# Geomagnetism package

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**Abstract**

In this note we derive all the equations used in the geomagntism package and in a companion Jupyter Notebook

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# Contents

# 1 Introduction

The package `geomagnetim` intends to serve pedagogical purposes rather to replace the well established FORTRAN or C progams develloped by the National Oceanic and Atmospheric Administration[1]. In contrast with these programs, which favor compactness and minimize time execution, the geomagnetic package, tentatively, focuses on lisibility. You can download geomagnetis using:

**pip install geomagnetism**

Use case examples are given in a companion Jupyter Notebook .

# 2 Geomagnetism calculation

As the terrestrial magnetic field obeys both $\nabla \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$, it can be shown that the magnetic field can be expressed as the gradient of a scalar potential $V$ which satisfies the Laplace equation:

$$\Delta \mathrm{V} = 0. \tag{2.1}$$

For a spherical geometry the geomagnetic potential is given by the following spherical harmonic expension (SH) [1,2,4]:

$$V(r,\theta,\phi,t) = a \sum_{n=1}^{N} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} \left[g_n^m(t)\cos(m\phi) + h_n^m(t)\sin(m\phi)\right] P_{(s),n}^{m}\left(\cos(\theta)\right) \tag{2.2}$$

where $P_{(s),n}^{m}(x)$ are the Schmidt quasi-normalized associated Legendre polynomials (for more details see section 3):

$$P_{(s),n}^{m}(x) = \begin{cases} \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n & : m = 0 \\ \sqrt{\frac{2(n-m)!}{(n+m)!}}\left(1-x^2\right)^m \frac{1}{2^n n!}\frac{d^{n+m}}{dx^n}\left(x^2-1\right)^n & : m > 0 \end{cases} \tag{2.3}$$

where $g_n^m$ and $h_n^m$ are the Gauss's coefficients. Note that the sum over $n$ begins with the the value $n = 1$ as the index $n = 0$ would correspond to a monopole. The dipole, quadrupole, octupole,... contribution correspond to $n = 1, 2, 3, \ldots$ These coefficients vary with time and are tabulated by the National Oceanic and Atmospheric Administration. The coefficient $a$ is the mean radius of the earth (6371.2 km); $r$ ,the radial distance from the center of the Earth ; $\theta$ , the geocentric colatitude ; $\phi$ , the east longitude measured from Greenwich. We note that the Condon-Shortley phase correction $(-1)^m$ is omitted in the definition of the associated Legendre polynomial and the polynomes are normalized using Schmidt quasi-normalization [5]. The relation $\mathbf{B} = -\nabla V$ leads to:

$$\begin{aligned}
X_c \equiv \\
&\mathbf{B}_x = -B_\theta = \frac{1}{r}\frac{\partial V}{\partial \theta} = \\
&\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m\cos(m\phi) + h_n^m\sin(m\phi)\right]\frac{dP_{(s),n}^{m}(\cos\theta)}{d\theta}, \\
Y_c \equiv \\
&\mathbf{B}_y = B_\phi = \frac{-1}{r\sin\theta}\frac{\partial V}{\partial \phi} = \\
&\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n} m\left[g_n^m\sin(m\phi) - h_n^m\cos(m\phi)\right]\frac{P_{(s),n}^{m}(\cos\theta)}{\sin\theta}, \\
Z_c \equiv \\
&\mathbf{B}_z = -B_r = \frac{\partial V}{\partial r} = \\
&\sum_{n=1}^{N}(n+1)\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m\cos(m\phi) + h_n^m\sin(m\phi)\right]P_{(s),n}^{m}(\cos\theta),
\end{aligned} \tag{2.4}$$

where $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{B}_z$ are the field components respectively in the northward, eastward and downward directions. Theses components are expressed in the geocentric referential as recall by the index $c$. The parameter $N$ stands for the order of the SH decomposition.

[1] In addition, a Python code is available on PyPI. Two inline calculators have been develloped by NOAA and by the International Geomagnetic Reference Field.

# 3 Spherical harmonics normalisation

In the field of geomagnetism the Schmidt quasi-normalized Legendre polynomials $P^m_{(s),n}$ are widely used [2]. They are proportional to the Legendre polynomial:

$$P^m_{(s),n} = N^m_n P^m_n \tag{3.2}$$

where the the associated Legendre polynomials[3] are defined as[4] [6]:

$$P^m_n(x) = \frac{(-1)^m}{2^n n!}\sqrt{(1-x^2)^m}\,\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n. \tag{3.3}$$

and where the normalization coefficients $N^m_n$ are equal to [5]:

$$\boxed{N_{n,m} = \begin{cases} (-1)^m\sqrt{\frac{(2-\delta^0_m)(n-m)!}{(n+m)!}} & : n-|m| \geq 0 \\ 0 & : n-|m| < 0 \end{cases}} \tag{3.4}$$

The Schmidt quasi-normalizated polynomials obey, for $\forall n$, $\forall N$, $\forall m$, $\forall M$, the identities [5]:

$$\begin{aligned} \tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C^m_n(\theta,\phi)C^M_N(\theta,\phi)\sin\theta d\theta d\phi &= \tfrac{1}{2n+1}\delta^N_n\delta^M_m \\ \tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} S^m_n(\theta,\phi)S^M_N(\theta,\phi)\sin\theta d\theta d\phi &= \tfrac{1}{2n+1}\delta^N_n\delta^M_m \\ \tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C^m_n(\theta,\phi)S^M_N(\theta,\phi)\sin\theta d\theta d\phi &= 0 \end{aligned} \tag{3.5}$$

where the following notations are used:

$$\begin{aligned} C^m_n(\theta,\phi) &\equiv P^m_{(s),n}\cos\theta\cos m\theta \quad &: m = 0,1,2\cdots n \\ S^m_n(\theta,\phi) &\equiv P^m_{(s),n}\cos\theta\sin m\theta \quad &: m = 1,2\cdots n \end{aligned} \tag{3.6}$$

For computational efficiency we define the matrice $\underline{\underline{\mathbf{P}}}$ and $\underline{\underline{\mathbf{P}}}_{(s)}$:

$$\underline{\underline{\mathbf{P}}} = \begin{bmatrix} P^0_0 & P^0_1 & \cdots & P^0_N \\ 0 & P^1_1 & \cdots & P^1_N \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P^M_N \end{bmatrix}, \tag{3.7}$$

$$\underline{\underline{\mathbf{P}}}_{(s)} = \begin{bmatrix} P^0_{(s),0} & P^0_{(s),1} & \cdots & P^0_{(s),N} \\ 0 & P^1_{(s),1} & \cdots & P^1_{(s),N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P^M_{(s),N} \end{bmatrix} \tag{3.8}$$

and we compute $\underline{\underline{\mathbf{P}}}_{(s)}$ as the matrix component-wise product:

$$\underline{\underline{\mathbf{P}}}_{(s)} = \underline{\underline{\mathbf{P}}} \odot \underline{\underline{\mathbf{N}}}, \tag{3.9}$$

In the `geomagnetism` package:

(a) The function `Norm_Schimdt(m,n)` computes the normalisation matrix (3.8) using coefficients (3.4).

(b) The function `Norm_Stacey(m,n)` computes the normalisation matrix using coefficients (3.1).

[2]Note that other authors in geophysics use different normalization factors For example, Stacey [4] uses:

$$N^{,m}_n = \begin{cases} (-1)^m\sqrt{(2-\delta^0_m)(2m+1)\frac{(n-m)!}{(n+m)!}} & : |m| \leq n \\ 0 \quad : |m| > n \end{cases} \tag{3.1}$$

.

[3]In the `geomaganetism` package,the associated Legendre polynomials be computeded by the scipy function `lpmn(M,N,x)`

[4]To stick with the `scipy` package conventions, these polynomials are defined using the Condon-Shortley phase correction $(-1)^m$.

```
# Exemple of computation of normalization matrix
geo.Norm_Stacey(3,4)
>> array([[ 1.          ,  1.          ,  1.          ,  1.          ,  1.          ],
       [ 0.          , -1.73205081, -1.          , -0.70710678, -0.54772256],
       [ 0.          ,  0.          ,  0.64549722,  0.28867513,  0.16666667],
       [ 0.          ,  0.          ,  0.          , -0.13944334, -0.05270463]])

geo.Norm_ Schimdt (3,4)
>> array([[ 1.          ,  1.          ,  1.          ,  1.          ,  1.          ],
       [ 0.          , -1.          , -0.57735027, -0.40824829, -0.31622777],
       [ 0.          ,  0.          ,  0.28867513,  0.12909944,  0.0745356 ],
       [ 0.          ,  0.          ,  0.          , -0.05270463, -0.01992048]])
```

# 4 Geotetic to geocentric referentials

The computation of the geomagnetic field is done in a geocentric coordinate system. So if we provide the geotetic coordinates we have to convert them into geocentric ones. In the following we deduce the transformation relation used in the function `geodetic_to_geocentric`.

## 4.1 Ellipse equation

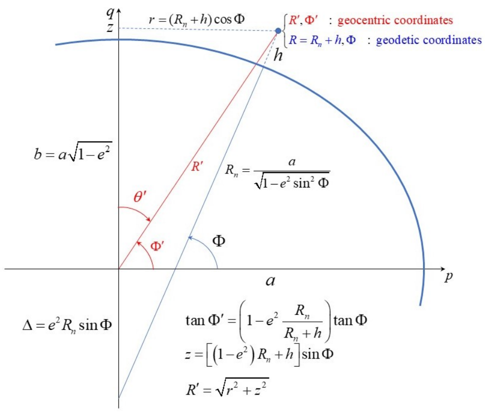



Figure 1: Ellipse notations convention.

Using the notation of the figure 1, the equation of the ellipse reads :

$$\frac{q^2}{b^2} + \frac{p^2}{a^2} = 1. \tag{4.1}$$

The geotetic latitude $\Phi$ can be be expressed through the derivative :

$$\cot\Phi = -\frac{dq}{dp} = \frac{b^2}{a^2}\frac{p}{q} \tag{4.2}$$

From equation (4.2) we can express $q$ as :

$$q = p\frac{b^2}{a^2}\tan\Phi \tag{4.3}$$

Using equations (4.1) and (4.3) we can express the ellipse coordinates $p$ and $q$ as a function of the geotetic latitude $\Phi$ as :

$$p = \frac{a\cos\Phi}{\sqrt{1-e^2\sin^2\Phi}} \tag{4.4a}$$

$$q = \frac{a(1-e^2)\sin\Phi}{\sqrt{1-e^2\sin^2\Phi}} \tag{4.4b}$$

where $e \equiv \sqrt{1 - \frac{b^2}{a^2}}$ is the eccentricity. The prime vertical curvature radius $R_n$ (see figure 1) can be deduced from $p$ as :

$$R_n = \frac{p}{\cos\Phi} = \frac{a}{\sqrt{1 - e^2 \sin^2\Phi}} \tag{4.5a}$$

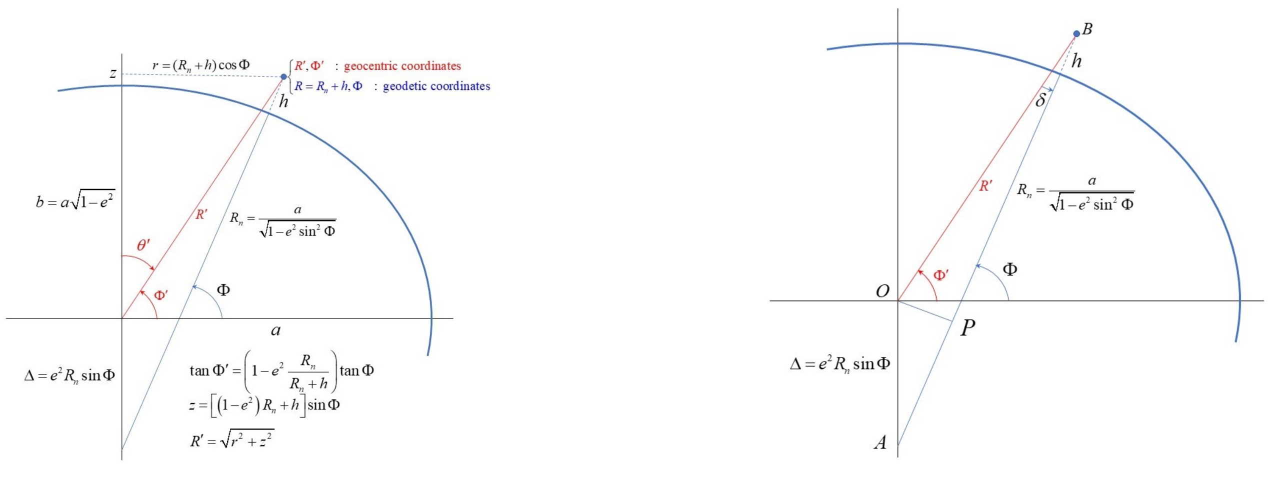
$$R_n = \frac{a^2}{\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi}} \tag{4.5b}$$

Using the prime vertical curvature we can re-express $p$ and $q$ as :

$$p = R_n \cos\Phi \tag{4.6a}$$

$$q = (1 - e^2) R_n \sin\Phi \tag{4.6b}$$

## 4.2 Geotetic to geocentric transformation




(a) Geotetic and geocentric notation

(b) Notations used to compute $\cos\delta$ and $\sin\delta$

Figure 2: Notation conventions.

In this section we derive the relation between the geotetic colatitude $\Phi$ and the geocentric colatitude $\Phi'$ Using the conventions of the figure 2a, we have:

$$\boxed{\frac{\tan\Phi'}{\tan\Phi} = \frac{z}{z + \Delta} = \frac{(1 - e^2)R_n + h}{(1 - e^2)R_n + h + e^2 R_n} = 1 - e^2 \frac{R_n}{R_n + h}} \tag{4.7}$$

The flattening $f$ is defined as follow:

$$f = \frac{a - b}{a} \tag{4.8}$$

Usually, the geodetic reference ellipsoid is specified by its reciprocal flattening $f^{-1}$. The reciprocal flattening is related to the eccentricity $e$ by:

$$\begin{aligned} e &= \sqrt{1 - \tfrac{b^2}{a^2}} = \sqrt{f(2 - f)} \\ 1 - e^2 &= \tfrac{b^2}{a^2} \end{aligned} \tag{4.9}$$

We can express $z$ and $r$ as :

$$\begin{cases} z = (R_n + h)\sin\Phi - \Delta = \left[(1 - e^2)R_n + h\right]\sin\Phi \\ r = (R_n + h)\cos\Phi \end{cases} \tag{4.10}$$

where $h$ is the height above the reference ellipsoid. Thanks to equation (4.10) we can obtain the geocentric radius as :

$$R' = \sqrt{z^2 + r^2} \tag{4.11}$$

Using the equations (4.5b) and (4.10) we express the geocentric as a function both the semi major and the semi minor axis. We obtain [3]:

$$\boxed{R'^2 = \frac{h^2 + 2h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi} + \left[a^4 - (a^4 - b^4)\sin^2\Phi\right]}{a^2 - (a^2 - b^2)\sin^2\Phi}} \tag{4.12}$$

Combining the equations (4.11) and (4.10) we can express the geocentric colatitude as:

$$\cos\theta = \frac{z}{\sqrt{r^2 + z^2}} = \frac{\sin\Phi}{\sqrt{\left[\frac{R_n + h}{(1-e^2)R_n + h}\right]^2 \cos^2\Phi + \sin^2\Phi}} \tag{4.13}$$

The relation (4.13) can be rewritted using the semi major and minor axis. Using equations (4.5b) and (4.13) we obtain [3]:

$$\cos\theta = \frac{\sin\Phi}{\sqrt{c\cos^2\Phi + \sin^2\Phi}} \quad \text{where} \quad c = \left[\frac{a^2 + h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi}}{b^2 + h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi}}\right]^2 \tag{4.14}$$

In the triangle AOB of the figure 2b the length of AB leads to the equality :

$$\Delta\sin\Phi + R'\cos\delta = R_n + h, \tag{4.15}$$

and, after rearanging:

$$\cos\delta = \frac{1}{R'}\left[h + R_n(1 - e^2(\sin\Phi)^2\right]. \tag{4.16}$$

Using the relation (4.5) equation (4.16) reads :

$$\boxed{\cos\delta = \frac{1}{R'}\left[h + \frac{a^2}{R_n}\right]}. \tag{4.17}$$

The length of the common side OP of the two rectangles triangle AOP and BOP of the figure 2b leads to the equality :

$$R'\sin\delta = \Delta\cos\Phi.$$

So:

$$\boxed{\sin\delta = \frac{R_n}{R'}e^2\cos\Phi\sin\Phi} \tag{4.18}$$

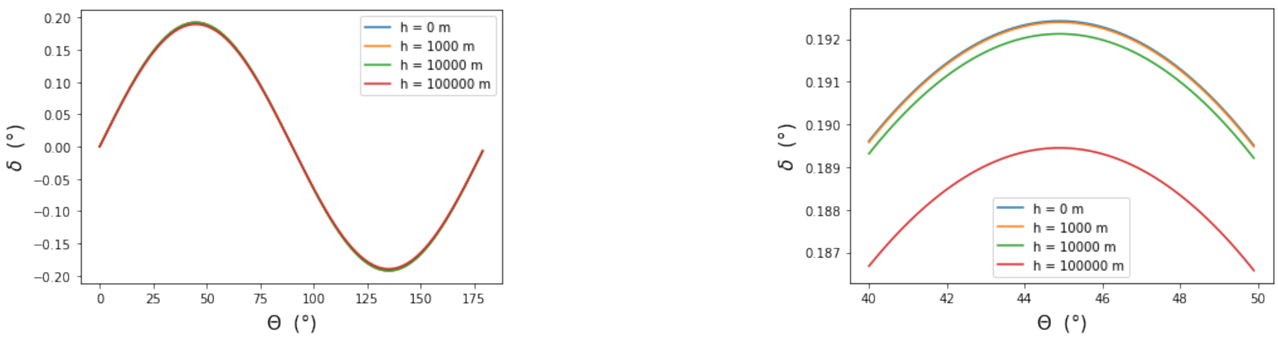
## 4.3 Computational aspect

The function **geodetic_to_geocentric(ellipsoid, co_latitude, height)** computes the geocentric colatitude and radius using respectively the equations (4.7) and (4.11). The angle $\delta = \theta' - \theta$ between the geocentric and the geotetic colattitude is also computed. The figure 3 shows the variation of $\delta$ versus the geotetic colatitude $\theta$ and the height $h$.

```
# Exemple of computation of r_geocentric, co_latitude_geocentric, delta
import geomagnetism as geo
ellipsoid = geo.geomagnetism.WGS84 # tuple (mean earth radius in meter, inverse flattening)
r_geocentric, co_latitude_geocentric, delta = geo.geodetic_to_geocentric(ellipsoid , 170,
                                                 100_000)
>> (6457402.34844737, 2.965925285681976, -0.0011344427083841424)
ellipsoid = geo.geomagnetism.GRS80
r_geocentric, co_latitude_geocentric, delta = geo.geodetic_to_geocentric(ellipsoid , 170,
                                                 100_000)
>> (6457402.348345751, 2.9659252856763882, -0.001134442713972117)
```

```
# Exemple of computation of r_geocentric, cos(co_latitude_geocentric), sin(
                                                 co_latitude_geocentric)
# cos(delta), sin(delta)
import geomagnetism as geo
ellipsoid = geo.geomagnetism.WGS84 # tuple (earth major axis in meter, earth minor axis in
                                                 meter)
ellipsoid = geo.geomagnetism.WGS84_
r, ct, st, cd,sd = geo.geodetic_to_geocentric_IGRF13(ellipsoid,170, 100_000)
```

```
>> (6457402.34844737 -0.9846101254413312 0.17476527366272146 0.9999993565199398 -0.
                                        0011344424650535443)
ellipsoid = geo.geomagnetism.GRS80_
r, ct, st, cd,sd = geo.geodetic_to_geocentric_IGRF13(ellipsoid,170, 100_000)
>> (6457402.348345758 -0.9846101254403548 0.17476527366822295 0.9999993565199334 -0.
                                        0011344424706410008)
```




(a) Variation of $\delta$ for $\theta$ varying from 0° to 180°

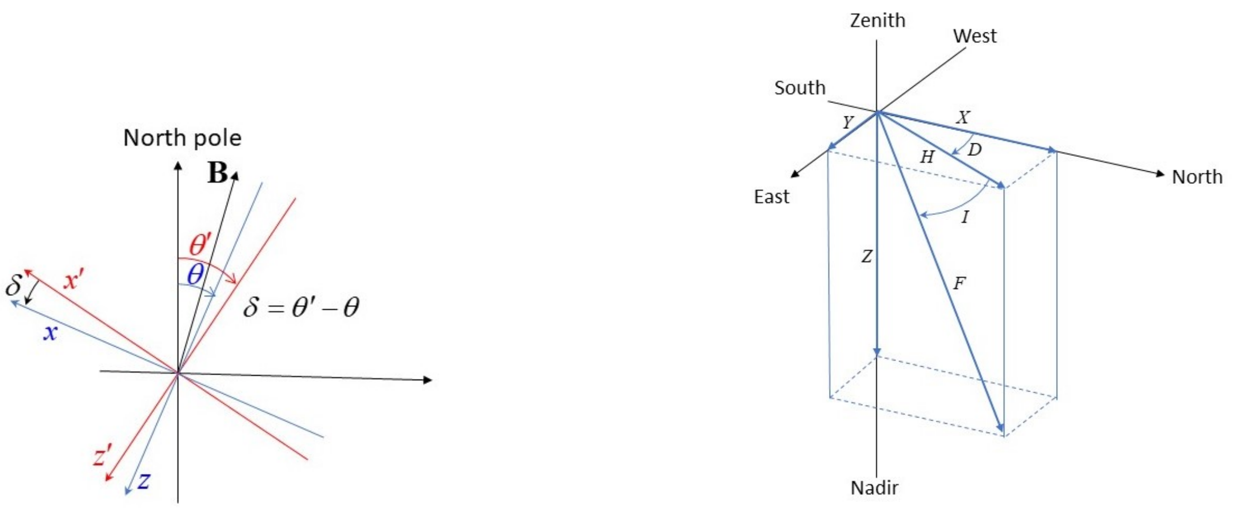
(b) Zoom of the variation of $\delta$ near a maximum.

Figure 3: Variation of $\delta$ versus the geotetic colatitude $\theta$ and the height $h$ above the Earth.

Alternatively, the function `geodetic_to_geocentric_IGRF13(ellipsoid, co_latitude, height)` is a translation of the FORTRAN routine where the authors compute the geocentric radius as well as $\cos\delta$ and $\sin\delta$ using respectivelly the equations (4.12), (4.17), (4.18).

# 5 Base transformation

As geomagnetic field components are computed in a geocentric referential (see section 2), we have to express $B_x$, $B_y$, $B_z$ in a geotetic referential.
Passing from geocentric to geotetic referential the magnetic field undergoes the following transfomation :




(a) Geotetic and geocentric referentials

(b) Field geomagnetic conventions and notations. Credit Chullia [2]

Figure 4: Notation conventions.

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geodetic}} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geocentric}} \tag{5.1}$$

with $\delta = \theta' - \theta = \Phi - \Phi'$ (Figure 4a) where $\cos\delta$ and $\sin\delta$ can be obtained thanks to the equations (4.17) and (4.18). Using Peddie notation [3] we have:

$$\boxed{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} X_c \\ Y_c \\ Z_c \end{pmatrix}} \tag{5.2}$$

Using the notations of the Figure 4b, the geomagnetic horizontal intensity $H$, total intensity $F$, declination $D$ and inclination $I$ can be obtained from:

$$\boxed{\begin{cases} H = \sqrt{X^2 + Y^2} \\ F = \sqrt{H^2 + Y^2} \\ D = \mathrm{atan2}(Y, X) \\ I = \mathrm{atan2}(Z, H) \end{cases}} \tag{5.3}$$

# 6 Geomagnetic field computation at North and South Pole

When the colatitude $\theta$ tends towards 0 (North Pole) or towards $\pi$, the equations (2.3) are numerically instabe. To deal with that problem we have to evaluate $\frac{dP^m_{(s),n}(\cos\theta)}{d\theta}$, $\frac{P^m_{(s),n}(\cos\theta)}{\sin\theta}$, $P^m_{(s),n}(\cos\theta)$ for $\theta = 0$ and for $\theta = \pi$.

**Identity 1.** $P^m_{(s),n}(1) = \delta^0_m$.

**Identity 2.** $P^m_{(s),n}(-1) = (-1)^n \delta^0_m$.

*Proof.* The associate Legendre polynomials can be defined as:

$$P^m_n(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x) \tag{6.1}$$

Using the change of variable $x = \cos\theta$ equation (6.1) reads:

$$P^m_n(\cos\theta) = (-1)^m (\sin\theta)^m \frac{d^m}{d(\cos\theta)^m} P_n(\cos\theta) \tag{6.2}$$

From the equation (6.2) we deduce that for $\theta = 0$ or for $\theta = \pi$, $P^m_n(\cos\theta)$ is not null iff $m = 0$. As $P_n(1) = 1$ [8, p. 752] we deduce the Identity 1. As $P_n(-1) = (-1)^n$ [8, p. 752] we deduce the Identity 2. □

**Identity 3.** $\lim\limits_{\theta\to 0} \frac{P^m_{(s),n}(\cos\theta)}{\sin\theta} = \delta^m_1 \sqrt{\frac{n(n+1)}{2}}$.

**Identity 4.** $\lim\limits_{\theta\to \pi} \frac{P^m_{(s),n}(\cos\theta)}{\sin\theta} = (-1)^n \delta^m_1 \sqrt{\frac{n(n+1)}{2}}$.

*Proof.* From equation (6.2) we deduce that $\frac{P^m_n(\cos\theta)}{\sin\theta}$ is not null iff $m = 1$. This condition leads to:

$$\lim_{\theta\to 0} \frac{P^1_n(\cos\theta)}{\sin\theta} = \frac{d}{dx} P_n(x)\Big|_{x=1} \tag{6.3}$$

As the Legendre polynomials $P_n(x)$ satisfy the differential equation :

$$(1 - x^2) \frac{d^2}{dx^2} P_n(x) - 2x \frac{d}{dx} P_n(x) + n(n+1) P_n(x) = 0, \tag{6.4}$$

and also satisfy the two identities $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ [8, p. 752], we have:

$$\frac{d}{dx} P_n(1) = \frac{n(n+1)}{2} \tag{6.5a}$$

$$\frac{d}{dx} P_n(-1) = (-1)^{n-1} \frac{n(n+1)}{2} \tag{6.5b}$$

Taking into account respectivelly equation (6.5a) and equation (6.5b) in conjonction with the Schmidt normalisation coefficients (3.4) we obtain the Identity 3 and the Identity 4. □

**Identity 5.** $\frac{dP^m_{(s),n}(\cos\theta)}{d\theta}\Big|_{\theta=0} = \delta^m_1 \sqrt{\frac{n(n+1)}{2}}$.

**Identity 6.** $\left.\frac{dP^m_{(s),n}(\cos\theta)}{d\theta}\right|_{\theta=\pi} = (-1)^n\delta^m_1\sqrt{\frac{n(n+1)}{2}}$.

*Proof.* The derivative versus $\theta$ of the equation (6.2)leads to the expression::

$$\frac{d}{d\theta}P^m_n(\cos\theta) = (-1)^m(\sin\theta)^{m-1}\left[m\cos\theta\frac{d^m}{d(\cos\theta)^m}P_n(\cos\theta) - (\sin\theta)^2\frac{d^{m+1}}{d(\cos\theta)^{m+1}}P_n(\cos\theta)\right] \tag{6.6}$$

showing that $\frac{d}{d\theta}P^m_n(\cos\theta)$ is not null iff $m = 1$. Taking into account respectivelly equation (6.5a) equation (6.5b) in conjonction with the Schmidt normalisation coefficients (3.4) we obtain the Identity 5 and the Identity 6. $\square$

Using the identities 1 to 6 we can derive the expression of the magnetic at the North and the South Pole as follow:

(a) Putting the identities 1, 3, 5 in equation (2.4) we obtain the following expressions of the magnetic field at the North Pole:

$$\boxed{\begin{cases} X_c(0) = \sum\limits_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}g^1_n\cos(\phi) + \sum\limits_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}h^1_n\sin(\phi) \\ Y_c(0) = \sum\limits_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}g^1_n\sin(\phi) - \sum\limits_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}h^1_n\cos(\phi) \\ Z_c(0) = \sum\limits_{n=1}^{N}(n+1)\left(\frac{a}{r}\right)^{n+2}g^0_n \end{cases}} \tag{6.7}$$

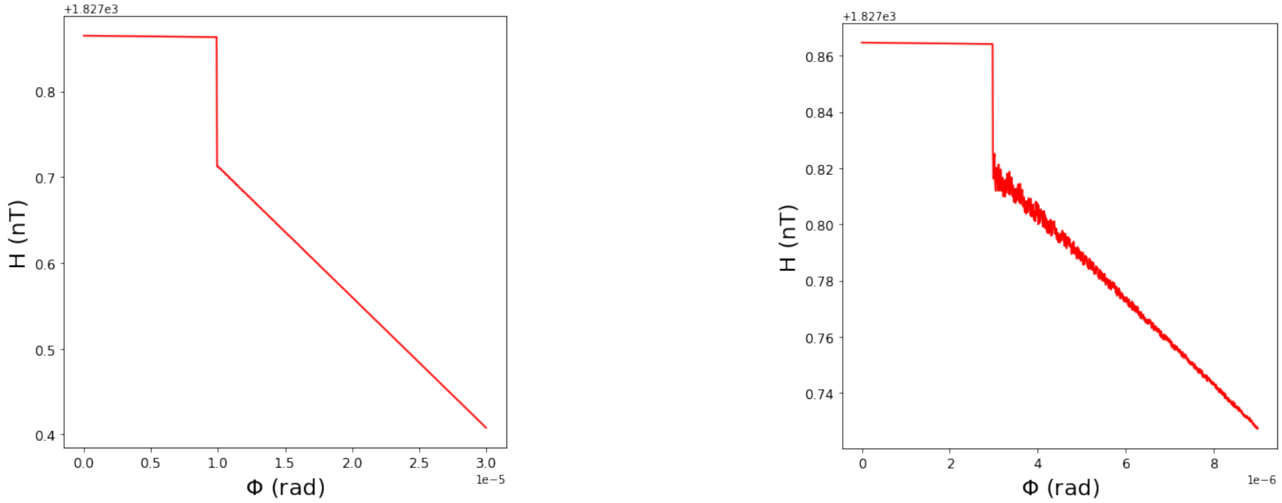
(b) Putting the identities 3, 4, 6 in equation (2.4) we obtain the following expressions of the magnetic field at the South Pole:

$$\boxed{\begin{cases} X_c(\pi) = \sum\limits_{n=1}^{N}\left(-\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}g^1_n\cos(\phi) + \sum\limits_{n=1}^{N}\left(-\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}h^1_n\sin(\phi) \\ Y_c(\pi) = \sum\limits_{n=1}^{N}\left(-\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}g^1_n\sin(\phi) - \sum\limits_{n=1}^{N}\left(-\frac{a}{r}\right)^{n+2}\sqrt{\frac{n(n+1)}{2}}h^1_n\cos(\phi) \\ Z_c(\pi) = -\sum\limits_{n=1}^{N}(n+1)\left(-\frac{a}{r}\right)^{n+2}g^0_n \end{cases}} \tag{6.8}$$

In the `geomagnetism` package the constant EPS monitors the choice of the equation used to compute the electromagnetic field according to Algorithm 1. The default value of EPS is set to EPS $= 10^{-5}$rad. At $\phi =$ EPS the electromagnetic field components undergo a discontinuity as shown in figure 5. The value of the constant EPS can be reaffected using:
`geo.geomagnetism.EPS=NewEps`. Nevertheless as shown by Figure 5b for EPS $\lesssim 8\times10^{-6}$rad numerical computations of equations (2.4) are noisy.

**Algorithm 1** geomagnetic field computation

$EPS \leftarrow 10^{-5}$
**if** $Eps - \phi \in [\pi, \pi - \text{EPS}]$ **then**
  use equations (6.8).
**else if** $\phi \in [0, \text{EPS}]$ **then**
  use equations (6.7).
**else**
  use equations (2.4).
**end if**




(a) EPS $= 10^{-5}$

(b) EPS $= 3 \times 10^{-6}$

Figure 5: Computational discontinuity of the component $H$ at $\phi =$ EPS.

# 7 Spherical harmonic coefficients

## 7.1 Gauss coefficients

To compute the geomagnetic fields, through the equation (2.4), we must provide: the values of the Gauss coefficients $h_n^m$ and $g_n^m$, the order $N$ of the SH development. As the Gauus coefficients vary with time we must also provide the values $\dot{h}_n^m$ and $\dot{g}_n^m$. All these values are stored in the following variables:

- `dic_dic_h` contains $h_n^m$ coefficients stored in a dict of dict as `{year : {(m,n):h,...},...}`
- `dic_dic_g` contains $g_n^m$ coefficients stored in a dict of dict as `{year : {(m,n):g,...},...}`
- `dic_dic_SV_h` contains $\dot{h}_n^m$ coefficients stored in a dict of dict as `{year : {(m,n):SV_h,...},...}`
- `dic_dic_SV_g` contains $\dot{g}_n^m$ coefficients stored in a dict of dict as `{year : {(m,n):SV_g,...},...}`
- `dic_N` dict containing the order $N$ of the SH decomposition as `dic_N[year]=N`
- `Years` contains the list of tabulated years

The function `read_IGRF13_COF` reads the `IGRF13.COF` coefficients.

```
file = 'IGRF13.COF' # downloaded from https://www.ngdc.noaa.gov/IAGA/vmod/coeffs/igrf13coeffs.
                                                    txt
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years = geo.read_IGRF13_COF(file)
```

The function `read_WMM` reads the `WMM_2020.COF` or the`WMM_2015.COF` coefficients.

```
file = 'WMM_2020.COF' # downloaded from https://www.ngdc.noaa.gov/geomag/WMM/wmm_ddownload.
                                                    shtml
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years= geo.read_WMM(file)
```

```
file = 'WMM_2015.COF' # downloaded from https://www.ngdc.noaa.gov/geomag/WMM/wmm_ddownload.
                                                    shtml
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years= geo.read_WMM(file)
```

## 7.2 Gauss coefficients secular variation

The Gauss coefficients $g_n^m$ and $h_n^m$ vary with time [7]. There secular variation $\dot{g}_n^m$ and $\dot{h}_n^m$ expressed in `nT/year` are tabulated every year and are considered to be constant during that current year. So, at time $t$, the Gauss coefficients can be expressed as:

$$\begin{cases} g_m^n(t) = g_m^n(t_0) + \dot{g}_m^n(t_0)(t - t_0) \\ h_m^n(t) = h_m^n(t_0) + \dot{h}_m^n(t_0)(t - t_0) \end{cases} \tag{7.1}$$

where $t_0$ stands for the time on the 1 January at 00:00 PM of the current year.Using $\dot{g}_n^m$ and $\dot{h}_n^m$ it is straightforward to compute the secular variation of the geomagnetic field componants. We have [2]:

$$\begin{aligned}
\dot{X}_c \equiv \\
& \dot{\mathbf{B}}_x = -\dot{B}_\theta = \frac{1}{r}\frac{\partial \dot{V}}{\partial \theta} = \\
& \sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[\dot{g}_n^m \cos(m\phi) + \dot{h}_n^m \sin(m\phi)\right]\frac{dP_{(s),n}^m(\cos\theta)}{d\theta}, \\
\dot{Y}_c \equiv \\
& \dot{\mathbf{B}}_y = \dot{B}_\phi = \frac{-1}{r\sin\theta}\frac{\partial \dot{V}}{\partial \phi} = \\
& \sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n} m\left[\dot{g}_n^m \sin(m\phi) - \dot{h}_n^m \cos(m\phi)\right]\frac{P_{(s),n}^m(\cos\theta)}{\sin\theta}, \\
\dot{Z}_c \equiv \\
& \dot{\mathbf{B}}_z = -\dot{B}_r = \frac{\partial \dot{V}}{\partial r} = \\
& \sum_{n=1}^{N}(n+1)\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[\dot{g}_n^m \cos(m\phi) + \dot{h}_n^m \sin(m\phi)\right]P_{(s),n}^m(\cos\theta),
\end{aligned} \tag{7.2}$$

The time derivative of the equation (5.2) can be expressed as:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} \dot{X}_c \\ \dot{Y}_c \\ \dot{Z}_c \end{pmatrix} \tag{7.3}$$

Finally, the time derivative of equation (5.3) reads [5]:

$$\boxed{\begin{cases} \dot{H} = \frac{X\dot{X}+Y\dot{Y}}{H} \\ \dot{F} = \frac{X\dot{X}+Y\dot{Y}+Z\dot{Z}}{F} \\ \dot{I} = \frac{H\dot{Z}-Z\dot{H}}{F^2} \\ \dot{D} = \frac{X\dot{Y}-Y\dot{X}}{H^2} \end{cases}} \tag{7.4}$$

# 8 The B_componants function

The function `B_componants` is defined as:

```
def B_components(
    phi_,
    theta_,
    altitude,
    Date,
    referential="geodetic",
    file_gauss_coeff="IGRF13.COF",
    ELLIPSOID=WGS84,
    SV=False,
):
```

The function `B_componants` takes the arguments:

`phi_`: Longitude $\Phi$ in `deg` $\Phi \in [0°, 360°[$ or West East longitude $\Phi \in [-180°, 180°[$;

`theta_`: Colatitude $\theta$ in `deg` $\theta \in [0°, 180°]$;

`altitude`: Elevation in metres;

`Date`: time used to compute the magnetic field. Date is a dictionary

[5] We use $\frac{\partial}{\partial x}\text{atan2}(y, x) = -\frac{y}{x^2+y^2}$ and $\frac{\partial}{\partial y}\text{atan2}(y, x) = \frac{x}{x^2+y^2}$

`Date["mode"]`

if `Date["mode"]=="ymd"` time is expressed as yyyy/mm/dd and is contained in `Date["year"]`

if `Date["mode"]=="dec"` we have:

`Date["year"]` = year

`Date["month"]` = month month = 1,2,...,12

`Date["day"]` = day of the month

`Date["hour"]` = hour of day

`Date["minute"]` = minute of the hour

`Date["second"]` = second of the minute

`referential`:

the colatitude is expressed in a geotetic referential if `referential = geotetic`;

the colatitude is expressed in a geotetic referential if `referential = geotetic`.

`file_gauss_coeff`: name of the file containing the Gauss coefficients $h$ and $g$.

`file = "IGRF13.COF"` covers the years from 1900 to 2015;

`file = "WMM_2015.COF"` covers the year 2015;

`file = "WMM_2020.COF"` (default value) covers the year 2020 with extrapolation allowed up to year 2025.

`ELLIPSOID`: tuple $(a, f^{-1})$ where $a$ is the Earth semi major axis in metres, and $f^{-1}$ the Earth reciprocal flattening. Conventional values are:

`WGS80 = (6378137, 298.257222100882711)`

`WGS84 = (6378137, 298.257223563)` (default value)

`SV`: if `SV==True` the computation of the field secular variation is done.

The function `B_componants` returns the dictionary `result`:

`result["D"]`$=D$: Declination in `deg`;

`result["F"]`$=F$: Total field intensity in `nT`;

`result["H"]`$=H$: horizontal field intensity in `nT`;

`result["I"]`$=I$: Inclination in `deg`;

`result["X"]`$=X$: North component in `nT` in the geocentric coordinate;

`result["Y"]`$=Y$: East component in `nT` in both geocentric and geotetic coordinate;

`result["Z"]`$=Z$: Down component in `nT` in the geocentric coordinate;

`result["Fd"]`$=\dot{F}$: Total field intensity secular variation in `nT/year` if `SV==True` and `None` otherwise;

`result["Hd"]`$=\dot{H}$: Horizontal field intensity secular variation in `nT/year` if `SV==True` and `None` otherwise;

`result["Xd"]`$=\dot{X}$: North component field intensity secular variation in `nT/year` if `SV==True` and `None` otherwise;

`result["Yd"]`$=\dot{Y}$: East component field intensity secular variation in `nT/year` if `SV==True` and `None` otherwise;

`result["Zd"]`$=\dot{Z}$: Down component field intensity secular variation in `nT/year` if `SV==True` and `None` otherwise;

`result["Id"]`$=\dot{I}$: Inclination secular variation in `deg/year` if `SV==True` and `None` otherwise;

`result["Dd"]`$=\dot{D}$: Declination secular variation in `deg/year` if `SV==True` and `None` otherwise.

For example:

```
height= 100_000.0
colatitude= 170.0
longitude= 240.0
Date={"mode":"dec","year":2022.5}

result = geo.B_components(longitude,colatitude,height,Date,referential="geodetic",
                                   file="WMM_2020.COF", SV=True)
>> {'X': 5814.9658886214675,
>> 'Y': 14802.966383932766,
>> 'Z': -49755.31199391833,
>> 'F': 52235.358844960836,
>> 'H': 15904.139148337288,
>> 'I': -72.27367389486136,
>> 'D': 68.55389056498416,
>> 'Xd': 28.038196182663352,
>> 'Yd': 1.3970624624335226,
>> 'Zd': 85.63095330312873,
>> 'Hd': 11.551824423488469,
>> 'Fd': -78.04814717528828,
>> 'Id': 0.040667257177207254,
>> 'Dd': -0.09217565861159664}
```

# 9 Benchmark

Using the latitude, longitude, time and height above the ellipsoid given in the Table 1, the Table 2 shows the geomagnetic field element obtained by both the geomagnetism package and those extracted from the Table 3b High-precision numerical example from [2].

| | | |
|---|---|---|
| Time | 2022.5 | yr |
| Height-above-Ellipsoid | 100 | km |
| Latitude | -80 | deg |
| Longitude | 240 | deg |

Table 1: parameters values used for the benchmark.

| notation | geomagnetism | WMM test value | relative error |
|---|---|---|---|
| D | 69.125 | 69.13 | -0.006 |
| Dd | -0.094 | -0.09 | 4.342 |
| F | 54912.078 | 54912.1 | 0.0 |
| Fd | -83.356 | -83.4 | -0.052 |
| H | 16884.992 | 16885.0 | 0.0 |
| Hd | 12.551 | 12.6 | -0.388 |
| I | -72.091 | -72.09 | 0.002 |
| Id | 0.041 | 0.04 | 4.462 |
| X | 6016.523 | 6016.5 | 0.0 |
| Xd | 30.379 | 30.4 | -0.068 |
| Y | 15776.705 | 15776.7 | 0.0 |
| Yd | 1.847 | 1.8 | 2.578 |
| Z | -52251.635 | -52251.6 | 0.0 |
| Zd | 91.656 | 91.7 | -0.047 |

Table 2: comparison of the geomagnetic field element obtained by both the geomagnetism package and those extract from the Table 3b High-precision numerical example from Chullia. [2]

# 10 Examples

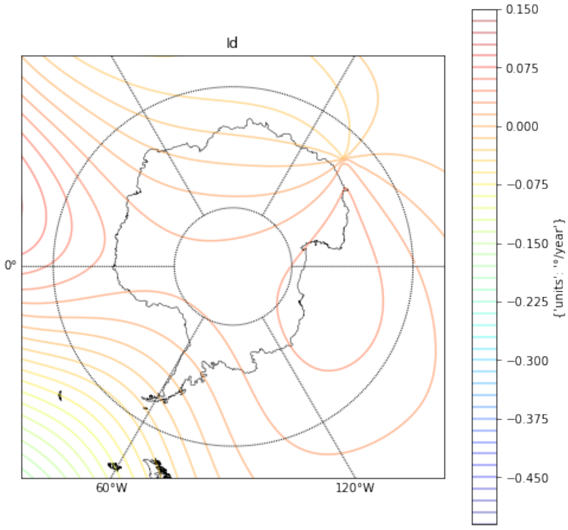
## 10.1 2D plots: Example 1

Using the following code generates the xarrays of: $X$, $Y$, $Z$, $F$, $H$, $\dot{X}$, $\dot{Y}$, $\dot{Z}$, $\dot{F}$,$\dot{H}$ on the grid defined by the two arrays `colalatitudes` and `longitudes`.

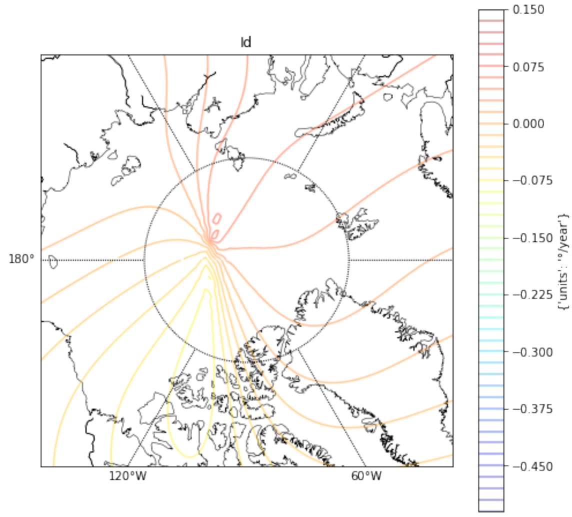
```
import numpy as np
import geomagnetism as geo
'''
compute the xarrays: dintensities containing X,Y,Z,F,H;
                     dangles containing I, D;
                     dintensities_sv containing the the time derivative of X,Y,Z,F,H;
                     dangles_sv containing the the time derivative of I, D;
for the colatitudes and longitudes specified in the arrays colatitudes, longitudes
'''

colatitudes = np.linspace(0,180,181)
longitudes = np.linspace(-180,179,360)

dintensities, dangles, dintensities_sv, dangles_sv = geo.grid_geomagnetic(colatitudes,
                                                longitudes)
```

Using the following code generates the the plots of Figure 6 and of the Figure 7.

```
'''
Examples of plots of geomagnetic components. Note that, to plot the geomagnetic field, we use
                                                three different map projection: \texttt{spstere
                                                } for the South Pole; \texttt{npstere} for the
                                                North Pole; \texttt{mill} for the whole Earth.
                                                Figure~\ref{fig:example2} shows an evolving
                                                dent in Earth's magnetic field over South
                                                America.
'''
component = 'F' # sould be 'X','Y','Z','F','H','I','D','Xd','Yd','Zd','Fd','Hd','Id','Dd'
geo.plot_geomagetism(dintensities,
                 dangles,
                 dintensities_sv,
                 dangles_sv,
                 component,
                 {'proj':'spstere',
                  'boundinglat':-55,
                  'lon_0':270})

geo.plot_geomagetism(dintensities,
                 dangles,
                 dintensities_sv,
                 dangles_sv,
                 component,
                 {'proj':'npstere',
                  'boundinglat':70,
                  'lon_0':270} )

geo.plot_geomagetism(dintensities,
                 dangles,
                 dintensities_sv,
                 dangles_sv,
                 component,
                 {'proj':'mill',
                  'llcrnrlat':-90,
                  'urcrnrlat': 90,
                  'llcrnrlon':0,
                  'urcrnrlon':360} )
```

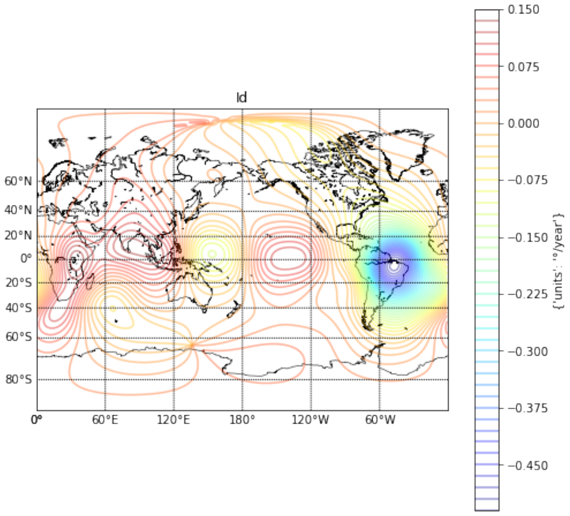



(a) $\dot{I}$ at South Pole

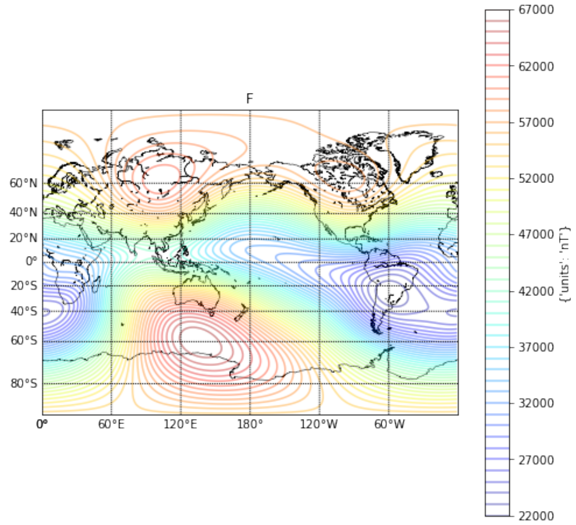



(b) $\dot{I}$ at North Pole

Figure 6: Examples of the secular variation of the geomagnetic field inclination at the North and South Pole.




(a) Secular variation $\dot{I}$ of geomagnetic inclination.



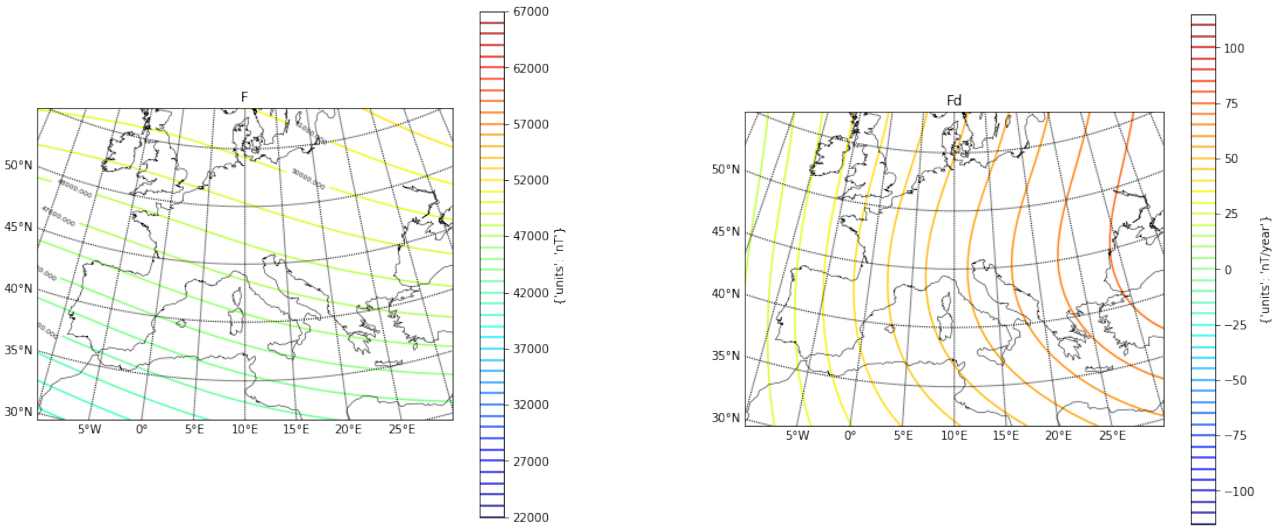

(b) Geomagnetic intensity $F$.

Figure 7: Examples of geomagnetic field computation for the whole Earth.

## 10.2 2D plots: Example 2

Using the following code generates the the plots of Figure 8 and Figure 9.

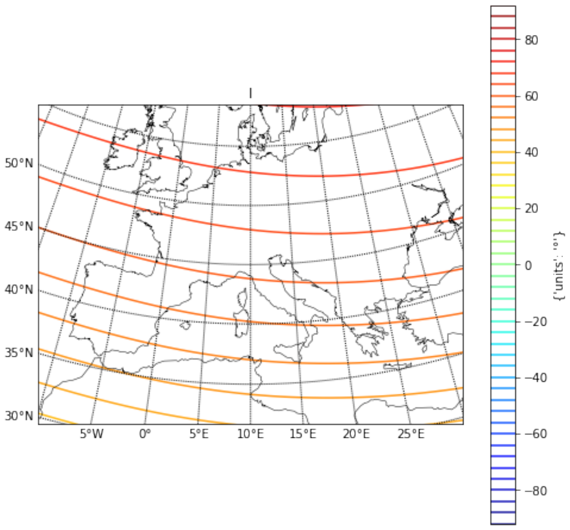
```
geo.plot_geomagetism(dintensities,
                  dangles,
                  dintensities_sv,
                  dangles_sv,
                  'F',
                  {'proj':'lcc',
                   'lat_1':45.,
                   'lat_2':55,
                   'lat_0':45,
                   'lon_0':10.})

geo.plot_geomagetism(dintensities,
                  dangles,
                  dintensities_sv,
                  dangles_sv,
                  'Fd',
                  {'proj':'lcc',
                   'lat_1':45.,
                   'lat_2':55,
                   'lat_0':45,
                   'lon_0':10.})
geo.plot_geomagetism(dintensities,
                  dangles,
                  dintensities_sv,
                  dangles_sv,
                  'I',
                  {'proj':'lcc',
                   'lat_1':45.,
                   'lat_2':55,
                   'lat_0':45,
                   'lon_0':10.})

geo.plot_geomagetism(dintensities,
                  dangles,
                  dintensities_sv,
                  dangles_sv,
                  'Id',
                  {'proj':'lcc',
                   'lat_1':45.,
                   'lat_2':55,
                   'lat_0':45,
                   'lon_0':10.})
```



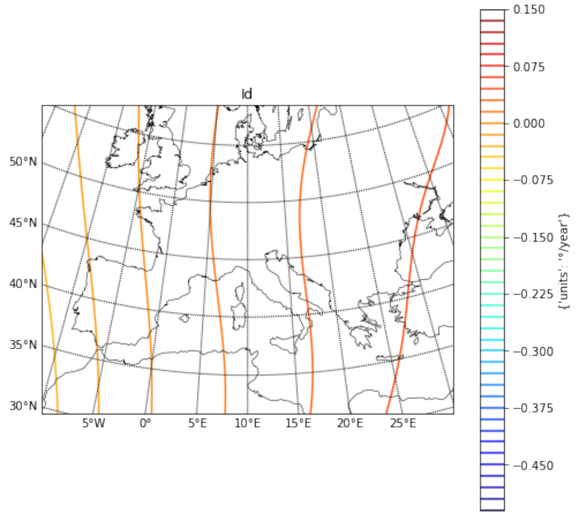

(a) Intensity $F$ of the geomagnetic field.

(b) Secular variation $\dot{F}$ of the geomagnetic field intensity.

Figure 8: Examples of field magnitude and field magnitude secular variation in South Western Europe.




(a) Inclination $I$ of the geomagnetic field.




(b) Secular variation $\dot{I}$ of the geomagnetic field inclination.

Figure 9: Examples of field inclination and inclination secular variation in South Western Europe.

# References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