1. Pf. $-\varepsilon u''(x) + u = -\varepsilon \cdot -\frac{1}{\varepsilon} \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1-e^{-\frac{2}{\sqrt{\varepsilon}}}} + x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1-e^{-\frac{2}{\sqrt{\varepsilon}}}}$

$= x$

So $u(x) = x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1-e^{-\frac{2}{\sqrt{\varepsilon}}}}$ is a solution

characteristic equation is $-\varepsilon r^2 + 1 = 0$, $r = \pm \frac{1}{\sqrt{\varepsilon}}$.

So general solution is $u(x) = C_1 e^{\frac{1}{\sqrt{\varepsilon}}x} + C_2 e^{-\frac{1}{\sqrt{\varepsilon}}x}$

$\because -\varepsilon u'' + u = x$ $\therefore u = x$ is a sol.

$\therefore u(x)$ can be expressed as $C_1 e^{\frac{1}{\sqrt{\varepsilon}}x} + C_2 e^{-\frac{1}{\sqrt{\varepsilon}}x} + x$.

$u(0) = C_1 + C_2 = 0$, $u(1) = C_1 e^{\frac{1}{\sqrt{\varepsilon}}} + C_2 e^{-\frac{1}{\sqrt{\varepsilon}}} + 1 = 0$

we get unique C_1, C_2. So the sol. is unique.

$\Rightarrow u(x) = x - \frac{e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}}{1-e^{-\frac{2}{\sqrt{\varepsilon}}}}$ is a uni. sol.

2. $-\varepsilon \delta_h \delta_{-h} u_i^h + u_i^h = f_i$,

$-\varepsilon \frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{2} + u_i^h = f_i$.

$-\frac{\varepsilon}{h^2} u_{i+1}^h + (1 + \frac{2\varepsilon}{h^2}) u_i^h - \frac{\varepsilon}{h^2} u_{i-1}^h = f_i$.

with $r = \frac{\varepsilon}{h^2}$, $s = 1 + \frac{2\varepsilon}{h^2}$, we have

$$\begin{cases} u_0^h = 0 \\ -r u_{i+1}^h + s u_i^h - r u_{i-1}^h = f_i \\ u_N^h = 0. \end{cases}$$

So $L^h = \begin{pmatrix} 1 & 0 & 0 & & & \\ -r & s & -r & & & \\ & -r & s & -r & & \\ & & & \ddots & & \\ & & & -r & s & -r \\ & & & 0 & 0 & 1 \end{pmatrix}$

3. $L^h R^h v = L^h (V_0, V_1, \cdots, V_N)^T$,

① $(L^h R^h v)_0 = V_0 = 0$ ②. $(R^h L v)_0 = Lv(x_0) = 0$

③ $|(L^h R^h v)_0 - (R^h L v)_0| = 0$. then we have $|(L^h R^h v)_N - (R^h L v)_N| = 0$.

For $i \in [1, N-1]$, $(L^h R^h v)_i = -\varepsilon \delta_h \delta_{-h} v_i + v_i$

$(R^h L v)_i = L v(x_i) = -\varepsilon v''(x_i) + v(x_i)$.

$|(L^h R^h v)_i - (R^h L v)_i| = O(h^2)$.

So L^h is consistent of $\alpha = 2$.

For any $V \in R^{N+1}$, if $|V_0| = \|V\|_\infty$, then

$$\|L^h V\|_\infty \geq |(L^h V)_0| = |V_0| = \|V\|_\infty$$

if $|V_N| = \|V\|_\infty$, then

$$\|L^h V\|_\infty \geq |(L^h V)_N| = |V_N| = \|V\|_\infty.$$

if $V_i = \|V\|_\infty$, for some $1 \leq i \leq N-1$,

$$(L^h V)_i = -r V_{i+1} + s V_i - r V_{i-1}$$

$$= -r(V_{i+1} - V_i) - r(V_{i-1} - V_i) + (s - 2r) V_i.$$

$r > 0$, $s > 0$, $s - 2r = 1$, we have $(L^h V)_i \geq V_i$,

So $\|L^h V\|_\infty \geq |(L^h V)_i| \geq |V_i| = \|V\|_\infty$.

if $-V_i = \|V\|_\infty$, $(L^h V)_i = -r V_{i-1} + s V_i - t V_{i+1}$

$$= -r(V_{i-1} - V_i) + t(V_{i+1} - V_i) + 2V_i$$

$$\leq 2V_i \leq 0.$$

$\|L^h V\|_\infty \geq |(L^h V)_i| \geq 2|V_i| = 2\|V\|_\infty$.

$\Rightarrow \|V\|_\infty \leq \|L^h V\|_\infty$, L^h is stable.

4. Comparing FEM error with CFD error, we can find CFD is better.