

Example 3 Given i.i.d $\{\alpha_i : i \in 1, 2, \dots, N\}$, we use

$$\bar{\alpha}_N = \frac{1}{N} \sum_{i=1}^N \alpha_i$$

as its estimator of the mean $\mathbb{E}[\alpha_1]$ and use

$$\beta_N = \frac{1}{N} \sum_{i=1}^N (\alpha_i - \bar{\alpha}_N)^2$$

as the estimator of $\text{Var}(\alpha_1)$. Suppose $\alpha_1 \in L^4$, then

- Prove β_N is biased.
- (optional) Prove that β_N is consistent in L^2 .
- Can you propose an unbiased estimator?

$$\text{Ex 3. (i)} \quad \mathbb{E}[\bar{\alpha}_N] = \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N \alpha_i\right] = \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\alpha_i] = \frac{1}{N} \cdot N \mathbb{E}[\alpha_1] = \mathbb{E}[\alpha_1]$$

$$\text{Var}[\bar{\alpha}_N] = \text{Var}\left[\frac{1}{N} \sum_{i=1}^N \alpha_i\right] = \frac{1}{N^2} \sum_{i=1}^N \text{Var}(\alpha_i) = \frac{1}{N^2} \cdot N \text{Var}(\alpha_1) = \frac{1}{N} \text{Var}(\alpha_1)$$

$$\mathbb{E}[\beta_N] - \text{Var}(\alpha_1)$$

$$= \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N (\alpha_i - \bar{\alpha}_N)^2\right] - \text{Var}(\alpha_1)$$

$$= \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\alpha_i^2 - 2\alpha_i \bar{\alpha}_N + \bar{\alpha}_N^2] - \text{Var}(\alpha_1)$$

$$= \frac{1}{N} \left(\sum_{i=1}^N \mathbb{E}[\alpha_i^2] - 2 \sum_{i=1}^N \mathbb{E}[\alpha_i \bar{\alpha}_N] + \sum_{i=1}^N \mathbb{E}[\bar{\alpha}_N^2] \right) - \text{Var}(\alpha_1) \quad (*)$$

$$\mathbb{E}[\alpha_i^2] = \text{Var}(\alpha_1) + \mathbb{E}[\alpha_1]^2, \quad \mathbb{E}[\bar{\alpha}_N^2] = \text{Var}(\bar{\alpha}_N) + \mathbb{E}[\bar{\alpha}_N]^2 = \frac{1}{N} \text{Var}(\alpha_1) + \mathbb{E}[\alpha_1]^2.$$

$$(*) = \frac{1}{N} \sum_{i=1}^N \left(\frac{N+1}{N} \text{Var}(\alpha_1) + 2 \mathbb{E}[\alpha_1]^2 \right) - \frac{2}{N} \mathbb{E}\left[\sum_{i=1}^N \alpha_i \bar{\alpha}_N\right] - \text{Var}(\alpha_1)$$

$$= \frac{1}{N} \text{Var}(\alpha_1) + 2 \mathbb{E}[\alpha_1]^2 - \frac{2}{N} \mathbb{E}[N \cdot \bar{\alpha}_N^2]$$

$$= \frac{1}{N} \text{Var}(\alpha_1) + 2 \mathbb{E}[\alpha_1]^2 - 2 \left(\text{Var}(\bar{\alpha}_N) + \mathbb{E}[\bar{\alpha}_N]^2 \right)$$

$$= \frac{1}{N} \text{Var}(\alpha_1) + 2 \mathbb{E}[\alpha_1]^2 - \frac{2}{N} \text{Var}(\alpha_1) - 2 \mathbb{E}[\alpha_1]^2$$

$$= -\frac{1}{N} \text{Var}(\alpha_1) \neq 0$$

So β_N is biased.

(iii) From (i), we have $E[\beta_N] - \text{Var}(\alpha_1) = -\frac{1}{N} \text{Var}(\alpha_1)$, $E[\beta_N] = \frac{N-1}{N} \text{Var}(\alpha_1)$.

$$\begin{aligned}\text{Let } \beta_N^* &= \frac{N}{N-1} \beta_N, \text{ so } E[\beta_N^*] - \text{Var}(\alpha_1) \\ &= \frac{N}{N-1} E[\beta_N] - \text{Var}(\alpha_1) \\ &= \text{Var}(\alpha_1) - \text{Var}(\alpha_1) \\ &= 0\end{aligned}$$

So β_N^* is an unbiased estimator.

Example 4 • Use β_{100} of Example 3 to estimate $MSE(\hat{\pi}_N)$ by repeating π_N of Example 1. One must write both pseudocode and python code.

• Repeat above estimation of $MSE(\hat{\pi}_N)$ for $N = 2^i : i = 5, \dots, 10$ and plot log-log chart.

Ex 4. Estimate $MSE(\hat{\pi}_N)$

```
1: produce  $MSE_{pi}(n, N)$ 
2:    $pi\_list = []$ 
3:    $sum\_pi\_list \leftarrow 0$ 
4:    $pi\_bar \leftarrow 0$ 
5:   for  $i = 1, 2, \dots, n$  do:
6:      $p = mcpi(N)$ 
7:      $pi\_list.append(p)$ 
8:      $sum\_pi\_list \leftarrow sum\_pi\_list + p$ 
9:    $pi\_bar \leftarrow sum\_pi\_list / n$ 
10:   $M \leftarrow 0$ 
11:  for  $i = 1, 2, \dots, N$  do:
12:     $M \leftarrow M + (pi\_list[i] - pi\_bar)^2$ 
13:  return  $\frac{M}{N}$ .
```

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

Back to our Example 1, we write

$$\alpha = \mathbb{E}[X] = \mathbb{E}[h(Y)],$$

where $X = h(Y)$ and $Y \sim U(0,1)$. In other words, although X -sampling is not directly available in python, one can use $U(0,1)$ random generator (see `numpy.random.uniform`) to produce Y_i , then compute $h(Y_i)$ for the sample X_i .

Algorithm 1 Integral by MC - Example 1

1: procedure MCINTEGRAL(N)	▷ N is total number of samples
2: $s \leftarrow 0$	▷ s is the sum of samples
3: for $i = 1 \dots N$ do	
4: generate two numbers Y from $U(0,1)$	▷ use <code>numpy.random.uniform</code>
5: $s \leftarrow s + h(Y)$	
6: return $\frac{s}{N}$	▷ return the average

Example 2 • Implement Algo 1 for estimator $\hat{\alpha}_N$;

- Estimate $MSE(\hat{\alpha}_N)$ for $N = 2^5 \dots, 2^{10}$ and plot log-log chart.

Ex 2

Implement Algo 1

1: produce MCINTEGRAL(N)

2: $S \leftarrow 0$

3: **for** $i = 1, 2, \dots, N$ **do**:

4: generate number Y from $U(0,1)$

5: **if** $Y \leq 0.01$:

6: $S = S + 100$

7: **else**:

8: $S = S + 1$

9: **return** $\frac{S}{N}$