PROCEEDINGS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Proc. R. Soc. Lond. A 1954 **225**, 505-515
doi: 10.1098/rspa.1954.0218

The stability of the plane free surface of a liquid in vertical periodic motion

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(*Communicated by Sir Geoffrey Taylor, F.R.S.—Received* 13 *April* 1954)

A vessel containing a heavy liquid vibrates vertically with constant frequency and amplitude. It has been observed that for some combinations of frequency and amplitude standing waves are formed at the free surface of the liquid, while for other combinations the free surface remains plane. In this paper the stability of the plane free surface is investigated theoretically when the vessel is a vertical cylinder with a horizontal base, and the liquid is an ideal frictionless fluid making a constant angle of contact of 90° with the walls of the vessel. When the cross-section of the cylinder and the frequency and amplitude of vibration of the vessel are prescribed, the theory predicts that the mth mode will be excited when the corresponding pair of parameters (p_m, q_m) lies in an unstable region of the stability chart; the surface is stable if none of the modes is excited. (The corresponding frequencies are also shown on the chart.) The theory explains the disagreement between the experiments of Faraday and Rayleigh on the one hand, and of Matthiessen on the other. An experiment was made to check the application of the theory to a real fluid (water). The agreement was satisfactory; the small discrepancy is ascribed to wetting effects for which no theoretical estimate could be given.

1. Introduction

When a vessel containing liquid is made to vibrate vertically, a pattern of standing waves is often observed at the free surface. These waves were first studied experimentally by Faraday (1831), who noticed that the frequency of the liquid vibrations was only half that of the vessel. The problem was next investigated by Matthiessen (1868, 1870), who found in his experiments that the vibrations were synchronous. The discrepancy between this and Faraday's result led Lord Rayleigh (1883 *b*) to make a further series of experiments which supported Faraday's view. Rayleigh (1883 *a*) also suggested that his theory of maintained vibrations (as applied to Melde's experiment, and originally restricted to systems having one degree of freedom) might be extended to explain Faraday's experiment. The theory of Melde's experiment leads to Mathieu's equation, while the theoretical treatment in the present paper leads to a system of Mathieu equations, and thus in a sense bears out Rayleigh's suggestion. It will be shown that the observations of Faraday and Rayleigh, and of Matthiessen, can both be explained by the present theory (see § 3). The present work has been made possible by the development of the theory of Mathieu functions since Rayleigh's time.

We shall consider a mass of liquid which is contained in a vertical cylindrical vessel of arbitrary cross-section closed by a horizontal plane at a depth h below the free surface (figure 1). The vessel is accelerated vertically in a simple harmonic motion, the maximum acceleration being f, and the period $2\pi/\omega$. We shall investigate whether the free surface remains plane during the motion, or whether it becomes unstable.

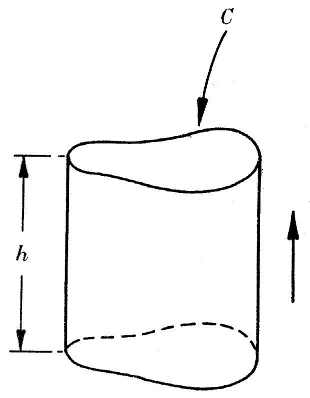

FIGURE 1. Diagram of the accelerated mass of liquid. The arrow shows the direction of the acceleration $f \cos \omega t$, and the curve C is the boundary of the free surface.

2. IDEAL-FLUID THEORY

We take Cartesian axes (x, y, z) moving with the vessel, such that the equation of the undisturbed free surface is $z = 0$, and the equation of the base of the vessel is $z = h > 0$. The axes move with a vertical acceleration $f \cos \omega t$, and so the motion relative to these axes is the same as if the vessel and the axes were at rest, and the gravitational acceleration were $(g - f \cos \omega t)$. We shall work out the theory for an ideal fluid in which viscosity and effects due to wetting of the walls are neglected. The internal stress in the fluid is then a pressure; the Eulerian equations of motion (Lamb 1932, §§4 to 7) are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + (g - f\cos \omega t),$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2·1}$$

The velocity components u, v, w and the pressure p are here expressed as functions of x, y, z and t. The density ρ is assumed to remain constant during the motion. If the motion was originally started from rest, there is a velocity potential $\phi(x, y, z, t)$ (Lamb, §17) such that $(u, v, w) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi$, and the equations of motion then have the integral

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \tfrac{1}{2}(u^2 + v^2 + w^2) - (g - f\cos \omega t)\, z = F(t), \tag{2·2}$$

where $F(t)$ is independent of x, y, z, and may be put equal to 0; and (2·1) gives

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{2·3}$$

The pressure at the free surface is

$$p=\gamma(\sigma_1+\sigma_2),$$

where γ is the surface tension, and σ_1, σ_2 are the principal curvatures of the surface (Lamb, §262). If the equation of the free surface is $z=\zeta(x,y,t)$, (2·2) gives

$$\frac{\gamma}{\rho}(\sigma_1+\sigma_2)+\frac{\partial\phi}{\partial t}+\tfrac{1}{2}(u^2+v^2+\omega^2)-(g-f\cos\omega t)\,\zeta=0 \tag{2·4}$$

at the free surface. The kinematical surface condition (Lamb, §9) is

$$\frac{\mathrm{D}}{\mathrm{D}t}(\zeta(x,y,t)-z)\equiv\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}-w=0 \tag{2·5}$$

at the free surface. The remaining boundary conditions require that the normal velocities (relative to the moving axes) at the walls and base of the vessel are zero, that is,

$$\frac{\partial\phi}{\partial n}=0 \tag{2·6}$$

on the walls, and

$$\frac{\partial\phi}{\partial z}=0 \tag{2·7}$$

on the base, $z=h$. Further progress can be made if (2·4) and (2·5) are linearized by omitting squares and products of u, v, w and ζ. This is justified if the deflexion and slope of the free surface are everywhere small. (In a real fluid this is not true near the walls.) The simplified equations are, from (2·5),

$$\frac{\partial\zeta}{\partial t}=w=\frac{\partial\phi}{\partial z}\quad\text{for } z=0; \tag{2·8}$$

and from (2·4)

$$\frac{\gamma}{\rho}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)+\left(\frac{\partial\phi}{\partial t}\right)_{z=0}-(g-f\cos\omega t)\,\zeta=0. \tag{2·9}$$

The approximation used here for the curvature is familiar in membrane theory (Rayleigh 1894, §194).

There is an important consequence of these boundary conditions. From (2·6) and (2·8) it follows that $\partial^2\zeta/\partial t\,\partial n=0$ at any point of the curve C bounding the free surface, whence $\partial\zeta/\partial n=$ its initial value $=0$. Thus the angle of contact at the walls is 90°. Also, by applying the operator $\partial/\partial n$ to (2·9), it is found that $\frac{\partial}{\partial n}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)=0$ on C. These boundary conditions show that ϕ, ζ and $\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)$ can each be expanded in terms of the complete orthogonal set of eigenfunctions $S_m(x,y)$, where

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_m^2\right)S_m(x,y)=0 \tag{2·10}$$

inside C, and $\partial S_m/\partial n = 0$ on C. This system has non-trivial solutions only when k_m^2 is an eigenvalue (necessarily real and positive, see Courant & Hilbert 1931, **1**, 257). The eigenfunction $S_0(x,y) \equiv$ constant, corresponding to the eigenvalue $k_0 = 0$, must be included in the set. Because $\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)$ satisfies the boundary conditions, its expansion may be obtained from the expansion of ζ by term-by-term differentiation (compare the argument in Weinstein 1949). It follows simply from (2·3), (2·7) and (2·8) that the required expansions are

$$\zeta(x,y,t) = \sum_0^\infty a_m(t)\, S_m(x,y),$$

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2} = -\sum_0^\infty k_m^2 a_m(t)\, S_m(x,y),$$

$$\phi(x,y,z,t) = -\sum_1^\infty \frac{\mathrm{d}a_m(t)}{\mathrm{d}t}\frac{\cosh k_m(h-z)}{k_m \sinh k_m h} S_m(x,y) + G(t), \tag{2·11}$$

where $G(t)$ is independent of x, y, z. Since the total volume of fluid remains constant, $a_0(t)$ is constant, and if the origin of ζ is adjusted so that $a_0(t) = 0$, it follows from (2·9) that $G(t) = 0$. Substitution of (2·11) and (2·9) then shows that

$$\sum_1^\infty \frac{S_m(x,y)}{k_m \tanh k_m h}\left[\frac{\mathrm{d}^2 a_m}{\mathrm{d}t^2} + k_m \tanh k_m h\left(\frac{k_m^2\gamma}{\rho}+g-f\cos\omega t\right)a_m\right] = 0,$$

and since the functions $S_m(x,y)$ are linearly independent, the coefficients $a_m(t)$ satisfy

$$\frac{\mathrm{d}^2 a_m}{\mathrm{d}t^2} + k_m \tanh k_m h\left(\frac{k_m^2\gamma}{\rho}+g-f\cos\omega t\right)a_m = 0. \tag{2·12}$$

If the parameters p_m and q_m are defined by the equations

$$p_m = \frac{4k_m \tanh k_m h}{\omega^2}\left(g+\frac{k_m^2\gamma}{\rho}\right), \quad q_m = \frac{2k_m f \tanh k_m h}{\omega^2}, \tag{2·13}$$

and if $T = \frac{1}{2}\omega t$, then (2·12) takes the form

$$\frac{\mathrm{d}^2 a_m}{\mathrm{d}T^2} + (p_m - 2q_m \cos 2T)\, a_m = 0, \tag{2·14}$$

which is the standard form of Mathieu's equation adopted by McLachlan (1947). If f is put equal to zero, (2·14) becomes an equation of simple harmonic motion relating to *free* vibrations of the liquid. The frequency (= 1/period) of these vibrations is

$$\frac{\omega_m}{2\pi} = \frac{1}{2\pi}\left[\tanh k_m h\left(\frac{k_m^3\gamma}{\rho}+k_m g\right)\right]^{\frac{1}{2}}, \tag{2·15}$$

and $p_m = \omega_m^2/\omega^2$. Note also that $q_m = 2k_m \tanh k_m h \times$ (amplitude of vibration).

3. Mathieu's equation and the stability conditions

To consider whether or not the motion of the free surface is stable we need to know the behaviour of the solutions of (2·14) for large values of T. In standard works on Mathieu functions (e.g. McLachlan 1947), it is shown that Mathieu's equation can have solutions of three kinds depending on the values of the parameters p and q, as written in (2·14). The majority of physical problems in which

Mathieu's equation arises are potential problems involving elliptical co-ordinates; in this class of problem p is determined so as to give rise to periodic solutions (Mathieu functions) with periodicity 2π or π in T. The special values of p, called characteristic numbers, are extensively tabulated. In the present problem, however, p is not necessarily a characteristic number, and we require solutions of a more general type, which, depending on the values of p and q, may be either stable or unstable, i.e. the solutions either remain finite or become unbounded as $T \to \infty$.

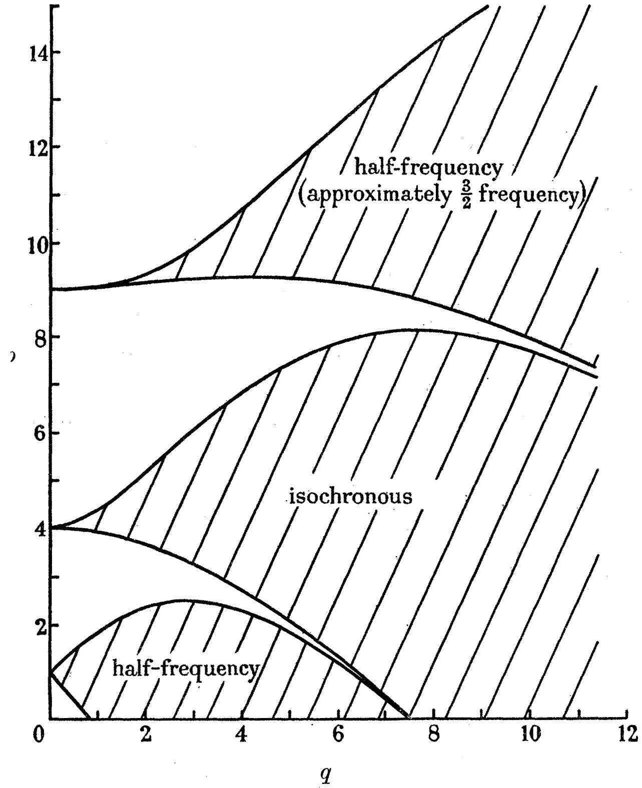

FIGURE 2. Stability chart for the solutions of Mathieu's equation

$$\frac{d^2a}{dT^2} + (p - 2q \cos 2T)\, a = 0.$$

When the loci of the characteristic numbers are plotted against q, the (p, q) plane is divided up into discrete regions as shown in figure 2. Only the first quadrant in the region of the origin is shown, since negative values of p are not required in the present problem, and the complete diagram is symmetrical about the p-axis. Complete diagrams, covering all four quadrants, are given in Jahnke & Emde (1938) and McLachlan (1947). It can be shown that if the point (p, q) lies in any of the regions shaded in figure 2, the solutions of Mathieu's equation are unstable; the solutions are then oscillatory, but with exponentially increasing amplitude. If (p, q) is in one of the unshaded regions, the solutions are stable, again being oscillatory though not regularly periodic. The unstable solutions have (apart from an exponential factor, Whittaker & Watson 1927, §19·4) an exact periodicity of 2π or

π in T, and an approximate periodicity indicated in figure 2. In particular, when (p, q) lies in the unstable region nearest the origin, the frequency of the solution is half the driving frequency (i.e. half $\omega/2\pi$).

When f and ω are given, (2·13) gives a point on the stability chart for each of the sequence of eigenvalues k_m. To determine whether or not the free surface is stable, it is therefore necessary to calculate (p, q) for each mode in turn, and observe its position on the stability chart. If one of the points lies in an unstable region, the free surface is unstable in the respective mode; this means that the mode increases in amplitude until eventually it is restrained by non-linear effects not considered here, or until the free surface disintegrates.

It may be argued that there is probably at least one of the points (p, q) in an unstable region, whatever f and ω. In practical cases, however, where the system is slightly dissipative, the higher-order modes tend to be suppressed, and an appreciable disturbance of the free surface occurs only when one of the lower-order modes is excited. The standing waves observed in practice are often caused by the superposition of various unstable modes, but with sufficient care it is possible to excite the modes separately. When the frequency of a *free* vibration of the liquid coincides with a subharmonic of the applied vibration, the parameter p takes the value n^2, where n is an integer, and figure 2 shows that instability can then occur for small values of q (i.e. small values of f). In particular, waves with half the frequency of the vessel are excited for small values of f when p is approximately 1, and (p, q) lies in the unstable region nearest the origin; and synchronous waves are excited when p is approximately 4, and (p, q) lies in the second unstable region.

4. Special cases: rectangular and circular cylinders

When the boundary curve is a rectangle with its corners at $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$, the surface harmonics S_m must satisfy

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_m^2\right) S_m(x, y) = 0,$$

$$\frac{\partial S_m}{\partial x} = 0 \quad \text{for } x = 0, x = a,$$

$$\frac{\partial S_m}{\partial y} = 0 \quad \text{for } y = 0, y = b.$$

It is convenient to arrange the harmonics and the eigenvalues k_m^2 in a double sequence, and it is readily shown that

$$S_{l,m} = \cos\left(\frac{l\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right),$$

$$k_{l,m}^2 = \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right),$$

where l and m include zero and all the positive integers.

When the boundary curve is a circle of radius R, it is convenient to use cylindrical polar co-ordinates (r, θ) in terms of which S_m must satisfy

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \theta^2}+k_m^2\right) S_m = 0.$$

It is again readily shown that

$$S_{l,m} = J_l(k_{l,m} r) \frac{\sin}{\cos} l\theta,$$

where $k_{l,m}$ is the mth zero of $J'_l(k_{l,m} R)$.

5. Experimental tests

While not being intended as a complete investigation, the following experiments serve to illustrate the effects under discussion. The experiments were confined to low-order circular wave modes, and to instabilities occurring in the first unstable region of the (p, q) chart; this partly avoided difficulties due to 'overlapping' of the modes, which became increasingly apparent for the higher-order modes (it appeared that once a particular mode became established with finite amplitude, it tended to suppress other modes which were unstable according to the first-order theory). The $(2, 1)$ mode (2 nodal circles and 1 nodal diameter) was found particularly convenient, its frequency being suitably far removed from those of adjacent modes, and results for this mode are given below.

In the experimental arrangement a circular Perspex tube, of internal diameter $2\frac{1}{8}$ in. ($= 5{\cdot}40$ cm), was first carefully cleaned, then filled with distilled water to a depth of about 10 in. and sealed off. The tube was mounted rigidly at the centre of a mild-steel beam 34 in. long and of cross-section $2\frac{3}{4} \times \frac{3}{8}$ in. which was lightly supported in a horizontal plane, the axis of the tube being vertical. By means of small rotating out-of-balance weights at the centre of the beam, driven by a variable-speed electric motor, suitable transverse vibrations of the beam could be excited, so that the attached tube was vibrated vertically. To ensure that the motion of the tube was strictly vertical, a flexible support was attached to the top of the tube; as a means of checking this adjustment a graticule was fixed to the side of the tube, and was viewed through a microscope. The beam was driven near resonance to ensure a good wave-form, and there was provision for altering the frequency of resonance by the addition of weights to the centre of the beam. Small adjustments of the amplitude of the vibrations were made by altering the driving frequency; this was preferable to the use of additional damping for controlling the amplitude, which would have resulted in a deterioration of the wave-form. The frequency was measured by comparison with a previously calibrated valve oscillator of the Wein-bridge type incorporating special measures to ensure frequency stability. An electrical signal at the frequency of the beam was obtained from a small magnetic pick-up, and a cathode-ray oscilloscope was used for the comparison. The amplitude was measured by the familiar method using a smoked glass slide and stylus.

In investigating the $(2, 1)$ mode, the amplitude of vibration required to make the free surface unstable was found in the following way. For each reading a different weight was added to the beam, thus altering the frequency of resonance and giving a different relation between amplitude and frequency; the amplitude was then adjusted (by varying the speed of the out-of-balance weights) until waves just began to build up on the free surface, and the critical amplitude and frequency were measured. In the stable region the free surface was found to remain calm for fairly strong vibrations of the beam, although minute ripples could always be seen. These ripples were possibly due to instability in a remote part of the stability chart, or to harmonics of the beam vibration. When instability occurred the waves grew slowly (during several seconds) to their maximum size, which was large enough to show the pattern of the mode quite clearly. Because the waves grew so slowly, it was necessary to increase the beam vibrations very gradually until instability occurred. The vibration was then reduced until the waves disappeared (usually quite suddenly), and the amplitude and frequency were again measured. The latter measurements were influenced by the finite size of the waves, and were found to be different from those corresponding to the initial growth of the waves (to which the present theory applies). Inspection with a stroboscope showed conclusively that the frequency of the waves was half that of the beam.

6. Discussion: comparison of theory and experiment

For a comparison with theory it is necessary to evaluate the parameters p and q for the $(2, 1)$ circular mode on the curve of neutral stability. We have $g = 981$, $\rho = 1$, $R = 2{\cdot}70$, $\gamma = 72{\cdot}5$ (pure water at $16°$ C) in c.g.s. units; $k_{2,1}R = 5{\cdot}331$ (Smith, Rodgers & Traub 1944), and the depth was so large that $\tanh k_{2,1}R = 1$. With these values it is found from (2·15) that $\omega_{2,1}/2\pi = 15{\cdot}87\,\mathrm{s}^{-1}$. For each measurement of ω and f the values of p and q were found from (2·16) and (2·13), and were seen to lie on a curve similar to that in figure 3, but lying very slightly above it. If, however, we take $\omega_{2,1}/2\pi = 15{\cdot}82\,\mathrm{s}^{-1}$, which is the experimental value corresponding to minimum f, we then have minimum q corresponding to $p = 1$, and the curve is brought into better agreement with the theoretical curve; the modified values of p and q are shown in figure 3, which includes part of the theoretical curve from figure 2. The small frequency difference may be ascribed to an effective surface tension of 70·4 dyn/cm, or to an effective radius exceeding R by a fraction of 1 mm; the latter mechanism would involve capillary effects at the wall. It will be seen that the theoretical and measured curves in figure 3 agree closely, except at the upper and lower ends of the experimental curve. At the upper end the presence of the $(1, 0)$ and $(1, 3)$ modes caused partial suppression of the $(2, 1)$ mode, while at the lower end the $(2, 0)$ mode became active in this way.

We conclude that the ideal-fluid theory predicts the stability of the plane free surface accurately in the instance checked in the present paper, and therefore probably in general. We will now consider whether the small differences between theory and experiment can be explained, and we recall that in deriving the theory it was found necessary to assume that

(i) the amplitude of the motion is infinitesimal;

(ii) the fluid is frictionless;

(iii) viscous effects and capillary contact effects between the fluid and the wall are negligible, i.e. the angle of contact is 90°.

These assumptions must now be examined critically.

(i) This is clearly justified when we are concerned with the critical curve (shown in figure 3) at which waves first appear when the amplitude of the applied vibration is increased, although it is not justified when we are concerned with the curve at which the waves disappear when the amplitude is decreased. Our theory does not apply to the latter case, but the following facts emerged from our experiments. It was found that the second curve (not shown in the present paper) lies outside the theoretical curve except for very small values of q, where apparently the sustaining

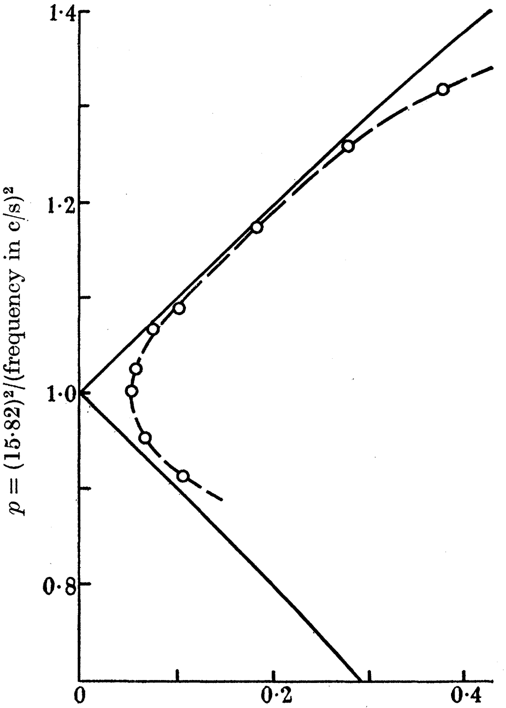

FIGURE 3. Experimental (- - -) and theoretical (—) stability curves for the (2, 1) mode in a circular cylinder of radius 2·70 cm.

effect of finite size is least. This effect appears to be more prominent for $p > 1$ than for $p < 1$, which suggests that finite size reduces the natural frequency of the waves (cf. Penney & Price 1952, p. 267, and Taylor 1952). Results for other modes were similar, but varied in the extent to which the experimental curve could be drawn before interference by adjacent modes became apparent.

(ii) Energy dissipation by viscosity causes the amplitude of the free motion to be damped exponentially, by a factor $e^{-\beta_m T}$, say, where β_m depends on the mode under consideration. Waves will develop only when their growth is rapid enough to overcome the damping factor. This explains qualitatively why the curve of neutral stability in figure 3 lies inside the unstable region for the ideal fluid. There is little difficulty in calculating the viscous dissipation near the walls by the method given by Ursell (1952), appendix B.* For the (2, 1) mode a value of $\beta_{2,1} = 0{\cdot}0014$ was obtained. The dissipation in the body of the fluid is negligible. To compare this with the measurements, we must find for the measured values of (p, q) in the unstable region the appropriate exponential factor $e^{\beta T}$ governing the growth of the waves. There are no published tables, but it is known (McLachlan, §4·92) that $\beta = \frac{1}{2}q$ when $p = 1$ and q is small. In figure 3, $q = 0{\cdot}053$ when $p = 1$, and so $\beta = 0{\cdot}027$ at this point. The measured value is thus nearly twenty times the value calculated from viscous dissipation.

(iii) As has just been seen, the actual energy dissipation is too large to be ascribed to viscosity inside the liquid. The calculation did not allow for the energy which is dissipated when the liquid surface rises and falls in contact with the walls of the vessel; and although little is known about the change in contact angle and the resulting stresses at the surface, the hypothesis that the greater part of this dissipation is due to these contact effects does not appear unreasonable. More experimental evidence is needed to throw light on this problem.

7. Conclusion

It has been shown how an ideal-fluid theory can be used to predict the stability of the plane free surface of liquid contained in a vessel vibrating vertically, and an experiment has been described which confirms the theory in the region of the stability chart where the driving frequency is twice the frequency of the liquid vibrations. It was not possible to account successfully for the small discrepancy between theory and experiment, since the available information about contact effects between the free surface and the wall is very meagre. More experimental work on contact is needed, and an investigation of other parts of the stability chart is also desirable.

A particular aspect of the present problem has already been treated by Taylor (1950), who considered the motion of the surface of separation between two fluids of different densities, which are given a constant downward acceleration perpendicular to the interface. To compare the present theory, the atmosphere above the free surface may be regarded as a fluid of zero density. The stability is determined by an equation similar to (2·12), with $f = 0$ and $(g - f_0)$ instead of g, which has unstable (exponential) solutions when

$$f_0 - g - k_m^2 \frac{\gamma}{\rho} > 0.$$

When surface tension is neglected, this corresponds to Taylor's result that the surface is unstable when the resultant acceleration is towards the fluid of greater density. It is seen that surface tension has a stabilizing effect. The influence of surface tension and viscosity in Taylor's problem has been discussed at length by Pennington & Bellman (1952). Note that in our problem instability may occur when f is much less than g.

* Owing to a misprint in this reference, the actual dissipation is twice the amount given at the foot of p. 95.

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The diffraction of waves by an irregular refracting medium

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(*Communicated by Sir Edward Appleton, F.R.S.—Received* 14 *April* 1954)

A method is described of calculating the diffraction effects produced by a thick stratum of an irregular refracting medium. It consists of evaluating the statistics of the phase irregularities in the wave-front after traversing the medium, and treating these irregularities as having been produced by a thin phase-changing screen. For a particular statistical model of the irregularities in the medium the result is shown to be identical with that obtained by Fejer using a different method.

In a recent paper, Fejer (1953) has discussed the diffraction of a plane electromagnetic wave in passing through a medium containing irregularities of the dielectric constant. He has also compared the results with those obtained in the case of diffraction by a thin phase-changing screen. The purpose of this note is to point out an alternative and physically simpler method of treatment of the problem of volume irregularities, which leads to the same results as obtained by Fejer for multiple scattering. The method shows more clearly the quantitative correspondence between the parameters of the thin screen and those of the thick medium having the same scattering properties. It consists of calculating the statistics of the phase irregularities in the wave-front after traversing the medium, and treating these irregularities as having been produced by a thin phase-changing screen. It has recently been applied