5. Proof

Assume there exists a point $\hat{\sigma}_1$, s.t. $f(\hat{\sigma}_1) = P$. $\hat{\sigma}_1 \neq \hat{\sigma}$.

Assume $\hat{\sigma}_1 > \hat{\sigma}$,

$P = f(\hat{\sigma}_1) < f(\hat{\sigma}) = P$ because of strictly increase. contradiction.

So there exists unique $\hat{\sigma}$, s.t. $f(\hat{\sigma}) = P$.

Since $|f(\sigma) - P| \geq 0$, $\min_{\sigma \in (0,\infty)} |f(\sigma) - P| \geq 0$.

$\hat{\sigma} = \arg\min_{\sigma \in (0,\infty)} |f(\sigma) - P|$ because of the uniqueness.

Q1. f_{min} and f_{max}

$S_0 = 100$, $r = 0.0475$, $K = 110$, $T = 1$.

according to the BSM put price formula:

$$P_0 = f(\sigma) = -S_0 \phi(-d_1) + Ke^{-rT} \phi(-d_2).$$

$$\text{where } d_1 = \frac{(r + \frac{1}{2}\sigma^2)T + \ln\frac{S_0}{K}}{\sigma\sqrt{T}}$$

$$= \frac{(0.0475 + \frac{1}{2}\sigma^2) + \ln\frac{100}{110}}{\sigma}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{(0.0475 - \frac{1}{2}\sigma^2) + \ln\frac{100}{110}}{\sigma}$$

$$P = f(0) = -100 + 110e^{-0.0475} = 4.897 = f_{min}.$$

$$\text{So } P = \lim_{\sigma \to \infty} f(\sigma) = 0 + 110e^{-0.0475} = 104.897 = f_{max}$$