Binomial regression

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Outline

- Models for count data
- Residual diagnostics in GLMs
- Other useful models

Questions about yesterday?

The binomial GLM

Data: r the number of successes in N trials
Parameters: probability p (now: π_i)
Goal: estimate π_i for each observation

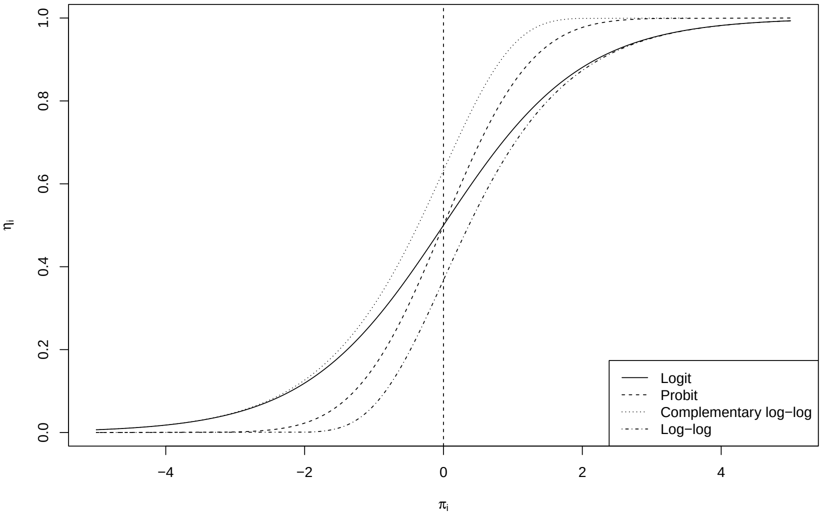
Binomial GLM use

- When a linear regression is not appropriate :)
- For binary data or counts of successes/failures

In ecology

- Predicting species' distributions
- Number of germinated plant seeds
- Prevalence of disease in a population
- Probability of observing a behavior
- Proportion of orchids 🙃

Binomial link functions (2)



Outline

Log-linear regression

Log-linear regression is a class of models that uses the log-link function:

$$\begin{aligned} \log\{\mathbb{E}(y_i|x_i)\} = \eta_i = \alpha + x_i\beta \\ \mathbb{E}(y_i|x_i) = \lambda_i = \exp(\alpha + x_i\beta) \end{aligned} \tag{1}$$

Log-linear regression is commonly used to analyse count data

Typical count cases

- Caught fish
- Plants at a site
- Pidgeons in a city
- Bigfoot reports
- Wrongful convictions
- Stars in the night sky

The Poisson GLM

Data: k_i the count
Parameters: mean λ
Goal: estimate λ_i for each observation

The Poisson distribution

$$\mathcal{L}(y_i;\Theta) = \text{exp}\{y_i \log(\lambda_i) - \lambda_i - \log(y_i!)\} \quad (2)$$

The Poisson paramater λ is the mean of the counting process

The Poisson distribution: rates

Alternatively we can write:

$$\mathcal{L}(y_i; \Theta) = \text{exp}\{y_i \log(rt) - rt - \log(y_i!)\} \tag{3}$$

so, $\lambda = rt$

- r is the rate at which counts occur, per time period t
- we could also record counts within a certain period, instead of the total

Catching fish

- We go fishing for an hour and catch $\lambda = 5$ fish
- On average we caught $r = \frac{5}{60}$ fish per $t = 1$

Is Poisson regression really a GLM?

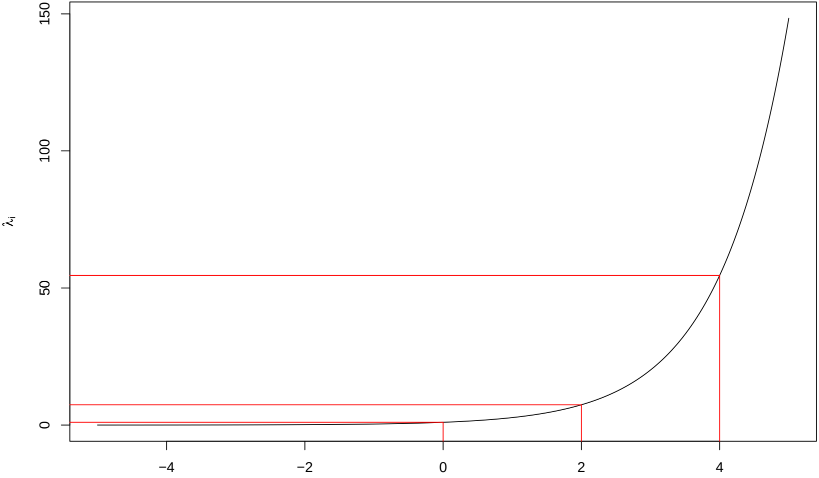
$$\mathcal{L}(y_i;\Theta) = \text{exp}\left\{\frac{y_i \log(\lambda) + \log(\lambda)}{1} + \log(y_i!)\right\} \tag{4}$$

All GLMs can be formulated as:

$$\mathcal{L}(y_i;\Theta) = \text{exp}\{\frac{y_i \eta_i - b(\eta_i)}{a(\phi)} + c(y_i,\phi)\} \tag{5}$$

Log-link function

So log is the canonical link. This looks like:



Counts: a multiplicative process

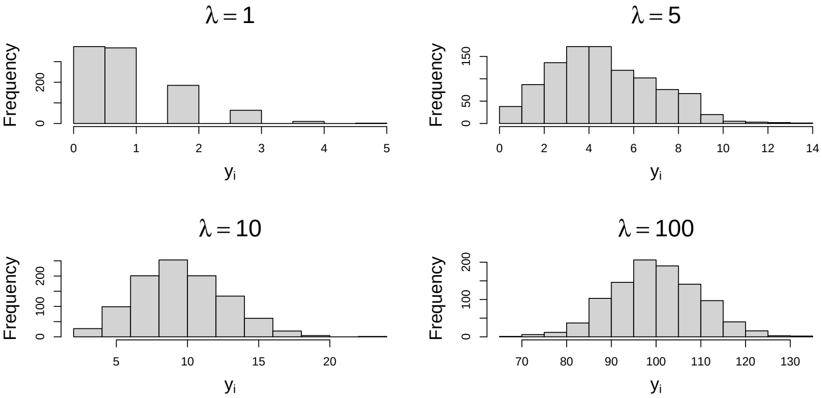
Say that we have the model:

$$\log(\lambda) = \alpha + x_i\beta \tag{6}$$

- with $\alpha = 1$ and $\beta = \log(2) \approx 0.693$
- x_i is either 0 or 1: either I was fishing or you were
- `exp(1) = 2.71828` the average number of fish I caught
- `exp(1+2) = exp(1)*2 = 5.437` the average number of fish you caught

So, you caught twice as many fish!

The Poisson distribution visually



Poisson assumptions

- An event can occur $0 \ldots \infty$ times
- Events are independent
- The rate of events is constant
- Events cannot occur simultaneously
- Variance equals the mean

Example: horseshoe crabs

Counts of male crabs ("satellites") near female crabs

Figure 1: nwf.org

Figure 2: uwm.edu

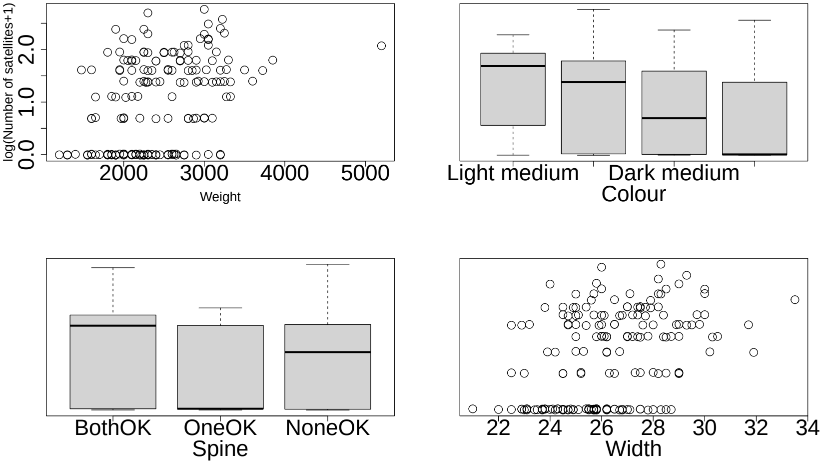
Horseshoe crabs: the data

Data originally from Brunswick (1996) via Agresti (2007) via Dunn and Smyth (2018)

- 173 observations
- 4 traits: colour, spine condition, width (cm), weight (g)

Col	Spine	Width	Sat	Wt
M	NoneOK	28.3	8	3050
DM	NoneOK	22.5	0	1550
LM	BothOK	26.0	9	2300
DM	NoneOK	24.8	0	2100
DM	NoneOK	26.0	4	2600
M	NoneOK	23.8	0	2100
LM	BothOK	26.5	0	2350

Horseshoe crabs: the data



What can we tell about the number of satellites?

Horseshoe crabs: fit the model

```
model <- glm(Sat ~ Spine + Colour + Width + Weight,
             family = "poisson", data = hcrabs)
```

Horseshoe crabs: interpreting parameters

```
##                      Estimate Std. Error z value Pr(>|z|)
## (Intercept)           -0.3600    0.97000   -0.37   0.7100
## SpineOneOK            -0.1500    0.21000   -0.70   0.4800
## SpineNoneOK            0.0870    0.12000    0.73   0.4700
## ColourMedium          -0.2600    0.17000   -1.60   0.1200
## ColourDark medium     -0.5100    0.20000   -2.60   0.0086
## ColourDark            -0.5300    0.23000   -2.30   0.0190
## Width                  0.0170    0.04900    0.34   0.7300
## Weight                 0.0005    0.00017    3.00   0.0028
```

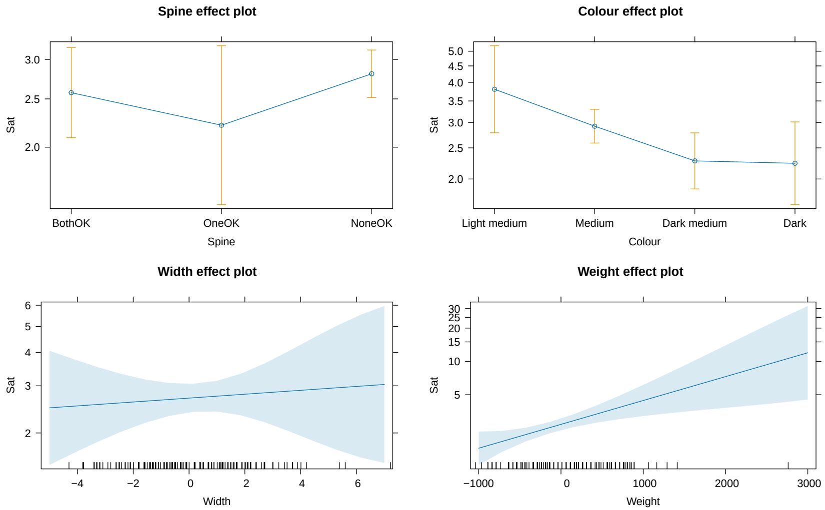
(Intercept) = Light-medium coloured females with both spines in good condition, Width and Weight = 0

Horseshoe crabs: interpreting parameters, centered

```
##                     Estimate Std. Error z value Pr(>|z|)
## (Intercept)           1.3000    0.15000    8.70  4.4e-18
## SpineOneOK           -0.1500    0.21000   -0.70  4.8e-01
## SpineNoneOK           0.0870    0.12000    0.73  4.7e-01
## ColourMedium         -0.2600    0.17000   -1.60  1.2e-01
## ColourDark medium    -0.5100    0.20000   -2.60  8.6e-03
## ColourDark           -0.5300    0.23000   -2.30  1.9e-02
## Width                 0.0170    0.04900    0.34  7.3e-01
## Weight                0.0005    0.00017    3.00  2.8e-03
```

- (Intercept) = Light-medium coloured females with both spines in good condition, Weight $\approx$ 2437g, Width $\approx$ 26.3cm
- Width: `exp(0.0170) = 1.017`, so double the number of satellites by `log(2)/0.01709 = 40.56`cm above average
- Weight: `exp(0.0005) = 1.0005`, so double the number of satellites by `log(2)/0.0005 = 1386.3`g above average

Horseshoe crabs: visual interpretation



Some other options

- Negative binomial (two types, with dispersion)
- Conway-Maxwell Poisson (with dispersion)
- Generalized Poisson (with dispersion)
- Skellam distribution (difference of counts)
- Binomial distribution (counts with a maximum)
- Truncated distributions
- Quasi-likelihood models

Do we have a good model?

More on this after the break

- Overdispersion or underdispersion
- Zero-inflation

Overdispersion

Our assumption: $\lambda = \text{var}(\mathbf{y})$
Reality: $\lambda \geq \text{var}(\mathbf{y})$

- Mean = variance
- If there is more variation, this assumption fails
- Consequences: CIs underestimate, biased parameter estimates, inflation in model selection

For our example: many females have few satellites, but some females have very many.

Underdispersion

Our assumption: $\lambda = \text{var}(\mathbf{y})$
Reality: $\lambda \leq \text{var}(\mathbf{y})$

Considerably less common than overdispersion.

Detecting overdispersion

- Residual diagnostics
- $D(\mathbf{y}; \hat{\boldsymbol{\mu}})/(n-k)$: should be close to 1
- `performance::check_overdispersion` (relies on asymptotics)
- Simulation (later today)

Dealing with dispersion: options

- Correct for it (calculate dispersion)
- Fit a different model
  - Negative binomial (overdispersion, `MASS package`)
  - Conway-Maxwell Poisson (over- and underdispersion.)
  - Generalized Poisson(over- and underdispersion)
  - Quasi-likelihood models
  - Mixed models (not covered here)

Quasi-likelihood models

Introduced by Wedderburn (1974)

- No “real” likelihood is specified for the data
- Means no AIC, but deviance exists
- Largely defined by its variance function

For Poisson responses: does not correct the parameter estimates

Negative-binomial

$$\mathcal{L}(y_i;\Theta) = \frac{\Gamma(y_i+\phi)}{\Gamma(\phi)y_i!}\left(\frac{\phi}{\mu_i+\phi}\right)^{\phi}\left(\frac{\mu_i}{\mu_i+\phi}\right)^{y_i} \tag{7}$$

- $\text{var}(\mathbf{y}) = \boldsymbol{\mu} + \frac{\boldsymbol{\mu}^2}{\phi}$
- For large ϕ Poisson!
- Requires more data/information due to extra parameter

Horseshoe crabs: Negative-binomial

```
modelnb <- MASS::glm.nb(Sat ~ Spine + Colour + Width + Weight,
                        data = hcrabs)
```

and compare the models:

```
AIC(model, modelnb)
```

```
##         df      AIC
## model    8 920.8833
## modelnb  9 763.3204
```

Horseshoe crabs: comparing estimates

```
##                      Poisson estimate NB estimate Poisson SE   NB SE
## (Intercept)                    1.3000      1.4000    0.15000 0.34000
## SpineOneOK                    -0.1500     -0.2400    0.21000 0.40000
## SpineNoneOK                    0.0870      0.0430    0.12000 0.25000
## ColourMedium                  -0.2600     -0.3200    0.17000 0.37000
## ColourDark medium             -0.5100     -0.6000    0.20000 0.42000
## ColourDark                    -0.5300     -0.5800    0.23000 0.47000
## Width                          0.0170     -0.0024    0.04900 0.10000
## Weight                         0.0005      0.0007    0.00017 0.00036
```

- SEs have doubled
- Coefficients have changed, but largely the same conclusions
- Except perhaps for the effect of "Width"

Summary

- Counts are analysed with log-linear models
- Due to the Poisson assumption, dispersion issues can arise
- This biases parameter and uncertainty estimates
- Negative-binomial models are useful for overdispersion problems
- Conway-Maxwell & Generalized Poisson can be used as well, and for underdispersion
  - `glmmTMB` or `VGAM` packages
- Zero-inflation (later)