Mathieu Functions and Numerical Solutions of the Mathieu Equation

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***Abstract*—We review the full spectrum of solutions to the Mathieu differential equation $y'' + [a - 2q\cos(2z)]y = 0$, and we describe a numerical algorithm which allows a flexible approach to the computation of all the Mathieu functions. We use an elegant and compact matrix notation which can be readily implemented on any computing platform. We give some explicit examples (written in the programming language *Scilab*) that provide a ready-to-use package for solving the Mathieu differential equation and related applications in several fields.**

***Index Terms*—Scilab, Mathieu Equation, Mathieu Functions**

I. Introduction

The Mathieu equation (ME) appears in several different situations in Physics: electromagnetic or elastic wave equations with elliptical boundary conditions as in waveguides or resonators, the motion of particles in alternated-gradient focussing or electromagnetic traps, the inverted pendulum, parametric oscillators, the motion of a quantum particle in a periodic potential, the eigenfunctions of the quantum pendulum, are just few examples.

Although Mathieu Functions (MFs) have a wide spectrum of applications, they are not commonly used and many books on "special functions" do not report them at all (but there are, of course, many exceptions [1] [2] [3]). This is likely due to the complexity of the solutions of the ME.

This is why, more than 100 years after their definition [4], still many papers are written on specific technical aspects of MFs and their computational problems [5] [6] [7] [8] [9] [10]. The reader is invited to refer to the papers by Ruby [11] and by Gutierrez-Vega et al. [12] for illustrations and examples of applications of MFs. The classic reference book is still the one by McLachlan [13] who has defined the standard notation that we use throughout this paper.

Some high-level mathematical language contains Mathieu functions, but these are defined as black-box commands: then, it is difficult to understand how these commands work and how to extend their use to other cases or specific applications. The present toolbox is the first implementation in the framework of an open-source software. This fact, together with the modularity and simplicity of the code and some specific application-oriented functions, allows easy improvements, extensions and applications to different specific cases.

In this paper we want first to give a brief guide to orient the reader in the "forest" of solutions the ME can have, and then show how with a few simple modules in the language Scilab it is possible to easily solve the ME in the various cases or to calculate the different kinds of MFs.

This method was inspired by the remark by Fëdorov [14] that the Mathieu equation is the Fourier transform of the Raman-Nath system of equations, and a matrix-based program had been written for the latter [15].

Our program is composed of 2 modules, essentially based on the Scilab command "spec" for the diagonalization of a matrix, one for MFs and one for non-periodic solutions of the ME, plus one for elliptical-to-cartesian coordinate transformation, which turns out to be necessary when plotting the results.

As contrasted with polynomial approximations and Runge-Kutta methods, this one does not accumulate errors as it is based on matrix diagonalization and Fourier transform. Moreover, the modular form of the program makes it more flexible, as different script programs can be made for specific problems using the same building blocks.

This approach is efficient in determining a whole range of values rather than a single one.

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II. Solutions to the Mathieu equation

In this Section we briefly review some known facts about the solutions of the Mathieu equation (for more details see [13], [2]). The Mathieu equation is a second-order homogeneous linear differential equation of the form

$$\frac{d^2y}{dz^2} + [a - 2q\cos(2z)]\, y = 0\,, \tag{1}$$

where the constants a, q are often referred as *characteristic number* and *parameter*, respectively. Most of the applications of eq. (1) deal only with real values of a and q, though one can also consider the more general case where a, q are complex numbers.

By substituting the independent variable $z \to iz$ in eq. (1) one obtains the so called *modified Mathieu equation*:

$$\frac{d^2y}{dz^2} - [a - 2q\cosh(2z)]\, y = 0\,. \tag{2}$$

Equation (2) describes, for instance, the radial component of the vibrations of an elliptic drum.

Let us first consider equation (1) (eq. (2) will be considered at the end of this Section). The most general solution $y(z)$ of eq. (1) can be written as a linear combination of two independent solutions $y_1(z)$ and $y_2(z)$, i.e. $y(z) = Ay_1(z) + By_2(z)$ with A, B arbitrary complex constants. According to Floquet's theorem, it is possible to choose $y_1(z)$ and $y_2(z)$ in a very simple and convenient form: indeed, there always exists a solution of eq. (1) in the form (*solutions of the first kind*)

$$y_1(z) = e^{i\nu z} p(z) \tag{3}$$

where the *characteristic exponent* ν depends on a and q, and $p(z)$ is a periodic function with period π. It is easy to check that a solution of the form in eq. (3) is bounded for $z \to \infty$, unless ν is a complex number, for which $y_1(z)$ is unbounded. If ν is real but not a rational number then eq. (3) is non periodic. If ν is a rational number, i.e. ν is a proper fraction as $\nu = s/p$, then eq. (3) is periodic of period at most $2\pi p$ (and not π or 2π). Finally if ν is a real integer then y_1 is a periodic function with period π or 2π.

Moreover if ν is not a real integer then a second independent solution is $y_2(z) = y_1(-z)$ whereas if ν is a real integer then a second independent solution is always of the form $y_2(z) = \tau\, z\, y_1(z) + f(z)$ (*Mathieu functions of the second kind*) where τ is a constant and $f(z)$ has the same periodicity properties of y_1. The results stated above are collected in the table I.

There are two ways of tackling the problem of solving eq. (1), according to the physical problem one is interested in:

1) The first case is when a, q are *independently given* constants (e.g. in the case of the parametric oscillator),

TABLE I: Stability and periodicity of two independent solutions of the Mathieu equation, at different values of the characteristic exponent ν.

VALUES of ν	STABILITY of y_1	PERIOD of y_1	$y_2(z)$
ν complex and not real	not bounded	not periodic	$y_1(-z)$
ν real and not rational	bounded	not periodic	$y_1(-z)$
ν rational and not integer	bounded	periodic (not π or 2π)	$y_1(-z)$
ν integer	bounded	periodic π or 2π	$\tau\, z\, y_1(z) + f(z)$

and then the general solution may be periodic or not, bounded or not depending on the corresponding ν values, as shown in Table I. Such a ν value can be determined by the following method [13]: by introducing in eq. (3) the Fourier representation of a periodic function with period π,

$$p(z) = \sum_{k=-\infty}^{+\infty} c_k e^{2ikz}$$

and by inserting the result in eq. (1), one obtains the following recurrence equation for c_k's

$$[(2k+\nu)^2 - a]c_k + q[c_{k+1} + c_{k-1}] = 0 \tag{4}$$

which can be written in matrix form,

$$(H_\nu - a\mathbb{I})\, \mathbf{c} = 0 \tag{5}$$

where $\mathbb{I}$ is the identity matrix and H_ν is a symmetric tridiagonal matrix as in eq. (6) and $\mathbf{c}$ is a column vector of the form $\mathbf{c} = (\ldots, c_{-1}, c_0, c_1, \ldots)$. The linear system eq. (5) has non trivial solution only if $\det(H_v - a\mathbb{I}) = 0$, which is an equation for ν. The solution is eq. (7), where $\Delta(\nu)$ is written as eq. (8).

Then, given a, q, one obtains a corresponding ν value by evaluating (7) and (8). Note that it is sufficient to consider a solution of eq. (7) in the interval $[0, 1]$ because, if $\bar{\nu}$ is a solution of eq. (7), then $\nu = \pm\bar{\nu} \pm 2n$ is a solution too (signs uncorrelated).

It is very easy to write a SCILAB computer program to evaluate (7) and (8); of course the matrix in (8) must be truncated to some order, say N, then obtaining the solution ν_N. It is sufficient to evaluate also $\nu_{N'}$ for $N' > N$: if $|\nu_N - \nu_{N'}|$ is less than a fixed-at will precision parameter, then keep it, else go further by increasing N.

$$H_\nu = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & (\nu-4)^2 & q & 0 & 0 & 0 & \vdots \\ \vdots & q & (\nu-2)^2 & q & 0 & 0 & \vdots \\ \vdots & 0 & q & \nu^2 & q & 0 & \vdots \\ \vdots & 0 & 0 & q & (\nu+2)^2 & q & \vdots \\ \vdots & 0 & 0 & 0 & q & (\nu+4)^2 & q \\ \cdots & \cdots & \cdots & \cdots & \cdots & q & \cdots \end{pmatrix} \tag{6}$$

$$\begin{cases} \nu = \frac{2}{\pi} \arcsin \sqrt{\Delta(0) \sin^2\left(\frac{\pi}{2}\sqrt{a}\right)} & \text{for a} \neq (2\text{k})^2 \\ \nu = \frac{1}{\pi} \arccos\left[2\,\Delta(1) - 1\right] & \text{for a} = (2\text{k})^2 \end{cases} \tag{7}$$

$$\Delta(\nu) = \det \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 1 & \frac{q}{(\nu-4)^2-a} & 0 & 0 & 0 & \vdots \\ \vdots & \frac{q}{(\nu-2)^2-a} & 1 & \frac{q}{(\nu-2)^2-a} & 0 & 0 & \vdots \\ \vdots & 0 & \frac{q}{\nu^2-a} & 1 & \frac{q}{\nu^2-a} & 0 & \vdots \\ \vdots & 0 & 0 & \frac{q}{(\nu+2)^2-a} & 1 & \frac{q}{(\nu+2)^2-a} & \vdots \\ \vdots & 0 & 0 & 0 & \frac{q}{(\nu+4)^2-a} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{8}$$

The coefficients c_k can be obtained, up to an arbitrary proportionality constant, by considering the eigenvector corresponding to the zero eigenvalue of the symmetric matrix $H_\nu - a\mathbb{I}$. The constant is determined by imposing suitable normalization conditions.

2) The second case is when, from the very beginning, one is interested only to *periodic solutions* with period π or 2π (e.g. in the case of the angular part of the wave equation in elliptical coordinates). Then a and q cannot be given independently: they must satisfy the equation (owing the periodicity) of the form $\nu(a,q) = n$, where n is an integer number. All a's values for which $\nu(a,q) = n$, are called *characteristic values:* the corresponding periodic solutions (3) are called *Mathieu functions* (or *Mathieu functions of the first kind*). In this case, the second solution y_2 (*Mathieu function of the second kind*) is usually rejected since it is not bounded.
It is easy to see from (3) that if $\nu(a,q) = n$, then y_1 is periodic with period π for even n and periodic with period 2π for odd n. Since a periodic function with period π it has also period 2π, then by inserting the Fourier representation of a 2π periodic function in (3), $y_1(z) = \sum_{k=-\infty}^{+\infty} c_k e^{ikz}$. One obtains the following recurrence relation,

$$\left(k^2 - a\right) c_k + q\left(c_{k+2} + c_{k-2}\right) = 0$$
$$\text{or}$$
$$(H - a\mathbb{I})\mathbf{c} = 0$$

where H is the symmetric pentadiagonal matrix as in eq. (9).
The linear system (9) has non trivial solution only if $\det(H - a\mathbb{I}) = 0$, which is the equation for the eigenvalues of H. By solving it one obtains all a's value we are looking for. If q is real then H is real symmetric and all the eigenvalues a's are real. Then they can be ordered in increasing order: it is usual to call them a_n and b_n by the eq. (10).
In particular, notice that when $q = 0$ one has $a_0 = 0$ and $a_k = b_k = k^2$, for $k = 0, 1, \ldots$. The non-zero

$$H = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & (-2)^2 & 0 & q & 0 & 0 & \cdots \\ \cdots & 0 & (-1)^2 & 0 & q & 0 & \cdots \\ \cdots & q & 0 & 0^2 & 0 & q & \cdots \\ \cdots & 0 & q & 0 & (1)^2 & 0 & \cdots \\ \cdots & 0 & 0 & q & 0 & (2)^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \; ; \; \mathbf{c} = \begin{pmatrix} \vdots \\ c_{-1} \\ c_0 \\ c_1 \\ \vdots \end{pmatrix} \tag{9}$$

$$\begin{cases} a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < \cdots & \text{for q} < 0 \\ a_0 < b_1 \le a_1 < b_2 \le a_2 < b_3 \le a_3 < \cdots & \text{for q} \ge 0 \end{cases} \tag{10}$$

solutions of eq. (9), corresponding to a fixed eigenvalue a_k or b_k, are given by the eigenvectors (i.e. the columns of the orthogonal matrix V which diagonalizes H) of $H = VDV^+$ (D diagonal matrix). The Mathieu functions so obtained are defined as $ce_n(z,q)$ (even solutions corresponding to a_n eigenvalues: cosine-elliptic), $se_n(z,q)$ (odd solutions corresponding to b_n eigenvalues: sine-elliptic). Moreover, ce_{2n} and se_{2n+1} have period π, and ce_{2n+1} and se_{2n+2} have period 2π. $ce_n(z,q)$ and $se_n(z,q)$ functions are illustrated in Fig.1.

The second linearly independent solution (as we said before, necessarily not periodic) are referred as $fe_n(z,q)$ for the $ce_n(z,q)$ case, and as $ge_n(z,q)$ for the $se_n(z,q)$.

For the sake of completeness, let us mention that when q is negative imaginary, $q = -is$ with s real and positive, in some book it is possible to find cer_n, cei_n, ser_n, sei_n functions. They are defined by

$$\begin{aligned} cer_n(z,q) &= \Re ce_n(z,-is) \\ cei_n(z,q) &= \Im ce_n(z,-is) \\ ser_n(z,q) &= \Re se_n(z,-is) \\ sei_n(z,q) &= \Im se_n(z,-is) \end{aligned}$$

The solutions of the modified Mathieu equation eq. (2), can be easily obtained by the following way; for a-values corresponding to $ce_m(z,q)$, $se_m(z,q)$ the first solutions of eq. (2) are derived by substituting iz for z, i.e.

$$\begin{aligned} Ce_n(z,q) &= ce_n(iz,q) \\ Se_n(z,q) &= -ise_n(iz,q) \end{aligned}$$

Ce_n and Se_n are called *modified Mathieu functions of the first kind* (illustrated in Fig. 2).

In the same way, one defines Cer_n, Cei_n, Ser_n, Sei_n, Ge_n, Fe_n corresponding to cer_n, cei_n, ser_n, sei_n, fe_n, ge_n respectively.

The functions Ce_n and Se_n can be written also in terms of Bessel functions $J_n(z)$, i.e

$$y_1(z) = \sum_{r=0}^{\infty} c_r J_r(2\sqrt{q}\cosh z) \tag{11}$$

Note that the Bessel expansion is equivalent to the exponential expansion but it is slightly better for numerical calculations. Moreover, an independent solution can be obtained directly by substituting J-Bessel functions with Y-Bessel functions, thus defining the functions Fey_n, Gey_n. In the same way, by using K-Bessel functions one defines Fek_n and Gek_n functions.

III. Organization of the program

The program is composed of four modules, two for the calculation of periodic Mathieu functions (hence q is given as an input, and the a's and b's are calculated as output), and two for the general aperiodic case (both a and q are given as arbitrary input):

- `[ab,c]=mathieuf(q,[mat_dimension])` calculates the characteristic values a_k, b_k and the coefficients c_k of the expansion of Mathieu functions, for a given matrix dimension (default 12).
- `mathieu('kind',order,arg,q,[precision])` uses the former program to calculate the Mathieu functions with a given precision (default 10^{-3}), by starting with a matrix size (depending on the order n and the parameter q) which is increased until the required precision is reached. Some inputs can also be entered

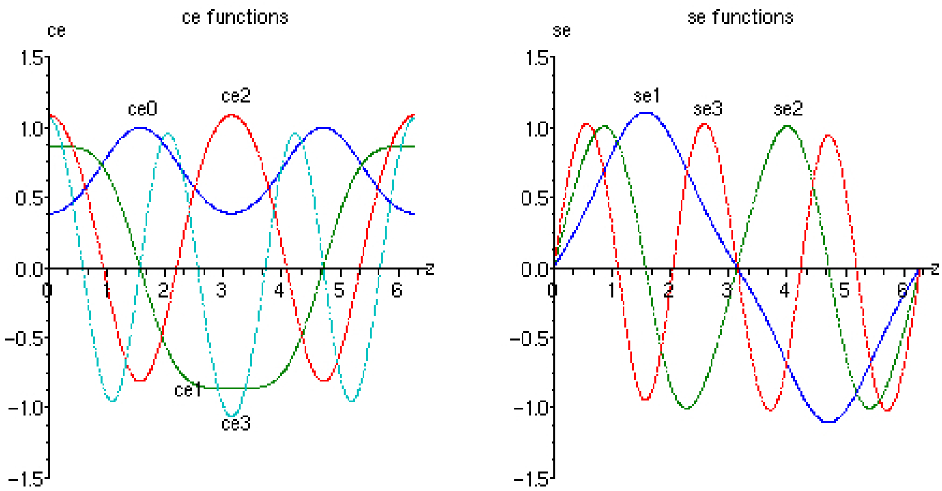

Fig. 1: The ce functions (left) and se functions (right), for q=1. Our Scilab commands are:
`plot(mathieu('ce',(0:3)',0:.1:2*%pi,1)); plot(mathieu('se',(1:3)',0:.1:2*%pi,1));`

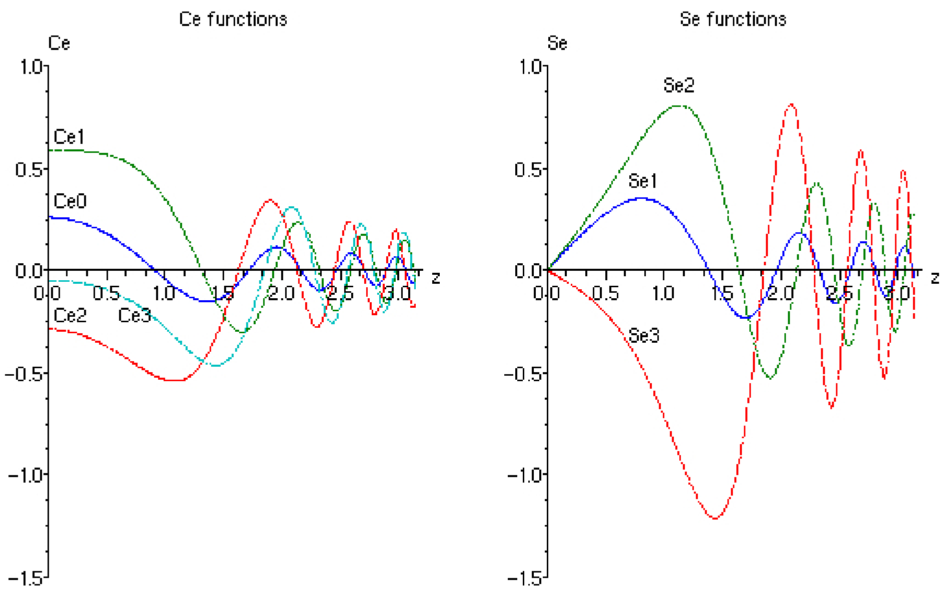

Fig. 2: The Ce functions (left) and Se functions (right), for q=1. Our Scilab commands are:
`plot(mathieu('Ce',(0:3)',0:.1:2*%pi,1)); plot(mathieu('Se',(1:3)',0:.1:2*%pi,1));`

interactively.

- `[nu,c] =mathieuexp(a,q,[mat_dimension])` calculates the characteristic exponent $\nu(a,q)$ of non-periodic solutions of the ME and the coefficients c_k of the expansion of the periodic factor.

- `mathieus(arg,a,q,[precision])` uses the former program and calculates solutions of ME with arbitrary a and q parameters.

A fifth module:

- `[x,y]=ell2cart(u,v,c)` has also been written for the transformation from cartesian to elliptical coordinates, which is necessary when plotting the results of a two-dimensional calculation (for ex. the elliptical waveguide).

The `mathieuf.sci` program, which is the basic building block called by the program `mathieu.sci`, is essentially made of commands for the formation and diagonalization of the matrix:

```
function [ab,c] = mathieuf(q,n)
d0=[-n:n].^2;
d1=q*ones(1,2*n-1);
m=diag(d0)+diag(d1,2)+diag(d1,-2);
[U,D]=spec(m);
[ab,I]=sort(diag(D));
ab=ab($:-1:1);
I=I($:-1:1);
U=U(:,I);
```

plus normalization and some arrangement of signs at the end (in order to agree with notations of [13])

```
c=U/(sqrt(2)*diag(diag(U*U')));
s=c(n+1,:)<0;    c(:,s)=-c(:,s);
s=c(n+2,:)<0;    c(:,s)=-c(:,s);
```

and that's all.

All these programs have been tested for a, q real values and real or complex z values. For complex q values, the program `mathieuf.sci` has some problem with the ordering of the characteristic eigenvalues (indeed, they are complex in this case): this problem seems not to be considered, by other programs for computation of numerical solutions of ME. In the following section we shall give some example, and in particular we will show how it is possible to give meaning to the ordering of characteristic eigenvalues corresponding to complex q values. We like stressing the great simplicity of such programs: just a few code-lines are sufficient to build the matrix H, to diagonalize and to find eigenvector thus obtaining all is needed for the solution. The limitation of this technique is given by the framework program language, which imposes restrictions both on numerical precision and on memory available.

IV. Examples

This section is a gallery of some output of our programs, with the commands used to calculate and display them.

Figure 3 shows the well-known stability curves, where the white areas show values of "a" and "q" with stable (periodic) solutions, and the coloured contours indicate the unstable solutions (the value expressed by the contour is the exponential envelope factor of the diverging function). The equations in this case might describe a parametric oscillator, i.e. a harmonic oscillator where the oscillating frequency ω_0 is modulated at a frequency $\omega_m = 1/\sqrt{a}\omega_0$ with a relative amplitude (max. freq. deviation divided by ω_0) $2q/a$. The $q = 0$ axis represents an unmodulated harmonic oscillator, with stable sinusoidal solutions for $a > 0$ and unstable hyperbolic solutions for $a < 0$. The stable (white) area below the a=0 axis indicates the conditions of stability of the "inverted pendulum" (stability of an otherwise unstable oscillator).The butterfly-like unstable region pointing to a=1 represent the parametric resonance when the oscillator is modulated at twice the natural frequency of the oscillator.

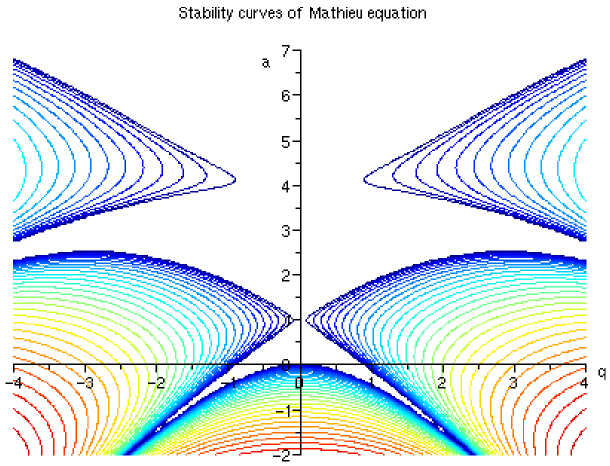

Fig. 3: Stability curves of the Mathieu equation as a function of 'a' and 'q'; our Scilab commands are:

```
a=[-2:0.05:7]; q=[-4:0.05:4];
for j=1:length(a)
for h=1:length(q)
aa(h,j)=mathieuexp(a(j),q(h),12);
end end
contour(q,a,imag(aa),30);
```

Fig. 4 is a plot over one period of $se_1(x, q)$ for various values of q, showing how the function, which is a sinusoid for $q = 0$, deviates from it more and more as q increases.

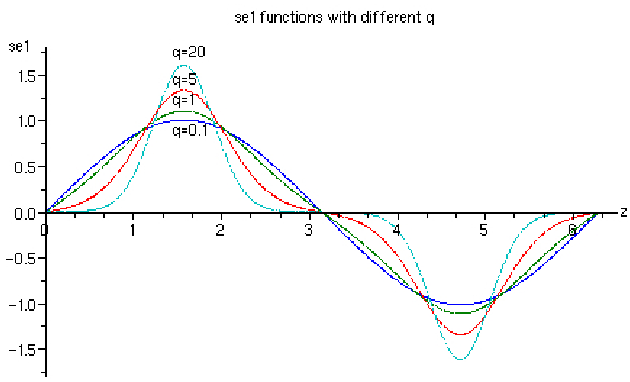

Fig. 4: The function 'se1' or different values of 'q'; our Scilab commands are:

```
for q=[.1,1,5,20] plot(mathieu('se',
1,0:.2:%pi,q)); end
```

Fig. 5 is the 3-D plot of the field of a higher-order mode of an elliptical waveguide (or vibrating drum). For this a program has been written, "`elliptical.sci`", which determines the zeros of the chosen radial function, in order to determine the ellipse which satisfies the boundary conditions; then com-

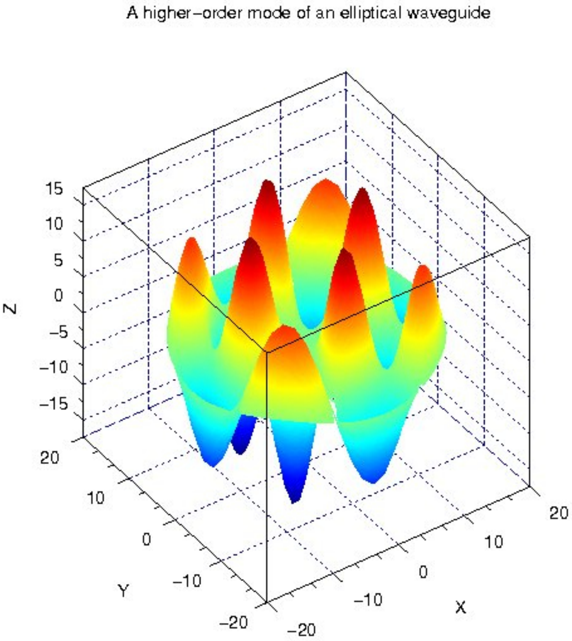

Fig. 5: A higher order mode of an elliptical waveguide (or drum); our Scilab command is
`elliptical("e",2,4,0.1);`

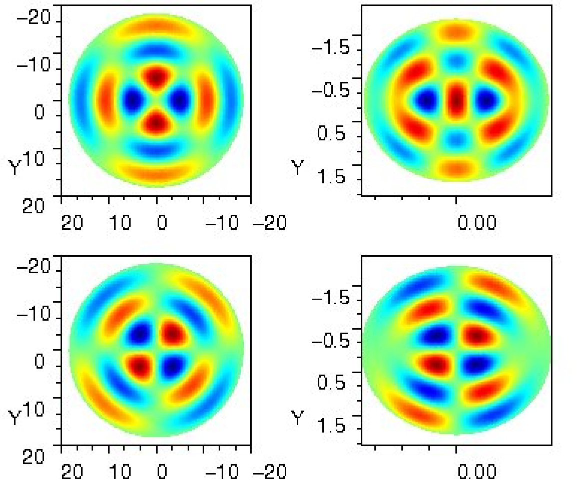

Fig. 6: Comparison of a mode with circular and slightly elliptical boundary; our Scilab commands are:

```
elliptical('e',3,2,0.1);
elliptical('e',3,2,8);
elliptical('o',3,2,0.1);
elliptical('o',3,2,8);
```

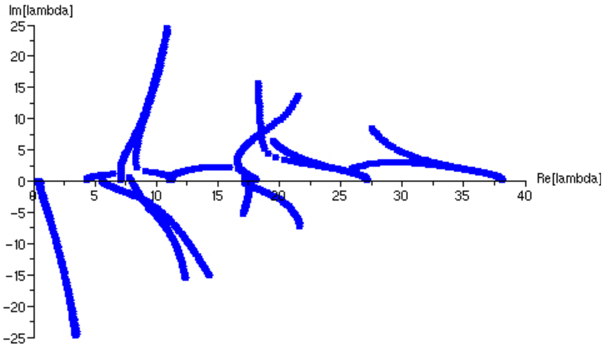

Fig. 7: Eigenvalues (imaginary vs. real part) for complex values of 'q' (with imaginary part from 0 to 15, real part=1); our Scilab commands are:

```
for t=0.1:0.2:15;
plot(real(2+mathieuf(2+t*%i,6)),
imag(mathieuf(2+t*%i,6)),'.'); end
```

putes the desired radial and angular Mathieu functions, and then makes a surface plot of their product.

In this case we need the auxiliary function `ell2cart.sci` in order to represent the results using cartesian plots. Fig. 6 shows some colour-coded 2-D plots of a high-order mode, showing the strong deformation due to a slight ellipticity. Fig. 7 is an analysis of MFs with a complex q, showing the behaviour of the function $se_n(q,x)$ when the imaginary part of q is varied. This illustrates well the problem of ordering of the solutions [16].

V. Conclusions

We hope that this tool we have developed can help to make the application of MFs and the ME more easy and more widely used. The three modules that we have written in Scilab have also been translated into Matlab, which has a rather similar syntax. Furthermore, scripts and function programs can be introduced to extend the functionality of this package or apply it to specific cases.

Due to the large number of different specific functions and of technical problems encountered in less common cases, in the spirit of open source software, this tool is permanently a work in progress, in which readers are invited to solve existing problems and to extend its use. In fact, while the program works well with the commonly used functions and ranges of variables, some problems still persist in the normalization of a few functions and in the calculation of some less used ones in some ranges of variables.

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