Geomagnetism package

François Bertin

11/2020

Abstract

In this note we derive all the equations used in the geomagntism package.

Contents

1 Introduction

The package `geomagnetim` intends to serve pedagogical purposes rather to replace the well established FORTRAN or C progam develloped by academic institutes. In contrast with these programs, which favor compactness and minimize time execution, the geomagnetic package, tentatively, focuses on lisibility. You can download geomagnetis using :

pip install geomagnetism

Use case examples are given in a companion Jupyter Notebook

2 Geomagnetism calculation

As the terrestrial magnetic field obeys both $\nabla \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$, it can be shown that the magnetic field can be expressed as the gradient of a scalar potential V which satisfies the Laplace equation:

$$\Delta \text{V} = 0. \tag{2.1}$$

For a spherical geometry the geomagnetic potential is given by the following spherical harmonic expension (SH) [1, 2, 4]:

$$V(r, \theta, \phi, t) = a \sum_{n=1}^{N} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} \left[g_n^m(t)\cos(m\phi) + h_n^m(t)\sin(m\phi)\right] P_{(s),n}^m\left(\cos(\theta)\right) \tag{2.2}$$

where $P_{(s),n}^m(x)$ are the Schmidt quasi-normalized associated Legendre polynomials (for more details see section 3):

$$P_{(s),n}^m(x) = \begin{cases} \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n & : m = 0 \\ \sqrt{\frac{2(n-m)!}{(n+m)!}}\left(1 - x^2\right)^m \frac{1}{2^n n!} \frac{d^{n+m}}{dx^n}\left(x^2 - 1\right)^n & : m > 0 \end{cases} \tag{2.3}$$

where g_n^m and h_n^m are the Gauss's coefficients. Note that the sum over n begins with the the value $n = 1$ as the index $n = 0$ would correspond to a monopole. The dipole, quadrupole, octupole,... contribution correspond to $n = 1, 2, 3, \ldots$ These coefficients vary with time and are tabulated by the National Oceanic and Atmospheric Administration. The coefficient a is the mean radius of the earth (6371.2 km); r ,the radial distance from the center of the Earth ; θ , the geocentric colatitude ; ϕ , the east longitude measured from Greenwich. We note that the Condon-Shortley phase correction $(-1)^m$ is omitted in the definition of the associated Legendre polynomial and the polynomes are normalized using Schmidt quasi-normalization [5]. The relation $\mathbf{B} = -\nabla V$ leads to:

$$\begin{aligned} X_c \equiv & \\ & \mathbf{B}_x = -B_\theta = \frac{1}{r}\frac{\partial V}{\partial \theta} = \\ & \sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m\cos(m\phi) + h_n^m\sin(m\phi)\right]\frac{dP_{(s),n}^m(\cos\theta)}{d\theta}, \\ Y_c \equiv & \\ & \mathbf{B}_y = B_\phi = \frac{-1}{r\sin\theta}\frac{\partial V}{\partial \phi} = \\ & \sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n} m\left[g_n^m\sin(m\phi) - h_n^m\cos(m\phi)\right]\frac{P_{(s),n}^m(\cos\theta)}{\sin\theta}, \\ Z_c \equiv & \\ & \mathbf{B}_z = -B_r = \frac{\partial V}{\partial r} = \\ & \sum_{n=1}^{N}(n+1)\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m\cos(m\phi) + h_n^m\sin(m\phi)\right]P_{(s),n}^m(\cos\theta), \end{aligned} \tag{2.4}$$

where $\mathbf{B}_x$, $\mathbf{B}_y$, $\mathbf{B}_z$ are the field components respectively in the northward, eastward and downward directions. Theses components are expressed in the geocentric referential as recall by the index c. The parameter N stands for the order of the SH decomposition.

3 Spherical harmonics normalisation

In the field of geomagnetism the Schmidt quasi-normalized Legendre polynomials $P^m_{(s),n}$ are widely used [1]. They are proportional to the Legendre polynomial:

$$P^m_{(s),n} = N^m_n P^m_n \tag{3.2}$$

where the the associated Legendre polynomials [2] are defined as[3] [6]:

$$P^m_n(x) = \frac{(-1)^m}{2^n n!}\sqrt{(1-x^2)^m}\frac{d^{n+m}}{dx^{n+m}}\left(x^2-1\right)^n. \tag{3.3}$$

and where the normalization coefficients N^m_n are equal to [5]:

$$\boxed{N_{n,m} = \begin{cases} (-1)^m\sqrt{\frac{(2-\delta^0_m)(n-m)!}{(n+m)!}} & : n-|m| \geq 0 \\ 0 & : n-|m| < 0 \end{cases}} \tag{3.4}$$

The Schmidt quasi-normalizated polynomials,for $\forall n$, $\forall N$, $\forall m$, $\forall M$, obey the following normalization [5]:

$$\begin{aligned}
\tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C^m_n(\theta,\phi)C^M_N(\theta,\phi)\sin\theta d\theta d\phi &= \tfrac{1}{2n+1}\delta^N_n\delta^M_m \\
\tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} S^m_n(\theta,\phi)S^M_N(\theta,\phi)\sin\theta d\theta d\phi &= \tfrac{1}{2n+1}\delta^N_n\delta^M_m \\
\tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} C^m_n(\theta,\phi)S^M_N(\theta,\phi)\sin\theta d\theta d\phi &= 0
\end{aligned} \tag{3.5}$$

where the following notations are used:

$$\begin{aligned}
C^m_n(\theta,\phi) &\equiv P^m_{(s),n}\cos\theta\cos m\theta \quad &: m = 0,1,2\cdots n \\
S^m_n(\theta,\phi) &\equiv P^m_{(s),n}\cos\theta\sin m\theta \quad &: m = 1,2\cdots n
\end{aligned} \tag{3.6}$$

For computational efficiency we define the matrice $\underline{\underline{\mathbf{P}}}$ and $\underline{\underline{\mathbf{P}}}_{(s)}$:

$$\underline{\underline{\mathbf{P}}} = \begin{bmatrix} P^0_0 & P^0_1 & \cdots & P^0_N \\ 0 & P^1_1 & \cdots & P^1_N \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P^M_N \end{bmatrix}, \tag{3.7}$$

$$\underline{\underline{\mathbf{P}}}_{(s)} = \begin{bmatrix} P^0_{(s),0} & P^0_{(s),1} & \cdots & P^0_{(s),N} \\ 0 & P^1_{(s),1} & \cdots & P^1_{(s),N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P^M_{(s),N} \end{bmatrix} \tag{3.8}$$

and we compute $\underline{\underline{\mathbf{P}}}_{(s)}$ as the matrix component-wise product:

$$\underline{\underline{\mathbf{P}}}_{(s)} = \underline{\underline{\mathbf{P}}} \odot \underline{\underline{\mathbf{N}}}, \tag{3.9}$$

In the `geomagnetism` package:

(a) The function `Norm_Schimdt(m,n)` computes the normalisation matrix (3.8) using coefficients (3.4).

(b) The function `Norm_Stacey(m,n)` computes the normalisation matrix using coefficients (3.1).

[1]Note that other authors in geophysics use different normalization factors For example, Stacey [4] uses :

$$N'^m_n = \begin{cases} (-1)^m\sqrt{(2-\delta^0_m)(2m+1)\frac{(n-m)!}{(n+m)!}} & : |m| \leq n \\ 0 \quad : |m| > n \end{cases} \tag{3.1}$$

.

[2]In the `geomaganetism` package,the associated Legendre polynomials be computeded by the scipy function `lpmn(M,N,x)`

[3]To stick with the `scipy` package conventions, these polynomials are defined using the Condon-Shortley phase correction $(-1)^m$.

```
# Exemple of computation of normalization matrix
geo.Norm_Stacey(3,4)
>> array([[ 1.         ,  1.         ,  1.         ,  1.         ,  1.         ],
       [ 0.         , -1.73205081, -1.         , -0.70710678, -0.54772256],
       [ 0.         ,  0.         ,  0.64549722,  0.28867513,  0.16666667],
       [ 0.         ,  0.         ,  0.         , -0.13944334, -0.05270463]])

geo.Norm_ Schimdt (3,4)
>> array([[ 1.         ,  1.         ,  1.         ,  1.         ,  1.         ],
       [ 0.         , -1.         , -0.57735027, -0.40824829, -0.31622777],
       [ 0.         ,  0.         ,  0.28867513,  0.12909944,  0.0745356 ],
       [ 0.         ,  0.         ,  0.         , -0.05270463, -0.01992048]])
```

4 Geotetic to geocentric transformation

The computation of the geomagnetic field is done in a geocentric coordinate system. So if we provide the geotetic coordinates we have to convert them into geocentric ones. In the following we deduce the transformation relation used in the function `geodetic_to_geocentric`.

4.1 Ellipse equation

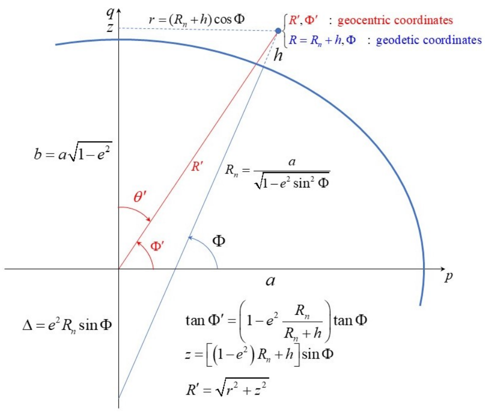

Figure 1: Ellipse notations convention.

Using the notation of the figure 1, the equation of the ellipse reads :

$$\frac{q^2}{b^2}+\frac{p^2}{a^2}=1. \tag{4.1}$$

The geotetic latitude Φ can be be expressed through the derivative :

$$\cot\Phi=-\frac{dq}{dp}=\frac{b^2}{a^2}\frac{p}{q} \tag{4.2}$$

From equation (4.2) we can express q as :

$$q=p\frac{b^2}{a^2}\tan\Phi \tag{4.3}$$

Using equations (4.1) and (4.3) we can express the ellipse coordinates p and q as a function of the geotetic latitude Φ as :

$$p=\frac{a\cos\Phi}{\sqrt{1-e^2\sin^2\Phi}} \tag{4.4a}$$

$$q=\frac{a(1-e^2)\sin\Phi}{\sqrt{1-e^2\sin^2\Phi}} \tag{4.4b}$$

where $e \equiv \sqrt{1-\frac{b^2}{a^2}}$ is the eccentricity. The prime vertical curvature radius R_n (see figure 1) can be deduced from p as :

$$R_n = \frac{p}{\cos\Phi} = \frac{a}{\sqrt{1-e^2\sin^2\Phi}} \tag{4.5a}$$

$$R_n = \frac{a^2}{\sqrt{a^2-(a^2-b^2)\sin^2\Phi}} \tag{4.5b}$$

Using the prime vertical curvature we can re-express p and q as :

$$p = R_n \cos\Phi \tag{4.6a}$$

$$q = (1-e^2)R_n \sin\Phi \tag{4.6b}$$

4.2 Geotetic to geocentric transformation

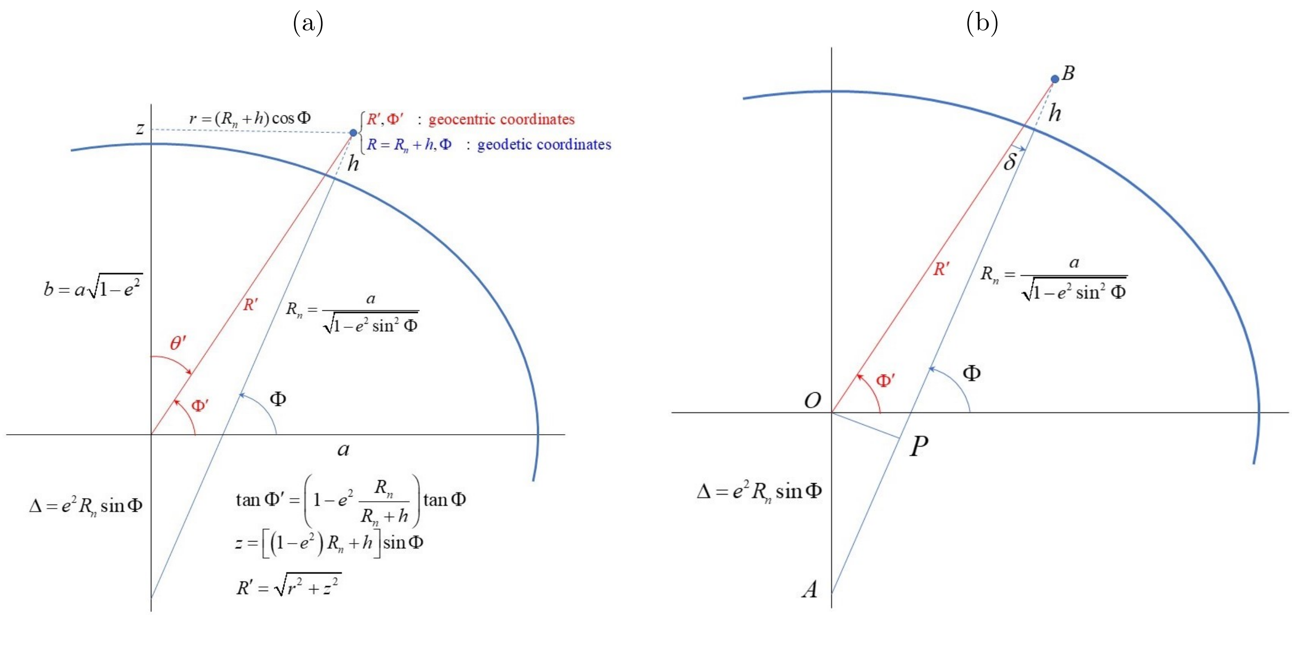

Figure 2: Notation conventions for: (a) geotetic and geocentric notation; (b) the computation of $\cos\delta$ and $\sin\delta$.

In this section we derive the relation between the geotetic colatitude Φ and the geocentric colatitude Φ' Using the conventions of the figure 4.2, we have:

$$\boxed{\frac{\tan\Phi'}{\tan\Phi} = \frac{z}{z+\Delta} = \frac{(1-e^2)R_n+h}{(1-e^2)R_n+h+e^2R_n} = 1-e^2\frac{R_n}{R_n+h}} \tag{4.7}$$

The flattening f is defined as follow:

$$f = \frac{a-b}{a} \tag{4.8}$$

Usually, the geodetic reference ellipsoid is specified by its reciprocal flattening f^{-1}. The reciprocal flattening is related to the eccentricity e by:

$$\begin{aligned} e &= \sqrt{1-\tfrac{b^2}{a^2}} = \sqrt{f(2-f)} \\ 1-e^2 &= \tfrac{b^2}{a^2} \end{aligned} \tag{4.9}$$

We can express z and r as :

$$\begin{cases} z = (R_n+h)\sin\Phi - \Delta = \left[(1-e^2)R_n+h\right]\sin\Phi \\ r = (R_n+h)\cos\Phi \end{cases} \tag{4.10}$$

where h is the height above the reference ellipsoid. Thanks to equation (4.10) we can obtain the geocentric radius as :

$$R' = \sqrt{z^2 + r^2} \tag{4.11}$$

Using the equations (4.5b) and (4.10) we express the geocentric as a function both the semi major and the semi minor axis. We obtain [3]:

$$\boxed{R'^2 = \frac{h^2 + 2h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi} + \left[a^4 - (a^4 - b^4)\sin^2\Phi\right]}{a^2 - (a^2 - b^2)\sin^2\Phi}} \tag{4.12}$$

Combining the equations (4.11) and (4.10) we can express the geocentric colatitude as:

$$\cos\theta = \frac{z}{\sqrt{r^2 + z^2}} = \frac{\sin\Phi}{\sqrt{\left[\frac{R_n + h}{(1-e^2)R_n + h}\right]^2 \cos^2\Phi + \sin^2\Phi}} \tag{4.13}$$

The relation (4.13) can be rewritted using the semi major and minor axis. Using equations (4.5b) and (4.13) we obtain [3]:

$$\cos\theta = \frac{\sin\Phi}{\sqrt{c\cos^2\Phi + \sin^2\Phi}} \quad \text{where} \quad c = \left[\frac{a^2 + h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi}}{b^2 + h\sqrt{a^2 - (a^2 - b^2)\sin^2\Phi}}\right]^2 \tag{4.14}$$

In the triangle `AOB` of the figure 4.2 the length of `AB` leads to the equality :

$$\Delta\sin\Phi + R'\cos\delta = R_n + h, \tag{4.15}$$

and, after rearanging:

$$\cos\delta = \frac{1}{R'}\left[h + R_n(1 - e^2(\sin\Phi)^2\right]. \tag{4.16}$$

Using the relation (4.5) equation (4.16) reads :

$$\boxed{\cos\delta = \frac{1}{R'}\left[h + \frac{a^2}{R_n}\right]}. \tag{4.17}$$

The length of the common side `OP` of the two rectangles triangle `AOP` and `BOP` of the figure 4.2 leads to the equality :

$$R'\sin\delta = \Delta\cos\Phi.$$

So:

$$\boxed{\sin\delta = \frac{R_n}{R'}e^2\cos\Phi\sin\Phi} \tag{4.18}$$

4.3 Computational aspect

The function **`geodetic_to_geocentric(ellipsoid, co_latitude, height)`** computes the geocentric colatitude and radius using respectively the equations (4.7) and (4.11). The angle $\delta = \theta' - \theta$ between the geocentric and the geotetic colattitude is also computed. The figure 3 shows the variation of δ versus the geotetic colatitude θ and the height h.

```
# Exemple of computation of r_geocentric, co_latitude_geocentric, delta
import geomagnetism as geo
ellipsoid = geo.geomagnetism.WGS84 # tuple (mean earth radius in meter, inverse flattening)
r_geocentric, co_latitude_geocentric, delta = geo.geodetic_to_geocentric(ellipsoid , 170,
                                                100_000)
>> (6457402.34844737, 2.965925285681976, -0.0011344427083841424)
ellipsoid = geo.geomagnetism.GRS80
r_geocentric, co_latitude_geocentric, delta = geo.geodetic_to_geocentric(ellipsoid , 170,
                                                100_000)
>> (6457402.348345751, 2.9659252856763882, -0.001134442713972117)
```

```
# Exemple of computation of r_geocentric, cos(co_latitude_geocentric), sin(
                                                co_latitude_geocentric)
# cos(delta), sin(delta)
import geomagnetism as geo
ellipsoid = geo.geomagnetism.WGS84 # tuple (earth major axis in meter, earth minor axis in
                                                meter)
ellipsoid = geo.geomagnetism.WGS84_
r, ct, st, cd,sd = geo.geodetic_to_geocentric_IGRF13(ellipsoid,170, 100_000)
>> (6457402.34844737 -0.9846101254413312 0.17476527366272146 0.9999993565199398 -0.
                                                0011344424650535443)
ellipsoid = geo.geomagnetism.GRS80_
r, ct, st, cd,sd = geo.geodetic_to_geocentric_IGRF13(ellipsoid,170, 100_000)
>> (6457402.348345758 -0.9846101254403548 0.17476527366822295 0.9999993565199334 -0.
                                                0011344424706410008)
```

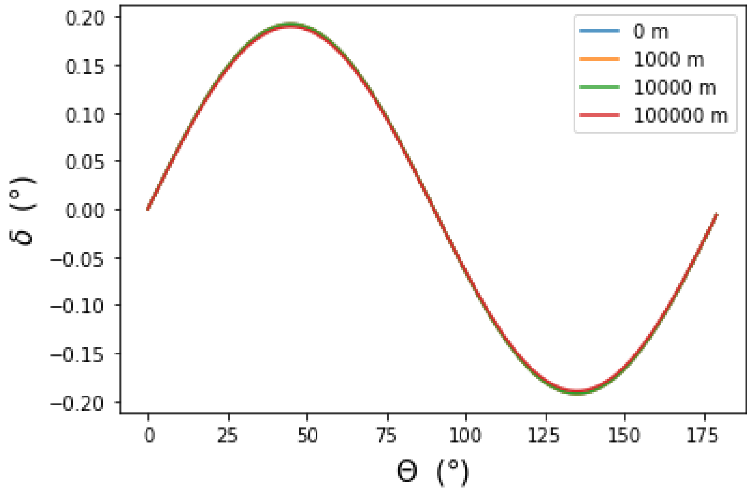

Figure 3: Variation of δ versus the geotetic colatitude θ and the height h.

Alternatively, the function `geodetic_to_geocentric_IGRF13(ellipsoid, co_latitude, height)` is a translation of the FORTRAN routine where the authors compute the geocentric radius as well as $\cos\delta$ and $\sin\delta$ using respectivelly the equations (4.12), (4.17), (4.18).

5 Base transformation

Passing from geocentric to geotetic referential the magnetic field undergoes the following transfomation :

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geodetic}} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{\text{geocentric}} \tag{5.1}$$

with $\delta = \theta' - \theta = \Phi - \Phi'$ (see Figure 5). Using Peddie notation [3] we have :

$$\boxed{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} X_c \\ Y_c \\ Z_c \end{pmatrix}} \tag{5.2}$$

Using the notations of the Figure 5, the geomagnetic horizontal intensity H , total intensity F , declination D and inclination I can be obtained from :

$$\boxed{\begin{cases} H = \sqrt{X^2 + Y^2} \\ F = \sqrt{H^2 + Y^2} \\ D = \operatorname{atan2}(Y, X) \\ I = \operatorname{atan2}(Z, H) \end{cases}} \tag{5.3}$$

(a) (b)

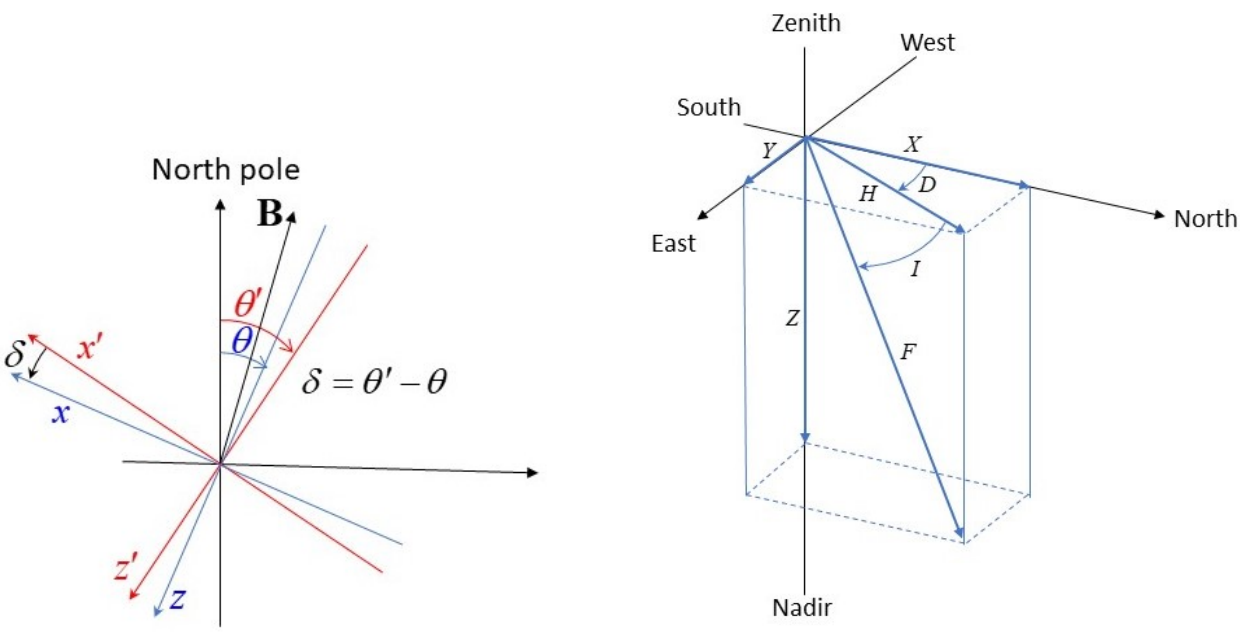

Figure 4: Notation conventions: (a) geotetic and geocentric referential; (b)Field geomagnetic conventions and notations. Credit Chullia [2]

6 Geomagnetic field computtion at North and South pole

When the colatitude θ tends towards 0 (North pole) or towards π, the equations (2.3) are numerically instabe. To deal with that problem we have to evaluate $\frac{dP^m_{(s),n}(\cos\theta)}{d\theta}$, $\frac{P^m_{(s),n}(\cos\theta)}{\sin\theta}$, $P^m_{(s),n}(\cos\theta)$ for $\theta = 0$ and for $\theta = \pi$.

Identity 1. $P^m_{(s),n}(1) = \delta^0_m$.

Identity 2. $P^m_{(s),n}(-1) = \delta^0_m(-1)^n$.

Proof. The associate Legendre polynomials can be definined as:

$$P^m_n(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x) \tag{6.1}$$

Using the change of variable $x = \cos\theta$ equation (6.1) reads:

$$P^m_n(\cos\theta) = (-1)^m(\sin\theta)^m\frac{d^m}{d(\cos\theta)^m}P_n(\cos\theta) \tag{6.2}$$

From the equation (6.2) we deduce that for $\theta = 0$ or for $\theta = \pi$, $P^m_n(\cos\theta)$ is not null iff $m = 0$. As $P_n(1) = 1$ we deduce the Identity1. As $P_n(-1) = (-1)^n$ we deduce the Identity2. □

Identity 3. $\lim\limits_{\theta\to 0}\frac{P^m_{(s),n}(\cos\theta)}{\sin\theta} = \delta^m_1\sqrt{\frac{n(n+1)}{2}}$.

Identity 4. $\lim\limits_{\theta\to \pi}\frac{P^m_{(s),n}(\cos\theta)}{\sin\theta} = \delta^m_1(-1)^n\sqrt{\frac{n(n+1)}{2}}$.

Proof. From equation (6.2)) we deduce that $\frac{P^m_n(\cos\theta)}{\sin\theta}$ is not null iff $m = 1$. This condition leads to:

$$\lim_{\theta\to 0}\frac{P^1_n(\cos\theta)}{\sin\theta} = \frac{d}{dx}P_n(x)\bigg|_{x=1} \tag{6.3}$$

As the Legendre polynomials $P_n(x)$ satisfy the differential equation :

$$(1-x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d}{dx}P_n(x) + n(n+1)P_n(x) = 0, \tag{6.4}$$

and also satisfy the two identities $P_n(1) = 1$ and $P_n(-1) = (-1)^n$, we have:

$$\frac{d}{dx} P_n(1) = \frac{n(n+1)}{2} \tag{6.5a}$$

$$\frac{d}{dx} P_n(-1) = (-1)^{n-1} \frac{n(n+1)}{2} \tag{6.5b}$$

Taking into account respectivelly equation (6.5a) and equation (6.5b)in conjonction with the Schmidt normalisation coefficients (3.4) we obtain the Identity3 and the Identity4. □

Identity 5. $\left.\frac{dP^m_{(s),n}(\cos\theta)}{d\theta}\right|_{\theta=0} = \delta^m_1 \sqrt{\frac{n(n+1)}{2}}$.

Identity 6. $\left.\frac{dP^m_{(s),n}(\cos\theta)}{d\theta}\right|_{\theta=\pi} = (-1)^n \delta^m_1 \sqrt{\frac{n(n+1)}{2}}$.

Proof. The derivative versus θ of the equation (6.2)leads to the expression::

$$\frac{d}{d\theta} P^m_n(\cos\theta) = (-1)^m (\sin\theta)^{m-1} \left[m\cos\theta \frac{d^m}{d(\cos\theta)^m} P_n(\cos\theta) - (\sin\theta)^2 \frac{d^{m+1}}{d(\cos\theta)^{m+1}} P_n(\cos\theta) \right] \tag{6.6}$$

showing that $\frac{d}{d\theta} P^m_n(\cos\theta)$ is not null iff $m = 1$. Taking into account respectivelly equation (6.5a) equation (6.5b)in conjonction with the Schmidt normalisation coefficients (3.4) we obtain the Identity5 and the Identity6. □

Using the identities1 to 6 we can derive the expression of the magnetic at the North and South pole as follow:

(a) Putting the identities1,3, 5 in equation (2.4) we obtain the following expressions of the magnetic field at the North pole:

$$\boxed{\begin{cases} X_c(0) = \sum\limits_{n=1}^{N} \left(\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} g^1_n \cos(\phi) + \sum\limits_{n=1}^{N} \left(\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} h^1_n \sin(\phi) \\ Y_c(0) = \sum\limits_{n=1}^{N} \left(\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} g^1_n \sin(\phi) - \sum\limits_{n=1}^{N} \left(\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} h^1_n \cos(\phi) \\ Z_c(0) = \sum\limits_{n=1}^{N} (n+1) \left(\frac{a}{r}\right)^{n+2} g^0_n \end{cases}} \tag{6.7}$$

(b) Putting the identities3,4, 6 in equation (2.4) we obtain the following expressions of the magnetic field at the South pole:

$$\boxed{\begin{cases} X_c(\pi) = \sum\limits_{n=1}^{N} \left(-\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} g^1_n \cos(\phi) + \sum\limits_{n=1}^{N} \left(-\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} h^1_n \sin(\phi) \\ Y_c(\pi) = \sum\limits_{n=1}^{N} \left(-\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} g^1_n \sin(\phi) - \sum\limits_{n=1}^{N} \left(-\frac{a}{r}\right)^{n+2} \sqrt{\frac{n(n+1)}{2}} h^1_n \cos(\phi) \\ Z_c(\pi) = -\sum\limits_{n=1}^{N} (n+1) \left(-\frac{a}{r}\right)^{n+2} g^0_n \end{cases}} \tag{6.8}$$

In the `geomagnetism` package the constant EPS monitors the choice of the equation used to compute the electromagnetic field:

- if $\phi \in [\pi, \pi - \mathrm{EPS}]$ we use the equations (6.8).
- if $\phi \in [0, \mathrm{EPS}]$ we use the equations (6.7).
- if $\phi \in]\mathrm{EPS}, \pi - \mathrm{EPS}[$ we use the equations (2.4).

The default value of EPS is set to $\mathrm{EPS} = 10^{-5}$rad. At $\phi = \mathrm{EPS}$ the electromagnetic field components undergo a discontinuity as shown in figure 5. The value of the constant EPS can be reaffected [4] using:
`geo.geomagnetism.EPS=NewEps`.

[4]Note that for $\mathrm{EPS} \lesssim 8 \times 10^{-6}$rad equations (2.4) results are noisy.

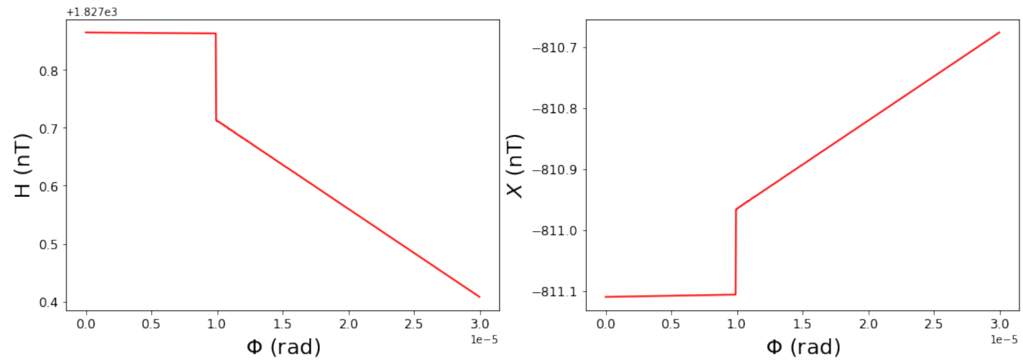

Figure 5: Discontinuity of the component H (a) and X (b) at $\phi = \text{EPS}$.

7 Spherical harmonic coefficients

7.1 Gauss coefficients

To compute the geomagnetic fields, through the equation (2.4), we must provide : the values of the Gauss coefficients h_n^m and g_n^m, the order N of the SH development. As the Gauus coefficients vary with time we must also provide the values $\dot{h}_n^m$ and $\dot{g}_n^m$. All these values are stored in the following variables:

- `dic_dic_h` contains h_n^m coefficients stored in a dict of dict as {`year : {(m,n):h,...},...`}
- `dic_dic_g` contains g_n^m coefficients stored in a dict of dict as {`year : {(m,n):g,...},...`}
- `dic_dic_SV_h` contains $\dot{h}_n^m$ coefficients stored in a dict of dict as {`year : {(m,n):SV_h,...},...`}
- `dic_dic_SV_g` contains $\dot{g}_n^m$ coefficients stored in a dict of dict as {`year : {(m,n):SV_g,...},...`}
- `dic_N` dict containing the order N of the SH decomposition as `dic_N[year]=N`
- `Years` contains the list of tabulated years

The function `read_IGRF13_COF` reads the `IGRF13.COF` coefficients.

```
file = 'IGRF13.COF' # downloaded from https://www.ngdc.noaa.gov/IAGA/vmod/coeffs/igrf13coeffs.
                                                                              txt
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years = geo.read_IGRF13_COF(file)
```

The function `read_WMM` reads the `WMM_2020.COF` or the`WMM_2015.COF` coefficients.

```
file = 'WMM_2020.COF' # downloaded from https://www.ngdc.noaa.gov/geomag/WMM/wmm_ddownload.
                                                                              shtml
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years= geo.read_WMM(file)
```

```
file = 'WMM_2015.COF' # downloaded from https://www.ngdc.noaa.gov/geomag/WMM/wmm_ddownload.
                                                                              shtml
dic_dic_h, dic_dic_g, dic_dic_SV_h, dic_dic_SV_g, dic_N, Years= geo.read_WMM(file)
```

7.2 Gauss coefficients secular variation

The Gauss coefficients g_n^m and h_n^m vary with time [7]. There secular variation $\dot{g}_n^m$ and $\dot{h}_n^m$ expressed in `nT/year` are tabulated every year and are considered to be constant during that current year. So, at time t, the Gauss coefficients can be expressed as:

$$\begin{cases} g_m^n(t) = g_m^n(t_0) + \dot{g}_m^n(t_0)(t - t_0) \\ h_m^n(t) = h_m^n(t_0) + \dot{h}_m^n(t_0)(t - t_0) \end{cases} \tag{7.1}$$

where t_0 stands for the time on the 1 January at 00:00 PM of the current year.Using $\dot{g}_n^m$ and $\dot{h}_n^m$ it is straightforward to compute the secular variation of the geomagnetic field componants. We have [2]:

$$
\begin{aligned}
\dot{X}_c \equiv \\
& \dot{\mathbf{B}}_x = -\dot{B}_\theta = \frac{1}{r}\frac{\partial \dot{V}}{\partial \theta} = \\
& \sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[\dot{g}_n^m \cos(m\phi) + \dot{h}_n^m \sin(m\phi)\right]\frac{dP_{(s),n}^m(\cos\theta)}{d\theta}, \\
\dot{Y}_c \equiv \\
& \dot{\mathbf{B}}_y = \dot{B}_\phi = \frac{-1}{r\sin\theta}\frac{\partial \dot{V}}{\partial \phi} = \\
& \sum_{n=1}^{N}\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n} m\left[\dot{g}_n^m \sin(m\phi) - \dot{h}_n^m \cos(m\phi)\right]\frac{P_{(s),n}^m(\cos\theta)}{\sin\theta}, \\
\dot{Z}_c \equiv \\
& \dot{\mathbf{B}}_z = -\dot{B}_r = \frac{\partial \dot{V}}{\partial r} = \\
& \sum_{n=1}^{N}(n+1)\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left[\dot{g}_n^m \cos(m\phi) + \dot{h}_n^m \sin(m\phi)\right]P_{(s),n}^m(\cos\theta),
\end{aligned}
\tag{7.2}
$$

The time derivative of the equation (5.2) can be expressed as:

$$
\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{pmatrix} \dot{X}_c \\ \dot{Y}_c \\ \dot{Z}_c \end{pmatrix} \tag{7.3}
$$

Finally, the time derivative of equation (5.3) reads [5]:

$$
\boxed{\begin{cases} \dot{H} = \frac{X\dot{X}+Y\dot{Y}}{H} \\ \dot{F} = \frac{X\dot{X}+Y\dot{Y}+Z\dot{Z}}{F} \\ \dot{I} = \frac{H\dot{Z}-Z\dot{H}}{F^2} \\ \dot{D} = \frac{X\dot{Y}-Y\dot{X}}{H^2} \end{cases}} \tag{7.4}
$$

8 Benchmark

Using the latitude, longitude, time and height above the ellipsoid given in the Table **??**, the Table **??**shows the geomagnetic field element obtained by both the geomagnetism package and those extracted from the Table 3b High-precision numerical example from [2].

Time	2022.5	yr
Height-above-Ellipsoid	100	km
Latitude	-80	deg
Longitude	240	deg

Table 1: parameters values used for the benchmark.

[5]We use $\frac{\partial}{\partial x}\text{atan2}(y,x) = -\frac{y}{x^2+y^2}$ and $\frac{\partial}{\partial y}\text{atan2}(y,x) = \frac{x}{x^2+y^2}$

notation	geomagnetism	WMM test value	relative error
D	69.125	69.13	-0.006
Dd	-0.094	-0.09	4.342
F	54912.078	54912.1	0.0
Fd	-83.356	-83.4	-0.052
H	16884.992	16885.0	0.0
Hd	12.551	12.6	-0.388
I	-72.091	-72.09	0.002
Id	0.041	0.04	4.462
X	6016.523	6016.5	0.0
Xd	30.379	30.4	-0.068
Y	15776.705	15776.7	0.0
Yd	1.847	1.8	2.578
Z	-52251.635	-52251.6	0.0
Zd	91.656	91.7	-0.047

Table 2: comparison of the geomagnetic field element obtained by both the geomagnetism package and those extract from the Table 3b High-precision numerical example from Chullia. [2]

9 Examples

Using the following code generates the xarrays of: X,Y,Z,F,H,$\dot{X}$,$\dot{Y}$,$\dot{Z}$,$\dot{F}$,$\dot{H}$.

```
import numpy as np
import geomagnetism as geo
'''
compute the xarrays: dintensities containing X,Y,Z,F,H;
                     dangles containing I, D;
                     dintensities_sv containing the the time derivative of X,Y,Z,F,H;
                     dangles_sv containing the the time derivative of I, D;
for the colatitudes and longitudes specified in the arrays colatitudes, longitudes
'''

colatitudes = np.linspace(0,180,181)
longitudes = np.linspace(-180,179,360)

dintensities, dangles, dintensities_sv, dangles_sv = geo.grid_geomagnetic(colatitudes,
                                                     longitudes)
```

Using the following code generates the the plots of figure 9.

```
'''
Examples of plots of geomagnetic components using different pojections (spstere, npstere, mill
                                                       )
'''
component = 'F' # sould be 'X','Y','Z','F','H','I','D','Xd','Yd','Zd','Fd','Hd','Id','Dd'
geo.plot_geomagetism(dintensities,
                  dangles,
                  dintensities_sv,
                  dangles_sv,
                  component,
                  {'proj':'spstere',
                   'boundinglat':-55,
                   'lon_0':270})

geo.plot_geomagetism(dintensities,
                  dangles,
                  dintensities_sv,
                  dangles_sv,
                  component,
                  {'proj':'npstere',
                   'boundinglat':70,
                   'lon_0':270} )

geo.plot_geomagetism(dintensities,
                  dangles,
                  dintensities_sv,
                  dangles_sv,
                  component,
                  {'proj':'mill',
```

```
                    'llcrnrlat':-90,
                    'urcrnrlat': 90,
                    'llcrnrlon':0,
                    'urcrnrlon':360} )
```

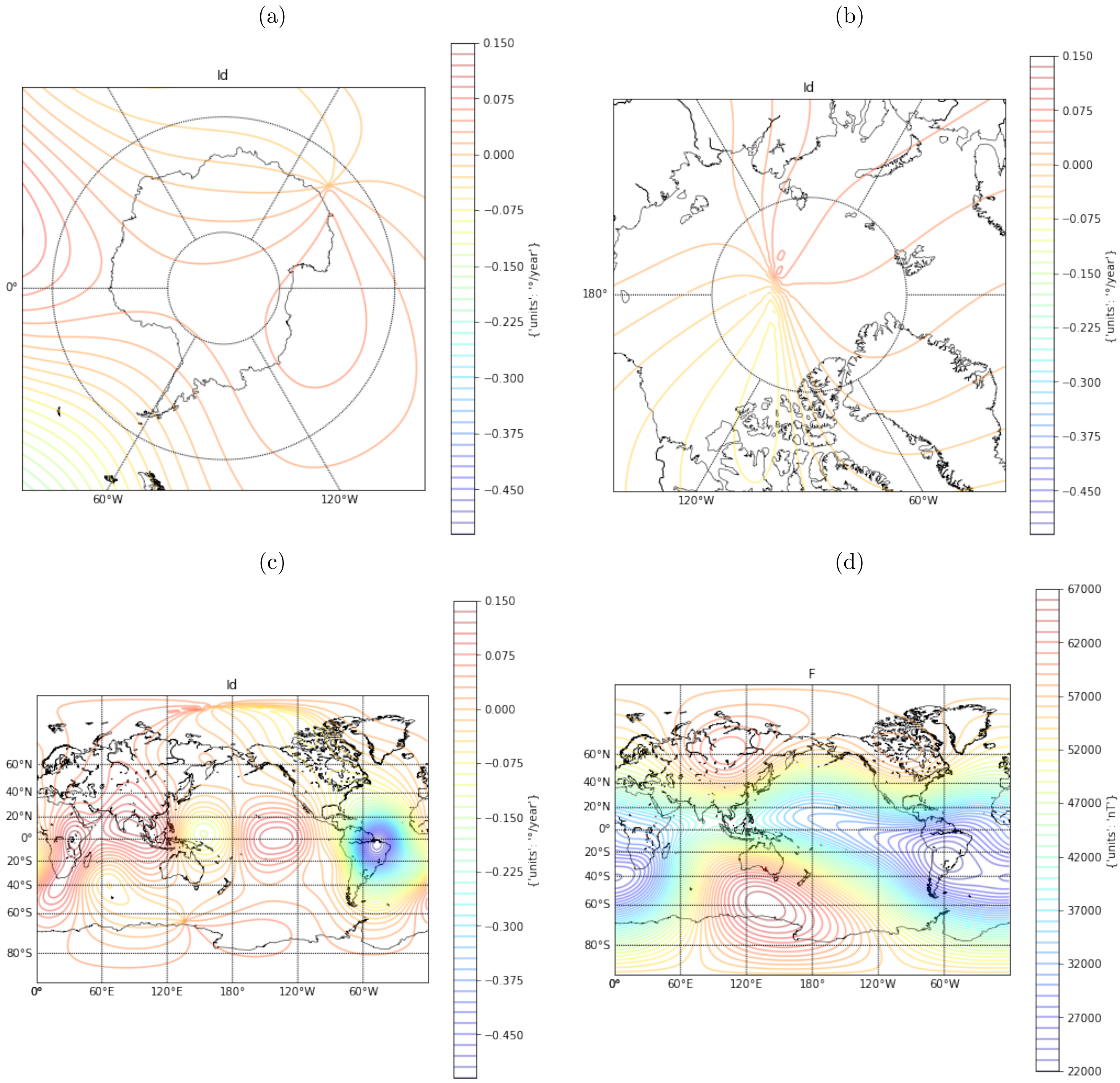

Figure 6: Examples: (a) $\dot{I}$ at South pole; (b) $\dot{I}$ at North pole; (c) $\dot{I}$ for the whole world;(d) F for the whole world.

References

[1] Campbell, W. H. (2007). *Introduction to Geomagnetic Fields* Cambridge University Press.

[2] Chullia, A et al. (2020). *The US/UK World Magnetic Model for 2020-2025*: Technical Report, National Centers for Environmental Information, NOAA

[3] Peddie, N. W. (1982).*International Geomagnetic Reference Field : the third generation.* J. Geomag. Geolectr 34: 309-326.

[4] Stacey, F. D. and P. M. Davis (2008). *Physics of the Earth* Cambridge University Press

[5] Winch, D. E., et al. (2005). *Geomagnetism and Schmidt quasi-normalization.* Geophys. J. Int. 160: 487-454.

[6] Zhang, S. and Jin, J. (1996). *Computation of Special Functions.* Wiley-Interscience.

[7] Glaßmeier, K.-H., et al. (2008) *Geomagnetic Field Variations (Advances in Geophysical and Environmental Mechanics and Mathematics).* Springer.