For C, $I = C\dfrac{du}{dt}$ $\qquad \tau = R_{Th}C$

For L, $u = L\dfrac{di}{dt}$ $\qquad \tau = \dfrac{L}{R_{Th}}$

For 一阶恒电源: for C, $V(t) = V(\infty) + (V(0) - V(\infty))e^{-\frac{t}{\tau}}$, $i(t) = C\dfrac{d(V(t))}{dt}$

⇓ 只含C,L其一 $\qquad$ for L, $I(t) = I(\infty) + (I(0) - I(\infty))e^{-\frac{t}{\tau}}$, $V(t) = L\dfrac{d(i(t))}{dt}$

For 一阶变电源, Use KCL, KVL 得 ODE

4. 一阶线性方程 形如 $\frac{dy}{dx} = P(x)\cdot y + Q(x)$，若 $Q(x)=0$，则为一阶齐次方程

① $\frac{dy}{dx} = P(x)\cdot y$

$\Rightarrow \frac{dy}{y} = P(x)\cdot dx$, $y = \underline{C}\cdot e^{\int P(x)dx}$

② $\frac{dy}{dx} = P(x)\cdot y + Q(x)$: 常数变易法

令从①出发，设 $\bar{y} = C(x)e^{\int P(x)dx}$ 是(1)的一个解

$C'(x)e^{\int P(x)dx} + C(x)\cdot P(x)\cdot e^{\int P(x)dx} = C(x)\cdot P(x)\cdot e^{\int P(x)dx} + Q(x)$

$\Rightarrow Q(x) = C'(x)\cdot e^{\int P(x)dx}$

$\Rightarrow C(x) = \int \frac{Q(x)}{e^{\int P(x)dx}}dx + C$

联系213解方程！

$$y = \underbrace{C\cdot e^{\int P(x)dx}}_{\text{齐次方程通解}} + \underbrace{e^{\int P(x)dx}\cdot\int Q(x)\cdot e^{-\int P(x)\cdot dx}dx}_{\text{非齐次方程特解}}$$

For 二阶无电源 LC串联

Solving the differential equation by using initial conditions,

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is,

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \omega_o = \frac{1}{\sqrt{LC}}$$

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Source-Free Series *RLC* Circuit

The characteristic equation can be written as,

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

carries two possible solutions,

$$i_1 = A_1e^{s_1t} \qquad i_2 = A_2e^{s_2t}$$

combining the solution,

$$\boxed{i(t) = A_1e^{s_1t} + A_2e^{s_2t}}$$

Here, we will discuss 4 possible cases for the solution

$\frac{\alpha}{2L}$

For 二阶无电源 LC并联

Source-Free Parallel *RLC* Circuit

Solving the differential equation by using initial conditions,

$$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

A more compact way of expressing the roots is,

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

$$\alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_o = \frac{1}{\sqrt{LC}}$$

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$\frac{1}{2RC}$

Source-Free Parallel *RLC* Circuit

The characteristic equation can be written as,

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

carries two possible solutions,

$$v_1 = A_1e^{s_1t} \qquad v_2 = A_2e^{s_2t}$$

combining the solution,

$$\boxed{v(t) = A_1e^{s_1t} + A_2e^{s_2t}}$$

For 二阶有恒电源LC串联

Step response to series RLC network

Forms of complete solution

Overdamped

$$\boxed{v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

Critically damped

$$\boxed{v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t}}$$

Underdamped

$$\boxed{v(t) = V_s + e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)}$$

For 二阶有恒电源LC并联

Step response to parallel RLC network

Forms of complete solution

Overdamped

$$\boxed{i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

Critically damped

$$\boxed{i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t}}$$

Underdamped

$$\boxed{i(t) = I_s + e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)}$$

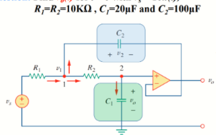
$w_d = \sqrt{w^2 - \alpha^2}$

For 二阶 amp

五. 2^{nd} order op-amp circuits

[eg].

Question: Find $v_o(t)$ for $t > 0$ with $v_s = 10u(t)$, $R_1 = R_2 = 10\text{K}\Omega$, $C_1 = 20\mu\text{F}$ and $C_2 = 100\mu\text{F}$



Applying the KCL at node 1,

$$\frac{v_s - v_1}{R_1} = C_2\frac{dv_2}{dt} + \frac{v_1 - v_o}{R_2} \qquad \text{Eq.1}$$

Applying the KCL at node 2,

$$\frac{v_1 - v_o}{R_2} = C_1\frac{dv_o}{dt} \qquad \text{Eq.2}$$

$$v_2 = v_1 - v_o \qquad \text{Eq.3}$$

Eliminating v_1 and v_2 from Eq.1-Eq.3 and using the given values for resistors and capacitors, we get

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + 5v_o = 5v_s$$

To find the transient response, we set $v_s = 0$, the characteristic equation becomes

$$s^2 + 2s + 5 = 0$$

with the roots $s_{1,2} = -1 \pm j2$

The transient response will be of the form,

$$v_{ot} = e^{-t}(A\cos 2t + B\sin 2t)$$

The steady state response will be of the form,

$$v_o(\infty) = v_1(\infty) = v_s$$

$$v_o(\infty) = v_s = 10\text{ mV}, \qquad t > 0$$

The complete response will be of the form,

$$v_o(t) = v_{ot} + v_{oss}$$

$$v_o(t) = 10 + e^{-t}(A\cos 2t + B\sin 2t)\text{ mV}$$

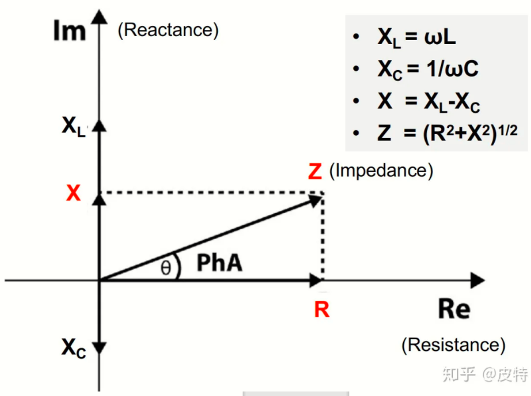
By comparing the values of initial conditions at $v_o(0)$ and $dv_o(0)/dt$, the values of coefficients are

$A = -10$ and $B = -5$

Thus, the complete solution will be ,

$$v_o(t) = 10 - e^{-t}(10\cos 2t + 5\sin 2t)\text{ mV}$$

RLC 电路相图：



- $X_L = \omega L$
- $X_C = 1/\omega C$
- $X = X_L - X_C$
- $Z = (R^2+X^2)^{1/2}$

Admittance

➢ The admittance **Y** is the reciprocal of impedance, measured in siemens (S)

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$$

The admittance can be represented in complex form as

$$\boxed{\mathbf{Y} = G + jB}$$

Conductance $G = Re(\mathbf{Y})$

susceptance $B = Im(\mathbf{Y})$

Impedance

➢ The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω)

➢ In the previous lecture, we obtained voltage-current relation for passive elements (**R**, **L** & **C**)

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

So, the phasor form of Ohm's law will be

$$\boxed{\text{Impedance} = \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}}$$

For phasor

Phasors

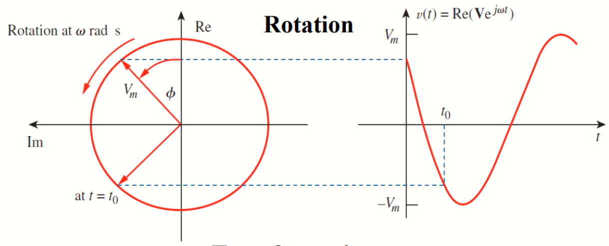
The complex numbers can also be represented by it's polar or exponential form

$$z = r\underline{/\phi} = re^{j\phi}$$

Thus, the complex number in three forms are,

$z = x + jy$	Rectangular form
$z = r\underline{/\phi}$	Polar form
$z = re^{j\phi}$	Exponential form

Phasors



Transformation

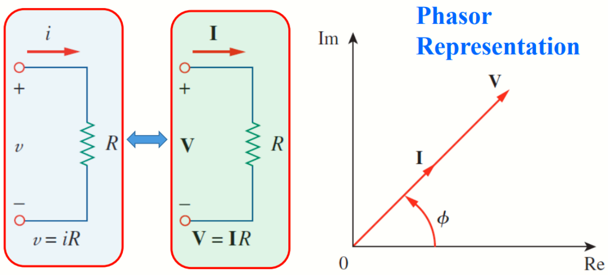
$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m\underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

Phasors – Representation

Time Domain		Phasor Domain
$V_m \cos(\omega t + \phi)$	$\Leftrightarrow$	$V_m\underline{/\phi}$
$V_m \sin(\omega t + \phi)$	$\Leftrightarrow$	$V_m\underline{/\phi - 90^\circ}$
$I_m \cos(\omega t + \theta)$	$\Leftrightarrow$	$I_m\underline{/\theta}$
$I_m \sin(\omega t + \theta)$	$\Leftrightarrow$	$I_m\underline{/\theta - 90^\circ}$
$\frac{dv}{dt}$	$\Leftrightarrow$	$j\omega\mathbf{V}$
$\int v\, dt$	$\Leftrightarrow$	$\frac{\mathbf{V}}{j\omega}$

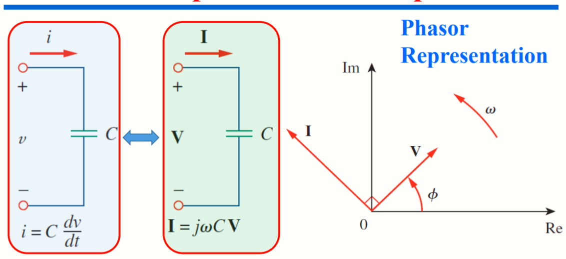
Phasor Representation – Resistor



$$v = iR = RI_m \cos(\omega t + \phi)$$

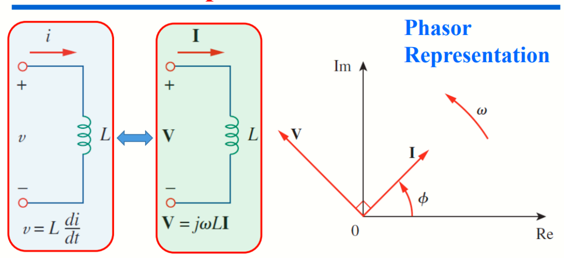
$$\mathbf{V} = RI_m\underline{/\phi} \Longrightarrow \mathbf{V} = R\mathbf{I}$$

Phasor Representation – Capacitor



$$i = C\frac{dv}{dt} \Longrightarrow \mathbf{I} = j\omega C\mathbf{V}$$

Phasor Representation – Inductor



$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \Longrightarrow \mathbf{V} = j\omega L\mathbf{I}$$

Phasors – Superposition

The weighted superposition of

$$k_1 f_1(t) + k_2 f_2(t)$$

of co-sinusoids

$$f_1(t) = \mathrm{Re}\{F_1 e^{jwt}\} \quad \& \quad f_2(t) = \mathrm{Re}\{F_2 e^{jwt}\}$$

with phasor F_1 and F_2 is also a co-sinusoid with a phasor

$$k_1 F_1 + k_2 F_2$$

Phasor Representation – Summary

Voltage – current relationship

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

For Average Power and max load power

Instantaneous and Average Power

The phasor form of *i(t)* ➡ $\mathbf{I} = I_m \angle \theta_i$

and *v(t)* ➡ $\mathbf{V} = V_m \angle \theta_v$

The power in the phasor form can be expressed in the form,

$$\boxed{P = \frac{1}{2}\mathrm{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

Maximum Average Power Transfer

- For **maximum average power transfer,** the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{\mathrm{Th}}$

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{\mathrm{Th}} - jX_{\mathrm{Th}} = \mathbf{Z}^*_{\mathrm{Th}}}$$

$$\cancel{P_{\max} = \frac{|\mathbf{V}_{\mathrm{Th}}|^2}{8R_{\mathrm{Th}}}}$$

For resonance

Resonance : Source – free LC circuit

The co-sinusoids of the signals will be

$$i(t) = Re\{Ie^{j\omega_o t}\} = |I|\cos(\omega_o t + \theta)$$

and,

$$v(t) = Re\left\{\frac{I}{j\omega_o C}e^{j\omega_o t}\right\} = \frac{|I|}{\omega_o C}\sin(\omega_o t + \theta)$$

are possible with any I and $\theta = \angle I$, and the oscillation frequency will be

$$\boxed{\omega_o = \frac{1}{\sqrt{LC}}}$$ **Resonant Frequency**

Summary – Resonance

Characteristic	Series circuit	Parallel circuit
Resonant frequency, ω_0	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$
Quality factor, Q	$\frac{\omega_0 L}{R}$ or $\frac{1}{\omega_0 RC}$	$\frac{R}{\omega_0 L}$ or $\omega_0 RC$
Bandwidth, B	$B = \frac{\omega_0 R}{\omega_0 L} = \frac{\omega}{\omega_0 RC}$ $\frac{\omega_0}{Q}$	$\frac{\omega_0}{Q}$
Half-power frequencies, ω_1, ω_2	$\omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$	$\omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$
For $Q \geq 10$, ω_1, ω_2	$\omega_0 \pm \frac{B}{2}$	$\omega_0 \pm \frac{B}{2}$

For Gain and Transfer function

Decibel Scale

- In communications systems, gain is measured in *bels*
- Historically, the bel is used to measure the ratio of two levels of power or power gain G

$$G = \text{Number of bels} = \log_{10}\frac{P_2}{P_1}$$

- The *decibel* (dB) provides us with a unit of less magnitude. It is 1/10th of a bel and is given by

$$\boxed{G_{\mathrm{dB}} = 10\log_{10}\frac{P_2}{P_1}}$$

Transfer function – Pole and Zero

$$\text{Transfer function} = \mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

Zero ➡ $H(\omega) = 0, at\ Y(\omega) = 0$

Pole ➡ $H(\omega) = \infty, at\ X(\omega) = 0$

For bode plot

$$H(\omega) = \frac{200j\omega}{(j\omega+2)(j\omega+10)}$$

Bode Plot – Example 1

Solution:

We first put $H(\omega)$ in the standard form by dividing out the poles and zeros. Thus,

$$H(\omega) = \frac{10j\omega}{\left(1+j\frac{\omega}{2}\right)\left(1+j\frac{\omega}{10}\right)}$$

$$= \frac{10|j\omega|}{\left|1+j\frac{\omega}{2}\right|\left|1+j\frac{\omega}{10}\right|}\angle\left(90^o - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{\omega}{10}\right)$$

Bode Plot – Example 1

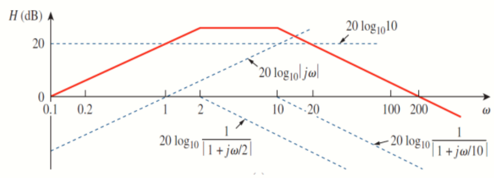
Hence , the magnitude and phase will be,

$$H_{dB} = 20log_{10}10 + 20log_{10}|j\omega| - 20log_{10}\left|1+\frac{j\omega}{2}\right| - 20log_{10}\left|1+\frac{j\omega}{10}\right|$$

$$\phi = 90^o - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{\omega}{10}$$

Bode Plot – Example 1

$$H_{dB} = 20log_{10}10 + 20log_{10}|j\omega| - 20log_{10}\left|1+\frac{j\omega}{2}\right| - 20log_{10}\left|1+\frac{j\omega}{10}\right|$$



Bode Plot – Example 1

$$\phi = 90^o - tan^{-1}\frac{\omega}{2} - tan^{-1}\frac{\omega}{10}$$