CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

3. Transition Path

Adv. Macro: Heterogenous Agent Models

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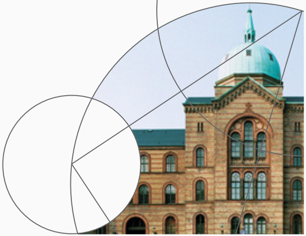

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Introduction

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  2. Example from GEModelToolsNotebooks/HANC
     (except stuff on *linearized solution* and *simulation*)

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- **Today:** *Transition path*
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- **Code:**
  1. Based on the GEModelTools package
  2. Example from GEModelToolsNotebooks/HANC
     (except stuff on *linearized solution* and *simulation*)
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools
     (except stuff on *linearized solution* and *simulation*)
  3. Kirkby (2017)

Ramsey

Ramsey: Summary

- **Simplified form:**

$$
\begin{aligned}
u'(C_t^{hh}) &= \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh}) \\
K_t &= (1-\delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}
\end{aligned}
$$

- **Production function:** $\Gamma_t K_t^{\alpha} L_t^{1-\alpha}$
- **Utility function:** $\frac{\left(C_t^{hh}\right)^{1-\sigma}}{1-\sigma}$
- **Steady state:**

$$
\begin{aligned}
K_{ss} &= \left(\frac{\left(\frac{1}{\beta} - 1 + \delta\right)}{\Gamma_{ss}\alpha}\right)^{\frac{1}{\alpha-1}} \\
C_{ss}^{hh} &= (1-\delta)K_{ss} + \Gamma_{ss}K_{ss}^{\alpha} - K_{ss}
\end{aligned}
$$

Ramsey: As an equation system

$$
\begin{bmatrix}
r_t^K - \alpha \Gamma_t K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t - (1-\alpha) \Gamma_t K_t^{\alpha} L_t^{-\alpha} \\
r_t - (r_t^K - \delta) \\
A_t - K_t \\
A_t^{hh} - \left((1+r_t) A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\
C_t^{hh,-\sigma} - \beta(1+r_{t+1}) C_{t+1}^{hh,-\sigma} \\
A_t - A_t^{hh} \\
L_t - L_t^{hh} \\
\forall t \in \{0,1,\dots\}, \text{given } K_{-1}
\end{bmatrix}
= \mathbf{0}
$$

Remember: Perfect foresight

Truncated, reduced vector form

$$\boldsymbol{H}\left(\boldsymbol{K},\boldsymbol{L},\boldsymbol{\Gamma},K_{-1}\right)=\begin{bmatrix} A_t-A_t^{hh} \\ L_t-L_t^{hh} \\ \forall t\in\{0,1,\dots,T-1\} \end{bmatrix}=\mathbf{0}$$

where $\boldsymbol{X}=(X_0,X_1,\dots,X_{T-1})$, $A_{-1}^{hh}=K_{-1}$ and

$$\begin{aligned}
r_t^K &= \alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K-\delta \\
C_t^{hh} &= \left(\beta(1+r_{t+1})\right)^{-\sigma} C_{t+1}^{hh}\,\text{(backwards)} \\
L_t^{hh} &= 1 \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh}+w_tL_t^{hh}-C_t^{hh}\,\text{(forwards)}
\end{aligned}$$

Truncation: $T<\infty$ fine when $\Gamma_t=\Gamma_{ss}$ for all $t>\underline{t}$ with $\underline{t}\ll T$

Further reduced

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1}) = \begin{bmatrix} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \dots, T-1\} \end{bmatrix} = \boldsymbol{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned} L_t = L_t^{hh} &= 1 \\ r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ A_t &= K_t \\ r_t &= r_t^K - \delta \\ C_t^{hh} &= (\beta(1+r_{t+1}))^{-\sigma} C_{t+1}^{hh} \text{ (backwards)} \\ A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \text{ (forwards)} \end{aligned}$$

Solution in sequence space

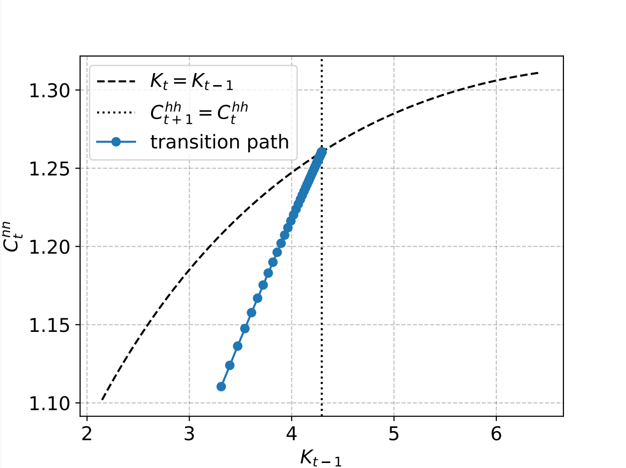
- **Truncation:** $T = 200$
- **Jacobian:** Find $\boldsymbol{H}_{\boldsymbol{K}}$ by *numerical differentiation*

$$\boldsymbol{H}_{\boldsymbol{K}} = \begin{bmatrix} \frac{\partial(A_0 - A_0^{hh})}{\partial K_0} & \frac{\partial(A_0 - A_0^{hh})}{\partial K_1} & \cdots \\ \frac{\partial(A_1 - A_1^{hh})}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

  Question: Are there any analytical zeros?
- **Transition path:** Given $\boldsymbol{\Gamma}$ and K_{-1} solve $\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1})$ with non-linear equation system solver (e.g. broyden)
- **Notebook:** *Ramsey.ipynb*

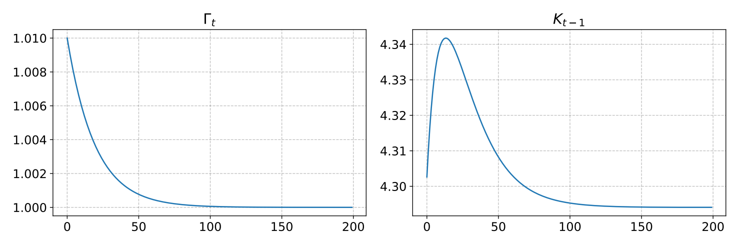
Example 1: Initially low capital

Initially away from steady state: $K_{-1} = 0.75K_{ss}$



Example 2: Technology shock

Technology shock: $\Gamma_t = 0.01\Gamma_{ss}0.95^t$ (exogenous, deterministic)



Terminology: MIT-shock

Transition path

Equation system

The model can be written as an **equation system**

$$\begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \boldsymbol{D}_t - \Pi_z \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t \boldsymbol{D}_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \mathbf{0}$$

where $\{\Gamma_t\}_{t \geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $x_t^* = x^* \left(\{r_\tau, w_\tau\}_{\tau \geq t} \right)$ and $X_t^{hh} = \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t$
2. Choice transition: $\Lambda_t = \Lambda \left(\{r_\tau, w_\tau\}_{\tau \geq t} \right)$

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over β_i, z_t and a_{t-1}
4. and the policy functions $\{a_t^*\}$, $\{\ell_t^*\}$ and $\{c_t^*\}$

are such that in all periods

1. Firms maximize profits (prices)
2. Household maximize expected utility (policy functions)
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Truncated, reduced vector form

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0\right) = \left[\begin{array}{c} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{array}\right] = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ A_t &= K_t \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t^{hh} &= \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\ & \forall t \in \{0, 1, \ldots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduction

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0\right) = \left[\begin{array}{c} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{array}\right] = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned}
L_t &= 1 \\
A_t &= K_t \\
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
\boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\
A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\
&\forall t \in \{0, 1, \ldots, T-1\}
\end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Could we solve it with a Newton method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$.
3. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below chosen tolerance
4. Calculate the Jacobians $\boldsymbol{H}^i_{\boldsymbol{K}} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$
5. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i - \left(\boldsymbol{H}^i_{\boldsymbol{K}}\right)^{-1} \boldsymbol{H}^i$
6. Increment i and return to step 2

Question: What is the problem?

Alternative: Use Broydens method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate the steady state Jacobian $\boldsymbol{H}_{\boldsymbol{K},ss} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}_{ss}, \boldsymbol{\Gamma}_{ss})$
3. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{K}^i, \boldsymbol{\Gamma})$.
4. Calculate Jacobian by
$$\boldsymbol{H}_{\boldsymbol{K}}^i = \begin{cases} \boldsymbol{H}_{\boldsymbol{K},ss} & \text{if } i = 0 \\ \boldsymbol{H}_{\boldsymbol{K}}^{i-1} + \frac{(\boldsymbol{H}^i - \boldsymbol{H}^{i-1}) - \boldsymbol{H}_{\boldsymbol{K}}^{i-1}(\boldsymbol{K}^i - \boldsymbol{K}^{i-1})}{\left|\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right|_2} \left(\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
5. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below tolerance
6. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i - \left(\boldsymbol{H}_{\boldsymbol{K}}^i\right)^{-1} \boldsymbol{H}^i$
7. Increment i and return to step 3

Question: What are the benefits? Are we only finding an approximate solution?

Bottleneck: How do we find the Jacobian?

1. **Naive approach:** For each $s \in \{0, 1, ..., T-1\}$ do
   1.1 Set $K_t = K_{ss} + \mathbf{1}\{t = s\} \cdot \Delta, \qquad \Delta = 10^{-4}$
   1.2 Find $\boldsymbol{r}$ and $\boldsymbol{w}$
   1.3 Solve household problem backwards along transition path
   1.4 Simulate households forward along transition path
   1.5 Calculate $\frac{\partial H_t}{\partial K_s} = \frac{K_t - A_t^{hh}}{\Delta}$ for all t

   Bottleneck: We need T^2 solution steps and simulation steps!

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   Bottleneck: We need T^2 solution steps and simulation steps!
2. **Fake news algorithm:** From household Jacobian to full Jacobian

$$\boldsymbol{H_K} = \boldsymbol{I} - \left(\mathcal{J}^{A^{hh},r}\mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K}\right)$$

   $\mathcal{J}^{r,K}, \mathcal{J}^{w,K}$: Fast from the onset - *only involve aggregates*
   $\mathcal{J}^{A^{hh},r}, \mathcal{J}^{A^{hh},w}$: Only requires T solution steps and simulation steps!

   $\Rightarrow$ *detailed discussed later today*

What have we found?

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What have we found?

- **Underlying assumption:** No aggregate uncertainty
- **»Shock«, Γ:** A fully unexpected non-recurrent event $\equiv$ *MIT shock*
- **Transition path, $\boldsymbol{K}$:** Non-linear perfect foresight response to
  1. Initial distribution, $\underline{\boldsymbol{D}}_0 \neq \boldsymbol{D}_{ss}$, *or to*
  2. Shock, $\Gamma_t \neq \Gamma_{ss}$ for some t (i.e. impulse-response)

The HANC example from GEModelToolsNotebooks

- **Presentation:** I go through the code

Interpreting the household Jacobians

- **Jacobian of consumption wrt. wage:** *What happens to consumption in period t when the wage (and thus income) increases in period s?*

$$\mathcal{J}^{C^{hh},w} = \begin{bmatrix} \frac{\partial C_0^{hh}}{\partial w_0} & \frac{\partial C_0^{hh}}{\partial w_1} & \cdots \\ \frac{\partial C_1^{hh}}{\partial w_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

- **Columns:** The full dynamic response to a shock in period s

Decomposition of GE response

- **GE transition path:** $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$
- **PE response of each:**
  1. Set $(\boldsymbol{r}, \boldsymbol{w}) \in \{(\boldsymbol{r}^*, \boldsymbol{w}_{ss}), (\boldsymbol{r}_{ss}, \boldsymbol{w}^*)\}$
  2. Solve household problem backwards along transition path
  3. Simulate households forward along transition path
  4. Calculate outcomes of interest
- **Additionally:** We can vary the initial distribution, $\underline{\boldsymbol{D}}_0$, to find the response of sub-groups

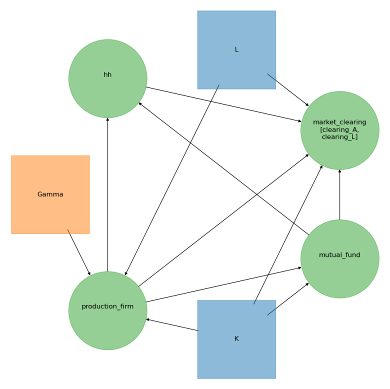
DAGs

General model class I

1. Time is discrete (index t).
2. There is a continuum of households (index i, when needed).
3. There is *perfect foresight* wrt. all aggregate variables, $\boldsymbol{X}$, indexed by $\mathcal{N}$, $\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t=0}^{\infty} = \{\boldsymbol{X}^j\}_{j\in\mathcal{N}} = \{X_t^j\}_{t=0,j\in\mathcal{N}}^{\infty}$, where $\mathcal{N} = \mathcal{Z} \cup \mathcal{U} \cup \mathcal{O}$, and $\mathcal{Z}$ are *exogenous shocks,* $\mathcal{U}$ are *unknowns*, $\mathcal{O}$ are outputs, and $\mathcal{H} \in \mathcal{O}$ are *targets.*
4. The model structure is described in terms of a set of *blocks* indexed by $\mathcal{B}$, where each block has inputs, $\mathcal{I}_b \subset \mathcal{N}$, and outputs, $\mathcal{O}_b \subset \mathcal{O}$, and there exists functions $h^o(\{\boldsymbol{X}^i\}_{i\in\mathcal{I}_b})$ for all $o \in \mathcal{O}_b$.
5. The blocks are *ordered* such that (i) each output is *unique* to a block, (ii) the first block only have shocks and unknowns as inputs, and (iii) later blocks only additionally take outputs of previous blocks as inputs. This implies the blocks can be structured as a *directed acyclical graph* (DAG).

DAG: Directed Acyclical Growth

- **Orange square:** Shocks (exogenous)
- **Purple square:** Unknowns (endogenous)
- **Green circles:** Blocks (with variables and targets inside)



General model class II

6. The number of targets are equal to the number of unknowns, and an *equilibrium* implies $\boldsymbol{X}^o = 0$ for all $o \in \mathcal{H}$. Equivalently, the model can be summarized by an *target equation system* from the unknowns and shocks to the targets,

   $$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0},$$

   and an *auxiliary model equation* to infer all variables

   $$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z}).$$

   A *steady state* satisfy

   $$\boldsymbol{H}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}) = 0 \text{ and } \boldsymbol{X}_{ss} = \boldsymbol{M}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss})$$

General model class III

7. The *discretized household block* can be written recursively as

$$
\begin{aligned}
\boldsymbol{v}_t &= v(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\underline{\boldsymbol{v}}_t &= \Pi(\boldsymbol{X}_t^{hh})\boldsymbol{v}_t \\
\boldsymbol{D}_t &= \Pi(\boldsymbol{X}_t^{hh})' \underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})' \boldsymbol{D}_t \\
\boldsymbol{a}_t^* &= \boldsymbol{a}^*(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\boldsymbol{Y}_t^{hh} &= \boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})' \boldsymbol{D}_t \\
&\underline{\boldsymbol{D}}_0 \text{ is given,} \\
&\boldsymbol{X}_t^{hh} = \{\boldsymbol{X}_t^i\}_{i \in \mathcal{I}_{hh}}, \boldsymbol{Y}_t^{hh} = \{\boldsymbol{X}_t^o\}_{o \in \mathcal{O}_{hh}},
\end{aligned}
$$

   where $\boldsymbol{Y}_t$ is aggregated outputs with $\boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})$ as individual level measures (savings, consumption labor supply etc.).

8. Given the sequence of shocks, $\boldsymbol{Z}$, there exists a *truncation period*, T, such all variables return to steady state beforehand.

Fake News Algorithm

Fake news algorithm

- **Household block:**

$$\boldsymbol{Y}^{hh} = hh(\boldsymbol{X}^{hh})$$

- **Goal:** Fast computation of

$$\mathcal{J}^{hh} = \frac{dhh(\boldsymbol{X}_{ss}^{hh})}{d\boldsymbol{X}^{hh}}$$

- **Naive approach:** Requires T^2 solution and simulation steps
- **Next slides:** *Sketch of much faster approach*
  (with $\Pi_t = \Pi_{ss}$ for notational simplicity)

Forward looking behavior

- **Notation:** $\bullet_t^{s,i}$ when there in period s is a shock to variable i
- **Time to shock:** Sufficient statistic for value and policy functions

$$\underline{\boldsymbol{v}}_t^{s,i} = \begin{cases} \underline{\boldsymbol{v}}_{ss} & \text{for } t > s \\ \underline{\boldsymbol{v}}_{t-1}^{s-1,i} & \text{for } t \leq s \end{cases} \text{ and } \boldsymbol{v}_t^{s,i} = \begin{cases} \boldsymbol{v}_{ss} & \text{for } t > s \\ \boldsymbol{v}_{t-1}^{s-1,i} & \text{for } t \leq s \end{cases}$$

$$\boldsymbol{y}_t^{s,i} = \begin{cases} \boldsymbol{y}_{ss} & t > s \\ \boldsymbol{y}_{T-1-(s-t)}^{T-1,i} & t \leq s \end{cases} \text{ and } \Lambda_t^{s,i} = \begin{cases} \Lambda_{ss} & t > s \\ \Lambda_{T-1-(s-t)}^{T-1,i} & t \leq s \end{cases}$$

- **Computation:** Only a single backward iteration required!
- **Note:** This is not an approximation

The first steps forward

- **Effect on output variable o in period 0:**

$$\mathcal{Y}_{0,s}^{o,i} \equiv \frac{dY_0^{o,s,i}}{dx} = \frac{\left(d\boldsymbol{y}_0^{o,s,i}\right)'}{dx} \left(\Pi_{ss}\right)' \underline{\boldsymbol{D}}_{ss}$$

- **Effect on beginning-of-period distribution in period 1:**

$$\underline{\mathcal{D}}_{1,s}^{i} \equiv \frac{d\underline{\boldsymbol{D}}_1^{s,i}}{dx} = \frac{\left(d\Lambda_0^{s,i}\right)'}{dx} \left(\Pi_{ss}\right)' \underline{\boldsymbol{D}}_{ss}$$

- **Expectation vector:** $\mathcal{E}_t^o \equiv \left(\Pi_{ss}\Lambda_{ss}\right)^t \Pi_{ss}\boldsymbol{y}_{ss}^o$,
- **Computational cost:**
  1. The cost of computing $\mathcal{Y}_{0,s}^{o,i}$ and $\underline{\mathcal{D}}_{1,s}^{i}$ for $s \in \{0, 1, \dots, T-1\}$ are similar to a full forward simulation for T periods.
  2. The cost of computing $\mathcal{E}_s^o$ is negligible in comparison and can be done recursively, $\mathcal{E}_t^o = \Pi_{ss}\Lambda_{ss}\mathcal{E}_{t-1}^o$ with $\mathcal{E}_0^o = \Pi_{ss}\boldsymbol{y}_{ss}^o$.

Main result

- **Result:** Tedious algebra imply the Jacobian can be constructed from the known objects as

$$\mathcal{F}_{t,s}^{,i,o} \equiv \begin{cases} \mathcal{Y}_{0,s}^{o,i} & t = 0 \\ \left(\mathcal{E}_{t-1}^{o}\right)' \underline{\mathcal{D}}_{1,s}^{i} & t \geq 1 \end{cases}$$

$$\mathcal{J}_{t,s}^{hh,i,o} = \sum_{k=0}^{\min\{t,s\}} \mathcal{F}_{t-k,s-k}^{i,o}$$

- **Intuition:** ???
- **Mathematically:** Use the chain-rule over and over again
- **Note:** Use linearity and that we start from steady state

Chain-rule unfolding $t = 0$

$$\mathcal{J}^{hh,i,o}_{0,s} = \mathcal{F}^{i,o}_{0,s} = \mathcal{Y}^{o,i}_{0,s} = \underbrace{\frac{dY_0^{o,s,i}}{dx}}_{\text{change in policy}}$$

Chain-rule unfolding $t = 1$

$$\mathcal{J}^{hh,i,o}_{1,0} = \mathcal{F}^{i,o}_{1,0} = (\mathcal{E}^{o}_{0})' \, \underline{\mathcal{D}}^{i}_{0} = \underbrace{(\boldsymbol{y}^{o}_{ss})' \, \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{0,i}_{1}}{dx}}_{\text{change in distribution}}$$

$$s \geq 1: \; \mathcal{J}^{hh,i,o}_{1,s} = \mathcal{F}^{i,o}_{1,s} + \mathcal{F}^{i,o}_{0,s-1} = \underbrace{(\boldsymbol{y}^{o}_{ss})' \, \Pi'_{ss} \Lambda'_{ss} \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{s,i}_{1}}{dx}}_{\text{change in distribution}} + \underbrace{\frac{dY^{o,s-1,i}_{0}}{dx}}_{\text{change in policy}}$$

Chain-rule unfolding $t = 2$

$$\mathcal{J}^{hh,i,o}_{2,0} = \mathcal{F}^{i,o}_{2,0} = \underbrace{\left(\boldsymbol{y}^{o}_{ss}\right)' \Pi'_{ss} \Lambda'_{ss} \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{0,i}_{1}}{dx}}_{\text{change in distribution}}$$

$$\mathcal{J}^{hh,i,o}_{2,1} = \mathcal{F}^{i,o}_{2,1} + \mathcal{F}^{i,o}_{1,0} = \underbrace{\left(\boldsymbol{y}^{o}_{ss}\right)' \Pi'_{ss} \Lambda'_{ss} \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{1,i}_{1}}{dx} + \left(\boldsymbol{y}^{o}_{ss}\right)' \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{0,i}_{1}}{dx}}_{\text{change in distribution}}$$

$$\begin{aligned} s \geq 2: \ \mathcal{J}^{hh,i,o}_{2,s} &= \mathcal{F}^{i,o}_{2,s} + \mathcal{F}^{i,o}_{1,s-1} + \mathcal{F}^{i,o}_{0,s-2} \\ &= \underbrace{\left(\boldsymbol{y}^{o}_{ss}\right)' \Pi'_{ss} \Lambda'_{ss} \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{s,i}_{1}}{dx} + \left(\boldsymbol{y}^{o}_{ss}\right)' \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{s-1,i}_{1}}{dx}}_{\text{change in distribution}} + \underbrace{\frac{dY^{o,s-2,i}_{0}}{dx}}_{\text{change in policy}} \end{aligned}$$

Bottlenecks

Bottlenecks

- **Small models:** Finding the stationary equilibrium
  - **Trick:** *(Modified) policy function iteration* (Howard improvement)
  - **Idea:** Multiple steps as once when finding the value function
    See e.g. Rendahl (2022) and Eslami and Phelan (2023)
- **Bigger models:** With many unknowns and targets both computing the Jacobian and solving the equation system can be costly
  $\Rightarrow$ *SSJ toolbox from Auclert et. al. (2021) has some methods for speeding this up not available in GEModelTools*

Exercises

Exercises: HANCGovModel

Same model. Your choice of τ_{ss}. New questions:

1. **Define the transition path.**
2. **Plot the DAG**
3. **How does the Jacobians look like?**
4. **Find the transition path for** $G_t = G_{ss} + 0.01 G_{ss} 0.95^t$
5. **What explains household savings behavior?**
6. **What happens to consumption inequality?**

Summary

Summary and next week

- **Today:**
  1. The concept of a transition path
  2. Details of the GEModelTools package
- **Homework:** Work on completing the model extension exercise
- **Next week:** Begin working on Assignment 1