
2. Goods market: $Y_t = C_t^{hh} + I_t + G_t$
3. Asset market: $K_t + B_t = A_t^{hh}$

2. Calibration

The parameters and steady state government behavior are as follows:

1. **Preferences and abilities:** $\beta = 0.96$, $\sigma = 2$, $\varphi_i \in \{0.9, 1.1\}$, $\nu = 1.0$, $\zeta_i \in \{0.9, 1.1\}$

$$\begin{aligned} \Pr[\varphi_i = 0.9, \zeta_i = 0.9] &= 0.25 \\ \Pr[\varphi_i = 1.1, \zeta_i = 0.9] &= 0.25 \\ \Pr[\varphi_i = 0.9, \zeta_i = 1.1] &= 0.25 \\ \Pr[\varphi_i = 1.1, \zeta_i = 1.1] &= 0.25 \end{aligned}$$

2. **Income:** $\rho_z = 0.96$, $\sigma_\psi = 0.15$

3. **Production:** $\Gamma = 1$, $\alpha = 0.3$, $\delta = 0.1$

4. **Government:** $G_{ss} = 0.30$, $\tau^a_{ss} = 0.1$, $\tau^\ell_{ss} = 0.30$

3. Finding the steady state

Let $\mathcal{K}_{ss} = K_{ss}/L_{ss}$ denote the steady state capital-labor ratio.
From a guess on $\mathcal{K}_{ss}$ we can derive:

1. Calculate $r^K_{ss} = \alpha\Gamma\left(\mathcal{K}_{ss}\right)^{\alpha-1}$

2. Calculate $w_{ss} = (1-\alpha)\Gamma\left(\mathcal{K}_{ss}\right)^{\alpha}$

3. Calculate $r_{ss} = r^B_{ss} = r^K_{ss} - \delta$

4. Solve and simulate household problem to obtain A^{hh}_{ss} and L^{hh}_{ss}

5. Calculate$B_{ss} = \frac{\tau^a_{ss} r_{ss} A^{hh}_{ss} + \tau^\ell_{ss} w_{ss} L^{hh}_{ss} - G_{ss}}{r^B_{ss}}$

6. Calculate $L_{ss} = L^{hh}_{ss}$

7. Calculate $K_{ss} = \mathcal{K}_{ss} L_{ss}$

We then only need to check the asset market clearing condition:

$$K_{ss} + B_{ss} - A^{hh}_{ss} = 0$$

We can also derive a *lower* and an *upper* bound on $\mathcal{K}_{ss}$. From ensuring a positive real interest rate, we get

$$r_{ss} > 0 \Leftrightarrow r^K_{ss} - \delta > 0 \Leftrightarrow \mathcal{K}_{ss} > \left(\frac{\delta}{\alpha\Gamma}\right)^{\frac{1}{\alpha-1}}$$

From ensuring the real interest rate is not too high relative to the household discount factor, we get

$$1 + r_{ss} < \frac{1}{\beta} \Leftrightarrow r^K_{ss} - \delta < \frac{1}{\beta} - 1 \Leftrightarrow \mathcal{K}_{ss} < \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha\Gamma}\right)^{\frac{1}{\alpha-1}}$$

4. Solving the household problem

The following provides a recipe for solving the household problem for fixed $\varphi_i = \varphi$ and $\zeta_i = \zeta$.

The envelope condition implies

$$\underline{v}_{a,t+1}(z_{t-1}, a_{t-1}) = \mathbb{E}\left[(1+\tilde{r}_t)c_t^{-\rho} \mid z_{t-1}, a_{t-1}\right] \tag{10}$$

The first order conditions imply

$$c_t = \left(\beta \underline{v}_{a,t+1}(z_t, a_t)\right)^{-\frac{1}{\sigma}} \tag{11}$$

$$\ell_t = \left(\frac{\tilde{w}_t \zeta_i z_t}{\varphi_i}\right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu} \tag{12}$$

The household problem can be solved with an extended EGM:

1. Calculate c_t and ℓ_t over end-of-period states from FOCs
2. Construct endogenous grid $m_t = c_t + a_t - \tilde{w}_t \ell_t \zeta_i z_t$
3. Use linear interpolation to find consumption $c^*(z_t, a_{t-1})$ and labor supply $\ell^*(z_t, a_{t-1})$ with $m_t = (1+\tilde{r}_t)a_{t-1}$
4. Calculate savings $a^*(z_t, a_{t-1}) = (1+\tilde{r}_t)a_{t-1} + \tilde{w}_t \ell_t^* \zeta_i z_t - c_t^*$
5. If $a^*(z_t, a_{t-1}) < 0$ set $a^*(z_t, a_{t-1}) = 0$ and search for ℓ_t such that $f(\ell_t) \equiv \ell_t - \left(\frac{\tilde{w}_t z_t}{\varphi_i}\right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu} = 0$ holds and $c_t = (1+\tilde{r}_t)a_{t-1} + \tilde{w}_t \ell_t \zeta_i z_t$. This can be done with a Newton solver with an update from step j to step $j+1$ by

$$\begin{aligned}
\ell_t^{j+1} &= \ell_t^j - \frac{f(\ell_t)}{f'(\ell_t)} \\
&= \ell_t^j - \frac{\ell_t - \left(\frac{\tilde{w}_t z_t}{\varphi_i}\right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu}}{1 - \left(\frac{\tilde{w}_t z_t}{\varphi_i}\right)^{\frac{1}{\nu}} (-\sigma/\nu) \frac{\partial c_t}{\partial \ell_t}} \\
&= \ell_t^j - \frac{\ell_t - \left(\frac{\tilde{w}_t z_t}{\varphi_i}\right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu}}{1 - \left(\frac{\tilde{w}_t z_t}{\varphi_i}\right)^{\frac{1}{\nu}} (-\sigma/\nu)\, c_t^{-\sigma/\nu-1} \tilde{w}_t \zeta_i z_t}
\end{aligned}$$

The next page contains a code snippet with $\zeta_i z_t = 1$ you can base your code on.

```python
# a. prepare
fac = (wt/varphi)**(1/nu)

# b. use FOCs
c_endo = (beta*vbeg_a_plus)**(-1/sigma)
ell_endo = fac*(c_endo)**(-sigma/nu)

# c. interpolation
m_endo = c_endo + a_grid - wt*ell_endo
m_exo = (1+rt)*a_grid
c = np.zeros(Na)
interp_1d_vec(m_endo,c_endo,m_exo,c)
ell = np.zeros(Na)
interp_1d_vec(m_endo,ell_endo,m_exo,ell)

a = m_exo + wt*ell - c

# d. refinement at borrowing constraint
for i_a in range(Na):

    if a[i_a] < 0.0:

        # i. binding constraint for a
        a[i_a] = 0.0

        # ii. solve FOC for ell
        elli = ell[i_a]

        it = 0
        while True:

            ci = (1+rt)*a_grid[i_a] + wt*elli

            error = elli - fac*ci**(-sigma/nu)
            if np.abs(error) < tol_ell:
                break
            else:
                derror = 1 - fac*(-sigma/nu)*ci**(-sigma/nu-1)*wt
                elli = elli - error/derror

            it += 1
            if it > max_iter_ell: raise ValueError('too many iterations')

        # iii. save
        c[i_a] = ci
        ell[i_a] = elli
```

Listing 1: Extended EGM