CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

4. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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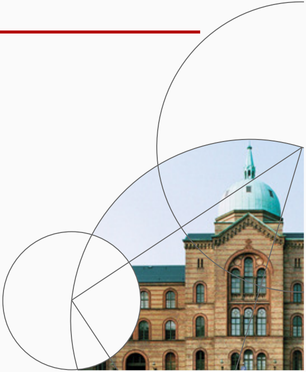
Danmarks
Grundforskningsfond
Danish National
Research Foundation



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  1. Is in active development
  2. You can help to improve interface
  3. You can help to find bugs
  4. You can help to add features

  Documentation: See GEModelToolsNotebooks

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- **Literature:** Aiyagari (1994)

HANC

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  3. The Standard Incomplete Market (SIM) model

Notation - aggregate variables

- **Aggregate variables (quantities and prices):**
  1. Output: Y_t
  2. Consumption: C_t
  3. Investment: I_t
  4. Technology: Γ_t
  5. Capital: K_t
  6. Labor: L_t
  7. Rental rate: r_t^k
  8. Real wage: w_t
  9. Real interest rate: r_t
  10. Profit: Π_t

Notation - idiosyncratic variables

- **Idiosyncratic variables:**
  1. Savings: a_t (end-of-period)
  2. Consumption: c_t
  3. Productivity: z_t
- **Distributions:**
  1. $\underline{\boldsymbol{D}}_t$ over z_{t-1} and a_{t-1}
  2. $\boldsymbol{D}_t$ over z_t and a_{t-1}

Firms

- **Production function:** $Y_t = \Gamma_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^k K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1},L_t} \Pi_t$
  1. Rental rate: $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^k = \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha\text{-}1}$
  2. Real wage: $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha}$

Households - formulation

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \mid z_t, a_t\right] \\
\text{s.t. } & a_t + c_t = (1 + r_t)a_{t-1} + w_t z_t + \Pi_t \geq 0 \\
& \log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_t] = 1
\end{aligned}
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with $r_t \equiv r_t^k - \delta$, where δ is the depreciation rate

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- **Aggregates:**

$$A_t^{hh} = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t = A^{hh}\left(\boldsymbol{D}_t, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}\right) = \boldsymbol{a}_t^{*\prime}\boldsymbol{D}_t$$

$$C_t^{hh} = \int c_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t = C^{hh}\left(\boldsymbol{D}_t, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}\right) = \boldsymbol{c}_t^{*\prime}\boldsymbol{D}_t$$

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- **Distributional dynamics** (with histogram method):

  1. Stochastic: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$
  2. Choices: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t, \quad \Lambda_t = \Lambda\left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}\right)$

Households - solution

- **Beginning-of-period value function:**

$$\underline{v}_t(z_{t-1}, a_{t-1}) = \mathbb{E}\left[v_t(z_t, a_{t-1}) \,|\, z_{t-1}, a_{t-1}\right]$$

Note: This re-formulation will be useful later in the course

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- **Envelope theorem:** Differentiate with fixed a_t choice

$$\underline{v}_{a,t} \equiv \frac{\partial \underline{v}_t}{\partial a_{t-1}} = \mathbb{E}\left[(1+r_t)c_t^{-\sigma} \mid z_{t-1}, a_{t-1}\right]$$

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- **EGM:**

  1. Find solution from FOC

$$c_t^{-\sigma} = \beta \underline{v}_{a,t+1} \Leftrightarrow c_t = \left(\beta \underline{v}_{a,t+1}\right)^{-\frac{1}{\sigma}}$$

  2. Calculate endogenous grid $m_t = a_t + c_t$
  3. Interpolate at $m_t = (1+r_t)a_{t-1} + w_t z_t + \Pi_t$

Resource constraint and market clearing

- **Law-of-motion for capital**

$$K_t = (1-\delta)K_{t-1} + I_t$$

- **Market clearing:**
  1. Labor market: $L_t = \int z_t d\boldsymbol{D}_t = 1$
  2. Goods market: $Y_t = C_t + I_t$
  3. Capital market: $K_{t-1} = \int a_{t-1} d\boldsymbol{D}_t$

Walras's Law

- **Assumption:** The *capital market clears*

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  2. Capital dynamics implies $I_t = K_t - (1-\delta) K_{t-1}$
  3. Aggregating across individuals

$$
\begin{aligned}
C_t &= \int c_t dD_t \\
&= \int \left[(1+r_t) a_{t-1} + w_t z_t - a_t \right] d\boldsymbol{D}_t \\
&= (1+r_t) K_{t-1} + w_t - K_t
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\end{aligned}
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  4. Combined: Then *the goods market clears*

$$
\begin{aligned}
C_t + I_t &= \left[(1+r_t)K_{t-1} + w_t - K_t\right] + \left[K_t - (1-\delta)K_{t-1}\right] \\
&= w_t + (r_t + \delta)K_{t-1} \\
&= Y_t
\end{aligned}
$$

Equation system

The model can be written as an **equation system**

$$\boldsymbol{H}\left(\{K_t, L_t; \Gamma_t\}_{t\geq 0}, \underline{\boldsymbol{D}}_0\right) = \left[\begin{array}{c} r_t - (\alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1} - \delta) \\ w_t - (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t - \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t' \boldsymbol{D}_t \\ K_t - \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t - 1 \\ \forall t \in \{0, 1, \dots\} \end{array}\right] = \mathbf{0}$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $a_t^* = a^*\left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau\geq t}\right)$
2. Choice transition: $\Lambda_t = \Lambda\left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau\geq t}\right)$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$H_{ss}\left(K_{ss}, L_{ss}; \Gamma_{ss}\right) = \begin{bmatrix} r_{ss} - (\alpha \Gamma_{ss}(K_{ss}/L_{ss})^{\alpha-1} - \delta) \\ w_{ss} - (1-\alpha)\Gamma_{ss}(K_{ss}/L_{ss})^{\alpha} \\ \boldsymbol{D}_{ss} - \Pi_z' \underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda_{ss}' \boldsymbol{D}_{ss} \\ K_{ss} - \boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss} \\ L_{ss} - 1 \end{bmatrix} = \mathbf{0}$$

Note I: Households still move around »inside« the distribution due to idiosyncratic shocks

Note II: Steady state for aggregates (quantities and prices) and the distribution as such

Stationary equilibrium - more verbal definition

For a given Γ_{ss}

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution $\boldsymbol{D}_{ss}$ over z_t and a_{t-1}
4. and the policy functions $a_{ss}^*(z_t, a_{t-1})$ and $c_{ss}^*(z_t, a_{t-1})$

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. The labor market clears
5. The capital market clears
6. The goods market clears

Direct implementation

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $r_{ss} = \alpha \Gamma_{ss} (K_{ss})^{\alpha - 1} - \delta$ and $w_{ss} = (1 - \alpha) \Gamma_{ss} (K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss}$

Direct implementation (alternative)

Root-finding problem in r_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $K_{ss} = \left(\frac{r_{ss}+\delta}{\alpha\Gamma_{ss}}\right)^{\frac{1}{\alpha-1}}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
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Indirect implementation

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{k} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

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Steady interest rate

- **Complete markets / representative agent:**
  Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1+r)C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1+r)C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1+r}$$

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- **Heterogeneous agents:** *No such equation exists*
  1. Euler-equation replaced by asset market clearing condition
  2. Idiosyncratic income risk affects the steady state interest rate

Calibration

How to choose parameters?

- **External calibration:** Set subset of parameters to the *standard values in the literature* or *directly from data estimates* (e.g. income process)

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  1. **Informal:** Roughly match targets by hand
  2. **Formal:**
     - 2a. Solve root-finding problem
     - 2b. Minimize a squared loss function
  3. **Estimation:** Formal with squared loss function + standard errors

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  3. **Estimation:** Formal with squared loss function + standard errors
- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

Exercises: Model extensions

1. **Households:** Solve

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] \\
\text{s.t. } a_t + c_t &= (1 + r_t) a_{t-1} + (1 - \tau_t) z_t \geq 0 \\
\log z_{t+1} &= \rho_z \log z_t + \psi_{t+1} \quad , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \, \mathbb{E}[z_t] = 1
\end{aligned}
$$

   where r_t is the real-interest rate and τ_t is a tax rate

2. **Government:** Set taxes and government bonds follows

$$
B_{t+1} = (1 + r_t) B_t - \int \tau_t z_t d\boldsymbol{D}_t
$$

3. **Bond market clearing:** $B_t = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
4. **Define and find the stationary equilibrium**
5. **What is the optimal level of τ_t?**

Summary

Summary and next week

- **Today:**
  1. The concept of a stationary equilibrium
  2. Introduction to the GEModelTools package
- **Next week:** More on models with interesting dynamics in the stationary equilibrium
- **Homework:**
  1. Work on completing the model extension exercise
  2. Read: Hubmer et al. (2021)