University of Illinois
ECE 310

Profs. Liang, Moon, Snyder

Midterm Exam 2

7:00-9:00pm, Wednesday, April 5, 2023

Name: ____________________________________

Section: 9:00 AM 12:00 PM 3:00 PM

NetID: ____________

Score: ____________

Problem	Pts.	Score
1	10	
2	3	
3	3	
4	5	
5	14	
6	8	
7	20	
8	10	
9	15	
10	12	
Total	100	

Instructions

- You may not use any books, calculators, or notes other than one handwritten two-sided sheet of 8.5" x 11" paper.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.

(10 Pts.)

1. Answer **True** or **False** to each of the following statements: *Grading:* Correct answer = 2 pt.; Incorrect answer = -1 pt. No answer = 0 pts.

   (a) By sampling a continuous-time signal $x_c(t) = \cos(\pi^3 t)$ with some sampling period T, it is possible to obtain a discrete time signal $x[n] = \cos(3\pi n/4)$. **True/False**

   (b) If the Nyquist sampling rate for a continuous-time signal $x_c(t)$ is F_s, then the Nyquist sampling rate for $y_c(t) = x_c(3t)$ is $F_s/3$. **True/False**

   (c) Suppose $x[n]$ is a finite-length signal with DTFT $X_d(\omega)$. We zero-pad $x[n]$ with some number of zeros to obtain $y[n]$ with DTFT $Y_d(\omega)$. It follows then that $X_d(\omega) = Y_d(\omega)$. **True/False**

   (d) If $x[n]$ is the inverse DFT of $\{1, 2, 3, 4\}$, then $x[n]$ must be zero for $n < 0$ or $n > 3$. **True/False**

   (e) If $x_c(t)$ is a bandlimited continuous-time signal, then there must exist a finite $\Omega_{\max}$ such that $X_a(\Omega) = 0$ for $|\Omega| > \Omega_{\max}$, where $X_a(\Omega)$ is the CTFT of $x_c(t)$. **True/False**

(3 Pts.)

2. Let $X_d(\omega)$ be the DTFT of a *real-valued* sequence $x[n]$. We further assume that $X_d(\omega) = j\omega$ for $0 \le \omega \le \pi$. We then have (*select one answer only*):

   (a) $X_d(\omega) = \omega$ for $-\pi \le \omega \le 0$

   (b) $X_d(\omega) = -\omega$ for $-\pi \le \omega \le 0$

   (c) $\boxed{X_d(\omega) = j\omega \text{ for } -\pi \le \omega \le 0}$

   (d) $X_d(\omega) = -j\omega$ for $-\pi \le \omega \le 0$

   (e) None of the above

(3 Pts.)

3. Let $X_d(\omega)$ be the DTFT of an *arbitrary* sequence $x[n]$. We further assume that $X_d(\omega) = j\omega$ for $0 \le \omega \le \pi$. We then have (*select one answer only*):

   (a) $X_d(\omega) = \omega$ for $2\pi \le \omega \le 3\pi$

   (b) $X_d(\omega) = -\omega$ for $2\pi \le \omega \le 3\pi$

   (c) $X_d(\omega) = j\omega$ for $2\pi \le \omega \le 3\pi$

   (d) $X_d(\omega) = -j\omega$ for $2\pi \le \omega \le 3\pi$

   (e) $\boxed{\text{None of the above}}$

(5 Pts.)

4. Calculate the inverse DTFT, $x[n]$, of $X_d(\omega) = 5e^{j\pi\omega}\delta(\omega - \omega_o)$, where ω_0 is a constant.

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 5e^{j\pi\omega}\delta(\omega-\omega_0)e^{j\omega n}d\omega \\ &= \frac{5}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(\pi+n)}\delta(\omega-\omega_0)d\omega \\ &= \boxed{\frac{5}{2\pi}e^{j\omega_0(\pi+n)}} \end{aligned}$$

(14 Pts.)

5. We have an LSI system with with the following unit pulse response given by $h[n]$:

$$h[n] = \delta[n] + \delta[n-6].$$

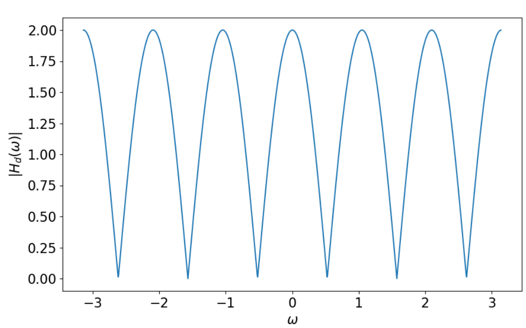
(a) Determine the frequency response $H_d(\omega)$ of this LTI system.

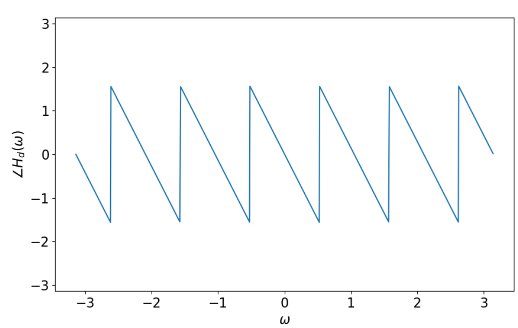
$$\begin{aligned} H_d(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= 1 + e^{-j6\omega} \\ &= e^{-j3\omega}(e^{j3\omega} + e^{j3\omega}) \\ &\boxed{= 2e^{-j3\omega}\cos(3\omega)}. \end{aligned}$$

(b) Plot the magnitude response $|H_d(\omega)|$ and phase response $\angle H_d(\omega)$ for $-\pi \leq \omega \leq \pi$. Make sure to carefully label your axes.
From the frequency response above, we may find the magnitude and phase responses:

$$\begin{aligned} |H_d(\omega)| &= 2|\cos(3\omega)| \\ \angle H_d(\omega) &= -3\omega + \angle\cos(3\omega). \end{aligned}$$

Plotting these responses gives the below plots:





(8 Pts.)

6. The frequency response of an LSI system is

$$H_d(\omega) = \omega e^{j\pi\cos\omega}, \qquad |\omega| \leq \pi\ .$$

Compute the response of this system $y[n]$ to the following input signal:

$$x[n] = 3 + e^{j\frac{\pi}{3}n} + \sin\left(\frac{\pi}{2}n + \frac{\pi}{4}\right).$$

$$\begin{aligned} H_d(0) &= 0 \\ H_d\left(\frac{\pi}{3}\right) &= \frac{\pi}{3}e^{j\frac{\pi}{2}} = j\frac{\pi}{3} \\ H_d\left(\frac{\pi}{2}\right) &= \frac{\pi}{2} \\ H_d\left(-\frac{\pi}{2}\right) &= -\frac{\pi}{2}. \end{aligned}$$

We may rewrite $x[n]$ as

$$x[n] = 3 + e^{j\frac{\pi}{3}n} + \frac{e^{j(\frac{\pi}{2}n+\frac{\pi}{4})} - e^{-j(\frac{\pi}{2}n+\frac{\pi}{4})}}{2j}$$

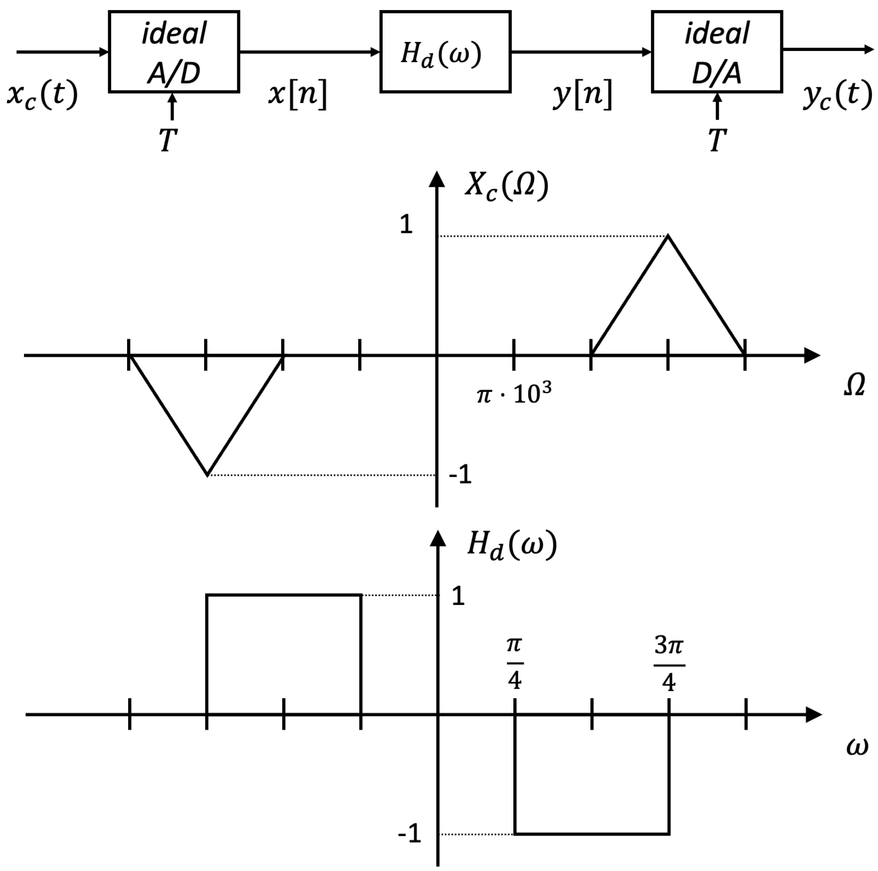
Using the above values of the frequency response,

$$y[n] = (3)(0) + \left(e^{j\frac{\pi}{3}n}\right)\left(j\frac{\pi}{3}\right) + \left(\frac{e^{j(\frac{\pi}{2}n+\frac{\pi}{4})} + e^{-j(\frac{\pi}{2}n+\frac{\pi}{4})}}{2j}\right)\left(\frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi}{3}e^{j(\frac{\pi}{3}n+\frac{\pi}{2})} - j\frac{\pi}{2}\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$$

(20 Pts.)

7. Consider the process of a bandlimited continuous-time signal $x_c(t)$ via the system shown below.



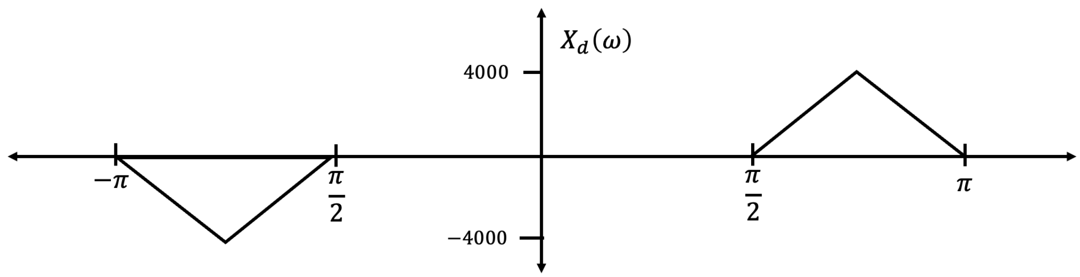
(a) What is the Nyquist rate for the signal $x_c(t)$ (i.e., no aliasing at the A/D converter)? Answer the rate in Hz.

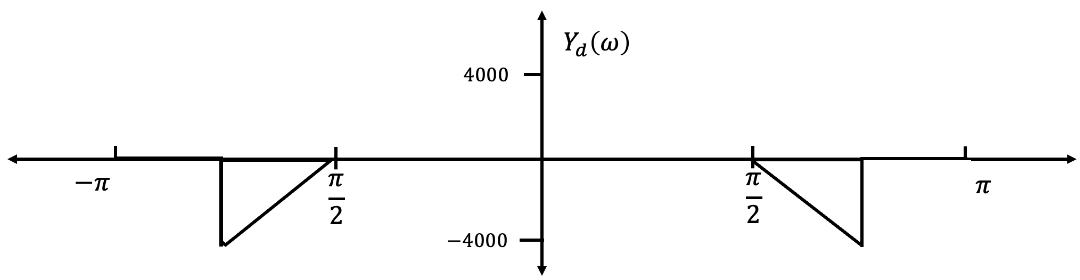
$$\text{Max radial frequency} = \Omega_{\max} = 4\pi \times 10^3 \text{ rad/s}$$

$$\text{Max linear frequency} = f_{\max} = \frac{4\pi \times 10^3}{2\pi} = 2kHz$$

$$\boxed{f_{\text{Nyquist}} > 2 \cdot 2kHz = 4kHz.}$$

(b) Sketch the DTFT $X_d(\omega)$ and $Y_d(\omega)$ at $T = \frac{1}{4} \cdot 10^{-3}$ for $-\pi \leq \omega \leq \pi$. Please carefully label your axes.





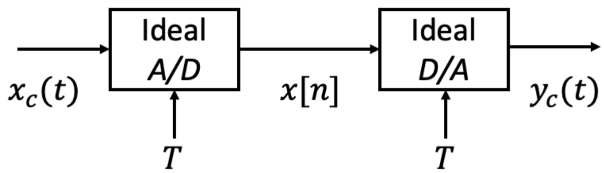
(c) Sketch the DTFT $X_d(\omega)$ and $Y_d(\omega)$ at $T = \frac{1}{3} \cdot 10^{-3}$ for $-\pi \leq \omega \leq \pi$. Please carefully label your axes.

We are sampling below the Nyquist rate in this part, thus there will be aliasing. The spectral copies overlap to alias after A/D conversion such that they cancel out and $x[n] = 0$. Thus both $X_d(\omega)$ and $Y_d(\omega)$ are zero for all ω.

(10 Pts.)

8. Consider the sampling and reconstruction system shown below. The input to the ideal A/D is

$$x_c(t) = 2\cos\left(100\pi t + \frac{\pi}{4}\right) + \cos\left(300\pi t\right)$$



(a) If the output of A/D is

$$x[n] = 2\cos\left(\frac{1}{4}\pi n + \frac{\pi}{4}\right) + \cos\left(\frac{3}{4}\pi n\right)$$

determine two choices for T consistent with the information.

Using $\omega = \frac{\pi}{4}$, for example,

$$\omega = \Omega T$$
$$\frac{\pi}{4} = 100\pi T$$
$$\boxed{T_1 = \frac{1}{400}\ \mathrm{s}}.$$

We know by the 2π periodicity of the DTFT that frequency $\frac{\pi}{4} \equiv \frac{\pi}{4} + 2\pi k$ for any integer k. Thus, we can find another T by using frequency $\omega = \frac{9\pi}{4}$:

$$\frac{9\pi}{4} = 100\pi T$$

$$\boxed{T_2 = \frac{9}{400}\ \mathrm{s.}}$$

In general,

$$T = \frac{1+8k}{400}\ \mathrm{s},\ \ k \geq 0$$

(b) If the output of the ideal D/A is

$$y_c(t) = 2\cos\left(100\pi t + \frac{\pi}{4}\right) + 1$$

determine the value of T.

We need a value of T that only aliases the 300π analog frequency to zero. Thus, we want $\omega = 300\pi T = 2\pi k$ for some multiple of 2π. Using $k = 1$:

$$2\pi = 300\pi T$$
$$T = \frac{1}{150}\ \mathrm{s.}$$

(15 Pts.)

9. For all parts of this question, let $\{x[n]\}_{n=0}^{5} = \{1, 2, -3, -4, 5, 6\}$ be a length-6 signal with DFT $\{X[k]\}_{k=0}^{5}$ given below:

$$X[k] = \{X_0, X_1, X_2, X_3, X_4, X_5\}.$$

(a) Compute X_0 and X_3.

$$\boxed{X[0] = X_0 = \sum_{n=0}^{N-1} x[n]e^{-j0} = \sum_{n=0}^{5} x[n] = 7}$$

$$\boxed{X[3] = X_3 = \sum_{n=0}^{N-1} x[n]e^{-j\pi n} = \sum_{n=0}^{5} x[n](-1)^n = -1}$$

(b) The DFT of another length-6 signal $\{y[n]\}_{n=0}^{5}$ is given by $\{Y[k]\}_{k=0}^{5}$:

$$Y[k] = \{X_4, -X_5, X_0, -X_1, X_2, -X_3\}.$$

Determine the signal $y[n]$.

$$Y[k] = e^{j\pi k} X[\langle k-2\rangle_6].$$

Applying the circular time-shift and circular frequency-shift properties:

$$\boxed{y[n] = e^{j\frac{2\pi}{3}n} x[\langle n-3\rangle_6].}$$

(c) Suppose we zero-pad $x[n]$ with 24 zeros to obtain $\{z[n]\}_{n=0}^{29}$ with corresponding DFT $\{Z[k]\}_{k=0}^{29}$. Which of the following relations is true? (Circle all that are true. More than one may be true.)

i. $\boxed{X[0] = Z[0]}$

ii. $X[1] = Z[4]$

iii. $\boxed{X[2] = Z[10]}$

iv. $\boxed{X[1] = Z^*[25]}$

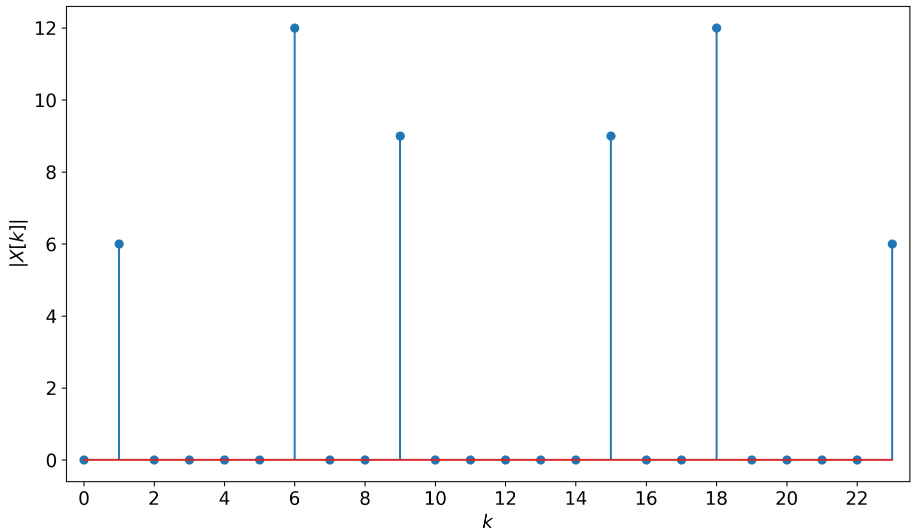
v. $X[2] = Z^*[22]$

(12 Pts.)

10. An analog signal $x_a(t)$ is known to be composed of M spectral components such that

$$x_a(t) = \sum_{m=1}^{M} A_m \cos(\Omega_m t).$$

The analog signal is sampled with sampling period $T = \frac{1}{12{,}000}$ s to create digital signal $\{x[n]\}_{n=0}^{23} = \{x_a(nT)\}_{n=0}^{23}$ with DFT $X[k]$. The DFT magnitude spectrum $|X[k]|$ is given in the below image. You may assume that $x_a(t)$ is sampled above the Nyquist rate and thus no aliasing occurs.



(a) Determine the number of spectral components M.

$$M = 3$$

(b) Determine the amplitudes of each spectral component $\{A_m\}_{m=1}^{M}$.

In general, each peak in the DFT has height $\frac{AN}{2}$. The three distinct spectral components have heights 6, 12, 9, respectively. Thus,

$$\frac{A_1 \cdot 24}{2} = 6 \implies A_1 = \frac{1}{2}$$
$$\frac{A_2 \cdot 24}{2} = 12 \implies A_1 = 1$$
$$\frac{A_2 \cdot 24}{2} = 9 \implies A_1 = \frac{3}{4}$$

(c) Determine the analog radial frequencies of each spectral component $\{\Omega_m\}_{m=1}^{M}$. (Please use consistent subscripts between parts (b) and (c))

The three spectral components occur at $k = 1, 6, 9$, respectively. The digital frequencies are given by $\omega_k = \frac{2\pi k}{N}$. Thus, using $\omega = \Omega T$,

$$\frac{2\pi \cdot 1}{24} = \frac{\pi}{12} \implies \Omega_1 = 1000\pi \text{ rad/s}$$
$$\frac{2\pi \cdot 6}{24} = \frac{\pi}{2} \implies \Omega_1 = 6000\pi \text{ rad/s}$$
$$\frac{2\pi \cdot 9}{24} = \frac{3\pi}{4} \implies \Omega_1 = 9000\pi \text{ rad/s}$$