HKN ECE 310 Review Session

Whomst: Bex, Arjun, Eric

Logistics + Shameless Plug

- Slides and Recording will be uploaded here:
- https://hkn.illinois.edu/services

Topics

- DTFT and Frequency Response
- Ideal Sampling and Reconstruction
- DFT & Circular Properties

DTFT

Given a discrete-time signal, we can find the DTFT using the following:

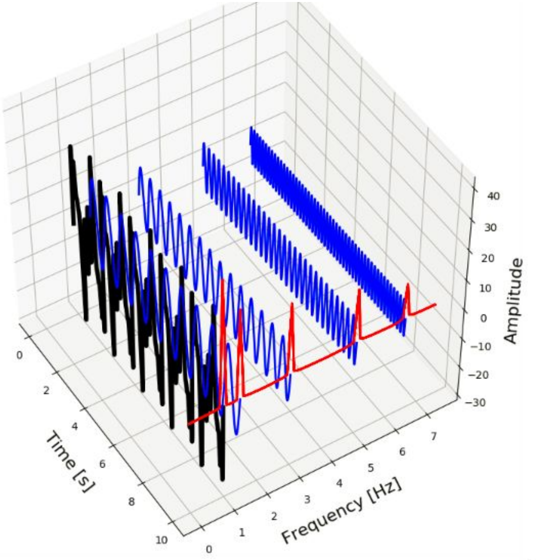
$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$X(\omega) = X(z)$ such that $z = exp(j\omega)$ if $X(z)$'s ROC includes the unit circle. (i.e. can be thought of as a slice of the z-transform of $x[n]$ around the unit circle)

DTFT

What does the DTFT tell us?

- Frequency content of a discrete signal
- Magnitude and phase response of a system



Conceptual Question

- The DTFT of a sequence always exists as long as its z-transform exists. T/F
- If the z-transform of the function does not include $|z|=1$, then the DTFT of the sequence does not exist. T/F

DTFT - cont'd

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega$$

Important Properties:

- **Periodicity by 2π!**
- Linearity
- Symmetries (Magnitude, angle, real part, imaginary part)
- Time shift and modulation
- Product of signals and convolution
- Parseval's Relation

Frequency Response

For any **stable** LSI system: $H_d(\omega) = H(z)|_{z=e^{j\omega}}$

- What is the physical interpretation of this?
  - The DTFT is the z-transform evaluated along the unit circle!
- Why is the frequency response nice to use in addition to the z-transform?
  - $e^{j\omega}$ is an *eigenfunction* of LSI systems
    - $h[n]*Ae^{j\omega_0} = \lambda Ae^{j\omega_0} = AH_d(\omega_0)e^{j\omega_0}$
  - By extension:
    - $x[n] = \cos(\omega_0 n + \theta) \rightarrow y[n] = |H_d(\omega_0)|\cos(\omega_0 n + \theta + \angle H_d(\omega_0))$

Magnitude and Phase Response

- Very similar to ECE 210
- Frequency response, and all DTFTs for that matter, are 2π periodic
- Magnitude response is fairly straightforward
  - Take the magnitude of the frequency response, remembering that $|e^{j\omega}| = 1$
- Limit your domain from -π to π.
- For **real-valued** signals and systems:
  - Magnitude response is even-symmetric
  - Phase response is odd-symmetric

DTFT Exercise 1

- Let our signal be

$$h[n] = \{\mathbf{1}, 2, 1\}.$$

a) Compute the DTFT of $h[n]$.
b) Plot the magnitude response.
c) Plot the phase response.

DTFT Exercise 1

Let our signal be $h[n] = \{\underset{\uparrow}{1}, 2, 1\}$.

Compute the DTFT of $h[n]$:

$$\begin{aligned}
\mathcal{F}(h[n]) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\
&= 1 + 2e^{-j\omega} + e^{-j2\omega} \\
&= e^{-j\omega}(2 + e^{j\omega} + e^{-j\omega}) \\
&= e^{-j\omega}(2 + 2\cos(\omega)).
\end{aligned}$$

DTFT Exercise 1

Let our signal be $h[n] = \{\underset{\uparrow}{1}, 2, 1\}$.

Plot the magnitude response:

$$H_d(\omega) = e^{-j\omega}(2 + 2\cos(\omega))$$

$$|H_d(\omega)| = \sqrt{H_d(\omega)H_d^*(\omega)} = |2 + 2\cos(\omega)|.$$

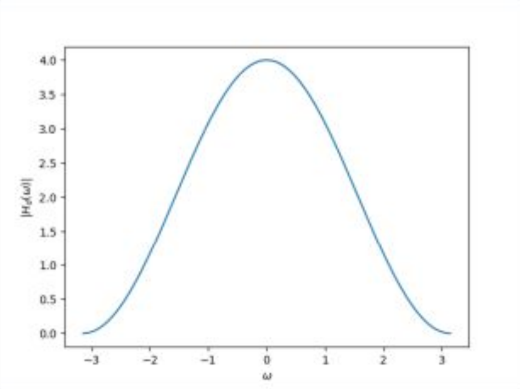


Figure 1: $|H_d(\omega)|$

DTFT Exercise 1

Let our signal be $h[n] = \{\underset{\uparrow}{1}, 2, 1\}$.

Plot the phase response:

Recall that we may decompose frequency responses as follows:

$$H_d(\omega) = \mathrm{Re}\{H_d(\omega)\} + j\mathrm{Im}\{H_d(\omega)\} = |H_d(\omega)|e^{j\angle H_d(\omega)}$$

Thus, we have

$$\angle H_d(\omega) = -\omega \text{ } (\pm\pi \text{ jumps where } H_d(\omega) \text{ changes signs.})$$

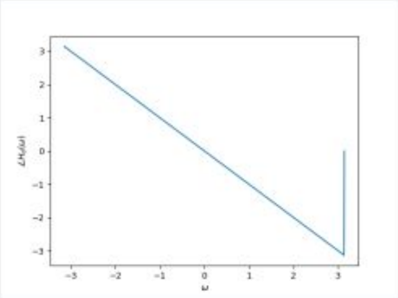


Figure 2: $\angle H_d(\omega)$

DTFT Exercise 2

- Suppose we have a new system defined by a real-valued impulse response $h[n]$ with corresponding DTFT $H_d(\omega)$. We also know the following about the magnitude and phase responses:

$$|H_d(\omega)| = \begin{cases} 1, & \pi \le \omega < \frac{3\pi}{2} \\ 2, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases} \qquad \angle H_d(\omega) = \begin{cases} -\frac{\pi}{4}, & \pi \le \omega < \frac{3\pi}{2} \\ \frac{\pi}{4}, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases}$$

a) Plot the magnitude response of $H_d(\omega)$ on the interval $-\pi$ to π.

b) Plot the phase response of $H_d(\omega)$ on the interval $-\pi$ to π.

DTFT Exercise 2

Given $h[n]$ real-valued and

$$|H_d(\omega)| = \begin{cases} 1, & \pi \le \omega < \frac{3\pi}{2} \\ 2, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases}, \quad \angle H_d(\omega) = \begin{cases} -\frac{\pi}{4}, & \pi \le \omega < \frac{3\pi}{2} \\ \frac{\pi}{4}, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases}$$

Plot $|H_d(\omega)| \in [-\pi, \pi]$:

We are given $h[n]$ is real-valued, thus Hermitian symmetry is our best friend here.

$$H_d(\omega) = H_d^*(-\omega), |H_d(\omega)| = |H_d(-\omega)|, \ \angle H_d(\omega) = -\angle H_d(-\omega)$$

Also, utilizing the 2π periodicity of the DTFT, we have:

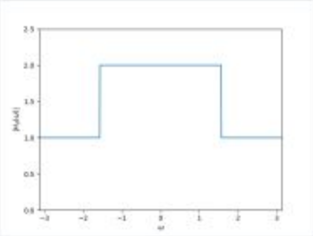


Figure 3: $|H_d(\omega)|$

DTFT Exercise 2

Given $h[n]$ real-valued and

$$|H_d(\omega)| = \begin{cases} 1, & \pi \le \omega < \frac{3\pi}{2} \\ 2, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases}, \quad \angle H_d(\omega) = \begin{cases} -\frac{\pi}{4}, & \pi \le \omega < \frac{3\pi}{2} \\ \frac{\pi}{4}, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases}$$

Plot $\angle H_d(\omega) \in [-\pi, \pi]$:

Similarly as in Part (a),

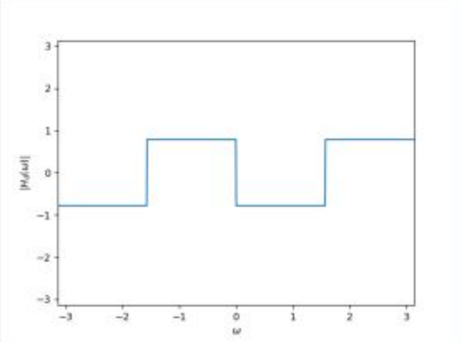


Figure 4: $\angle H_d(\omega)$

Sinusoidal Response Exercise 1

- We have an LSI system defined by the following LCCDE:

$$y[n] = x[n] - 2x[n-1] + x[n-2].$$

a) Find $H(z)$.
b) Find $H_d(\omega)$.
c) Find the output $y[n]$ to each of the following inputs:
   i) $x_1[n] = 2 + \cos(\pi n)$
   ii) $x_2[n] = e^{j\frac{\pi}{4}n} + \sin\left(-\frac{\pi}{2}n\right)$

Sinusoidal Response Exercise 1a

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

Find transfer function $H(z)$:

$$\begin{aligned} y[n] &= x[n] - 2x[n-1] + x[n-2] \\ \overset{\mathcal{Z}\{\cdot\}}{\leftrightarrow} Y(z) &= X(z)(1 - 2z^{-1} + z^{-2}) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= 1 - 2z^{-1} + z^{-2}. \end{aligned}$$

Sinusoidal Response Exercise 1b

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

Find frequency response $H_d(\omega)$:

The solution in part (a) is clearly BIBO stable since we have no poles. Thus,

$$
\begin{aligned}
H_d(\omega) &= H(z)\Big|_{z=e^{j\omega}} \\
&= 1 - 2e^{-j\omega} + e^{-j2\omega} \\
&= e^{-j\omega}(e^{j\omega} + e^{-j\omega} - 2) \\
&= e^{-j\omega}(2\cos(\omega) - 2).
\end{aligned}
$$

$$|H_d(\omega)| = |2\cos(\omega) - 2|, \ \angle H_d(\omega) = -\omega - \pi$$

Sinusoidal Response Exercise 1c.i

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

Find the output $y[n]$ for signals:

From our solution in (b), we have:

$$|H_d(\omega)| = |2\cos(\omega) - 2|, \;\; \angle H_d(\omega) = -\omega - \pi$$

$x_1[n] = 2 + \cos(\pi n)$:

$$|H_d(0)| = 0, \;\; \angle H_d(0) = 0, \;\; |H_d(\pi)| = 4, \;\; \angle H_d(\pi) = 0$$

$$y_1[n] = 2(0)\cos(0n + 0) + (4)\cos(\pi n \;\;)$$
$$= \;\; 4\cos(\pi n)$$

$2\cos(\pi)$ - 2 is -4, so phase should be $-\pi + \pi = 0$!

Sinusoidal Response Exercise 1c.ii

Given LSI system $y[n] = x[n] - 2x[n-1] + x[n-2]$:

Find the output $y[n]$ for signals:

From our solution in (b), we have:

$$|H_d(\omega)| = |2\cos(\omega) - 2|,\ \angle H_d(\omega) = -\omega - \pi$$

$x_2[n] = e^{j\frac{\pi}{4}n} + \sin\left(-\frac{\pi}{2}n\right)$:

$$H_d\left(\frac{\pi}{4}\right) = (\sqrt{2}-2)e^{-j\frac{\pi}{4}},\ \left|H_d\left(-\frac{\pi}{2}\right)\right| = 2,\ \angle H_d\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

$$\begin{aligned} y_2[n] &= \left((\sqrt{2}-2)e^{-j\frac{\pi}{4}}\right)e^{j\frac{\pi}{4}n} + (2)\sin\left(-\frac{\pi}{2}n - \frac{\pi}{2}\right) \\ &= (\sqrt{2}-2)e^{j\left(\frac{\pi}{4}n - \frac{\pi}{4}\right)} + 2\sin\left(-\frac{\pi}{2}n - \frac{\pi}{2}\right) \end{aligned}$$

$2\cos(\pi/4) - 2$ is -2, so phase should be $\pi/2 - \pi = -\pi/2$!

Ideal A/D Conversion

- Sampling via an impulse train will yield infinitely many copies of the analog spectrum in the digital frequency domain

$$X_d(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X_a\left(\frac{\omega - 2\pi k}{T}\right)$$

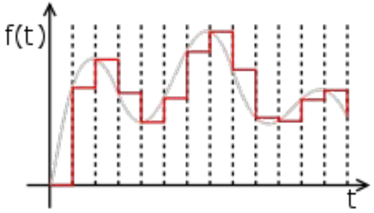
- Important relations to recall:
  - Nyquist Sampling Theorem:

$$\frac{1}{T} = f_s > 2B = 2f_{max}$$

  - Relationship between digital ω and analog frequencies Ω:

$$\omega = \Omega T$$

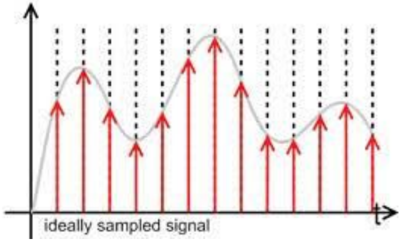
Ideal D/A Conversion



- Recall that our DTFT has infinitely many copies of our sampled analog spectrum

- Ideal D/A conversion requires we perfectly recover only the central copy between -π and π.
    - We suppose that we have an ideal low-pass analog filter (“interpolation filter”) with cutoff frequency corresponding to $\frac{\pi}{T}$.
    - Sinc in time, rect in frequency

- But ideal D/A can be expensive…what now?
    - Zero-Order Hold
        - Rect in time, sinc in frequency



Sampling Exercise 1

- Suppose we sampled some analog signal defined by

$$x_a(t) = \cos(\Omega_0 t)$$

with sampling period $T = \frac{1}{1000} s$ to obtain the digital signal $x[n] = \cos(\frac{\pi}{4} n)$.
Which of the following are possible values for Ω_0? (There may be more than one!)

a) 250π rad/s
b) $\frac{\pi}{4000}$ rad/s
c) -1750π rad/s
d) 4250π rad/s
e) $\frac{1}{8}$ rad/s

Sampling Exercise 1

For analog signal $x_a(t) = \cos(\Omega_0 t)$ sampled at $T = \frac{1}{1000}$s to obtain $x[n] = \cos\left(\frac{\pi}{4}n\right)$

What are possible values for Ω_0?

Always remember $\omega_d = \Omega_a T$! Moreover, by 2π periodicity of the DTFT,

$$x[n] = \cos\left(\frac{\pi}{4}n\right) \equiv \cos\left(\left(\frac{\pi}{4} + 2\pi k\right)n\right), \; k \in \mathbb{Z}.$$

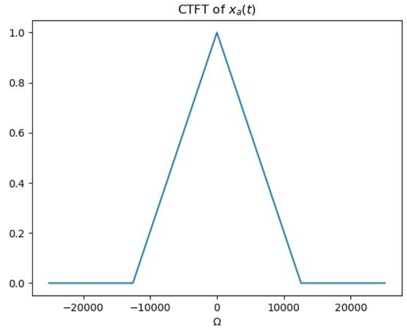
Thus,

$$\Omega_0 T = \left(\frac{\pi}{4} + 2\pi k\right)$$
$$\Omega_0 = 250\pi + 2000\pi k, \; k \in \mathbb{Z}.$$

Possible values for Ω_0 are then 250π (choice a), -1750π (choice c) and 4250π (choice d).

Sampling Exercise 2

- We have an analog signal $x_a(t)$ with $CTFT$ $X_a(\Omega)$ with maximum frequency 4000π.



For each of the following sampling periods T, draw the sampled $DTFT$ spectrum $Xd(\omega)$ on the interval -3π to 3π.

a) $T_1 = \frac{1}{8000}\ s$

b) $T_2 = \frac{1}{4000}\ s$

c) $T_3 = \frac{1}{2000}\ s$

Sampling Exercise 2a

The maximum frequency of $X_a(\Omega)$ is 4000π.

Sketch DTFT spectrum when $T = \frac{1}{8000}$

In each part, we should focus on where the maximum frequency maps. This will help us identify aliasing. If there is no aliasing, the shape of the spectrum will not change. By $\omega_d = \Omega_a T$,

$$\omega_d = \frac{4000\pi}{8000} = \frac{\pi}{2}.$$

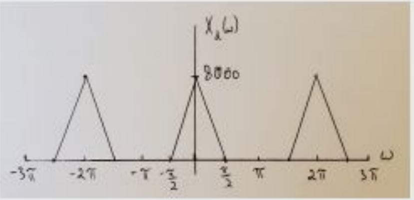


Figure 5: $T_1 = \frac{1}{8000} s$

Sampling Exercise 2b

The maximum frequency of $X_a(\Omega)$ is 4000π.

Sketch DTFT spectrum when $T = \frac{1}{4000}$

By $\omega_d = \Omega_a T$,

$$\omega_d = \frac{4000\pi}{4000}$$
$$= \pi.$$

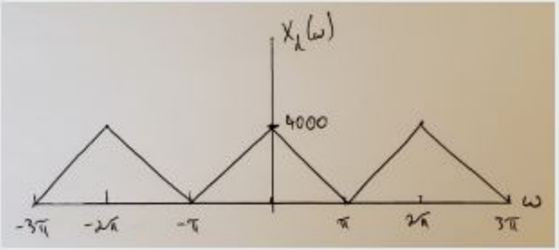


Figure 6: $T_1 = \frac{1}{4000}s$

Sampling Exercise 2c

The maximum frequency of $X_a(\Omega)$ is 4000π.

Sketch DTFT spectrum when $T = \frac{1}{2000}$

By $\omega_d = \Omega_a T$,

$$\begin{aligned}\omega_d &= \frac{4000\pi}{2000} \\ &= 2\pi.\end{aligned}$$

Aliasing!

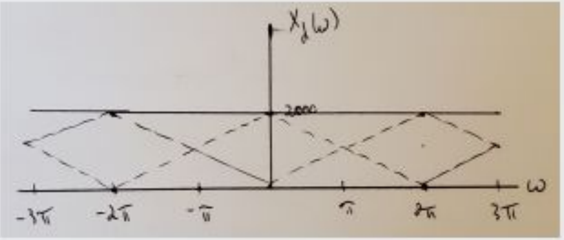
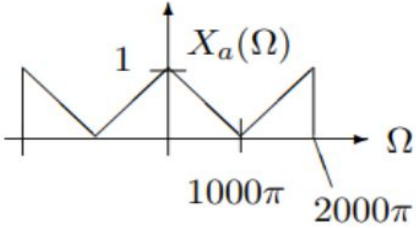
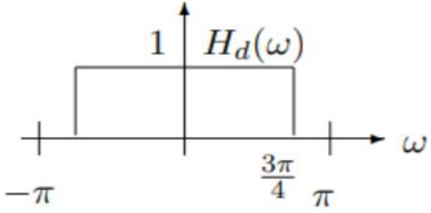


Figure 7: $T_1 = \frac{1}{2000} s$

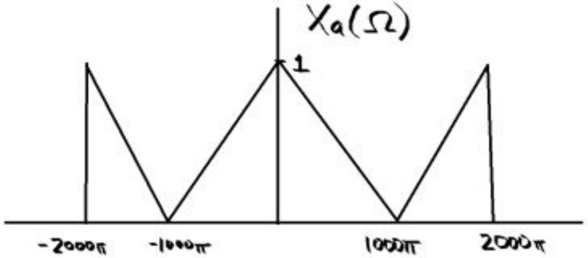
Sample Exercise 3

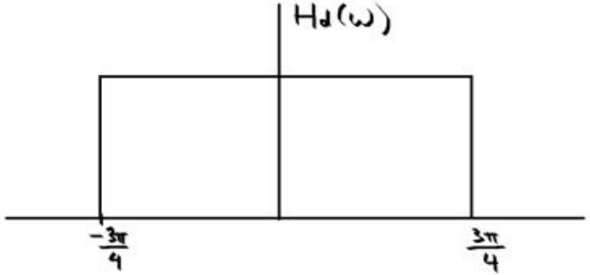
For the digital system in problem 3, assume $T = 0.5$ msec, with





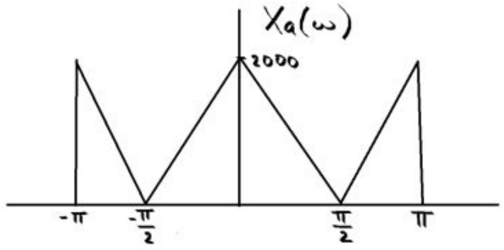
(a) Sketch $X_d(\omega)$, $Y_d(\omega)$, and $Y_a(\Omega)$.

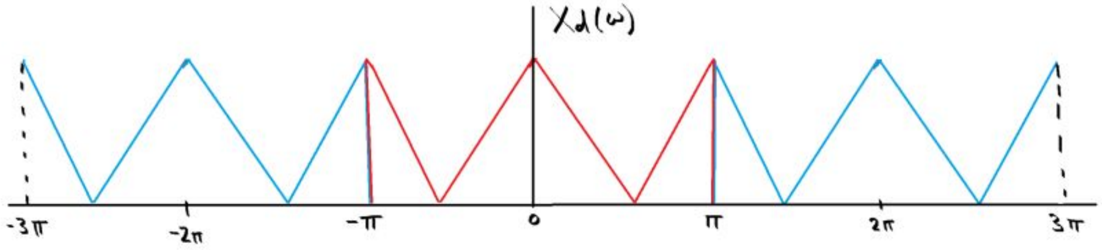


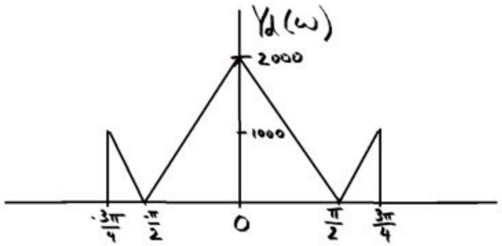


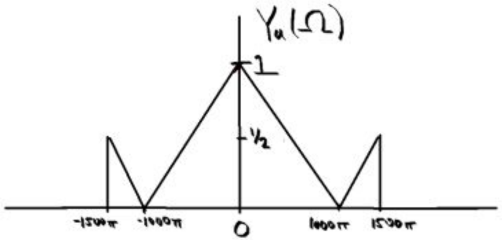
$$\boxed{\omega = \Omega T, \quad T = \frac{1}{2000}}$$

Sample Exercise 3 (cont)









Upsampling

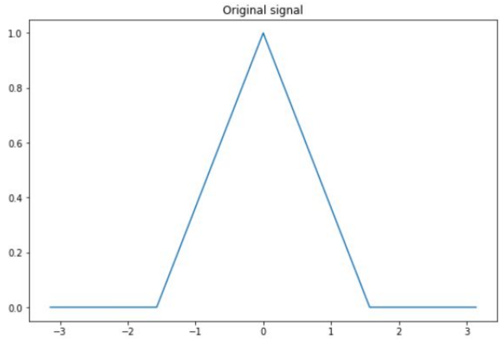
- In the DFT, we are adding zeros between every value we already have
  - If we upsample by U, we want there to be U times as many signal entries, so we add U-1 0s in between
- Example
  - Signal: [0, 1, 2, 3]
  - Upsampled by 3: [0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0]
- DTFT Effects:
  - Contracts DTFT (copies move in, will have to LPF those away)

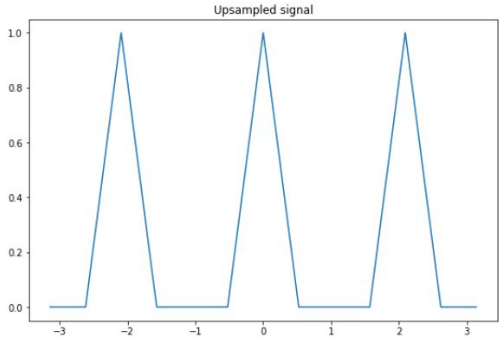
$$y[n] = \begin{cases} x\left[\frac{n}{U}\right], & n = \pm U, 2U, \ldots \\ 0, & \text{else} \end{cases}$$

$$Y_d(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n U}$$

$$Y_d(\omega) = X_d(U\omega).$$

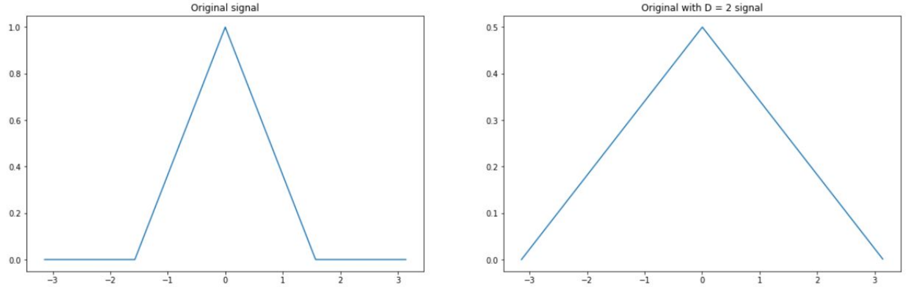




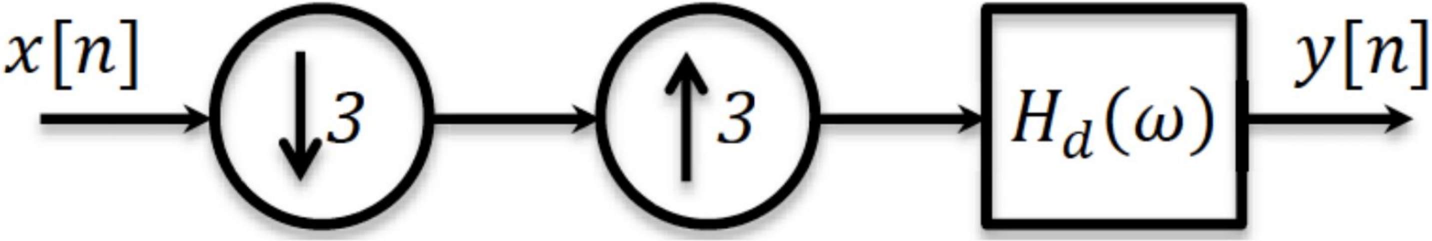
Downsampling

- The opposite of upsampling (wild)
  - $x_d[n] = x[Dn]$ with downsampling factor D
  - Effectively, keep only every D values from input signal
- Example
  - Signal: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
  - Downsampled by 3: [0, 3, 6, 9]
- DTFT Effects:
  - Expands DTFT (susceptible to aliasing !!!)
    - Solution to aliasing: low pass filter first

$$Y_d(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X_d\left(\frac{\omega - 2\pi k}{D}\right).$$



Upsampling and Downsampling in DFT Exercise

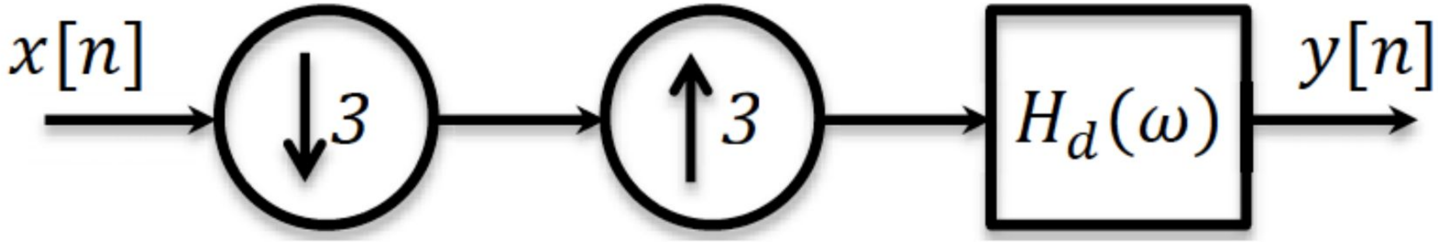


$H_d(w)$ is ideal LPF with cutoff $2\pi/5$

A. $x[n] = cos(0.2\pi n)$ $y[n] = ?$

B. $x[n] = cos(0.6\pi n)$ $y[n] = ?$

Upsampling and Downsampling in DFT Exercise



A. $y[n] = \frac{1}{3}\cos(0.2\pi n) = \frac{1}{3}x[n]$

B. $y[n] = \frac{1}{3}\cos(\frac{\pi}{15}n)$

Upsampling and Downsampling in DTFT Exercise

$$X_d(\omega) = \begin{cases} 1 - \frac{3|\omega|}{\pi}, & |\omega| \leq \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < |\omega| \leq \pi \end{cases}$$

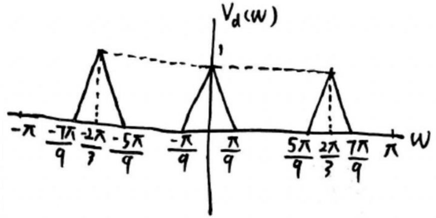
Given $V_d(w)$ is $X_d(w)$ upsampled by L, sketch $V_d(w)$ if L=3

Given $W_d(w)$ is $X_d(w)$ downsampled by D, sketch $V_d(w)$ if D=3

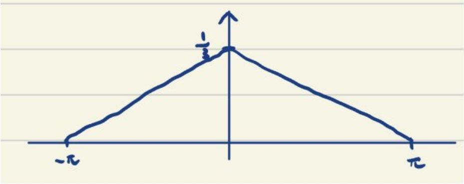
Upsampling and Downsampling in DTFT Exercise

$$X_d(\omega) = \begin{cases} 1 - \frac{3|\omega|}{\pi}, & |\omega| \leq \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < |\omega| \leq \pi \end{cases}$$

Given $V_d(w)$ is $X_d(w)$ upsampled by L, sketch $V_d(w)$ if L=3



Given $W_d(w)$ is $X_d(w)$ downsampled by D, sketch $V_d(w)$ if D=3



A small problem...

- “Oh no, we’re on computers. Things have to be discrete!”
- Fine, let’s sample. Behold: the digital domain!
- “Ok, that’s cool. How do we do things with it?”
- Behold: The DTFT! It allows us to do cool things to sampled signals.
  - But it’s continuous.
- ...
- ...
- “Oh no, we’re on computers. Things have to be discrete!”

- Introducing... DFT! It is literally just sampling the DTFT. It worked one time, so we might as well do it again.
  - This intuition alone will allow you to take care of all DFT problems.

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}, 0 \leq k \leq \boxed{N-1}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi kn}{N}}, 0 \leq n \leq \boxed{N-1}$$

Not defined at k = N! In other words, there is no point at 2π! X[N-1] is the final sample.

- What is the relationship between the DTFT and the DFT?

$$\omega_k = \frac{2\pi k}{N}$$

$$\omega_k \in \left\{0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{2\pi(N-1)}{N}\right\}$$

DFT Properties

The DTFT was periodic. The DFT is a sampled DTFT. So the DFT is also periodic.

- Periodicity by N

$$X[k] = X_d[(\frac{2\pi}{N}) \times k] \leftrightarrow x[<n>_N]$$

- *Circular* shift

$$x[n]W_N^{-k_0 n} \leftrightarrow X[<k - k_0>_N]$$

- *Circular* modulation

$$x[n]\cos((\frac{2\pi}{N})k_0 n) \leftrightarrow (\frac{1}{2})X[<k + k_0>_N] + (\frac{1}{2})X[<k - k_0>_N]$$

- *Circular* convolution - Don't worry about this for the exam!

$$x[n] Ⓝ h[n] \leftrightarrow X[k]H[k]$$

- We must amend our *DTFT* properties with the "circular" term because the *DFT* is defined over a finite length signal and assumes periodic extension of that finite signal.

More DFT Properties

- If the signal is symmetric, then its DFT's magnitude has even symmetric. Its DFT's phase has odd symmetric.

- The DFT of the DFT (cuz why not) is the original signal reversed, then multiplied by N.

Circular Convolution Exercise 1

Compute the linear AND circular convolution $y[n] = x[n] ⊛_4 h[n]$ where

$\{x[n]\}^3_{n=0} = \{ 2 , 4, 6, 8 \}$ & $h[n] = \{ -1, -1, 1 \}$

Table Method: h[n] rows and x[n] columns → y[n] = {-4, 2, -8, -10}

ℓ =	**0**	**1**	**2**	**3**	**y[n]**
$x[\ell]$ =	**2**	**4**	**6**	**8**	
$h(<0-\ell>_4)$	-1	0	1	-1	(-1)*2+6-8 = -4
$h(<1-\ell>_4)$	-1	-1	0	1	-2-4+8 = 2
$h(<2-\ell>_4)$	1	-1	-1	0	2-4-6 = -8
$h(<3-\ell>_4)$	0	1	-1	-1	4-6-8 = -10

Linear result: y[n] = {2, 2, 0, -2, -14, -8}

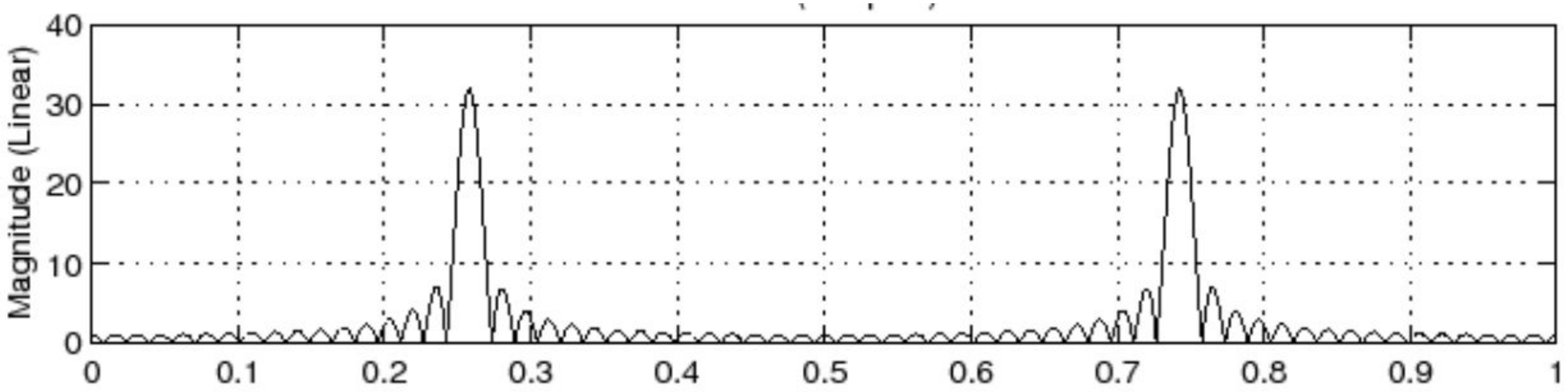
Zero-Padding

- We can improve the **resolution** of the DFT simply by adding zeros to the end of the signal.
- This doesn’t change the frequency content of the DTFT!
  - No information/energy is being added.
- Instead, it increases the number of samples the DFT takes of the DTFT.
- This can be used to improve spectral resolution.

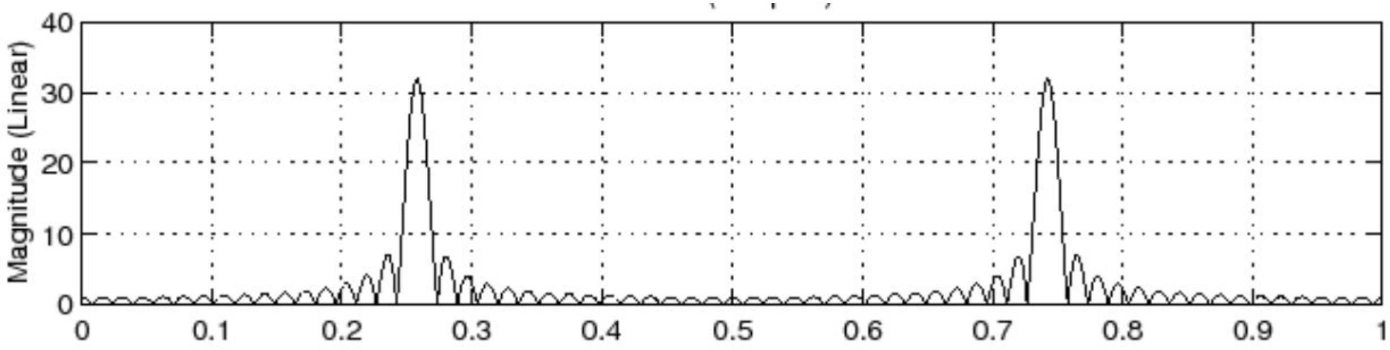
DFT Spectral Analysis

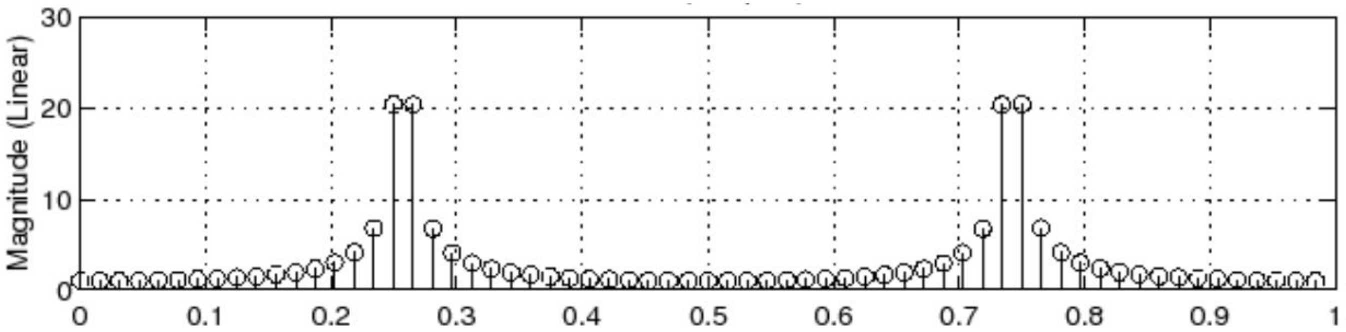
Recall the magnitude of the DTFT of a sinusoidal.

DFT samples this. But what if the sample doesn't land on the peak perfectly?



DFT Spectral Analysis





Wat do :(

DFT Exercise 1

- Surprisingly, we have another signal

$$x[n] = \cos\left(\frac{\pi}{3}n\right), 0 \le n < 18.$$

a) For which value(s) of k is the DFT of $x[n]$, $X[k]$, largest?

b) Suppose now that we zero-pad our sequence with 72 zeros to obtain $y[n]$. For which value(s) of k is $Y[k]$ largest?

$$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$$

DFT Exercise 1a

Given $x[n] = \cos\left(\frac{\pi}{3}n\right),\ 0 \leq n < 18,$

For which value(s) of k is $X[k]$ largest?

$$\omega_k = \frac{2\pi k}{N}$$
$$\frac{\pi}{3} = \frac{\pi}{9}k$$
$$k_1 = 3.$$

We also have energy at $\omega = -\frac{\pi}{3}$:

$$-\frac{\pi}{3} = \frac{\pi}{9}k$$
$$k_2 = -3$$

Instead by N periodicity of the DFT, we should say

$$k_2 \equiv k_2 + N = 15$$

DFT Exercise 1b

Given $x[n] = \cos\left(\frac{\pi}{3}n\right),\ 0 \le n < 18,$

For which k is $X[k]$ largest if $x[n]$ is padded with 72 zeros?

New length is $N = 90$:

$$\omega_k = \frac{2\pi k}{N}$$
$$\frac{\pi}{3} = \frac{\pi}{45}k$$
$$k_1 = 15.$$

$$-\frac{\pi}{3} = \frac{\pi}{45}k$$
$$k_2 = -15$$
$$\implies k_2 = 75.$$

Windowing

- Recall that the *DFT* implies infinite periodic extension of our signal.
- This extension can lead to artifacts known as “spectral leakage”
- Window functions help with these artifacts
  - Rectangular window
  - Hamming window
  - Hanning window - NOT IMPORTANT FOR EXAM
  - Kaiser window - NOT IMPORTANT FOR EXAM

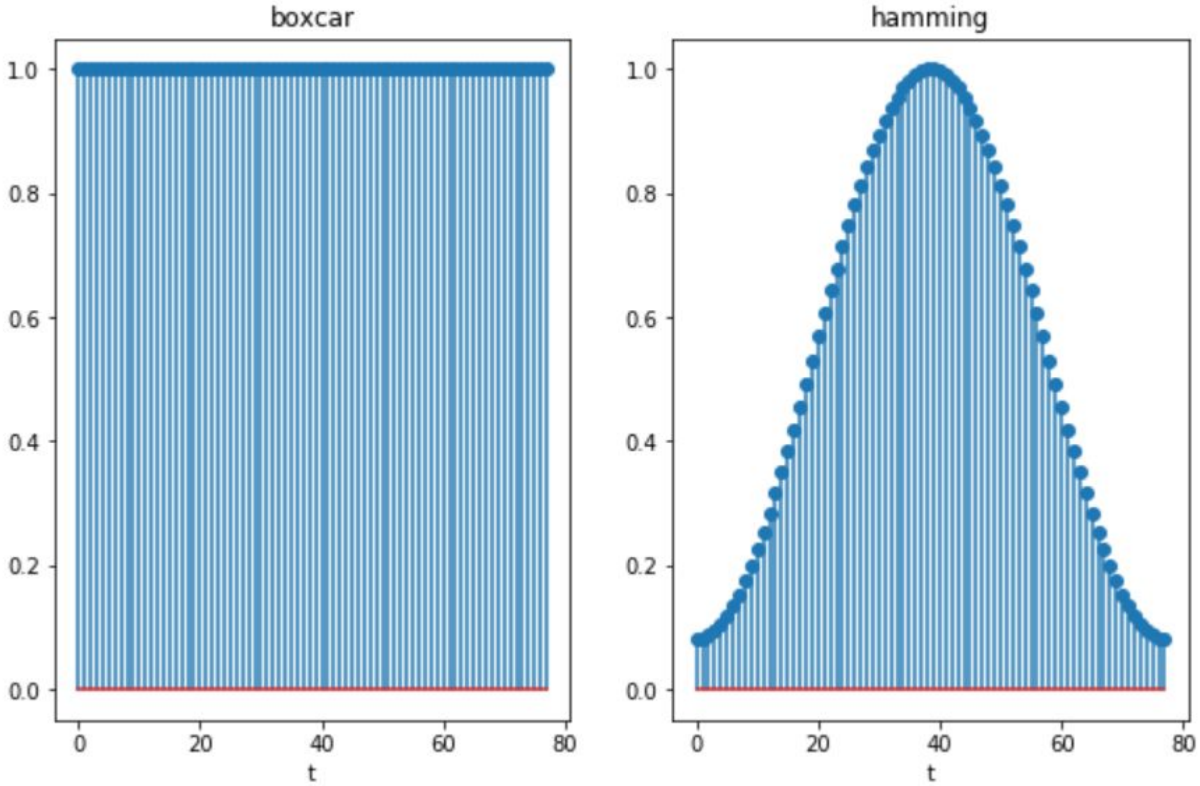
Windowing - Cont'd

- Windowing is just multiplication in the time domain

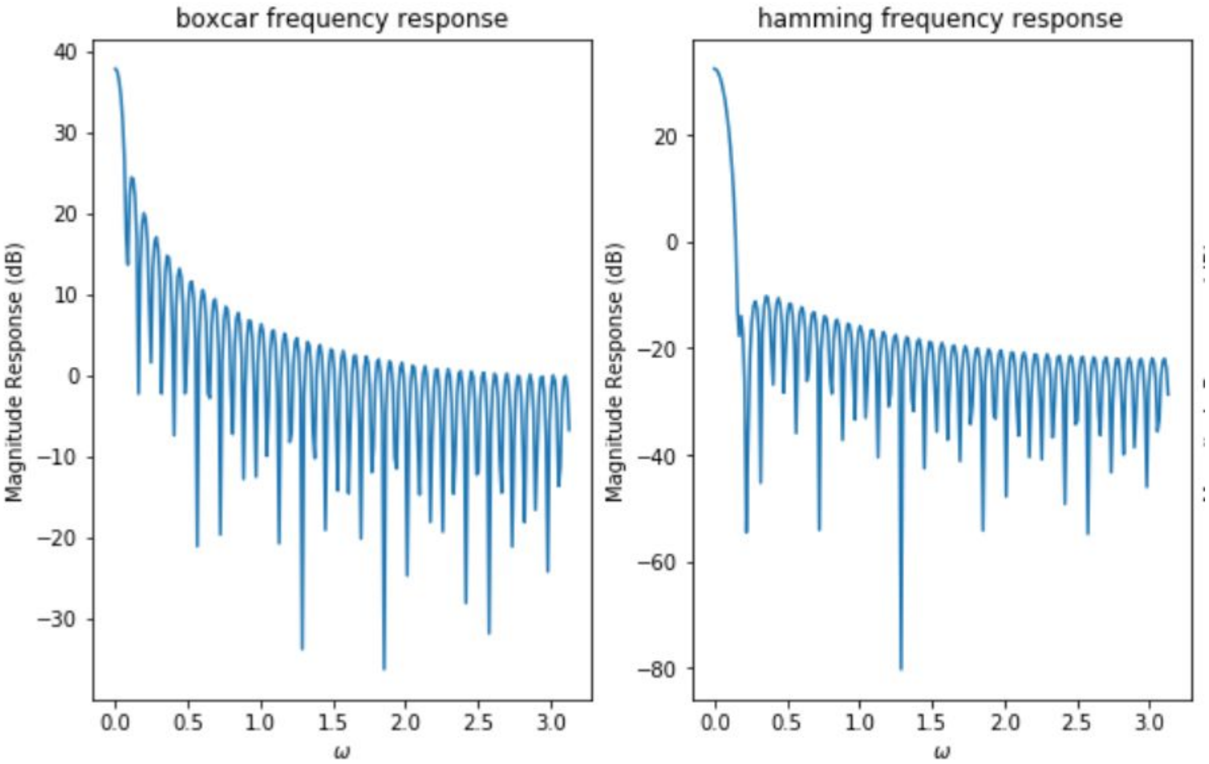
$$x_w = x[n]w[n]$$

- We multiply the window functions from last slide to the input signal in the time domain to help with spectral leakage
- We care about the main lobe width and side lobe attenuation of these windows.
  - In particular, know the tradeoffs between the Rectangular and Hamming windows
  - Hamming has more side lobe attenuation but wider main lobe

Windowing



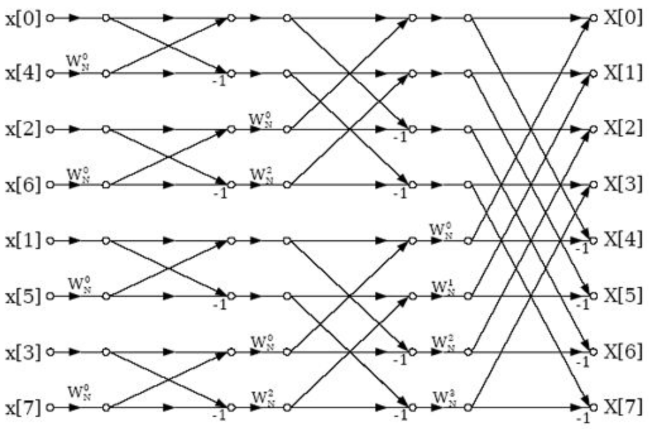
Windowing



boxcar frequency response
Magnitude Response (dB)
40
30
20
10
0
−10
−20
−30
0.0
0.5
1.0
1.5
2.0
2.5
3.0
ω
hamming frequency response
Magnitude Response (dB)
20
0
−20
−40
−60
−80
0.0
0.5
1.0
1.5
2.0
2.5
3.0
ω

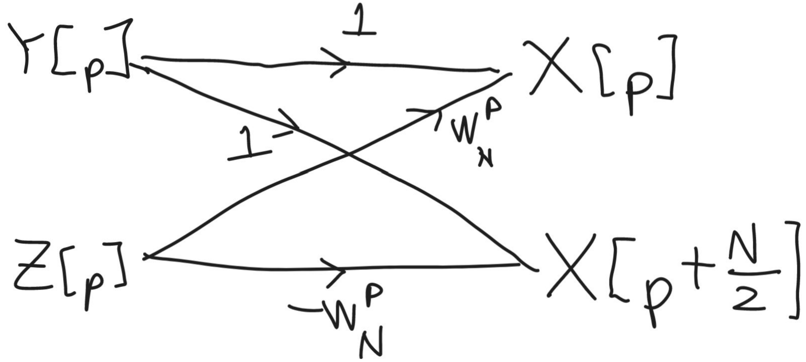
Fast Fourier Transform

- Class of “divide-and-conquer” algorithms to compute the DFT efficiently.
- Naïve DFT takes $O(n^2)$ operations, FFT takes $O(n \log n)$.
- Decimation in Time vs. Decimation in Frequency
- Butterfly diagrams

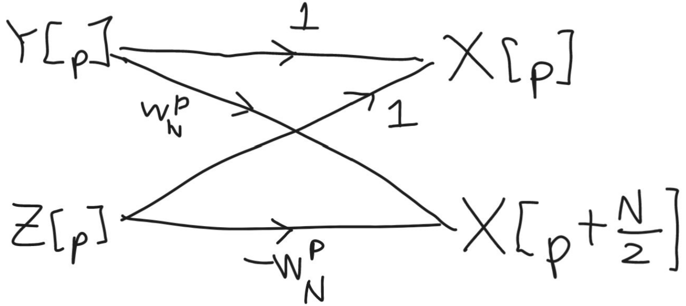


FFT Butterflies

$$W_N = e^{-j\frac{2\pi}{N}}$$



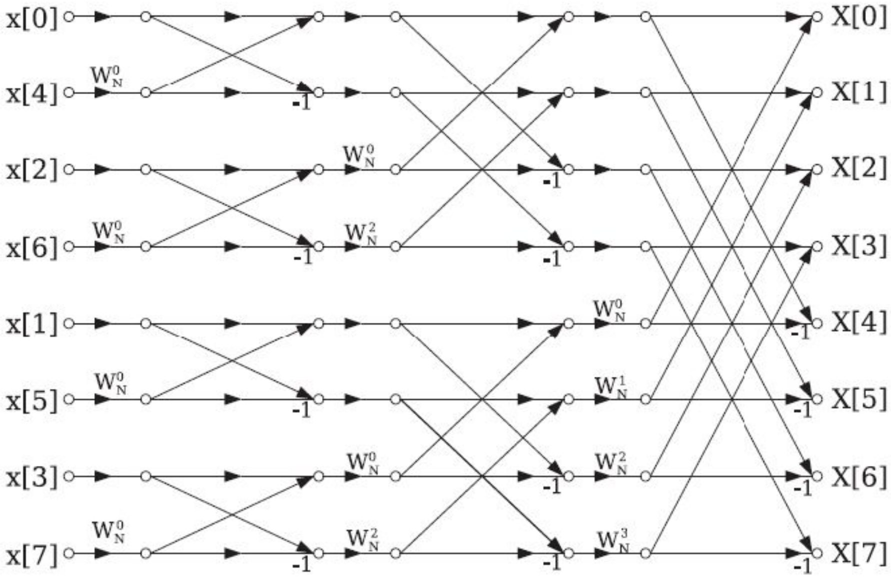
Decimation in Time



Decimation in Frequency

Slightly different!

FFT Butterflies



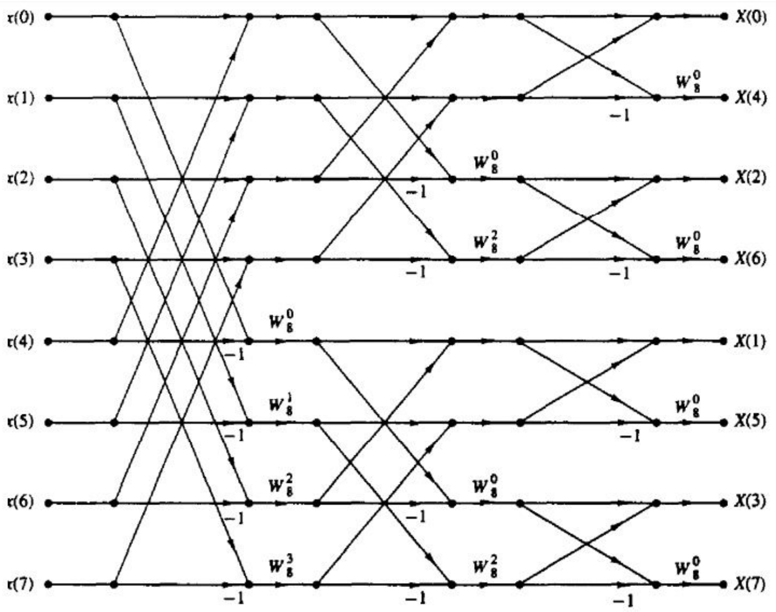


Figure 6.11 $N = 8$-point decimation-in-frequency FFT algorithmn.

Decimation in Time

The time points are "decimated" to do the first FFTs.

Decimation in Frequency

The frequency points are "decimated" at the end.

Fast Linear Convolution via FFT

- Convolution in the time domain requires $O(n^2)$ operations.
- By convolution theorem, perhaps we can do better in the frequency domain?
- Don't forget multiplication in DFT domain is *circular* convolution in time.
- To avoid aliasing, we adopt the following procedure
- Given signal x and filter h of lengths N and L, respectively:

1. Zero-pad x and h to length $N + L - 1$
2. Take their FFTs
3. Multiply in frequency domain
4. Take the inverse FFT

This procedure takes $O(n \log n)$ operations.

FFT Practice 1

Suppose you want to convolve two real-number signals of length 7000 and length 1100.

How many real-number multiplications does this take using classic convolution?

How many real-number multiplications does this take using using FFT?

Classic Convolution

Partial overlap at the beginning: (1 + 2 + … + 1099) = (1099)(1100)/2

Partial overlap at the end: (1099 + 1098 + … + 2 + 1) = (1099)(1100)/2

Full overlap in the middle: (1100)(7000 - 1100 + 1)

Total: 7,700,000 multiplications.

FFT

Zero-pad both signals to length 8192. (0)

2x FFT.

- Multiplication by 1 is free.
- Each left-side of the butterfly is multiplied by one complex number.
- This is done log_2 times, one for each butterfly phase.
- Total: (**2** * 8192 * log_2(8192) * 4)

Multiply the signals together (8192 * 4)

1x IFFT: (1 * 8192 * log_2(8192) * 4)

Total: 1,310,720 multiplications.

Some Hard Numbers...

Convolving length 7000 with length 1100:

- Classic convolution: 7,700,000 multiply operations

- FFT: 1,310,720 multiply operations

That's 83% faster.

So what :|

- A modern-day computer can do around 1e9 operations per second.

- 7,700,000 operations = 7.7 milliseconds.
- 1,310,720 operations = 1.3 milliseconds.

- Ok that doesn't really do anything.

Fine. Some Hard(er) Numbers...

The average MP3 file seems to be about 4MB.

Convolving length 4e6 with length 4e6 (apparently to add reverb or filter or something):

- Classic convolution: 1.6e13 multiply operations
  - About 4.5 hours.
- FFT: 1.12e9 multiply operations
  - About a second.

Ok that's pretty good.

FFT: The Most Important Algorithm of the Modern World

- Fast large-integer and polynomial multiplication
- Solve partial differential equations
- Filtering algorithms
- Digital recording, sampling, and pitch correction
- JPEG and MP3
- Used in 5G, LTE, Wi-Fi, and other communication systems

These are ALL made viable by FFT. Without it, these would be so slow…

Trivia Time!

Who was the first to come up with the FFT?

Answer: Gauss. Yes, that Gauss. And in 1805.

Yes. This predates Fourier.

And he didn't even publish it :(

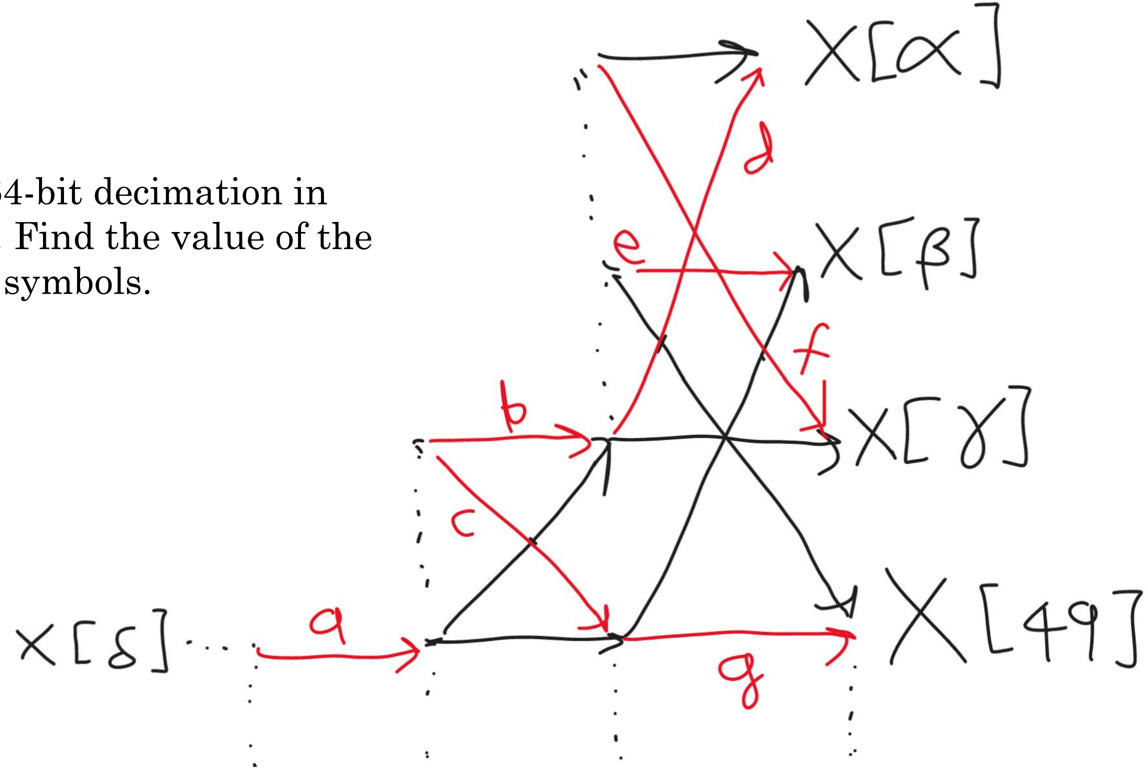
FFT Practice

What is the first step when drawing the entire FFT diagram of the following signal?

$$x[n] = \{1, 2, 3\}$$

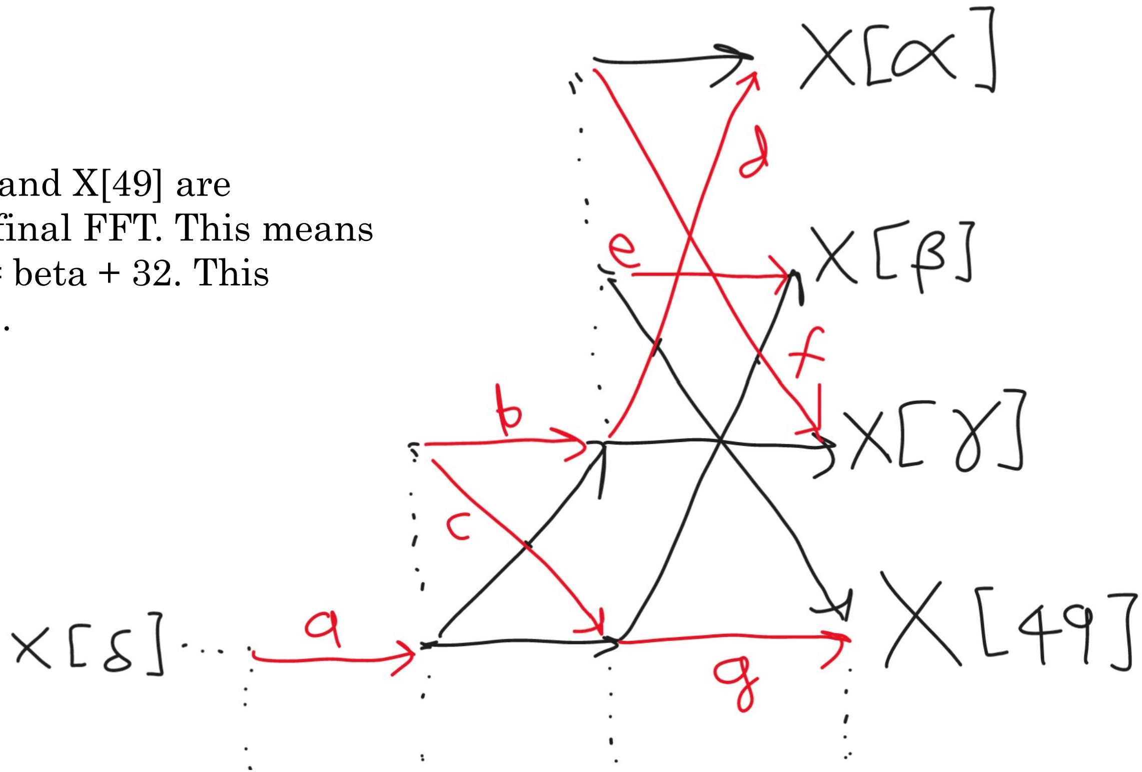
FFT Practice

This is part of a 64-bit decimation in time radix-2 FFT. Find the value of the letters and greek symbols.



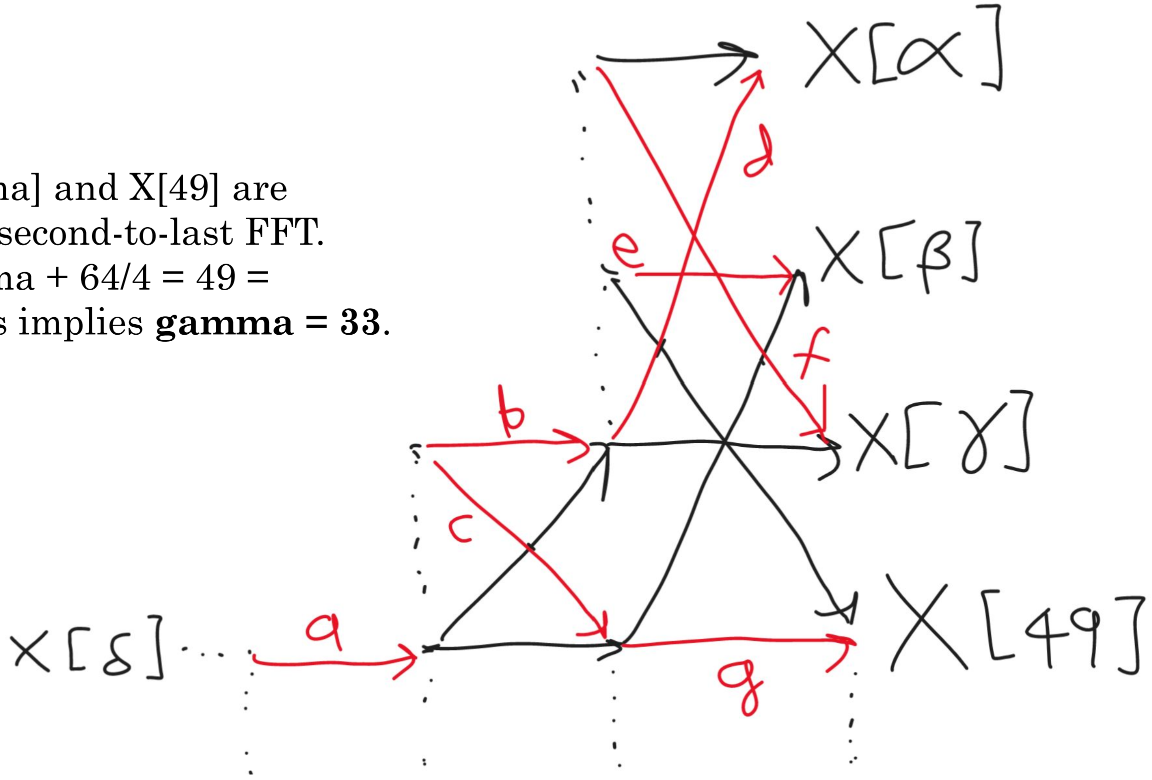
FFT Practice

We know X[beta] and X[49] are connected by the final FFT. This means beta + 64/2 = 49 = beta + 32. This implies **beta = 17**.



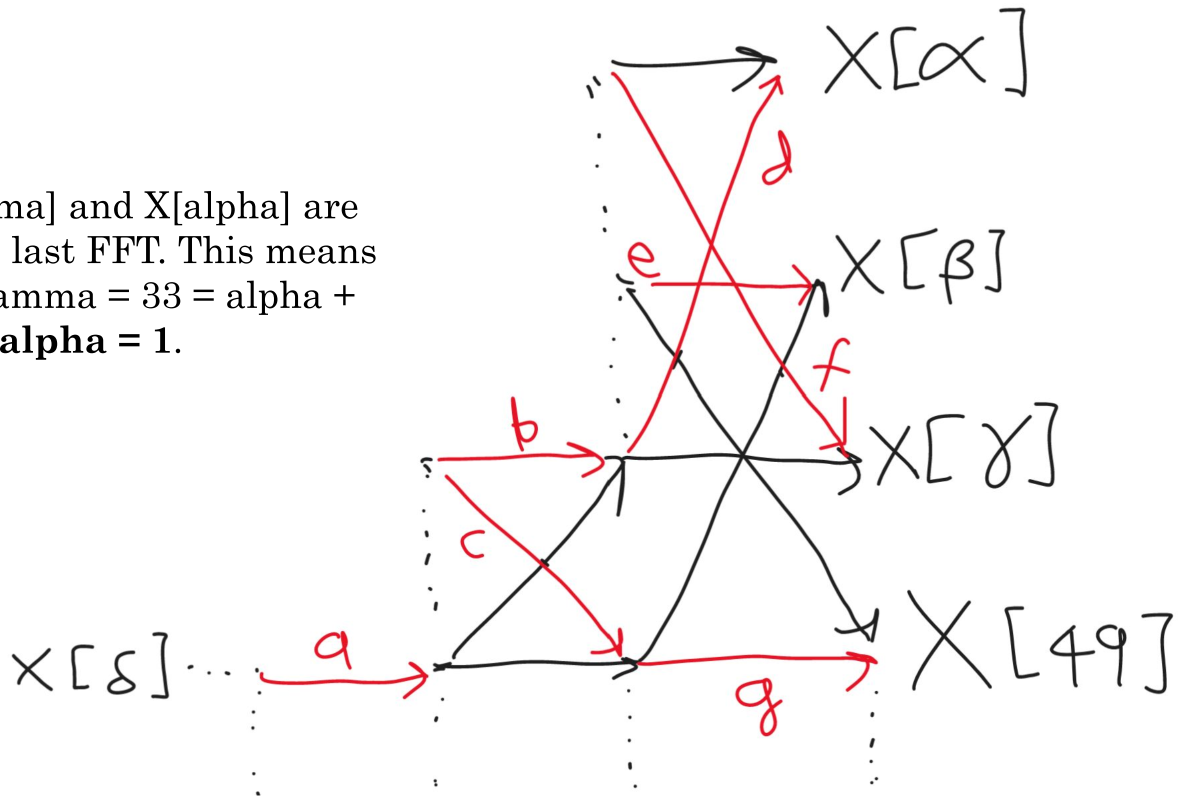
FFT Practice

We know X[gamma] and X[49] are connected by the second-to-last FFT. This means gamma + 64/4 = 49 = gamma + 16. This implies **gamma = 33**.



FFT Practice

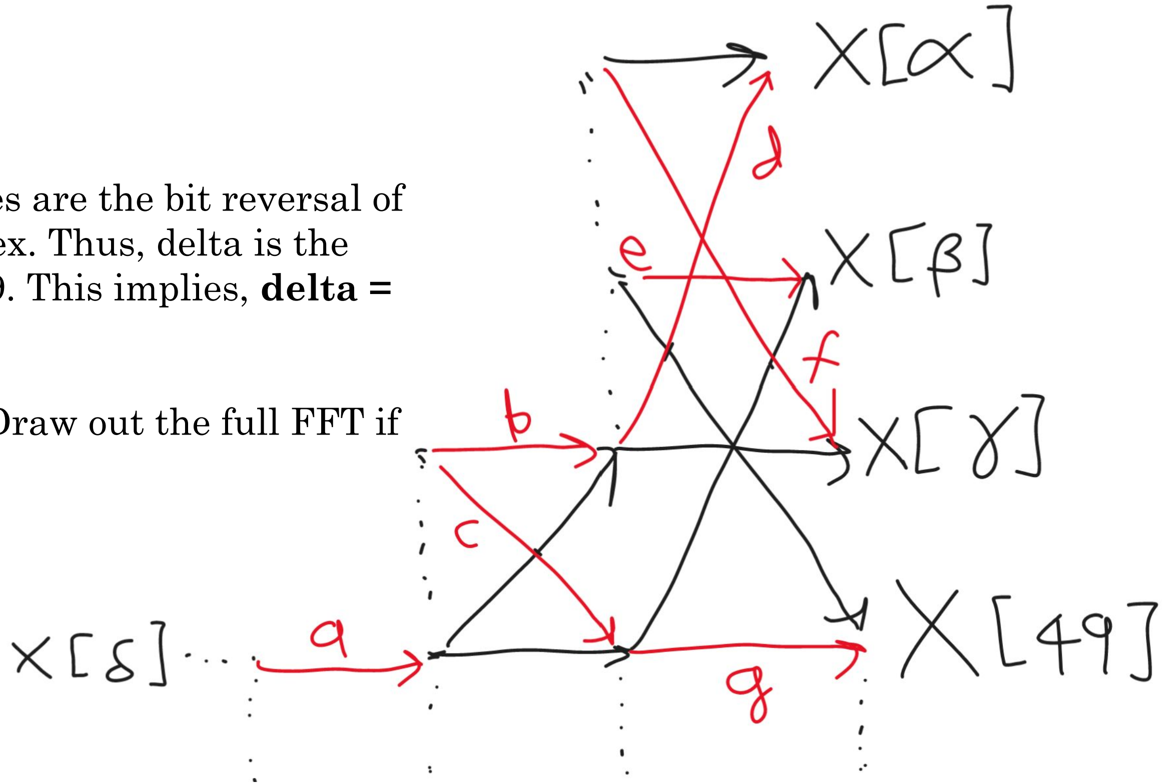
We know X[gamma] and X[alpha] are connected by the last FFT. This means alpha + 64/2 = gamma = 33 = alpha + 32. This implies **alpha = 1**.



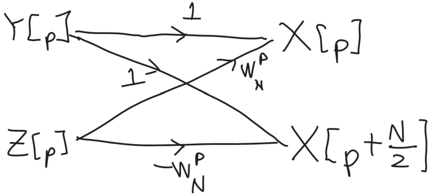
FFT Practice

FFT input indices are the bit reversal of their output index. Thus, delta is the bit-reversal of 49. This implies, **delta = 35**.

Not convinced? Draw out the full FFT if you wish.



FFT Practice



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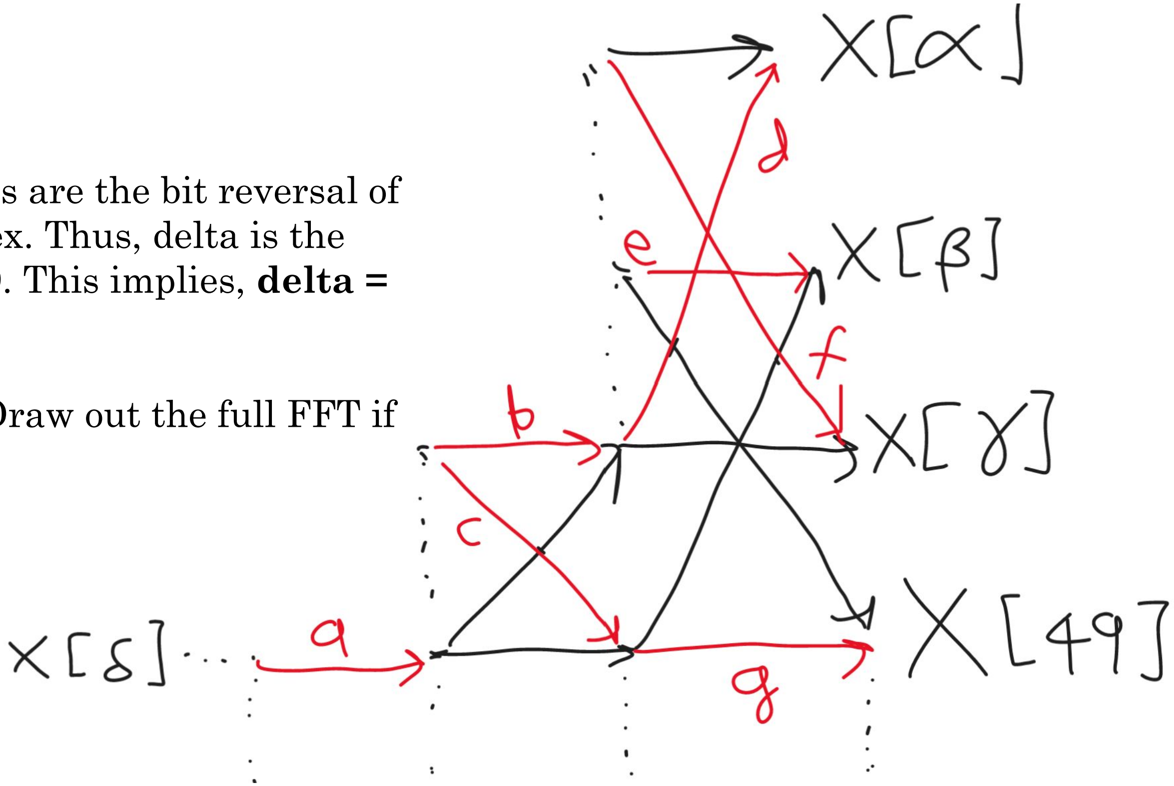
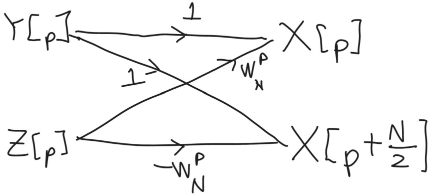
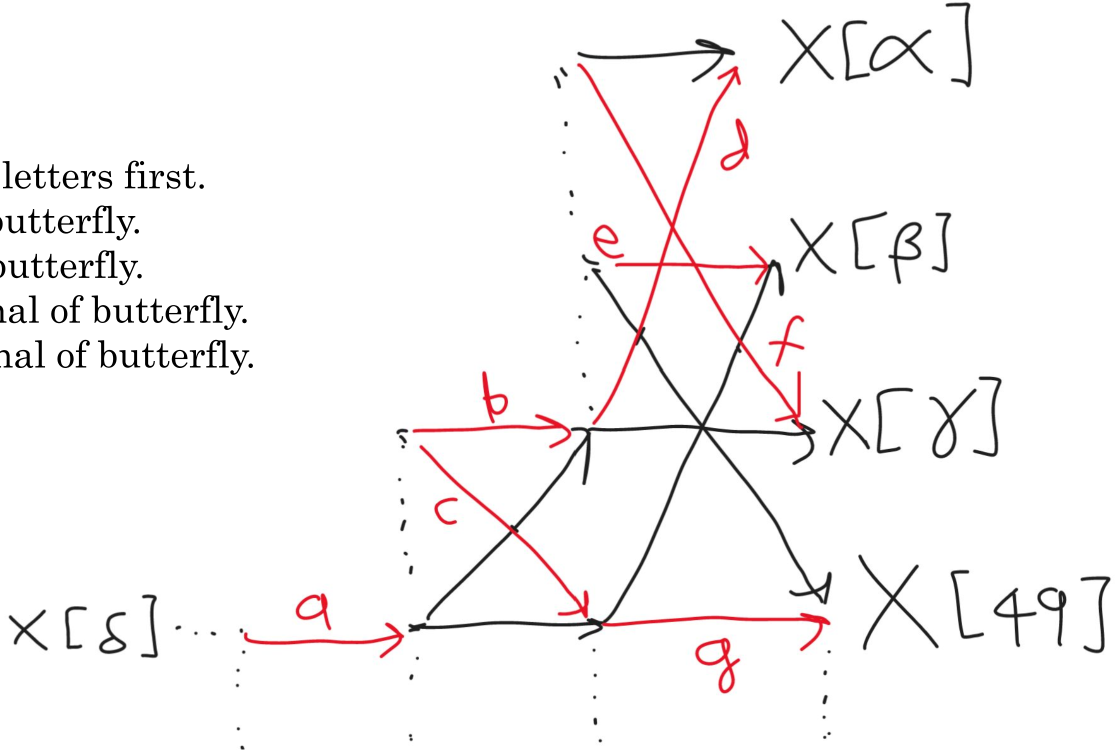


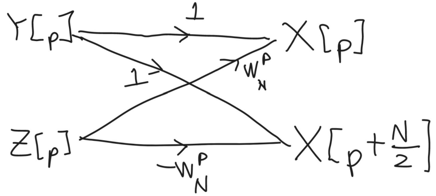
FFT Practice



Let's do the easier letters first.
e = 1: Top part of butterfly.
b = 1: Top part of butterfly.
f = 1: Down-diagonal of butterfly.
c = 1: Down-diagonal of butterfly.



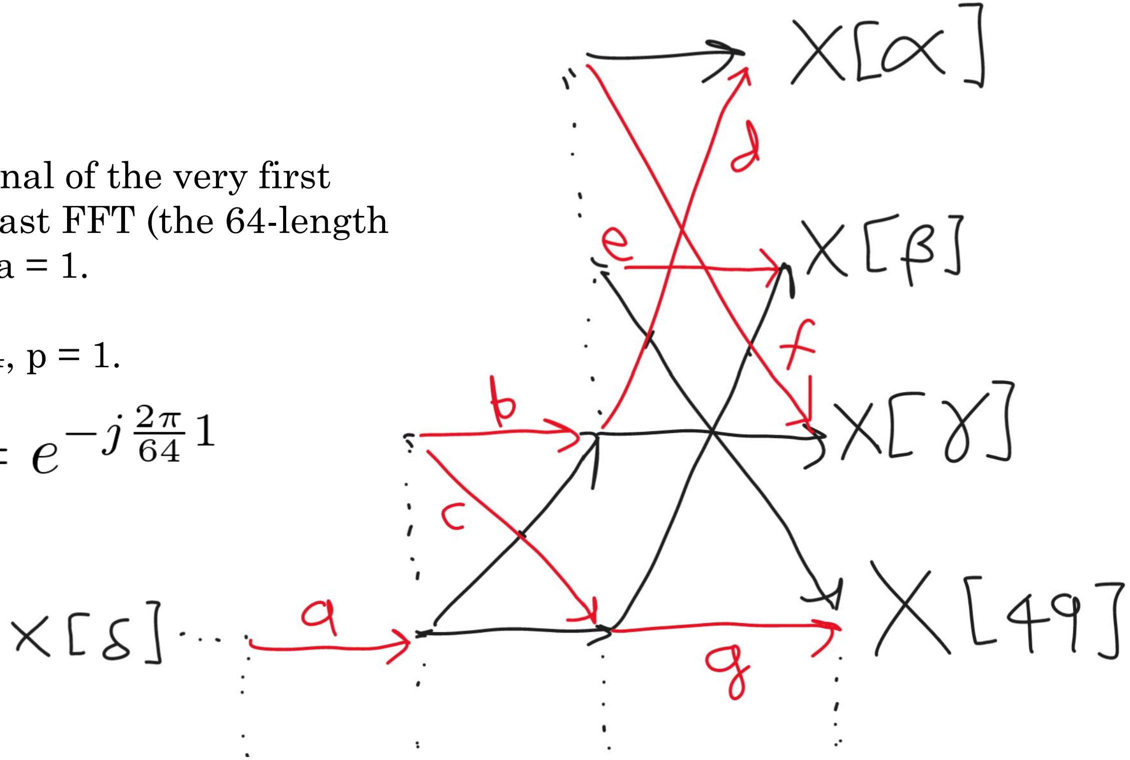
FFT Practice



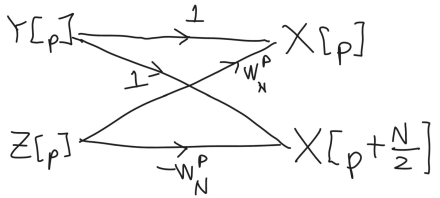
d is the up-diagonal of the very first butterfly in the last FFT (the 64-length FFT), since alpha = 1.

Thus, use N = 64, p = 1.

$$d = W_{64}^{1} = e^{-j\frac{2\pi}{64}1}$$



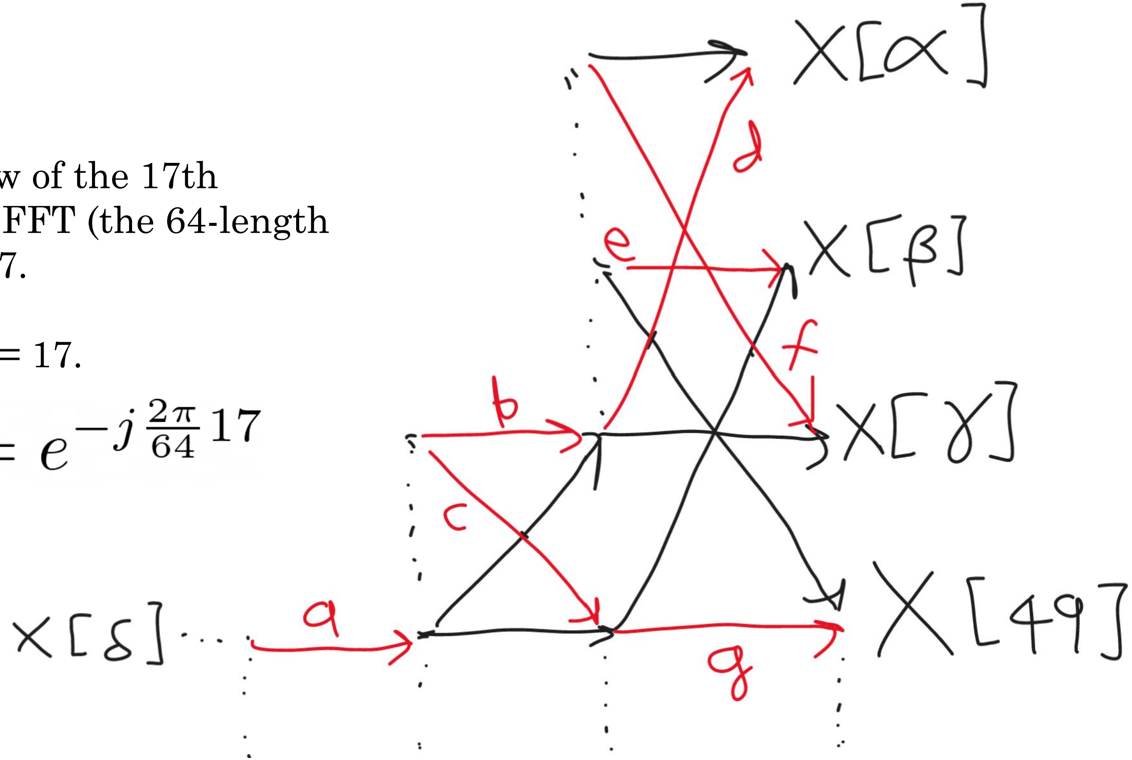
FFT Practice



g is the bottom arrow of the 17th butterfly in the last FFT (the 64-length FFT), since beta = 17.

Thus, use N = 64, p = 17.

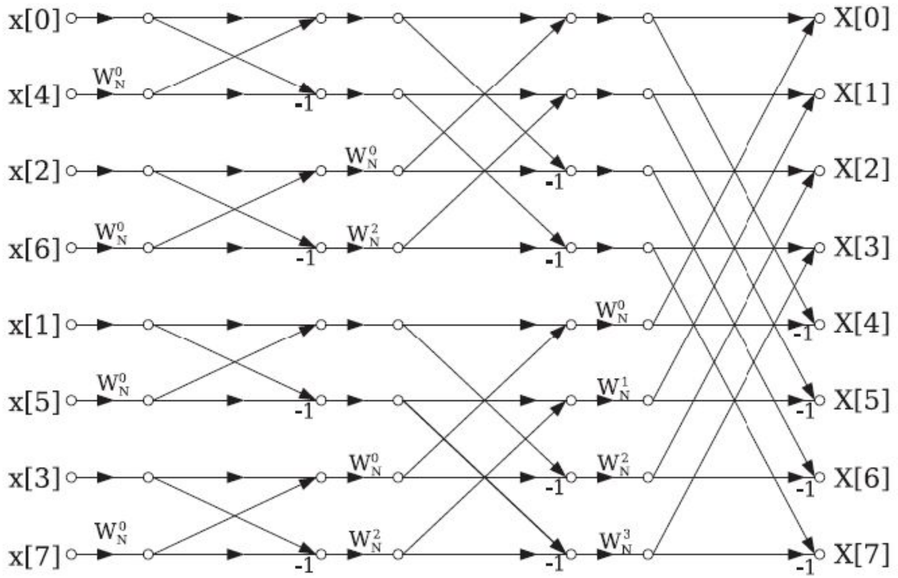
$$g = -W_{64}^{17} = e^{-j\frac{2\pi}{64}17}$$



A reminder...

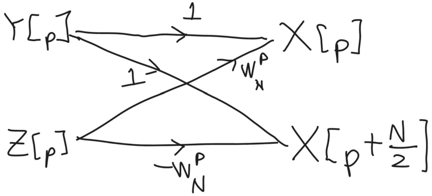
In the last FFT, we're doing 8-length butterflies.
In the second-to-last FFT, we're doing 4-length butterflies.
In the third-to-last FFT, we're doing 2-length butterflies.

"Final elements" 0 and 1 are grouped into one butterfly. 2 and 3 are grouped. 4 and 5 are grouped. 6 and 7 are grouped.



Decimation in time for length-8 signal

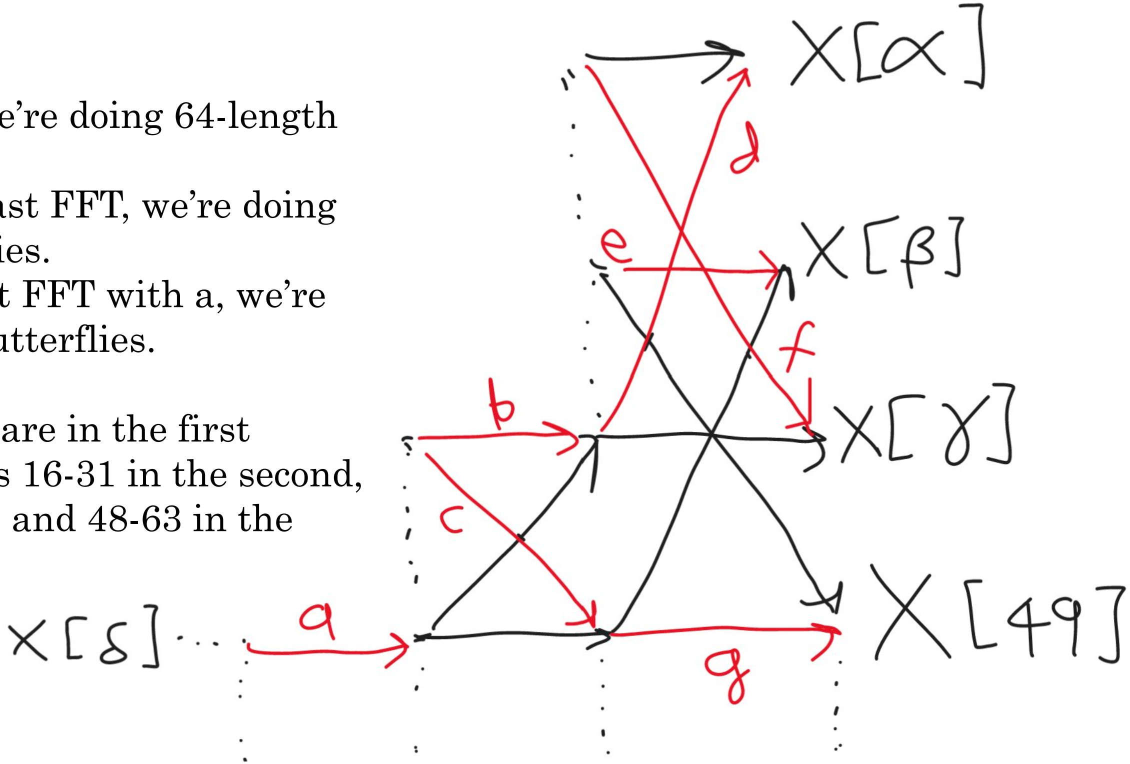
FFT Practice



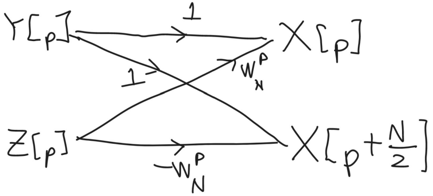
Analogously,

In the last FFT, we're doing 64-length butterflies.
In the second-to-last FFT, we're doing 32-length butterflies.
In the third-to-last FFT with a, we're doing 16-length butterflies.

So, elements 0-15 are in the first butterfly, elements 16-31 in the second, 32-47 in the third, and 48-63 in the fourth.



FFT Practice

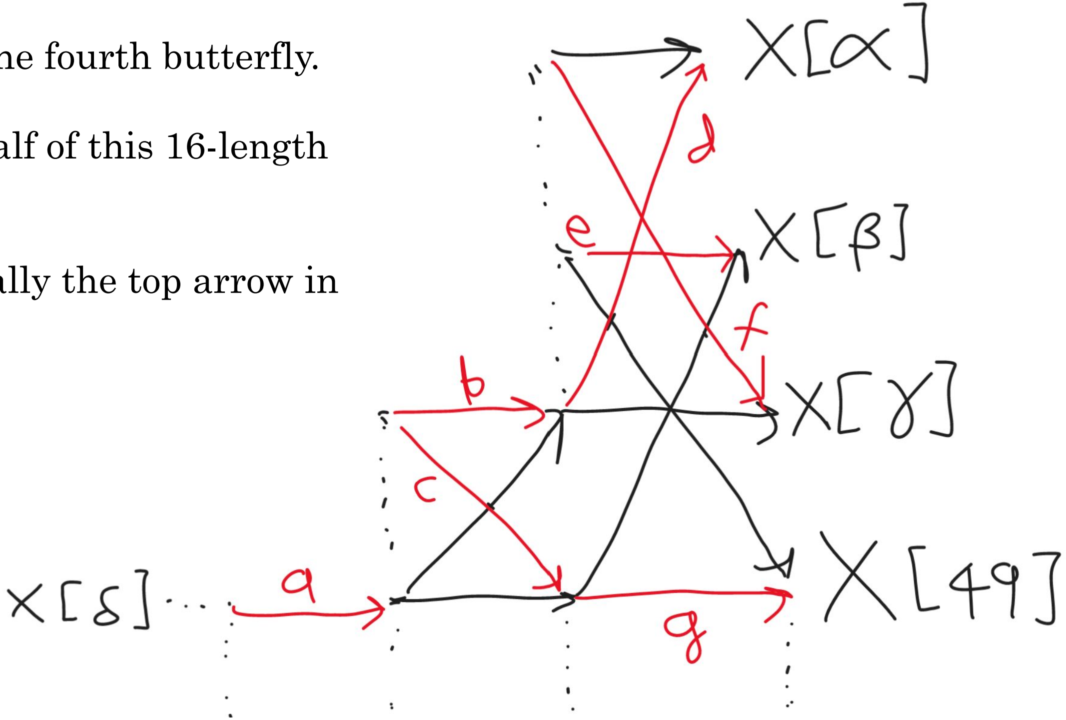


Elements 48-63 in the fourth butterfly.

49 is in the upper-half of this 16-length butterfly.

Therefore, a is actually the top arrow in the butterfly.

a = 1.



Thank you!

Good luck on Midterm 2!

Let us know if we were helpful: