Consider 2-d PDE

$$\frac{1}{2}\Delta v(x) - v(x) + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2} = 0,\ x \in O = (0,1)^2$$

with its boundary data

$$v(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2,\ x \notin O.$$

- Show that exact solution is

$$v(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2.$$

- Identify γ, ℓ^h, p^h in the CFD solution given by

$$v(x) = \gamma\Big\{\ell^h(x) + \sum_{i=1}^{d} p^h(x + he_i|x)v(x + he_i) + p^h(x - he_i|x)v(x - he_i)\Big\}.$$

- For $\{h = 2^{-i}, i = 2, 3, 4, 5\}$, compute CFD solution and find maxnorm of error.

- $V(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2, \ x \notin O$

$\partial_i V(x) = 2x_1 - 1 + 2x_2 - 1, \ \partial_{ii} V(x) = 2 + 2 = 4$

$\frac{1}{2}\Delta V(x) = \frac{1}{2}\sum_{i=1}^{2} \partial_{ii} V(x) = \frac{1}{2} \cdot 4 = 2$

$LHS = \frac{1}{2}\Delta V(x) - V(x) + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2} = 0$

$= 2 - (x_1 - \frac{1}{2})^2 - (x_2 - \frac{1}{2})^2 + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2}$

$= 2 - x_1^2 + x_1 - \frac{1}{4} - x_2^2 + x_2 - \frac{1}{4} + x_1^2 + x_2^2 - x_1 - x_2$

$= 0 = RHS$.

So $V(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2, \ x \notin O$ is exact solution.

- $\frac{1}{2}\Delta V(x) - V(x) + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2} = 0$

$$\begin{cases} \partial_i V(x) = [V(x+he_i) - V(x-he_i)]/2h \\ \partial_{ii} V(x) = [V(x+he_i) - 2V(x) + V(x-he_i)]/h^2 \\ V_i^+ = V(x+he_i), \ V_i^- = V(x-he_i) \end{cases}$$

we have $\frac{1}{2}\sum_{i=1}^{2} \partial_{ii} V(x) - V(x) + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2} = 0$

$$\frac{1}{2}\sum_{i=1}^{2}\frac{V_i^+ - 2V + V_i^-}{h^2} - V + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2} = 0$$

$$\left(\frac{2}{h^2} + 1\right)V = \sum_{i=1}^{2}\left(\frac{V_i^+}{2h^2} + \frac{V_i^-}{2h^2}\right) + x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2}$$

$$\left(\frac{2+h^2}{2}\right)V = \sum_{i=1}^{2}\left(\frac{V_i^+}{4} + \frac{V_i^-}{4}\right) + \frac{h^2}{2}\left(x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2}\right)$$

$$V = \frac{2}{2+h^2}\left\{\frac{h^2}{2}\left(x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2}\right) + \sum_{i=1}^{2}\left(\frac{V_i^+}{4} + \frac{V_i^-}{4}\right)\right. \quad (*)$$

Compared (*) with $v(x) = r\{ l^h(x) + \sum_{i=1}^{d} p^h(x+he_i|x)v(x+he_i) + p^h(x-he_i|x)v(x-he_i)$

we get $r = \frac{2}{2+h^2}$, $l^h = \frac{h^2}{2}(x_1^2 + x_2^2 - x_1 - x_2 - \frac{3}{2})$, $p^h = \frac{1}{4}$

- Showed in code.