Contents

Introduction

The package geomagnetim intends to serve pedagogical purpose rather to replace well established FORTRAN or C progam develloped by academic institutions. In contrast with these programs, which favor compactness and time execution, the geomagnetic package ,tentatively, focus on lisibility.

Geomagnetism calculation

As the terrestrial magnetic field obeys both $\nabla \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$, it can be shown that the magnetic field can be expressed as the gradient of a scalar potential V which satisfies the Laplace equation:

$$\Delta \mathrm{V} = 0 \tag{1.1}$$

For a spherical geometry the geomagnetic potential is given by the spherical harmonic expension (SH) (Stacey and Davis , Campbell 2007):

$$V(r,\theta,\phi,t) = a\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left[g_n^m(t)\cos(m\phi)+h_n^m(t)\sin(m\phi)\right]P_n^m\left(\cos(\theta)\right) \tag{1.2}$$

with:

$$P_n^m\left(x\right) = \begin{cases} \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n & : m = 0 \\ \sqrt{\dfrac{2(n-m)!}{(n+m)!}}\left(1-x^2\right)^m \dfrac{1}{2^n n!}\dfrac{d^{n+m}}{dx^n}\left(x^2-1\right)^n & : m > 0 \end{cases} \tag{1.3}$$

g_n^m and h_n^m are the Gauss's coefficients. Note that the sum over n begins with the the value $n = 1$ as the index $n = 0$ would correspond to a monopole. The dipole, quadrupole, octupole,... contribution correspond to $n = 1, 2, 3, \ldots$ These coefficients varies with time and are tabulated (https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html). The coefficient a is the mean radius of the earth

(6371 km); r,θ,ϕ are the geocentric spherical coordinates (radius,geocentric colatitude and longitude). We note that the Condon-Shortley phase correction $(-1)^m$ is omitted in the definition of the associated Legendre polynomial and the polynomes are normalized using Schmidt quasi-normalization(Winch, Ivers et al. 2005). The relation $\mathbf{B}=-\nabla V$ leads to:

$$\begin{cases} B_x = -B_\theta = \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} = \displaystyle\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m\cos(m\phi)+h_n^m\sin(m\phi)\right]\frac{dP_n^m(\cos\theta)}{d\theta} \\ B_y = B_\phi = \dfrac{-1}{r\sin\theta}\dfrac{\partial V}{\partial \phi} = \displaystyle\sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n} m\left[g_n^m\sin(m\phi)-h_n^m\cos(m\phi)\right]\frac{P_n^m(\cos\theta)}{\sin\theta} \\ B_z = -B_r = \dfrac{\partial V}{\partial r} = \displaystyle\sum_{n=1}^{N}(n+1)\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m\cos(m\phi)+h_n^m\sin(m\phi)\right]P_n^m(\cos\theta) \end{cases} \quad (1.4)$$

Where $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{B}_z$ are the field components respectively in the northward, eastward and downward directions.

Spherical harmonics normalisation

If we define:

$$\begin{aligned} C_n^m(\theta,\phi) &\equiv P_n^m(\cos\theta)\cos m\theta \quad : m=0,1,2\cdots n \\ S_n^m(\theta,\phi) &\equiv P_n^m(\cos\theta)\sin m\theta \quad : m=1,2\cdots n \end{aligned} \quad (1.5)$$

the Schmidt quasi-normalization conditions reads (Winch, Ivers et al. 2005):

$$\begin{aligned} &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C_n^m(\theta,\phi)C_N^M(\theta,\phi)\sin\theta d\theta d\phi = \frac{1}{2n+1}\delta_n^N\delta_m^M \\ &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} S_n^m(\theta,\phi)S_N^M(\theta,\phi)\sin\theta d\theta d\phi = \frac{1}{2n+1}\delta_n^N\delta_m^M \\ &\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C_n^m(\theta,\phi)S_N^M(\theta,\phi)\sin\theta d\theta d\phi = 0 \quad :\forall n,N,m,M \end{aligned} \quad (1.6)$$

The associated Legendre polynomial value are generated by the scipy function M,Mp=lpmn(M,N,x) where:

$$P_n^m = \begin{bmatrix} P_0^0 & P_1^0 & \cdots & P_N^0 \\ 0 & P_1^1 & \cdots & P_N^1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_N^M \end{bmatrix}, \quad M[m,n]=P_n^m \quad (1.7)$$

with :

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}\sqrt{\left(1-x^2\right)^m}\frac{d^{n+m}}{dx^n}\left(x^2-1\right)^n \quad (1.8)$$

To normalize we use:

$$M \leftarrow M \odot Norm\_Schmidt \tag{1.9}$$

where $\odot$ is the component-wise multiplication and :

$$N_{n,m} = \begin{cases} (-1)^m \sqrt{\dfrac{\left(2-\delta_m^0\right)(n-m)!}{(n+m)!}} & : n-|m| \geq 0 \\ 0 \quad : n-|m| < 0 \end{cases} \tag{1.10}$$

Note that other authors in geophysics use different normalization factors. Stacey (Stacey and Davis) use :

$$N_n'^m = \begin{cases} (-1)^m \sqrt{\left(2-\delta_m^0\right)\left(2m+1\right)\dfrac{(n-m)!}{(n+m)!}} & : |m| \leq n \\ 0 \quad : |m| > n \end{cases} \tag{1.11}$$

The function `Norm_Schimdt(m, n)` computes the normalisation matrix (1.10)

The function `Norm_ Stacey (m, n)` computes the normalisation matrix (1.11)

Exemples :

```
geo.Norm_Stacey(3,4)
>> array([[ 1.        ,  1.        ,  1.        ,  1.        ,  1.        ],
       [ 0.        , -1.73205081, -1.        , -0.70710678, -0.54772256],
       [ 0.        ,  0.        ,  0.64549722,  0.28867513,  0.16666667],
       [ 0.        ,  0.        ,  0.        , -0.13944334, -0.05270463]])


geo.Norm_ Schimdt (3,4)
>> array([[ 1.        ,  1.        ,  1.        ,  1.        ,  1.        ],
       [ 0.        , -1.        , -0.57735027, -0.40824829, -0.31622777],
       [ 0.        ,  0.        ,  0.28867513,  0.12909944,  0.0745356 ],
       [ 0.        ,  0.        ,  0.        , -0.05270463, -0.01992048]])
```

Geotetic to geocentric transformation

The computation of the geomagnetic field is done in a geocentric coordinate system. So if we provide The geotetic coordinates we have to convert them into geocentric ones. In the following we deduce the transformation relation used in the function `geodetic_to_geocentric`.

Ellipse equation

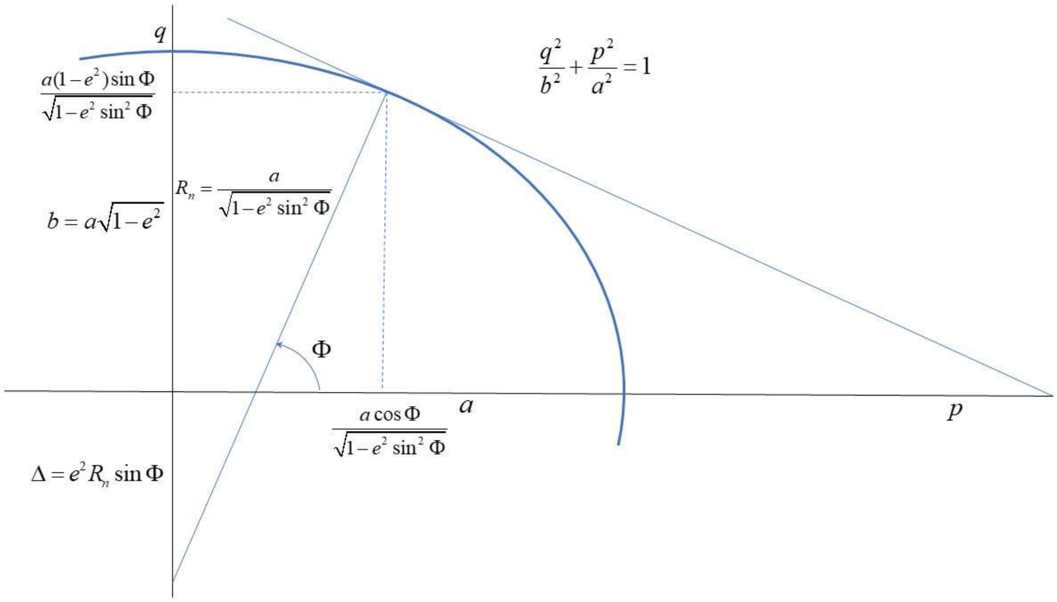

Figure 1 : Ellipse notation convention.

The equation of the ellipse read :

$$\frac{q^2}{b^2}+\frac{p^2}{a^2}=1 \tag{1.12}$$

The geotetic latitude Φ can be be expressed through the derivative :

$$\cot\Phi=-\frac{dq}{dp}=\frac{b^2}{a^2}\frac{q}{p} \tag{1.13}$$

From (1.13) we can express q as :

$$q=p\frac{a^2}{b^2}\tan\Phi \tag{1.14}$$

Using (1.12) and (1.14) we obtain :

$$\begin{cases} p=\dfrac{a\cos\Phi}{\sqrt{1-e^2\sin^2\Phi}} \\ q=\dfrac{a(1-e^2)\sin\Phi}{\sqrt{1-e^2\sin^2\Phi}} \end{cases} \tag{1.15}$$

The prime vertical curvature radius R_n can be deduced from p as :

$$R_n = \frac{p}{\cos\Phi} = \frac{a}{\sqrt{1-e^2\sin^2\Phi}}$$
$$R_n = \frac{a^2}{\sqrt{a^2-(a^2-b^2)\sin^2\Phi}} \tag{1.16}$$

Using the prime vertical curvature we can re-express p and q as :

$$\begin{cases} p = R_n \cos\Phi \\ q = (1-e^2)R_n \sin\Phi \end{cases} \tag{1.17}$$

Geotetic to geocentric transformation

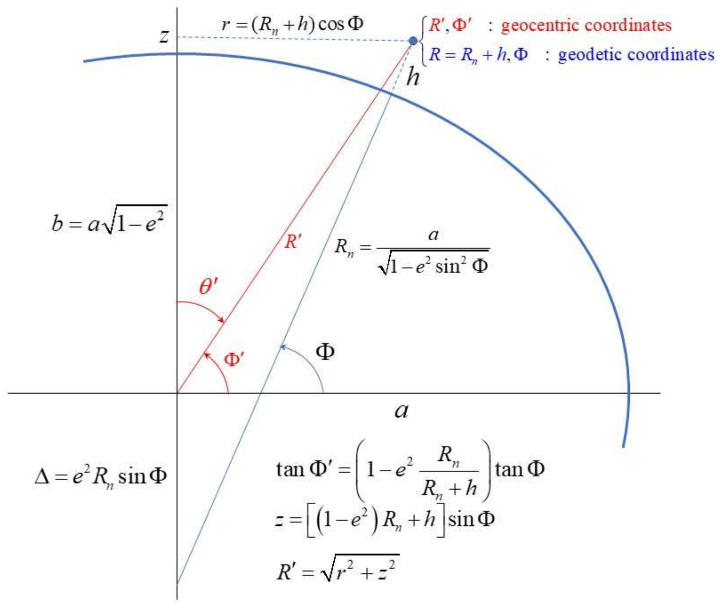

Figure 2 : Geotetic and geocentric notation convention.

Relation between the geotetic colatitude Φ and the geocentric colatitude Φ' :

Using the conventions of Figure 2 we have

$$\frac{\tan\Phi'}{\tan\Phi} = \frac{z}{z+\Delta} = \frac{(1-e^2)R_n+h}{(1-e^2)R_n+h+e^2R_n} = 1-e^2\frac{R_n}{R_n+h} \tag{1.18}$$

The flattening is defined as follow:

$$f = \frac{a-b}{a} \tag{1.19}$$

The geodetic reference ellipsoids are specified by giving the reciprocal flattening f^{-1}

As usual the eccentricity reads:

$$
\begin{aligned}
&e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{f(2-f)} \\
&1 - e^2 = \frac{b^2}{a^2}
\end{aligned} \tag{1.20}
$$

We can express z and r as :

$$
\begin{cases}
z = (R_n + h)\sin\Phi - \Delta = \left[(1-e^2)R_n + h\right]\sin\Phi \\
r = (R_n + h)\cos\Phi
\end{cases} \tag{1.21}
$$

Using (1.21) the geocentric radius is equal to :

$$
R' = \sqrt{z^2 + r^2} \tag{1.22}
$$

Using (1.22) we can express the geocentric radius using the semi major and the semi minor axis. We obtain (Peddie 1982) :

$$
R'^2 = \frac{h^2 + 2h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi} + \left[a^4 - (a^4 - b^4)\sin^2\Phi\right]}{a^2 - (a^2 - b^2)\sin^2\Phi} \tag{1.23}
$$

Computational aspect

The function `geodetic_to_geocentric(ellipsoid, co_latitude, height)` computes the geocentric and the geocentric colatitude using respectively (1.22) and (1.18). The angle

$$
\delta = \theta' - \theta \tag{1.24}
$$

between the geocentric and the geotetic colattitude is also computed (Figure 3).

The routine uses the tuple ellipsoid = $\left(a, f^{-1}\right)$. Depending on the selected convention ellipsoid can be set to

GRS80 = geo.geomagnetism.GRS80

WGS84 = geo.geomagnetism.WGS84

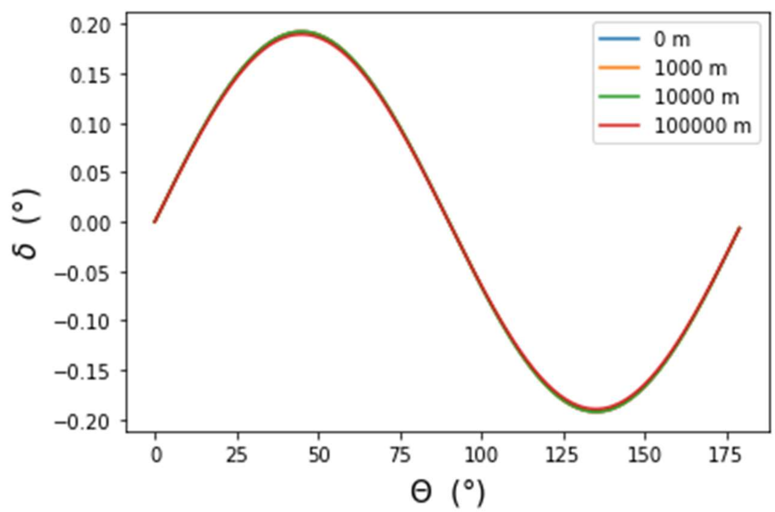

Figure 3 : Variation of δ versus the geotetic colatitude θ and the height h.

Base transformation

Passing from geocentric to geotetic referential the magnetic field undergoes the following transfomation :

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geodetic}} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geocentric}} \tag{1.25}$$

with $\delta = \theta' - \theta = \Phi - \Phi'$ (Figure 4).

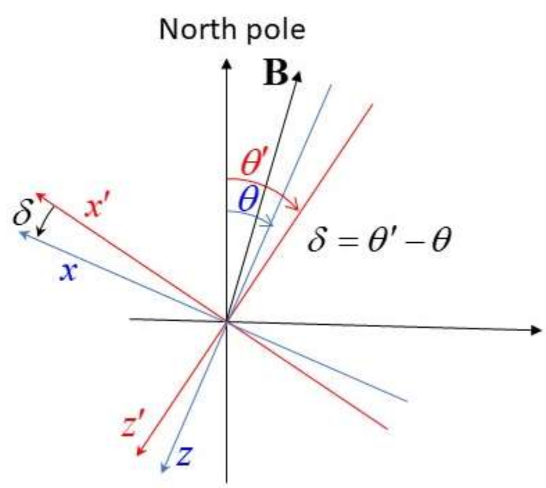

Figure 4 : Variation of δ versus the geotetic colatitude θ and the height h.

Spherial harmonic coefficients

The main ingredient for the computing of the geomagnetic field are the spherical harmonic coefficients g_m^n and h_n^m. These coefficients depend on the geocentic latitude on the longitude, and on the time. The package `geomagnetism` provide several functions to read the tabulated cofficients

Other programs

- In line calculators

  https://www.ngdc.noaa.gov/geomag/calculators/magcalc.shtml

  http://wdc.kugi.kyoto-u.ac.jp/igrf/gggm/index.html

permits the computation of the magnetic declination.

- a FORTRAN code available can be downloaded from :

  https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html .

Qote: "This code is is a synthesis routine for the 13th generation IGRF as agreed in December 2019 by IAGA Working Group V-MOD. It is valid 1900.0 to 2025.0 inclusive. Values for dates from 1945.0 to 2015.0 inclusive are definitive, otherwise they are non-definitive. Reference radius remains as 6371.2 km - it is NOT the mean radius (= 6371.0 km) but 6371.2 km is what is used in determining the coefficients.'

- a C code available along with the Geomag 7.0 software (Windows version) :

  https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html

- a PYTHON code :

  https://pypi.org/project/geomag/#files

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Peddie, N. W. (1982). "International Geomagnetic Reference Field : the third generation." J. Geomag. Geolectr **34**: 309-326.

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