CS222 Homework 6

NP and Computational Intractability

Exercises for Algorithm Design and Analysis by Li Jiang, 2018 Autumn Semester

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1 Problem 1

1.1 Question

For each of the two questions below, decide whether the answer is (i) "Yes," (ii)"No," or (iii)"Unknown, because it would resolve the question of whether P = NP." Give a brief explanation of your answer.

(a) Let's define the decision version of the Interval Scheduling Problem from Chapter 4 as follows: Given a collection of intervals on a time-line, and a bound k, does the collection contain a subset of nonoverlapping intervals of size at least k? Question: Is it the case that Interval Scheduling $\leq_p$Vertex Cover?

(b) Question: Is it the case that Independent Set $\leq_p$ Interval Scheduling?

1.2 Answer

(a) Yes. Vertex cover is a NP-complete problem, and for every problem X $\in$ NP, X $\leq_p$ NP. Because Interval Scheduling problem is a P problem, and Interval Scheduling $\in$ NP, Interval Scheduling $\leq_p$Vertex Cover.

(b) Unknown, because it would resolve the question of whether P = NP. Vertex cover $\equiv$ Independent Set $\Rightarrow$ Vertex Cover $\leq_p$ Interval Scheduing $\Rightarrow$ Interval Scheduling $\equiv$ Vertex Cover $\Rightarrow$ P=NP

2 Problem 2

2.1 Question

PARTITION : Given a finite set A and a size $s(a) \in Z$ for each $a \in A$, is there a subset $A' \subseteq A$ such that $\Sigma_{a\in A'} s(a) = \Sigma_{a\in A-A'} s(a)$?

SUBSET SUM: Given a finite set A, a size $s(a) \in Z$ for each $a \in A$ and an integer B, is there a subset $A' \subseteq A$ such that $\Sigma_{a\in A'} s(a) = B$?

KNAPSACK: Given a finite set A, a size $s(a) \in Z$ and a value $v(a) \in Z$ for each $a \in A$ and integers B and K, is there a subset $A' \subseteq A$ such that $\Sigma_{a\in A'} s(a) \leq B$ and $\Sigma_{a\in A'} v(a) \geq K$

(a) Prove PARTITION $\leq_p$ SUBSET SUM.

(b) Prove SUBSET SUM $\leq_p$ KNAPSACK.

2.2 Answer

(a) let $B = \Sigma_{a\in A-A'} s(a)$, then we can use the black box which solves SUBSET SUM to solve PARTITION, so that PARTITION $\leq_p$ SUBSET SUM.

(b) let $v = s$ and $B = K$, then KNAPSACK is $\Sigma_{a\in A'} s(a) \leq B$ and $\Sigma_{a\in A'} s(a) \geq B \Rightarrow \Sigma_{a\in A'} s(a) = B$, so that SUBSET SUM $\leq_p$ KNAPSACK.

3 Problem 3

3.1 Question

Since the 3-Dimensional Matching Problem is NP-complete, it is natural to expect that the corresponding 4-Dimensional Matching Problem is at least as hard. Let us define 4-Dimensional Matching as follows. Given sets W, X, Y, and Z, each of size n, and a collection C of ordered 4-tuples of the form $w_i; x_j; y_k; z_l$, do there exist n 4-tuples from C so that no two have an element in common?

Prove that 4-Dimensional Matching is NP-complete.

3.2 Answer

We only need to prove 3-Dimensional Matching $\leq_p$ 4-Dimensional Matching.

Define sets W, X, Y, Z of size n, and a collection C' of 4-tuple, which $(x_j, y_k, z_l) \in C$, and for i in range(1, n), there is a 4-tuple (w_i, x_j, y_k, z_l). If $A = (x_j, y_k, z_l) \in C$, assign $f(A) = (w_j, x_j, y_k, z_l) \in C'$, if $B = (w_i, x_j, y_k, z_l) \in C'$n=, assign $f'(B) = (x_j, y_k, z_l) \in C$.

$\forall i \in (0, n)$, if A_i in C, then A'_i in C', similarly, if B_i in C', then $f'(B)$ in C, so if there is a perfect 4-Dimensional matching, then we can solve the 3-Dimensional matching, then 3-Dimensional Matching $\leq_p$ 4-Dimensional Matching, so that 4-Dimensional Matching is a NP-complete problem.

4 Problem 4

4.1 Question

You are given a directed graph G = (V, E) with weights w_e on its edges $e \in E$. The weights can be negative or positive. The Zero-Weight-Cycle Problem is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0.

Prove that this problem is NP-complete.

4.2 Answer

We can calculate whether the sum of the edge weights on this cycle is 0, so Zero-Weight-Cycle Problem is a NP problem.

Subset sum problem is a NP-complete problem, we only need to prove Subset sum $\leq_p$ Zero-Weight-Cycle.

For Zero-Weight-Cycle problem, there is n nodes, and $\sum_1^{n-1} w_i = W$, $w_n = -W$, so the problem can be reduced to whether is a subset of size n-1, whose weight is added up to W. For any cycle if it contains w_n, and the left weight is added up to W so that it is a Zero-Weight-Cycle, then the subset without w_n solves the Subset sum problem. Therefore, Subset sum $\leq_p$ Zero-Weight-Cycle, and Zero-Weight-Cycle is a NP-complete problem.