Chapter 2. Axioms of Probability

$\sigma - algebra$

- Let Ω be abstract space, that is with no special structure.
- Let 2^{Ω} denotes all subsets of Ω , including the empty set denoted by $\emptyset$.

With $\mathcal{A}$ being a subset of 2^{Ω} , we consider the following properties:

1. $\emptyset \in \mathcal{A} \quad and \quad \Omega \in \mathcal{A}$
2. If $A \in \mathcal{A}$ then $A^C \in \mathcal{A}$, where A^C denotes the complement of A
3. $\mathcal{A}$ is closed under finite unions and finite intersections:
    - that is, if $A_1, \dots, A_n$ are all in $\mathcal{A}$, then $\cup_{i=1}^{n} A_i$ and $\cup_{i=1}^{n} A_i$ are in $\mathcal{A}$ as well
    - $A_1, \dots, A_n \in \mathcal{A} \qquad \Rightarrow \qquad \cup_{i=1}^{n} A_i \quad and \quad \cap_{i=1}^{n} A_i \in \mathcal{A}$
4. $\mathcal{A}$ is closed under countable unions and intersections:
    - that is, if $A_1, A_2 \dots$ is countable sequence of events in $\mathcal{A}$, then $\cup_{i=1}^{n} A_i$ and $\cap_{i=1}^{n} A_i$ are both also in $\mathcal{A}$

Definition 2.1

$\mathcal{A}$ is an **algebra** if it satisfies (1),(2) and (3) above

$\mathcal{A}$ is an $\sigma - algebera$ (or a $\sigma - field$), if is satisfies (1),(2) and (4) above

- Under (2), (1) can replaced by either (1'): $\emptyset \in \mathcal{A}$ or by (1''): $\Omega \in \mathcal{A}$
- (1)+(4) $\Rightarrow$ (3) hence any $\sigma - algebra$ is an $algebra$ (but there are $algebra$ that are not $\sigma - algebra$)

Exercise 2.17 Suppose that Ω is an infinite set (countable or not), and let $\mathcal{A}$ be the family of all subsets which are either finite or have a finite complement. Show that $\mathcal{A}$ is an $algebra$, but not a $\sigma - algebra$

Solution:

Under the definition, $\mathcal{A}$ is the family of subsets which are either finite or have a finite complement.

1. $\emptyset$ is a finite subset $\Rightarrow \quad \emptyset \in \mathcal{A}$
2. if $A \in \mathcal{A} \Rightarrow \quad A$ is finite or A^C is finite

   so $A^C \in \mathcal{A}$
3. if $A_i \in \mathcal{A}, i = 1, 2, \ldots, n \quad \Rightarrow \quad A_i$ is finite or A^C is finite

   1. if $\quad \exists i_0$ so that A_{i_0} is finite

$$\begin{gathered} \cap_{i=1}^n A_i \subset A_{i_0} \quad \text{is finite} \\ \Downarrow \\ \text{so that} \quad \cap_{i=1}^n A_i \quad \text{is finite} \end{gathered}$$

   2. $\forall i = (1, 2, \ldots, n)$, A_i is infinite

      $\Rightarrow A_i^C$ is finite

$$\begin{gathered} \left(\cap_{i=1}^n A_i\right)^C = \cup_{i=1}^n A_i^C \quad \text{finite} \\ \Downarrow \\ so \quad \left(\cap_{i=1}^n A_i\right)^C \in \mathcal{A} \end{gathered}$$

4. Ω is infinite , there exist $\omega \in \Omega \quad i = 1, 2, \cdots$

   记 $A_i = \{\omega_i\} \quad i = 1, 2, \cdots \quad \Rightarrow \quad A_i \in A$

   Consider $\cup_{m=1}^{\infty} = \{\omega_{2m}, \omega_{2m+2}, \cdots\}$ is infinite

   Moreover , for each fixed $m = 1, 2, \cdots$, we have $\omega_{2m-1} \in A_{2m}^C$ for every $m = 1, 2, \cdots$

   So $A_{2m-1} \subset A_{2m}^C$ for every $m = 1, 2, \cdots$

   $\cup_{m=1}^{\infty} A_{2m-1} \subset A_{2m}^C$

   $\therefore \quad \text{infinite} \quad \cup_{m=1}^{\infty} A_{2m-1} \subset \cap_{m=1}^{\infty} A_{2m}^C = \left(\cup_{m=1}^{\infty} A_{2m}^C\right)^C$ is infinite

   $\therefore \quad \cup_{m=1}^{\infty} A_{2m}$ if infinite

   $\therefore \quad \left(\cup_{m=1}^{\infty} A_{2m}\right)^C$ is infinite

Definition 2.2

If $\mathcal{C} \subset 2^{\Omega}$, **The** $\sigma - algebra$ **generated by** $\mathcal{C}$, and written as $\sigma(\mathcal{C})$.

- 2^{Ω} is a $\sigma - algebra$
- **Exercise 2.2** Let $(\mathcal{G}_\alpha)_{\alpha \in A}$ be an arbitrary family of $\sigma - algebras$ defined on abstract space Ω. Show that $\mathcal{H} = \cap_{\alpha \in A} \mathcal{G}_\alpha$ is also a $\sigma - algebra$.

Example

1. $\mathcal{A} = \{\emptyset, \Omega\}$ (the trivial $\sigma - algebra$)
2. A is a subset, then $\sigma(A) = \{\emptyset, A, A^C, \Omega\}$
3. if $\Omega = \mathbb{R}$ (the real numbers) (or more generally if Ω is a space with a topology, a case we treat in Chapter 8), the **Borel** $\sigma - algebra$ is the $\sigma - algebra$ generated by the open sets (or by the closed sets, which is equivalent)

trivial 平凡的

Borel $\sigma - algebra$

博雷尔集---由所有开集生成的 σ 代数，或包含所有闭集生成的σ 代数

Wolfram Mathworld: A sigma-algebra which is related to the topology of a set. The Borel sigma-algebra is defined to be the sigma-algebra generated by the open sets (or equivalently, by the closed sets).

Wikipedia:

- The Borel algebra on X is the smallest σ-algebra containing all open sets (or, equivalently, all closed sets).
- Borel sets are important in measure theory, since any measure defined on the open sets of a space, or on the closed sets of a space, must also be defined on all Borel sets of that space. Any measure defined on the Borel sets is called a Borel measure. Borel sets and the associated Borel hierarchy also play a fundamental role in descriptive set theory.

作业：（补补补）

Exercise 2.2 Let $(\mathcal{G}_\alpha)_{\alpha \in A}$ be an arbitrary family of $\sigma - algebras$ defined on abstract space Ω. Show that $\mathcal{H} = \cap_{\alpha \in A} \mathcal{G}_\alpha$ is also a $\sigma - algebra$.

- Ω 是不是 $\mathcal{H}$
- $\forall B \in \mathcal{H} = \cap_{\alpha \in \Lambda} \mathcal{G}_\alpha$

$$\forall \alpha \in A \quad B \in \mathcal{G}_\alpha \quad \text{is sigma algebra}$$

$$\text{so} \quad B^c \in \mathcal{G}_\alpha$$

$$B^c \in \cap_{\alpha \in \Lambda} \mathcal{G}_\alpha = \mathcal{H}$$

  - 2^Ω 样本空间的最大σ 代数
  - $\mathcal{C}$ 是集合类
  - $\sigma(\mathcal{C})$ 是最小的σ 代数

相关概念:

1. **内点**:如果存在 P_0 的某一邻域 $U(P_0)$,使得 $U(P_0) \subset E$,则称 P_0 为 E 的内点
2. **外点**:如果P_0是E^c 的内点,则称P_0为E 的外点(这里的余集是针对全空间R^n 来作的
3. **界点**:如果P_0 既非 E 的内点又非E 的外点,即:P_0 的任一邻域内既有属于E 的点,也有不属于E 的点
4. **聚点**:设E 是 R^n 中一点集,P_0 为R^n 中一定点,如果P_0 的任一邻域内都含有无穷多个属于E 的点,则称P_0 为E 的一个聚点
5. **孤立点**:如果P_0 属于E 但不是E 的聚点,则P_0 为E 的孤立点

- **开核**:E 的全体内点所组成的集合,记为$\mathring{E}$
$$\mathring{E} = \{x : \exists U(x) \subset E\}$$
- **导集**:E 的全体聚点所组成的集合,记为E'
$$E' = \{x : \forall U(x), U(x) \cap E \backslash \{x\} \neq \emptyset\}$$
- **边界**:E 的全体界点所组成的集合,记为∂E
$$\partial E = \{x : \forall U(x), U(x) \cap E \neq \emptyset, U(x) \cap E^c \neq \emptyset\}$$
- **闭包**:$E \cup E'$ 记为E 的闭包,记为 $\overline{E}$
$$\begin{aligned} \overline{E} &= \{x : \forall U(x), U(x) \cap E \neq \emptyset\} \\ &= E \cup \partial E \\ &= \mathring{E} \cup \partial E \\ &= E' \cup \{E\text{的孤立点}\} \end{aligned}$$

1. **开集**:设$E \subset R^n$,如果 E 的每一个点都是E 的内点,则称E 为开集
2. **闭集**:设$E \subset R^n$,如果 E 的每一个聚点都属于E ,则称E 为闭集

   开集与闭集的对偶性:
$$E\text{为开集,则}E^c\text{为闭集}$$

Theorem 2.1

The Borel $\sigma - algebra$ of $\mathbb{R}$ is generated by intervals of the form $(-\infty, a]$, where $a \in \mathbb{Q}$

$(\mathbb{Q} = rationals)$

Proof.

Let $\mathcal{C}$ denote all open intervals. Since every open set in $\mathcal{R}$ is the countable union of open intervals, we have $\sigma(\mathcal{C}) =$ the Borel $\sigma - algebra$ of $\mathbb{R}$

Let $\mathcal{D}$ denote all intervals of the form $(-\infty, a]$, where $a \in \mathbb{Q}$.

Let $(a, b) \in \mathcal{C}$

Let $(a_n)_{n \geq 1}$ be a sequence of rationals decreasing to a

Let $(b_n)_{n \geq_1}$ be a sequence of rationals increasing strictly to b.

Then

$$\begin{aligned}(a, b) &= \cup_{n=1}^{\infty} (a_n, b_n] \\ &= \cup_{n=1}^{\infty} \left((-\infty, b_n] \cap (-\infty, a_n]^c \right)\end{aligned}$$

Therefore , $\mathcal{C} \subset \sigma(\mathcal{D})$, where $\sigma(\mathcal{C}) \subset \sigma(\mathcal{D})$

However, since each element of $\mathcal{D}$ is a closed set, it is also a Borel set, and therefore $\mathcal{D}$ is contained in the Borel sets $\mathcal{B}$, Thus we have

$$\mathcal{B} = \sigma(\mathcal{C}) \subset \sigma(\mathcal{D}) \subset \mathcal{B}$$

and hence $\sigma(\mathcal{D}) = \mathcal{B}$

这个定理给出了更容易验证 $Borel \sigma - algebra$ 的一种方法，给出了验证博雷尔集的一个充要条件

Definition 2.3

A probability measure defined on a $\sigma - algebra \quad \mathcal{A}$ of Ω is a function $P : \mathcal{A} \to [0,1]$ that satisfies:

- $P(\Omega) = 1$ 规范性
- For every countable sequence $(A_n)_{n\geq 1}$ of elements of $\mathcal{A}$, pairwise disjoint (that is, $A_n \cap A_m = \emptyset$ Whenever $n \neq m$), one has:
  $P\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P\left(A_n\right)$

Axiom 2 above is called **countable additivity**; the number $P(A)$ is called the probability of the event A

- 有限可加 additivity
- 可列可加 countable additivity

概率是 $\mathcal{A}$ 到 Ω 上的一个映射，是任何一个样本空间的子集

Remark: In definition 2.3 one might imagine a more naive condition than 2, namely

$$A, B \in \mathcal{A}, A \cap B = \emptyset \quad \Rightarrow \quad P(A \cup B) = P(A) + P(B)$$

This property is called additivity (or "finite additivity") and, by an elementary induction, it implies that for every finite $A_1, \cdots, A_m$ of pairwise disjoint events $A_i \in \mathcal{A}$ we have

$$P\left(\cup_{n=1}^{m} A_N\right) = \sum_{n=1}^{\infty} P(A_n)$$

Theorem 2.2

If P is a probability measure on $(\Omega, \mathcal{A})$, then:

i. We have $P(\emptyset)=0$

ii. P is additive

> ↑ 去年考的

Proof.

If in Axiom (2) we take $A_n=\emptyset$ for all n, we see that the number $a=P(\emptyset)$ is equal to an infinite sum of itself;

Since $0 \leq a \leq 1$, this is possible only if $a=0$, and we have (i).

For (ii) if suffices to apply Axiom (2) with $A_1=A$ and $A_2=B$ and $A_3=A_4=\cdots=\emptyset$, plus the fact that $P(\emptyset)=0$, to obtain the additivity of P.

Axiom (1): $P(\Omega)=1$ 规范性

Axiom (2):

For every countable sequence $(A_n)_{n \geq 1}$ of elements of $\mathcal{A}$, pairwise disjoint (that is, $A_n \cap A_m=\emptyset$ Whenever $n \neq m$), one has:

$$P\left(\cup_{n=1}^{\infty} A_n\right)=\sum_{n=1}^{\infty} P\left(A_n\right)$$

Remark

Conversely, countable additivity is not implied by additivity.

In fact, in spite of its intuitive appeal, additivity is not enough to handle the mathematical problems of the theory, even in such a simple example as tossing a coin, as we will see later.

Theorem 2.3

Let $\mathcal{A}$ be a $\sigma - algebra$.

Suppose that $P : \mathcal{A} \to [0, 1]$ satisfies (1) and is additive.

Then the following are equivalent:

i. Axiom (2) of Definition 2.3 (i.e. P is countable additivity)

ii. If $A_n \in \mathcal{A}$ and $A_n \downarrow \emptyset$, then $P(A_n) \downarrow 0$

iii. If $A_n \in \mathcal{A}$ and $A_n \downarrow A$, then $P(A_n) \downarrow P(A)$

iv. If $A_n \in \mathcal{A}$ and $A_n \uparrow \Omega$, then $P(A_n) \uparrow 1$

v. If $A_n \in \mathcal{A}$ and $A_n \uparrow A$, then $P(A_n) \uparrow P(A)$

Proof.

The notation $A_n \downarrow A$ means that $A_{n+1} \subset A_n$, each n, and $\cap_{n=1}^{\infty} A_n = A$.

The notation $A_n \uparrow A$ means that $A_n \subset A_{n+1}$, each n, and $\cup_{n=1}^{\infty} A_n = A$

Note that is $A_n \downarrow A$, then $A_n^c \uparrow A^c$, and by the finite additivity axiom $P(A_n^c) = 1 - P(A_n)$.

Therefore (ii) is equivalent to (iv) and similarly (iii) is equivalent to (v).

Moreover by choosing A to be Ω we have that (v) implies (iv)

只要证明了 (ii)，就可以推出 (iv)；只要证明了 (iii)，就可以推出 (v)；在 (v) 里面，取 $A \to \Omega$，则能证明 (iv)。

Suppose now that we have (iv).

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Let $A_n \in \mathcal{A}$ with $A_n \uparrow A$.

Set $B_n = A_n \cup A^c$.

Then $B_n$ increase to $\Omega$, hence $P(B_n)$ increase to $1$

Since $A_n \subset A$ we have $A_n \cap A^c = \emptyset$. Thus
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$$1 = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} \{P(A_n) + P(A^c)\}$$

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hence $\lim_{n\to\infty} P(A_n) = 1 - P(A^c) = P(A)$, and we have (v)  (It means (iv)$\to$ (v))

It remains to show that (i) $\Leftrightarrow$ (v)
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Suppose we have (v).

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Let $A_n \in \mathcal{A}$ be pairwise disjoint:

    that is, if $n \neq m$, then $A_n \cap A_m = \emptyset$

Define $B_n = \cup_{1\le k\le n} A_k$ and $B = \cup_{k=1}^{n} P(A_k)$
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Then by the definition of a Probability Measure we have $P(B_n) = \sum_{k=1}^{n} P(A_k)$ which increase with n to $\sum_{k=1}^{n} P(A_n)$, and also $P(B_n)$ increases to $P(B)$ by (v)

We deduce $\lim_{n\to\infty} P(B_n) = P(B)$ and we have

$$P(B) = P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

and thus we have (i)

Finally assume we have (i), and we wish to establish (v)

Let $A_n \in \mathcal{A}$, with A_n increasing to A.

We construct a new sequence as follows:

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 - A_1 = A_2 \cap A_1^c \\ &\vdots \\ B_n &= A_n - A_{n-1} \end{aligned}$$

Then $\cup_{n=1}^{\infty} B_n = A$ and the events $(B_n)_{n\geq 1}$ are pairwise disjoint.

Therefore by (i) we have

$$P(A) = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$

But also $\sum_{k=1}^{n} P(B_k) = P(A_n)$, hence we deduce $\lim_{n\to\infty} P(A_n) = P(A)$ and we have (v)

Remark： 示性函数

If $A \in 2^{\Omega}$, we define the indicator function by

$$1_A(\omega) = \begin{cases} 1 & if\, \omega \in A \\ 0 & if\, \omega \notin A \end{cases}$$

We often do not explicitly write the ω, and just write 1_A

We can say that $A_n \in \mathcal{A}$ converges to A (we write $A_n \to A$) if $\lim_{n\to\infty} 1_{A_n}(\omega) = 1_A(\omega)$ for all $\omega \in \Omega$

Note it also tends to A in the above sense.

补几个概念：

数列极限：数列 $\{a_n\}$ 收敛于 a 的定义：

对于每一个给定的 $\varepsilon > 0$ ，存在 N ，使得对满足条件 $n > N$ 的每个自然数 n ，成立不等式 $|a_n - a| < \varepsilon$

$$\forall \varepsilon > 0, \exists N, \forall n > N \quad \Rightarrow \quad |a_n - a| < \varepsilon$$

函数收敛：函数 $F(x)$ 在 x 趋于 a 时以 A 为极限：

设 $a, A \in \mathbb{R}$ ，函数 f 在点 a 的一个邻域中有定义，若对每一个给定的 $\varepsilon > 0$ ，存在 $\delta > 0$ ，使得当 $x \in O_\delta(a) - \{a\}$ (即$0 < |x - a| < \delta$) 时，成立 $|f(x) - A| < \varepsilon$

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in O_\delta(a) - \{a\} \quad \Rightarrow \quad |f(x) - A| < \varepsilon$$

函数的一致连续性：函数 f 在区间I 上为一致连续：

如果对每一个 $\varepsilon > 0$ ，存在 $\delta > 0$ ，使得当$x', x'' \in I$ 且 $|x' - x''| < \delta$ 时，成立 $|f(x') - f(x'')| < \delta$

函数列的一致收敛：设一个函数列 $\{f_n\}$ 在数集 E 上收敛于其极限函数 $f(x)$. 称 $\{f_n\}$ 于 E 上一致收敛：

如果对于每个正数 $\varepsilon > 0$，存在 N， 使得对每个正整数 $n > N$ ，和每个 $x \in E$，均成立 $|f_n(x) - f(x)| < \varepsilon$

$$\begin{aligned}
&for\ any\ fix\ x \in E \\
&\forall \varepsilon > 0, \exists N, \forall n > N \\
&\Downarrow \\
&|f_n(x) - f(x)| < \varepsilon
\end{aligned}$$

Theorem 2.4

Let P be a probability measure

Let A_n be a sequence of events in $\mathcal{A}$ which converges to A

Then.

$$A \in \mathcal{A} \quad and \quad \lim_{n \to \infty} P(A_n) = P(A)$$

Proof.

Let us define

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{m\geq n} A_m$$

$$\liminf_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{M\geq n} A_m$$

> 令 $B_n = \cup_{m\geq n} A_m \quad C_n = \cap_{m\geq n} A_m$
>
> $\exists \omega \in \cup_{n=1}^{\infty} \cap_{m\geq n} A_m = \cup_{n=1}^{\infty} C_n$
>
> 一定存在一个 n_0 ，使得 $\omega \in C_{n_0} = \cap_{m\geq n_0} A_m$
>
> 则对每个 $m \geq n_0$ ，有 $\omega \in A_m$
>
> 则
>
> $$\begin{aligned} &\omega \in \cup_{m\geq 1} A_m = B_1 \\ &\omega \in \cup_{m\geq 2} A_m = B_2 \\ &\vdots \\ &\omega \in \cup_{m\geq n} A_m = B_n \end{aligned}$$
>
> 则 $\omega \in \cap_{n=1}^{\infty} B_n = \cap_{n=1}^{\infty} \cup_{m\geq n} A_m$
>
> 说明
>
> $$\liminf_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{M\geq n} A_m \qquad \subset \qquad \limsup_{n\to\infty} A_m = \cap_{n=1}^{\infty} \cup_{m\geq n} A_m$$

Since $\mathcal{A}$ is a σ-algebra, we have

$$\limsup_{n\to\infty} A_n \in \mathcal{A} \qquad \liminf_{n\to\infty} A_n \in \mathcal{A}$$

> 这一步是由于上面上极限和下极限的定义，可以看出 A_m 是 $\sigma-$代数，则其集合的运算也是 $\sigma-$代数

By hypothesis A_n converges to A, which means $\lim_{n\to\infty} 1_{A_n} = 1_A$ for all ω.

> $A_n \in \mathcal{A}$ converges to A if $\lim_{n\to\infty} 1_{A_n}(\omega) = 1_A(\omega)$ for all $\omega \in \Omega$

This is equivalent to saying that

$$A = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$$

> 用极限逼近的思想去证明
>
> $\forall \omega \in A \qquad 1_A(\omega) = 1$
>
> $\lim_{n\to\infty} 1_{A_n}(\omega) = 1_A(\omega) = 1$
>
> $$\begin{aligned} &\exists n_0 \quad s.t. \quad n \geq n_0 \quad 1_{A_n}(\omega) = 1 \\ &so. \quad \omega \in A_n, n \geq n_0 \\ &\omega \in \cap_{n\geq n_0} A_n \subset \cup_{n_0\geq 1} \cap_{n\geq n_0} A_n = \liminf_{n\to\infty} A_n \\ \\ &\Rightarrow A \subset \liminf_{n\to\infty} A_n \end{aligned}$$
>
> $\forall \omega \in A^c \qquad 1_{A_n}(\omega) = 0$
>
> $\lim_{n\to\infty} 1_{A_n}(\omega) = 1_A(\omega) = 0$

$$\begin{aligned}
&\exists n_1 \quad s.t. \quad n \geq n_1 \quad 1_{A_n}(\omega) = 0 \\
&so. \quad \omega \in A_n^c, n \geq n_1 \\
&\omega \in \cap_{n \geq n_0} A_n^c \subset \cup_{n_0 \geq 1} \cap_{n \geq n_0} A_n^c = (\cap_{n_1 \geq 1} \cup_{n \geq n_1} A_n)^c \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (\limsup_{n \to \infty} A_n)^c \\
&so. \quad A^c \subset (\limsup_{n \to \infty} A_n)^c \\
&\Rightarrow \liminf_{n \to \infty} A_n \subset A
\end{aligned}$$

$$\limsup A_n \subset A \subset \liminf A_n$$

so

$$A = \limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n$$

Therefore $A \in \mathcal{A}$

Now, let $B_n = \cup_{m \geq n} A_m \quad C_n = \cap_{m \geq n} A_m$

Then

B_n decreases to A : B_n 是并集，随着 n 的增大，并集的个数减小，所以 B_n 会减小到 A

C_n increases to A : C_n 是交集，随着 n 的增大，交集的个数减小，所以 C_n 会增大到 A

> 注意这里我的写法和书上不一致，主要是为了和前面的保持一致

Thus (by theorem 2.3)

$\lim_{n \to \infty} P(B_n) = \lim_{n \to \infty} P(C_n) = P(A)$

However $C_n \subset A_n \subset B_n$, therefore

$$P(C_n) \leq P(A_n) \leq P(B_n)$$

So.

$$\lim_{n \to \infty} P(A_n) = P(A)$$

Exercise 2.6

Let $\mathcal{A}$ be a σ-algebra of subsets of Ω

Let $B \in \mathcal{A}$

1. Show that $\mathcal{F} = \{A \cap B : A \in \mathcal{A}\}$ is a σ-algebra of subsets of B.
2. Is it still true when B is a subset of Ω that does not belong to $\mathcal{A}$

Proof.

1. 证明空集和全集(此时全集为B)在 $\mathcal{F}$ 上

$$\begin{aligned}\emptyset = \emptyset \cap B \quad \emptyset \in \mathcal{A} \quad \Rightarrow \emptyset \in \mathcal{F} \\ B = B \cap B \quad B \in \mathcal{A} \quad \Rightarrow B \in \mathcal{F}\end{aligned}$$

2. 证明对于任意的集合 C 在 $\mathcal{F}$ 里面，它的补集也在 $\mathcal{F}$ 里面

$$\begin{aligned}\forall C \in \mathcal{F} \quad \exists A \in \mathcal{A} \quad s.t. C = AB \\ B - C = B \cap C^c &= B \cap (A \cap B)^c \\ &= B \cap (A^c + B^c) \\ &= BA^c + BB^c \\ &= BA^c \\ \because A \in \mathcal{A} \qquad \therefore A^c \in \mathcal{A} \\ \therefore C^c = B - C \in \mathcal{F}\end{aligned}$$

3. 证明对于可数并和可数交，其仍在 $\mathcal{F}$ 里面

$$\begin{aligned}if\, C_n \in \mathcal{F} \quad \exists A_n \in \mathcal{A} \quad s.t. C_n = A_n B \\ \cup_{n=1}^{\infty} C_n = \cup_{n=1}^{\infty}(A_n B) = \left(\cup_{n=1}^{\infty} A_n\right) B \\ \\ \therefore \cup_{n=1}^{\infty} A_n \in \mathcal{A} \\ \therefore \cup_{n=1}^{\infty} C_n \in \mathcal{F}\end{aligned}$$

Exercise 2.7

Exercise 以前讲过

Chapter 3 以后补。。。