Chapter 1. Introduction

Introduction

Random Experiments

1. Output can not be surely predicted in advance;
2. When one repeats the same experiment a large number of times one can observe some "regularity" in the average output

The state space

This is the set of all possible outcomes of the experiment, and it is usually denoted by Ω

The events

An "event" is a property which can be observed either to hold or not to hold after the experiment is done.

In mathematical terms, an event is a subset of Ω. If A and B are two events, then,

- the contrary event is interpreted as the complement set A^C
- the event " A or B " is interpreted as the union $A \cup B$
- the event " A and B " is interpreted as the intersection $A \cap B$
- the sure event is Ω
- an **elementary event** is a "singleton"
  - i.e. a subset $\{\omega\}$ containing a single outcome ω of Ω
- $\mathcal{A}$: the family of all events.
- 2^Ω :the set of all subsets of Ω

The family $\mathcal{A}$ should be "stable" by the logical operations described above:

If $A, B \in \mathcal{A}$, then we must have

$$A^C \in \mathcal{A}, \qquad A \cap B \in \mathcal{A}, \qquad A \cup B \in \mathcal{A}$$

and also

$$\Omega \in \mathcal{A} \quad and \quad \emptyset \in \mathcal{A}$$

The probability

With each event A one associates a number denoted by $P(A)$ and called the "probability of A"

This number measures the likelihood of the event A to be realized a **priori** , before performing the experiment. It is chosen between 0 and 1, and the most likely the event is, the closer to 1 this number is.

To get an idea of the properties of these numbers, one can imagine that they the limits of the "frequency" with which the events are realized:

Let us repeat the same experiment n times; the n outcomes might of course be different (think of n successive tosses of the same dice, for instance). Denote by $f_n(A)$ the frequency with which the event A is realized (i.e. the number of times the event occurs, divided bu n)

Intuitively we have:

$$P(A) = \lim_{n \to \infty} f_n(A)$$

Form the obvious properties of frequencies, we immediately deduce that:

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $P(A \cup B) = P(A) + P(B) \quad if \quad A \cap B = \emptyset$

A mathematical model for our experiment is thus a triple $(\Omega, \mathcal{A}, P)$, consisting of

- the space Ω
- the family $\mathcal{A}$ of all events
- the family of all $P(A)$ for $A \in \mathcal{A}$

Hence we can consider that P is a map from $\mathcal{A}$ into $[0,1]$, which satisfies at least the properties (2) and (3) above (plus in fact an additional property, more difficult to understand, which will be given in the next Chapter)

Random variable

A random variable is a quantity which depends on the outcome of the experiment.

In mathematical terms, this is a map from Ω into a space E ,where often

$$E = \mathbb{R} \quad or \quad E = \mathbb{R}^d$$

Let $\mathcal{X}$ be such a random variable, mapping Ω into E . One can then transport" the probabilistic structure onto the target space E , by setting

$$P^{\mathcal{X}}(B) = P\left(X^{-1}(B)\right) \quad for \quad B \subset E$$

Where $\mathcal{X}^{-1}(B)$ denotes the pre-image of B by $\mathcal{X}$

- i.e. the set of all $\omega \in \Omega$ such that $X(\omega) \in B$

This formula defines a new probability, denoted by $P^{\mathcal{X}}$, but on the space E instead of Ω . This probability $P^{\mathcal{X}}$ is called the **law of the variable $\mathcal{X}$**

Example (toss of two dice)

We have seen that $\Omega = \{(i,j) : 1 \leq i,j \leq 6\}$, and it is natural to take here $\mathcal{A} = 2^{\Omega}$ and

$$P(A) = \frac{\sharp(A)}{36} \quad if \quad A \subset \Omega$$

Where $\sharp(A)$ denotes the number of points in A

One easily verifies the properties (1),(2),(3) above, and

$$P(\{\omega\}) = \frac{1}{36}$$

for each singleton

The map $X : \Omega \to \mathbb{N}$ defined by

$$X(i, j) = i + j$$

is the random variable "sum of the two dices" , and its law is

$$P^{\mathcal{X}}(B) = \frac{\text{number of pairs } (i, j) \text{ such that } i + j \in B}{36}$$

(for example, $P^{\mathcal{X}}(\{2\}) = P(\{1, 1\}) = \frac{1}{36}$, $P^{\mathcal{X}}(\{3\}) = \frac{2}{36}$, etc ...)