



## 2. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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Jeppe Druedahl & Patrick Moran

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# Introduction

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- **Equilibrium-concept:** *Stationary equilibrium*
- **Code:** Based on the **GEModelTools** package
  1. Is in active development
  2. You can help to improve interface
  3. You can help to find bugs
  4. You can help to add features

**Documentation:** See **GEModelToolsNotebooks**

**Original package:** **SSJ** + **course** (*more complicated back-end*)

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- **Literature:** Aiyagari (1994)



## Ramsey-recap

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# Ramsey: Firms

- **Production function:**  $Y_t = F(K_{t-1}, L_t)$  [note timing of capital]
- **Profits:**  $\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1}$
- **Profit maximization:**  $\max_{K_{t-1}, L_t} \Pi_t$ 
  1. Rental rate:  $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^K = F_K(K_{t-1}, L_t)$
  2. Real wage:  $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = F_L(K_{t-1}, L_t)$

Zero profits:  $\Pi_t = 0 \Rightarrow$

$$Y_t = w_t L_t + r_t^K K_{t-1} \text{ [functional income distribution]}$$

# Ramsey: Zero-profit mutual fund

- Owns all capital
- Capital depreciate with rate  $\delta \in (0, 1)$ ,

$$K_t = (1 - \delta)K_{t-1} + I_t$$

- Deposits (from households),  $A_{t-1}$ : The rate of return is

$$r_t = r_t^K - \delta$$

- Balance sheet:

$$A_{t-1} = K_{t-1}$$

- **Utility maximization:**

$$v_0(A_{-1}^{hh}) = \max_{\{C_t^{hh}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t^{hh})$$

s.t.

$$A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}$$

Exogenous labor supply:  $L_t^{hh} = 1$

- **Euler-equation** (implied by Lagrangian):

$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$

# Ramsey: Market Clearing

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- **Labor market:**  $L_t = L_t^{hh} = 1$
- **Goods market:**  $Y_t = C_t^{hh} + I_t$
- **Walras:** Capital and labor market clears  $\Rightarrow$  goods market clears

$$\begin{aligned}C_t^{hh} + I_t &= [(1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - A_t^{hh}] + (K_t - (1 - \delta)K_{t-1}) \\&= [(1 + r_t)K_{t-1} + w_t L_t - K_t] + (K_t - (1 - \delta)K_{t-1}) \\&= r_t^K K_{t-1} + w_t L_t \\&= Y_t\end{aligned}$$



- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh})$$

$$K_t = (1 - \delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}$$

# Ramsey: Summary

- Simplified form:

$$u'(C_t^{hh}) = \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1 - \delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}$$

- Extended form:

$$r_t^K = F_K(K_{t-1}, L_t)$$
$$w_t = F_L(K_{t-1}, L_t)$$
$$r_t = r_t^K - \delta$$
$$A_t = K_t$$
$$A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}$$
$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$
$$A_t = A_t^{hh}$$
$$L_t = L_t^{hh}$$

## Ramsey: As an equation system

$$\begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - ((1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}) \\ u'(C_t^{hh}) - \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{ given } K_{-1} \end{bmatrix} = 0$$

**Note I:** There is *perfect foresight*.

**Note II:** This is the so-called *sequence-space* formulation.

# Ramsey: Steady state

- **Euler-equation** can be solved for  $K_{ss}$ :

$$u'(C_{ss}) = \beta(1 + F_K(K_{ss}, 1) - \delta)u'(C_{ss}) \Leftrightarrow$$
$$F_K(K_{ss}, 1) = \frac{1}{\beta} - 1 + \delta$$

- **Accumulation equation** then implies  $C_{ss}$ :

$$K_{ss} = (1 - \delta)K_{ss} + F(K_{ss}, 1) - C_{ss} \Leftrightarrow$$
$$C_{ss} = (1 - \delta)K_{ss} + F(K_{ss}, 1) - K_{ss}$$

**HANC**

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- **Model blocks:**

1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
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3. The Standard Incomplete Market (SIM) model

# Heterogeneous households

- **Utility maximization** for household  $i$ :

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$\ell_{it} = z_{it}$$

$$a_{it} = (1 + r_t)a_{it-1} + w_t z_{it} - c_{it}$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \quad \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \quad \mathbb{E}[z_{it}] = 1$$

$$a_{it} \geq 0$$

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1. *Ex ante* due to different preferences,  $\beta_i$

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- **Where are there heterogeneity?**

1. *Ex ante* due to different preferences,  $\beta_i$
2. *Ex post* due to stochastic productivity,  $z_{it}$

- **Incomplete markets due to borrowing constraint**

(fancy words: partial self-insurance, lack of Arrow-Debreu securities)



# Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = *(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

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$$\Lambda_t = \Lambda(\{r_\tau, w_\tau\}_{\tau \geq t})$$

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3. Savings transition:  $\underline{D}_{t+1} = \Lambda_t D_t$  where again

$$\Lambda_t = \Lambda(\{r_\tau, w_\tau\}_{\tau \geq t})$$

- **Aggregate consumption and savings:**

$$X_t^{hh} = \int x_t^*(\beta_i, z_{it}, a_{it-1}) dD_t = X^{hh}(\{r_\tau, w_\tau\}_{\tau \geq t}, \underline{D}_0) \text{ for } x \in \{a, \ell, c\}$$

# Equation system

$$\begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \underline{D}_t - \Pi_z \underline{D}_t \\ \underline{D}_{t+1} - \Lambda_t \underline{D}_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{ given } \underline{D}_0 \end{bmatrix} = 0$$

where  $K_{-1} = \int a_{it-1} d\underline{D}_0$

1. **Perfect foresight** wrt. aggregate variables
2. **Stationary equilibrium:** Time-constant solution.
3. **Transition path:** Time-varying solution due to e.g. initial conditions or temporary deviations of exogenous variables.

- **Must be solved *numerically*:**
- **Household problem:**
  1. Discretize and evaluate with interpolation
  2. Make recursion until convergence
- **Transition path:**
  1. Find the stationary equilibrium
  2. Find Jacobian around stationary equilibrium (*next time*)
  3. Solve using quasi-Newton solver (*next time*)



- **Beginning-of-period value function:**

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E} [v_t(\beta_i, z_{it}, a_{it-1}) \mid \beta_i, z_{it-1}, a_{it-1}]$$

# Solution of household problem

- **Beginning-of-period value function:**

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- **Envelope theorem:** Differentiate with fixed  $a_t$  choice

$$\underline{v}_{a,t}(\beta_i, z_{it-1}, a_{it-1}) \equiv \frac{\partial \underline{v}_t}{\partial a_{it-1}} = \mathbb{E} [(1 + r_t)c_{it}^{-\sigma} \mid \beta_i, z_{it-1}, a_{it-1}]$$

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- **EGM:** Separately for each  $\beta_i$  and  $z_{it}$

1. Find solution from FOC for each  $\tilde{a}_{it}$  in exogenous grid

$$\tilde{c}_{it}^{-\sigma} = \beta_i \underline{v}_{a,t+1}(\beta_i, z_{it}, \tilde{a}_{it}) \Leftrightarrow \tilde{c}_{it} = (\beta_i \underline{v}_{a,t+1}(\beta_i, z_{it}, \tilde{a}_{it}))^{-\frac{1}{\sigma}}$$

2. Calculate endogenous grid  $\tilde{m}_{it} = \tilde{a}_{it} + \tilde{c}_{it}$
3. Interpolate at  $m_{it} = (1 + r_t)a_{it-1} + w_t z_{it} + \Pi_t$  to get optimal  $a_{it}$
4. Enforce constraint by  $a_{it} = \max\{a_{it}, 0\}$
5. Consumption is  $c_{it} = m_{it} - a_{it}$

# Market clearing

- **Capital market:**  $K_t = A_t = \int a_t^*(\beta_i, z_{it}, a_{it-1}) d\mathbf{D}_t$
- **Labor market:**  $L_t = \int \ell_t^*(\beta_i, z_{it}, a_{it-1}) d\mathbf{D}_t = \int z_{it} d\mathbf{D}_t = 1$
- **Goods market:**  $Y_t = C_t^{hh} + I_t$
- **Walras:** Capital and labor market clears  $\Rightarrow$  goods market clears

$$\begin{aligned} C_t^{hh} + I_t &= \int c_{it}^* d\mathbf{D}_t + [K_t - (1 - \delta)K_{t-1}] \\ &= \int [(1 + r_t)a_{it-1} + w_t z_{it} - a_{it}] d\mathbf{D}_t \\ &= [(1 + r_t)K_{t-1} + w_t L_t - K_t] + [K_t - (1 - \delta)K_{t-1}] \\ &= r_t^K K_{t-1} + w_t L_t \\ &= Y_t \end{aligned}$$

# Stationary Equilibrium

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# Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$\begin{bmatrix} r_{ss}^K - (\alpha \Gamma_{ss} (K_{ss}/L_{ss})^{\alpha-1}) \\ w_{ss} - (1 - \alpha) \Gamma_{ss} (K_{ss}/L_{ss})^{\alpha} \\ r_{ss} - (r_{ss}^K - \delta) \\ A_{ss} - K_{ss} \\ \underline{D}_{ss} - \Pi'_z \underline{D}_{ss} \\ \underline{D}_{ss} - \Lambda'_{ss} \underline{D}_{ss} \\ A_{ss} - A_{ss}^{hh} \\ L_{ss} - L_{ss}^{hh} \end{bmatrix} = 0$$

**Note I:** Households still move around »inside« the distribution due to idiosyncratic shocks

**Note II:** Steady state for aggregates (quantities and prices) and the distribution as such

# Stationary equilibrium - more verbal definition

For a given  $\Gamma_{ss}$

1. Quantities  $K_{ss}$  and  $L_{ss}$ ,
2. prices  $r_{ss}$  and  $w_{ss}$  (always  $\Pi_{ss} = 0$ ),
3. the distribution  $\mathbf{D}_{ss}$  over  $\beta_i$ ,  $z_{it}$  and  $a_{it-1}$
4. and the policy functions  $a_{ss}^*$ ,  $\ell_{ss}^*$  and  $c_{ss}^*$

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3.  $\mathbf{D}_{ss}$  is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

**Root-finding problem** in  $K_{ss}$  with the objective function:

1. Set  $L_{ss} = 1$  (and  $\Pi_{ss} = 0$ )
2. Calculate  $r_{ss} = \alpha \Gamma_{ss}(K_{ss})^{\alpha-1} - \delta$  and  $w_{ss} = (1 - \alpha) \Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find  $\mathbf{a}_{ss}^*$
4. Simulate households *forwards* until convergence, i.e. find  $\mathbf{D}_{ss}$
5. Return  $K_{ss} - \mathbf{a}_{ss}^* \mathbf{D}_{ss}$



## Direct implementation (alternative)

**Root-finding problem** in  $r_{ss}$  with the objective function:

1. Set  $L_{ss} = 1$  (and  $\Pi_{ss} = 0$ )
2. Calculate  $K_{ss} = \left( \frac{r_{ss} + \delta}{\alpha \Gamma_{ss}} \right)^{\frac{1}{\alpha-1}}$  and  $w_{ss} = (1 - \alpha) \Gamma_{ss} (K_{ss})^\alpha$
3. Solve infinite horizon household problem *backwards*, i.e. find  $\mathbf{a}_{ss}^*$
4. Simulate households *forwards* until convergence, i.e. find  $\mathbf{D}_{ss}$
5. Return  $K_{ss} - \mathbf{a}_{ss}^{*'} \mathbf{D}_{ss}$

# Indirect implementation

1. Choose  $r_{ss}$  and  $w_{ss}$
2. Solve infinite horizon household problem *backwards*, i.e. find  $\mathbf{a}_{ss}^*$
3. Simulate households *forwards* until convergence, i.e. find  $\mathbf{D}_{ss}$
4. Set  $K_{ss} = \mathbf{a}_{ss}^{*'} \mathbf{D}_{ss}$
5. Set  $L_{ss} = 1$  (and  $\Pi_{ss} = 0$ )
6. Set  $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^\alpha}$
7. Set  $r_{ss}^k = \alpha \Gamma_{ss} (K_{ss})^{\alpha-1}$
8. Set  $\delta = r_{ss}^k - r_{ss}$

**Code**

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- **Preferences:**  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ 
  1. Discount factors:  $\beta \in \{0.965, 0.975, 0.985\}$  in equal pop. shares
  2. Relative risk aversion:  $\sigma = 2$
- **Income:**
  1. AR(1):  $\rho_z = 0.95$
  2. Std.:  $\sigma_\psi = 0.30\sqrt{(1 - \rho_z^2)}$
- **Technology:**  $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$ 
  1. Capital share:  $\alpha = 0.36$
  2. TFP:  $\Gamma_{ss} = 1.082$
  3. Depreciation:  $\delta = 0.193$
- **Steady state:**
  1. Prices:  $r_{ss} = 0.01$  and  $w_{ss} = 1$
  2. Quantities:  $K_{ss}/Y_{ss} = 1.776$

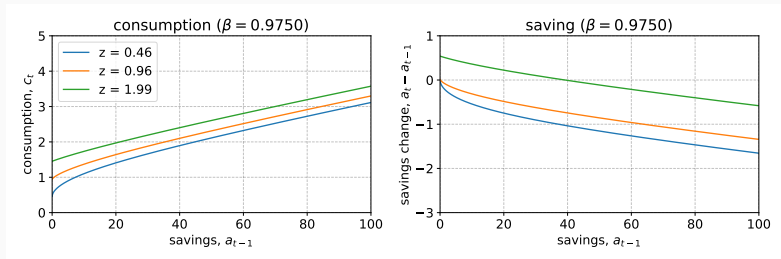
# Consumption function

- Euler-equation still necessary for  $a_{it} > 0$ :

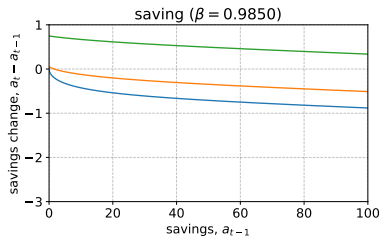
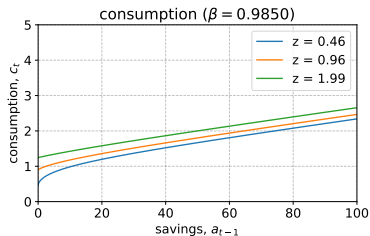
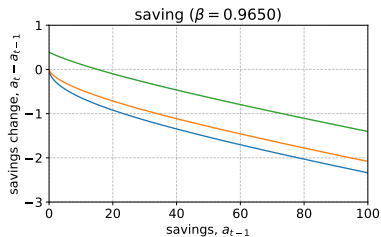
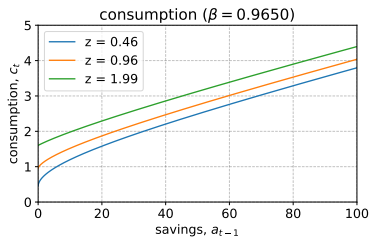
$$c_{it}^{-\sigma} = \beta_i(1 + r_{t+1})\mathbb{E}_t [c_{it+1}^{-\sigma}]$$

- Precautionary saving:

1. Low consumption for low cash-on-hand  $\rightarrow$  *buffer-stock target*
2. Steep slope for low cash-on-hand  $\rightarrow$  *high MPC*

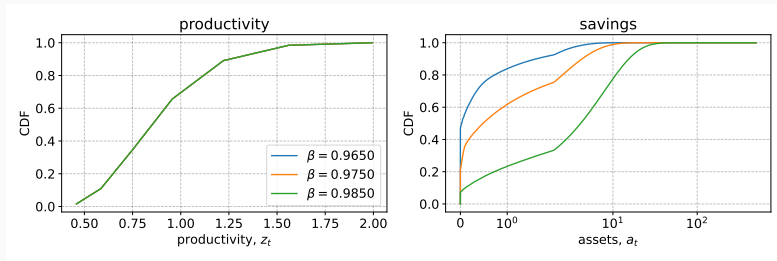


# Low vs. high $\beta_i$



# Distribution, $D_t$

- **Productivity:** Marginal distribution over only  $z_{it}$
- **Savings:** Marginal distribution over  $a_{it}$  cond. on  $\beta_i$



- **Drivers of wealth inequality:**
  1. Stochastic income
  2. Heterogeneous patience  $\rightarrow$  savings behavior

# Steady state interest rate

- **Representative agent / complete markets:**

Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$



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- **Heterogeneous agents:** *No such equation exists*

1. Euler-equation replaced by asset market clearing condition
2. Idiosyncratic income risk affects the steady state interest rate

$\sigma_\psi$	PE ( $r_{ss} = 1\%$ ), $A^{hh}$	GE, $r_{ss}$	GE, $A^{hh}$
0.09	2.78	1.00%	2.78
0.14	7.39	0.12%	2.97
0.19	13.68	-1.11%	3.30

**Partial Equilibrium:** Same interest rate.

**General Equilibrium:** Capital+labor market clearing.

# Calibration

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  1. **Informal:** Roughly match targets by hand
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    - 2a. Solve root-finding problem
    - 2b. Minimize a squared loss function
  3. **Estimation:** Formal with squared loss function + standard errors

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- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

# Exercises

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## Exercise: HANCGovModel

- **No production.** No physical savings instrument
- **Households:** Get stochastic endowment  $z_{it}$  of consumption good
- **Government:**
  1. Choose government spending
  2. Collect taxes,  $\tau_t$ , proportional to endowment
  3. Bonds: Pays 1 consumption good next period. Price is  $p_t^B < 1$

$$p_t^B B_t = B_{t-1} + G_t - \int \tau_t z_{it} d\mathbf{D}_t$$

# Exercise: Households

## Households:

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t [v_{it+1}(z_{it+1}, a_{it})]$$

$$\text{s.t. } p_t^B a_{it} + c_{it} = a_{it-1} + (1 - \tau_t) z_{it} \geq 0$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \mathbb{E}[z_{it}] = 1$$

## Euler-equation:

$$c_t^{-\sigma} = \beta \frac{v_{a,t+1}(z_{it}, a_{it})}{p_t^B}$$

## Envelope condition:

$$v_{a,t}(z_{it-1}, a_{it-1}) = c_{it}^{-\sigma}$$



## Exercise: Questions

1. **Define the stationary equilibrium**
2. **Solve and simulate the household problem**  
with  $p_{ss}^B = 0.975$  and  $\tau_{ss} = 0.12$ .
3. **Find the stationary equilibrium**  
with  $G_{ss} = 0.10$  and  $\tau_{ss} = 0.12$ .
4. **What happens for  $\tau_{ss} \in (0.11, 0.15)$ ?**
5. **When is average household utility maximized?**

# Summary

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# Summary and next week

- **Today:**

1. The concept of a stationary equilibrium
2. Introduction to the **GEModelTools** package

- **Next week:** *Transition path*

- **Homework:**

1. Work on completing the HANCGovModel exercise (solution in repository folder HANCGovModel)
2. Read documentation for GEModelTools (*except on linearized solution and simulation*)
3. Glance at Auclert et. al. (2021) and Kirkby (2017)