Example 3 *Given i.i.d* $\{\alpha_i : i \in 1, 2, \ldots, N\}$, *we use*

$$\bar{\alpha}_N = \frac{1}{N}\sum_{i=1}^{N}\alpha_i$$

as its estimator of the mean $\mathbb{E}[\alpha_1]$ *and use*

$$\beta_N = \frac{1}{N}\sum_{i=1}^{N}(\alpha_i - \bar{\alpha}_N)^2$$

as the estimator of $Var(\alpha_1)$. *Suppose* $\alpha_1 \in L^4$, *then*

- *Prove* β_N *is biased.*
- *(optional) Prove that* β_N *is consistent in* L^2.
- *Can you propose an unbiased estimator?*

Ex 3. (i) $\mathbb{E}[\bar{\alpha}_N] = \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N}\alpha_i] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[\alpha_i] = \frac{1}{N}\cdot N\mathbb{E}[\alpha_1] = \mathbb{E}[\alpha_1]$

$Var[\bar{\alpha}_N] = Var[\frac{1}{N}\sum_{i=1}^{N}\alpha_i] = \frac{1}{N^2}\sum_{i=1}^{N}Var(\alpha_i) = \frac{1}{N^2}\cdot N\,Var(\alpha_1] = \frac{1}{N}Var(\alpha_1)$

$\mathbb{E}[\beta_N] - Var(\alpha_1)$

$= \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N}(\alpha_i - \bar{\alpha}_N)^2] - Var(\alpha_1)$

$= \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[\alpha_i^2 - 2\alpha_i\bar{\alpha}_N + \bar{\alpha}_N^2] - Var(\alpha_1)$

$= \frac{1}{N}\left(\sum_{i=1}^{N}\mathbb{E}[\alpha_i^2] - 2\sum_{i=1}^{N}\mathbb{E}[\alpha_i\bar{\alpha}_N] + \sum_{i=1}^{N}\mathbb{E}[\bar{\alpha}_N^2]\right) - Var(\alpha_1)$ (*).

$\mathbb{E}[\alpha_i^2] = Var\,\alpha_1 + \mathbb{E}[\alpha_1]^2$, $\mathbb{E}[\bar{\alpha}_N^2] = Var\,\bar{\alpha}_N + \mathbb{E}[\bar{\alpha}_N]^2 = \frac{1}{N}Var(\alpha_1) + \mathbb{E}[\alpha_1]^2$.

(*) $= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{N+1}{N}Var\,\alpha_1 + 2\mathbb{E}[\alpha_1]^2\right) - \frac{2}{N}\mathbb{E}[\sum_{i=1}^{N}\alpha_i\bar{\alpha}_N] - Var(\alpha_1)$

$= \frac{1}{N}Var\,\alpha_1 + 2\mathbb{E}[\alpha_1]^2 - \frac{2}{N}\mathbb{E}[N\cdot\bar{\alpha}_N^2]$

$= \frac{1}{N}Var\,\alpha_1 + 2\mathbb{E}[\alpha_1]^2 - 2(Var(\bar{\alpha}_N) + \mathbb{E}[\bar{\alpha}_N]^2]$

$= \frac{1}{N}Var\,\alpha_1 + 2\mathbb{E}[\alpha_1]^2 - \frac{2}{N}Var(\alpha_1) - 2\mathbb{E}[\alpha_1]^2$

$= -\frac{1}{N}Var\,\alpha_1 \neq 0$

So β_N is biased.

(iii) From (i), we have $E[\beta_N] - Var(\alpha_1) = -\frac{1}{N} Var\alpha_1$, $E[\beta_N] = \frac{N-1}{N} Var\alpha_1$.

Let $\beta_N^* = \frac{N}{N-1}\beta_N$, so

$$\begin{aligned} E[\beta_N^*] - Var\alpha_1 &= \frac{N}{N-1} E[\beta_N] - Var\alpha_1 \\ &= Var\alpha_1 - Var\alpha_1 \\ &= 0 \end{aligned}$$

So β_N^* is an unbiased estimator.

Example 4 • *Use β_{100} of Example 3 to estimate $MSE(\hat{\pi}_N)$ by repeating π_N of Example 1. One must write both pseudocode and python code.*

• *Repeat above estimation of $MSE(\hat{\pi}_N)$ for $N = 2^i : i = 5, ...10$ and plot log-log chart.*

Ex4. Estimate $MSE(\hat{\pi}_N)$

```
procedure MSEpi(n, N)
    pi_list = []
    sum_pi_list ← 0
    pi_bar ← 0
    for i = 1, 2, ..., n do:
        p = mcpi(N)
        pi_list.append(p)
        sum_pi_list ← sum_pi_list + p.
    pi_bar ← sum_pi_list / n
    M ← 0
    for i = 1, 2, ..., N do:
        M ← M + (pi_list[i] - pi_bar)^2
    return M/N.
```

$$h(x) = 100 \cdot I_{(0,1/100]}(x) + 1 \cdot I_{(1/100,1)}(x).$$

Back to our Example 1, we write

$$\alpha = \mathbb{E}[X] = \mathbb{E}[h(Y)],$$

where $X = h(Y)$ and $Y \sim U(0,1)$. In other words, although X-sampling is not directly available in python, one can use $U(0,1)$ random generator (see *numpy.random.uniform*) to produce Y_i, then compute $h(Y_i)$ for the sample X_i.

Algorithm 1 Integral by MC - Example 1

procedure MCINTEGRAL(N) ▷ N is total number of samples
    $s \leftarrow 0$ ▷ s is the sum of samples
    for $i = 1...N$ **do**
        generate two numbers Y from $U(0,1)$ ▷ use *numpy.random.uniform*
        $s \leftarrow s + h(Y)$
    return $\frac{s}{N}$ ▷ return the average

Example 2 • *Implement Algo 1 for estimator* $\hat{\alpha}_N$*;*

- *Estimate* $MSE(\hat{\alpha}_N)$ *for* $N = 2^5 \ldots, 2^{10}$ *and plot log-log chart.*

Ex2.

Implement Algo 1

procedure MCINTEGRAL (N)
    $S \leftarrow 0$
    for $i = 1, 2, \cdots, N$ do:
        generate number Y from $U(0,1)$
        if $Y <= 0.01$:
            $S = S + 100$
        else:
            $S = S + 1$
    return $\frac{S}{N}$