

Geomagnetism package notes

FRANCOIS BERTIN

Contents

Chapitre d'équation 1 Section 1	1
Introduction.....	1
Geomagnetism calculation.....	1
Spherical harmonics normalisation.....	2
Geotetic to geocentric transformation	3
Ellipse equation	4
Geotetic to geocentric transformation	5
Computational aspect	7
Base transformation.....	8
Spherial harmonic coefficients.....	10
Other programs	11
Bibliography.....	12

Introduction

The package geomagnetim intends to serve pedagogical purpose rather to replace well established FORTRAN or C program developed by academic institutions. In contrast with these programs, which favor compactness and time execution, the geomagnetic package ,tentatively, focus on lisibility.

You can download geomagnetis using :

```
pip install geomagnetism
```

Geomagnetism calculation

As the terrestrial magnetic field obeys both $\nabla \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$, it can be shown that the magnetic field can be expressed as the gradient of a scalar potential V which satisfies the Laplace equation:

$$\Delta V = 0 \quad (1.1)$$

For a spherical geometry the geomagnetic potential is given by the spherical harmonic expansion (SH) (Stacey and Davis , Campbell 2007):

$$V(r, \theta, \phi, t) = a \sum_{n=1}^N \left(\frac{a}{r} \right)^{n+1} \sum_{m=0}^n \left[g_n^m(t) \cos(m\phi) + h_n^m(t) \sin(m\phi) \right] P_n^m(\cos(\theta)) \quad (1.2)$$

with:

$$P_n^m(x) = \begin{cases} \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n & : m = 0 \\ \sqrt{\frac{2(n-m)!}{(n+m)!}} (1-x^2)^m \frac{1}{2^n n!} \frac{d^{n+m}}{dx^n} (x^2 - 1)^n & : m > 0 \end{cases} \quad (1.3)$$

g_n^m and h_n^m are the Gauss's coefficients. Note that the sum over n begins with the value $n = 1$ as the index $n = 0$ would correspond to a monopole. The dipole, quadrupole, octupole,... contribution correspond to $n = 1, 2, 3, \dots$. These coefficients varies with time and are tabulated (<https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html>). The coefficient a is the mean radius of the earth (6371.2 km); r , the radial distance from the center of the Earth ; θ , the geocentric colatitude ; ϕ , the east longitude measured from the Greenwich. We note that the Condon-Shortley phase correction $(-1)^m$ is omitted in the definition of the associated Legendre polynomial and the polynomes are normalized using Schmidt quasi-normalization (Winch, Ivers et al. 2005). The relation $\mathbf{B} = -\nabla V$ leads to:

$$\begin{cases} X_c = \mathbf{B}_x = -B_\theta = \frac{1}{r} \frac{\partial V}{\partial \theta} = \sum_{n=1}^N \left(\frac{a}{r} \right)^{n+2} \sum_{m=0}^n \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right] \frac{dP_n^m(\cos \theta)}{d\theta} \\ Y_c = \mathbf{B}_y = B_\phi = \frac{-1}{r \sin \theta} \frac{\partial V}{\partial \phi} = \sum_{n=1}^N \left(\frac{a}{r} \right)^{n+2} \sum_{m=0}^n m \left[g_n^m \sin(m\phi) - h_n^m \cos(m\phi) \right] \frac{P_n^m(\cos \theta)}{\sin \theta} \\ Z_c = \mathbf{B}_z = -B_r = \frac{\partial V}{\partial r} = \sum_{n=1}^N (n+1) \left(\frac{a}{r} \right)^{n+2} \sum_{m=0}^n \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right] P_n^m(\cos \theta) \end{cases} \quad (1.4)$$

Where \mathbf{B}_x , \mathbf{B}_y , \mathbf{B}_z are the field components respectively in the northward, eastward and downward directions. Theses components are expressed in the geocentric referential as recall by the index c

Spherical harmonics normalisation

If we define:

$$\begin{aligned} C_n^m(\theta, \phi) &\equiv P_n^m(\cos \theta) \cos m\theta & : m = 0, 1, 2 \dots n \\ S_n^m(\theta, \phi) &\equiv P_n^m(\cos \theta) \sin m\theta & : m = 1, 2 \dots n \end{aligned} \quad (1.5)$$

the Schmidt quasi-normalization conditions reads (Winch, Ivers et al. 2005):

$$\begin{aligned} \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi C_n^m(\theta, \phi) C_N^M(\theta, \phi) \sin \theta d\theta d\phi &= \frac{1}{2n+1} \delta_n^N \delta_m^M \\ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi S_n^m(\theta, \phi) S_N^M(\theta, \phi) \sin \theta d\theta d\phi &= \frac{1}{2n+1} \delta_n^N \delta_m^M \\ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi C_n^m(\theta, \phi) S_N^M(\theta, \phi) \sin \theta d\theta d\phi &= 0 \quad : \forall n, N, m, M \end{aligned} \quad (1.6)$$

The associated Legendre polynomial value are generated by the scipy function $M, Mp = \text{pmn}(M, N, x)$ where:

$$P_n^m = \begin{bmatrix} P_0^0 & P_1^0 & \dots & P_N^0 \\ 0 & P_1^1 & \dots & P_N^1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_N^M \end{bmatrix}, \quad M[m, n] = P_n^m \quad (1.7)$$

with :

$$P_n^m(x) = \frac{(-1)^m}{2^n n!} \sqrt{(1-x^2)^m} \frac{d^{n+m}}{dx^n} (x^2-1)^n \quad (1.8)$$

To normalize we use:

$$M \leftarrow M \odot \text{Norm_Schmidt} \quad (1.9)$$

where \odot is the component-wise multiplication and :

$$N_{n,m} = \begin{cases} (-1)^m \sqrt{\frac{(2-\delta_m^0)(n-m)!}{(n+m)!}} & : n-|m| \geq 0 \\ 0 & : n-|m| < 0 \end{cases} \quad (1.10)$$

Note that other authors in geophysics use different normalization factors. Stacey (Stacey and Davis) use :

$$N_n^m = \begin{cases} (-1)^m \sqrt{(2-\delta_m^0)(2m+1)} \frac{(n-m)!}{(n+m)!} & : |m| \leq n \\ 0 & : |m| > n \end{cases} \quad (1.11)$$

The function `Norm_Schmidt(m, n)` computes the normalisation matrix (1.10)

The function `Norm_Stacey(m, n)` computes the normalisation matrix (1.11)

Examples :

```
geo.Norm_Stacey(3,4)
>> array([[ 1.,          , 1.,          , 1.,          , 1.,          ],
          [ 0.,          , -1.73205081, -1.,          , -0.70710678, -0.54772256],
          [ 0.,          , 0.,          , 0.64549722, 0.28867513, 0.16666667],
          [ 0.,          , 0.,          , 0.,          , -0.13944334, -0.05270463]])
```

```
geo.Norm_Schmidt(3,4)
>> array([[ 1.,          , 1.,          , 1.,          , 1.,          ],
          [ 0.,          , -1.,          , -0.57735027, -0.40824829, -0.31622777],
          [ 0.,          , 0.,          , 0.28867513, 0.12909944, 0.0745356 ],
          [ 0.,          , 0.,          , 0.,          , -0.05270463, -0.01992048]])
```

Geotetic to geocentric transformation

The computation of the geomagnetic field is done in a geocentric coordinate system. So if we provide The geotetic coordinates we have to convert them into geocentric ones. In the following we deduce the transformation relation used in the function `geotetic_to_geocentric`.

Ellipse equation

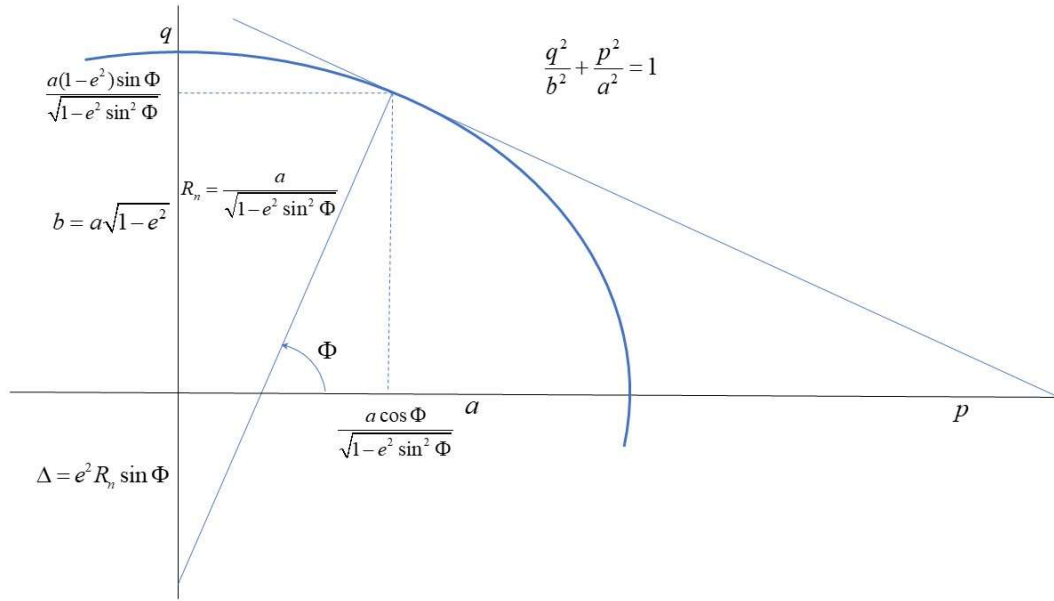


Figure 1 : Ellipse notation convention.

The equation of the ellipse read :

$$\frac{q^2}{b^2} + \frac{p^2}{a^2} = 1 \quad (1.12)$$

The geotectic latitude Φ can be expressed through the derivative :

$$\cot \Phi = -\frac{dq}{dp} = \frac{b^2}{a^2} \frac{q}{p} \quad (1.13)$$

From (1.13) we can express q as :

$$q = p \frac{a^2}{b^2} \tan \Phi \quad (1.14)$$

Using (1.12) and (1.14) we obtain :

$$\begin{cases} p = \frac{a \cos \Phi}{\sqrt{1 - e^2 \sin^2 \Phi}} \\ q = \frac{a(1 - e^2) \sin \Phi}{\sqrt{1 - e^2 \sin^2 \Phi}} \end{cases} \quad (1.15)$$

The prime vertical curvature radius R_n can be deduced from p as :

(1.16)

$$\begin{cases} p = R_n \cos \Phi \\ q = (1 - e^2) R_n \sin \Phi \end{cases} \quad (1.17)$$
[illegible]
$$\frac{\tan \Phi'}{\tan \Phi} = \frac{z}{z + \Delta} = \frac{(1 - e^2)R_n + h}{(1 - e^2)R_n + h + e^2 R_n} = 1 - e^2 \frac{R_n}{R_n + h} \quad (1.18)$$
$$f = \frac{a-b}{a} \quad (1.19)$$

5

As usual the eccentricity reads:

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{f(2-f)} \quad (1.20)$$

$$1 - e^2 = \frac{b^2}{a^2}$$

We can express z and r as :

$$\begin{cases} z = (R_n + h) \sin \Phi - \Delta = [(1 - e^2)R_n + h] \sin \Phi \\ r = (R_n + h) \cos \Phi \end{cases} \quad (1.21)$$

Using (1.21) the geocentric radius is equal to :

$$R' = \sqrt{z^2 + r^2} \quad (1.22)$$

Using (1.22) we can express the geocentric radius using the semi major and the semi minor axis. We obtain (Peddie 1982) :

$$R'^2 = \frac{h^2 + 2h\sqrt{a^2 - (a^2 - b^2)\sin^2 \Phi} + [a^4 - (a^4 - b^4)\sin^2 \Phi]}{a^2 - (a^2 - b^2)\sin^2 \Phi} \quad (1.23)$$

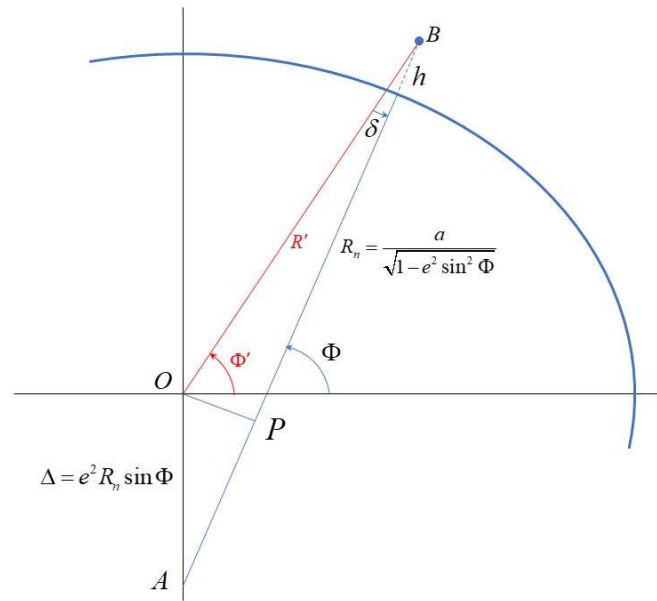


Figure 3 : Computation of $\cos \delta$ and $\sin \delta$.

Using (1.21) and (1.22), the geocentric colatitude can be deduced from :

$$\cos \theta = \frac{z}{\sqrt{r^2 + z^2}} = \frac{\sin \Phi}{\sqrt{\left[\frac{R_n + h}{(1 - e^2)R_n + h} \right]^2 \cos^2 \Phi + \sin^2 \Phi}} \quad (1.24)$$

The relation (1.24) can be expressed using the semi major and minor axis. Using (1.16), (1.20) we obtain (Peddie, 1982) :

$$\cos \theta = \frac{\sin \Phi}{\sqrt{c \cos^2 \Phi + \sin^2 \Phi}}; \quad c = \left[\frac{a^2 + h\sqrt{a^2 - (a^2 - b^2)\sin^2 \Phi}}{b^2 + h\sqrt{a^2 - (a^2 - b^2)\sin^2 \Phi}} \right]^2 \quad (1.25)$$

Using the triangle AOB of the Figure 3 we obtain the relation :

$$\Delta \sin \Phi + R' \cos \delta = R_n + h \quad (1.26)$$

After rearranging :

$$\cos \delta = \frac{1}{R'} \left[h + R_n(1 - e^2) \right] \quad (1.27)$$

Using (1.16) we have :

$$\cos \delta = \frac{1}{R'} \left[h + \frac{a^2}{R_n} \right] \quad (1.28)$$

The length of OP is equal to

$$R' \sin \delta = \Delta \cos \Phi \quad (1.29)$$

So :

$$\sin \delta = \frac{R_n}{R'} e^2 \cos \Phi \sin \Phi \quad (1.30)$$

Computational aspect

The function `geodetic_to_geocentric(ellipsoid, co_latitude, height)` computes the geocentric and the geocentric colatitude using respectively (1.22) and (1.18). The angle

$$\delta = \theta' - \theta \quad (1.31)$$

between the geocentric and the geotetic colatitude is also computed (Figure 4).

The routine uses the tuple `ellipsoid = (a, f-1)`. Depending on the selected convention ellipsoid can be set to

`GRS80 = geo.geomagnetism.GRS80`

`WGS84 = geo.geomagnetism.WGS84`

Exemple :


```

r_geocentric, co_latitude_geocentric, delta =
geo.geodetic_to_geocentric(geo.geomagnetism.WGS84 , 170, 100_000)
>> (6457402.34844737, 2.965925285681976, -0.0011344427083841424)

```

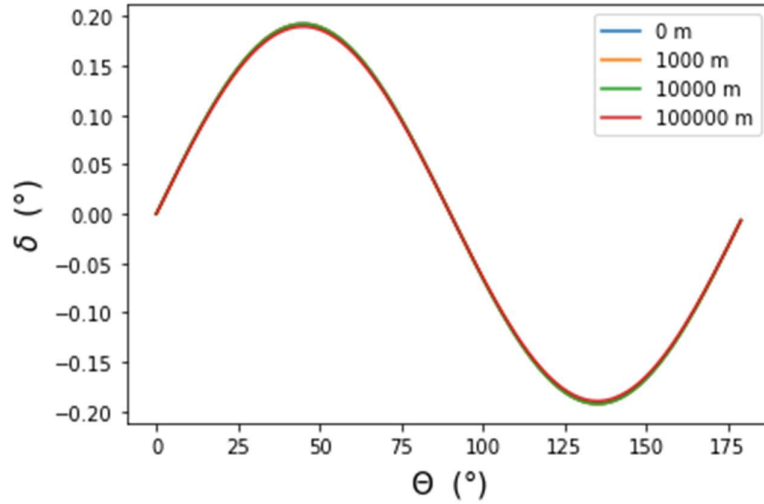


Figure 4 : Variation of δ versus the geotetic colatitude θ and the height h .

The function `geodetic_to_geocentric_IGRF13(ellipsoid, co_latitude, height)` is a translation of a FORTRAN (<https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html>) where the authors use (1.23), (1.28), (1.30) to compute the geocentric radius the cosine and sine of the angle δ .

The function use the tuple `ellipsoid=(a,b)`. Depending on the selected convention ellipsoid can be set to

```

GRS80_ = geo.geomagnetism.GRS80_
WGS84_ = geo.geomagnetism.WGS84_

```

Base transformation

Passing from geocentric to geotetic referential the magnetic field undergoes the following transformation :

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geotetic}} = \begin{bmatrix} \cos \delta & 0 & -\sin \delta \\ 0 & 1 & 0 \\ \sin \delta & 0 & \cos \delta \end{bmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geocentric}} \quad (1.32)$$

with $\delta = \theta' - \theta = \Phi - \Phi'$ (Figure 5). Using Peddie notation (Peddie, 1982) we have :

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos \delta & 0 & -\sin \delta \\ 0 & 1 & 0 \\ \sin \delta & 0 & \cos \delta \end{bmatrix} \begin{pmatrix} X_c \\ Y_c \\ Z_c \end{pmatrix} \quad (1.33)$$

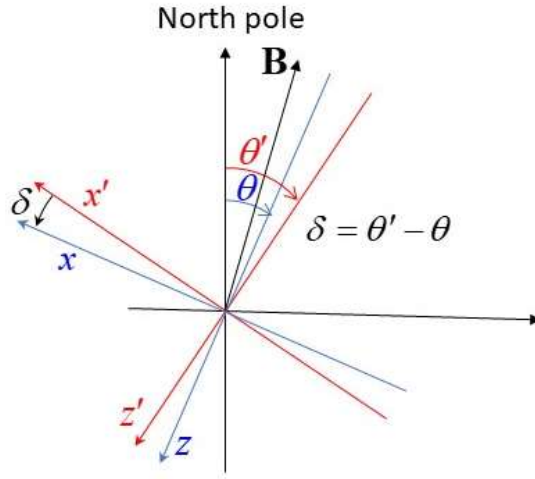


Figure 5 : Variation of δ versus the geotetic colatitude θ and the height h .

Using the notations of the Figure 6, the geomagnetic horizontal intensity H , total intensity F , declination D and inclination I can be obtained from :

$$\begin{cases} H = \sqrt{X^2 + Y^2} \\ F = \sqrt{H^2 + Z^2} \\ D = \tan^{-1} \frac{Y}{X} \\ I = \tan^{-1} \frac{Z}{H} \end{cases} \quad (1.34)$$

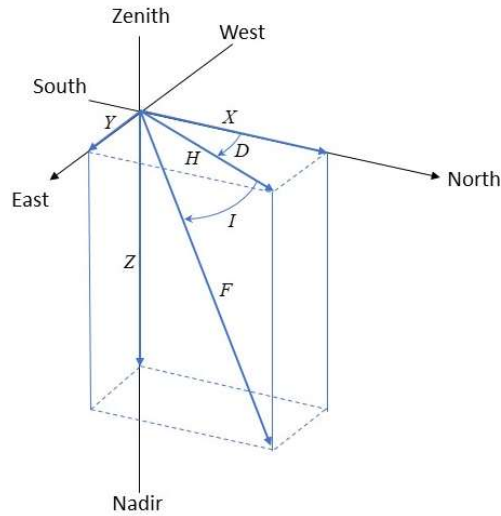


Figure 6 : Geomagnetic conventions and notations from (Chullia, 2020)

Spherial harmonic coefficients

Gauss coefficients

The main ingredient for the computing of the geomagnetic field are the spherical harmonic coefficients g_m^n and h_m^n . These coefficients depend on the geocentric latitude on the longitude, and on the time.

The package `geomagnetism` provide several functions to read tabulated coefficients

```
import geomagnetism as geo
file = "IGRF13.COF" # downloaded from
https://www.ngdc.noaa.gov/IAGA/vmod/coeffs/igrf13coeffs.txt
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years=
geo.read_IGRF13_COF(file)
m=1
n=3
years="1965"
h = dic_dic_h[years][(m,n)]
print(f'Spherial harmonic coefficients h(m={m},n={n}) for year = {years} :{h}')
>>Spherial harmonic coefficients h(m=1,n=3) for year = 1965 :-404.0
```

Tableau 1 : Gauss coefficients from different sources.

File name	Deb	End	Source
WMM_2020.COF	2020	2020	https://www.ngdc.noaa.gov/geomag/WMM/soft.shtml
WMM_2015.COF	2015	2015	https://www.ngdc.noaa.gov/geomag/WMM/soft.shtml
IGRF13.COF	1900	2020	https://www.ngdc.noaa.gov/IAGA/vmod/coeffs/igrf13coeffs.txt
IGRF13coeffs.txt	1900	2020	https://www.ngdc.noaa.gov/IAGA/vmod/coeffs/igrf13coeffs.txt
FORTTRAN_1900_1995.txt	1900	1995	https://www.ngdc.noaa.gov/IAGA/vmod/igrf13.f

IGRF13.COF and IGRF13coeffs.txt contain the same data with a different format.

Gauss coefficients secular variation

As the Gauss coefficients of the geomagnetic field vary with time their secular variation \dot{g}_m^n and \dot{h}_m^n expressed in nT/year are tabulated. We have :

$$\begin{cases} g_m^n(t) = g_m^n(t_0) + \dot{g}_m^n(t_0)(t - t_0) \\ h_m^n(t) = h_m^n(t_0) + \dot{h}_m^n(t_0)(t - t_0) \end{cases} \quad (1.35)$$

The secular vaiation are stored in the dictionaries `dic_dic_SV_h`, `dic_dic_SV_g`. Using these coefficients it is straightforward to compute the secular variation of the geomagnetic field componants. We have (Chullia, 2020):

$$\begin{cases} \dot{X}_c = \dot{\mathbf{B}}_x = -\dot{B}_\theta = \frac{1}{r} \frac{\partial \dot{V}}{\partial \theta} = \sum_{n=1}^N \left(\frac{a}{r} \right)^{n+2} \sum_{m=0}^n \left[\dot{g}_m^n \cos(m\phi) + \dot{h}_m^n \sin(m\phi) \right] \frac{dP_n^m(\cos \theta)}{d\theta} \\ \dot{Y}_c = \dot{\mathbf{B}}_y = \dot{B}_\phi = \frac{-1}{r \sin \theta} \frac{\partial \dot{V}}{\partial \phi} = \sum_{n=1}^N \left(\frac{a}{r} \right)^{n+2} \sum_{m=0}^n m \left[\dot{g}_m^n \sin(m\phi) - \dot{h}_m^n \cos(m\phi) \right] \frac{P_n^m(\cos \theta)}{\sin \theta} \\ \dot{Z}_c = \dot{\mathbf{B}}_z = -\dot{B}_r = \frac{\partial \dot{V}}{\partial r} = \sum_{n=1}^N (n+1) \left(\frac{a}{r} \right)^{n+2} \sum_{m=0}^n \left[\dot{g}_m^n \cos(m\phi) + \dot{h}_m^n \sin(m\phi) \right] P_n^m(\cos \theta) \end{cases} \quad (1.36)$$

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} \cos \delta & 0 & -\sin \delta \\ 0 & 1 & 0 \\ \sin \delta & 0 & \cos \delta \end{bmatrix} \begin{pmatrix} \dot{X}_c \\ \dot{Y}_c \\ \dot{Z}_c \end{pmatrix} \quad (1.37)$$

$$\begin{cases} \dot{H} = \frac{X\dot{X} + Y\dot{Y}}{H} \\ \dot{F} = \frac{X\dot{X} + Y\dot{Y} + Z\dot{Z}}{F} \\ \dot{I} = \frac{H\dot{Z} - Z\dot{H}}{F^2} \\ \dot{D} = \frac{X\dot{Y} - Y\dot{X}}{H^2} \end{cases} \quad (1.38)$$

Other programs

- In line calculators

<https://www.ngdc.noaa.gov/geomag/calculators/magcalc.shtml>

<http://wdc.kugi.kyoto-u.ac.jp/igrf/gggm/index.html>

permits the computation of the magnetic declination.

- a FORTRAN code available can be downloaded from :

<https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html> .

Quote: "This code is a synthesis routine for the 13th generation IGRF as agreed in December 2019 by IAGA Working Group V-MOD. It is valid 1900.0 to 2025.0 inclusive. Values for dates from 1945.0 to 2015.0 inclusive are definitive, otherwise they are non-definitive. Reference radius remains as 6371.2 km - it is NOT the mean radius (= 6371.0 km) but 6371.2 km is what is used in determining the coefficients."

- a C code available along with the Geomag 7.0 software (Windows version) :

<https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html>

- a PYTHON code :

<https://pypi.org/project/geomag/#files>

Bibliography

Campbell, W. H. (2007). Introduction to Geomagnetic Fields Cambridge University Press.

Chullia, A et al. (2020) "The US/UK World Magnetic Model for 2020-2025: Technical Report, National Centers for Environmental Information, NOAA"

Peddie, N. W. (1982). "International Geomagnetic Reference Field : the third generation." J. Geomag. Geoelectr **34**: 309-326.

Stacey, F. D. and P. M. Davis Physics of the Earth, Cambridge University Press.

Winch, D. E., et al. (2005). "Geomagnetism and Schmidt quasi-normalization." Geophys. J. Int. **160**: 487-454.