1. Prove or disprove: Suppose f is convex and X is submartingale, prove that $g(t) = \mathbb{E}[f(X_t)]$ is increasing.

Pf. Assume $f(x) = -x$,

$\because f(\frac{x_1+x_2}{2}) = \frac{1}{2}(f(x_1)+f(x_2))$, for $\forall x_1, x_2 \in \mathbb{R}$. $\therefore f(x)$ is convex

$\therefore \forall s \le t$, $\mathbb{E}[f(X_t) - f(X_s) | \mathcal{F}_s] = \mathbb{E}[-X_t + X_s | \mathcal{F}_s] = X_s - \mathbb{E}[X_t | \mathcal{F}_s]$

since X_s is $\mathcal{F}_s$-measurable.

$\because X_t$ is submartingale, $\mathbb{E}[X_t | \mathcal{F}_s] \ge \mathbb{E}[X_s | \mathcal{F}_s] = X_s$

$\therefore X_s - \mathbb{E}[X_t | \mathcal{F}_s] \le 0$, which means $\mathbb{E}[f(X_t) - f(X_s) | \mathcal{F}_s] \le 0$

$\mathbb{E}[f(X_t)] \le \mathbb{E}[f(X_s)]$. for $s \le t$, so $g(t)$ is decreasing.

disproved.

2. Let $t \to e^{-rt} S_t$ be a martingale, then prove that

$$C(t) = \mathbb{E}[e^{-rt}(S_t - k)^+] \text{ is increasing.}$$

Pf. $t \to e^{-rt} S_t$ is a martingale, So $\mathbb{E}[e^{-rt} S_t | \mathcal{F}_s] = \mathbb{E}[e^{-rs} S_s | \mathcal{F}_s]$

$= e^{-rs} S_s$. $\forall s \le t$.

① $S_t > k$, $\mathbb{E}[e^{-rt}(S_t - k)^+ | \mathcal{F}_s] = \mathbb{E}[e^{-rt}(S_t - k) | \mathcal{F}_s]$

$= \mathbb{E}[e^{-rt} S_t - e^{-rt} k | \mathcal{F}_s]$

$= \mathbb{E}[e^{-rt} S_t - k(e^{-rt} + e^{-rs} - e^{-rs}) | \mathcal{F}_s]$

$= \mathbb{E}[e^{-rt}(S_s - k) - k\mathbb{E}[e^{-rt} - e^{-rs} | \mathcal{F}_s] \ge \mathbb{E}[e^{-rt}(S_s - k) | \mathcal{F}_s]$

$\therefore C(t) \ge C(s)$. for $\forall t \ge s$

② $S_t \le k$. $\mathbb{E}[e^{-rt}(S_t - k)^+ | \mathcal{F}_s] = 0$.

So $C(t)$ is increasing.

3. Suppose $r = 0$ and S is martingale, prove that $P(t) = \mathbb{E}[(S_t - k)^-]$ is increasing.

Pf Since S is a martingale with the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$,

$$E[S_t|\mathcal{F}_s] = E[S_s|\mathcal{F}_s] = S_s, \text{ for } \forall s \le t.$$

① $S_t < K$,

$$E[(S_t-K)^-|\mathcal{F}_s] = E[(K-S_t)|\mathcal{F}_s] = K - S_s \geq (S_s-K)^-$$

② $S_t \geq K$,

$$E[(S_t-K)^-|\mathcal{F}_s] = 0 \geq (S_s-K)^-$$

$$E[E[(S_t-K)^-|\mathcal{F}_s]] \geq E[(S_s-K)^-|\mathcal{F}_s], \text{ for } \forall s \le t.$$

So $p(t)$ is increasing.