CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

1. Introduction

Adv. Macro: Heterogenous Agent Models

Jeppe Druedahl & Patrick Moran

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Danmarks Grundforskningsfond
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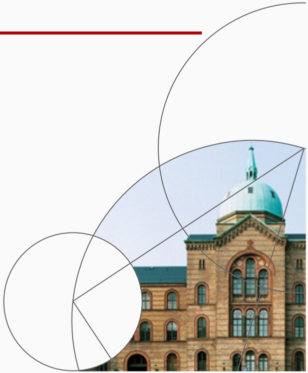

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  Prerequisite: *Intro. to Programming and Numerical Analysis*
  Complicated: *Close to the research frontier*

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  Prerequisite: *Intro. to Programming and Numerical Analysis*
  Complicated: *Close to the research frontier*
- **Plan for today:**
  1. More about the course
  2. Consumption-saving models
  3. Numerical dynamic programming

Macroeconomic Models with Heterogeneous Agents

- **Model components:**
  1. Optimizing individual agents (households + firms)
  2. Idiosyncratic and aggregate risk
  3. Information flows (who knows what when $\Rightarrow$ often everything)
  4. Market clearing (Walras vs. search-and-match)

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- **HANK:** Heterogeneous Agent *New Keynesian* model
  (i.e. include price and wage setting frictions)

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  $\sim$2 hours of »normal« lecture
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  Web: sites.google.com/view/numeconcph-advmacrohet/

  Git: github.com/numeconcopenhagen/adv-macro-het

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- **Code:**
  1. We provide code you will build upon
  2. Based on the GEModelTools package

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- **Exam:**
  1. Hand-in 3×**assignments**
  2. **48 hour take-home:** Programming of new extension
     + analysis of model + interpretation of results

Python

1. **Assumed knowledge:** From Introduction to Programming and Numerical Analysis you are assumed to know the basics of
   1.1 Python
   1.2 VSCode
   1.3 git
2. **Updated Python:** Install (or re-install) newest Anaconda
3. **Packages:** `pip install quantecon, EconModel, consav`
4. **GEMoodel tools:**
   4.1 Clone the `GEModelTools` repository
   4.2 Locate repository in command prompt
   4.3 Run `pip install -e .`

Course plan

See CoursePlan.pdf in repository

Knowledge

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Skills

1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Competencies

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

Consumption-Saving

Generations of models

1. Permanent income hypothesis (Friedman, 1957) or life-cycle model (Modigliani and Brumburg, 1954)
2. Buffer-stock consumption model
   (Deaton, 1991, 1992; Carroll, 1992, 1997)
3. Multiple-asset buffer-stock consumption models
   (e.g. Kaplan and Violante (2014))

Consumption-saving

$$v_0(a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**
  Consumption: c_t
  Productivity: z_t
  End-of-period savings: a_t (*no debt at death*)
- **Parameters:**
  Discount factor: β
  Wage: w
  Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

$$v_0(a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **It is a *static* problem:**
  1. **Information:** z_t is known for all t at $t=0$
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
  3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0^\star, c_1^\star, \ldots, c_{T-1}^\star$

IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$
\begin{aligned}
a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
&= R^2 a_{T-3} + Rwz_{T-2} - Rc_{T-1} + wz_{T-1} - c_{T-1} \\
&= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t}(wz_t - c_t)
\end{aligned}
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\end{aligned}
$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$
R^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0
$$

where $s_0 \equiv Ra_{-1}$ (after-interst assets)
and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

FOC and Euler-equation

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t u'(c_t) - \lambda (1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda \left(\beta R\right)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda \left(\beta R\right)^{-t}}{-\lambda \left(\beta R\right)^{-(t+k)}} = \left(\beta R\right)^k$$

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned}\sum_{t=0}^{T-1} R^{-t}(\beta R)^{t/\sigma} c_0 &= s_0 + h_0 \Leftrightarrow \\ c_0^\star &= \frac{1 - \frac{(\beta R)^{1/\rho}}{R}}{1 - \left(\frac{(\beta R)^{1/\rho}}{R}\right)^T}(s_0 + h_0)\end{aligned}$$

Infinite horizon

- **Infitnite horizon** for $\beta R < 1$: Let $T \to \infty$ to get

$$c_0^\star = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)$$

- **Interesting properties** are e.g.:
    1. Interest rate sensitivity: $\frac{\partial c_0}{\partial r}$
    2. MPC of permanent income change: $\frac{\partial c_0}{\partial w}$
    3. MPC of future income: $\frac{\partial c_0}{\partial z_t}$
    4. MPC of windfall income: $\frac{\partial c_0}{\partial s_0}$
- **No borrowing constraints or uncertainty**
- **Other simplifications:** No age life-cycle, bequests etc.

Liquidity/borrowing constraints

- Implied period 0 **savings** are:

$$a_0 = Ra_{-1} + wz_0 - c_0$$

- **Borrowing constraint:** $a_0 \geq -w \cdot b$
- **Maximum consumption:** $\overline{c}_0 = Ra_{-1} + wz_0 + wb$
- **Optimal consumption:** Constrained or unconstrained.

$$c_0^{\star} = \min\left\{\overline{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)\right\}$$

- **Empirical realism.** Incl. high MPC of constrained.
- **Technical issue:** Borrowing constraints further in the future complicated the analytical solution considerably.

Uncertainty

$$v_0(a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$z_t \sim \mathcal{Z}(z_{t-1})$$

$$a_t \geq -wb$$

$$\lim_{t\to\infty} (1+r)^{-t} a_t \geq 0 \quad \text{[No-Ponzi game]}$$

- **Stochastic income** from 1st order Markov-process, $\mathcal{Z}$
- ***A true dynamic problem:***
  1. **Information:** z_t is revealed period-by-period
  2. **Target:** *Expected* discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
  3. **Behavior:** Choose c_t *sequentially* as information is revealed
  4. **Solution:** Sequence of consumption *functions*, $c_t^\star(z_t, a_{t-1})$

IBC

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \quad \Leftrightarrow \quad b_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

- **What if $T \to \infty$?** We must have $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} = 0$
  1. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
  2. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \to \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t} c_t = (1+r)a_{-1} + \sum_{t=0}^{\infty} R^{-t} w z_t$$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
  1. **Feasible:** Yes, if $a_t > -\underline{a}$
  2. **Utility change:** $u'(c_t) + \beta (-R) \mathbb{E}_t [u'(c_{t+1})] > 0$

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- **Case II:** If $u'(c_t) < \beta(1+r) \mathbb{E}_t [u'(c_{t+1})]$:
  Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
  1. **Feasible:** Yes (always)
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  1. **Feasible:** Yes (always)
  2. **Utility change:** $u'(c_t) + \beta R \mathbb{E}_t [u'(c_{t+1})] > 0$
- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = -\underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t [u'(c_{t+1})]$, or
  2. **Unconstrained:** $a_t > -\underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$

Special case I: Qudratic utility

- **Quadratic utility:** $u(c_{it}) = -\frac{1}{2}(\overline{c} - c)^2$ with $\beta R = 1$
- **Euler-equation:** *Consumption = expected future consumption*

$$(\overline{c} - c_t) = \mathbb{E}_t\left[(\overline{c} - c_{t+k})\right] \Leftrightarrow c_t = \mathbb{E}_t\left[c_{t+k}\right]$$

- Use **IBC** in expectation to get **consumption function:**

$$\sum_{t=0}^{\infty} R^{-t}\mathbb{E}_0\left[c_t\right] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w\mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c_0 = ra_{-1} + \frac{r}{R}\sum_{t=0}^{T} R^{-t} w\mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c^{\star}(a_{t-1}, z_t) = ra_{t-1} + \frac{r}{R}\sum_{k=0}^{\infty} R^{-k} w\mathbb{E}_t\left[z_{t+k}\right]$$

- **Certainty equivalence:** *Only expected income matter.*

Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha}e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$z_t = z_{it-1} + \psi_{it}$$
$$\psi_{it} \sim \mathcal{N}(0, \sigma_\psi^2)$$

- **Consumption function** (see proof)**:**

$$c_t^\star(a_{it-1}, z_{it}) = ra_{it-1} + wz_{it} - \frac{\log(\beta R)^{\frac{1}{\alpha}} + \alpha\frac{\sigma_\psi^2}{2}}{r^2}$$

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^\star \downarrow$ for given a_{t-1} and z_t
$\Rightarrow$ *accumulation of buffer-stock*

Further resources

1. Lecture notes by Christopher Carroll
2. Lecture notes by Pierre-Olivier Gourinchas
3. The Economics of Consumption, Jappelli and Pistaferri (2017)
4. Liquidity constraints and precautionary saving, Carroll, Holm, Kimball (JET, 2021)
5. Theoretical Foundations of Buffer Stock Saving, Carroll (QE, forthcomming)

Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*. (See Bellman, 1957, Chap. III.3.)

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- **In math:**
  1. **Value function, v_t:** Defined recursively from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

  with $v_T(\bullet) = 0$.

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     with $v_T(\bullet) = 0$.

  2. **Policy function, $c_t^\star$:** Is the same as

$$c_t^\star(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

1. **State variables:** z_t and a_{t-1}
2. **Control variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Note: Conceptually straightforward to extend to more goods, more assets or other states, more complex uncentainty, bounded rationality etc.

Infinite horizon: $T \to \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy function converge to $v(z_t, a_{t-1})$ and $c^\star(z_t, a_{t-1})$ for large enough T*
- **Maximum upper limit for β:** $\frac{1}{1+r}$
- **In practice:** Solve backwards until value and policy functions does not change anymore (given some tolerance)

One analytical result: Natural borrowing limit

- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$

One analytical result: Natural borrowing limit

- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$
- **Thought experiment:** Assumptions
  1. $a_{t-1} = -\frac{w\underline{z}}{r} + \Delta$
  2. $z_t = \underline{z}$, $\forall t$
  3. $c_t = 0$, $\forall t$

One analytical result: Natural borrowing limit

- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$
- **Thought experiment:** Assumptions
  1. $a_{t-1} = -\frac{w\underline{z}}{r} + \Delta$
  2. $z_t = \underline{z}$, $\forall t$
  3. $c_t = 0$, $\forall t$
- **Implication:** For $\Delta < 0$ assets will be *decreasing without bound!*

$$
\begin{aligned}
a_t &= (1+r)\left(-\frac{w\underline{z}}{r} + \Delta\right) + w\underline{z} = -\frac{w\underline{z}}{r} + (1+r)\Delta \\
a_{t+1} &= -\frac{w\underline{z}}{r} + (1+r)^2\Delta \\
&\dots \\
a_{t+k} &= -\frac{w\underline{z}}{r} + (1+r)^k\Delta \to -\infty
\end{aligned}
$$

One analytical result: Natural borrowing limit

- **Minimum income:** $\underline{z} = \min_{z \in \mathcal{G}_z} z$
- **Thought experiment:** Assumptions
  1. $a_{t-1} = -\frac{w\underline{z}}{r} + \Delta$
  2. $z_t = \underline{z}$, $\forall t$
  3. $c_t = 0$, $\forall t$
- **Implication:** For $\Delta < 0$ assets will be *decreasing without bound!*

$$
\begin{aligned}
a_t &= (1+r)\left(-\frac{w\underline{z}}{r} + \Delta\right) + w\underline{z} = -\frac{w\underline{z}}{r} + (1+r)\Delta \\
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- **Natural borrowing constraint:** $a_t > w \max\left\{-b, -\frac{\underline{z}}{r}\right\}$

Numerical solution

Timing of shocks

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- **End-of-period value function** (after realization)**:**

$$\begin{aligned} v_t(z_t, a_{t-1}) &= \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(z_t, a_t) \\ &\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq wb \end{aligned}$$

Discretization and linear interpolation

- **Discretization:** All state variables belong to discrete sets $\equiv$ *grids*,

$$z_t \in \mathcal{G}_z = \{z^0, z^1, \ldots, z^{\#_z - 1}\}$$
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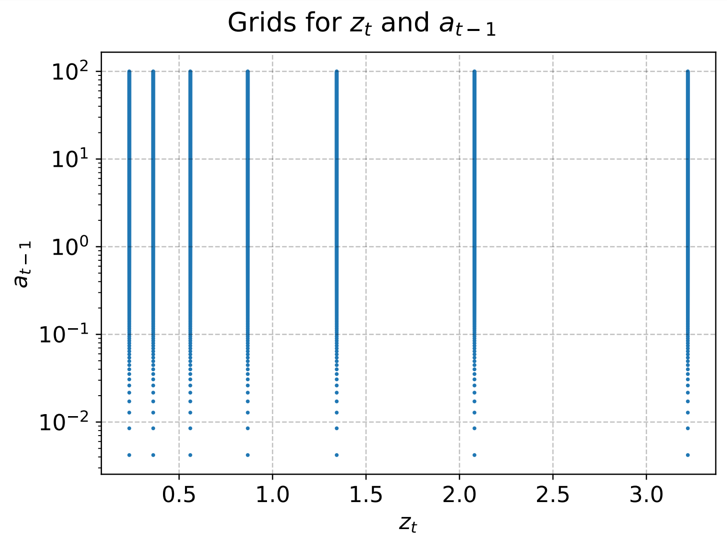
- **Linear interpolation** (function approximation):
  1. Assume $\underline{v}_{t+1}$ is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Evaluate $\underline{v}_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$\underline{\breve{v}}_{t+1}(z^{i_z}, a) = \underline{v}_{t+1}(z^{i_z}, a^{\iota}) + \omega_i (a - a^{\iota})$$
$$\omega_i \equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{\iota_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}}$$
$$\iota \equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a$$

Grids



Deriving transition probabilities

- **Specification:** Assume

$$z_t = \tilde{z}_t \xi_t, \ \ \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \ \ \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

  where μ_ξ and μ_ψ ensures $\mathbb{E}[\xi_t] = 1$, $\mathbb{E}[\tilde{z}_t] = 1$ and $\mathbb{E}[z_t] = 1$

- **Discretization of $\tilde{z}_t$:** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-}, i_{\tilde{z}}}$ given ρ_z and σ_ψ (using a method such as Tauchen (1986) or Rouwenhorst (1995))
- **Discratization of ξ_t:** Derive $\mathcal{G}_\xi$ and $\pi_{i_{\xi-}, i_\xi}$ given σ_ξ (using Gauss-Hermite quadrature, see next slides)
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of $\tilde{z}_t$ and ξ_t to get transition probabilities π_{i_{z-}, i_z} (kronecker product)

Transition probability matrix

Transition probabilities

z_{t-1} \ z_t	0.23	0.36	0.56	0.87	1.34	2.08	3.22
0.23	0.886	0.108	0.006	0.000	0.000	0.000	0.000
0.36	0.018	0.888	0.090	0.004	0.000	0.000	0.000
0.56	0.000	0.036	0.889	0.072	0.002	0.000	0.000
0.87	0.000	0.001	0.054	0.889	0.054	0.001	0.000
1.34	0.000	0.000	0.002	0.072	0.889	0.036	0.000
2.08	0.000	0.000	0.000	0.004	0.090	0.888	0.018
3.22	0.000	0.000	0.000	0.000	0.006	0.108	0.886

Details: Gauss-Hermite I

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x)g(x)dx$$

  - $f : \mathbb{R} \rightarrow \mathbb{R}$ some function
  - $g(x)$ is the probability distribution function (PDF) for x

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$$\mathbb{E}(f(x)) \approx \sum_{i=1}^{S} \omega_i f(x_i)$$

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- **How to choose S and the *nodes* (x_i) and *weights* (ω_i)?**
  Answer: Guassian quadrature

Details: Gauss-Hermite II

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \sum_{i=1}^{S} \omega_i f(x_i) + \frac{S!\sqrt{\pi}}{s^S(2S)!} f^{(2S)}(\epsilon)$$

for some ϵ and where the (x_i, ω_i)'s can be easily found

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- **Example: Random normal variable:** $Y \sim \mathcal{N}(\mu, \sigma^2)$ so that

$$\begin{aligned}\mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y)e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{S} \omega_i f(\sqrt{2}\sigma x_i + \mu)\end{aligned}$$

Value function iteration

- **Beginning-of-period value function:**

$$\underline{v}_t(z^{i_{z-}}, a^{i_{a-}}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_{z-},i_z} v_t(z^{i_z}, a^{i_{a-}})$$

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- **End-of-period value-of-choice:**

$$v_t(z^{i_z}, a^{i_{a-}} | c_t) = u(c_t) + \beta \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} \check{v}_{t+1}(z^{i_{z+1}}, a_t)$$

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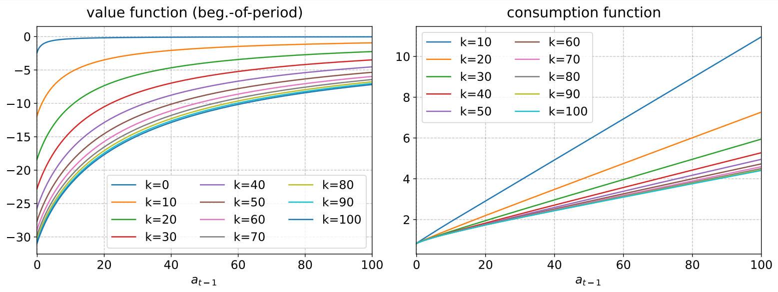
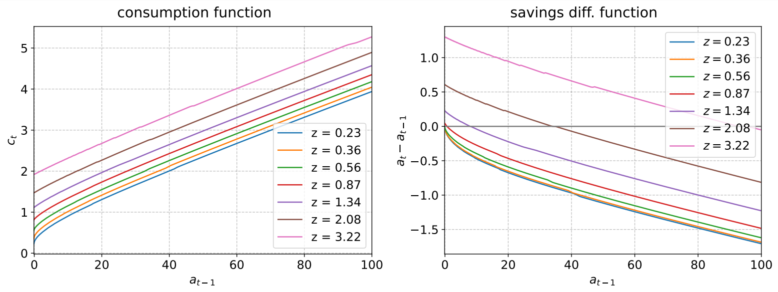
- **Nested loops:**
    1. **Outer loop:** Backwards in time from $t = T-1$ (note $\underline{v}_T$ is known)
    2. **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^\star(z_t, a_{t-1})$ and therefore $v_t^\star(z_t, a_{t-1})$ with a *numerical optimizer*

Convergence



Converged policy functions



Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \ldots, N-1\}$

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  2. Use linear interpolation to evaluate

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- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness

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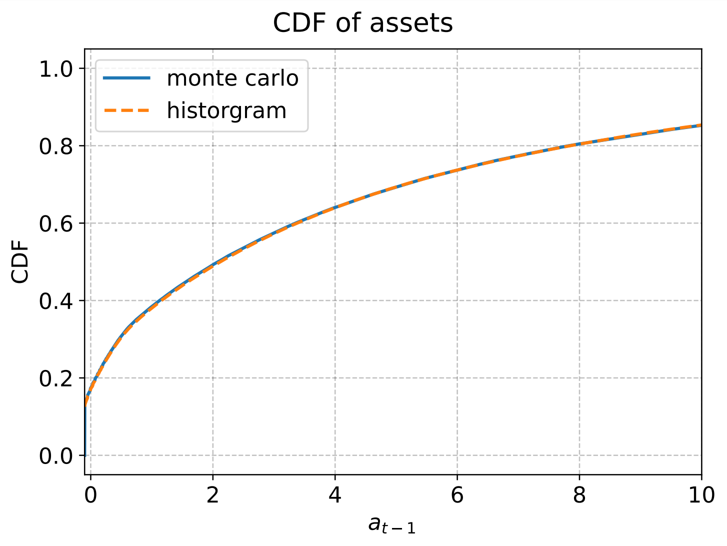
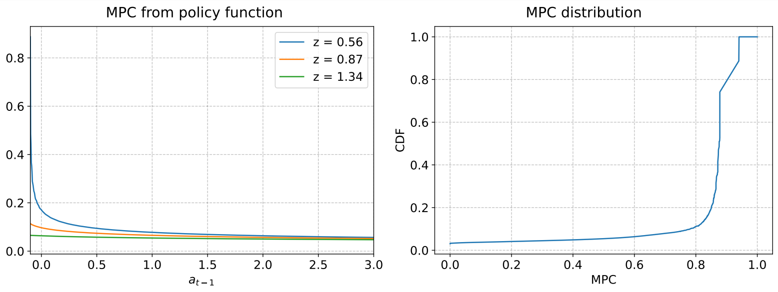
- **Review:**
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

CDF of savings in final period



MPCs



MPC from policy function
z = 0.56
z = 0.87
z = 1.34
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
a_{t-1}
MPC distribution
CDF
MPC
0.0
0.2
0.4
0.6
0.8
1.0

Side-note: Matrix formulation

- The histogram method can be written in **matrix form**:

$$\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$$
$$\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$$

  where

  $\underline{\boldsymbol{D}}_t$ is vector of length $\#_z \times \#_a$

  $\boldsymbol{D}_t$ is vector of length $\#_z \times \#_a$

  Π_z' is derived from the π_{i_{z-},i_z}'s

  Λ_t' is derived from the ι's and ω's

- **Note:** Example shown in notebook
- **Further details:** Young (2010), Tan (2020), Ocampo and Robinson (2022)

EGM

Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z - 1} \pi_{i_z, i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

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2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

Endogenous grid-point method (EGM)

Alternative to value function iteration:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z,i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

2. **Invert Euler-equation:**

$$c(z^{i_z}, a^{i_a}) = (\beta(1+r)q(z^{i_z}, a^{i_a}))^{-\frac{1}{\sigma}}$$

3. **Endogenous cash-on-hand:**

$$m(z^{i_z}, a^{i_a}) = a^{i_a} + c_+(z^{i_z}, a^{i_a})$$

Endogenous grid-point method (EGM)

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4. **Consumption function:** Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$
   If $m \leq m(z^{i_z}, a^0)$: $c^\star(z^{i_z}, a^{i_{a-}}) = m + wb$
   Else: $c^\star(z^{i_z}, a^{i_{a-}}) =$ interpolate $m(z^{i_z}, :)$ to $c(z^{i_z}, :)$ at m

Practice

In practice

- **EconModel:** Go through notebook 01. Using the EconModelClass (except part on C++)
- **ConSav:** Look at the 04. Tools folder.
- **Todays notebook:** *Consumption-Saving Model* show implementation of solution and simulation methods.

Summary

Summary and next week

- **Today:**
  1. Introduction to course
  2. Consumption-saving models
  3. Numerical dynamic programming
- **Next week:** Stationary equilibrlium
- **Homework:**
  1. **Work on:** Familiarize your self with the code
  2. **Read:** Aiyagari (1994), »Uninsured Idiosyncratic Risk and Aggregate Saving«