University of Illinois
ECE 310

Profs. Do, Snyder, Moustakides

Midterm Exam 2

7:00-9:00pm, Wednesday, November 1, 2023

Name: ________________________________

Section: 9:00 AM 12:00 PM 3:00 PM

NetID: ____________

Score: ____________

Problem	Pts.	Score
1	16	
2	6	
3	6	
4	18	
5	20	
6	16	
7	18	
Total	100	

Instructions

- You may not use any books, calculators, or notes other than **two** handwritten two-sided sheets of 8.5" x 11" paper.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.

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(16 Pts.)

1. Select **True** or **False** to each of the following statements:

   (a) A discrete-time signal $x[n]$ with z-transform $X(z)$ always has a finite DTFT $X_d(\omega)$ given by $X(z)$ evaluated at $z = e^{j\omega}$, i.e. evaluate $X(z)$ along the unit-circle. **True**/**<u>False</u>**

   (b) Let $x[n]$ be a discrete-time signal with DTFT $X_d(\omega) = e^{-j\frac{\omega}{2}}$, $-\pi \leq \omega \leq \pi$. The corresponding signal $x[n] = \delta\left[n - \frac{1}{2}\right]$. **True**/**<u>False</u>**

   (c) The DTFT of a real-valued signal is always Hermitian symmetric. **<u>True</u>**/**False**

   (d) Every continuous-time signal has a minimum sampling frequency to avoid aliasing during analog-to-digital conversion. **True**/**<u>False</u>**

   (e) A discrete-time signal is given by $x[n] = 4\delta[n-2]$. We pass $x[n]$ to an ideal D/A converter with sampling period T to recover $x_a(t)$. The recovered continuous-time signal is given by $x_a(t) = 4\delta(n - 2T)$. **True**/**<u>False</u>**

   (f) Let $\{x[n]\}_{n=0}^{N-1}$ be a length-N signal with DTFT $X_d(\omega)$ and $\{y[n]\}_{n=0}^{M+N-1}$ be $x[n]$ zero-padded with M zeros. The DTFT of $y[n]$ given by $Y_d(\omega)$ is equal to $X_d(\omega)$. **<u>True</u>**/**False**

   (g) The DFT of $\{x[n]\}_{n=0}^{23} = \cos\left(\frac{\pi}{8}n\right)$ will only have two non-zero values. **True**/**<u>False</u>**

   (h) The DTFT and DFT of a length-14 sequence $x[n]$ are given by $X_d(\omega)$ and $X[k]$, respectively. We know then that $X[11] = X_d\left(-\frac{3\pi}{7}\right)$. **<u>True</u>**/**False**

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(6 Pts.)

2. Suppose that $x[n] = u[n+2] - u[n-3]$, where $u[n]$ is the step sequence. Calculate $X_d(0)$ and $X_d(\pi)$.

   We may write $x[n] = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$.

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X_d(0) = \sum_{n=-\infty}^{\infty} x[n], \quad X_d(\pi) = \sum_{n=-\infty}^{\infty} x[n](-1)^n$$

$$\boxed{X_d(0) = 5, \quad X_d(\pi) = 1}$$

(6 Pts.)

3. Suppose we sample $x_a(t) = \sin(\Omega_0 t)$ at sampling frequency $f_s = 400$ Hz to obtain $x[n] = \sin\left(\frac{\pi}{2}n\right)$. Which of the following are possible values of Ω_0? (Circle all that apply)

   (a) $\boxed{-600\pi}$

   (b) -400π

   (c) $\boxed{200\pi}$

   (d) 400π

   (e) 800π

   (f) 1200π

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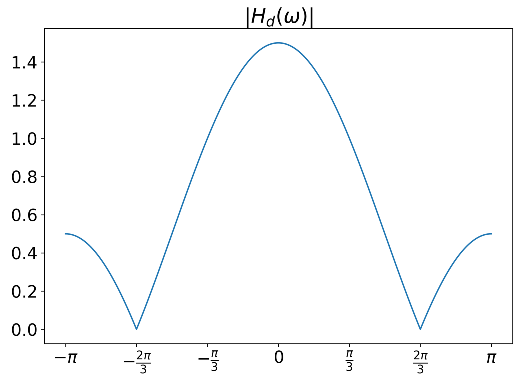
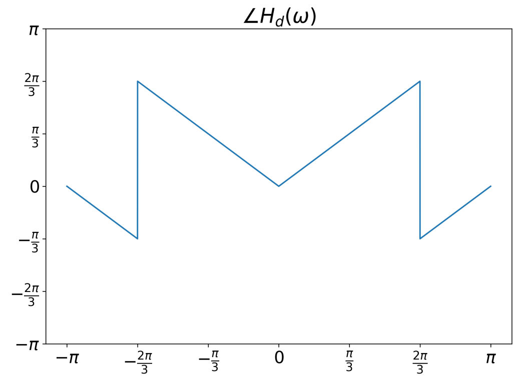
(18 Pts.)

4. Consider an LTI system with frequency response $H_d(\omega)$ given by

$$H_d(\omega) = \left(\frac{1}{2} + \cos(\omega)\right) e^{j|\omega|}, \quad -\pi \leq \omega \leq \pi.$$

(a) Plot the magnitude response $|H_d(\omega)|$ and phase response $\angle H_d(\omega)$. Label your axes carefully.





(b) Compute the output of this system to input signal $x[n] = 4j - 2e^{j\left(\frac{2\pi}{3}n + \frac{\pi}{4}\right)} + 3\sin\left(\frac{3\pi}{4}n\right)$.

The phase response of this system is not odd-symmetric, thus we can infer that the corresponding impulse response $h[n]$ is not real-valued. Therefore, we must decompose the sin term into each of its complex exponentials. Evaluating the frequency response at each given frequency:

$$H_d(0) = \frac{3}{2}, \ H_d\left(\frac{2\pi}{3}\right) = 0, \ H_d\left(\frac{3\pi}{4}\right) = H_d\left(-\frac{3\pi}{4}\right) = \frac{\sqrt{2}-1}{2} e^{-j\frac{\pi}{4}}$$

Following the eigenfunction property of LTI systems, we obtain the answer:

$$\boxed{y[n] = x[n] * h[n] = 6j + \frac{3(\sqrt{2}-1)}{4j} e^{j\left(\frac{3\pi}{4}n - \frac{\pi}{4}\right)} - \frac{3(\sqrt{2}-1)}{4j} e^{-j\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)}}$$

This answer may also be simplified as

$$\boxed{y[n] = 6j + \frac{3(\sqrt{2}-1)}{4} e^{j\left(\frac{3\pi}{4}n - \frac{3\pi}{4}\right)} - \frac{3(\sqrt{2}-1)}{4} e^{-j\left(\frac{3\pi}{4}n + \frac{3\pi}{4}\right)}}$$

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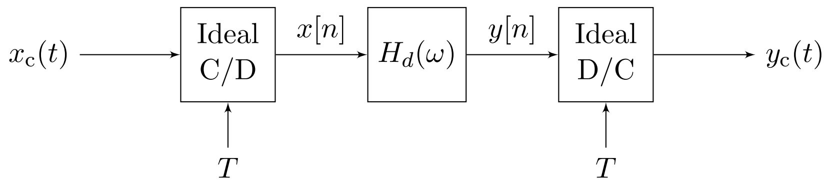
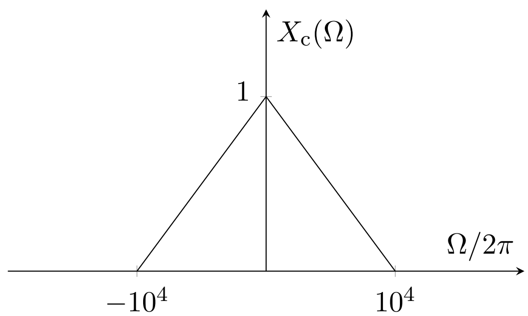
(20 Pts.)

5. Consider the following signal processing chain with an ideal continuous-to-discrete (C/D) or analog-to-digital converter (ADC), followed by a discrete-time filter $H_d(\omega)$, and then followed by an ideal discrete-to-continuous (D/C) or digital-to-analog converter (DAC). Both C/D and D/C have the same sampling interval T.



Most of human speech can be considered to be band-limited to 10 kHz. To be precise, consider a speech signal $x_c(t)$ that has the following continuous-time Fourier transform (CTFT):



(a) Suppose we want to **low-pass** filter $x_c(t)$ by passing only the frequencies up to 5 kHz in $x_c(t)$ into $y_c(t)$. Find the **largest** sampling period T and the **corresponding** $H_d(\omega)$ for which the above system can perform the desired low-pass filtering operation.

We need to ensure that no frequencies within the [-5 kHz, 5 kHz] experience aliasing since the low-pass digital filter $H_d(\omega)$ will be able to perfectly discard all frequencies outside this range. Thus, we may accept some aliasing in frequencies in the [5 kHz, 10 kHz] range. We must guarantee then that the digital frequency corresponding to 5 kHz, given by $\omega_0 = 2\pi \cdot 5{,}000 \cdot T$, is less than the left edge of the spectral copy in $X_d(\omega)$ centered at $\omega = 2\pi$. This left edge will lie at $2\pi - 2\pi \cdot 10{,}000 \cdot T$. Solving this inequality, we find

$$10{,}000\pi T < 2\pi - 20{,}000\pi T$$

$$\boxed{T < \frac{1}{15{,}000}\ \text{s} = 66.67\ \mu\text{s}}$$

The cutoff frequency of the desired low-pass filter $H_d(\omega)$ will be a function of the sampling period according to the $\omega = \Omega T$ relation. Thus, for $T = 1/15{,}000$ we have

$$\boxed{H_d(\omega) = \begin{cases} 1, & |\omega| \le \frac{2\pi}{3} \\ 0, & \frac{2\pi}{3} < |\omega| \le \pi \end{cases}}$$

(b) Now suppose we want to **high-pass** filter $x_c(t)$ by passing only the frequencies above 5 kHz in $x_c(t)$ into $y_c(t)$. Find the **largest** sampling period T and the **corresponding** $H_d(\omega)$ for which the above system can perform the desired high-pass filtering operation.

Aliasing will first occur at higher frequencies, i.e. starting from $\omega = \pi$ moving towards $\omega = 0$. Thus, if our high-pass filter is retaining the highest frequencies, we cannot allow any aliasing at all. Thus, we must sample at least as fast as the Nyquist rate. The maximum frequency is 10 kHz, thus

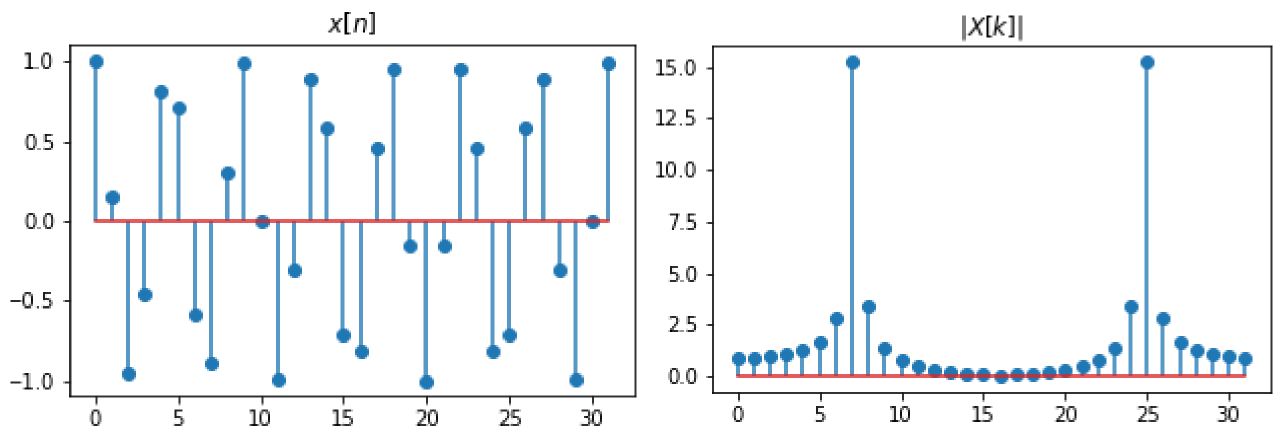
$$\boxed{T < \frac{1}{20{,}000} \text{ s} = 50\ \mu\text{s}}$$

For $T = \frac{1}{20{,}000}$s, we will have the following representation for the high-pass $H_d(\omega)$:

$$\boxed{H_d(\omega) = \begin{cases} 0, & |\omega| \leq \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}}$$

(16 Pts.)

6. Suppose that you are given a signal $\{x[n]\}_{n=0}^{31}$, where $x[n] = \cos(\lambda_0 n)$ for an unknown frequency λ_0. The signal and its DFT (of length 32) plots are as follows.



(a) From the DFT plot, estimate the unknown frequency λ_0.

First peak value is at $k = 7$. Given $\omega_k = \frac{2\pi k}{N}$,

$$\omega_7 = \frac{2\pi \cdot 7}{32}$$

$$\boxed{\lambda_0 \approx \frac{7\pi}{16}.}$$

(b) Find the smallest possible interval $[a, b]$ such that you would be sure that the unknown frequency λ_0 has to be: $a \leq \lambda_0 \leq b$.

The value $|X[8]|$ is slightly larger than $|X[6]|$. We can conclude then that the true peak in the DTFT of $x[n]$ must lie between $k = 7$ and $k = 8$. Furthermore, since $|X[7]| > |X[8]|$, we know that the ideal value of k is closer to 7 than 8. In other words, let k^* be the ideal (non-integer) value of k that will exactly correspond to λ_0:

$$k^* - 7 < 8 - k^*$$
$$k^* < \frac{15}{2}.$$

We cannot know how close k^* is to $k = 7$, thus we can at best say $7 \leq k \leq \frac{15}{2}$. Plugging into the expression for ω_k:

$$\boxed{a = \frac{7\pi}{16},\ b = \frac{15\pi}{32},\ \frac{14\pi}{32} \leq \lambda_0 \leq \frac{15\pi}{32}}$$

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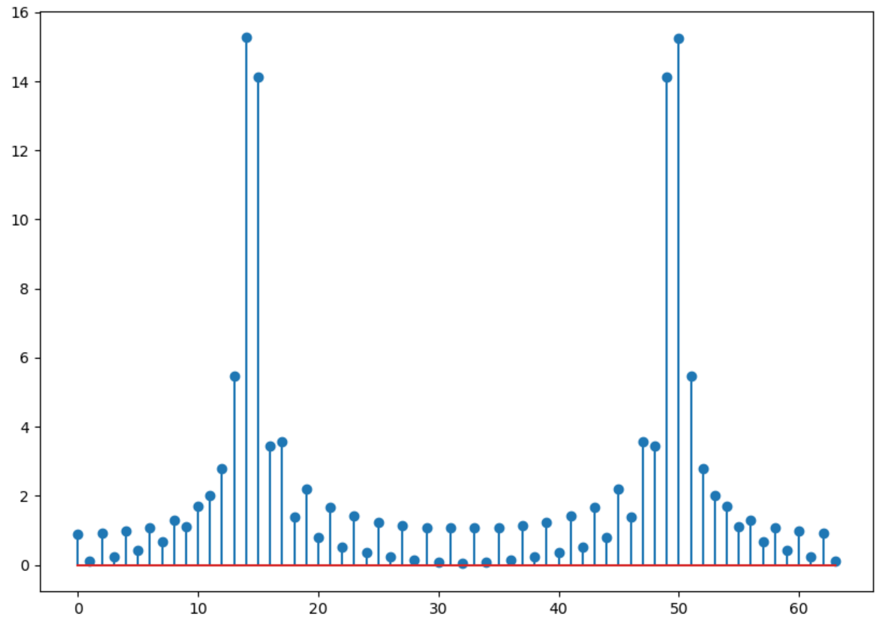
(c) Your friend applies some processing to the input signal and obtains the following DFT plot. What processing steps did she apply? Justify your answer.



The length of the DFT has been increased from 32 to 64. This may be accomplished by zero-padding 32 zeros to $x[n]$. Furthermore, the heights of the peak and side lobe values are unchanged, thus an additional window function was not applied. In summary, your friend only padded 32 zeros to $x[n]$.

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(18 Pts.)

7. The three parts of this problem are related.

(a) Let $\{b[n]\}_{n=0}^{N-1}$ be a length-N **complex-valued** sequence and let $\{B[k]\}_{k=0}^{N-1}$ be the corresponding DFT of $b[n]$. The conjugation property of the DFT states:

$$\{b^*[n]\}_{n=0}^{N-1} \overset{\text{DFT}}{\longleftrightarrow} \{B^*[N-k]\}_{k=0}^{N-1}.$$

Prove that the conjugation property holds. *Hint:* You may consider using the definition of the DFT, applying conjugation, and replacing the integer k to show that the property holds.

$$\begin{aligned}
\text{DFT}\{b^*[n]\} &= \sum_{n=0}^{N-1} b^*[n]e^{-j\frac{2\pi k}{N}n} \\
&= \left(\sum_{n=0}^{N-1} b[n]e^{j\frac{2\pi k}{N}n}\right)^* \\
&= \left(\sum_{n=0}^{N-1} b[n]e^{j\frac{2\pi k}{N}n}e^{-j\frac{2\pi N}{N}n}\right)^* \\
&= \left(\sum_{n=0}^{N-1} b[n]e^{-j\frac{2\pi(N-k)}{N}n}\right)^* \\
&= (B[N-k])^* \\
&= B^*[N-k]
\end{aligned}$$

(b) Specify the value $B^*[N]$ in terms of the original DFT $\{B[k]\}_{k=0}^{N-1}$.

By the N-periodicity of the DFT, we may say $\boxed{B^*[N] = B^*[0].}$

(c) We now will use the conjugation property to compute the DFTs of **two** real-valued signals from a **single** DFT of a complex-valued signal.

Let $\{x[n]\}_{n=0}^{N-1}$ and $\{y[n]\}_{n=0}^{N-1}$ be two **real-valued** sequences with correspondings DFTs $X[k]$ and $Y[k]$, respectively. Let $z[n] = x[n] + jy[n]$ with DFT $Z[k]$. Given that we can write

$$x[n] = \frac{1}{2}\big(z[n] + z^*[n]\big), \quad y[n] = \frac{1}{2j}\big(z[n] - z^*[n]\big),$$

use the linearity of the DFT and the property provided in part (a) to write each of the two DFTs $X[k]$ and $Y[k]$ in terms of the single DFT $Z[k]$.

$$\boxed{X[k] = \frac{1}{2}Z[k] + \frac{1}{2}Z^*[N-k]}$$

$$\boxed{Y[k] = \frac{1}{2j}Z[k] - \frac{1}{2j}Z^*[N-k]}$$