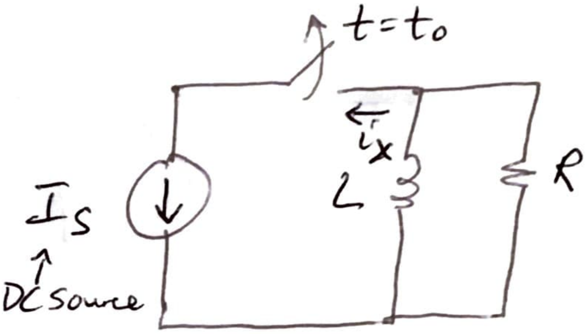
Response of 1st-order RL & RC ckts <Chpt. 7>

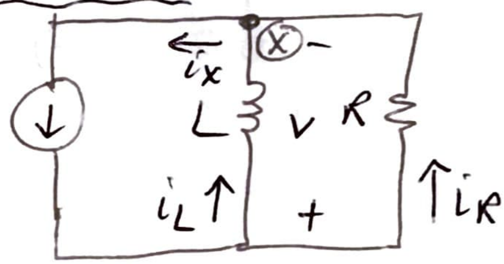
Natural response of RL ckt ("Natural response" means source is removed)

Consider the following ckt ⇒



Suppose that, prior to opening the switch at time t_o, the ckt was in the shown state for a very long time.

$t \ll t_o$ ckt ⇒



For the $t \ll t_o$ ckt, we have

$V_R = V_L = V$ (R & L are in parallel)

⇒ $V_R = V = R i_R$ & $V_L = V = L\frac{di_L}{dt}$

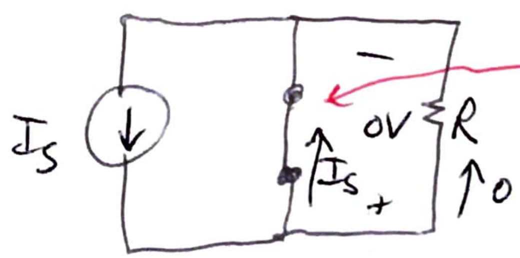
⇒ Since ckt was in shown state for a very long time & I_S is a DC source, we have that $\frac{di_L}{dt} = 0$ ⇐ The inductor current is not changing wrt time anymore

⇒ Since $\frac{di_L}{dt} = 0$, we have that $V = L\frac{di_L}{dt}^{\to 0} = 0$

⇒ Since $V = 0$, $V_R = V = Ri_R = 0$ ⇒ $i_R = V/R = 0/R = 0$

⇒ KCL @ node X says that $i_L + i_R = i_x = I_S$, & since $i_R = 0$, we have that $i_L + \cancel{i_R}^{0} = I_S$ ⇒ $i_L = I_S$ ⇒ $\boxed{i_R = 0 \text{ \& } i_L = I_S}$

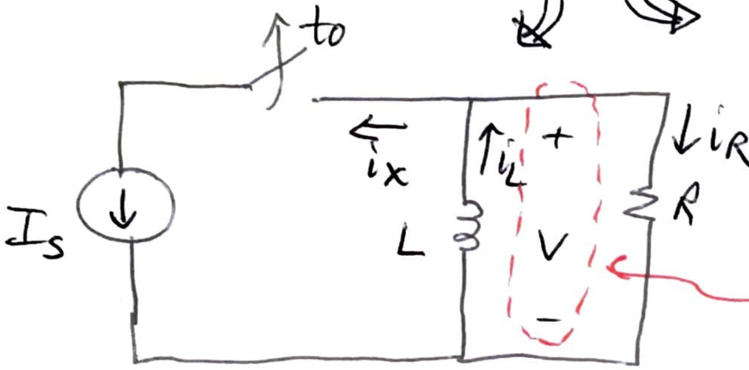
⇒ Key takeaway: inductor "looks like" a short after a very long time in a DC ckt, i.e., we can model the $t \ll t_o$ ckt as

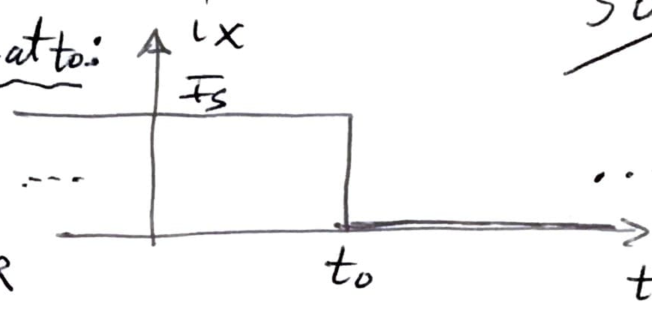


inductor modeled as a short in the DC steady-state ckt at $t \ll t_o$.

→

Now, let's open the switch at t_0:





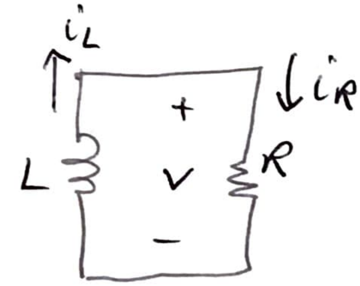
why am I flipping voltage V polarity here? Because inductor is releasing energy to the ckt after switch is opened up.

Side note

⇒ When DC inputs of 1st-order RC & RL ckts (& 2nd-order RLC ckts in Chpt. 8) are removed from a ckt, we call the resulting response the "natural response"

⇒ Conversely, when DC inputs are applied, we call the resulting response the "step response"

OK, here's the ckt after the switch is opened



$i_L = i_R$; what is i_L?

First, some notation: (1) $i_L(t_0^-)$ means the current in the inductor just before the switch is opened (or closed, or whatever); (2) $i_L(t_0^+)$ means the current in the inductor just after the switch is opened (or closed, or whatever); (3) $i_L(\infty)$ means the current in the inductor after a very long time has passed since the switch was opened (or closed, or whatever).

⇒ Now, $i_L(t_0^-) = I_s$ <from previous analysis, right before switch was opened>

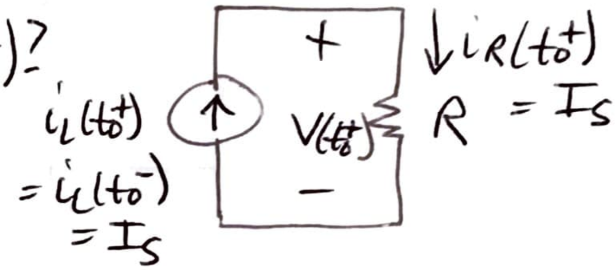
$i_L(t_0^+) = ?$ ⇒ $V = L\frac{di_L}{dt}$ ← for $\frac{di_L}{dt}$ to exist, i_L cannot change in value instantaneously from time t_0^- to time t_0^+

⇒ So, we have that current in an inductor cannot change instantaneously and that $i_L(t_0^+) = i_L(t_0^-) = I_s$

→

⇒ At $t = t_0^+$, we can model the inductor as a constant current source ⇒ ckt model at $t = t_0^+$ ⇒

What is $v(t_0^+)$? What is $v(t_0^-)$?



From previous analysis, $v(t_0^-) = 0$ & from ckt to the right, we have that $v(t_0^+) = R I_s$

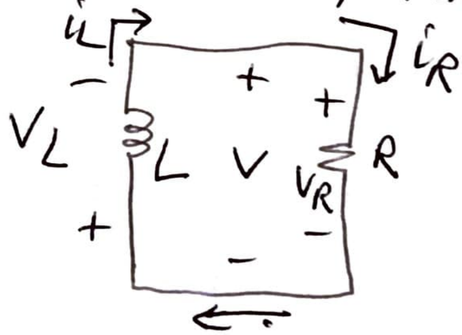
⇒ Now, what happens when ckt with switch opened stays in that state for a very long time? (i.e., as $t \to \infty$) ⇒

$$\Rightarrow \frac{di_L}{dt} = 0, \quad v(\infty) = L\cancelto{0}{\frac{di_L}{dt}} = 0$$

$$i_R(\infty) = \frac{\cancelto{0}{v(\infty)}}{R} = 0, \text{ \& by KCL } i_L = i_R = 0$$

↑ inductor looks like a short

So, we know the details of ckt response @ times t_0^-, t_0^+, and $t \to \infty$. How do we determine, say, $i_L(t)$ for time values between t_0^+ & $t \to \infty$? ⇒ solve Differential equation:



At $t \geq t_0^+$, $i_L = i_R = i$ ... let's analyze the $t \geq t_0^+$ ckt:

KVL around the loop ⇒ $v_L + v_R = 0$

$$\Rightarrow L\frac{di}{dt} + Ri = 0 \quad (1)$$

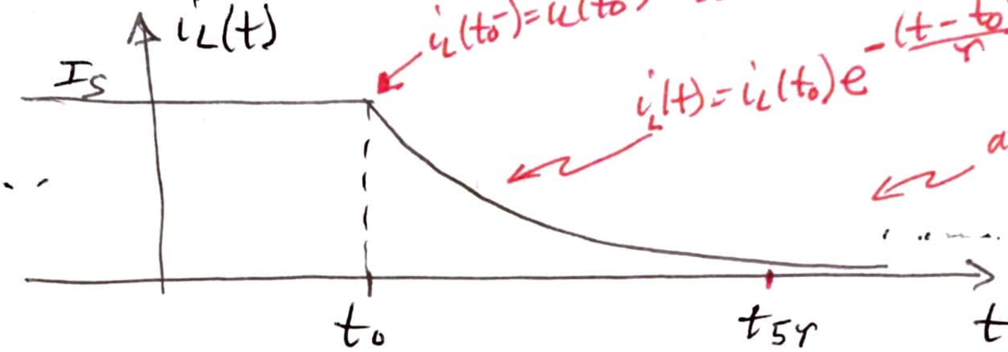
Note that (1) does not contain any derivative terms higher than first-order, which is why we say that this is a first-order RL ckt

$$\Rightarrow L\frac{di}{dt} = -Ri \Rightarrow \frac{L\,di}{i} = -R\,dt$$

$$\Rightarrow \frac{di}{i} = -\frac{R}{L}dt \Rightarrow \int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_0}^{t} dr \Rightarrow \ln\left[\frac{i(t)}{i(t_0)}\right] = -\frac{R}{L}[t - t_0]$$

$$\Rightarrow i(t) = i(t_0)\exp[-(t-t_0)/\tau], \quad t \geq t_0^+, \quad \tau = \frac{L}{R} = \text{time constant for RL ckt}$$

So, $i_L(t)$ can be graphed, as follows:



= time such that $t - t_0 = 5\tau$

$\Rightarrow$ time constant $\tau = \frac{L}{R}$ [sec] tells us how "fast" or "slow" the exponential decay happens

$\Rightarrow$ "low" τ, "fast" decay

$\Rightarrow$ "high" τ, "slow" decay

(can also get $V(t) = L\frac{di_L}{dt} = Ri_L(t)$

$\Rightarrow V(t) = Ri(t_0)\exp\left[-\frac{(t-t_0)}{\tau}\right]$, $t \geq t_0$)

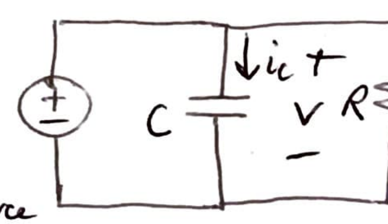
$\Rightarrow$ Rule of Thumb: A natural/step response will decay to $\simeq$ the final/steady-state value in 5τ (5τ = 5 time constants)

What about the natural response of an RC ckt?

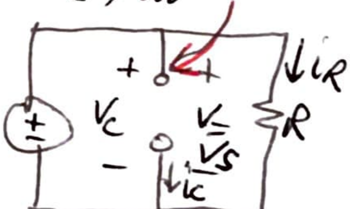
$i_c = C\frac{dV_c}{dt} = 0$

(1) $t < t_0$ ckt $\Rightarrow$

$\left\langle \frac{dV_c}{dt} = 0 \right\rangle$



$\Rightarrow$



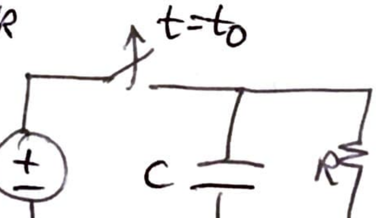
$i_c(t_0^-) = 0$, $i_R(t_0^-) = V_s/R$

Cap looks like an open

$V_c(t_0^-) = V_s$

$\Rightarrow$ Constant Voltage source at switching time

(2) $t = t_0^+$ ckt $\Rightarrow$

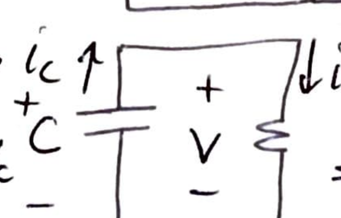


$\Rightarrow$

$V_c(t_0^+) = V_c(t_0^-) = V_s$

Cap Voltage cannot change instantaneously $i = C\frac{dv}{dt}$

(3) $t \to \infty$ ckt $\Rightarrow$



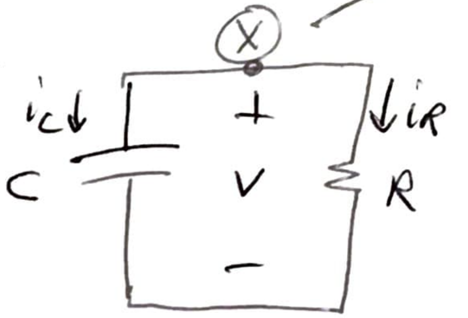
$\frac{dV_c}{dt} = 0 \Rightarrow$

cap looks like an open

$\Rightarrow i_c(\infty) = i_R(\infty) = 0$

& $V(\infty) = 0$

(4) For time $t > t_0$: Derive & solve differential equation

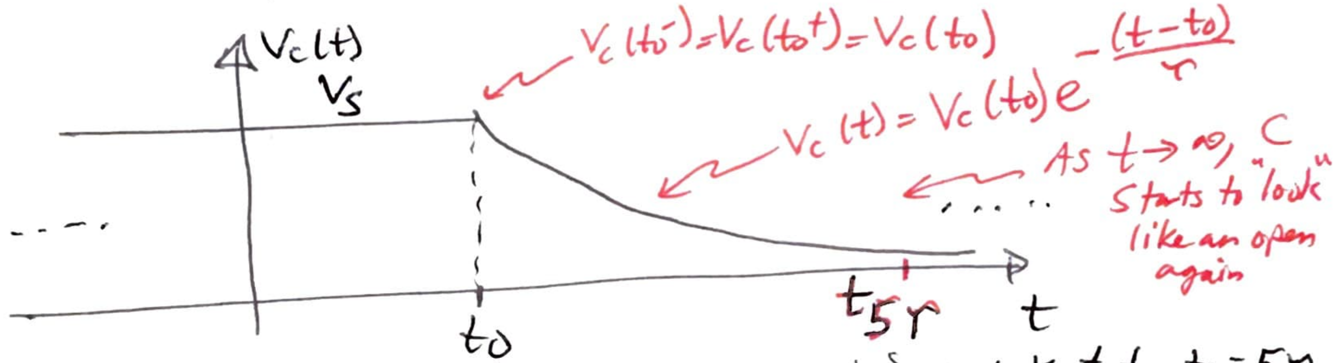


KCL @ (X): $i_C + i_R = 0$

$$\Rightarrow C\frac{dv}{dt} + v/R = 0$$

Sol'n $\Rightarrow V(t) = V(t_0)\exp\left[-(t-t_0)/\tau\right],\ t \geq t_0,\ \tau = RC$

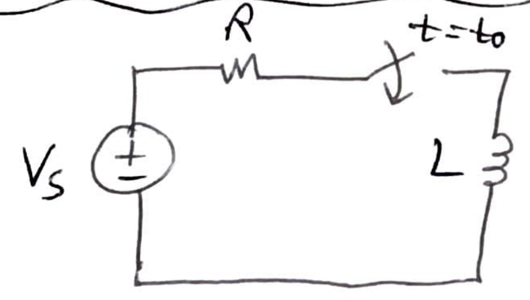
$\Rightarrow$ For RC ckt, time constant $\tau = RC$ [see]



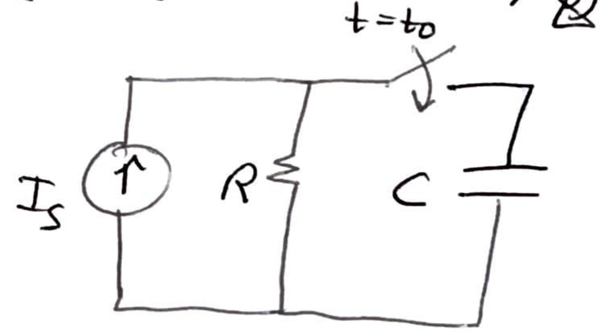
$= $ time such that $t - t_0 = 5\tau$

(Can also get $i_R(t) = C\frac{dv}{dt} = \frac{V_c(t)}{R} = \frac{V_c(t_0)}{R}\exp\left[\frac{-(t-t_0)}{\tau}\right], t > t_0$)

Step response of RC OR RL ckts: e.g., ckts of this type



OR



How to solve? Rather than treating these as a special case, let's discuss the general approach for finding either the natural response OR the step response of an RC OR RL ckt.

(*) General form of solution

$$\Rightarrow x(t) = x(\infty) + [x(t_0^+) - x(\infty)]\, e^{-\left[\frac{t-t_0}{\tau}\right]},$$

where $x(t)$ is the current or voltage of interest.

(I) Pick x: For RC ckts, convenient to choose cap voltage.
For RL ckts, convenient to choose inductor current.
$\Rightarrow$ Determine $x(t_0^-)$

(II) Determine $x(t_0^+)$, i.e., initial value

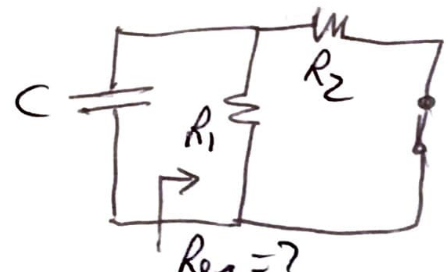
(III) Determine $x(\infty)$, i.e., final value

(IV) Determine time constant τ $\Rightarrow$ $\tau = \frac{L}{R_{eq}}$ for RL ckts

& $\tau = R_{eq}C$ for RC ckts, where R_{eq} is the equivalent resistance "seen" by L or C.

R_{eq}? Consider the following example: Suppose this is the ckt you want to analyze after switch has been closed/open to determine time constant τ:

$\Rightarrow$ Deactivate independent source(s),
& determine resistance "seen" by C (or L, if RL ckt):



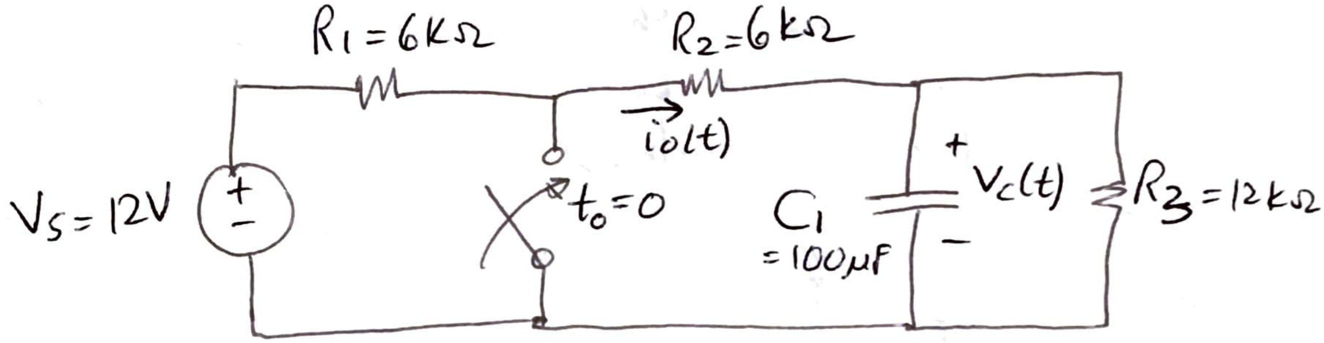
$\Rightarrow R_{eq} = R_1 // R_2$, so

$$\boxed{\tau = R_{eq}C = (R_1 // R_2)C}$$

Note if the deactivated ckt for determining τ contains dependent source(s), then you need to remove C (or L) & replace with test source & compute $R_{eq} = V_T / I_T$.

Example[1] for HW 12 | ELEN 2425

In the following circuit, find $i_o(t)$ and $v_c(t)$ for $t > 0$.



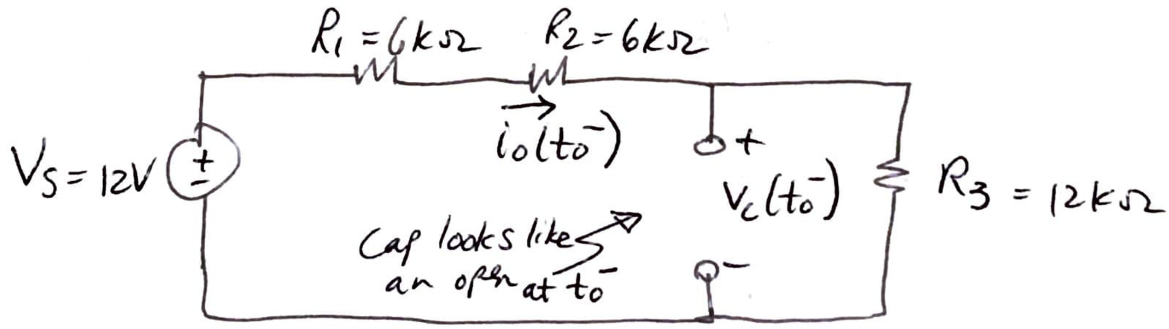
We know that $i_o(t)$ & $V_c(t)$ have the following forms:

$$i_o(t) = i_o(\infty) + [i_o(t_o^+) - i_o(\infty)]e^{-[t-t_o]/\tau}, \quad t > 0 \quad (1)$$

and $$V_c(t) = V_c(\infty) + [V_c(t_o^+) - V_c(\infty)]e^{-[t-t_o]/\tau}, \quad t \geq 0 \quad (2)$$

$\Rightarrow$ Note that (1) is only valid for $t > 0$, and that (2) is valid for $t \geq 0$ (why? look in your notes from class).

I) Analyze the above circuit at $t = t_o^-$:



Thus $\underbrace{i_o(t_o^-) = \frac{V_S}{R_1+R_2+R_3} = 500\mu A}$, and

$$\underbrace{V_c(t_o^-) = \left(\frac{R_3}{R_1+R_2+R_3}\right)V_S = 6V}$$

$\rightarrow$

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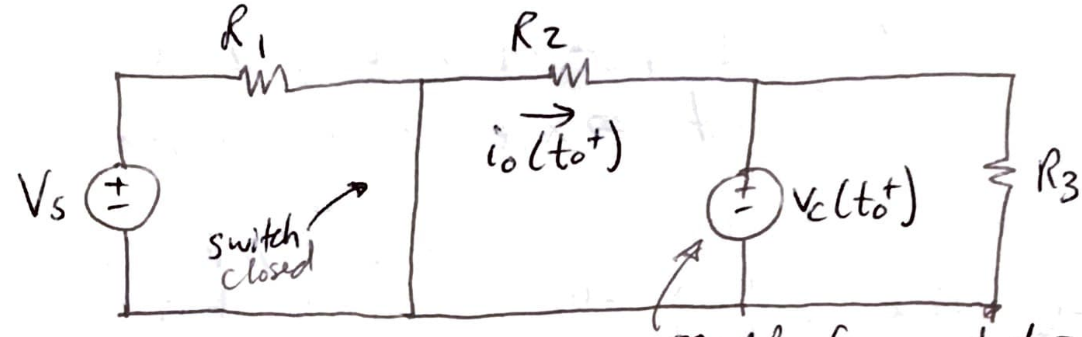
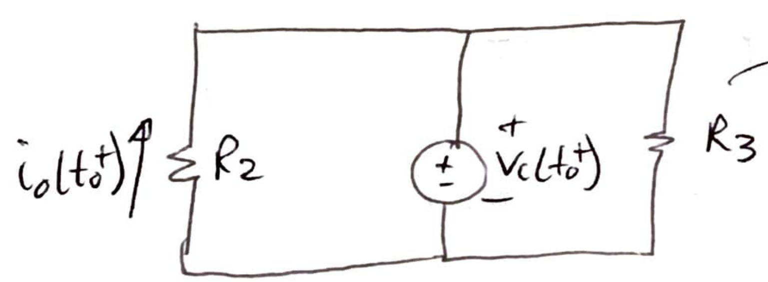
II Analyze the circuit at $t = t_0^+$:



→ $V_c(t_0^+) = V_c(t_0^-)$, since cap voltage cannot change instantaneously. $\Rightarrow$ $\underbrace{V_c(t_0^+) = V_c(t_0^-) = 6V}$

→ $i_o(t_0^+) = ?$ Let's redraw the circuit, removing those parts that do not affect operation anymore:

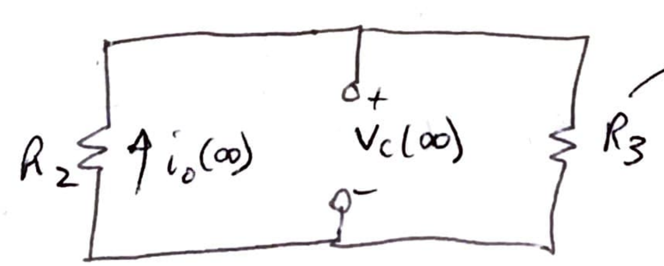


→ So, $i_o(t_0^+) = \dfrac{-V_c(t_0^+)}{R_2}$

$\Rightarrow$ $\underbrace{i_o(t_0^+) = -1mA}$

(notice that $i_o(t_0^+) \neq i_o(t_0^-)$)

III Analyze the circuit for $t > t_0^+$ to find values as $t \to \infty$:



→ No independent sources, so $\underbrace{i_o(\infty) = 0}$, and $\underbrace{V_c(\infty) = 0}$

→

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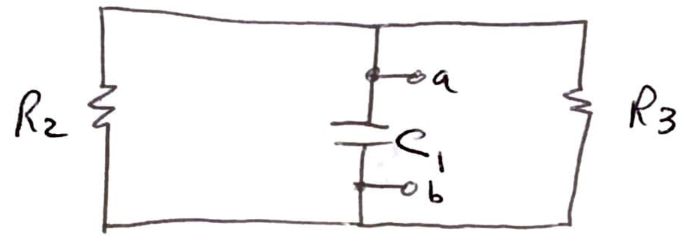
IV Find τ:

$\tau = R_{eq} C_1$, what is R_{eq}?



R_{eq} is the equivalent resistance ~~at~~ seen by the capacitor, or, equivalently, at terminals a & b. [Note, that if the circuit had sources, you would need to handle those the way you would in doing a Thevenin/Norton equivalent resistance calculation.]

$$\Rightarrow R_{eq} = R_2 // R_3 = 6k // 12k = 4k\Omega,$$

$$\tau = R_{eq} C_1 = (4k\Omega)(100\mu F) = 400ms$$

(also note that $1/\tau = 2.5s^{-1}$)

So, $t_0 = 0s$ and

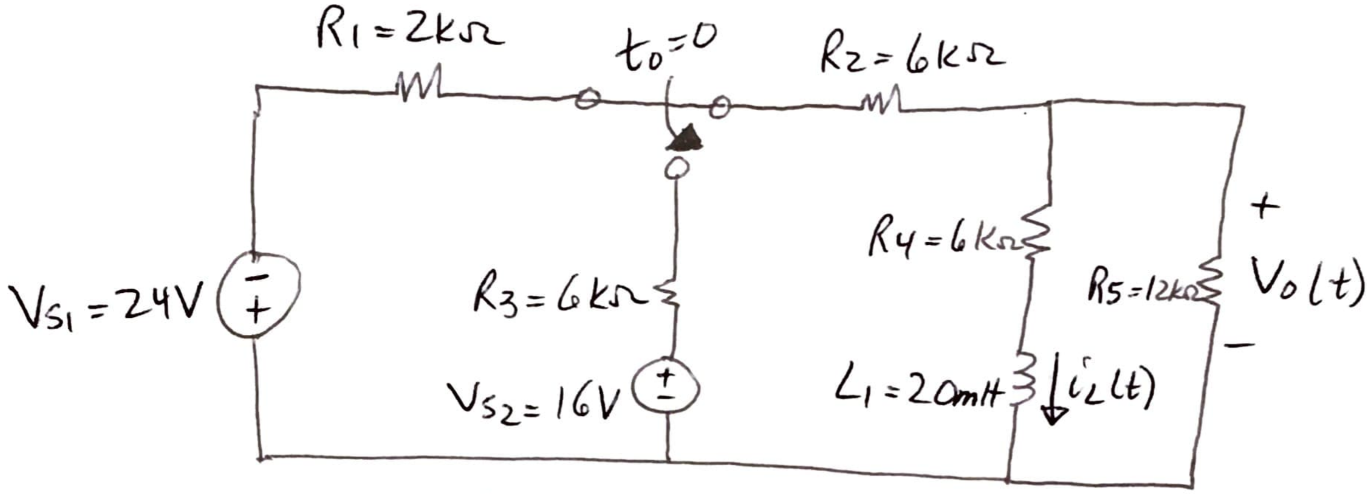
$$\boxed{\begin{array}{l} V_c(t) = 6e^{-2.5t}\ V,\ t \geq 0, \text{ and} \\ i_o(t) = -e^{-2.5t}\ mA,\ t > 0 \end{array}}$$

Example 2 starts on the next page

Example 2 for Hw 12

ELEN 2425

In the following circuit, find $i_L(t)$ and $V_o(t)$ for $t > 0$.



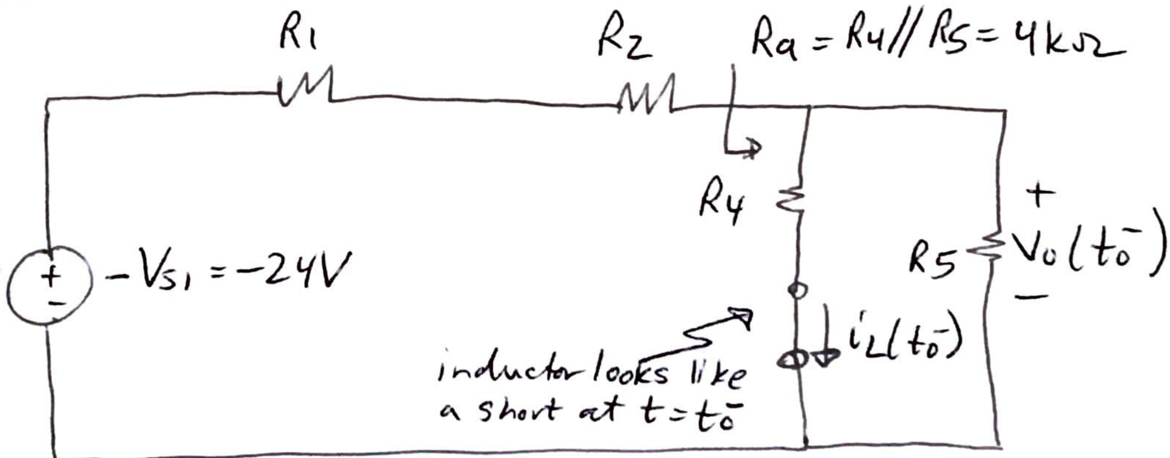
We know that $i_L(t)$ and $V_o(t)$ have the following forms:

$$i_L(t) = i_L(\infty) + [i_L(t_0^+) - i_L(\infty)]e^{-[t-t_0]/\tau}, \quad t \geq 0 \quad (1)$$

$$V_o(t) = V_o(\infty) + [V_o(t_0^+) - V_o(\infty)]e^{-[t-t_0]/\tau}, \quad t > 0 \quad (2)$$

$\Rightarrow$ Note that (2) is only valid for $t > 0$, while (1) is valid for $t \geq 0$ (why? review your notes from class).

(I) Analyze $t = t_0^-$ circuit:



I flipped polarity, and am now referring to this source as $-V_{S1} = -24V$

$$\underbrace{V_o(t_0^-) = \left(\frac{R_a}{R_a + R_1 + R_2}\right)(-V_{S1}) = -8V}$$

and $\underbrace{i_L(t_0^-) = V_o(t_0^-)/R_4 = -1.33\,mA}$

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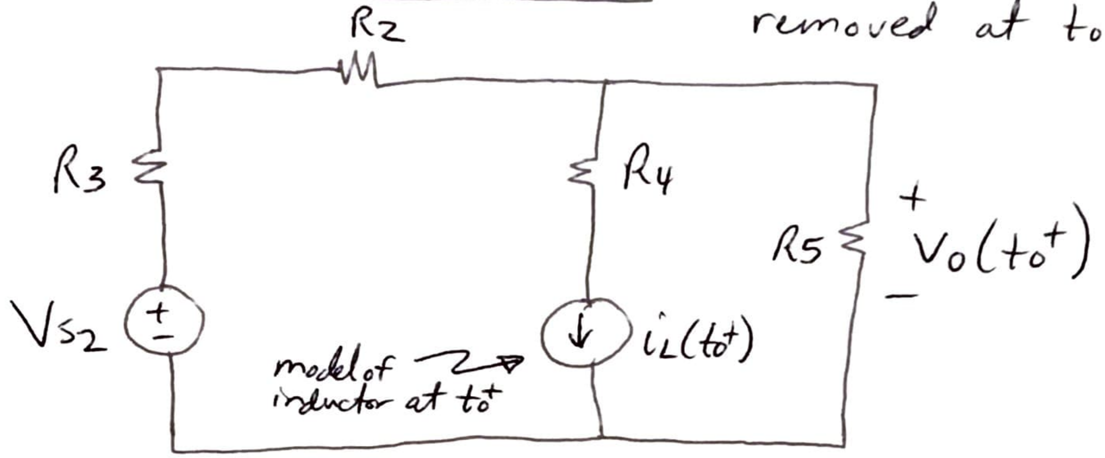
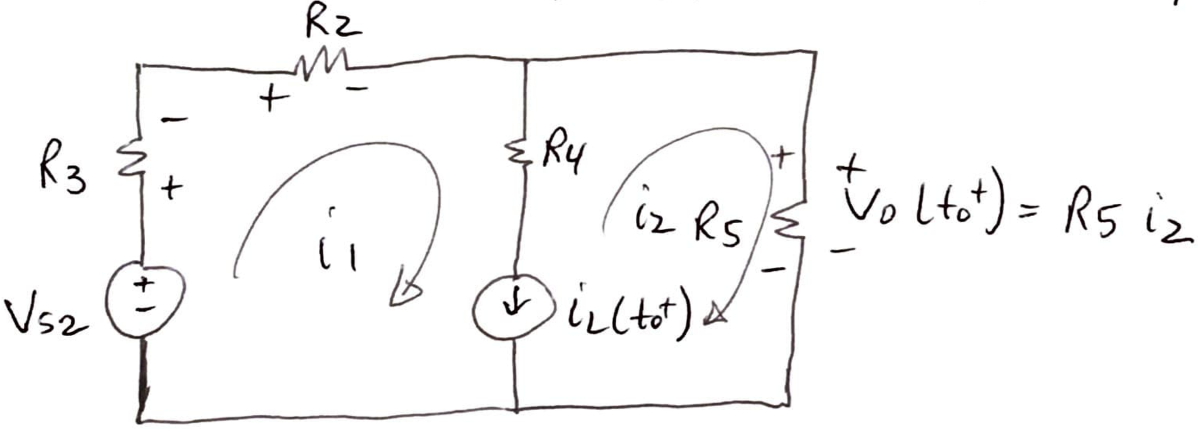
(II) Analyze $t = t_0^+$ circuit: V_{S1} and R_1 are effectively removed at t_0^+

$\Rightarrow$



$\rightarrow i_L(t_0^+) = i_L(t_0^-)$, since inductor current cannot change instantaneously $\Rightarrow$ $\underbrace{i_L(t_0^+) = i_L(t_0^-) = -1.33\text{mA}}$

$\rightarrow V_o(t_0^+) = ?$ We can choose mesh or nodal analysis to determine $V_o(t_0^+)$; I'll choose mesh analysis:



$\rightarrow i_1 - i_2 = i_L(t_0^+) = -1.33\text{mA}$ (3)

$\rightarrow$ KVL, supermesh: $-V_{S2} + R_3 i_1 + R_2 i_1 + R_5 i_2 = 0$

$\Rightarrow (R_2 + R_3) i_1 + R_5 i_2 = V_{S2}$ (4)

Solving (3) & (4) yields $i_1 = 0$, $i_2 = 1.33\text{mA}$, and so

$\underbrace{V_o(t_0^+) = R_5 i_2 = 16\text{ V}}$

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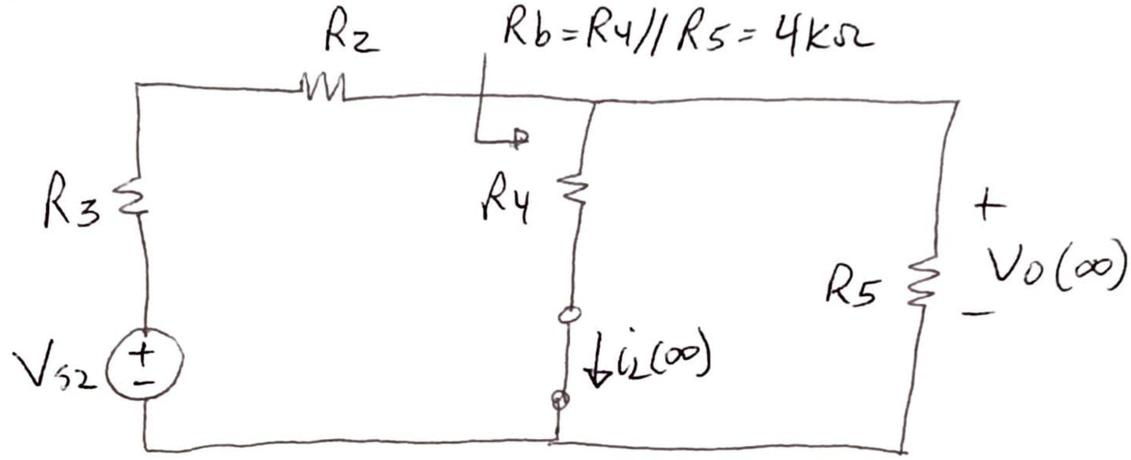
III Analyze $t > t_0^+$ circuit to find values as $t \to \infty$:



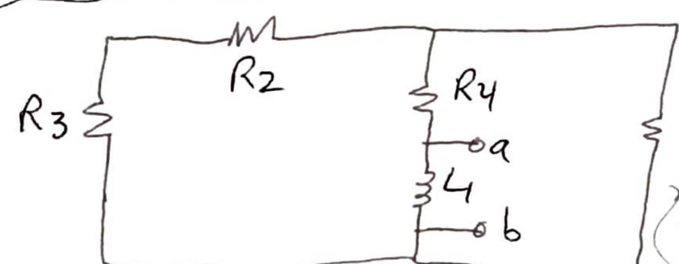
$$\Rightarrow V_o(\infty) = \left(\frac{R_b}{R_b + R_3 + R_2}\right) V_{s2} = 4V$$

and $i_L(\infty) = \dfrac{V_o(\infty)}{R_4} = 666.67\ \mu A$

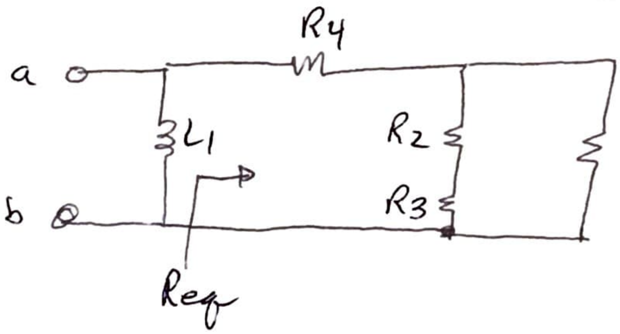
IV Find τ:

$\tau = \dfrac{L_1}{R_{eq}}$, what is R_{eq}? R_{eq} is the equivalent resistance seen by L_1

Deactivate independent sources



$\Rightarrow$



Side Note: If circuit contained dependent sources, you would need to drive terminals a & b with a test voltage or current, and find Req that way; if you needed to do this, you remove L_1 from the analysis.

So, $R_{eq} = R_4 + \left[(R_2 + R_3) // R_5\right] = 12\,k\Omega$

and so $\tau = \dfrac{L_1}{R_{eq}} = \dfrac{20\,mH}{12\,k\Omega} \cong 1.67\,\mu s$ (also note that $\frac{1}{\tau} = 600k\,s^{-1}$)

this says s^{-1}

cont'd

So, $t_0 = 0s$ and

$$i_L(t) = i_L(\infty) + [i_L(t_0^+) - i_L(\infty)]e^{-t/\tau}$$

$$= 666.67\mu A + [-1.33mA - 666.67\mu A]e^{-600kt}$$

$$\Rightarrow \boxed{i_L(t) = 666.67\mu A - 2e^{-600kt} mA, \; t \geq 0}$$

and $$V_o(t) = V_o(\infty) + [V_o(t_0^+) - V_o(\infty)]e^{-t/\tau}$$

$$= 4 + [16 - 4]e^{-600kt}$$

$$\Rightarrow \boxed{V_o(t) = 4V + 12e^{-600kt} V, \; t > 0}$$