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The Marginal Propensity to Consume in a Two-Asset Life Cycle Model

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Term Paper in Dynamic Programming and Structural Econometrics

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Abstract

We examine consumption and saving behavior in a two-asset life cycle model featuring a liquid and an illiquid asset. Our model and solution methods are inspired by Druedahl (2021), although we adapt the model to a life cycle setting and add stochastic illiquid asset returns. After solving the two-asset model using the Nested Value Function Iteration (NVFI) and Nested Endogenous Grid Method (NEGM) algorithms and briefly illustrating the Simulated Method of Moments (SMM) approach to structural estimation, we compare the simulation results with a benchmark one-asset model in the style of Carroll (1997). Specifically, we focus on the Marginal Propensity to Consume (MPC). Consistent with recent contributions by Kaplan and Violante (2014; 2022), we find that the two-asset model creates larger MPCs, which better matches empirical estimates. However, the average yearly MPC in our two-asset model is smaller than what is typically found in the empirical literature. Notably, because our application includes a relatively time-consuming pre-computation step, solving the model using either NVFI or NEGM is approximately equally fast.

Keywords: Precautionary-Savings Models; Nested Endogeneous Grid Method; Marginal Propensity to Consume; Two-Asset Models

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1 Introduction

In this term paper, we aim to compute the marginal propensity to consume (MPC) of a transitory income change. The size of the MPC is of essential importance for central macroeconomic concepts such as the fiscal multiplier, distributional effects, monetary policy transmission, portfolio choice and the propagation of shocks throughout the economy (Kaplan & Violante, 2022). Therefore, the ability of macroeconomic models to match empirical evidence on MPCs is crucial for their credibility, validity and relevance for policy advice. The motivation for our focus is the ongoing shift in contemporary macroeconomic research, where the widely criticized Representative Agent New Keynesian (RANK) model is gradually being crowded out in favor of models with many heterogeneous agents. Unlike RANK models, Heterogeneous Agent New Keynesian (HANK) models feature household blocks similar to precautionary savings models like those in this term paper (see e.g. Kaplan et al., 2018). The richer modeling framework makes household behavior in HANK models better aligned with empirical heterogeneous microeconometric consumption evidence, including the MPC (Kaplan & Violante, 2018).

We solve and simulate two life cycle models, namely a standard one-asset model similar to that of Carroll (1997), and an extended two-asset model with both liquid and illiquid assets. In the latter model, households place their saving either in liquid assets yielding a known interest rate or in durable goods stock, which yields both an uncertain financial return and a utility service flow. In each period, households decide whether to adjust the stock of durable goods subject to a transaction cost. After solving the models, we compare the simulation results between the two models and relate to the literature.

While the analytical exercise in our term paper is inspired by the recent work of Kaplan and Violante (2022), we draw heavily on Druedahl (2021) with respect to the model structure and solution methods. We choose to interpret the durable as housing and due to the discrete choice of either keeping or adjusting the housing stock, the model structure becomes more complex. As such, the standard Endogeneous Grid Method (EGM) algorithm is no longer a viable solution method due to the presence of non-convexities. Hence, we draw on modified versions of the Value Function Iteration (VFI) and EGM algorithms introduced in the course in order to solve the model within a reasonable amount of time. Specifically, we apply the Nested VFI (NVFI) and Nested EGM (NEGM) algorithms developed by Druedahl (2021). As the name suggests, these algorithms utilize a latent nesting structure present in models like those in this paper together with fast interpolation methods in order to substantially reduce computational costs.

Inspecting the life cycle profiles of the two-asset model, our results suggest that households use the illiquid asset to smooth consumption, in the same way they use liquid assets to smooth consumption in the one-asset model. The choice to additionally adjust and accumulate housing stock however affects the accumulation of the non-durable asset and therefore also the MPCs, resulting in larger MPCs. After solving and simulating both models, we find an annual MPC of 13.91 pct. in the one-asset model and MPCs in the range $19.23 - 26.47$ pct. in the two-asset model depending on the computation method and calibration of asset returns.

There is a large empirical literature on consumption behavior to income changes with estimates of quarterly MPCs of transitory income ranging from 15 up to 35 pct. Although our simulated MPCs are consistent with Kaplan and Violante (2022) in that the two-asset model is able to produce larger MPCs than the one-asset model, our annual MPCs are still low as compared to the empirical evidence on quarterly MPCs. We furthermore find that the size of the MPC is sensitive to the spread between return rates on durable and non-durable goods; a larger spread implies a larger MPC, as a larger return on durable goods increases the incentive to accumulate durable rather than non-durable goods, such that households are more liquidity constrained.

Finally, we compare the performance of VFI, NVFI and NEGM. While solving our two-asset model with standard VFI requires upwards of an estimated 13 hours model given our computational resources, NVFI and NEGM get the job done in 7.3 and 7 minutes respectively.

2 Model Description

2.1 A Two-Asset Life Cycle Model

We base our model on the consumption-saving model in Druedahl (2021), which features both durable and non-durable consumption goods, where adjustments in the stock of the latter are subject to a cost. Hence, the model is an extension of the standard one-asset buffer stock model covered in the course.[1] We choose to interpret the durable good as housing, such that it becomes natural to think of the durable both as a good and as an asset.

The decision variables in the model are housing stock chosen *within each period*, d_t and non-durable consumption, c_t. The state variables are permanent income, p_t, housing stock at the *beginning of each period*, n_t and cash-on-hand, m_t. Note that due to the presence of non-convexities, the usual trick of normalizing the model with respect to permanent income is not applicable. Hence, the extended model is comprised of two decision variables and three state variables, unlike the standard model with only one decision and state variable.

[1] A description of the simple one-asset buffer stock model is given in appendix A.1.

Utility follows CRRA preferences with a Cobb-Douglas aggregate over consumption, c_t and housing services, h_t.

$$u\left(c_t, h_t\right) = \frac{\left(c_t^{\alpha} h_t^{1-\alpha}\right)^{1-\rho}}{1-\rho} \tag{2.1}$$

Future utility is discounted with a factor $\beta \in (0,1)$. Housing services are proportional to the chosen level of housing stock, d_t, where $\underline{d} > 0$ is a floor under the housing stock.

$$h_t = \phi\left(d_t + \underline{d}\right), \quad \phi \in (0,1) \tag{2.2}$$

Thus, we can interpret $\underline{d}$ as the utility derived from a minimum standard housing option being provided by the government. Otherwise, the Cobb-Douglas specification would imply that even the poorest households are forced to purchase housing stock to avoid negative infinite utility.[2]

The income process covers permanent income, p_t, and transitory income, y_t. Permanent income is subject to constant exogeneous growth reflecting technological growth in the economy, $G = (1+g)$, a deterministic income profile path including retirement, l_t, and a shock, ψ_t. Transitory income is then determined by permanent income as well as a transitory shock, ξ_t. Both shocks, ψ_t, ξ_t are log-normally distributed.

$$p_{t+1} = \psi_{t+1} p_t G l_t, \quad \log \psi_{t+1} \sim N\left(-0.5\sigma_{\psi}^2, \sigma_{\psi}^2\right) \tag{2.3}$$

$$y_{t+1} = \xi_{t+1} p_{t+1}, \quad \log \xi_{t+1} \sim N\left(-0.5\sigma_{\xi}^2, \sigma_{\xi}^2\right) \tag{2.4}$$

The fact that labor income risk is idiosyncratic and uninsurable, combined with the CRRA utility function displaying prudence (i.e. $u_c''' > 0 \Rightarrow -(u_c'''/u_c'')c > 0$), means that households have a precautionary savings motive (Kimball, 1990).[3] Moreover, Carroll and Kimball (2001) shows that households who face occasionally binding liquidity constraints, such as in our model, will engage in precautionary saving to avoid hitting the constraint. Thus, consumers in our model build a buffer to self-insure against future negative income shocks. In addition, they will accumulate assets to consume out of when retired.

In each period the representative agent can choose to either keep her existing housing stock, $d_t = n_t$, or adjust her housing stock, $d_t \neq n_t$. The latter choice corresponds to selling the house "in full" and endure a proportional adjustment cost $\tau \in (0,1)$, such that

[2] See also the discussion in Section 6.

[3] This linkage between convexity of marginal utility and precautionary saving behavior is often disregarded in New Keynesian DSGE models because the solution typically is approximated by log-linearizing the system (including the Euler equation) around the steady state, whereby the convexity disappears.

the available liquid resources becomes

$$x_t = m_t + (1 - \tau)n_t \tag{2.5}$$

The end-of-period assets, a_t, thus depends on whether the agent chooses to keep or adjust.

$$a_t = \begin{cases} m_t - c_t & \text{if } d_t = n_t \\ x_t - c_t & \text{if } d_t \neq n_t \end{cases} \tag{2.6}$$

The next-period cash-on-hand, m_{t+1}, is determined by transitory income, y_{t+1}, and assets, where $R = (1 + r)$ is the interest rate on liquid assets.

$$m_{t+1} = Ra_t + y_{t+1} \tag{2.7}$$

The evolution of the housing stock is given by

$$n_{t+1} = (1 - \delta)R_h d_t \tag{2.8}$$

where $\delta > 0$ can be interpreted as the maintenance cost associated with housing (depreciation) and $R_h = R + r_h > R$ is the return on housing, reflecting that the value of houses tend to increase over time. We set $R_h > R$ such that there is an incentive to buy housing stock.

2.2 Extensions

The model we present above is an extension of the model in Druedahl (2021) by including growth to permanent income, G, a deterministic life cycle path for income, l_t (see Section 4), housing services in the utility function, h_t, and the return on housing, R_h. The implementation of these extensions are all somewhat trivial.

We further extend the model by making the return on housing capital stochastic. We want to implement a shock structure such that housing returns are larger than zero in expectation, $\mathbb{E}_t(r_h) > 0$, but is subject to large negative shocks with a low probability. We draw inspiration from Svensson (2003) and define:

$$r_h = 0.05 + \zeta_t, \quad \zeta_t = \varepsilon_t + \eta_t \tag{2.9}$$

ε_t denotes a normal-size shock to housing and follows a normal distribution. η_t denotes a large negative shock with probability γ.

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$
$$\eta_t = \begin{cases} 0 & \text{with prob. } 1-\gamma \\ \pi & \text{with prob. } \gamma \end{cases}$$

We then obtain a shock structure with a low-probability extreme event, such that in a high probability event $r_h = 0.05 + \varepsilon_t$, and in a low probability event $r_h = 0.05 + \varepsilon_t + \pi_t$.[4] The equations (2.1)-(2.9) thus complete the model description.

2.3 Bellman Equation

In order to solve the above model using the methods of Dynamic Programming, we need to write the model in recursive form. The general Bellman equation is given as

$$\begin{aligned} v_t\left(p_t, n_t, m_t\right) &= \max\left\{v_t^{keep}\left(p_t, n_t, m_t\right), v_t^{adj.}\left(p_t, x_t\right)\right\} \\ &\text{s.t.} \\ x_t &= m_t + (1-\tau)n_t \end{aligned} \tag{2.10}$$

where the two Bellman equations for the *adjust* and *keep* problem are given as

$$\begin{aligned} v_t^{keep}\left(p_t, n_t, m_t\right) &= \max_{c_t}\left\{u\left(c_t, h_t\right) + \beta\mathbb{E}_t\left[v_{t+1}\left(p_{t+1}, n_{t+1}, m_{t+1}\right)\right]\right\} \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ m_{t+1} &= Ra_t + y_{t+1} \\ n_{t+1} &= (1-\delta)R_h n_t \\ a_t &\geq 0 \end{aligned} \tag{2.11}$$

$$\begin{aligned} v_t^{adj.}\left(p_t, x_t\right) &= \max_{c_t, h_t}\left\{u\left(c_t, h_t\right) + \beta\mathbb{E}_t\left[v_{t+1}\left(p_{t+1}, n_{t+1}, m_{t+1}\right)\right]\right\} \\ &\text{s.t.} \\ a_t &= x_t - c_t - d_t \\ m_{t+1} &= Ra_t + y_{t+1} \\ n_{t+1} &= (1-\delta)R_h d_t \\ a_t &\geq 0 \end{aligned} \tag{2.12}$$

[4]When implementing the shock we consolidate ε_t and η_t, such that $\zeta_t = \begin{cases} \varepsilon_t & \text{with prob. } 1-\gamma \\ \varepsilon_t + \pi & \text{with prob. } \gamma \end{cases}$

Given the current state vector $s_t = (p_t, n_t, m_t)$, the agent decides the amount of durable and non-durable consumption entering the decision vector $e_t = (d_t, c_t)$, which in turn determines utility $u(c_t, h_t)$ and the next-period state jointly with the realized shocks (ψ_t, ξ_t, ζ_t). Thus, the solution to the problem is a decision rule $\lambda = \{\lambda_0, ..., \lambda_T\}$ for consumption $e_t(s_t)$ that solves the *Markov Decision Process* (MDP)

$$v(s) = \max_{\lambda=\{\lambda_0,...,\lambda_T\}} \mathbb{E}_\lambda \left[\sum_{t=0}^{T} \beta^t u_t(s_t, e_t) | s_0 = s \right] \tag{2.13}$$

In our specific case, the utility function is constant over time and independent of state variables, i.e. $u_t(s_t, e_t) = u(e_t)$.

3 Solution Methods

3.1 Backwards Induction and Discretization

As we use a model with a finite time-horizon, we can compute the optimal decision rule λ using backwards induction. In particular, since we assume that agents know their lifespan with certainty, it is optimal to consume all resources in the final period because there is "no tomorrow" and the continuation value thus is zero. Therefore, conditioning on keeping and adjusting respectively, we have

$$\begin{aligned}
v_T^{keep}(s_T) &= \max_{c_T \in (0, m_T)} u(c_T, h_T) = u(m_T, \phi(n_T + \underline{d})) \\
v_T^{adj.}(s_T) &= \max_{d_T \in (0, x_T)} u(c_T, h_T) = u(x_T - d_T, \phi(d_T + \underline{d})) \\
v_T(s_T) &= \max\{v_T^{\text{keep}}(s_T), v_T^{\text{adj.}}(s_T)\}
\end{aligned}$$

Knowing these terminal values, we can obtain the value function for each period by recursion on the Bellman equations given by (2.10)-(2.12) $\forall t \in \{0, 1, ..., T\}$.

For the above problem to be solvable using numerical methods, we first need to discretize the continuous state space and approximate the three-dimensional integral reflecting the expectation with respect to the shocks via either Monte Carlo draws or quadrature methods. We denote the discretized state space

$$\Psi_s = \{p^{i_p}\}_{i_p=1}^{n_p} \times \{n^{i_n}\}_{i_n=1}^{n_n} \times \{m^{i_m}\}_{i_m=1}^{n_m} \tag{3.1}$$

and choose the points in the grid such that it becomes relatively more dense for small values. Because we assume (log-) normal distributions, we use Gauss-Hermite quadrature (see Appendix A.3 for a brief description of this method) to obtain nodes and weights for integrating out the expectation. When choosing the density of the grid as well as the number of nodes, we face an inherent trade-off between precision and computational cost. We provide an overview of the chosen number of grids points and nodes in Table 5 in Appendix A.2.

The most basic implementation of backwards induction is equivalent to value function iteration (VFI), and can be summarized as follows for the keeper problem. For each period $t \in \{T, T-1, ..., 0\}$ and each combination of (p_t, n_t, m_t)

- If $t = T$, use the terminal conditions. Otherwise, set the lower and higher bound of non-durable consumption (i.e. $c_t \in (0, m_t)$). Then, for each point on a grid covering this interval:[5]

  1. Loop through all combinations of quadrature nodes and use the transition rules to compute the next period states $s_{t+1} = (p_{t+1}, n_{t+1}, m_{t+1})$ conditional on the given shock node $\chi_{ijk} = (\psi_i, \xi_j, \zeta_k)$.
  2. Interpolate on the next period conditional value functions to find $v_t^{keep}(s_{t+1})$ and $v_t^{adj.}(s_{t+1})$ and decide whether keeping or adjusting yields the highest value function (i.e. find $v_{t+1}(s_{t+1})$)
  3. Compute the continuation value as the weighted sum across quadrature nodes and weights ω_{ijk}: $\mathbb{E}_t v_{t+1}(s_{t+1}) \approx \sum_{i,j,k} \omega_{ijk} v_{t+1}(s_{t+1}|\chi_{ijk})$
  4. Compute the current period value function for the given level of non-durable consumption $v_t^{keep}(s_t) = u(c_t, h_t) + \beta \mathbb{E}_t v_{t+1}(s_{t+1})$

- After repeating steps 1-4 above for all points on the non-durable consumption grid, find the optimal consumption choice $c_t^*(s_t)$ given the current state and the associated value function $v_t^{keep\ *}(s_t)$.

- Move on to the next node in the state space and repeat the above procedure

Solving the adjuster problem involves precisely the same steps as above, except that the optimization is done with respect to both c_t and d_t. Despite being a valid solution method, basic VFI runs slowly for several reasons. The keep problem (2.11) is computationally costly due to the three-dimensional state space, and the adjust problem (2.12) is costly because of the two-dimensional decision space. In both problems however, the main cost comes from the computation of the continuation value $\beta \mathbb{E}_t \left[v_{t+1}(p_{t+1}, n_{t+1}, m_{t+1}) \right]$. Druedahl (2021) notes that the reason behind the continuation value becoming the main bottleneck is the high number of interpolations on the next period value function, as this needs to be done for each point in the state space and for each quadrature node used to

[5] We use the `golden_section_search.optimizer` function from the ConSav package developed by Jeppe Druedahl, which automatically determines the number of grid points in decision space when numerically solving for the optimal consumption choice.

compute the expected continuation value. With the computational power available to us, solving the baseline model with VFI is not practically feasible, see Section 5.3.

3.2 Nested Value Function Iteration

Druedahl (2021) demonstrates that substantial speed gains are possible when applying methods that utilize the structure of the problem in clever ways. We briefly cover these methods in the following. First, Druedahl notes that knowing the post-decision states p_t, d_t, a_t is sufficient to compute the continuation value and defines the *post-decision value function* as

$$\begin{aligned} w_t(p_t, d_t, a_t) &\equiv \beta \mathbb{E}_t \left[v_{t+1}(p_{t+1}, n_{t+1}, m_{t+1}) \right] \\ &= \beta \mathbb{E}_t \left[v_{t+1}(\underbrace{\psi_{t+1} p_t G l_{t+1}}_{p_{t+1}}, \underbrace{(1-\delta) R_h d_t}_{n_{t+1}}, \underbrace{R a_t + \xi_{t+1} \psi_{t+1} p_t G l_{t+1}}_{m_{t+1}}) \right] \end{aligned} \tag{3.2}$$

From the above, it is evident that given p_t, d_t, a_t, the next-period state variables $p_{t+1}, n_{t+1}, m_{t+1}$ are known in expectation, which enables us to rewrite the keep problem as

$$\begin{aligned} v_t^{\text{keep}}(p_t, n_t, m_t) = \max_{c_t} \, & u(c_t, n_t) + w_t(p_t, d_t, a_t) \\ & \text{s.t.} \\ & a_t = m_t - c_t \geq 0 \\ & d_t = n_t \end{aligned} \tag{3.3}$$

Importantly, we can calculate the post-decision value function without knowledge of current period consumption c_t. Hence, specifying a grid for a_t, we can loop through the shock realizations and compute w_t *only once* in a prior step, and then subsequently make only one interpolation on w_t per state space node instead of one interpolation on v_{t+1} for each possible combination of shock nodes. For a reasonably dense post-decision grid and a reasonable number of quadrature nodes, this drastically reduces the number of interpolations.

Second, a further speed up of the adjust problem is possible from recognizing that the model has a nesting structure. Specifically, we can consider the adjust problem as sequential; the household first decides the level of housing consumption, d_t, and then decides the level of consumption, c_t, given the chosen d_t. The consumption choice in the adjust problem is therefore the same as in the keep problem for a given amount of cash-on-hand left over after the durable consumption choice. Thus, we can reformulate the adjust problem into

$$
\begin{aligned}
v_t^{\mathrm{adj}}(p_t, x_t) &= \max_{d_t} v_t^{\mathrm{keep}}(p_t, d_t, m_t) \\
&\text{s.t.} \\
m_t &= x_t - d_t \\
d_t &\in [0, x_t]
\end{aligned} \tag{3.4}
$$

Using this nesting structure lowers the computational cost of the adjust problem in part by reducing its dimensionality and in part due to the fact that the evaluation of the utility function and multiple evaluations of the continuation value is replaced with a single interpolation of the value function from the keep problem. Druedahl (2021) names the algorithm resulting from the combination of the post decision value function and nesting the keep problem within the adjust problem Nested Value Function Iteration (NVFI).

3.3 The Endogeneous Grid Method With an Upper Envelope

In the NVFI algorithm, solving the keep problem still requires iterating over a three-dimensional state space. However, as the keep problem only involves a non-durable consumption decision, it is possible to solve it analytically with a generalized version of the Endogenous Grid Method (EGM) originally proposed by Carroll (2006). This method relies on deriving the household's Euler equation from the Bellman equation, and then back out the current period consumption choice and cash-on-hand for a fixed grid of end-of-period assets a_t. Due to the non-convex nature of the budget set arising from the discrete choice between adjusting or keeping, the Bellman equation (2.10) is not globally concave, and thus not everywhere differentiable. Therefore, we cannot derive the Euler equation in the usual manner, adhering to the basic version of the Envelope theorem.[6]. Instead, we can motivate that the Euler-equation still must be a necessary, although not sufficient, condition satisfied in all interior solutions using a variational argument.

Romer (2018) presents the variational argument for consumption smoothing as follows. If the household decides to reduce consumption today by an infinitesimal amount Δc to increase consumption tomorrow by Δc, then the change in marginal utility in the two periods must be equal for the choice to be optimal. That is, the marginal disutility of reducing consumption today by Δc must be equal to marginal utility of increasing consumption tomorrow by Δc. Conversely, the same argument could be made if the household were to increase consumption today by lowering future consumption. Equating the marginal disutility of reducing current consumption with the marginal utility of increasing future

[6]In the one-asset model outlined in Appendix A.1, we apply the standard method.

consumption we can thus obtain the Euler equation as

$$\Delta c u'_{t,c} = \Delta c \beta R \mathbb{E}_t[u'_{t+1,c}]$$
$$\alpha c_t^{\alpha(1-\rho)-1}(\phi(d_t+\underline{d}))^{(1-\alpha)(1-\rho)} = \beta R \mathbb{E}_t \left[\alpha c_{t+1}^{\alpha(1-\rho)-1}(\phi(d_{t+1}+\underline{d}))^{(1-\alpha)(1-\rho)}\right]$$

Next, we define $q_t(p_t, d_t, a_t) \equiv \beta R \mathbb{E}_t \left[\alpha c_{t+1}^{\alpha(1-\rho)-1}(\phi(d_{t+1}+\underline{d}))^{(1-\alpha)(1-\rho)}\right]$ as the post decision marginal value of cash following Druedahl (2021). We can therefore finally rewrite the Euler equation exclusively in terms of current period variables as

$$\alpha c_t^{\alpha(1-\rho)-1}(\phi(d_t+\underline{d}))^{(1-\alpha)(1-\rho)} = q_t(p_t, d_t, a_t) \tag{3.5}$$

From which we can back out consumption and cash-on-hand given by

$$c_t = \left[\frac{q_t(p_t, d_t, a_t)}{\alpha[\phi(d_t+\underline{d})]^{(1-\alpha)(1-\rho)}}\right]^{\frac{1}{\alpha(1-\rho)-1}} \tag{3.6}$$
$$m_t = a_t + c_t \tag{3.7}$$

As mentioned, the non-convexities present in the model implies that the Euler equation is no longer a sufficient, but only a necessary condition. Consequentially, a naive implementation of the EGM algorithm will most likely generate both optimal as well as non-optimal consumption choices, the latter not being points on the appropriate consumption function. Druedahl (2021) suggests extending the EGM algorithm with an upper envelope which removes these non-optimal points. The upper envelope step consists of a nested for-loop where the endogenous consumption and value function points are held up against the consumption and value of choice implied by linear interpolation over an exogenously created grid for cash on hand. By iteratively updating the consumption and value of choice associated with the exogenous grid to be the best yet found, non-optimal points gets removed. Adding q_t to the pre-computation step and solving the keep problem with EGM including the upper envelope, we arrive at the Nested EGM (NEGM) algorithm.

3.3.1 Loop-reordering and fast interpolation

Druedahl (2021) further points out that the computational time associated with w_t and q_t can be sped up by re-ordering the loops and interpolation. The standard approach is to loop over each combination of post-decision states (p_t, n_t, a_t) as well as the shocks (ψ_t, ξ_t, ζ_t) in order to calculate the values for $p_{t+1}, n_{t+1}, y_{t+1}, m_{t+1}$, which can then be used in the interpolation for the next-period value function v_{t+1} and next-period marginal utility $u'_{t+1,c}$.[7]

[7] We use linear interpolation, which in multiple dimensions corresponds to calculating a weighted average of a hypercube around the desired point.

Instead, Druedahl (2021) utilizes that the m_{t+1} vector is monotonically increasing for given $(\psi_{t+1}, \xi_{t+1}, \zeta_{t+1})$, when the a_t vector is chosen as monotonically increasing. Thus, given $p_t, n_t, \psi_{t+1}, \xi_{t+1}, \zeta_{t+1}$ we can calculate p_{t+1}, n_{t+1}, and therefore need only loop through the a_t grid to calculate m_{t+1} and subsequently w_t and q_t after interpolating to determine the next-period value function and marginal utility. Therefore, we only search through $\{p^{i_p}\}_{i_p=1}^{n_p} \times \{n^{i_n}\}_{i_n=1}^{n_n}$ once rather than for each m_t. Additionally, reordering the loops makes it possible to use the monotonicity to speed up the search through the a_t (and therefore also the m_t) grid.

4 Calibration and Simulation

Having presented the model setup and solution methods, we now move on to calibrating and simulating the model. Table 1 provides a description of all the model parameters along with their chosen calibration values, which we will go through in the following. We think of each period t as being of a one-year duration and let the consumer enter the model at age 25, retire at age 65 and exit the economy at age 80. The consumer experiences a one-time drop in income at retirement, reflecting the transition from wages to pension benefits, after which income remains low and stable throughout retirement. Additionally, during retirement income is no longer subject to permanent or transitory income shocks. First, with respect to preferences we set the subjective discount factor $\beta = 0.965$, implying a yearly subjective discount rate slightly above 3.6 pct., and the degree of relative risk aversion $\rho = 2$. Arguably, both values are within the range of standard choices in macroeconomic applications. α and ϕ are somewhat arbitrarily chosen.

Next, we calibrate the parameters associated with asset returns and the deterministic part of the income profiles. As mentioned, we set the return on housing to be higher (in expectation) than the return on safe liquid assets, such that there is an incentive to accumulate housing stock beyond the mere utility derived from housing services. This savings motive is in line with the dual role of durables as both goods and assets. Moreover, based on the rule of thumb suggested by the admittedly not very academic reference Bolius, we set the proportional adjustment cost $\tau = 0.05$. Finally, we choose a low value of $\delta = 0.02$, reflecting that houses depreciate slowly as stated by Piazzesi and Schneider (2016).

The growth in permanent income that is taken to reflect technological progress is set to 2 pct. such that $G = 1.02$. Next, we specify l_t for $t \in \{1, 2, ..., T_r - 1\}$ to be equally spaced points on the interval $(1, 1/G)$, $l_{T_r} = 0.67/G$ and then $1/G$ afterwards. Thus, the consumer experiences diminishing permanent income growth throughout her time on the labor market followed by a substantial drop in permanent income when retiring, and a subsequent flat income profile when retired (see e.g. Figure 3a).

Lastly, we must specify the mean and variance of the shocks. Here, we largely follow Carroll (1997) with respect to the income process. Furthermore, we select $\gamma = 0.05$ and $\pi = -0.25$, equivalent to a house price crash of approximately $(E[r_h] - \pi) * 100 = -17$ pct. occurring once every 20 years. Finally, we set the variance of ε to be rather small such that the return on housing in normal times is $r_h \in (0.00, 0.16)$ with 95 pct. certainty.

Table 1: Summary of Model Parameters and Calibrated Values

Parameter	**Description**	**Calibration**
	Preferences	
β	Subjective discount factor	0.965
ρ	Risk aversion	2
α	Intratemporal consumption share in utility	0.9
ϕ	Scaling of housing services	0.8
	Income profiles and returns	
R	Interest rate on liquid assets, $R = 1 + r$	1.03
$\mathbb{E}[R_h]$	Expected housing returns, $R + \mathbb{E}[r_h] > R$.	1.08
τ	Adjustment cost of housing stock	0.05
δ	Depreciation rate of housing, i.e. maintenance costs	0.02
G	Constant growth in permanent income, $G = 1 + g$	1.02
	Shocks	
ξ	Transitory income shock	$\log \xi \sim N(-0.5\sigma_\xi^2, \sigma_\xi^2)$
ψ	Permanent income shock	$\log \psi \sim N(-0.5\sigma_\psi^2, \sigma_\psi^2)$
σ_ξ	Standard deviation of transitory income shock	0.1
σ_ψ	Standard deviation of permanent income shock	0.1
ε	Normal-size shock to r_h	$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$
σ_ϵ^2	Standard deviation of normal-size shock to r_h	0.04
π	Extreme event shock to r_h	-0.25
γ	Probability of extreme event	0.05

In order to simulate the model, we first draw shock realizations in each period for $N =$ 10.000 consumers. Next, we can utilize the model solution to compute income profiles, optimal consumption paths, adjustment behavior and so on and so forth. Importantly, even though the shocks are drawn from the same distributions, households are exposed to different shock realizations within each period as well as throughout the life cycle. Hence, the simulated economy will feature wealth heterogeneity and the consumers will differ with respect to their liquidity constraints, which is crucial for obtaining sizable and heterogeneous MPCs. After a slight detour in the next subsection, we present various simulation output from both models in Section 5.

4.1 Structural Estimation of (Some) Model Parameters

In this subsection we present a mock example illustrating the Simulated Method of Moments (SMM) approach to structural estimation (Adda & Cooper, 2003, Ch. 4). Let $\Lambda^d = \frac{1}{N^d} \sum_{i=1}^{N^d} \Lambda_i^d$ be a vector of empirical moments, θ be a $J \times 1$ vector of parameters to be estimated and $\Lambda^m(\theta) = \frac{1}{SN^s} \sum_{j=1}^{S} \sum_{k=1}^{N^s} \Lambda_{jk}^s(\theta)$ the corresponding model moments averaged over S simulations. Then the SMM estimate is the vector that minimizes the squared distance between the empirical and simulated moments:

$$\hat{\theta} = \arg\min_{\theta}(\Lambda^d - \Lambda^m(\theta))'W(\Lambda^d - \Lambda^m(\theta)) \tag{4.1}$$

where W is a symmetric and positive semi-definite weighting matrix. Given some high level regularity conditions and assuming that our model is correctly specified, the SMM estimator is consistent and asymptotically normal, i.e.

$$\sqrt{N_d}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, (1 + S^{-1})V)$$
$$V = (G'WG)^{-1}G'W\Omega W'G(G'WG)^{-1}$$

where $G = \frac{\partial(\Lambda^d - \Lambda^m)}{\partial\theta}$ is the Jacobian of the moment condition with respect to the parameters and Ω is the variance-covariance matrix of the empirical moment conditions. Notably, consistency and asymptotic normality holds for a fixed number of simulations, albeit a small S yields a larger variance.

For ease of implementation, we create artificial empirical moments through by simulating data from our model using a different seed number. We choose cross-sectional means of a_t as the moments to be matched. Because all moments are measured on the same scale, we let W be the identity matrix. For the estimation routine to run within a reasonable amount of time, we set $S = J = 1,\ T_r = 10,\ T = 15$ and loop through a relatively coarse grid with candidate values of $\theta = \beta$, for which we solve and simulate the model and compute the objective function. Our estimate $\hat{\theta}$ is thus the value of β that minimizes (4.1), which we illustrate in Figure 5 in Appendix A.4. Unsurprisingly, as our empirical data is merely alternative simulations, we are able to exactly estimate the original calibrated value of β.[8]

5 Results

5.1 One-Asset Model

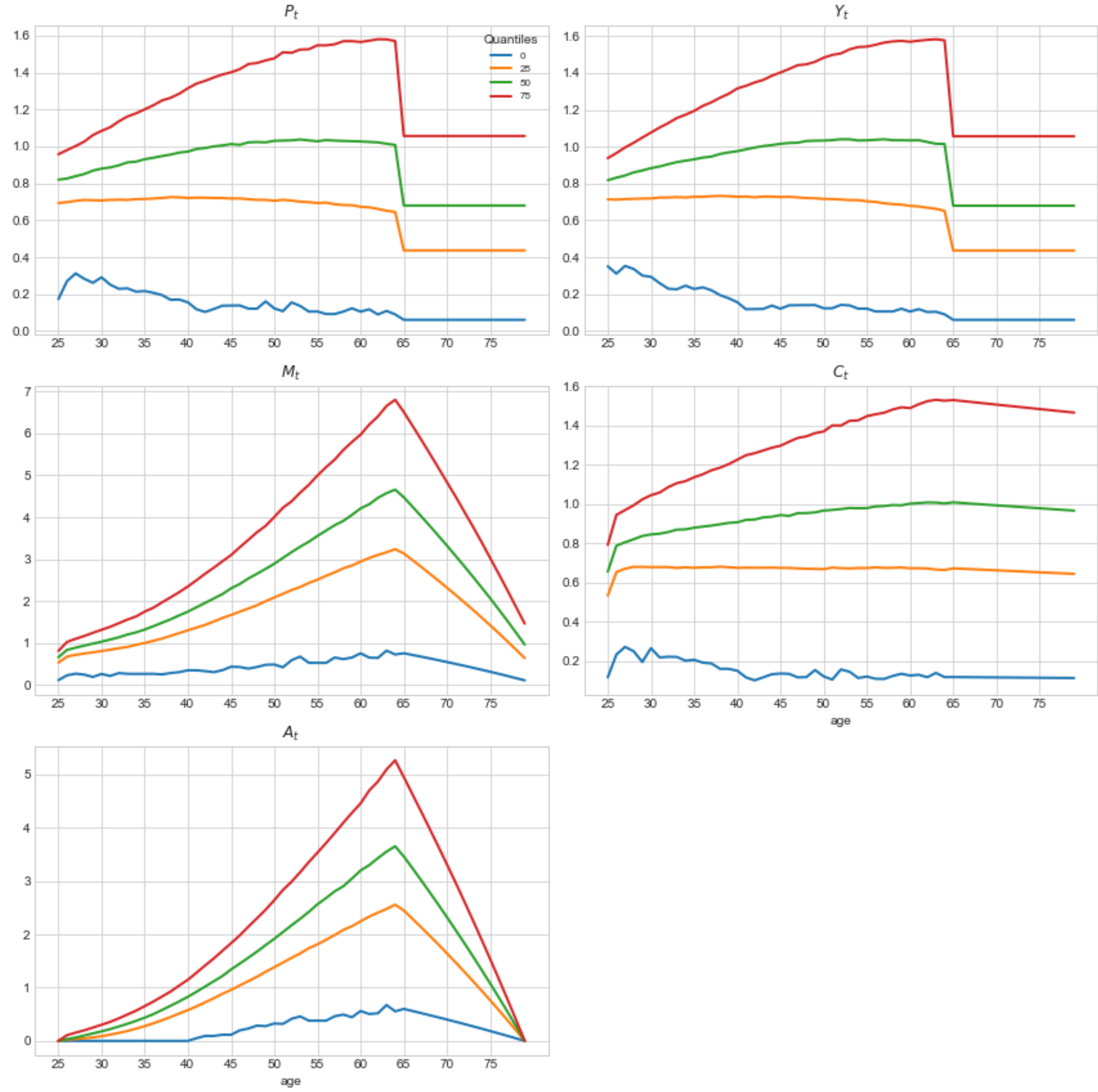
We first solve and simulate a simple one-asset model using standard EGM.[9] In Figure 1, we show the simulated life cycle for permanent income, transitory income, cash-on-hand, consumption and end-of-period assets.

[8]Concerning standard errors, the natural approach would be to compute these using a bootstrap routine as it is impossible to differentiate the moment condition to obtain G. Because bootstrapping would require (a lot) more model solutions and simulations and thus is quite computationally intensive, we refrain from computing standard errors.

[9]The full model description is given in Appendix A.1.

Both permanent, P_t, and transitory income, Y_t, follow very similar paths with an upward trend over the work life, and we can clearly observe the drop at retirement. After retirement both income shocks (ψ_{t+1}, ξ_{t+1}) are removed such that transitory and permanent income are effectively equal, and the profiles are completely flat and deterministic. For the poorest individual in the simulation (the blue line in the bottom of the figures), the drop at retirement is hardly observable because permanent income is already at a very low level.

Figure 1: One-Asset Model Life Cycle



Lines represent 0th, 25th, 50th and 75th quantiles.

End-of-period assets, A_t, and cash-on-hand, M_t, likewise follow very similar paths. Agents accumulate assets during their time on the labor market, and then gradually deplete their assets when retired. Consumption, C_t, is increasing over the work life and decreasing slightly after retirement. Despite the drop in income and kink in asset accumulation, the consumption profile for all individuals is smoothed over the life cycle. This is a result of the precautionary savings motives present in the model.

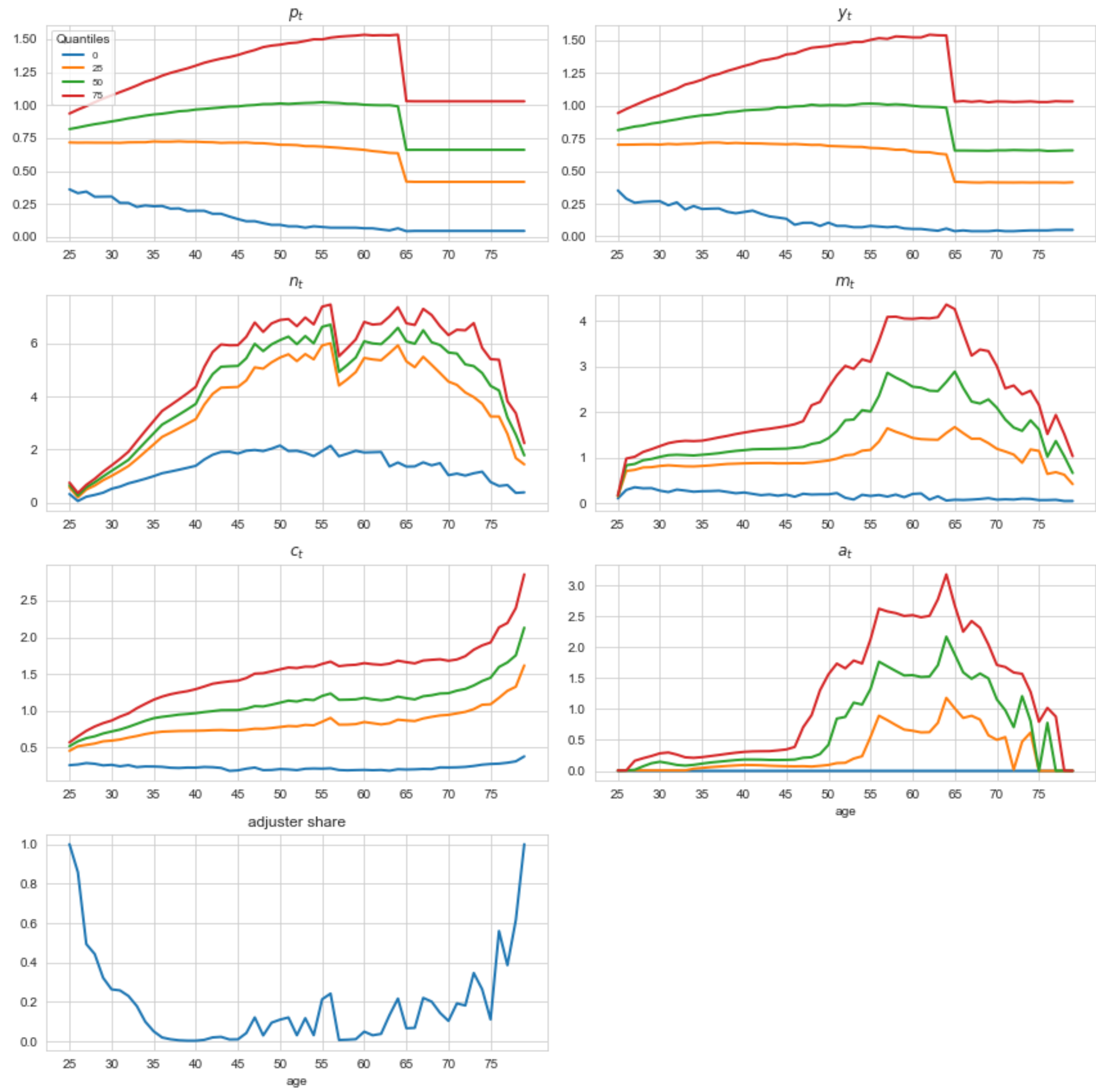
5.2 Two-Asset Model

Having solved the baseline one-asset model, we move on to solve and simulate the extended two-asset model using NEGM. We report the simulated life cycle for permanent income, transitory income, housing stock, cash-on-hand, consumption, end-of-period assets, and the share of households that adjust their housing stock in each period in Figure 2.

The income profiles for p_t and y_t are very similar to those in the one-asset model as they have identical calibrations. The remaining profiles offer more interesting results. As in the one-asset model, consumption is fairly smooth and increasing over the life cycle. However, towards the end of the model, consumption strongly increases due to households selling off their housing stock. This is also reflected in the housing stock profile; overall, households continue to increase their housing stock until the maximum is reached, and then sell off their housing stock impending death, enabling them to increase consumption further towards the end of their life. This consumption behavior shows that given our parameterization of the model, the return on housing is sufficiently strong to make it optimal for households to first max out on housing investments and postpone a substantial part of consumption until the very last periods of life. Moreover, the agents' housing stock is subject to a crash at approx. age 57, causing a substantial drop in housing. We note that the housing stock appears to be bounded by the maximal value of the housing grid $\bar{n}$. We face computational issues when setting $\bar{n}$ higher, and can therefore not simulate life cycle profiles where the value of the housing stock exceeds eight times mean income. This issue is further discussed in Section 6.

Cash-on-hand and end-of-period assets again follow a similar pattern to the one-asset model, with a gradual increase towards retirement followed by a decrease. However, the possibility of buying more housing effectively suppresses the increase in m_t and a_t for the first half of the life cycle approximately. Only when n_t approaches the maximum $\bar{n}$ do agents start properly accumulating cash-on-hand.

Figure 2: Two-Asset Model Life Cycle



Lines represent 0th, 25th, 50th and 75th quantiles except for adjuster share.

The adjuster share starts off high but then reveals that households stop buying housing at approximately age 40, implying that the increase in n_t thereafter reflects compounded housing returns rather than adjustments in the housing stock. Towards the end of life the share of households adjusting becomes much more volatile as the households incrementally sell off housing to increase consumption. As evident from Figure 7 in Appendix A.4, increasing the adjustment cost τ makes the households adjust less throughout their life, while the increase in adjuster share towards the end of life becomes steeper and less uneven.

5.3 Comparing Solution Method Performance

Table 2: Solution Method Speed

	All periods			*Average pr. period*		
	VFI	NVFI	NEGM	VFI	NVFI	NEGM
Total	≈13.5h	437.8	418.2	900	7.96	7.60
Pre-computation step	N/A	253.7	407.6	N/A	4.61	7.41
Keep	≈13h	182.3	9.4	876	3.31	0.17
Adjust	≈22 min.	1.4	1.2	24.2	0.03	0.02

Note: We report time in seconds unless otherwise stated. We solve the model using a MacOS Monterey Macbook Pro with a 1st gen. M1 chip (8-core CPU with 4 performance cores and 4 efficiency cores; 8-core GPU) and 8 GB of RAM. We terminate the VFI algorithm after solving 4 periods and infer the total time using the average pr. period. The table draws inspiration from Table 2 in Druedahl (2021).

Having solved the two-asset model using NEGM, we now compare the performance of different solution methods. We present computation speeds for the different solution methods both as a sum over all periods and an average pr. period in Table 2. First, we (try to) solve the model using standard VFI. Note that with VFI there is no pre-computation step as we do not calculate the post-decision value function w_t or the post-decision marginal value of cash, q_t. As we mention in Section 3.2, VFI is by far the slowest algorithm. After running for 60 minutes, VFI only solves 4 periods and therefore, to avoid wasting everybody's time, we move on to NVFI.

Using NVFI we can take advantage of the pre-computation of the post-decision value function w_t and the nesting structure of v^{keep} and v^{adjust}, which dramatically lowers the computation time. In total the model is solved in 437.8 sec., while the pre-computation of w_t step, keep problem and adjust problem take 253.7, 182.3, and 1.2 sec. respectively. The timing of the sub-problems clearly demonstrates that the nesting structure substantially speeds up the computation of the adjust problem, while the pre-computation step becomes the bottleneck.

Finally, using NEGM, we can obtain further speed gains by analytically computing the solution for c_t and m_t in the keep problem from the Euler equation for a given a_t. NEGM should therefore, as Druedahl (2021) argues, be considerably faster than NVFI. As is evident from Table 2 however, NEGM and NVFI are approximately equally fast in our case. Although NEGM speeds up the computation of the keep problem twentyfold, the addition of q_t leads to a twofold slowdown in the pre-computation step. Because the absolute time necessary to solve the pre-computation is considerably larger than for the keep problem, the speed gain from NEGM is rather modest in our particular case.

5.4 Marginal Propensities to Consume

From our solutions, we obtain (points on) the optimal consumption rule for each household as a function of m_t and (p_t, n_t, m_t) respectively in the one and two-asset models. Interpolating on these functions, we can compute the Marginal Propensities to Consume out of a small transitory income transfer (MPC) reminiscent of expansionary fiscal policy in terms of stimulus payments issuance. We draw inspiration from Kaplan and Violante (2022) (henceforth KV22), who also compare MPCs in various precautionary savings models, and use the following formula to compute the MPC for a household in a given time period:

$$\mathcal{MPC}_t^{keep}(\epsilon_m, s_t,) = \frac{c_t(p_t, n_t, m_t + \epsilon_m) - c_t(p_t, n_t, m_t)}{\epsilon_m}$$
$$\mathcal{MPC}_t^{adj.}(\epsilon_m, s_t) = \frac{c_t(p_t, x_t + \epsilon_m) - c_t(p_t, x_t)}{\epsilon_m}$$

We set $\epsilon_m = 0.00739$ as the income transfer thus is proportional to the \$500 used by KV22 (our mean income is 0.99 while their mean income is approximately \$67.000).[10] Computing the MPCs in our two-asset model is a bit more involved than in the one-asset model KV22 uses, as we need to make an assumption regarding the timing of the transfer and take into account that it may induce the consumers to alter their discrete choice. We distinguish between two scenarios in which the households receive the transfer before and after choosing whether to keep or adjust. In the former case, if the optimal discrete choice after receiving a transfer differs from that with no windfall, we cross-compute the MPCs based on the difference in optimal consumption between $e_t(d_t = n_t, c_t)$ and $e_t(d_t \neq n_t, c_t)$. In the latter case, we compute the MPCs exactly as stated in the above formulas.

We first calculate the average MPC over the full life cycle, during working life and during retirement for both the one-asset and two-asset model and report the results in Table 3. In line with KV22 we find that the MPC is larger in the two-asset model as compared to the one-asset model. However, the difference in size between MPCs in the two models is not consistent with KV22; while we find a comparable MPC at approximately 14 pct. in the one-asset model, the MPC for the two-asset model is much higher in KV22 at 41.2 pct.[11] We suspect this is due to the computational limitation to $\bar{n}$ as pointed out in 5.2. As expected, the MPC is larger during retirement than working life in both models. This is due to the decrease in precautionary savings, partly due to households' impending death, and partly due to the absence of income shocks after retirement.

[10] The mean income is slightly higher in the one-asset model at 1.003. We therefore set $\epsilon_m = 0.00749$ in the one-asset model.

[11] As KV22 use infinite horizon models, there is no natural comparison for the work life and retirement MPCs.

Table 3: Average Marginal Propensities to Consume

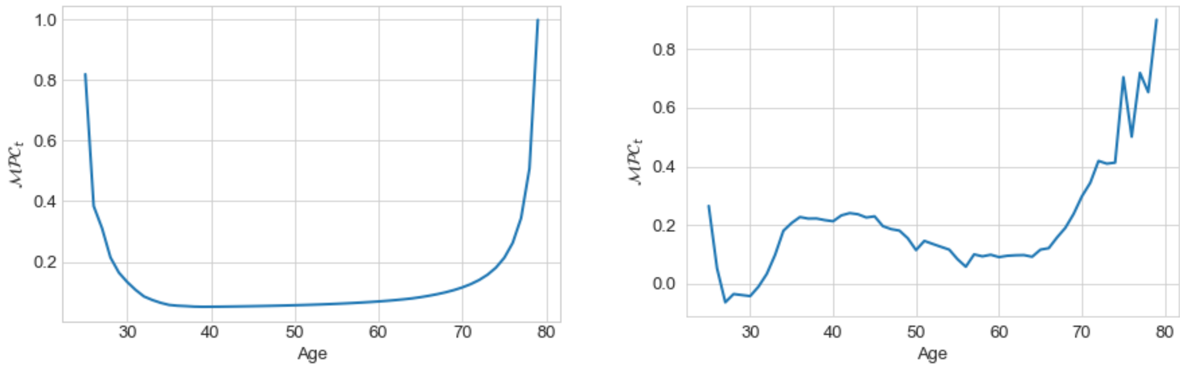
	One-Asset Model	*Two-Asset Model*	
		Cross-computed	Not Cross-computed
Full lifecycle	13.91	20.69	24.18
Working life	10.40	13.08	17.59
Retirement	23.44	41.24	42.02

Note: MPCs are reported in percentages and computed as means taken over $N =$ 10.000 individuals and time T for $T_{lifecycle} = 55$, $T_{working} = 40$, and $T_{retirement} =$ 15 periods.

In the two-asset model, we furthermore observe that the cross-computed MPCs are systemically lower than the not cross-computed counterparts. This makes good sense, as changing the discrete choice and moving to a lower level of consumption is not possible in the latter case. In fact, when cross-computing, we obtain a small number of large negative MPCs. This reflects that under some very particular circumstances, receiving a windfall can tip the optimal discrete choice for households such that they are willing to endure a substantially lower level of consumption in exchange for not having to pay the transaction cost associated with adjusting.

Next, we plot the average MPC over the life cycle in the two models in Figure 3. Notably, the MPC is much more smooth over the life cycle in the one-asset model in comparison to the two-asset model. The possibility of holding housing stock in the two-asset model affects the MPC since households become more liquidity constrained the more they increase their housing stock, and therefore more inclined to consume out of additional income. When households reach the maximum $\bar{n}$ and instead start accumulating liquid assets, the MPC falls as they become less liquidity constrained. Additionally, like the adjuster share the MPC is much more volatile as it increases toward one at the end of life as households sell off housing in increments. Lastly, we note that for the two-asset model, there is a short period early in the life cycle where the average MPC actually is negative. We suspect that this is due to the strong incentive to accumulate housing stock, the compounded returns from which households can benefit greatly from later in life.

Figure 3: MPC over Life Cycle

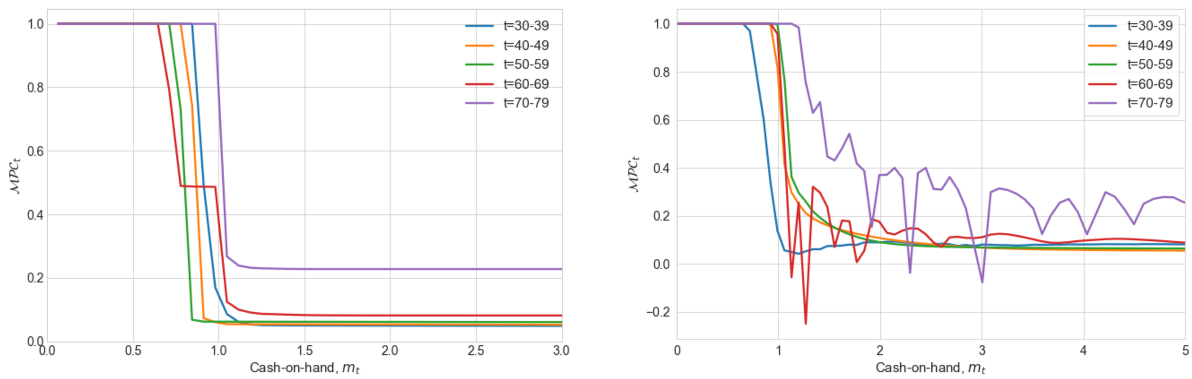


(a) One-Asset Model (b) Two-Asset Model

In the one-asset model households have no housing stock to sell of in order to achieve a rapid increase in liquid assets. The MPC therefore follows a smooth U-shape, being relatively low yet modestly increasing during working life. Towards the end of life the average MPC increases drastically and converges to one as households' saving incentive falls and they consume all remaining assets. When households enter the one-asset model, the average MPC is high at over 0.8, but quickly decreases to under 0.1. The start-of-life MPC is dependent on the simulated starting values. If households all start off relatively wealthy and unconstrained, the start-of-life MPC is much lower. Starting with a more heterogeneous distribution with a larger proportion of more liquidity constrained households increases the start-of-life MPC. This same mechanism is only to a much smaller degree present in the two-asset model, as households enter the model with some housing stock that they can choose to sell off if they are liquidity constrained early in life.

We further investigate consumption behavior by plotting MPCs for 10-year age intervals over the wealth distribution, shown in Figure 4. Starting with the one-asset model, all age brackets display unit MPCs for the lowest levels of cash-on-hand and then drop fairly quickly in the region of $m_t \in (0.5, 1)$ towards a stable lower MPC. With the chosen calibration we therefore see a transition from being constrained hand-to-mouth consumers before $m_t = 0.5$ to being poor consumers close to their constraint around $m_t \in (0.5, 1)$ to finally stabilizing as rich unconstrained consumers for $m_t > 1$.

Figure 4: MPC Over Wealth Distribution



(a) One-Asset Model (b) Two-Asset Model

Additionally, the higher the age bracket, the lower the wealth level required to no longer be hand-to-mouth is as households have had more time to accumulate assets. The exception to this is households in the last age bracket $70-79$, who remain hand-to-mouth consumers for the highest levels of m_t as the precautionary saving motive drastically diminishes toward the end of life. It is therefore also the $70-79$ age bracket that has the largest MPCs as rich unconstrained consumers.

We note that there is a slight kink for poor households close to their liquidity constraint in the age bracket $60-69$, as the retirement age 65 is included in this age bracket. Figure 6 in Appendix A.4 shows the MPC over wealth distribution for each age between $60-69$. The MPCs for ages up to 64 all drop at $m_t \approx 0.75$, while the MPCs for ages after 65 drop at $m_t \approx 1$, causing the kink in between $m_t = (0.5, 1)$ when taking the mean over the age bracket.

Moving on to the two-asset model, we observe many of the same general tendencies as in the one-asset model, although the picture is far from as clear-cut. All age brackets display unit MPCs for low values of m_t, although the drop happens for larger cash-on-hand values as compared to the one-asset model. The three youngest age brackets are fairly smooth and comparable to those in the one-asset model. The two oldest age brackets however are much more erratic and occasionally negative. The increase in volatility for the older age brackets is consistent with the evidence from the life cycle profiles in Figure 2. Although we cannot with certainty explain the presence of negative MPCs, the largest drop for $60-69$ could reflect the transition from working to retirement, while the unstable MPC for the oldest age bracket can be attributed to the deleveraging of housing stock.

5.4.1 Sensitivity Check - Changing the Spread of Asset Returns

KV22 point out that the average MPC in two-asset models are sensitive to the size of the spread between the return on liquid and illiquid assets. In particular, larger spreads leads to a higher proportion of hand-to-mouth households and thus a higher average MPC. To see if this also is holds in our case, we solve the model for varying levels of liquid and illiquid returns. Inspecting Table 4, we observe that the MPCs are increasing for higher spreads, consistent with KV2022. Additionally, we find that the MPCs for a given spread are decreasing for higher *levels* of returns. The latter effect is likely because higher returns in absolute values cause relatively rich households *ceteris paribus* approach the cap of durable stocks earlier in the life cycle, which in turn allows them to accumulate liquids assets earlier and thus be less liquidity constrained for a larger part of their life.

Table 4: Sensitivity of Cross-Computed MPCs With Respect to Asset Returns

	Full life cycle			*Working*			*Retired*		
Spread	$R = 1.01$	$R = 1.02$	$R = 1.03$	$R = 1.01$	$R = 1.02$	$R = 1.03$	$R = 1.01$	$R = 1.02$	$R = 1.03$
0.04	20.00	20.34	19.23	12.42	12.40	11.44	40.37	41.63	40.27
0.05	23.33	22.65	20.69	16.49	15.63	13.08	41.98	41.63	41.24
0.06	26.47	23.78	21.75	19.43	15.83	12.86	45.82	45.27	43.24

Note: MPCs are reported as percentages and are cross-computed, which corresponds to assuming that households receive the transfer *before* deciding whether to keep or adjust. We report a similar table for not cross-computed MPCs in Appendix A.4

6 Discussion

There is a plethora of papers exploring consumption behavior in response to income changes and the size of MPCs under different conditions. Among the empirical evidence there is large variation in the size of MPCs depending on the size of the windfall, whether the income change is permanent or transitory, and whether there is a positive or negative income shock.[12] Standard macroeconomic models featuring no heterogeneity in assets or agents generally have a hard time rationalizing these relatively high observed MPCs. A large area of macroeconomic research is therefore focused on incorporating heterogeneity in macroeconomic models in order to better replicate empirical evidence. Kaplan and Violante (2014; 2022), among others, show that a larger heterogeneity enables models to simulate MPCs in the empirically observed size range. Qualitatively, we are able to replicate the results of higher MPCs in a two-asset model as compared to the one-asset model. There are however several other issues and discrepancies between our simulated MPCs and the literature in general.

First, our simulations of annual MPCs between 19.23-26.47 pct. appear to be lower than what is typically found in the literature. Although KV22 emphasize that annual MPCs cannot simply be inferred from scaling up quarterly MPCs as this method causes upward

[12]Havranek and Sokolova (2020) examine the results of 144 different empirical studies, finding a mean quarterly MPC of 21 pct. However, they also find that the MPC can be heavily biased due to empirical approaches, publication bias, or the presence of liquidity constraints.

bias, our annual MPCs are substantially lower than the annual MPCs found by KV22 of approximately 40 pct. Despite incorporating illiquid assets and weath heterogeneity, we are thus not able to simulate MPCs in a size range comparable to the empirical evidence, suggesting that our model would benefit from extensions adding more realism.

Second, we find negative MPCs at the start-of-life (Figure 3b) and for some areas in the wealth distribution for the oldest and youngest age brackets (Figure 4b). Intuitively, it makes sense that MPCs are more volatile at the beginning and end of life as a result of changing motives for precautionary savings, and therefore might be more prone to becoming negative. Also, the lowest MPCs are found for the age bracket $60 - 69$, and therefore might reflect the transition from working to retirement. Despite these rationalizations, we cannot be entirely sure of whether some delicate computational technicalities disturb the output.

Third, our two-asset model seems sensitive to the calibration of parameters, grid points, and nodes. For certain parameterizations, we run into computational issues in the sense that the model produces a negative post decision marginal value of cash, i.e. negative marginal utility. A particularly unfortunate limitation in this regard is that the maximal value of the n_t grid cannot be set "too high". This causes some issues with respect to the dynamics created in the simulation results because richer households approach the cap of durable stock quite early in the life cycle. While we cannot be certain, we find it plausible that this is what drives the subsequent accumulation of liquid assets, which in turn drives down the average MPC. The fact that the simulation results seem to depend on a set of, to some extent, arbitrarily chosen boundary values of the state space is of course not desirable and an issue that should be addressed if we were to continue working with this model.

Fourth, our interpretation of n_t as housing stock is not straightforward. Realistically, housing should be quite large compared to cash-on-hand and annual income. Although the ratio of n_t to income y_t is realistic in our model simulation, the ratio of n_t to m_t should be much larger. As explained above however, setting $\bar{n}$ too high causes numerical issues, even if the number of nodes in the n_t grid is increased.

Fifth, although our two-asset model arguably is a better description of consumption behavior than the one-asset model, it is still a stylized framework. In the literature, there exist several extensions with the potential to make the model more realistic and perhaps improve its ability to match empirical evidence. First, our model does not allow any type of borrowing. This is at odds with the typical institutional framework in most countries, where households can pledge their house as collateral and engage in mortgage contracts with amortization plans spanning multiple periods (see e.g. Moran & Kovacs, 2022). Furthermore, such a framework would make it possible, and meaningful, for households to simultaneously possess positive gross debt and assets (Druedahl & Jørgensen, 2018), instead of our current formulation where only net worth matters. Finally, instead of the somewhat artificial floor on housing stock $\underline{d}$, an arguably more realistic modeling approach could include a rental market such that poor households can choose between obtaining utility flows from housing services from either owning or renting housing stock (see e.g. Kaplan & Violante, 2014). Such extensions, however, come with a cost in terms of requiring additional state variables, making the model more convoluted and time consuming to solve.

7 Concluding Remarks

To conclude, in this term paper we specify and solve a standard one-asset and an extended two-asset precautionary savings model using various solution methods. In the analysis of the models, we mainly focus on the MPC due to its importance for contemporary macroeconomic research. Consistent with recent contributions by Kaplan and Violante (2014; 2022), we find that the two-asset model creates larger MPCs, which better match empirical estimates. However, the average yearly MPC in our two-asset model is smaller than what is found in the empirical literature. Lastly, because our application includes a relatively time-consuming pre-computation step, solving the model using either NVFI or NEGM is approximately equally fast.

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A Appendices

A.1 The Standard One-Asset Buffer Stock Model

As a benchmark, we use a simple version of a one-asset model from Carroll (1997), with one decision variable C_t, and two state variables P_t, M_t. Utility follows CRRA preferences, and the agent decides consumption C_t given permanent income P_t and cash-on-hand M_t. The income structure is given as

$$
\begin{aligned}
Y_{t+1} &= \xi_{t+1} P_{t+1}, \quad \xi_t \sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\
P_{t+1} &= G l_t P_t \psi_t, \qquad \psi_t \sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2)
\end{aligned}
$$

where Y_t denotes transitory income, $G = (1+g)$ is a constant growth factor, and l_t denotes the deterministic income profile including retirement. Permanent income is subject to a shock ψ_t, and transitory income is subject to a transitory shock ξ_t. After the consumption choice the end-of-period assets are given as

$$A_t = M_t - C_t$$

where $A_t \geq 0$ such that there is no borrowing. Cash-on-hand evolves according to

$$M_{t+1} = RA_t + Y_{t+1}$$

This completes the model description. Using a slight abuse of notation, we normalize the model using the definition $x_t \equiv \frac{X_t}{P_t}$. By normalizing wrt. permanent income we thus economize on the state space, such that there is only one state variable m_t. The normalized Bellman is then

$$
\begin{aligned}
v_t(m_t) &= \max_{c_t} \left\{ \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G l_t \psi_{t+1})^{1-\rho} v_{t+1}(m_{t+1}) \right] \right\} \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{1}{G l_t \psi_{t+1}} R a_t + \xi_{t+1} \\
a_t &\geq 0
\end{aligned} \tag{A.1}
$$

In order to derive the closed form Euler equation, we first derive the first-order-condition

(FOC) wrt. consumption. Substituting in the constraints, the Bellman becomes

$$\Rightarrow v_t\left(m_t\right) = \max_{c_t}\left\{\frac{c_t^{1-\rho}}{1-\rho} + \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{1-\rho} v_{t+1}\left(\underbrace{\frac{1}{Gl_t\psi_{t+1}}R\left(m_t - c_t\right) + \xi_{t+1}}_{\equiv m_{t+1}}\right)\right]\right\}$$

We can then derive the FOC as

$$\begin{aligned} u'(c_t) - \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{1-\rho}\frac{\partial v_{t+1}\left(m_{t+1}\right)}{\partial m_{t+1}}\frac{R}{Gl_t\psi_{t+1}}\right] &= 0 \\ c_t^{-p} - \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{1-\rho}\frac{\partial v_{t+1}\left(m_{t+1}\right)}{\partial m_{t+1}}\frac{R}{Gl_t\psi_{t+1}}\right] &= 0 \\ \Leftrightarrow c_t^{-p} = \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{-\rho} R\frac{\partial v_{t+1}\left(m_{t+1}\right)}{\partial m_{t+1}}\right] \end{aligned}$$

We use the Envelope Theorem in order to derive $\frac{\partial v_t}{\partial m_t}$. First, we define

$$F(m_t, c_t) = u(c_t) + \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{1-\rho} v_{t+1}\left(\frac{1}{Gl_t\psi_{t+1}}R(m_t - c_t) + \xi_{t+1}\right)\right]$$

such that the policy function $c_t^*(m_t)$ satisfies $v_t(m_t) = F(m_t, c_t^*(m_t))$. We can then determine $v_t'(m_t)$:

$$\begin{aligned} \frac{\partial v_t(m_t)}{\partial m_t} &= \frac{\partial F(m_t, c_t^*(m_t))}{\partial m_t} + \frac{\partial F(m_t, c_t^*(m_t))}{\partial c_t^*(m_t)}\frac{\partial c_t^*(m_t)}{\partial m_t} \\ &= \underbrace{\beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{1-\rho} v_{t+1}'\left(\frac{1}{G\psi_{t+1}l_t}R(m_t - c_t) + \xi_{t+1}\right)\frac{1}{G\psi_{t+1}l_t}R\right]}_{\frac{\partial F(m_t, c_t^*(m_t))}{\partial m_t}} \\ &+ \underbrace{u'(c_t^*(m_t))\frac{c_t^*(m_t)}{\partial m_t} - \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{1-\rho} v_{t+1}'\left(\frac{1}{G\psi_{t+1}l_t}R(m_t - c_t) + \xi_{t+1}\right)\frac{1}{G\psi_{t+1}l_t}R\frac{\partial c_t^*(m_t)}{\partial m_t}\right]}_{\frac{\partial F(m_t, c_t^*(m_t))}{\partial c_t^*(m_t)}\frac{\partial c_t^*(m_t)}{\partial m_t}} \end{aligned}$$

First, we note that $\frac{\partial c_t^*(m_t)}{\partial m_t} = 1$ from $a_t = m_t - c_t$. From the FOC we know that $u'(c_t) - \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{1-\rho}\frac{\partial v_{t+1}(m_{t+1})}{\partial m_{t+1}}\frac{R}{Gl_t\psi_{t+1}}\right] = 0$, i.e. households are already in optimum. Therefore the two last terms denoting $\frac{\partial F(m_t, c_t^*(m_t))}{\partial c_t^*(m_t)}\frac{\partial c_t^*(m_t)}{\partial m_t}$ cancel out.

Thus, we know

$$\begin{aligned} \text{From FOC:}\quad & u'(c_t^*) = \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{-\rho} R\frac{\partial v_{t+1}\left(m_{t+1}\right)}{\partial m_{t+1}}\right] \\ \text{From Envelope Theorem:}\quad & \frac{\partial v_t(m_t)}{\partial m_t} = \beta\mathbb{E}_t\left[\left(Gl_t\psi_{t+1}\right)^{-\rho} R v_{t+1}'\left(\frac{1}{Gl_t\psi_{t+1}}R(m_t - c_t) + \xi_{t+1}\right)\right] \end{aligned}$$

Therefore, the condition

$$u'(c_t^*) = \frac{\partial v_t(m_t)}{\partial m_t}$$

must be satisfied in each period. This then yields the Euler equation:

$$u'(c_t^*) = \beta R \mathbb{E}_t \left[(G l_t \psi_{t+1})^{-p} u'(c_{t+1}^*) \right]$$

Inserting the policy function $c_{t+1}^*(m_{t+1})$ we get the closed form solution for c_t to be used in the EGM algorithm:

$$\Leftrightarrow c_t^{-\rho} = \beta R \mathbb{E}_t \left[(G l_t \psi_{t+1})^{-p} \left(c_{t+1}^* \left(\frac{1}{G l_t \psi_{t+1}} R a_t + \xi_{t+1} \right) \right)^{-p} \right]$$

$$\Leftrightarrow c_t = \beta R \mathbb{E}_t \left[(G l_t \psi_{t+1})^{-p} \left(c_{t+1}^* \left(\frac{1}{G l_t \psi_{t+1}} R a_t + \xi_{t+1} \right) \right)^{-p} \right]^{-\frac{1}{p}} \tag{A.2}$$

A.2 Implementation Details

In the following, we state some more details with respect to allocating and solving the two-asset model. First, an overview over grids and nodes can be found in Table 5. Second, we note that when solving the model, we follow Druedahl (2021) and actually work with the (negative) inverse of the value functions, the post-decision value function and the post-decision marginal value of cash defined as follows:

$$\begin{aligned}
\tilde{v}_t^{keep}(p_t, n_t, m_t) &\equiv -\frac{1}{v_t^{keep}(p_t, n_t, m_t)} \\
\tilde{v}_t^{adj.}(p_t, x_t) &\equiv -\frac{1}{v_t^{adj.}(p_t, x_t)} \\
\tilde{u}_{c,t}^{keep}(p_t, n_t, m_t) &\equiv \frac{1}{u_c\left(c_t^{keep}(p_t, n_t, m_t), n_t\right)} \\
\tilde{u}_{c,t}^{adj}(p_t, x_t) &\equiv \frac{1}{u_c\left(c_t^{adj}(p_t, x_t), d_t^{adj}(p_t, x_t)\right)}
\end{aligned}$$

These inverse functions are always positive, increasing, and approaches zero in the limit for m_t or x_t going to zero. As such, the inverse functions never go towards infinity, making them more stable to work with numerically compared with the original functions.

Lastly, initial values for the simulations of the two-asset model are drawn according to:

$$\log p_0 \sim N(-0.2, 0.2)$$
$$\log d_0 \sim N(-0.16, 0.2)$$
$$\log a_0 \sim N(-0.04, 0.1)$$

Likewise, initial values for the simulations of the one-asset model are drawn according to:

$$\log m_0 \sim N(-0.2, 0.2)$$
$$\log p_0 \sim N(-0.2, 0.2)$$

Table 5: Grid points and nodes

Parameter	**Description**	**Value**
	Grids	
N_p	Number of grid points for p_t	50
$\underline{p}$	Minimum p_t	1e-4
$\bar{p}$	Maximum p_t	3.0
N_n	Number of grid points for n_t	100
$\underline{n}$	Minimum n_t	0
$\bar{n}$	Maximum n_t	8.0
N_m	Number of grid points for m_t	100
$\underline{m}$	Minimum m_t	0
$\bar{m}$	Maximum m_t	10.0
N_x	Number of grid points for x_t	100
$\underline{x}$	Minimum x_t	0
$\bar{x}$	Maximum x_t	$\bar{n} + \bar{m}$
N_a	Number of grid points for a_t	100
$\underline{a}$	Minimum a_t	0
$\bar{a}$	Maximum a_t	11.0
	Shocks	
ξ_i	Nodes in transitory income shock	5
ψ_j	Nodes in permanent income shock	5
ε_k	Nodes in shock to housing returns r_h	10[a]
	Dimensions	
T_{min}	Age when entering the model	25
T_r	Retirement age	65
T	Duration of life	80
N	Number of consumers	10,000

[a] To be fully precise, we use 5 nodes to approximate ε. Then we multiply the associated weights with $(1-\gamma)$ and append 5 more identical nodes from which we subtract the low probability shock and multiply the weights with γ.

A.3 Gauss-Hermite Quadrature

In order to obtain the expected future value conditional on current states and choices, we need to solve integrals of dimensions equal to the number of shocks. The shocks in our two-asset model specification leaves us with three-dimensional integrals that are only feasible to solve through numerical integration procedures including discretization, Monte Carlo integration or quadrature integration methods. Gauss-Hermite quadrature falls within the latter category, where the conditional transition density is approximated by setting a non-negative weighting function and choosing the quadrature nodes such that the approximation error of the integral is 0 given the weighting function

$$\int_a^b f(x)w(x)dx = \sum_{i=1}^{n} \omega_i f(x_i) + \text{approximation error} = \sum_{i=1}^{n} \omega_i^* f(x_i)$$

with ω_i^* being the optimal quadrature weights. In particular, the Gauss-Hermite quadrature method uses $exp(-x_i^2)$ as the weighting function with the associated nodes, $x_i \in [-\infty, \infty]$, being the roots of the Hermite polynomial. Choosing the number of quadrature nodes for each dimension involves a trade-off between precision and speed and there are in general no guidelines as to which number will suffice in order to achieve a good approximation. The quadrature method is guaranteed to converge for large N^d but increasing the number also slows down the solution methods significantly (Judd, 1998). We use the Gauss-Hermite quadrature method since it approximates expectations with (log-) normal distributions well. This is due to the fact that the weighting function is proportional to the (log-) normal density.

A.4 Additional Output

Table 6: Sensitivity of Not Cross-Computed MPCs With Respect to Asset Returns

	Full life cycle			*Working*			*Retired*		
Spread	$R = 1.01$	$R = 1.02$	$R = 1.03$	$R = 1.01$	$R = 1.02$	$R = 1.03$	$R = 1.01$	$R = 1.02$	$R = 1.03$
0.04	22.35	21.88	20.37	14.34	14.01	12.67	43.51	42.68	40.91
0.05	25.22	23.88	22.07	18.22	16.75	14.99	43.72	43.01	41.13
0.06	27.61	25.60	23.53	20.83	18.71	16.73	45.77	44.28	42.01

Note: MPCs are reported as percentages and are not cross-computed, which corresponds to assuming that households receive the transfer *after* deciding whether to keep or adjust.

Figure 5: Estimation Output - Simulated Method of Moments

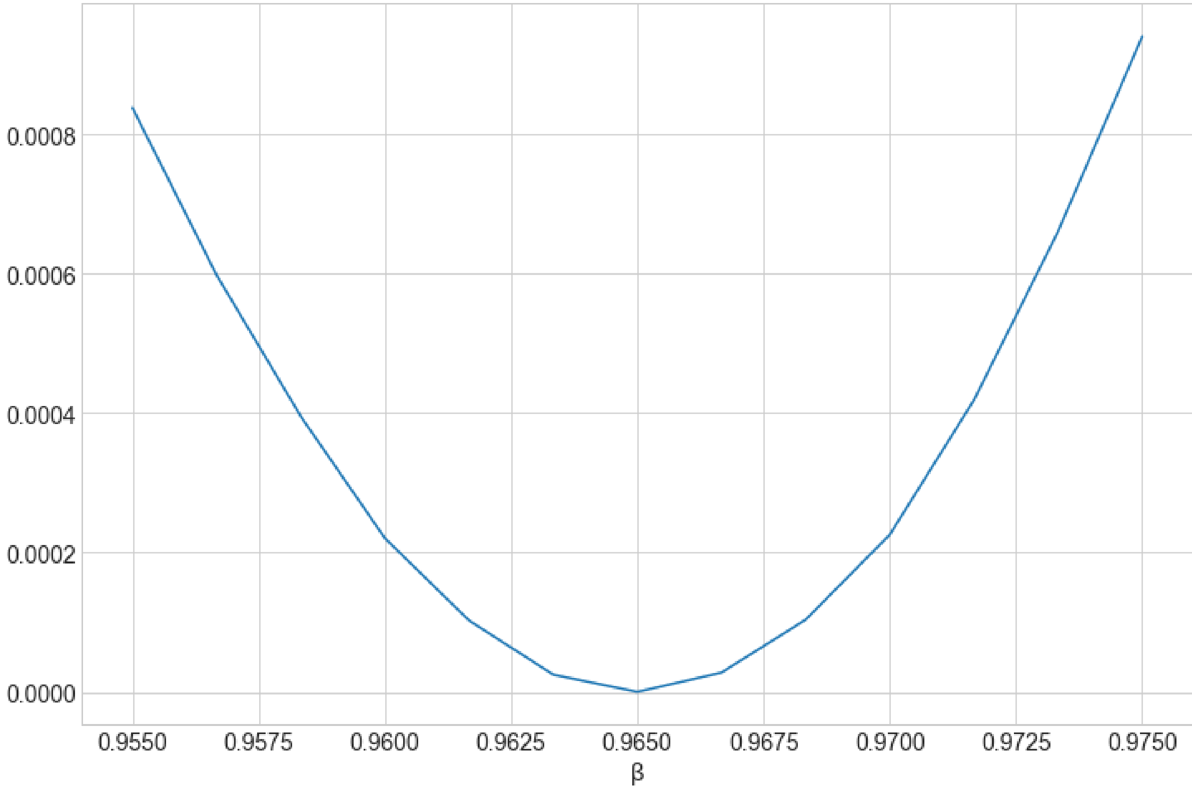

Figure 6: MPC over Wealth Distribution, t = 60-69

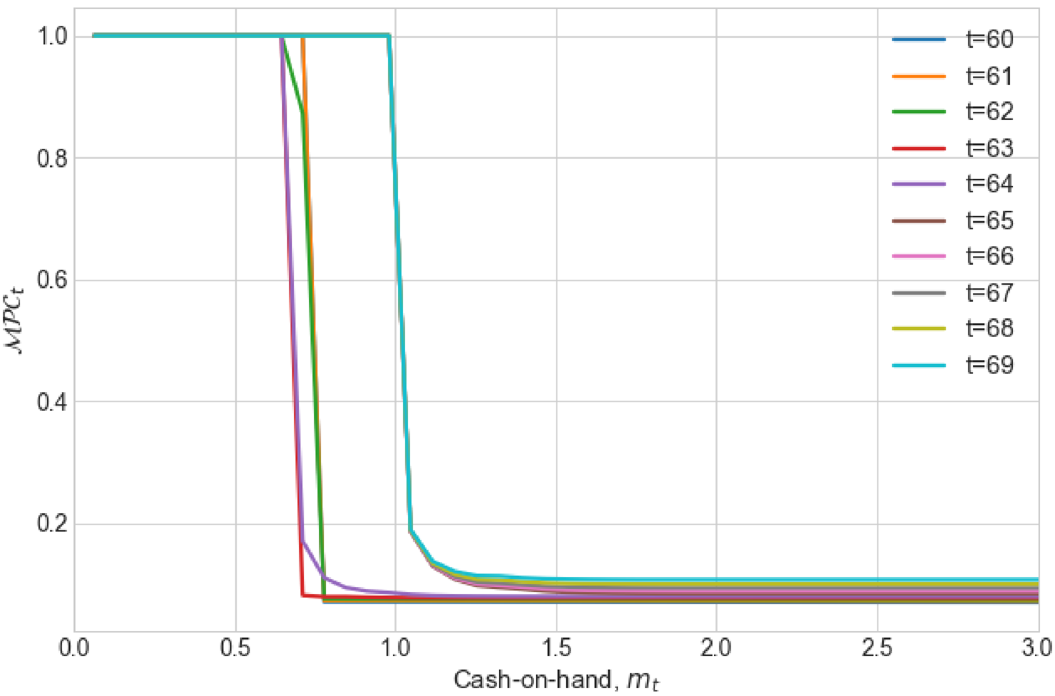

Figure 7: Adjuster share for alternative adj cost $\tau = 0.1$