Algorithmic Methods for Mathematical Models

– COURSE PROJECT –

We are in charge of designing a scientific experiment for finding the period of a sinusoidal sound wave. According to the theory of acoustics, it is known that the period must be a multiple of T seconds. In order to gauge its precise value, a sensor is located close to the source of the wave, to avoid interferences. This sensor can be activated at a concrete instant of time to measure the sound pressure at that instant. Then the period of the sound wave can be found as follows.

For example, let us assume that the sensor is activated at times 0, T, $2T$, $3T$ and $4T$, and that the respective values of the pressure are p_0, p_1, p_2, p_3 and p_4. Then:

- If the period is T, then $p_0 = p_1 = p_2 = p_3 = p_4$.
- If the period is $2T$, then $p_0 = p_2 = p_4$ and $p_1 = p_3$.
- If the period is $3T$, then $p_0 = p_3$, $p_1 = p_4$.
- If the period is $4T$, then $p_0 = p_4$.
- If the period is longer than $4T$, then it is not possible to infer its value.

However, it is possible to cover longer periods with the same number of measurements if the instants are not equidistant. For example, let us assume that the sensor is activated at times 0, T, $4T$, $9T$ and $11T$ (again, 5 measurements), and that the respective values of the pressure are p_0, p_1, p_4, p_9 and p_{11}. Then:

- If the period is T, then $p_0 = p_1$.
- If the period is $2T$, then $p_9 = p_{11}$.
- If the period is $3T$, then $p_1 = p_4$.
- If the period is $4T$, then $p_0 = p_4$.
- If the period is $5T$, then $p_4 = p_9$.
- If the period is $7T$, then $p_4 = p_{11}$.
- If the period is $8T$, then $p_1 = p_9$.
- If the period is $9T$, then $p_0 = p_9$.
- If the period is $10T$, then $p_1 = p_{11}$.
- If the period is $11T$, then $p_0 = p_{11}$.
- If the period is $6T$ or longer than $11T$, then it is not possible to infer its value.

So with this experiment one can infer periods up to $11T$, unless the period is $6T$.

This design makes an efficient use of the measurements because no two different pairs of numbers in the set $\{0, 1, 4, 9, 11\}$ are the same distance apart. Another set of 5 numbers with the same property is $\{0, 1, 3, 7, 12\}$. It is not difficult to see that it can cover periods up to $12T$, unless the period is $8T$ or $10T$. Note that in this case there are 2 periods ($8T$ and $10T$) that would be missed. Clearly, the larger the difference between the maximum and the minimum numbers in the set, the more periods will be missed. Therefore the goal is to find, for a given number n, a set of n different natural numbers $t_0, t_1, \ldots, t_{n-1}$ such that no two different pairs of numbers in the set are the same distance apart, and such that the difference between the maximum and the minimum numbers in the set is minimized. For example, it can be proved that the set $\{0, 1, 4, 9, 11\}$ is optimum when $n = 5$. Another optimum set when $n = 5$ is $\{0, 2, 7, 8, 11\}$.

1. **Work to be done:**

   (a) State the problem formally. Specify the inputs and the outputs, as well as any auxiliary sets of indices that you may need, and the objective function.

   (b) Build an integer linear programming model for the problem and implement it in OPL.

   (c) Because of the complexity of the problem, heuristic algorithms can also be applied. Here we will consider the following:

      i. a greedy constructive algorithm,

      ii. a greedy constructive + a local search procedure,

      iii. GRASP as a meta-heuristic algorithm. You can reuse the local search procedure that you developed in the previous step.

   Design the three algorithms and implement them in the programming language you prefer.

   (d) Tuning of parameters and instance generation:

   Given an instance of input to the problem, the value of n is the *size* of the instance.

      i. Tune the α parameter of the GRASP constructive phase with a set of instances of large enough size.

      ii. Generate problem instances with increasingly larger size. Solving each instance with CPLEX should take from 1 to 30 min.

   (e) Compare the performance of CPLEX with the heuristic algorithms, both in terms of computation time and of quality of the solutions as a function of the size of the instances.

   (f) Prepare a report and a presentation of your work on the project.

2. **Report:**

   Prepare a report (8-10 pages) in PDF format including:

   - The formal problem statement.
   - The integer linear programming model, with a definition and a short description of the variables, the objective function and the constraints. Do not include OPL code in the document, but rather their mathematical formulation.
   - For the meta-heuristics, the pseudo-code of your constructive, local search, and GRASP algorithms, including equations for describing the greedy cost function(s) and the RCL.
   - Tables or graphs with the tuning of parameters, and with the comparative results.

   Together with the report, you should give all sources (OPL code, programs of the meta-heuristics) and instructions on how to use them, so that results can be reproduced.

3. **Presentation:**

   You are expected to make a presentation of your work (7-10 minutes long) at the end of the course. The slots of Monday 29/05/23 and Thursday 01/06/23 will be devoted to these presentations. The schedule will be announced in its due time.

   The slides of the presentation in PDF format should be delivered with the report by **25/05/23**.

   The slides can contain figures, plots, equations, algorithms, etc. with a very short text that helps to understand them. It is expected that you give a full explanation of those contents during your presentation. On the other hand, the report should contain that explanation in a well-organized manner as a text.