



11. Introducing HANK

Adv. Macro: Heterogenous Agent Models

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2023



Introduction

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- **Today:**
 1. Linearized Impulse Response Function (IRF)
 2. Linearized simulation with aggregate risk
- **Relevance:** Business cycle analysis
- **Literature:**
 1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
 2. Boppart et. al. (2018), »Exploiting MIT shocks in heterogeneous-agent economies: The impulse response as a numerical derivative«
 3. Documentation for GEModelTools

IRFs and simulation

Reminder of model class

- Unknowns: U
- Shock: Z
- Additional variables: X
- Target equation system:

$$H(U, Z) = 0$$

- Auxiliary model equations:

$$X = M(U, Z)$$

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$$\begin{aligned} X = M(U, Z) \Rightarrow dX &= M_U dU + M_Z dZ \\ &= \underbrace{(-M_U H_U^{-1} H_Z + M_Z)}_{\equiv G} dZ \end{aligned}$$

- **Computation:** Same for Z as for U

- **Limitations:**

1. Imprecise for *large* shocks
2. Imprecise in models with *aggregate non-linearities*
(direct in aggregate equations or through micro-behavior)

- **Aggregate stochastic variables:** \mathbf{Z} follow some known process with innovations ϵ . *State space form:* RHS is what is known today

$$\begin{bmatrix} \underline{D}_{t+1} \\ X_t \\ Z_t \end{bmatrix} = \mathcal{M} \left(\begin{bmatrix} \underline{D}_t \\ X_{t-1} \\ Z_{t-1} \end{bmatrix}, \epsilon_t \right)$$

\neq perfect foresight wrt. future agg. variables in *sequence-space*

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- **Observation:** Linearization of aggregate variables imply *certainty equivalence* with respect to these

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- **Insight:** *The IRF from an MIT shock is equivalent to the IRF in a model with aggregate risk, which is linearized in the aggregate variables (Boppart et. al., 2018)*

Comparisons

- **State-space approach with linearization:** Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)

Con:

1. Harder to implement in my view
2. Valuable to be able to interpret Jacobians

Pro:

1. More similar to standard approaches for RBC and NK models
 2. Easier path to 2nd and higher order approximations
- **Global solution:** The distribution of households is a state variable for each household \Rightarrow *explosion in complexity*
 1. Original: Krusell and Smith (1997, 1998); Algan et al. (2014);
 2. Deep learning: Fernández-Villaverde et al. (2021); Maliar et al. (2021); Han et al. (2021); Kase et al. (2022); Azinovic et al. (2022); Gu et al. (2023); Chen et al. (2023)
 - **Discrete aggregate risk:** Lin and Peruffo (2023)

Basic linearized simulation

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 1. Draw time series of innovations, $\tilde{\epsilon}_t$
 2. Calculate the time series of shocks as $d\tilde{\mathbf{Z}}_t = \sum_{s=0}^{T-1} d\mathbf{Z}_s \tilde{\epsilon}_{t-s}$
Note: $d\mathbf{Z}_s \tilde{\epsilon}_{t-s}$ = effect of shock s periods ago today

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$$d\tilde{\mathbf{X}}_t = \sum_{s=0}^{T-1} d\mathbf{X}_s \tilde{\epsilon}_{t-s}$$

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- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:** Same result if we linearize all aggregated equations and write the model in $MA(\infty)$ form

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 1. The IRF for grid point i_g in a policy function can be calculated as

$$da_{i_g,s}^* = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \mathbf{X}^{hh}} \frac{\partial a_{i_g}^*}{\partial X_{s'-s}^{hh}} dX_{s'-s}^{hh}.$$

where $\partial a_{i_g}^* / \partial X_k^{hh}$ is the derivative to a k -period ahead shock to input X^{hh} (calculated in fake news algorithm)

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$$a_{i_g,t}^* = \sum_{s=0}^{T-1} da_{i_g,s}^* \tilde{\epsilon}_{t-s}$$

3. Distribution can then be simulated forwards

Calculating moments - variance

- Identical and independent distributed innovations:

$$\mathbb{E} \left[\epsilon_t^i \epsilon_{t'}^j \right] = \begin{cases} \sigma_i & \text{if } t = t' \text{ and } i = j \\ 0 & \text{el} \end{cases}$$

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- **Calculating moments such as $\text{var}(dC_t)$ from the IRFs:**

$$\begin{aligned} \text{var}(dC_t) &= \mathbb{E} \left[\left(\sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} dC_s \epsilon_{t-s}^i \right)^2 \right] \\ &= \sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} \mathbb{E} \left[\epsilon_{t-s}^i \epsilon_{t-s}^i \right] (dC_s^i)^2 \\ &= \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1} (dC_s^i)^2 \end{aligned}$$

where dC_s^i is the IRF to a unit-shock to i after s periods and σ_i is the standard deviation of shock i

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

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- **Covariance decomposition:**

$$\frac{\text{contribution from one shock}}{\text{contributions from all shocks}} = \frac{\sigma_j^2 \sum_{s=0}^{T-1-k} dC_s^j dY_{s+k}^j}{\sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i}$$

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- **Speed:** For a new set of parameters?
 1. Only shock processes change \Rightarrow *same Jacobians* (\mathbf{G}_U, \mathbf{G})
 2. Only need to re-compute Jacobian of aggregate variables? (only single block?)
 3. Also need to re-compute Jacobian of household problem?
 4. Also need to find stationary equilibrium again?

Sticky prices

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- **Central bank:** Set nominal interest rate

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- **Static** problem for representative final good firm:

$$\max_{y_{jt} \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t , and input prices, p_{jt}

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- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- FOC wrt. y_{jt}

$$0 = P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu}-1} - p_{jt} \Leftrightarrow$$

$$\frac{p_{jt}}{P_t} = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu-1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow$$

$$\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu-1}} = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow$$

$$y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

- **Dynamic problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

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$$\log(1 + \pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \quad \pi_t \equiv P_t / P_{t-1} - 1$$

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- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)

Intermediary goods firms

- **Dynamic problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** derived from FOC wrt. p_{jt} and envelope condition:

$$\log(1 + \pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \quad \pi_t \equiv P_t / P_{t-1} - 1$$

- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)
- **Implied dividends:** $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t$

- FOC wrt. p_{jt} :

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right) \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1} \frac{w_t}{Z_t} \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}} \\ - \frac{\mu}{\mu - 1} \frac{1}{\kappa} \frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

Derivation of NKPC

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- FOC + Envelope + Symmetry + $\pi_t = P_t/P_{t-1} - 1$

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right) \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1} \frac{w_t}{Z_t} \frac{Y_t}{P_t} \\ + \frac{\mu}{\mu - 1} \frac{1}{\kappa} \log(1 + \pi_t) \frac{Y_t}{P_t} + \frac{\frac{\mu}{\mu-1} \frac{1}{\kappa} \log(1 + \pi_{t+1}) \frac{Y_{t+1}}{P_t}}{1 + r_{t+1}}$$

- **Household problem:** Distribution, \mathbf{D}_t , over z_{it} and a_{it-1}

$$\begin{aligned} v_t(z_{it}, a_{it-1}) &= \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_{it}^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t [v_{t+1}(z_{it+1}, a_{it})] \\ \text{s.t. } a_{it} &= (1 + r_t)a_{it-1} + (w_t \ell_{it} - \tau_t + d_t)z_{it} - c_{it} \geq 0 \\ \log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \mathbb{E}[z_{it}] = 1 \end{aligned}$$

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- **Effective labor-supply:** $n_{it} = z_{it} \ell_{it}$

- **Beginning-of-period value function:**

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- **Endogenous grid method:** Vary z_t and a_t to find

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- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_{it}, a_{it-1}) \text{ and } \ell^*(z_{it}, a_{it-1}) \text{ with } m_{it} = (1 + r_t)a_{it-1}$$

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1. Stop if $f(\ell_{it}^*) = \ell_{it}^* - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} (c_{it}^*)^{-\frac{\sigma}{\nu}} < \text{tol.}$ where

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$$\ell_{it}^* = \frac{f(\ell_{it}^*)}{f'(\ell_{it}^*)} = \frac{f(\ell_{it}^*)}{1 - \left(\frac{w_t z_{it}}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c_{it}^*)^{-\frac{\sigma}{\nu}} w_t z_{it}}$$

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3. Return to step 1

- **Monetary policy:** Follow Taylor-rule:

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- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Assets: $B_{ss} = \int a_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t$
2. Labor: $N_t = \int n_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t$ (in effective units)
3. Goods: $Y_t = \int c_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t + G_t + \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(1 + \pi_t)]^2 Y_t$

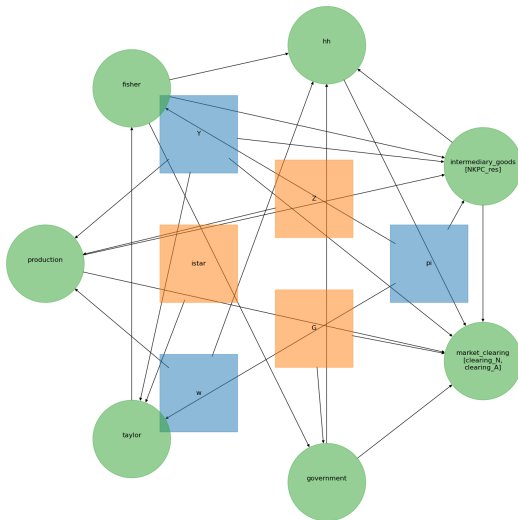
As an equation system

$$\begin{aligned} H(\pi, w, Y, i^*, \Gamma, \underline{D}_0) &= 0 \\ \left[\begin{array}{c} \log(1 + \pi_t) - \left[\kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}} \right] \\ N_t - \int n_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t \\ B_{ss} - \int a_t^*(z_{it}, a_{it-1}) d\mathbf{D}_t \end{array} \right] &= 0 \end{aligned}$$

The rest of the model is given by

$$X = M(\pi, w, Y, i^*, \Gamma)$$

As a DAG



Steady state

- **Chosen:** B_{ss} , G_{ss} , r_{ss}
- **Analytically:**
 1. **Normalization:** $Z_{ss} = N_{ss} = 1$
 2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i_{ss}^* = (1 + r_{ss})(1 + \pi_{ss}) - 1$
 3. **Firms:** $Y_{ss} = Z_{ss}N_{ss}$, $w_{ss} = \frac{Z_{ss}}{\mu}$ and $d_{ss} = Y_{ss} - w_{ss}N_{ss}$
 4. **Government:** $\tau_{ss} = r_{ss}B_{ss} + G_{ss}$
 5. **Assets:** $A_{ss} = B_{ss}$
- **Numerically:** Choose β and φ to get market clearing

Transmission mechanism to monetary policy shock

1. **Monetary policy shock:** $i_t^* \downarrow \Rightarrow i_t = i_t^* + \phi\pi_t \downarrow$
2. **Real interest rate:** $r_t = \frac{1+i_t-1}{1+\pi_t} \downarrow$
3. **Taxes:** $\tau_t = r_t B_{ss} \downarrow$
4. **Household consumption,** $C_t^{hh} \uparrow$, due to $r_t \downarrow$ and $\tau_t \downarrow$
5. **Firms production,** $Y_t \uparrow$, and **labor demand,** $N_t \uparrow$
6. **Inflation,** $\pi_t \uparrow$, and **wage,** $w_t \uparrow$ and **dividends,** $d_t \downarrow$
7. **Household labor supply,** $N_t^{hh} \uparrow$, due to $w_t \uparrow$ and $d_t \downarrow$,
but dampened $\tau_t \downarrow$
8. **Nominal rate,** $i_t \uparrow$ due to $\pi_t \uparrow$ implying $r_t \uparrow$
9. **Household consumption,** $C_t^{hh} \uparrow$, due to $w_t \uparrow$
but dampened by $d_t \downarrow$ and $r_t \uparrow$

Summary

- **Today:**

1. Aggregate risk and linearized dynamics (IRF and simulation)
2. Calculating aggregate moments (for calibration or estimation)
3. HANK with sticky prices

Exercise

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1. Compute the non-linear response to a temporary increase in government spending
2. Compute the linearized IRF to the same shock and compare
3. Sketch the transmission mechanism of government spending
4. Analyze how the aggressiveness of monetary policy affects the effectiveness of fiscal policy
5. Compare you previous results with the effects of a public transfer

Summary

Summary and next week

- **Today:**

1. Aggregate risk and linearized dynamics (IRF and simulation)
2. Calculating aggregate moments (for calibration or estimation)
3. A baseline HANK (sticky prices)

- **Next week:** More on HANK models

- **Homework:**

1. Work on exercise
2. Skim-read Auclert et al. (2023),
»The Intertemporal Keynesian Cross«