University of Illinois
ECE 310 **Profs. Liang, Moon, Snyder**

Midterm Exam

7:00-9:00pm, Wednesday, March 1, 2023

Name: ______________________________

Section: 9:00 AM 12:00 PM 3:00 PM

NetID: __________

Score: __________

Problem	Pts.	Score
1	10	
2	12	
3	15	
4	6	
5	20	
6	20	
7	9	
8	4	
9	4	
Total	100	

Instructions

- You may not use any books, calculators, or notes other than one <u>handwritten</u> two-sided sheet of 8.5" x 11" paper.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.

(10 Pts.)

1. Answer **True** or **False** to each of the following statements: *Grading:* Correct answer = 2 pt.; Incorrect answer = -1 pt. No answer = 0 pts.

   (a) If the system response $y[n]$ of a discrete-time system to any possible input signal $x[n]$ is fully described by its unit pulse response, then the system must be LTI. **<u>True</u>/False**

   (b) If a system is BIBO stable, any unbounded input will produce an unbounded output. **True/<u>False</u>**

   (c) If a system has a right-sided unit pulse response, then it must be causal. **True/<u>False</u>**

   (d) The DTFT, $X_d(\omega)$, of a sequence $x[n]$ is always related to its z-transform $X(z)$ by $X_d(\omega) = X(z)|_{z=e^{j\omega}}$. **True/<u>False</u>**

   (e) If $x[n]$ is a real-valued sequence, then $|X_d(\omega)|$ is an even function. **<u>True</u>/False**

(12 Pts.)

2. For each of the systems with input $x[n]$ and output $y[n]$ shown in the table, indicate by "**yes**" or "**no**" whether the properties indicated apply to the system. Note: you do not need to provide proofs/justification.

	Linear	Shift-Invariant	Causal	Stable
$y[n] = x[n] * (-1)^n u[n]$	**Yes**	**Yes**	**Yes**	**No**
$y[n] = \frac{x[n]}{x[2]}$	**No**	**No**	**No**	**No**
$y[n] = \cos^2\left(\frac{\pi}{2}n\right) x[n]$	**Yes**	**No**	**Yes**	**Yes**

(15 Pts.)

3. For each of the following parts, compute the convolution $x[n] * h[n]$ between the given sequences.

(a) $x[n] = \{\underset{\uparrow}{1}, -2, 2\}$, $h[n] = \{3, 1, \underset{\uparrow}{0}, 3, 1, 1\}$

$$
\begin{aligned}
y[n] &= \mathbf{H}x \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \\
&\boxed{= \{3, -5, \underset{\uparrow}{4}, 5, -5, 5, 0, 2\}}
\end{aligned}
$$

(b) $x[n] = \left(-\frac{1}{2}\right)^n u[n]$, $h[n] = \left(\frac{2}{3}\right)^n u[n-1]$

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} \left(-\frac{1}{2}\right)^k u[k] \left(\frac{2}{3}\right)^{n-k} u[n-k-1] \\
&= \left[\left(\frac{2}{3}\right)^n u[n-1]\right] \sum_{k=0}^{n-1} \left(-\frac{3}{4}\right)^k \\
&= \left(\frac{2}{3}\right)^n u[n-1] \left(\frac{1-\left(-\frac{3}{4}\right)^n}{1-\left(-\frac{3}{4}\right)}\right) \\
&\boxed{= \frac{4}{7}\left[\left(\frac{2}{3}\right)^n - \left(-\frac{1}{2}\right)^n\right] u[n-1]}
\end{aligned}
$$

(c) $x[n] = \log(|n|+1)$, $h[n] = u[n+1] - u[n-2]$

$$
\begin{aligned}
h[n] &= \delta[n+1] + \delta[n] + \delta[n-1] \\
x[n] * h[n] &= x[n+1] + x[n] + x[n-1] \\
&\boxed{= \log(|n+1|+1) + \log(|n|+1) + \log(|n-1|+1)}
\end{aligned}
$$

(6 Pts.)

4. An LTI system with unit pulse response $h[n]$ has the following input-output relationships:

$$\begin{aligned} x_1[n] * h[n] &= y_1[n] \\ x_2[n] * h[n] &= y_2[n], \end{aligned}$$

where

$$x_1[n] = \{\underset{\uparrow}{1}, 3, 6, -3\}$$

$$x_2[n] = \{\underset{\uparrow}{1}, 2, -1, 0\}.$$

Determine $h[n]$ in terms of $y_1[n]$ and $y_2[n]$.

$$x_1[n] = \{\underset{\uparrow}{1}, 3, 6, -3\}$$

$$3x_2[n-1] = \{\underset{\uparrow}{0}, 3, 6, -3\}$$

$$x_1[n] - 3x_2[n-1] = \delta[n]$$

$$\begin{aligned} h[n] &= h[n] * \delta[n] \\ &= h[n] * (x_1[n] - 3x_2[n-1]) \\ &\boxed{= y_1[n] - 3y_2[n-1]} \end{aligned}$$

(20 Pts.)

5. Consider a causal LTI system described by the following LCCDE:

$$y[n] = y[n-1] + \frac{3}{4}y[n-2] + x[n] - 4x[n-2].$$

(a) Determine the transfer function $H(z)$ and state the poles, zeros, and the ROC of this system.

$$\begin{aligned} Y(z)(1 - z^{-1} - \frac{3}{4}z^{-2}) &= X(z)(1 - 4z^{-2}) \\ \frac{Y(z)}{X(z)} = H(z) &= \frac{1 - 4z^{-2}}{1 - z^{-1} - \frac{3}{4}z^{-2}} \\ &= \frac{1 - 4z^{-2}}{(1 - \frac{3}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} \end{aligned}$$

Poles at $z = \frac{3}{2}, -\frac{1}{2}$, zero at $z \pm 2$, ROC=$|z| > \frac{3}{2}$.

(b) Calculate the system's response $y[n]$ to input $x[n] = 2\delta[n] - 3\delta[n-1]$.

$$\begin{aligned} X(z) &= 2 - 3z^{-1} \\ Y(z) &= X(z)H(z) \\ &= \frac{(2 - 3z^{-1})(1 - 4z^{-2})}{(1 - \frac{3}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} \\ &= \frac{2(1 - \frac{3}{2}z^{-1})(1 - 4z^{-2})}{(1 - \frac{3}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} \\ &= 2 \cdot \frac{1}{1 + \frac{1}{2}z^{-1}} - 8z^{-2} \cdot \frac{1}{1 + \frac{1}{2}z^{-1}} \end{aligned}$$

$$\boxed{y[n] = 2\left(-\frac{1}{2}\right)^n u[n] - 8\left(-\frac{1}{2}\right)^{n-2} u[n-2]}$$

(c) Is the system given by $H(z)$ BIBO stable? Justify your reasoning.

No, the system is not BIBO stable because the ROC does not contain the unit circle.

(20 Pts.)

6. Suppose that the input $x[n]$ to **a causal and stable LTI system** produces the output $y[n]$. The z-transform of $x[n]$ and $y[n]$ is given below:

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})}$$

$$Y(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})(1-\frac{1}{4}z^{-1})}$$

(a) Find the transfer function $H(z)$ and its ROC.

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})(1-\frac{1}{4}z^{-1})}}{\frac{1}{(1-2z^{-1})(1-z^{-1})}} \\ &\boxed{= \frac{1-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}, \quad |z| > \frac{1}{2}} \end{aligned}$$

(b) Find the unit pulse response $h[n]$.

$$\begin{aligned} \frac{1-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} &= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{4}z^{-1}} \\ 1-2z^{-1} &= A(1-\frac{1}{4}z^{-1}) + B(1-\frac{1}{2}z^{-1}) \\ A &= -6 \\ B &= 7 \end{aligned}$$

$$\boxed{h[n] = -6\left(\frac{1}{2}\right)^n u[n] + 7\left(\frac{1}{4}\right)^n u[n]}$$

(c) Determine the difference equation of the system.

$$\boxed{y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-1] + x[n] - 2x[n-1]}$$

(9 Pts.)

7. The transfer function of a causal LTI system is given below:

$$H(z) = \frac{z-3}{z-4}, \quad \text{ROC: } |z| > 4$$

   (a) Find a **bounded** input $x[n]$ that will produce an **unbounded** output $y[n]$.

$$\begin{aligned} x[n] &= \delta[n] \\ y[n] &= 4^n u[n] - 3(4)^{n-1} u[n-1] \end{aligned}$$

   (b) Find an **unbounded** input $x[n]$ that will produce a **bounded** output $y[n]$.

$$\begin{aligned} X(z) &= \frac{z-4}{z-3} \\ x[n] &= 3^n u[n] - 4(3)^{n-1} u[n-1] \\ y[n] &= \delta[n] \end{aligned}$$

   (c) Find a **bounded** input $x[n]$ that will produce a **bounded** output $y[n]$.

$$\begin{aligned} X(z) &= 1 - 4z^{-1} \\ x[n] &= \delta[n] - 4\delta[n-1] \\ y[n] &= \delta[n] - 3\delta[n-1] \end{aligned}$$

(4 Pts.)

8. Determine the signal $x[n]$ whose DTFT is $X_d(\omega) = 1 + 2\cos(2\omega) - 2j\sin(4\omega)$. (Circle one of the following) **Note**: The arrow indicates $n = 0$.

   (a) $x[n] = \{0.5\pi, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, -0.5\pi\}$

   (b) $x[n] = \{0.5, 0, 0, 0, \underset{\uparrow}{1}, -3j, 0, 0, -0.5\}$

   (c) $x[n] = \{-j, 0, 1, 0, \underset{\uparrow}{j}, 0, 1, 0, j\}$

   (d) $\boxed{x[n] = \{-1, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, 1\}}$

   (e) $x[n] = \{-1, 0, j, 0, \underset{\uparrow}{1}, 0, j, 0, 1\}$

   (f) None of the above

(4 Pts.)

9. Consider the sequence $\{x[n]\}_{n=-1}^{2} = \{1-j,\ 1,\ -1-j,\ 2j\}$. Determine the values of A, B, C, and D of the following calculations without explicitly evaluating $X_d(\omega)$ for every ω.

   (a) $X_d(0) = A + jB$.

   (b) $X_d(\frac{\pi}{2}) = C + jD$.

$$
\begin{aligned}
X_d(0) &= \sum_{n=-1}^{2} x[n] \\
&= (1-j) + (1) + (-1-j) + (2j) \\
&= 1
\end{aligned}
$$

$$\boxed{A = 1,\ B = 0}$$

$$
\begin{aligned}
X_d\left(\frac{\pi}{2}\right) &= \sum_{n=-1}^{2} x[n]e^{-j\frac{\pi}{2}n} \\
&= \sum_{n=-1}^{2} x[n](-j)^n \\
&= ((-j)^{-1})(1-j) + (1) + (-j)(-1-j) + (-1)(2j) \\
&= (j+1) + (1) + (j-1) + (-2j) \\
&= 1
\end{aligned}
$$

$$\boxed{C = 1,\ D = 0}$$