University of Illinois **Fall 2021**

ECE 310 **Profs. Zhao, Katselis, and Kamalabadi**

Midterm Exam 2 - Solution

8:30-10:00 P.M. Tuesday, Nov. 9, 2021.

Please do not start the exam until the starting time.

- No collaboration allowed: You are not allowed to share or collaborate on this exam and that all work should be your own.
- You can use your handwritten notes in paper, printouts of your tablet notes, and printouts of the instructor's slides or notes.
- You can also use your notes in your tablet but you can only scroll through them, you cannot search through them by typing or writing.
- Calculators and other electronic ways to do calculations, like Wolfram alpha, are not allowed. Neither is searching online.
- Please follow the rules for online examinations detailed on the course website.

GOOD LUCK!

1. **(15 Pts.)** Answer **True** or **False** to each of the following statements:

   (a) $x[n] = \cos(\omega_0 n)$ is an eigenfunction of a stable LTI system. **True/False**

   (b) Two different sequences of same length can have the same DFT. **True/False**

   (c) The ideal reconstruction filter is causal. **True/False**

   (d) Assume $x[n]$ is a finite-duration sequence of length 20, and $y[n]$ is obtained by zero-padding $x[n]$ to length 32. That is, $y[n] = x[n]$, for $n = 0, 1, \ldots, 19$, and $y[n] = 0$, $n = 20, 21, \ldots, 31$. Let $\{X[m]\}_{m=0}^{19}$ and $\{Y[m]\}_{m=0}^{31}$ be the DFT of $\{x[n]\}_{n=0}^{19}$ and $\{y[n]\}_{n=0}^{31}$, respectively, then $X[10] = Y[16]$. **True/False**

   (e) Increasing the sampling period shrinks the corresponding DTFT. **True/False**

Solution:

(a) **False**: cosine can not be an eigenfunction because if a filter only covers one of the impulses in the frequency domain, the output would not be the scaled version of the cosine in the time domain.

(b) **False**: DFT is a one-to-one thing; in other words, every different signal should correspond to a unique DFT.

(c) **False**: Ideal reconstruction filter is non-causal.

(d) **True.**

(e) **False:** The corresponding DTFT would be expanded.

2. **(15 Pts.)** A causal LTI system is described by the difference equation: $y[n] = y[n-2] + x[n] - x[n-1]$.

   (a) Determine the system's transfer function $H(z)$.

   (b) Determine the system's unit impulse response $h[n]$.

   (c) Determine the system's frequency response $H_d(\omega)$; is $H_d(\omega) = H(z)|_{z=e^{j\omega}}$? If not, explain why.

Solution:

(a)

$$\mathcal{Z}\{y[n] = y[n-2] + x[n] - x[n-1]\} \tag{1}$$

$$=> \quad Y(z) = z^{-2}Y(z) + X(z) - z^{-1}X(z) \tag{2}$$

$$=> \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - z^{-2}} = \frac{1}{1 + z^{-1}} \tag{3}$$

(b)

$$h[n] = \mathcal{Z}^{-1}\{\frac{1}{1 + z^{-1}}\} = (-1)^n u[n], \quad \text{where } |z| > |-1| \tag{4}$$

(c) Because the ROC of $H(z)$ does not include the unit circle, $H_d(w) \neq H(z)|_{z=e^{jw}}$.

$$H_d(w) = DTFT\{h[n]\} = DTFT\{e^{j\pi n}u[n]\}$$

$$H_d(w) = \frac{1}{1 - e^{-j(w-\pi)}} + \pi \sum_{k=-\infty}^{\infty} \delta(w - \pi - 2k\pi)$$

3. **(10 Pts.)** Let $X[k]$ be the 8 point DFT of $x[n] = \{\underset{\uparrow}{1}, 3, 2, 0, 1, 3, 2, 0\}$.

(a) Compute $X[k]$ for $k = 0, 2, 4, 6$.

Solution:

$$X[k] = \sum_{n=0}^{7} x[n]e^{-j\frac{2nk\pi}{8}} = (1 + e^{-jk\pi})(1 + 3e^{-j\frac{k\pi}{4}} + 2e^{-j\frac{k\pi}{2}})$$

$$= (1 + (-1)^k)(1 + 3e^{-j\frac{k\pi}{4}} + 2e^{-j\frac{k\pi}{2}})$$

$$X[0] = 12$$
$$X[2] = -2 - 6j$$
$$X[4] = 0$$
$$X[6] = -2 + 6j$$

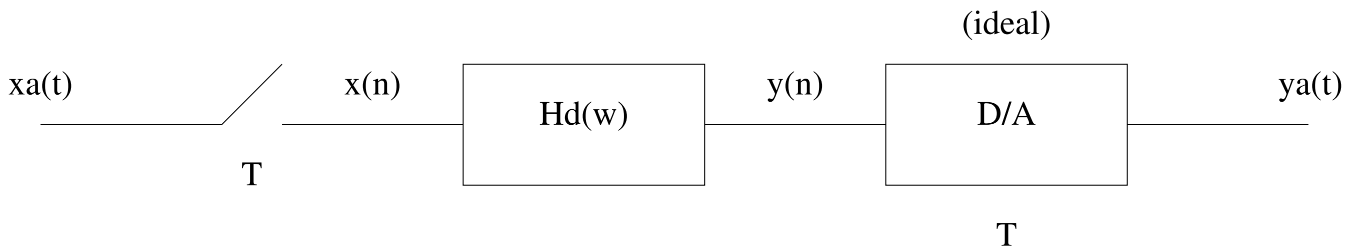
(b) Use properties of DFT to determine $Y[k]$ for $k = 0, 2, 4, 6$, where the corresponding sequence $y[n]$ is

$$y[n] = x[\langle 3 - n\rangle_8] \cdot (j)^n.$$

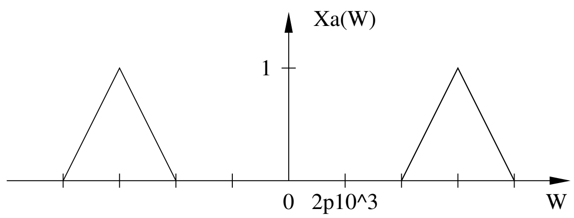
Solution:

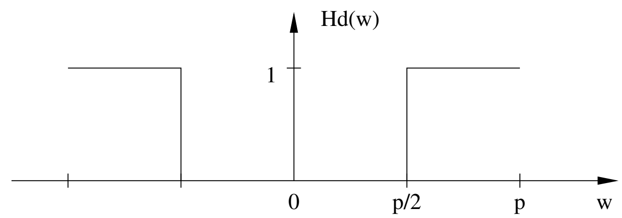
$$X[k] = \{12, 0, -2 - 6j, 0, 0, 0, -2 + 6j, 0\}$$
$$z[n] = x[\langle 3 - n\rangle_8] \leftrightarrow Z[k] = W_8^{3k}X[-k]$$
$$y[n] = z[n]W_8^{-2n} \leftrightarrow Y[k] = Z[\langle k - 2\rangle_8]$$
$$Y[k] = \{-6 + 2j, 0, 12, 0, -6 - 2j, 0, 0, 0\}$$

4. **(15 Pts.)** Consider the following digital system ($p = \pi$, $w = \omega$, and $W = \Omega$ in the figure)



with





Sketch $X_d(\omega)$, $Y_d(\omega)$, and $Y_a(\Omega)$ and clearly label the axes, for:

1) $T = \frac{1}{8 \times 10^3}$

2) $T = \frac{1}{4 \times 10^3}$

Solution: Recall the relation between angular frequency Ω and normalized frequency ω:

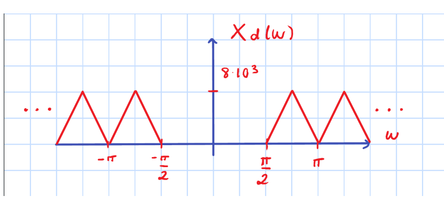
$$\omega = \Omega T,$$

where T is the sampling period. Then, for each case, we have the following frequency values:

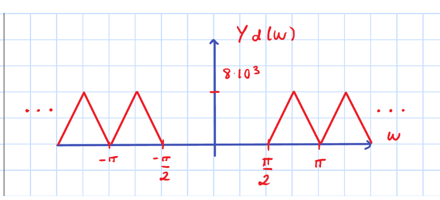
1) $\Omega_0 = 4\pi \cdot 10^3 \ \frac{\text{rad}}{\text{s}} \Rightarrow \omega_0 = \frac{\pi}{2} \frac{\text{rad}}{\text{sample}}$, $\Omega_1 = 8\pi \cdot 10^3 \ \frac{\text{rad}}{\text{s}} \Rightarrow \omega_1 = \pi \frac{\text{rad}}{\text{sample}}$

2) $\Omega_0 = 4\pi \cdot 10^3 \ \frac{\text{rad}}{\text{s}} \Rightarrow \omega_0 = \pi \frac{\text{rad}}{\text{sample}}$, $\Omega_1 = 8\pi \cdot 10^3 \ \frac{\text{rad}}{\text{s}} \Rightarrow \omega_1 = 2\pi \frac{\text{rad}}{\text{sample}}$

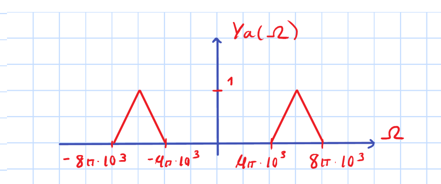
Figure 1 shows the frequency spectrum for each sampling period.



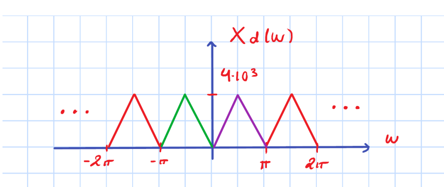
(a) 1a



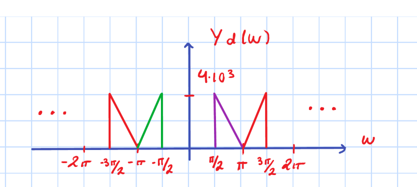
(b) 1b



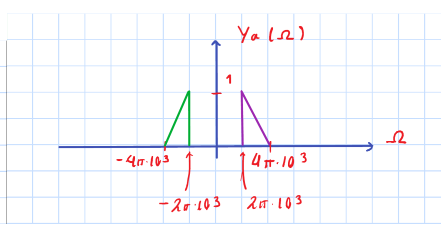
(c) 1c



(d) 2a



(e) 2b



(f) 2c

Figure 1: Question 4. Frequency spectrum for each sampling period T.

5. **(15 Pts.)** Consider the filtering system in Problem 4. Suppose that $x_a(t)$ is bandlimited to 4000 Hz. The system produces an output $y_a(t)$ such that $Y_a(\Omega) = H_a(\Omega)X_a(\Omega)$, where

$$H_a(\Omega) = \begin{cases} 1 - \frac{|\Omega|}{4000\pi}, & |\Omega| \leq 4000\pi \\ 0, & |\Omega| > 4000\pi \end{cases}$$

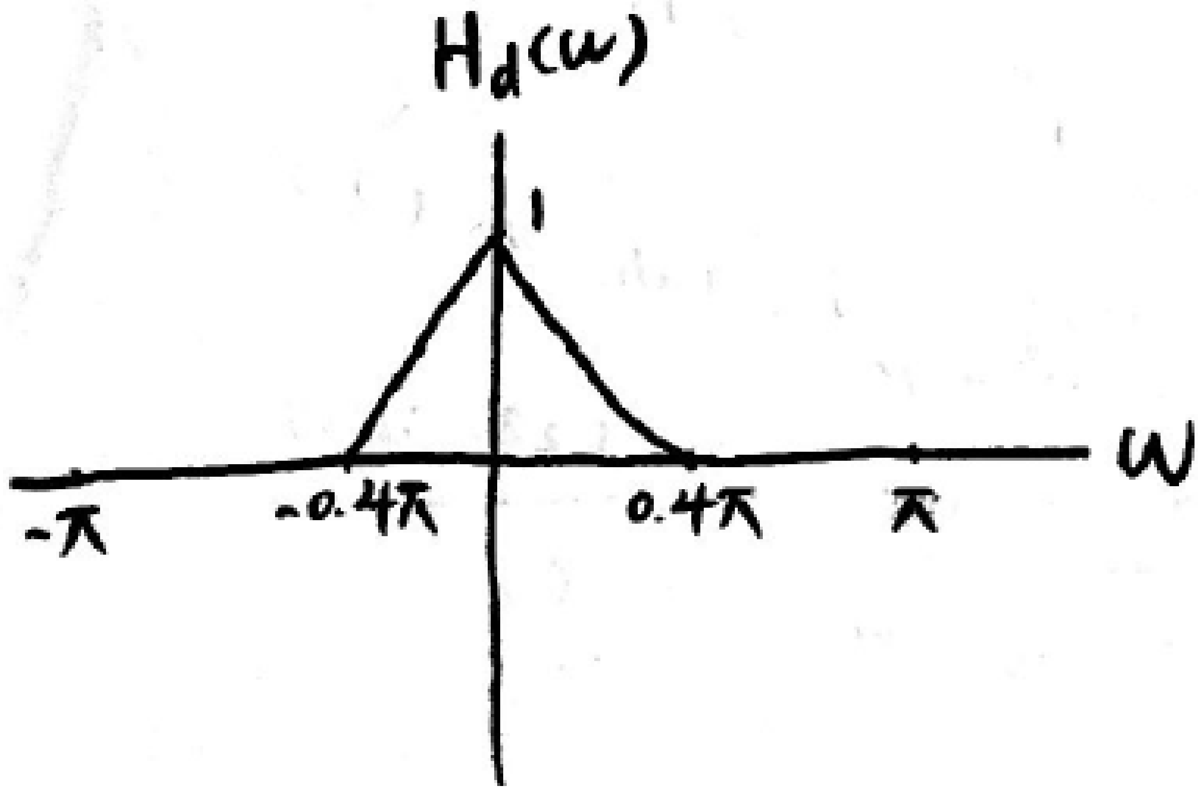
(a) Determine the largest sampling period such that no aliasing occurs at the output of the sampler (ideal A/D converter).

Solution: According to the Nyquist Theorem, $F_s \geq 2F_{max} = 8000Hz$. Hence, $T_s \leq \frac{1}{8000} =$ 0.125 msec. The largest sampling period is 0.125 msec.

(b) Determine and plot $H_d(\omega)$ for $T = 10^{-4}$ sec.

Solution: $T = 0.1$ msec, which is less than 0.125 msec. Hence, no aliasing happens at the A/D converter. Let's also assume that we have an ideal D/A converter at the output end. The scaling effects from the A/D and D/A converters will cancel out with each other. Therefore, $H_d(\omega)$ just needs to reproduce the scaling effects of $H_a(\Omega)$. $H_a(0) = 1$ and $\Omega = 0$ in the CTFT domain corresponds to $\omega = 0$ in the DTFT domain. As a result, $H_d(0) = 0$. When $|\Omega| = 4000\pi$, $H_a(\Omega) = 0$. $|\Omega| = 4000\pi$ in the CTFT domain corresponds to $|\omega| = 4000\pi \times 10^{-4} = 0.4\pi$. As a result, $H_d(0.4\pi) = H_d(-0.4\pi) = 0$. $H_a(\Omega)$ decreases linearly as Ω goes from 0 to 4000π or -4000π. Correspondingly, $H_d(\omega)$ decreases linearly as ω goes from 0 to 0.4π or -0.4π. Combining these infos together, we get:

$$H_d(\omega) = \begin{cases} 1 - \frac{|\omega|}{0.4\pi}, & |\omega| \leq 0.4\pi \\ 0, & |\omega| > 0.4\pi \end{cases}$$



6. **(15 Pts.)** Consider the two finite-length sequences:

$$x = \{\underset{\uparrow}{-2}, 3, 0, 3\} \text{ and } h = \{\underset{\uparrow}{-2}, 5, 1, 5, -3\}$$

(a) Let $Y_d(\omega) = X_d(\omega)H_d(\omega)$, where $X_d(\omega), H_d(\omega)$ denote the DTFT of $x[n], h[n]$. Compute $y[n]$.

Solution:

$$Y_d(w) = X_d(w)H_d(w) \quad \leftrightarrow \quad y[n] = h[n] * x[n]$$
$$y[n] = \{4, -16, 13, -13, 36, -6, 15, -9\}$$

(b) Let $Y[k] = X[k]H[k]$, where $X[k], H[k]$ denote the 6-point DFT of $x[n], h[n]$. Find $y[n]$.

Solution:

$$Y[k]_6 = X[k]_6 H[k]_6 \quad \leftrightarrow \quad y[n] = h[< n >_6] \circledast x[< n >_6]$$
$$y[n] = \{19, -25, 13, -13, 36, -6\}$$

(c) **True or False**: Zero padding both sequences to length $N = 7$ is adequate to guarantee that linear and circular convolutions coincide.

Solution: **False** Need to zero-pad to length 8.

7. **(15 Pts.)** A continuous-time signal $x_c(t) = \cos(10\pi t)$ is sampled at a rate of 100 Hz for 5 seconds to produce a discrete-time signal $x[n]$ with length $L = 500$.

(a) Let $X[k]$ be the L-point DFT of $x[n]$. At what value(s) of k will $X[k]$ have the greatest magnitude?

Solution: The ideal discrete-time sequence corresponds to:

$$x[n] = x_c(nT) = \cos(\frac{\pi n}{10}), \forall n \in \mathbb{Z}$$

Since the sequence is sampled for 5 seconds only, it is truncated to $n \in \{0, \ldots, 499\}$.
While the ideal cosine has a frequency response composed by pair of delta functions, the DTFT of its truncated version has the deltas replaced with sinc-like functions. Given the resemblance between the DTFT of the cosine and its truncated version, the greatest magnitudes for both DTFT correspond to the same normalized frequencies ω_{max}:

$$\omega_{\text{max}} = \pm \frac{\pi}{10} + 2\pi k,\ k \in \mathbb{Z}.$$

Taking into account the frequency range $\omega \in [0, 2\pi]$, the frequencies with the greatest magnitude correspond to $-\frac{\pi}{10} + 2\pi = \frac{19\pi}{10}$ and $\frac{\pi}{10}$. Then, using the relation between DFT and DTFT:

$$X[k] \triangleq X_d\left(\frac{2\pi k}{500}\right),\ k \in \{0, \ldots, 499\}$$

$\omega = \frac{19\pi}{10}$ and $\omega = \frac{\pi}{10}$ are mapped to $k = 475$ and $k = 25$, respectively.

(b) Suppose that $x[n]$ is zero-padded to a total length of $N = 1024$. At what value(s) of k does the N-point DFT have the greatest magnitude?

Solution: By taking $N = 1024$ frequency samples and considering the uniform sampling of the DFT ($\omega_k = \frac{2\pi k}{N}$), the frequency positions in which the maximum values appear ($\frac{19\pi}{10}$ and $\frac{\pi}{10}$) do not correspond to particular samples k. In other words, there are no integer values k that correspond to $\frac{19\pi}{10}$ and $\frac{\pi}{10}$.
Thus, we approximate the frequencies with the highest frequency response values by rounding them to the nearest sampled frequencies ω_k:

$$\frac{19\pi}{10} = \frac{2\pi k}{1024} \Rightarrow\ k = 972.8 \approx 973$$
$$\frac{\pi}{10} = \frac{2\pi k}{1024} \Rightarrow\ k = 51.2 \approx 51$$