1. $u(x+h) = \sum_n \frac{h^n}{n!} u^n(x)$, $u(x+2h) = \sum_n \frac{(2h)^n}{n!} u^n(x)$

$$u(x+h) = u(x) + hu'(x) + \tfrac{1}{2}h^2u''(x) + \tfrac{1}{6}h^3u'''(x) + \tfrac{1}{24}h^4u^4(x) + O(h^5) \quad ①$$

$$u(x-h) = u(x) - hu'(x) + \tfrac{1}{2}h^2u''(x) - \tfrac{1}{6}h^3u'''(x) + \tfrac{1}{24}h^4u^4(x) + O(h^5) \quad ②$$

$$u(x+2h) = u(x) + 2hu'(x) + 2h^2u''(x) + \tfrac{4}{3}h^3u'''(x) + \tfrac{2}{3}h^4u^4(x) + O(h^5) \quad ③$$

$$u(x-2h) = u(x) - 2hu'(x) + 2h^2u''(x) - \tfrac{4}{3}h^3u'''(x) + \tfrac{2}{3}h^4u^4(x) + O(h^5) \quad ④.$$

$$① - ② \Rightarrow u(x+h) - u(x-h) = 2hu'(x) + \tfrac{1}{3}h^3u'''(x) + O(h^5) \quad ⑤$$

$$③ - ④ \Rightarrow u(x+2h) - u(x-2h) = 4hu'(x) + \tfrac{8}{3}h^3u'''(x) + O(h^5) \quad ⑥.$$

$$⑤ \times 8 - ⑥ \Rightarrow 8u(x+h) - 8u(x-h) - u(x+2h) + u(x-2h) = 12hu'(x) + O(h^5).$$

$$u'(x) = \frac{8u(x+h) - 8u(x-h) - u(x+2h) + u(x-2h)}{12h} + O(h^4).$$

2. (1) $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0-h)}{2h} = \lim_{h\to 0} \frac{0}{2h} = 0.$$

(2) $$f''(x) \simeq \delta_h f'(x) = \frac{f'(x+h) - f'(x)}{h}$$

$$\delta_{-h} f(x) = \frac{f(x) - f(x-h)}{h}.$$

$$f''(x) \simeq \delta_h\delta_{-h} f(x) = \frac{\frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-h)}{h}}{h} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

$$f''(0) = a_h = \frac{f(h) + f(-h)}{h^2} = \frac{2f(h)}{h^2}$$

$$\left|\frac{2f(h)}{h^2} - f''(0)\right| = \left|\frac{f(h) + f(h)}{h^2} - f''(0)\right| = \left|\frac{f(h) + f(-h)}{h^2} - f''(0)\right|$$

$$= \left|\frac{f(0) + \cancel{f'(0)h} + f''(0)\frac{h^2}{2} + \cancel{f'''(0)\frac{h^3}{6}} + O(h^4) + f(0) - \cancel{f'(0)h} + f''(0)\frac{h^2}{2} - \cancel{f'''(0)\frac{h^3}{6}} + O(h^4)}{h^2} - f''(0)\right|$$

$$= \left|\frac{2f(0) + f''(0)h^2 + O(h^4)}{h^2} - f''(0)\right|$$

$$= \frac{O(h^4)}{h^2} = O(h^2)$$

(3) $$f(h) = f(0+h) = f(0) + f'(0)h + \tfrac{1}{2}f''(0)h^2 + \tfrac{h^3}{6}f'''(0) + \tfrac{h^4}{24}f^4(0) + \tfrac{h^5}{120}f^5(0) + O(h^6) \quad ①$$

$$f(-h) = f(0-h) = f(0) - f'(0)h + \tfrac{1}{2}f''(0)h^2 - \tfrac{h^3}{6}f'''(0) + \tfrac{h^4}{24}f^4(0)$$

$$-\frac{h^5}{120}f^5(0) + O(h^6) \quad ②$$

$$f(2h) = f(0+2h) = f(0) + 2f'(0)h + 2f''(0)h^2 + \frac{4}{3}f'''(0)h^3$$
$$+\frac{1}{3}f^4(0)h^4 + \frac{32}{120}f^5(0)h^5 + O(h^6) \quad ③$$

$$f(-2h) = f(0-2h) = f(0) - 2f'(0)h + 2f''(0)h^2 - \frac{4}{3}f'''(0)h^3$$
$$+\frac{1}{3}f^4(0)h^4 - \frac{32}{120}f^5(0)h^5 + O(h^6) \quad ④$$

$$① + ② \Rightarrow f(h) + f(-h) = 2f(h) = f''(0)h^2 + \frac{h^4}{12}f^4(0) + O(h^6) \quad ⑤$$

$$③ + ④ \Rightarrow f(2h) + f(-2h) = 2f(2h) = 4f''(0)h^2 + \frac{2}{3}f^4(0)h^4 + O(h^6) \quad ⑥$$

$$⑤ \times 8 - ⑥ \Rightarrow 16f(h) - 2f(2h) = 4f''(0)h^2 + O(h^6).$$

$$f''(0) = \frac{8f(h) - f(2h) - O(h^6)}{2h^2}$$

$$= \frac{4f(h) - \frac{1}{2}f(2h)}{h^2} - O(h^4)$$

$$\therefore C_1 = 4,\ C_2 = -\frac{1}{2}$$

(4) $$f''(x) = \lim_{h\to 0} \delta_h \delta_{-h} f(x) = \lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$x = 0,\ f''(x) = \lim_{h\to 0} \frac{f(h) - 0 + f(-h)}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0.$$