5. Proof

Assume there exists a point $\hat{\sigma}_1$, s.t. $f(\hat{\sigma}_1) = P$. $\hat{\sigma}_1 \neq \hat{\sigma}$.

Assume $\hat{\sigma}_1 > \hat{\sigma}$,

$P = f(\hat{\sigma}_1) < f(\hat{\sigma}) = P$ because of strictly increase. contradiction.

So there exists unique $\hat{\sigma}$, s.t. $f(\hat{\sigma}) = P$.

Since $|f(\sigma) - P| \geq 0$, $\min\limits_{\sigma \in (0,\infty)} |f(\sigma) - P| \geq 0$.

$\hat{\sigma} = \arg\min\limits_{\sigma \in (0,\infty)} |f(\sigma) - P|$ because of the uniqueness.

Q1. f_{min} and f_{max}

$S_0 = 100$, $r = 0.0475$, $K = 110$, $T = 1$.

according to the BSM put price formula:

$$P_0 = f(\sigma) = -S_0 \phi(-d_1) + Ke^{-rT} \phi(-d_2).$$

$$\text{where } d_1 = \frac{(r + \frac{1}{2}\sigma^2)T + \ln\frac{S_0}{K}}{\sigma\sqrt{T}}$$

$$= \frac{(0.0475 + \frac{1}{2}\sigma^2) + \ln\frac{100}{110}}{\sigma}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{(0.0475 - \frac{1}{2}\sigma^2) + \ln\frac{100}{110}}{\sigma}$$

$\lim\limits_{\sigma \to 0} d_1 = -\infty$, $\lim\limits_{\sigma \to 0} d_2 = -\infty$.

$$\text{So } P = f(0) = -100 + 110e^{-0.0475} = 4.897 = f_{min}.$$

$\lim\limits_{\sigma \to \infty} d_1 = +\infty$, $\lim\limits_{\sigma \to \infty} d_2 = -\infty$.

$$\text{So } P = \lim_{\sigma \to \infty} f(\sigma) = 0 + 110e^{-0.0475} = 104.897 = f_{max}$$

Q2. f is strictly increasing on $(0, \infty)$.

$$\frac{\partial P_0}{\partial \sigma} = -S_0 \frac{\partial \phi(-d_1)}{\partial \sigma} + Ke^{-rT} \frac{\partial \phi(-d_2)}{\partial \sigma}$$

$$= -S_0 \frac{\partial \phi(-d_1)}{\partial(-d_1)} \frac{\partial(-d_1)}{\partial \sigma} + Ke^{-rT} \frac{\partial \phi(-d_2)}{\partial(-d_2)} \frac{\partial(-d_2)}{\partial \sigma}$$

$$= S_0 \frac{\partial \phi(d_1)}{\partial(d_1)} \frac{\partial(d_1)}{\partial \sigma} - ke^{-rT} \frac{\partial \phi(d_2)}{\partial(d_2)} \frac{\partial(d_2)}{\partial \sigma}$$

$$\frac{\partial P}{\partial \sigma} = S_t \sqrt{T-t} \frac{\partial N(d_1)}{\partial d_1} > 0.$$

So f is strictly increasing on $(0, \infty)$