Second-Order Circuits

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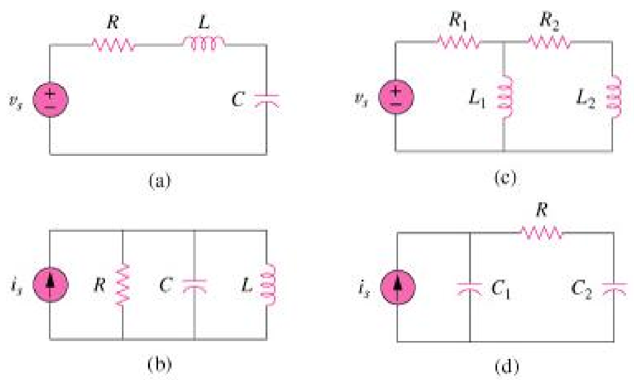
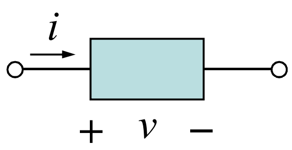
Introduction

- A second-order circuit is characterized by a second-order differential equation.
- It consists of resistors and the equivalent of two energy storage elements.



Finding Initial and Final Values

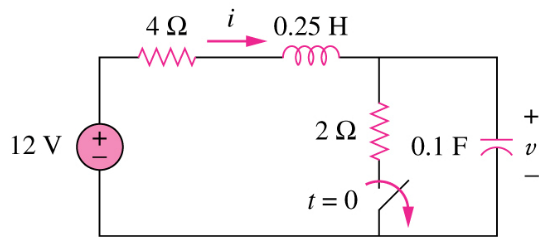
- $v(0)$, $i(0)$, $dv(0)/dt$, $di(0)/dt$, $v(\infty)$, and $i(\infty)$
- Two key points:
  - v and i are defined according to the ***passive sign convention***.



  - Continuity properties:
    - Capacitor voltage: $v_C(0^+) = v_C(0^-)$ (V_S-like)
    - Inductor current: $i_L(0^+) = i_L(0^-)$ (I_S-like)

Example

Q : Find
(a) $i(0^+), v(0^+)$,
(b) $di(0^+)/dt, dv(0^+)/dt$,
(c) $i(\infty), v(\infty)$.

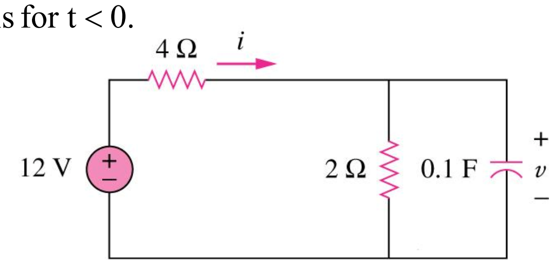


Sol : (a) Apply dc analysis for t < 0.

$$i(0^-) = \frac{12}{4+2} = 2\text{ A}$$

$$v(0^-) = 2i(0^-) = 4\text{ V}$$

$$\Rightarrow \begin{cases} i(0^+) = i(0^-) = 2\text{ A} \\ v(0^+) = v(0^-) = 4\text{ V} \end{cases}$$



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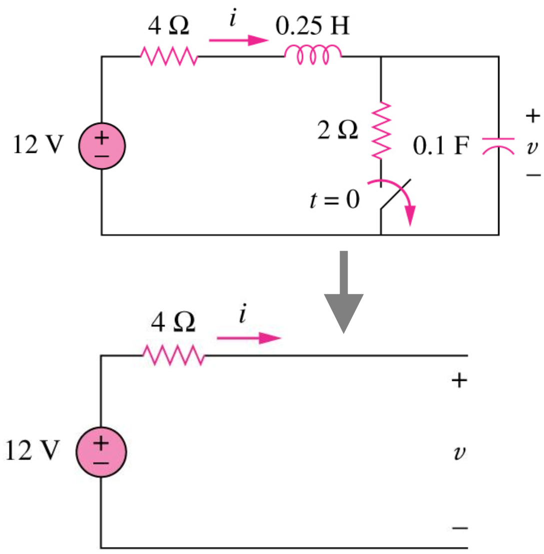
Sol : (c)

Apply dc analysis

for $t > 0$.



$$\Rightarrow i(\infty) = 0 \text{ A}$$

$$v(\infty) = 12 \text{ V}$$

Cont'd

Sol : (b) To find $\dfrac{dv(0^+)}{dt}$:

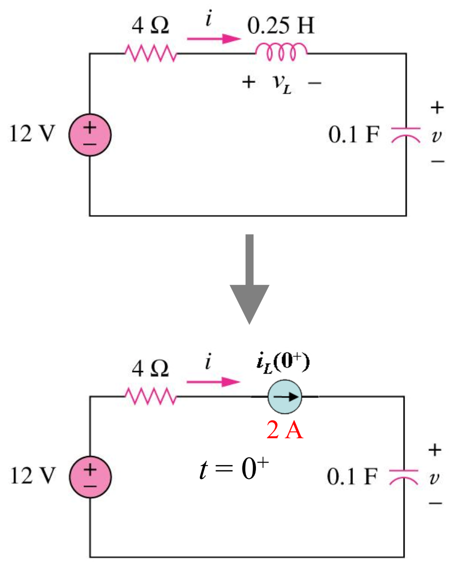
$$\because C\frac{dv}{dt} = i_C \Rightarrow \frac{dv}{dt} = \frac{i_C}{C}$$

- Since the inductor current cannot change abruptly.

  $\Rightarrow$ The inductor can be treated as a current source in this case.
- We can easily find

  $i_C(0^+) = i(0^+) = 2 \text{ A}$

$$\Rightarrow \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = 20 \text{ V/s}$$



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$\text{Sol: (b) To find } \dfrac{di(0^+)}{dt}\textbf{:}$

$$\because L\frac{di}{dt} = v_L \Rightarrow \frac{di}{dt} = \frac{v_L}{L}$$

- Since the capacitor voltage cannot change abruptly.

$\Rightarrow$ The capacitor can be treated as a voltage source in this case.

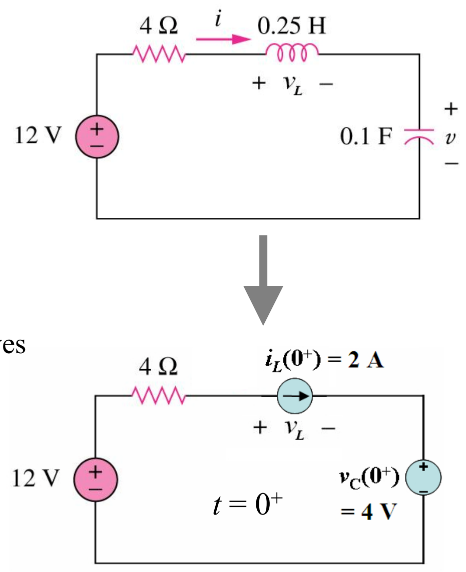
- To obtain $v_L(0^+)$, applying KVL gives

$$-12 + 4i(0^+) + v_L(0^+) + v_C(0^+) = 0$$

$$\Rightarrow v_L(0^+) = 12 - 8 - 4 = 0$$

- Thus we have

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0 \text{ A/s}$$



The Source-Free Series *RLC* Circuit

Assumed initial conditions:

$$\begin{cases} i(0) = I_0 & \text{(1a)} \\ v_C(0) = \dfrac{1}{C}\displaystyle\int_{-\infty}^{0} i\,dt = V_0 & \text{(1b)} \end{cases}$$

Applying KVL gives

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0 \qquad (2)$$

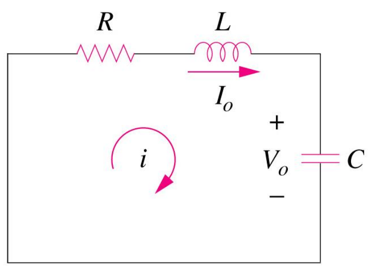
$$\Rightarrow \boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0} \qquad (3)$$

To solve (3), $di(0)/dt$ is required.

(1) and (2) gives

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

$$\Rightarrow \frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \qquad (4)$$



$$\boxed{\begin{array}{l} \dfrac{d^2 i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{i}{LC} = 0 \\ \text{Initial conditions:} \\ \begin{cases} i(0) = I_0 \\ \dfrac{di(0)}{dt} = -\dfrac{1}{L}(RI_0 + V_0) \end{cases} \end{array}}$$

Cont'd

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

Initial conditions:

$$\begin{cases} i(0) = I_0 \\ \dfrac{di(0)}{dt} = -\dfrac{1}{L}(RI_0 + V_0) \end{cases}$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Let $i = Ae^{st}$: A and s are constants.

$$\Rightarrow As^2 e^{st} + \frac{AR}{L} s e^{st} + \frac{A}{LC} e^{st} = 0$$

$$\Rightarrow Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

$$\Rightarrow \boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$ Characteristic equation

$$\begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{cases}$$ Natural frequencies

where $\begin{cases} \alpha = \dfrac{R}{2L} & \text{Damping factor} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} & \text{Resonant frequency} \end{cases}$

(or undamped natural frequency)

Summary

Characteristic equation:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{cases}$$

where $\begin{cases} \alpha = \dfrac{R}{2L} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$

Two solutions (if $s_1 \neq s_2$):

$$i_1 = A_1 e^{s_1 t}, \quad i_2 = A_2 e^{s_2 t}$$

A general solution:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where A_1 and A_2 are determined from the initial conditions.

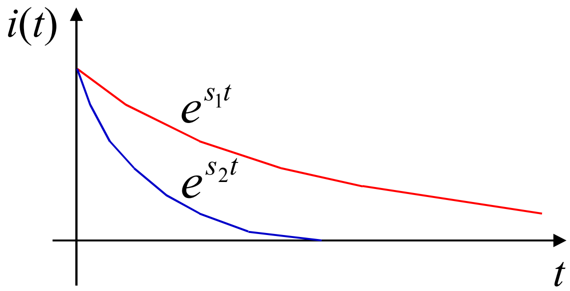
- Three cases discussed
  - Overdamped case (distinct real roots) : $\alpha > \omega_0$
  - Critically damped case (repeated real root) : $\alpha = \omega_0$
  - Underdamped case (complex-conjugate roots): $\alpha < \omega_0$

Overdamped Case ($\alpha > \omega_0$)

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}} \Rightarrow C > \frac{4L}{R^2}$$

Both s_1 and s_2 are negative and real.

$$\Rightarrow \boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$



Critically damped Case ($\alpha = \omega_0$)

Let $C = 4L/R^2$

$\Rightarrow s_1 = s_2 = -\alpha = -L/2R$

$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$

Single constant can't satisfy two initial conditions!

Back to the original differential equation.

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \alpha^2 i = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0$$

Let $f = \dfrac{di}{dt} + \alpha i$

$$\Rightarrow \frac{df}{dt} + \alpha f = 0 \Rightarrow f = A_1 e^{-\alpha t}$$

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1$$

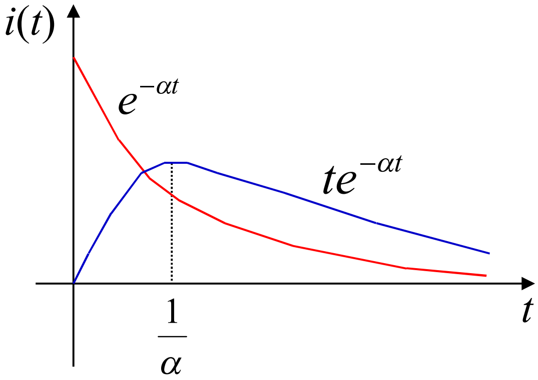
$$\Rightarrow \frac{d}{dt}\left(e^{\alpha t} i\right) = A_1$$

$$\Rightarrow e^{\alpha t} i = A_1 t + A_2$$

$$\Rightarrow \boxed{i(t) = \left(A_1 t + A_2\right) e^{-\alpha t}}$$

Critically damped Case (*Cont'd*)

$$i(t) = (A_1 t + A_2)e^{-\alpha t}$$



Underdamped Case ($\alpha < \omega_0$)

Let $\quad C < 4L/R^2$

$$\begin{cases} s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \\ s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \end{cases}$$

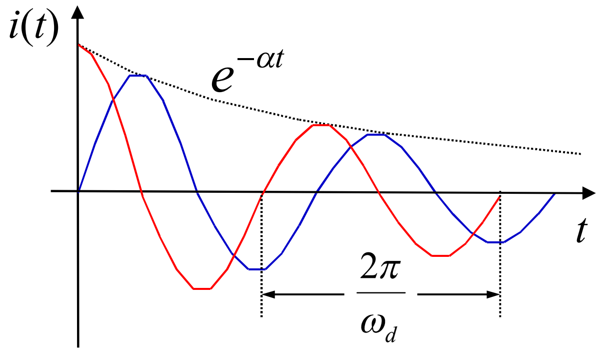
where $\quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$$\begin{aligned} i(t) &= B_1 e^{-(\alpha - j\omega_d)t} + B_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(B_1 e^{j\omega_d t} + B_2 e^{-j\omega_d t}) \\ &= e^{-\alpha t}[B_1(\cos\omega_d t + j\sin\omega_d t) + B_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(B_1 + B_2)\cos\omega_d t + j(B_1 - B_2)\sin\omega_d t] \end{aligned}$$

$$\Rightarrow \boxed{i(t) = e^{-\alpha t}(A_1\cos\omega_d t + A_2\sin\omega_d t)} \quad \text{where} \quad \begin{cases} A_1 = B_1 + B_2 \\ A_2 = j(B_1 - B_2) \end{cases}$$

Underdamped Case (*Cont'd*)

$$i(t) = \left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)e^{-\alpha t}, \quad \alpha = \frac{R}{2L}$$



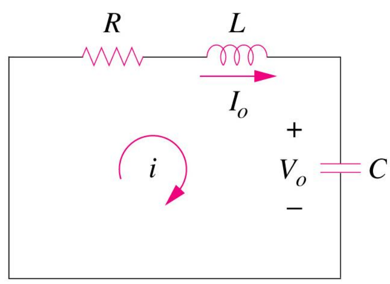
Finding The Constants $A_{1,2}$

To determine A_1 and A_2,
we need $i(0)$ and $di(0)/dt$.

1. $\boxed{i(0) = I_0}$
2. KVL at $t = 0$ gives

$$L\frac{di(0)}{dt} + RI_0 + V_0 = 0$$

or $\boxed{\dfrac{di(0)}{dt} = -\dfrac{1}{L}(RI_0 + V_0)}$

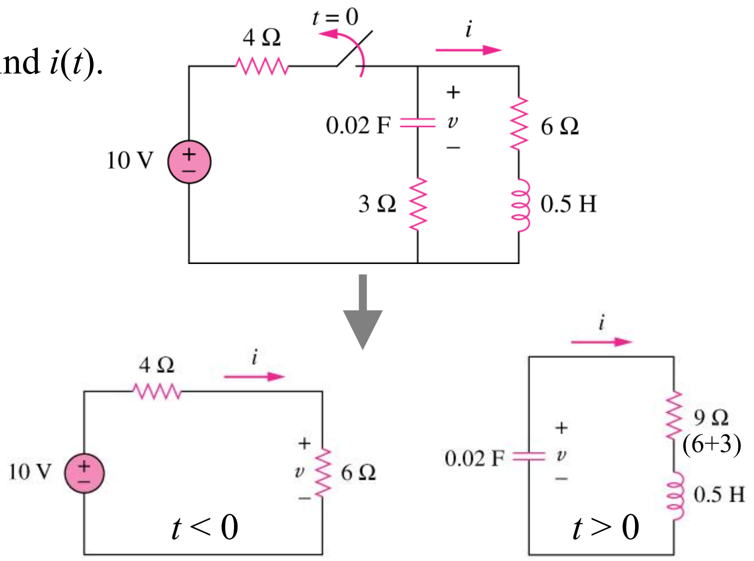


Conclusions

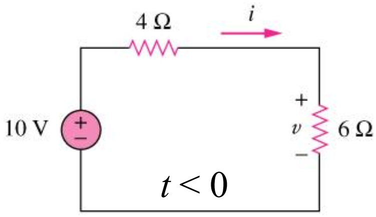
- The concept of ***damping***
  - The gradual loss of the initial stored energy
  - Due to the resistance ***R***
- **Oscillatory response is possible.**
  - The energy is transferred between L and C.
  - ***Ringing*** denotes the damped oscillation in the underdamped case.
- With the same initial conditions, the overdamped case has the longest settling time. The underdamped case has the fastest decay. (If a constant ω_0 is assumed.)

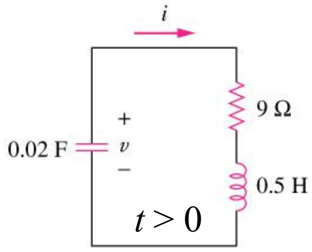
Example

Find $i(t)$.



Example (*Cont'd*)





$$(a)\ i(0)=\frac{10}{4+6}=1\,\text{A},\ v(0)=6i(0)=6\ \text{V}$$

$$(b)\ \alpha=\frac{R}{2L}=9,\ \omega_0=\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{0.01}}=10$$

$$\Rightarrow s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}=-9\pm\sqrt{81-100}$$

$$=-9\pm j4.359$$

$$\Rightarrow i(t)=e^{-9t}\left(A_1\cos 4.359t+A_2\sin 4.359t\right)$$

Initial conditions :

$$\begin{cases} i(0)=1 \\ \dfrac{di(0)}{dt}=-\dfrac{1}{L}\left(Ri(0)-v(0)\right) \\ \qquad\quad =-2\left(9(1)-6\right)=-6 \end{cases}$$

$$\Rightarrow \begin{cases} A_1=1 \\ A_2=0.6882 \end{cases}$$

The Source-Free Parallel *RLC* Circuit

Assumed initial conditions:

$$\begin{cases} i(0)=I_0=\dfrac{1}{L}\displaystyle\int_{\infty}^{0} v(t)dt & \text{(1a)} \\ v(0)=V_0 & \text{(1b)} \end{cases}$$

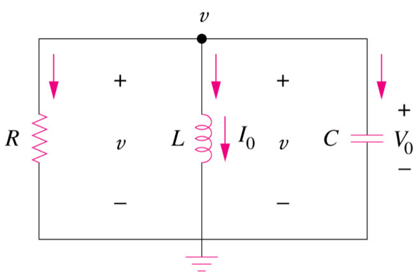
Applying KCL gives

$$\frac{v}{R}+\frac{1}{L}\int_{-\infty}^{t} vdt+C\frac{dv}{dt}=0 \qquad (2)$$

$$\Rightarrow \frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0 \qquad (3)$$

Let $v(t)=Ae^{st}$, the characteristic equation becomes

$$\boxed{s^2+\frac{1}{RC}s+\frac{1}{LC}=0}$$



$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

where $\begin{cases} \alpha=\dfrac{1}{2RC} \\ \omega_0=\dfrac{1}{\sqrt{LC}} \end{cases}$

Summary

- Overdamped case : $\alpha > \omega_0$

  $$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Critically damped case : $\alpha = \omega_0$

  $$v(t) = (A_1 + A_2 t)e^{-\alpha t}$$

- Underdamped case : $\alpha < \omega_0$

  $$s_{1,2} = -\alpha \pm j\omega_d$$

  where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

  $$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

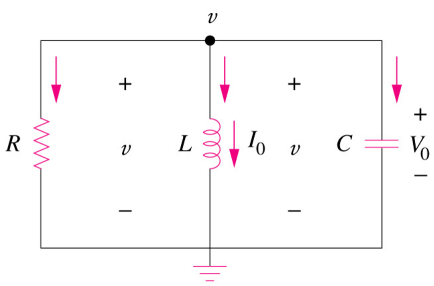
Finding The Constants $A_{1,2}$

To determine A_1 and A_2,
we need $v(0)$ and $dv(0)/dt$.

1. $\boxed{v(0) = V_0}$
2. KCL at $t = 0$ gives

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or $$\boxed{\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}}$$



Comparisons

- Series *RLC* Circuit

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\text{where } \begin{cases} \alpha = \dfrac{R}{2L} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$$

Initial conditions:

$$\begin{cases} i(0) = I_0 \\ \dfrac{di(0)}{dt} = -\dfrac{(V_0 + RI_0)}{L} \end{cases}$$

- Parallel *RLC* Circuit

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

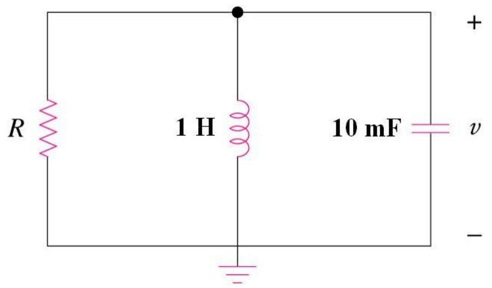
$$\text{where } \begin{cases} \alpha = \dfrac{1}{2RC} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$$

Initial conditions:

$$\begin{cases} v(0) = V_0 \\ \dfrac{dv(0)}{dt} = -\dfrac{(V_0 + RI_0)}{RC} \end{cases}$$

Example 1

Find $v(t)$ for t > 0.
$v(0) = 5$ V, $i(0) = 0$
Consider three cases:
$R = 1.923\ \Omega$
$R = 5\ \Omega$
$R = 6.25\ \Omega$



Case 1: $R = 1.923\,\Omega$

$$\alpha = \frac{1}{2RC} = 26, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t}$$

Initial conditions:

$$\begin{cases} v(0) = 5 \\ \dfrac{dv(0)}{dt} = -\dfrac{v(0) + Ri(0)}{RC} = -260 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -0.2083 \\ A_2 = 5.208 \end{cases}$$

Example 1 (*Cont'd*)

Case 2 : $R = 5\,\Omega$

$$\alpha = \frac{1}{2RC} = 10,\quad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -10$$

$$v(t) = (A_1 + A_2 t)e^{-10t}$$

Initial conditions :

$$\begin{cases} v(0) = 5 \\ \dfrac{dv(0)}{dt} = -\dfrac{v(0) + Ri(0)}{RC} = -100 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = 5 \\ A_2 = -50 \end{cases}$$

Case 3 : $R = 6.25\,\Omega$

$$\alpha = \frac{1}{2RC} = 8,\quad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

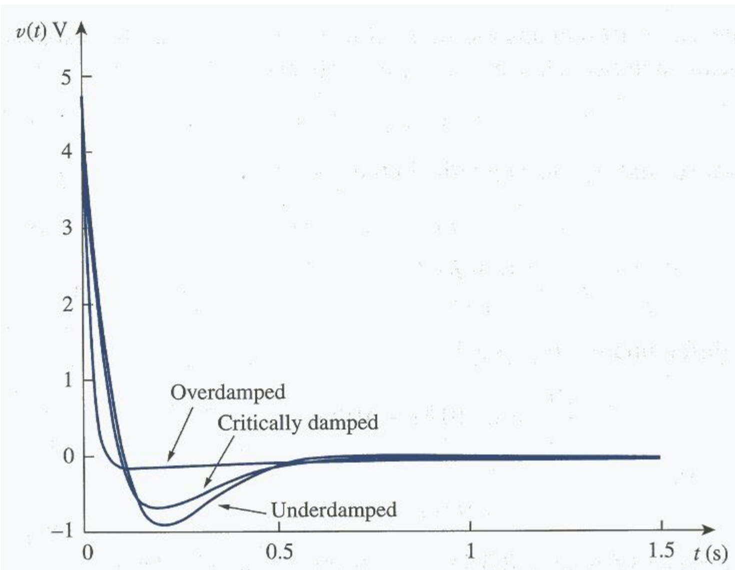
$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$$

Initial conditions :

$$\begin{cases} v(0) = 5 \\ \dfrac{dv(0)}{dt} = -\dfrac{v(0) + Ri(0)}{RC} = -80 \end{cases}$$

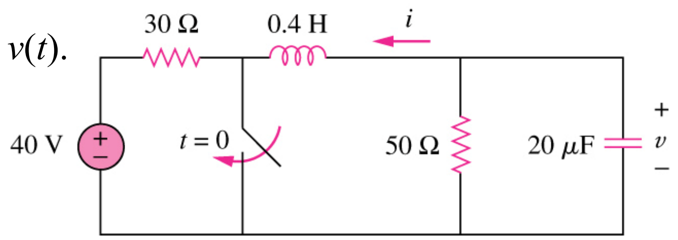
$$\Rightarrow \begin{cases} A_1 = 5 \\ A_2 = -6.667 \end{cases}$$

Example 1 (*Cont'd*)

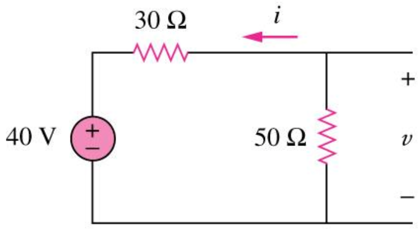


Example 2

Find $v(t)$.

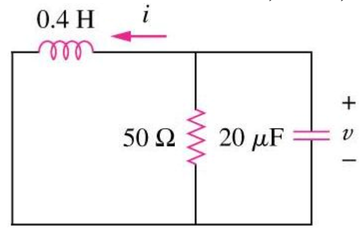


Get $x(0)$.



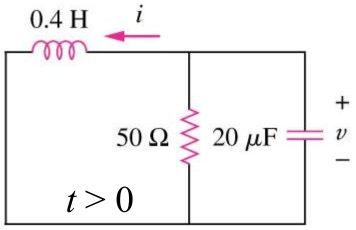
$t < 0$

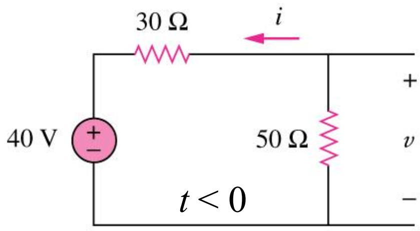
Get $x(\infty)$, $dx(0)/dt$, $s_{1,2}$, $A_{1,2}$.



$t > 0$

Example 2 (*Cont'd*)





$$\begin{cases} \alpha = \dfrac{1}{2RC} = 500 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 354 \end{cases}$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$
$$= -854, -146$$

$$v(t) = A_1 e^{-854t} + A_2 e^{-146t}$$

From the initial conditions :

$$\begin{cases} v(0) = \dfrac{50}{30+50}(40) = 25\ V \\ i(0) = -\dfrac{40}{30+50} = -0.5\ A \\ \dfrac{dv(0)}{dt} = -\dfrac{v(0)+Ri(0)}{RC} = \dfrac{25 - 50\times 0.5}{50\times 20\times 10^{-6}} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -5.156 \\ A_2 = 30.16 \end{cases}$$

Step Response of A Series *RLC* Circuit

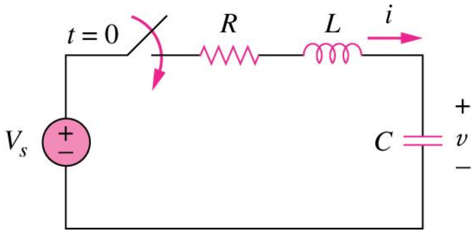
Applying KVL for $t > 0$,

$$Ri + L\frac{di}{dt} + v = V_S \quad (1)$$

But $i = C\frac{dv}{dt}$

$$\Rightarrow \boxed{\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_S}{LC}} \quad (2)$$

(2) has the same form as in the source-free case.



$v(t) = v_t(t) + v_{ss}(t)$

where

$$\begin{cases} v_t : \text{the transient response} \\ v_{ss} : \text{the steady-state response} \end{cases}$$

Characteristic Equation

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v - V_S}{LC} = 0$$

Let $v' = v - V_S$,

$$\Rightarrow \frac{d^2v'}{dt^2} + \frac{R}{L}\frac{dv'}{dt} + \frac{v'}{LC} = 0$$

The characteristic equation becomes

$$\Rightarrow s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Same as in the source-free case.

Summary

$$\Rightarrow v(t) = v_t(t) + v_{ss}(t)$$

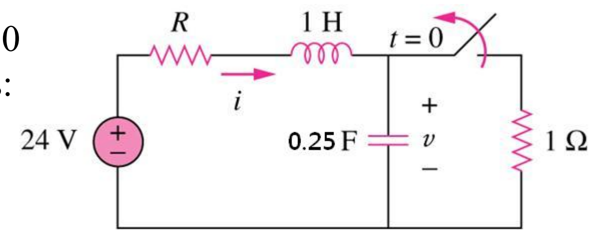
$$\text{where } \begin{cases} v_t(\infty) = 0 \\ v_{ss}(\infty) = v(\infty) = V_S \end{cases}$$

$$v_t(t) = \begin{cases} A_1 e^{s_1 t} + A_2 e^{s_2 t} & \text{(Overdamped)} \\ (A_1 + A_2 t)e^{-\alpha t} & \text{(Critically damped)} \\ (A_1 \cos\omega_d t + A_2 \sin\omega_d t)e^{-\alpha t} & \text{(Underdamped)} \end{cases}$$

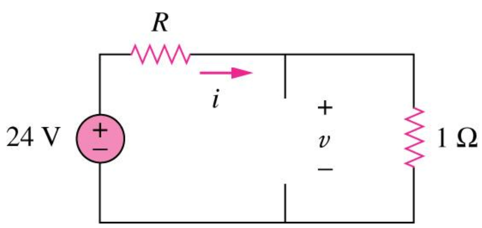
where $A_{1,2}$ are obtained from $v(0)$ and $dv(0)/dt$.

Example

Find $v(t)$, $i(t)$ for t > 0
Consider three cases:
$R = 5\ \Omega$
$R = 4\ \Omega$
$R = 1\ \Omega$

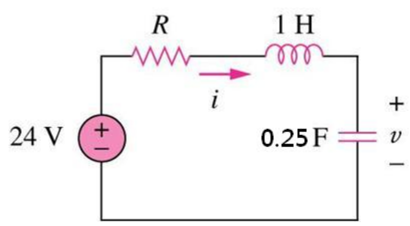


Get $x(0)$.



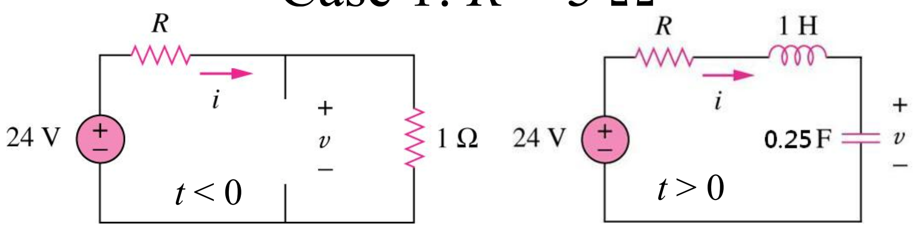
$t < 0$

Get $x(\infty)$, $dx(0)/dt$, $s_{1,2}$, $A_{1,2}$.



$t > 0$

Case 1: $R = 5\ \Omega$



$$\begin{cases} \alpha = \dfrac{R}{2L} = \dfrac{5}{2(1)} = 2.5 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 2 \end{cases}$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$
$$= -1, -4$$
$$v(t) = v_{ss} + \left(A_1 e^{-t} + A_2 e^{-4t}\right)$$
$$i(t) = C\frac{dv}{dt}$$

$$v_{ss} = v(\infty) = 24\ \text{V}$$

Initial conditions:

$$\begin{cases} i(0) = \dfrac{24}{5+1} = 4\ \text{A}, \quad v(0) = 1i(0) = 4\ \text{V} \\ i(0) = C\dfrac{dv(0)}{dt} \Rightarrow \dfrac{dv(0)}{dt} = \dfrac{4}{C} = 16 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -64/3 \\ A_2 = 4/3 \end{cases}$$

Case 2: $R = 4\ \Omega$

$$\begin{cases} \alpha = \dfrac{R}{2L} = \dfrac{4}{2(1)} = 2 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 2 \end{cases}$$

$$\Rightarrow s_{1,2} = -\alpha = -2$$
$$v(t) = v_{ss} + \left(A_1 + A_2 t\right)e^{-2t}$$
$$i(t) = C\frac{dv}{dt}$$

$$v_{ss} = v(\infty) = 24\ \text{V}$$

Initial conditions :

$$\begin{cases} i(0) = \dfrac{24}{4+1} = 4.8\ \text{A}, \quad v(0) = 1i(0) = 4.8\ \text{V} \\ i(0) = C\dfrac{dv(0)}{dt} \Rightarrow \dfrac{dv(0)}{dt} = \dfrac{4.8}{C} = 19.2 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -19.2 \\ A_2 = -19.2 \end{cases}$$

Case 3: $R = 1\ \Omega$

$$\begin{cases} \alpha = \dfrac{R}{2L} = \dfrac{1}{2(1)} = 0.5 \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = 2 \end{cases}$$

$$\Rightarrow s_{1,2} = -0.5 \pm j1.936$$

$$v(t) = v_{ss} + \begin{pmatrix} A_1 \cos 1.936t \\ + A_2 \sin 1.936t \end{pmatrix} e^{-0.5t}$$

$$i(t) = C\frac{dv}{dt}$$

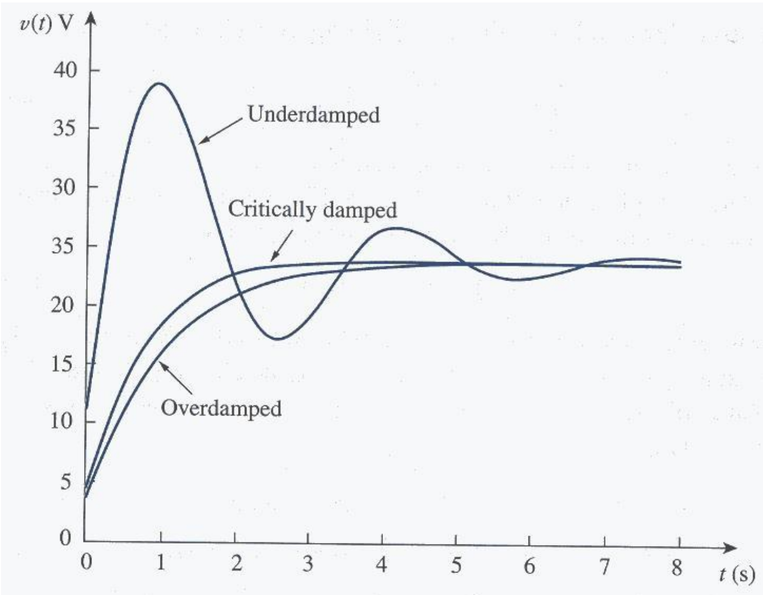
$$v_{ss} = v(\infty) = 24$$

Initial conditions :

$$\begin{cases} i(0) = \dfrac{24}{1+1} = 12\text{ A}, \quad v(0) = 1i(0) = 12\text{ V} \\ i(0) = C\dfrac{dv(0)}{dt} \Rightarrow \dfrac{dv(0)}{dt} = \dfrac{12}{C} = 48 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -12 \\ A_2 = 21.694 \end{cases}$$

Example (*Cont'd*)



Step Response of A Parallel RLC Circuit

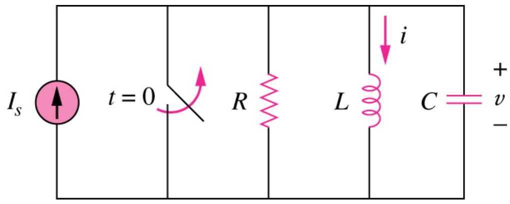
Applying KCL for $t > 0$,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_S \qquad (1)$$

But $v = L\frac{di}{dt}$

$$\Rightarrow \boxed{\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_S}{LC}} \quad (2)$$

(2) has the same form as in the source-free case.



$$\boxed{\begin{array}{l} i(t) = i_t(t) + i_{ss}(t) \\ \text{where} \\ \begin{cases} i_t : \text{the transient response} \\ i_{ss} : \text{the steady-state response} \end{cases} \end{array}}$$

Characteristic Equation

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i - I_S}{LC} = 0$$

Let $i' = i - I_S$,

$$\Rightarrow \frac{d^2 i'}{dt^2} + \frac{1}{RC}\frac{di'}{dt} + \frac{i'}{LC} = 0$$

The characteristic equation becomes

$$\Rightarrow s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Same as in the source-free case.

Summary

$$\Rightarrow i(t) = i_t(t) + i_{ss}(t)$$

$$\text{where } \begin{cases} i_t(\infty) = 0 \\ i_{ss}(t) = i(\infty) = I_S \end{cases}$$

$$i_t(t) = \begin{cases} A_1 e^{s_1 t} + A_2 e^{s_2 t} & \text{(Overdamped)} \\ (A_1 + A_2 t)e^{-\alpha t} & \text{(Critically damped)} \\ (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} & \text{(Underdamped)} \end{cases}$$

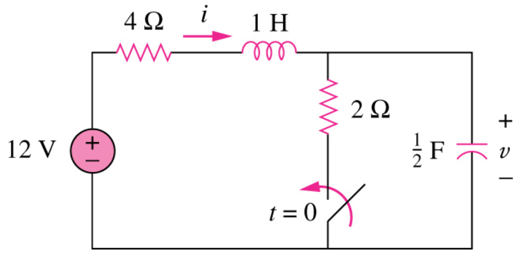
where $A_{1,2}$ are obtained from $i(0)$ and $di(0)/dt$.

General Second-Order Circuits

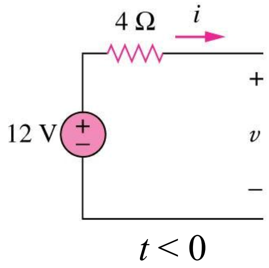
- Steps required to determine the step response.
  - Determine $x(0)$, $dx(0)/dt$, and $x(\infty)$.
  - Find the transient response $x_t(t)$.
    - Apply KCL and KVL to obtain the differential equation.
    - Determine the characteristic roots ($s_{1,2}$).
    - Obtain $x_t(t)$ with two unknown constants ($A_{1,2}$).
  - Obtain the steady-state response $x_{ss}(t) = x(\infty)$.
  - Use $x(t) = x_t(t) + x_{ss}(t)$ to determine $A_{1,2}$ from the two initial conditions $x(0)$ and $dx(0)/dt$.

Example

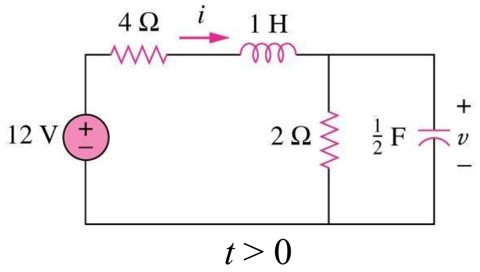
Find v, i
for t > 0.



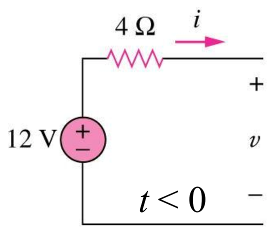
Get $x(0)$.

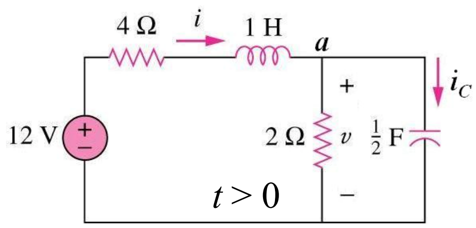


Get x(∞), $dx(0)/dt$, $s_{1,2}$, $A_{1,2}$.



Example (*Cont'd*)





Initial conditions:

$$\begin{cases} v(0^+) = v(0^-) = 12\text{ V} & \text{(1a)} \\ i(0^+) = i(0^-) = 0 & \text{(1b)} \end{cases}$$

Applying KCL at node a ($t > 0$),

$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2}$$

$$\Rightarrow i_C(0^+) = -6\text{ A}$$

$$\Rightarrow \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -12\text{ V/s} \quad \text{(1c)}$$

Final values for $t \to \infty$:

$$\begin{cases} i(\infty) = \dfrac{12}{4+2} = 2\text{ A} \\ v(\infty) = 2i(\infty) = 4\text{ V} \end{cases}$$

Example (*Cont'd*)

Applying KCL at node a gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \qquad (2)$$

Applying KVL to the left mesh gives

$$4i + 1\frac{di}{dt} + v = 12 \qquad (3)$$

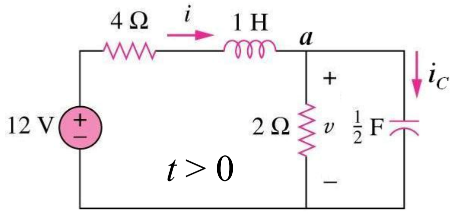
Substituting (2) into (3) gives

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 12$$

$$\Rightarrow \frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 24 \quad (4)$$

Characteristic equation :

$$s^2 + 5s + 6 = 0$$



$$\Rightarrow s = -2, -3$$

$$v(t) = v_{ss} + v_t(t)$$

$$\text{where } \begin{cases} v_{ss} = v(\infty) = 4 \\ v_t(t) = A_1 e^{-2t} + A_2 e^{-3t} \end{cases}$$

From (1a) and (1c) we obtain

$$\Rightarrow A_1 = 12, \ A_2 = 8$$

$i(t)$ can be obtain by using (2)

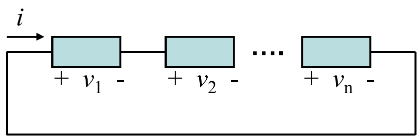
Duality

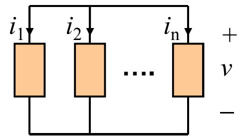
- Duality means the same characterizing equations with dual quantities interchanged.

Table for dual pairs

Resistance R	Conductance G
Inductance L	Capacitance C
Voltage v	**Current** i
Voltage source	Current source
Node	Mesh
Series path	**Parallel path**
Open circuit	Short circuit
KVL	**KCL**
Thevenin	Norton

A Case Study





$$v_1 = f_1(i)$$
$$v_2 = f_2(i)$$
$$\cdots$$
$$v_n = f_n(i)$$
$$\text{KVL}: v_1 + v_2 + \cdots + v_n = 0$$

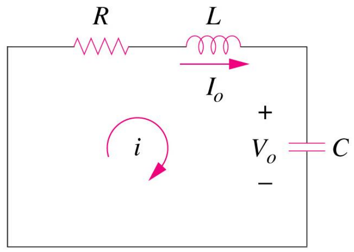
$$i_1 = f_1(v)$$
$$i_2 = f_2(v)$$
$$\cdots$$
$$i_n = f_n(v)$$
$$\text{KCL}: i_1 + i_2 + \cdots + i_n = 0$$

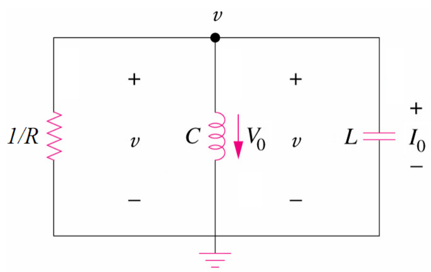
Element Transformations

- $v = Ri \quad \Leftrightarrow i = Rv \qquad (\text{Conductance} = R)$
- $i = C\dfrac{dv}{dt} \Leftrightarrow v = C\dfrac{di}{dt} \quad (\text{Inductance} = C)$
- $v = L\dfrac{di}{dt} \Leftrightarrow i = L\dfrac{dv}{dt} \quad (\text{Capacitance} = L)$
- $v = V_S \quad \Leftrightarrow i = V_S \qquad (\text{Current} = V_S)$
- $i = I_S \quad \Leftrightarrow v = I_S \qquad (\text{Voltage} = I_S)$

Example 1

- Series *RLC* Circuit
- Parallel *RLC* Circuit

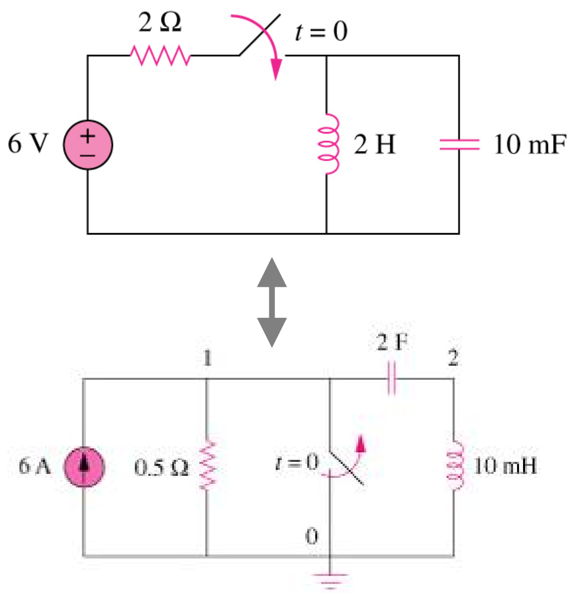




$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Example 2



Application: Smoothing Circuits