Chapter 6 Construction of a Probability Measure

概率测度的构造 adveprobab 2

对于 $\mathcal{B}$ 是 $\{\omega\}_{\omega\in\Omega}$ ，即 $\mathcal{B}$ 是单点集组成的事件类

如果概率 P 建立在 $\mathcal{B}$ 上，这样构造的概率比较好计算，现在的想法是：想通过有限集合或者可数集合，来进行拓展，将概率测度扩展开

即将 P 扩展到 $\mathcal{A}=2^{\Omega}$

核心：

$$finite \qquad \Rightarrow \qquad uncountable$$

- Assume given Ω (countable or uncountable) and a σ-algebra $\mathcal{A}\in 2^{\Omega}$
- $(\Omega,\mathcal{A})$ is called a measurable space. 构造P
- Want to construct probability measure on $\mathcal{A}$
  - When Ω is finite or countable, we have already seen this is simple to do.
  - When Ω is uncountable, the same technique does not work. 失效
    - Indeed, a "typical" probability P will have $P(\{\omega\})=0$ for all ω, and thus the family of all numbers $P(\{\omega\})$ for $\omega\in\Omega$ does not characterize probability P in general.
    - 怎么理解上一点？对于不可数集合。。。

- In many "concrete" situations --- that it is often relatively simple to construct a "probability" on an algebra which generates the σ-algebra $\mathcal{A}$, and the problem at hand is then to extend this probability to the σ-algebra itself.

  在由 $\mathcal{A}$ 生成的 sigma 代数里面构建一个代数上的概率是相对比较容易的，现在要将其拓展到 sigma 代数里面

- Suppose: 在一个代数上建立概率 P
- $\mathcal{A}_0$ is an algebra and, $\mathcal{A}=\sigma(\mathcal{A}_0)$.

- Given a probability P on the algebra $\mathcal{A}_0$: that is, a set function $P : \mathcal{A}_0 \to [0,1]$ satisfying
  - $P(\Omega) = 1$
  - (Countable Additivity) for any sequence $\{A_n\}_{n\geq 1} \subset \mathcal{A}_0$, pairwise disjoint, and such that $\cup_{n\geq 1} A_n \in \mathcal{A}_0$, we have

$$P\left(\cup_{n\geq 1} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

It might seem natural to use for $\mathcal{A}$, the set of all subsets of Ω, as we did in the case where Ω was countable.

We do not do so for the following reason, illustrated by an example: 不可数的不满足

- Suppose $\Omega = [0,1]$, and define a set function P on intervals of the form $P((a,b]) = b - a$, where $0 \leq a \leq b \leq 1$
- It is a natural "probability measure" that assigns the usual length of an interval as its probability.

  概率表示为长度
- Suppose we want to extend P in a unique way to $2^{\Omega} = 2^{[0,1]}$ = all subsets of $[0,1]$ such that
  - $P(\Omega) = 1$
  - $P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ for any sequence of subsets $\{A_n\}_{n\geq 1}$ with $A_n \cap A_m = \emptyset$ for $n \neq m$
- One can prove that no such P exists! 怎么证明其不存在性?
- The collection of sets $2^{[0,1]}$ is simply **too big** for this to work.

Borel realized that we can however do this on a smaller collection of sets, namely the smallest σ-algebra containing intervals of the form $(a,b]$.

Borel set :

- the sigma-algebra generated by the open sets 所有开集生成的 sigma 代数
- the sigma-algebra generated by the open intervals 所有开区间生成的 sigma 代数
- the sigma-algebra generated by the $(-\infty, a], a \in \mathbb{Q}$ 所有 $(-\infty, a]$ 生成的 sigma 代数

  这三个定义构造的 Borel set 越来越小，事件类越小，构建概率 P 越简单

由定理2.1：在$\mathbb{R}$上的博雷尔集是由$(-\infty, a]$这种形式的区间所生成的sigma代数

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Proof.
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Let $\mathcal{C}$ denote all open intervals. Since every open set in $\mathcal{R}$ is the countable union of open intervals, we have $\sigma(\mathcal{C}) =$ the Borel $\sigma - algebra$ of $\mathbb{R}$

Let $\mathcal{D}$ denote all intervals of the form $(-\infty, a]$, where $a \in \mathbb{Q}$.

Let $(a,b) \in \mathcal{C}$

Let $(a_n)_{n\geq 1}$ be a sequence of rationals decreasing to a

Let $(b_n)_{n\geq 1}$ be a sequence of rationals increasing strictly to b.

Then

$$\begin{aligned}(a,b) &= \cup_{n=1}^{\infty}(a_n,b_n] \\ &= \cup_{n=1}^{\infty}\left((-\infty,b_n]\cap(-\infty,a_n]^c\right)\end{aligned}$$

Therefore , $\mathcal{C}\subset\sigma(\mathcal{D})$, where $\sigma(\mathcal{C})\subset\sigma(\mathcal{D})$

However, since each element of $\mathcal{D}$ is a closed set, it is also a Borel set, and therefore $\mathcal{D}$ is contained in the Borel sets $\mathcal{B}$, Thus we have

$$\mathcal{B}=\sigma(\mathcal{C})\subset\sigma(\mathcal{D})\subset\mathcal{B}$$

and hence $\sigma(\mathcal{D})=\mathcal{B}$

这个定理给出了更容易验证 $Borel\sigma-algebra$ 的一种方法，给出了验证博雷尔集的一个充要条件

这个定理的证明核心在于 $(a,b)=\cup_{n=1}^{\infty}\left((-\infty,b_n]\cap(-\infty,a_n]^c\right)$

需要证明左边的开区间(a,b)所构成的 sigma 代数$\mathcal{B}$和右边的 $(-\infty,a]$这种形式构成的sigma代数$\sigma(\mathcal{D})$是相等的，则需要证明两个包含关系

Theorem 6.1 概率延拓定理（唯一延拓）

Each probability P defined on the algebra $\mathcal{A}_0$ has a **unique** extension (also call P) on $\mathcal{A}$.

> 每一个定义在代数 $\mathcal{A}_0$ 上的概率，在 $\mathcal{A}$ 上都有唯一的拓展
>
> We will show only the uniqueness. For the existence one can consult any standard text on measure theory.

Definition 6.1

- A class $\mathcal{C}$ of subsets of Ω is **closed under finite intersections** $\mathcal{C}$

  对有限并封闭 （代数满足对有限并封闭）
  - If for when $A_1, A_2, \ldots, A_n \in \mathcal{C}$,
    then $A_1 \cap A_2 \cap \cdots \cap A_n \in \mathcal{C}$ as well

    (n arbitrary but finite)
- A class $\mathcal{C}$ is **closed under increasing limits** $\mathcal{C}$

  对单增并封闭 （ σ-代数满足，且 σ-代数更小）
  - If wherever $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ is a sequence of events in $\mathcal{C}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{C}$ as well.
- A class $\mathcal{C}$ is **closed under differences** $\mathcal{C}$

  对差封闭 （代数满足对差封闭，但满足对差封闭的不一定是代数）
  - If whenever $A, B \in \mathcal{C}$ with $A \subset B$, then $B - A \in \mathcal{C}$.

⋆⋆⋆⋆⋆**Monotone Class Theorem 单调类定理**

Let $\mathcal{C}$ be a class of subsets of Ω, closed under finite intersections and containing Ω.

> $\mathcal{C}$ 满足两个条件：包含 Ω、对有限交封闭

Let $\mathcal{B}$ be the smallest class containing $\mathcal{C}$ which is closed under increasing limits and by difference.

> $\mathcal{B}$ 满足两个条件：包含$\mathcal{C}$ 的最小类、对单增并、差封闭

Then $\mathcal{B} = \sigma(C)$

> Proof.
>
> Note that
>
> - The intersection of classed of sets closed under increasing limits and differences is again a class of that type.
>
>   一列事件，对单增并封闭，则把之交起来也对单增并封闭
>
>   > Proof.
>   >
>   > 题设：
>   >
>   > - 有若干个事件类 $\mathcal{A}_\alpha, \alpha \in \mathcal{I}$
>   > - 对每个 $\alpha \in \mathcal{I}$， $\mathcal{A}_\alpha$ 对单增并封闭
>   >
>   > 推导：
>   >
>   > 若 $A_1 \subset A_2 \subset \cdots$ 且 $A_i \in \cap_{\alpha \in I} \mathcal{A}_\alpha$
>   >
>   > 对任意固定的 $\alpha \in \mathcal{I}$， $A_i \in \mathcal{A}_\alpha$， $, i = 1, 2, \ldots$
>   >
>   > 则 $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_\alpha$ ，于是 $\bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ （每个都属于 $\mathcal{A}_\alpha$ ，则属于 $\mathcal{A}_\alpha$ 的交）
>
>   一列事件，对差封闭，则把之交起来也对差封闭
>
>   > 自己证明一下

> So, by taking the intersection of all such classed,

- there always exists a smallest class containing $\mathcal{C}$ which is closed under increasing limits and by differences.

  总存在一个最小的类，满足：包含 $\mathcal{C}$ ，且对单增并封闭、差封闭

  所有满足（对单增并封闭、差封闭、包含 $\mathcal{C}$ ）的都交起来，最小

For each set B, denote $\mathcal{B}_B$ to be the collection of sets A such that $A \in \mathcal{B}$ and $A \cap B$, $i.e.$

$$\mathcal{B}_B = \{A : A \in \mathcal{B}, A \cap B \in \mathcal{B}\}$$

看上去 $\mathcal{B}_B$ 跟 B 的选择有关，但实际上无关

Given the properties of $\mathcal{B}$, one easily checks that $\mathcal{B}_B$ is closed under increasing limits and by differences.

证明对单调并封闭，即证明： 若$A_1 \subset A_2 \subset \ldots$ 且 $A_i \in \mathcal{B}_B$ ，证明 $\cup_{i=1}^{\infty} A_i \in \mathcal{B}_B$

则证明其满足两条：

- $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$
- $(\cup_{i=1}^{\infty} A_i)B \in \mathcal{B}$

证明第一条：$\cup_{i=1}^{\infty} A_i \in \mathcal{B}$

$\because$ $\mathcal{B}$ 对单增并封闭（由定义）

$\therefore$ $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$

证明第二条：$(\cup_{i=1}^{\infty} A_i)B \in \mathcal{B}$

$(\cup_{i=1}^{\infty} A_i)B = \cup_{i=1}^{\infty}(A_i B)$ 且 $A_1B \subset A_2B \subset \ldots$

则 $\cup_{i=1}^{\infty}(A_iB) \subset \mathcal{B}$ （定义）

则说明 $\cup_{i=1}^{\infty} A_i \in \mathcal{B}_B$ ，即 $\mathcal{B}_B$ 对单增并封闭

仿照这个证明，对差封闭的自己证明一下

Let $B \in \mathcal{C}$; 给定 $B \in \mathcal{C}$;

For each $C \in \mathcal{C}$ one has $B \cap C \subset \mathcal{C} \subset \mathcal{B}$ and $C \in \mathcal{B}$, thus $C \in \mathcal{B}_B$

Hence $\mathcal{C} \subset \mathcal{B}_B \subset \mathcal{B}$. 后一个是定义

$\because \mathcal{B}_B$ 是包含了 $\mathcal{C}$ 的对有限并、差封闭的

$\therefore \mathcal{B} = \mathcal{B}_B$ (by the properties of $\mathcal{B}$ and of $\mathcal{B}_B$)

说明在$B \in \mathcal{C}$时是可以证明$\mathcal{B} = \mathcal{B}_B$的

下面将其拓展到 $\mathcal{B}$ 上

Now let $B \in \mathcal{B}$.

For each $C \in \mathcal{C} \subset \mathcal{B}$

由上一条可知，在$B \in \mathcal{C}$时是可以证明$\mathcal{B} = \mathcal{B}_B$的，则在$C \in \mathcal{C}$时是可以证明$\mathcal{B} = \mathcal{B}_C$的，则 $B \in \mathcal{B} = \mathcal{B}_C$

we have $B \in \mathcal{B}_C$, and because of the preceding, $B \cap C \in \mathcal{B}$, hence $C \in \mathcal{B}_B$, whence $\mathcal{C} \subset \mathcal{B}_B \subset \mathcal{B}$, hence $\mathcal{B} = \mathcal{B}_B$

$$\begin{array}{lcl} B \in \mathcal{B}_C & & C \in \mathcal{B} \\ \Downarrow \text{定义} & & CB \in \mathcal{B} \\ 1. B \in \mathcal{B} & \Rightarrow & \Downarrow \text{定义} \\ 2. BC \in \mathcal{B} & & C \in \mathcal{B}_B \end{array}$$

$$\Downarrow$$

$$\begin{array}{l} \mathcal{C} \subset \mathcal{B}_B \subset \mathcal{B} \\ \mathcal{B} = \mathcal{B}_B \end{array}$$

即$\mathcal{B}_B$ 跟 B 无关，但是B 必须在 $\mathcal{B}$ 里面

Since $\mathcal{B} = \mathcal{B}_B$ for all $B \in \mathcal{B}$, we conclude $\mathcal{B}$ is closed by finite intersections.

$\because \mathcal{B}_B$ 对有限并封闭

$\forall A, B \in \mathcal{B}$

$\because B \in \mathcal{B} \qquad A \in \mathcal{B} = \mathcal{B}_B$

$\Rightarrow AB \in \mathcal{B}$

即 $\mathcal{B}$ 对有限并封闭

可以由定义推导

Furthermore $\Omega \in \mathcal{B}$, and $\mathcal{B}$ is closed by difference, hence also under complementation.

Since $\mathcal{B}$ is closed by increasing limits as well, we conclude $\mathcal{B}$ is a σ−algebra, and it is clearly the smallest such containing $\mathcal{C}$.

证明：$\mathcal{B}$ 是 σ-代数

缺一个可列并封闭：

证明：

若 $A_i \in \mathcal{B}$

令 $B_n = \cup_{i=1}^n A_i$，则 $\cup_{i=1}^n A_i = \cup_{i=1}^n B_i$ ，且 $B_1 \subset B_2 \subset \ldots$

$B_n^c = \cap_{i=1}^n A_i^c \in \mathcal{B}$

- $A_i \in \mathcal{B}$
- $A_i^c \in \mathcal{B}$
- A_i^c 有限并 $\in \mathcal{B}$

$\Rightarrow \quad B_n \in \mathcal{B}$

则 $\mathcal{B}$ 对可列并封闭

$\mathcal{B}$ 是 σ-代数，$\mathcal{B} \supset \mathcal{C} \qquad \Rightarrow \quad \mathcal{B} \supset \sigma(\mathcal{C})$

由 $\sigma(\mathcal{C}) \supset \mathcal{C}$ 且 $\sigma(\mathcal{C})$ 对单增并、差封闭

由 $\mathcal{B}$ 的最小性，可得 $\sigma(\mathcal{C}) \supset \mathcal{B}$

则 $\mathcal{B} = \sigma(\mathcal{C})$

The proof of the uniqueness in Theorem 6.1 is an immediate consequence of the following Corollary 6.1, itself a consequence of the Monotone Class Theorem

Corollary 6.1 推论：概率延拓定理

Let P and Q be two probabilities defined on $\mathcal{A}$

Suppose P and Q agree on a class $\mathcal{C} \subset \mathcal{A}$ which is closed under finite intersections.

If $\sigma(\mathcal{C}) = \mathcal{A}$, we have $P = Q$

这个定理说明：两个定义在对交封闭的 $\mathcal{C}$ 上相等的概率测度 P, Q ， 可以将其延拓到 $\sigma(\mathcal{C})$ 上

Proof. We can assume $w.l.o.g.$ that $\Omega \in \mathcal{C}$, since

- $\Omega \in \mathcal{A}$, because $\mathcal{A}$ is a σ−algebra
- $P(\Omega) = Q(\Omega) = 1$, because they are both probabilities.

Let

$$\mathcal{B} = \{A \in \mathcal{A} : P(A) = Q(A)\}$$

By the definition of a Probability measure and Theorem 2.3, $\mathcal{B}$ is closed by difference and by increasing limits.

Also $\mathcal{B}$ contains $\mathcal{C}$ by hypothesis.

Therefore since $\sigma(\mathcal{C}) = \mathcal{A}$, we have $\mathcal{B} = \mathcal{A}$ by the Monotone Class Theorem.

不妨设 $\Omega \in \mathcal{C}$ （若不在里面，定义 $\mathcal{C}' = \mathcal{C} \cup \{\Omega\}$ ）

$\mathcal{B} = \{A \in \mathcal{A} : P(A) = Q(A)\}$

证明： $\mathcal{B} = \mathcal{A}$ ，则在 $\mathcal{A}$ 上 $P(A) = Q(A)$

$\because$ $\mathcal{C}$ 上 $P(A)=Q(A)$ $\quad\therefore \mathcal{C}\subset\mathcal{B}$

若 $A_1\subset A_2\subset\dots$ $\quad A_i\in\mathcal{B}$

则 $\cup_{i=1}^{\infty}A_i$

$$P\left(\cup_{i=1}^{\infty}A_i\right)=\lim_{n\to\infty}P(A_n)=\lim_{n\to\infty}Q(A_n)=Q\left(\cup_{i=1}^{\infty}A_i\right)$$

$\therefore \cup_{i=1}^{\infty}A_i\in\mathcal{B}$

对差也一样

又 $\mathcal{B}\supset\mathcal{C}$ 且 $\mathcal{B}$ 对单增并、差封闭

$\therefore\quad \sigma(\mathcal{C})=\mathcal{A}\subset\mathcal{B}\subset\mathcal{A}$

$\therefore\quad \mathcal{A}=\mathcal{B}$

Definition 6.2

Let P be a probability on $\mathcal{A}$.

A **null set** (or **negligible set**) for P is a subset A of Ω such that there exists a $B\in\mathcal{A}$ satisfying $A\subset B$ and $P(B)=0$

可忽略的集合

$\mathcal{A}$ 是一个概率测度

因为 概率一定满足规范性和可列可加性

A 不一定在 $\mathcal{A}$ 上

Remark

- We say that a property holds almost surely ($a.s.$ in short) if it holds outside a negligible set.
- This notion clearly depends on the probability, so we say sometimes P-almost surely , or P-a.s.

  几乎处处收敛

$$\begin{aligned}\xi_n\to_{a.s.}\xi\Leftrightarrow\ &\exists A\subset B\\ &s.t.\ B\subset\mathcal{A}\text{ and }P(B)=0\\ &lim_{n\to\infty}\xi_n(\omega)=\xi\end{aligned}$$

  r 收敛

$$\xi_n\to_r\xi\Leftrightarrow E|\xi_n-\xi|^r\to 0\quad n\to\infty$$

Remark

- The negligible set are not necessarily in $\mathcal{A}$
- Nevertheless it is natural to say that they have probability zero.
- In the following theorem, we extend the probability to the σ-algebra which is generated by $\mathcal{A}$ and all P-negligible sets.

目标：将概率 P 延拓到 $\mathcal{A}\cup\{\text{all}P\text{-negligible}\}$

Theorem 6.4

Let P be a probability on $\mathcal{A}$ and let $\mathcal{N}$ be the class of all P-negligible sets.

Then

$$\mathcal{A}' = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$$

is a σ-algebra, called the P-completion of $\mathcal{A}$

- $\mathcal{A}'$ is the smallest σ-algebra containing $\mathcal{A}$ and $\mathcal{N}$
- P extends uniquely as a probability (still denoted by P) on $\mathcal{A}'$,

  by setting $P(A \cup N) = P(A)$ for $A \in \mathcal{A}$ and $N \in \mathcal{N}$

Proof.

由 Corollary 6.1 可知，存在 $\mathcal{A} \cup \mathcal{N}$

$\forall A \in \mathcal{A} \cup \mathcal{N}$ 即 $A \in \mathcal{A}$ *or* $A \in \mathcal{N}$

下面证明 $\mathcal{A} \cup \mathcal{N}$ 对有限交封闭

$$
\begin{aligned}
&1.\ 若\ A \in \mathcal{A}, B \in \mathcal{A} \quad \Rightarrow AB \in \mathcal{A} \subset \mathcal{A} \cup \mathcal{N} \\
&2.\ 若\ A \in \mathcal{A}, B \in \mathcal{N} \quad AB \in \mathcal{N} \subset \mathcal{A} \cup \mathcal{N}
\end{aligned}
$$

$$
\begin{aligned}
&\because B \in \mathcal{N} \rightarrow \exists C \in \mathcal{A} \quad s.t. \quad B \subset C \text{ and } P(C) = 0 \\
&\therefore AB \subset B \subset C \\
&\Rightarrow AB \in \mathcal{N}
\end{aligned}
$$

$$3.\ 若\ A \in \mathcal{N}, B \in \mathcal{N} \quad \Rightarrow AB \in \mathcal{N} \subset \mathcal{A} \cup \mathcal{N}$$

由 123 说明 $\mathcal{A} \cup \mathcal{N}$ 对有限交封闭

证明 $\mathcal{A} \cup \mathcal{N} \subset \mathcal{A}'$

$\forall A \in \mathcal{A} \cup \mathcal{N}$ ， 即 $A \in \mathcal{A}$ *or* $A \in \mathcal{N}$

$$
\left.
\begin{aligned}
&\because \emptyset \in \mathcal{A} \text{ and } \emptyset \in \mathcal{N} \\
&\forall B \in \mathcal{A}, \quad B = B \cup \emptyset \in \mathcal{A} \Rightarrow \mathcal{A} \subset \mathcal{A}' \\
&\forall N \in \mathcal{N}, \quad N = N \cup \emptyset \in \mathcal{A}' \Rightarrow \mathcal{N} \subset \mathcal{A}'
\end{aligned}
\right\} \Rightarrow \mathcal{A} \cup \mathcal{N} \subset \mathcal{A}'
$$

证明$\mathcal{A}' = \sigma(\mathcal{A} \cup \mathcal{N})$

Denote $\sigma(\mathcal{A} \cup \mathcal{N}) = \mathcal{A}''$

1.	2.
$\mathcal{A} \subset \mathcal{A}'$ *and* $\mathcal{N} \subset \mathcal{A}'$已证	$\forall B \in \mathcal{A}' \quad B = A \cup N, A \in \mathcal{A}, N \in \mathcal{N}$
$\Updownarrow$	$\because \mathcal{A}''$ is σ-algebra
$\mathcal{A} \cup \mathcal{N} \subset \mathcal{A}'$	and $A \in \mathcal{A} \subset \mathcal{A}'', N \in \mathcal{A} \subset \mathcal{A}''$
$\Rightarrow \mathcal{A}'' \subset \mathcal{A}'$	$B = A \cup N \in \mathcal{A}'' \Rightarrow \mathcal{A}' \subset \mathcal{A}''$

$$\Downarrow$$

$$\mathcal{A}' = \mathcal{A}'' = \sigma(\mathcal{A} \cup \mathcal{N})$$

证明 $\mathcal{A}'$ 是一个 σ-algebra

$$1.\ \Omega \in \mathcal{A}' \quad \Omega = \Omega \cup \emptyset, \quad \Omega \in \mathcal{A}, \emptyset \in \mathcal{N}$$

2. 若 $B \in \mathcal{A}'$ ，则 $\exists A \in \mathcal{A}, N \in \mathcal{N}$

$$\begin{aligned}
&s.t.\\
&\quad B = A \cup N\\
&B^C = A^C \cap N^C = (A^C \cap C^C) \cup (A^C \cap N^C \cap C)\\
&\qquad (A^C \cap C^C) \in \mathcal{A}\\
&\qquad (A^C \cap N^C \cap C) \subset C \in \mathcal{N}\\
&\quad \Downarrow\\
&B^C \in \mathcal{A}'
\end{aligned}$$

3. 可列可加

   $B_n \in \mathcal{A}', n = 1, 2, \ldots$

   $N_n \subset C_n \in \mathcal{A}, P(C_n) = 0$

   $P(\cup_{n=1}^{\infty} C_n) \le \sum_{n=1}^{\infty} P(C_n) = 0$

$$\begin{aligned}
B_n &= A_n \cup N_n \qquad A_n \in \mathcal{A}, N_n \in \mathcal{N}\\
\cup_{n=1}^{\infty} B_n &= \left(\cup_{n=1}^{\infty} A_n\right) \cup \left(\cup_{n=1}^{\infty} N_n\right) \in \mathcal{A}'\\
&\cup_{n=1}^{\infty} A_n \in \mathcal{A}\\
&\cup_{n=1}^{\infty} N_n \in \mathcal{N}
\end{aligned}$$

则 $\mathcal{A}'$ 是一个 σ-algebra ， $\mathcal{A}' = \sigma(\mathcal{A} \cup \mathcal{N})$

由 corollary 6.1 知, The uniqueness of the extension is straightforward.

现在想知道，若 $\mathcal{A}$ 上有不一样的分解，其概率是否会于分解有关?

Suppose now that $A_1 \cup N_1 = A_2 \cup N_2$ with $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}$

The symmetrical difference $A_1 \triangle A_2 = (A_1 \cap A_2^c) \cup (A_1^c \cap A_2)$ is contained in $N_1 \cup N_2$

Proof.

$$\begin{aligned}
A_1 A_2^c &\subset (A_1 \cup N_1) A_2^c = (A_2 \cup N_2) A_2^c = A_2 A_2^c \cup N_2 A_2^c = N_2 A_2^c \subset N_2\\
A_1^c A_2 &\subset A_1^c (A_2 \cup N_2) = A_1^c (A_1 \cup N_1) = A_1^c A_1 \cup A_1^c N_1 = A_1^c N_1 \subset N_1
\end{aligned}$$

$$\therefore A_1 A_2^c \cup A_1^c A_2 \quad \subset \quad N_1 \cup N_2$$

$\exists C_i \in \mathcal{A} \quad P(C_i) = 0, \quad N_i \subset C_i, \quad i = 1, 2, \ldots \quad A_i \in \mathcal{A}$

$A_1 \triangle A_2 \subset N_1 \cup N_2 \subset C_1 \cup C_2 \quad \Rightarrow \quad A_1 \triangle A_2 \in \mathcal{N}, \quad P(C_1 \cup C_2) \le P(C_1) + P(C_2) = 0$

$$\begin{aligned}
P(A_1) &= P(A_1 A_2 + A_1 A_2^c)\\
&= P(A_1 A_2) + P(A_1 A_2^c)\\
&= P(A_1 A_2)\\
&A_1 A_2^c \subset A_1 \triangle A_2 \subset N_1 \cup N_2 \subset C_1 \cup C_2 \quad \Rightarrow P(A_1 A_2^c) \le P(C_1 \cup C_2) = 0\\
P(A_1) &= P(A_1 A_2)\\
P(A_2) &= P(A_1 A_2)\\
\\
&\Rightarrow P(A_1) = P(A_2)\\
&\Rightarrow Q(A_1 \cup N_1) = Q(A_2 \cup N_2)
\end{aligned}$$

则不管怎么分解，延拓的概率都相等

现在证明 Q 是概率，概率需要验证两条:

1. $Q(\Omega) = Q(\Omega \cup \emptyset) = P(\Omega) = 1$

2. 假设 $B_n \in \mathcal{A}'$ 且 B_n 两两互斥 $n=1,2,...$

$$
\begin{aligned}
Q(\cup_{i=1}^{\infty} B_n) &= Q\left\{\cup_{i=1}^{\infty}(A_n \cup N_n\right\} \\
&= Q\left\{\cup_{i=1}^{\infty} A_n \cup \cup_{i=1}^{\infty} N_n\right\} \\
&= P\left\{\cup_{i=1}^{\infty} A_n\right\} \\
&\quad A_n \text{ 是} B_n \text{ 的一部分，则 } B_n \text{ 两两互斥得到 } A_n \text{ 两两互斥} \\
&= \sum_{n=1}^{\infty} P\left\{A_n\right\}
\end{aligned}
$$

$$
\begin{aligned}
Q(B_n) = Q(A_n \cup N_n) &= P(A_n) \\
\therefore Q(\cup_{i=1}^{\infty} B_n) &= \sum_{n=1}^{\infty} Q(B_n)
\end{aligned}
$$

则 Q 是 $\mathcal{A}'$ 上的概率测度，仍然记 Q 为 P

定理6.4证明完毕

回顾一下定理6.1以及其证明过程：

定理6.1： 概率 P 是 $\mathcal{A}$ 上的概率， $\mathcal{N}$ 是所有的可忽略集合类

定义 $\mathcal{A}' = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$

则可以得到：

- $\mathcal{A}'$ 是一个 σ-代数
- $\mathcal{A}' = \sigma(\mathcal{A} \cup \mathcal{N})$
- 概率 P （原来定义在 $\mathcal{A}$ 上）可在 $\mathcal{A}'$ 上唯一延拓

定理6.1把概率又往前延拓，现在是可以延拓到一个 σ-代数和一个可忽略集的并集构成的 σ-代数上了。

定理的证明需要证明几步：

1. $\mathcal{A}'$ 是一个 σ-代数 $\begin{cases} \emptyset\ \Omega \text{在里面} \\ A\text{在}\mathcal{A}'\text{里面，则 } A^c\text{也在}\mathcal{A}'\text{里面} \\ \text{对可列并封闭} \end{cases}$
2. 证明 $\mathcal{A}' = \sigma(\mathcal{A} \cup \mathcal{N})$ ，由两个包含关系可以证明

   其中， 证明 $\mathcal{A} \in \mathcal{N} \subset \mathcal{A}'$ 就是为了证明这一步
3. 证明概率 P 延拓到 $\mathcal{A}'$ 上之后（记为 Q)仍然是一个概率（由概率的定义来证明）
4. 证明唯一性，延拓之后的概率是唯一的，则仍然可以记为 P.

   唯一性的证明由性质6.1可以得到，需要证明

   - $\mathcal{A} \cup \mathcal{N}$ 对有限交封闭
   - $\mathcal{A}' = \sigma(\mathcal{A} \cup \mathcal{N})$
   - 概率与分解无关

     性质6.1： 概率 P 和 概率 Q 是定义在$\mathcal{A}$ 上的两个概率，若 P, Q 在 $\mathcal{C} \subset \mathcal{A}$ 上相等，并且对有限交封闭，如果 $\sigma(\mathcal{C}) = \mathcal{A}$ ，则 $P = Q$