

# Monty Hall et al.

---

Alberto Garfagnini

Università di Padova

AA 2022/2023 - Stat Lect. 13a



## The Monty Hall Problem

---

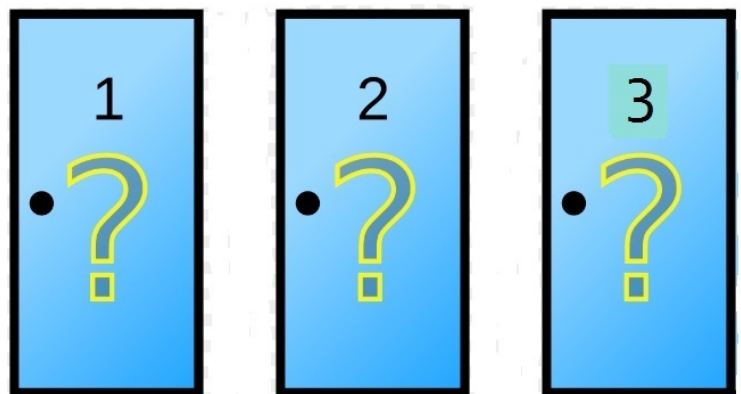
### The Game Show

- there are 3 door, closed
- behind one door there is a prize, an expensive car
- but behind the other doors there is a goat

### The Rules of the Game

- you select one door, but you cannot open it, yet
- the **game show host**, that **knows where the car is**, open one of the other two doors, revealing a goat behind it
- you are given the opportunity to change your choice of door, before opening it

What is your choice ?



# The Monty Hall Problem Solution

## The Game Propositions

- we select door number 1
- the host opens door number 2
- we are asked to choose between door 1 and 3

$W$  : the CAR is behind door 1

$C$  : we select the car by changing door

$$\begin{aligned} P(C|I) &= P(CW|I) + P(C\bar{W}|I) \\ &= P(C|W) \cdot P(W|I) + P(C|\bar{W}) \cdot P(\bar{W}|I) \end{aligned}$$

## Our Knowledge

$$P(W|I) = 1/3 \rightarrow P(\bar{W}|I) = 1 - P(W|I) = 2/3$$

$$P(C|W) = 0 \rightarrow P(C|\bar{W}) = 1$$

- therefore

$$P(C|I) = 2/3$$

# The Monty Hall Problem - Variation I

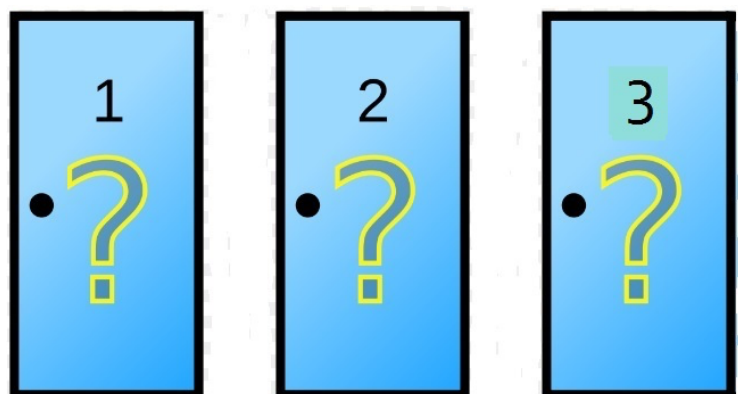
## The Game Show

- there are 3 door, closed
- behind one door there is a prize, an expensive car
- but behind the other doors there is a goat

## The Rules of the Game

- you select one door, but you cannot open it, yet
- the game show host, that **does NOT know which door hides the prize**, opens one of the other two doors. The door happens to have a goat behind it
- you are given the opportunity to change your original choice, switching to the other unopened door, before opening it

What is your choice ?



# The Monty Hall Problem Variation - Solution

## The Game Propositions

- we have selected door number 1
- the host opens door number 2, revealing a goat
- we are asked to choose between door 1 and 3

$G_k$  : a goat is behind door  $k$

$C_k$  : a car is behind door  $k$

- we need to evaluate the probability that door 3 hides a car, if door 2 hides a goat

$$P(C_3 \mid G_2 I) = \frac{P(G_2 \mid C_3 I) P(C_3 \mid I)}{\sum_{j=1}^3 P(G_2 \mid C_j I) P(C_j \mid I)}$$

## Our Knowledge

$$P(G_2 \mid C_1) = 1 \quad P(G_2 \mid C_2) = 0 \quad P(G_2 \mid C_3) = 1$$

$$P(C_1 \mid I) = 1/3 \quad P(C_2 \mid I) = 1/3 \quad P(C_3 \mid I) = 1/3$$

$$\rightarrow \text{therefore: } P(C_3 \mid G_2 I) = \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1}{2}$$

## The Monty Hall Problem generalization

- it is easy to generalize the problem to the case of  $n$  doors
- the game show host opens  $k$  doors, revealing as many goats ( $0 \leq k \leq n - 2$ )
- there is still ONE car

$\rightarrow$  what is the probability of winning if we switch to another closed door, randomly chosen ?

$C$  : we select the CAR by changing door

$W$  : the CAR is behind door 1

we have:

$$P(W \mid I) = 1/n \quad P(\overline{W} \mid I) = 1 - 1/n = (n - 1)/n$$

and

$$P(C \mid W I) = 0 \quad P(C \mid \overline{W} I) = 1/(n - k - 1)$$

- therefore

$$P(C \mid I) = \frac{1}{n - k - 1} \frac{n - 1}{n}$$

- the probability of winning is increased from  $1/n$  whenever one or more doors are opened.  $\rightarrow$  we should always switch doors

# Three prisoner's dilemma

---

## The problem

- Three prisoners are in jail and condemned to death
- Since on the next day it will be the King's birthday, his majesty has decided to release one prisoner
- Only the chief guard knows who is going to be freed, but is not allowed to tell anybody who he is
- Prisoner **A** begs the chief of the guards to tell him the name of one of the two prisoners that will be executed.
- He hopes that, by interrogating the guard, his chance of surviving increases
- The chief guard agrees to tell him the name of one of the two prisoners condemned to death. So they have the following deal:
  - if **B** is going to be pardoned, give me **C** name
  - if **C** is going to be pardoned, give me **B** name
  - if I am (**A**) going to be pardoned, flip a coin and tell me either **B** or **C** name
- The guard says: **B is going to be executed**.  
Has **A** probability of surviving increased or not ?

## Three prisoner's dilemma (2)

---

### The solution

- we define the propositions:
  - **A** will be freed
  - **B** will be freed
  - **C** will be freed
  - $\beta$  the guard tells A that **B** will be executed

We want to compute

$$\begin{aligned} P(A \mid \beta) &= \frac{P(\beta \mid A)P(A)}{P(\beta)} \\ &= \frac{P(\beta \mid A)P(A)}{P(\beta \mid A)P(A) + P(\beta \mid B)P(B) + P(\beta \mid C)P(C)} \end{aligned}$$

- we know that
  - $P(A) = P(B) = P(C) = 1/3$
  - $P(\beta \mid A) = 1/2$ ,  $P(\beta \mid B) = 0$  and  $P(\beta \mid C) = 1$
- therefore:

$$P(A \mid \beta) = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3}$$

# Three prisoner's dilemma (3)

---

## The solution

- and what about prisoner **C** ? Does his survival probability changes ?

We want to compute

$$\begin{aligned}P(C \mid \beta) &= \frac{P(\beta \mid C)P(C)}{P(\beta)} \\&= \frac{P(\beta \mid C)P(C)}{P(\beta \mid A)P(A) + P(\beta \mid B)P(B) + P(\beta \mid C)P(C)} \\&= \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{2}{3}\end{aligned}$$

□ we know that

- $P(A) = P(B) = P(C) = 1/3$
- $P(\beta \mid A) = 1/2$ ,  $P(\beta \mid B) = 0$  and  $P(\beta \mid C) = 1$

key points:

- **A** is freed: the guard can say that **B** or **C** will be executed
- **C** is freed: the guard can only say that **B** will be executed