⑥ $\left[(s/\omega_p)^2 + 2\zeta(s/\omega_p) + 1\right]$ terms in the denominator of $H(s)$:

4 possible scenarios: $\langle \omega_p$ = natural frequency, ζ = damping factor/coefficient, also related to Q = quality factor $= \frac{1}{2\zeta} \rangle$

1) $\zeta = 0$, $Q \to \infty$ $\Rightarrow$ undamped; roots $s_{1,2} = \pm j\omega_p \Leftarrow$ oscillatory, unstable (not covered in this course)

2) $\zeta = 1$, $Q = 0.5 \Rightarrow$ critically damped; real, repeated roots; this is just like having $(1+s/\omega_m)^2$ in the denominator

3) $\zeta > 1$, $0 < Q < 0.5 \Rightarrow$ overdamped; two distinct, real roots; this is just like having $(1+s/\omega_{m_1})(1+s/\omega_{m_2})$ in the denominator

4) $0 < \zeta < 1$, $0.5 < Q < \infty \Rightarrow$ underdamped; complex-conjugate pole pair
↳ this is the main focus here; for example, consider transfer function

$$H(s) = \frac{30(s+10)}{s^2+3s+50} = \frac{(30)(10)(1+s/10)}{(50)\left[s^2/50 + 3s/50 + 1\right]}$$

(⇐ putting $H(s)$ into proper Bode approximation form)

K term

$$\Rightarrow H(s) = \frac{6(1+s/10)}{(s/\sqrt{50})^2 + 2\zeta(s/\sqrt{50}) + 1}$$

← zero term

$\Rightarrow \omega_p = \sqrt{50}$ rps, $\zeta = ?$

$$\Rightarrow \text{Set } \frac{3s}{50} = 2\zeta(s/\sqrt{50}) \Rightarrow \zeta = \frac{3\omega_p}{2\omega_p^2} = \frac{3\sqrt{50}}{2(50)} \simeq 0.2121 \langle\text{unitless}\rangle$$

$\Rightarrow$ Note that $\zeta = 0.2121$ satisfies $0 < \zeta < 1 \Rightarrow$ underdamped; so we have a complex-conjugate pole pair in the denominator of transfer function $H(s)$; poles are

$$s_{1,2} = -\zeta\omega_p \pm \omega_p\underbrace{\sqrt{\zeta^2-1}}_{<0} = -\zeta\omega_p \pm j\omega_p\sqrt{1-\zeta^2}$$

$\Rightarrow$ in this particular example, we have $\omega_p \simeq 7.0711$ rps, $\zeta \simeq 0.2121$ with poles $s_{1,2} \simeq -1.5 \pm j6.91$ rps & $Q \simeq 2.3570$

2 →

OK, So how do we sketch a Bode approximation of a complex-conjugate pole pair?

$\Rightarrow$ Let's start with Bode approximation of the magnitude response:

(here, $\omega_0 = \omega_p$) $$|H(j\omega)| = \left|\frac{1}{(j\omega/\omega_0)^2 + 2\zeta(j\omega/\omega_0)+1}\right| = \frac{1}{|-(\omega/\omega_0)^2 + j2\zeta(\omega/\omega_0)+1|}$$

$$= \frac{1}{\sqrt{(1-(\omega/\omega_0)^2)^2 + (2\zeta\,\omega/\omega_0)^2}}$$

$$\Rightarrow |H(j\omega)|_{dB} = 20\log_{10}\left(\frac{1}{\sqrt{(1-(\omega/\omega_0)^2)^2+(2\zeta\,\omega/\omega_0)^2}}\right)$$

$$= -20\log_{10}\left(\sqrt{(1-(\omega/\omega_0)^2)^2 + (2\zeta\,\omega/\omega_0)^2}\right)$$

$\Downarrow$

Now, let's consider 3 ranges of ω values:

1) $\underbrace{\omega \ll \omega_0}_{\text{low-frequencies}}$, 2) $\underbrace{\omega \gg \omega_0}_{\text{high-frequencies}}$, 3) $\underbrace{\omega \simeq \omega_0}_{\omega \simeq \text{ natural frequency}}$

1) Low-frequencies, $\omega \ll \omega_0$: $|H|_{dB} \simeq -20\log_{10}(1) = 0\,dB$ $\Leftarrow$ makes sense, like a "regular" pole.

2) High-frequencies, $\omega \gg \omega_0$: $|H|_{dB} \simeq -20\log_{10}((\omega/\omega_0)^2) = -40\log_{10}(\omega/\omega_0)\,dB$

$\Rightarrow$ the high-frequency approximation is a straight-line with a slope of -40 dB/decade ($\simeq -12$dB/octave), going through ω_0 at 0dB $\Leftarrow$ -40dB/decade is as expected, since a 2nd-order denominator term has (essentially) 2 poles activated at high frequencies, at -20dB/decade per pole.

3) $\omega \simeq \omega_0$: It can be shown that a magnitude peak occurs near the natural frequency ω_0; here, we make the approximation that a peak exists only when $0 < \zeta < 0.5$, or alternatively $1 < Q < \infty$, and that the peak occurs at ω_0 with a height of $Q = \frac{1}{2\zeta}$ ($\Leftarrow$ not in dB!); to be clear, no appreciable peak at ω_0 for $0.5 \le \zeta < 1$, or alternatively $0.5 < Q \le 1$.

(*) To draw a piecewise-linear Bode approximation of the magnitude response (in dB!) for a complex-conjugate pole pair: Use the low-frequency asymptote (0 dB) up to the natural (or corner, or break) frequency and use the high-frequency asymptote (with slope -40 dB/decade $\simeq$ -12 dB/octave) thereafter. If $0<\zeta<0.5$, or alternatively $1<Q<\infty$, then draw a peak of amplitude $20\log_{10}(Q) = Q\,dB$; draw a smooth curve between the low- and high-frequency asymptote that goes through the peak value. (It can also be shown that the -3 dB frequency will be $\omega_0\sqrt{x^*}$, where $x^* = 1-2\zeta^2+\sqrt{(2\zeta^2-1)^2+1}$.)

$\Rightarrow$ Now, let's show the details of sketching a Bode approximation of the phase response for a complex-conjugate pole pair:

$$\angle H(j\omega) = \angle \frac{1}{(j\omega/\omega_0)^2+2\zeta\left(\frac{j\omega}{\omega_0}\right)+1} = -\angle\left(1-(\omega/\omega_0)^2+j2\zeta(\omega/\omega_0)\right)$$

$$= -\tan^{-1}\left(\frac{2\zeta\,\omega/\omega_0}{1-(\omega/\omega_0)^2}\right)$$

$(\pm 180°, \text{ if } 1-(\omega/\omega_0)^2<0)$

$\Downarrow$

Again, we consider 3 ranges of angular frequency ω values:

1) Low-frequencies, $\omega \ll \omega_0$: $\angle H(j\omega) \simeq -\tan^{-1}\left(2\zeta\frac{\omega}{\omega_0}\right) \simeq -\tan^{-1}(0) = 0° \Leftarrow$ as expected (like a "regular" pole)

2) High-frequencies, $\omega \gg \omega_0$: $\angle H(j\omega) \simeq -180° \Leftarrow$ as expected, for (essentially) 2 poles activated at high frequencies

3) $\omega=\omega_0$: $\angle H(j\omega_0) = -90° \Leftarrow$ as expected, since one pole has a phase rolloff of -45° at the pole frequency, and here we have two poles

(*) There are several approaches to drawing a piecewise-linear Bode approximation of the phase response (in degrees!) for a complex-conjugate pole pair; here is my suggestion: Connect low-frequency asymptote at 0° to the high-frequency asymptote at -180° starting at $\omega_1 = \frac{\omega_0}{10^{\zeta}} = \omega_0 10^{-\zeta}$ and ending at $\omega_2 = \omega_0 10^{\zeta}$; $\Rightarrow$

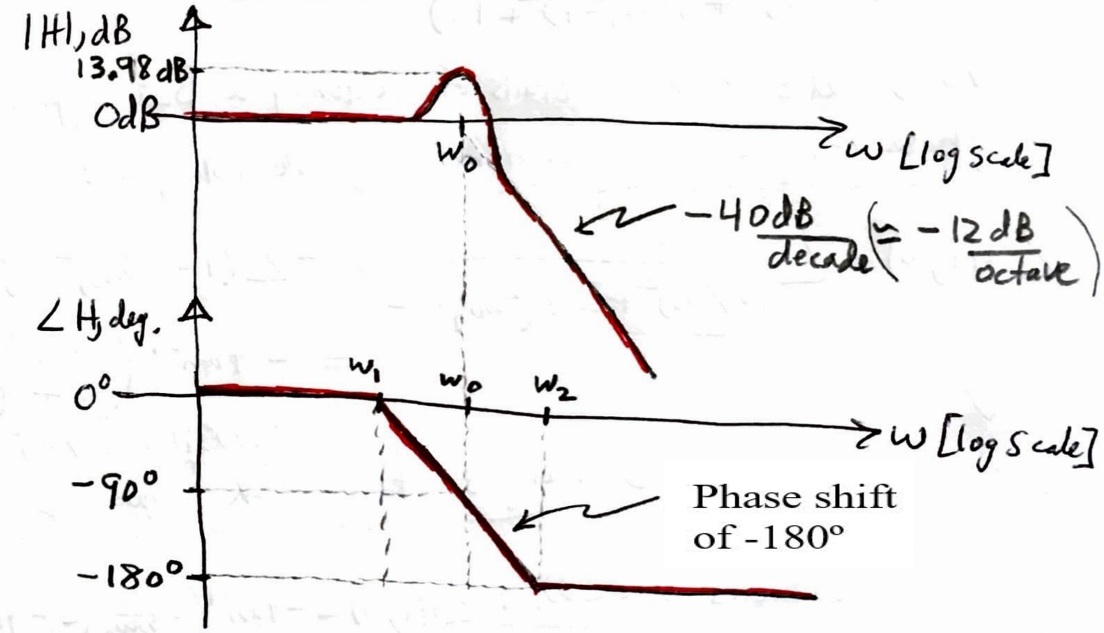
the slope of the line between ω_1 and ω_2 will depend on the ω_1 and ω_2 values, but the total phase roll-off from ω_1 to ω_2 will be -180°. If $\zeta<$ 0.02, just draw a vertical line at ω_0 (i.e., omit ω_1 and ω_2 and draw a vertical line at ω_0).

An example: Suppose $H(s) = \dfrac{1}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$, with $\zeta = 0.1$ and $\omega_0 = 10$ rps. Sketch Bode approximation of the frequency response, $H(j\omega)$. $\Rightarrow$ Note that $0 < \zeta < 0.5$, so there will be an appreciable peak in the magnitude response at ω_0 of Q_{dB}; what is Q?

$\Rightarrow Q = \dfrac{1}{2\zeta} = \dfrac{1}{2(0.1)} = 5$, and $Q_{dB} = 20\log_{10}(Q) \cong 13.98\,dB$.

$\Rightarrow \omega_1 = \dfrac{10}{10^{0.1}} \simeq 7.94$ rps & $\omega_2 = 10(10^{0.1}) \simeq 12.59$ rps

$\Rightarrow$ Here is the Bode approximation sketch of $H(j\omega)$:



⑦ $[(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1]$ terms in the numerator of $H(s)$:

To Bode approximate this kind of term, the magnitude and phase responses flip around in the usual, expected way, when we take a term in the denominator & move it to the numerator; consider the previous example (with $\omega_0 = 10$ rps, $\zeta = 0.1$), with the complex-conjugate pole pair in the numerator of $H(s)$ instead of the denominator: