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Geomagnetism package notes

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Introduction

The package geomagnetim intends to serve pedagogical purpose rather to replace well established FORTRAN or C progam develloped by academic institutions. In contrast with these programs, which favor compactness and time execution, the geomagnetic package ,tentatively, focus on lisibility.
You can download geomagnetis using :

pip install geomagnetism

Geomagnetism calculation

As the terrestrial magnetic field obeys both $\nabla \mathbf{B}=0$ and $\nabla \times \mathbf{B}=0$, it can be shown that the magnetic field can be expressed as the gradient of a scalar potential V which satisfies the Laplace equation:

$$\Delta \mathrm{V}=0 \tag{1.1}$$

For a spherical geometry the geomagnetic potential is given by the spherical harmonic expension (SH) (Stacey and Davis , Campbell 2007):

$$V(r,\theta,\phi,t)=a\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left[g_n^m(t)\cos(m\phi)+h_n^m(t)\sin(m\phi)\right]P_n^m\left(\cos(\theta)\right) \tag{1.2}$$

with:

$$P_n^m\left(x\right)=\begin{cases}\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n & : m=0\\ \sqrt{\dfrac{2(n-m)!}{(n+m)!}}\left(1-x^2\right)^m\dfrac{1}{2^n n!}\dfrac{d^{n+m}}{dx^n}\left(x^2-1\right)^n & : m>0\end{cases} \tag{1.3}$$

g_n^m and h_n^m are the Gauss's coefficients. Note that the sum over n begins with the the value $n=1$ as the index $n=0$ would correspond to a monopole. The dipole, quadrupole, octupole,... contribution correspond to $n=1,2,3,\ldots$ These coefficients varies with time and are tabulated (

https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html). The coefficient a is the mean radius of the earth (6371.2 km); r ,the radial distance from the center of the Earth ; θ, the geocentric colatitude ; ϕ, the east longitude measured from the Greewich. We note that the Condon-Shortley phase correction $(-1)^m$ is omitted in the definition of the associated Legendre polynomial and the polynomes are normalized using Schmidt quasi-normalization (Winch, Ivers et al. 2005). The relation $\mathbf{B} = -\nabla V$ leads to:

$$\begin{cases} X_c = \mathbf{B}_x = -B_\theta = \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} = \sum_{n=1}^{N}\left(\dfrac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\dfrac{dP_n^m(\cos\theta)}{d\theta} \\ Y_c = \mathbf{B}_y = B_\phi = \dfrac{-1}{r\sin\theta}\dfrac{\partial V}{\partial \phi} = \sum_{n=1}^{N}\left(\dfrac{a}{r}\right)^{n+2}\sum_{m=0}^{n} m\left[g_n^m \sin(m\phi) - h_n^m \cos(m\phi)\right]\dfrac{P_n^m(\cos\theta)}{\sin\theta} \\ Z_c = \mathbf{B}_z = -B_r = \dfrac{\partial V}{\partial r} = \sum_{n=1}^{N}(n+1)\left(\dfrac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]P_n^m(\cos\theta) \end{cases} \tag{1.4}$$

Where $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{B}_z$ are the field components respectively in the northward, eastward and downward directions. Theses components are expressed in the geocentric referential as recall by the index c

Spherical harmonics normalisation

If we define:

$$\begin{aligned} C_n^m(\theta,\phi) &\equiv P_n^m\left(\cos\theta\right)\cos m\theta \quad : m = 0,1,2\cdots n \\ S_n^m(\theta,\phi) &\equiv P_n^m\left(\cos\theta\right)\sin m\theta \quad : m = 1,2\cdots n \end{aligned} \tag{1.5}$$

the Schmidt quasi-normalization conditions reads (Winch, Ivers et al. 2005):

$$\begin{aligned} &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C_n^m(\theta,\phi)C_N^M(\theta,\phi)\sin\theta d\theta d\phi = \frac{1}{2n+1}\delta_n^N\delta_m^M \\ &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} S_n^m(\theta,\phi)S_N^M(\theta,\phi)\sin\theta d\theta d\phi = \frac{1}{2n+1}\delta_n^N\delta_m^M \\ &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C_n^m(\theta,\phi)S_N^M(\theta,\phi)\sin\theta d\theta d\phi = 0 \quad : \forall n,N,m,M \end{aligned} \tag{1.6}$$

The associated Legendre polynomial value are generated by the scipy function M,Mp=lpmn(M,N,x) where:

$$P_n^m = \begin{bmatrix} P_0^0 & P_1^0 & \cdots & P_N^0 \\ 0 & P_1^1 & \cdots & P_N^1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_N^M \end{bmatrix}, \quad M[m,n] = P_n^m \tag{1.7}$$

with :

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}\sqrt{\left(1-x^2\right)^m}\frac{d^{n+m}}{dx^n}\left(x^2-1\right)^n \tag{1.8}$$

To normalize we use:

$$M \leftarrow M \odot Norm\_Schmidt \tag{1.9}$$

where $\odot$ is the component-wise multiplication and :

$$N_{n,m} = \begin{cases} (-1)^m\sqrt{\dfrac{\left(2-\delta_m^0\right)(n-m)!}{(n+m)!}} & : n-|m| \geq 0 \\ 0 \quad : n-|m| < 0 \end{cases} \tag{1.10}$$

Note that other authors in geophysics use different normalization factors. Stacey (Stacey and Davis) use :

$$N_n'^m = \begin{cases} (-1)^m\sqrt{\left(2-\delta_m^0\right)(2m+1)\dfrac{(n-m)!}{(n+m)!}} & : |m| \leq n \\ 0 \quad : |m| > n \end{cases} \tag{1.11}$$

The function `Norm_Schimdt(m, n)` computes the normalisation matrix (1.10)

The function `Norm_ Stacey (m, n)` computes the normalisation matrix (1.11)

Exemples :

```
geo.Norm_Stacey(3,4)
>> array([[ 1.        ,  1.        ,  1.        ,  1.        ,  1.        ],
       [ 0.        , -1.73205081, -1.        , -0.70710678, -0.54772256],
       [ 0.        ,  0.        ,  0.64549722,  0.28867513,  0.16666667],
       [ 0.        ,  0.        ,  0.        , -0.13944334, -0.05270463]])


geo.Norm_ Schimdt (3,4)
>> array([[ 1.        ,  1.        ,  1.        ,  1.        ,  1.        ],
       [ 0.        , -1.        , -0.57735027, -0.40824829, -0.31622777],
       [ 0.        ,  0.        ,  0.28867513,  0.12909944,  0.0745356 ],
       [ 0.        ,  0.        ,  0.        , -0.05270463, -0.01992048]])
```

Geotetic to geocentric transformation

The computation of the geomagnetic field is done in a geocentric coordinate system. So if we provide The geotetic coordinates we have to convert them into geocentric ones. In the following we deduce the transformation relation used in the function `geodetic_to_geocentric`.

Ellipse equation

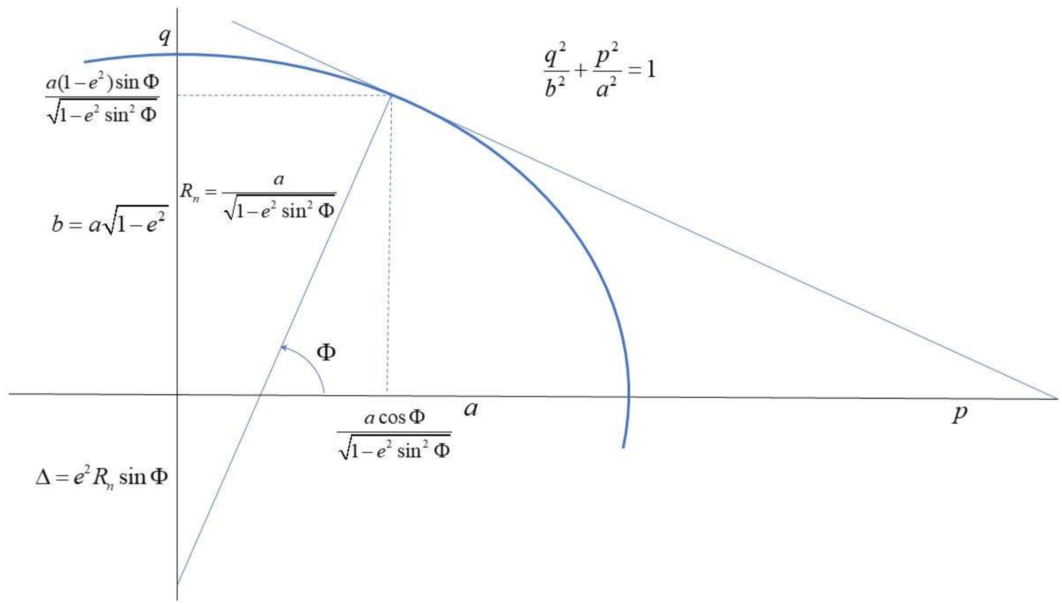

Figure 1 : Ellipse notation convention.

The equation of the ellipse read :

$$\frac{q^2}{b^2}+\frac{p^2}{a^2}=1 \tag{1.12}$$

The geotetic latitude Φ can be be expressed through the derivative :

$$\cot\Phi=-\frac{dq}{dp}=\frac{b^2}{a^2}\frac{q}{p} \tag{1.13}$$

From (1.13) we can express q as :

$$q=p\frac{a^2}{b^2}\tan\Phi \tag{1.14}$$

Using (1.12) and (1.14) we obtain :

$$\begin{cases} p=\dfrac{a\cos\Phi}{\sqrt{1-e^2\sin^2\Phi}} \\ q=\dfrac{a(1-e^2)\sin\Phi}{\sqrt{1-e^2\sin^2\Phi}} \end{cases} \tag{1.15}$$

The prime vertical curvature radius R_n can be deduced from p as :

$$R_n = \frac{p}{\cos\Phi} = \frac{a}{\sqrt{1-e^2\sin^2\Phi}}$$
$$R_n = \frac{a^2}{\sqrt{a^2-(a^2-b^2)\sin^2\Phi}} \tag{1.16}$$

Using the prime vertical curvature we can re-express p and q as :

$$\begin{cases} p = R_n\cos\Phi \\ q = (1-e^2)R_n\sin\Phi \end{cases} \tag{1.17}$$

Geotetic to geocentric transformation

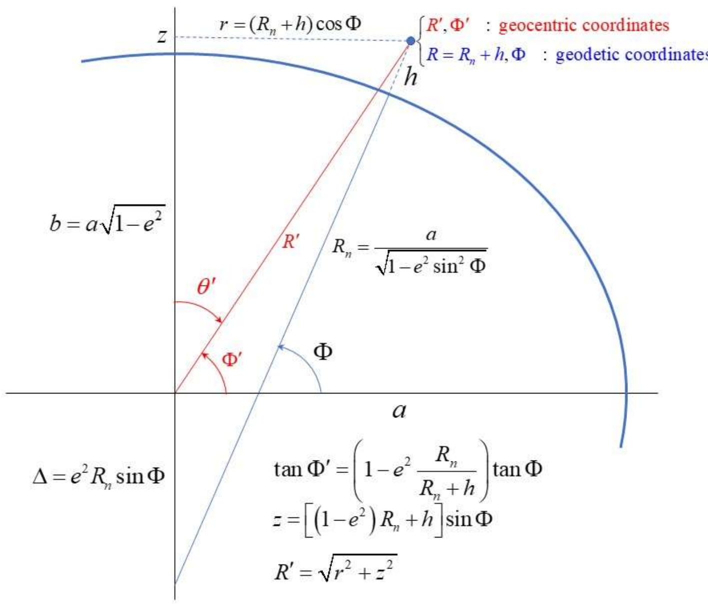

Figure 2 : Geotetic and geocentric notation convention.

Relation between the geotetic colatitude Φ and the geocentric colatitude Φ' :

Using the conventions of Figure 2 we have

$$\frac{\tan\Phi'}{\tan\Phi} = \frac{z}{z+\Delta} = \frac{(1-e^2)R_n+h}{(1-e^2)R_n+h+e^2R_n} = 1-e^2\frac{R_n}{R_n+h} \tag{1.18}$$

The flattening is defined as follow:

$$f = \frac{a-b}{a} \tag{1.19}$$

The geodetic reference ellipsoids are specified by giving the reciprocal flattening f^{-1}

As usual the eccentricity reads:

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{f(2-f)}$$
$$1 - e^2 = \frac{b^2}{a^2} \tag{1.20}$$

We can express z and r as :

$$\begin{cases} z = (R_n + h)\sin\Phi - \Delta = \left[(1-e^2)R_n + h\right]\sin\Phi \\ r = (R_n + h)\cos\Phi \end{cases} \tag{1.21}$$

Using (1.21) the geocentric radius is equal to :

$$R' = \sqrt{z^2 + r^2} \tag{1.22}$$

Using (1.22) we can express the geocentric radius using the semi major and the semi minor axis. We obtain (Peddie 1982) :

$$R'^2 = \frac{h^2 + 2h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi} + \left[a^4 - (a^4 - b^4)\sin^2\Phi\right]}{a^2 - (a^2 - b^2)\sin^2\Phi} \tag{1.23}$$

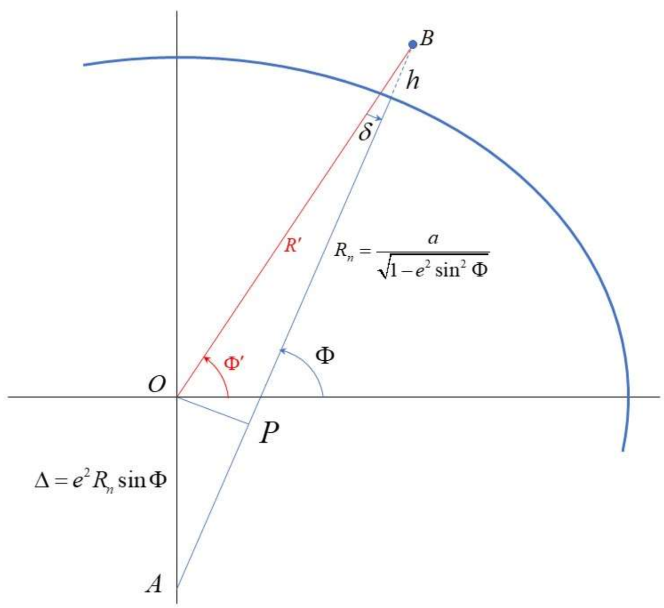

Figure 3 : Computation of $\cos\delta$ *and* $\sin\delta$ *.*

Using (1.21) and (1.22), the geocentric colatitude can be deduced from :

$$\cos\theta = \frac{z}{\sqrt{r^2+z^2}} = \frac{\sin\Phi}{\sqrt{\left[\frac{R_n+h}{(1-e^2)R_n+h}\right]^2 \cos^2\Phi + \sin^2\Phi}} \tag{1.24}$$

The relation (1.24) can be expressed using the semi major and minor axis. Using (1.16), (1.20) we obtain (Peddie, 1982) :

$$\cos\theta = \frac{\sin\Phi}{\sqrt{c\cos^2\Phi + \sin^2\Phi}}; \quad c = \left[\frac{a^2 + h\sqrt{a^2-(a^2-b^2)\sin^2\Phi}}{b^2 + h\sqrt{a^2-(a^2-b^2)\sin^2\Phi}}\right]^2 \tag{1.25}$$

Using the triangle AOB of the Figure 3 we obtain the relation :

$$\Delta\sin\Phi + R'\cos\delta = R_n + h \tag{1.26}$$

After rearanging :

$$\cos\delta = \frac{1}{R'}\left[h + R_n(1-e^2)\right] \tag{1.27}$$

Using (1.16) we have :

$$\cos\delta = \frac{1}{R'}\left[h + \frac{a^2}{R_n}\right] \tag{1.28}$$

The length of OP is equal to

$$R'\sin\delta = \Delta\cos\Phi \tag{1.29}$$

So :

$$\sin\delta = \frac{R_n}{R'} e^2 \cos\Phi\sin\Phi \tag{1.30}$$

Computational aspect

The function `geodetic_to_geocentric(ellipsoid, co_latitude, height)` computes the geocentric and the geocentric colatitude using respectively (1.22) and (1.18). The angle

$$\delta = \theta' - \theta \tag{1.31}$$

between the geocentric and the geotetic colattitude is also computed (Figure 4).

The routine uses the tuple ellipsoid = (a, f^{-1}). Depending on the selected convention ellipsoid can be set to

GRS80 = geo.geomagnetism.GRS80

WGS84 = geo.geomagnetism.WGS84

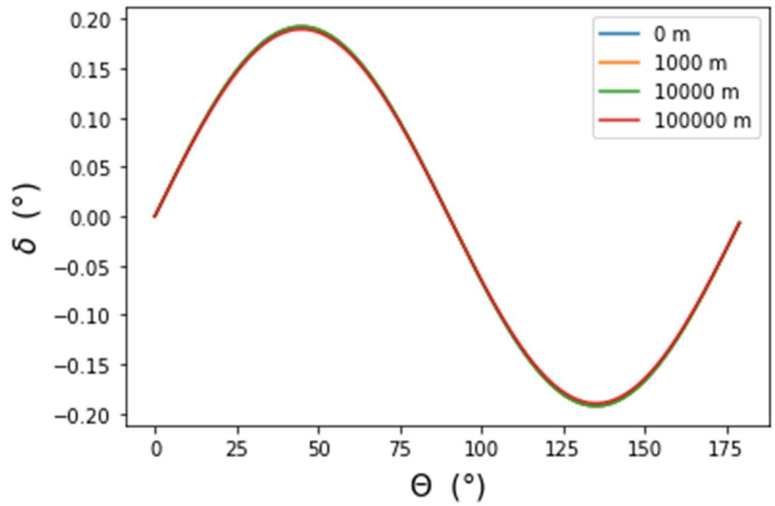

Figure 4 : Variation of δ versus the geotetic colatitude θ and the height h.

The function `geodetic_to_geocentric_IGRF13(ellipsoid, co_latitude, height)` is a translation of a FORTRAN (https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html) where the authors use (1.23), (1.28), (1.30) to compute the geocentric radius the cosine and sine of the angle δ.
The function use the tuple ellipoid=(a,b). Depending on the selected convention ellipsoid can be set to

GRS80_ = geo.geomagnetism.GRS80_
WGS84 _= geo.geomagnetism.WGS84 _

Base transformation

Passing from geocentric to geotetic referential the magnetic field undergoes the following transfomation :

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geodetic}} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geocentric}} \quad (1.32)$$

with $\delta = \theta' - \theta = \Phi - \Phi'$ (Figure 5). Using Peddie notation (Peddie, 1982) we have :

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} X_c \\ Y_c \\ Z_c \end{pmatrix} \quad (1.33)$$

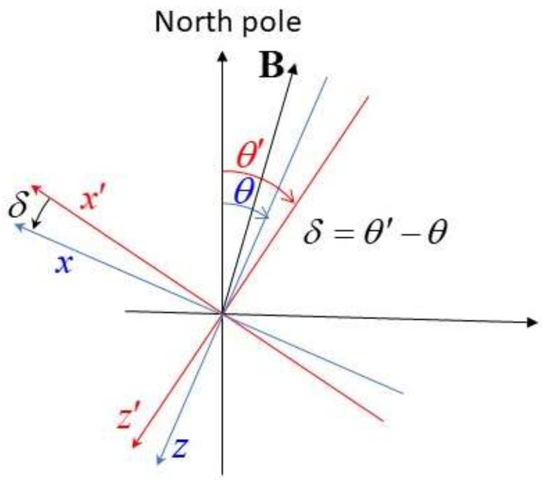

Figure 5 : Variation of δ versus the geotetic colatitude θ and the height h .

The geomagnetic horizontal intensity H , total intensity F , declination D and inclination I can be obtained from :

$$\begin{cases} H = \sqrt{X^2 + Y^2} \\ F = \sqrt{H^2 + Y^2} \\ D = \tan^{-1} \dfrac{Y}{X} \\ I = \tan^{-1} \dfrac{Z}{H} \end{cases} \qquad (1.34)$$

Spherial harmonic coefficients

The main ingredient for the computing of the geomagnetic field are the spherical harmonic coefficients g_m^n and h_n^m . These coefficients depend on the geocentic latitude on the longitude, and on the time. The package `geomagnetism` provide several functions to read tabulated cofficients

```
import geomagnetism as geo
file = "IGRF13.COF" # downloaded from
https://www.ngdc.noaa.gov/IAGA/vmod/coeffs/igrf13coeffs.txt
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years=
geo.read_IGRF13_COF(file)
m=1
n=3
years="1965"
h = dic_dic_h[years][(m,n)]
print(f'Spherial harmonic coefficients h(m={m},n={n}) for year = {years} :{h}')
>>Spherial harmonic coefficients h(m=1,n=3) for year = 1965 :-404.0
```

Other programs

- In line calculators

https://www.ngdc.noaa.gov/geomag/calculators/magcalc.shtml

http://wdc.kugi.kyoto-u.ac.jp/igrf/gggm/index.html

permits the computation of the magnetic declination.

- a FORTRAN code available can be downloaded from :

https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html .

Qote: "This code is is a synthesis routine for the 13th generation IGRF as agreed in December 2019 by IAGA Working Group V-MOD. It is valid 1900.0 to 2025.0 inclusive. Values for dates from 1945.0 to 2015.0 inclusive are definitive, otherwise they are non-definitive. Reference radius remains as 6371.2 km - it is NOT the mean radius (= 6371.0 km) but 6371.2 km is what is used in determining the coefficients.'

- a C code available along with the Geomag 7.0 software (Windows version) :

https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html

- a PYTHON code :

https://pypi.org/project/geomag/#files

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