University of Illinois
ECE 310

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Final Exam

8:00-11:00am, Friday, May 5, 2023

Problem	Pts.	Score
1	10	
2	6	
3	4	
4	4	
5	5	
6	4	
7	5	
8	6	
9	12	
10	8	
11	6	
12	4	
13	4	
14	6	
15	9	
16	7	
Total	100	

Name: ______________________________

Section: 9:00 AM 12:00 PM 3:00 PM

NetID: __________

Score: __________

Instructions

- You may not use any books, calculators, or notes other than three handwritten two-sided sheets of 8.5" x 11" paper.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.

(10 Pts.)

1. Answer **True** or **False** to each of the following statements: *Grading:* Each part is worth 1 pt.

   (a) The FFT is an efficient algorithm for computing the DFT of a length-N discrete-time signal in $\mathcal{O}(N \log N)$ time. **<u>True</u>/False**

   (b) All FIR filters have generalized linear phase. **True/<u>False</u>**

   (c) If the ADC sampling period T is 2, the amplitude of DTFT will always be half of its CTFT. **True/<u>False</u>**

   (d) If a bandlimited signal is sampled above the Nyquist rate, the discrete-time signal from the ADC is free of aliasing. **<u>True</u>/False**

   (e) Consider a radix-2 decimation in time FFT with size $N = 128$. Then the FFT will have seven stages of computation. **<u>True</u>/False**

   (f) Direct Form II implementations require fewer delay units per output sample than Direct Form I implementations. **<u>True</u>/False**

   (g) The response $y[n]$ of a BIBO *unstable* LTI system to any non-zero input $x[n]$ is always unbounded. **True/<u>False</u>**

   (h) Downsampling a digital signal may cause aliasing, thus we apply an anti-aliasing filter after downsampling to fix any aliasing when implementing a decimator system. **True/<u>False</u>**

   (i) $\frac{\omega_c}{\pi}\text{sinc}(\omega_c n) * \frac{\omega_c}{\pi}\text{sinc}(\omega_c n) = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n)$ **<u>True</u>/False**

   (j) Circular convolution can be applied to two arbitrary periodic sequences. **True/<u>False</u>**

(6 Pts.)

2. For each of the systems shown in the table, indicate by "**yes**" or "**no**" whether the properties indicated apply to the system. (Each correct answer receives 0.5 pt; each wrong answer receives -0.5 pt. no negative points for the whole problem.)

	Linear	Time-invariant	Causal	BIBO stable
$y[n] = 5x[4n-3]$	**Yes**	**No**	**No**	**Yes**
$y[n] = \frac{1}{4}y[n-1] + x[n+1]$	**Yes**	**Yes**	**No**	**Yes**
$y[n] = x[n] * u[n-2]$	**Yes**	**Yes**	**Yes**	**No**

(4 Pts.)

3. For each of the filters shown in the table, indicate by "**yes**" or "**no**" whether the properties indicated apply to the filter. (In each case, the remaining terms of the unit pulse response $h[n]$ of the filter are zero.) (Each correct answer receives 1 pt; each wrong answer receives -1 pt. no negative points for the whole problem.)

$h[n]$	GLP Type-1	GLP Type-2
$\{h_n\}_{n=0}^{4} = \{2, 1, 6, 1, 2\}$	**Yes**	**No**
$\{h[n]\}_{n=0}^{21} = (-1)^n \frac{1}{6}\text{sinc}\frac{\pi}{6}\left(n - \frac{21}{2}\right)$	**No**	**No**

(4 Pts.)

4. Consider the following discrete-time system where α is a real-valued constant:

$$x[n] \longrightarrow \boxed{\quad e^{j\alpha\omega} \quad} \longrightarrow y[n]$$

   (a) Specify the condition on α under which the system is LTI.

   Any choice of α works.

   (b) Specify the condition on α under which the system is causal.

   $\alpha \leq 0$

(5 Pts.)

5. Let $y[n] = x[3n + 2]$. Determine $Y_d(\omega)$ in terms of $X_d(\omega)$, where $X_d(\omega)$ and $Y_d(\omega)$ are the DTFT of $x[n]$ and $y[n]$, respectively.

   We can first apply shift property to shift left by 2, then downsample by 3. Let $z[n] = x[n+2]$; thus, $y[n] = z[3n]$.

$$\begin{aligned} Z_d(\omega) &= X_d(\omega)e^{j2\omega} \\ Y_d(\omega) &= \frac{1}{3}\sum_{k=0}^{2} Z_d\left(\frac{\omega - 2\pi k}{3}\right) \\ &= \boxed{\frac{1}{3}\sum_{k=0}^{2} X_d\left(\frac{\omega - 2\pi k}{3}\right) e^{j2\left(\frac{\omega-2\pi k}{3}\right)}.} \end{aligned}$$

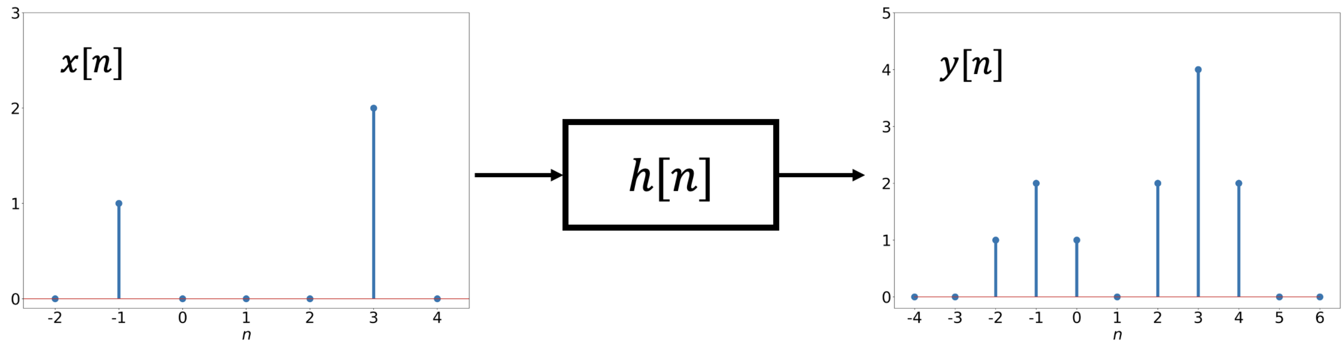
(4 Pts.)

6. Assume that the unit pulse response, $h[n]$, of an LTI system is bounded. Is the system described by $h[n]$ guaranteed to be BIBO stable? Justify your answer.

   No, the unit pulse response being bounded does not guarantee it is absolutely summable, which would guarantee stability. For example $h[n] = u[n]$ is a bounded unit pulse response, but it is not a BIBO stable system.

(5 Pts.)

7. Suppose we have an LTI system described by unit pulse response $h[n]$. For an input signal $x[n]$ shown below we obtain the corresponding system response $y[n]$.



   (a) Determine the unit pulse response $h[n]$ of the above system.

$$\boxed{h[n] = \{1, \underset{\uparrow}{2}, 1\}}$$

   (b) Is this system causal?

   No, this system is **non-causal**.

(6 Pts.)

8. Consider a stable and causal LTI system. The transfer function of the system is

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - \frac{1}{2}z^{-1}}$$

(a) Suppose the input is a unit step sequence, $u[n]$. Find the z-transform of the output, $Y(z)$ and its ROC.

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= z^{-1}\left(\frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \\ &= \boxed{\frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}, \text{ ROC } = |z| > \frac{1}{2}.} \end{aligned}$$

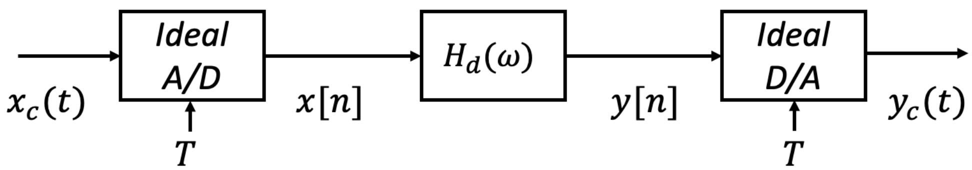
(b) Find the unit pulse response of the system.

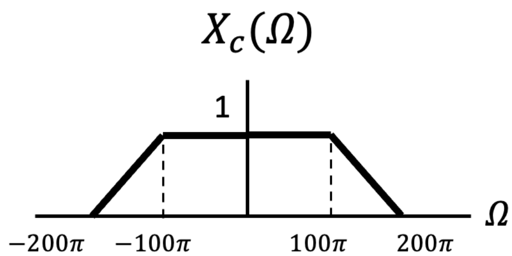
$$H(z) = \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}} - \frac{z^{-2}}{1 - \frac{1}{2}z^{-1}}$$

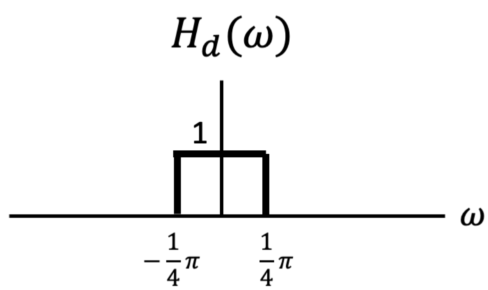
$$\boxed{h[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] - \left(\frac{1}{2}\right)^{n-2} u[n-2]}$$

(12 Pts.)

9. Consider the following system with the given input and digital filter:







(a) Find the maximum value of T allowed without aliasing errors in $x[n]$.

$$f_s > 2f_{\max} = 200 \text{ Hz}$$

$$\boxed{T < \frac{1}{200} \text{ s}}$$

(b) Find the maximum value of T allowed without aliasing errors in $y[n]$. Need to avoid aliasing for $\omega \in [-\pi/4, \pi/4]$. Thus, need

$$\frac{\pi}{4} < 2\pi - 200\pi T$$

$$\implies \boxed{T < \frac{7}{800} \text{ s}}$$

(c) Sketch $Y_d(\omega)$ and $Y_c(\Omega)$ for $T = \frac{5}{800}$. Please carefully label the frequencies and amplitudes.

(8 Pts.)

10. Let $x[n]$ and $h[n]$ be two length-5 sequences given below where A and B are unknown constants.

$$x[n] = \{\underset{\uparrow}{A}, -2, B, -2, 1\}, \quad h[n] = \{\underset{\uparrow}{2}, 3, 1, -1, -2\}.$$

Instead, we know that $y[n] = x[n] \circledast_5 h[n]$ is given by

$$y[n] = \{\underset{\uparrow}{-6}, -19, -1, 5, 9\}.$$

(a) Solve for A and B.

$$\begin{bmatrix} 2 & -2 & -1 & 1 & 3 \\ 3 & 2 & -2 & -1 & 1 \\ 1 & 3 & 2 & -2 & -1 \\ -1 & 1 & 3 & 2 & -2 \\ -2 & -1 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} A \\ -2 \\ B \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -19 \\ -1 \\ 5 \\ 9 \end{bmatrix}$$

From here, there are multiple systems of two equations that can be used to find A and B leading to:

$$\boxed{A = -4, \quad B = 3}$$

(b) Suppose we have another sequence $v[n]$:

$$v[n] = \{\underset{\uparrow}{B}, -2, 1, A, -2\}.$$

Determine $z[n] = v[n] \circledast_5 h[n]$. $v[n] = x[\langle n-3\rangle_5]$. Therefore,

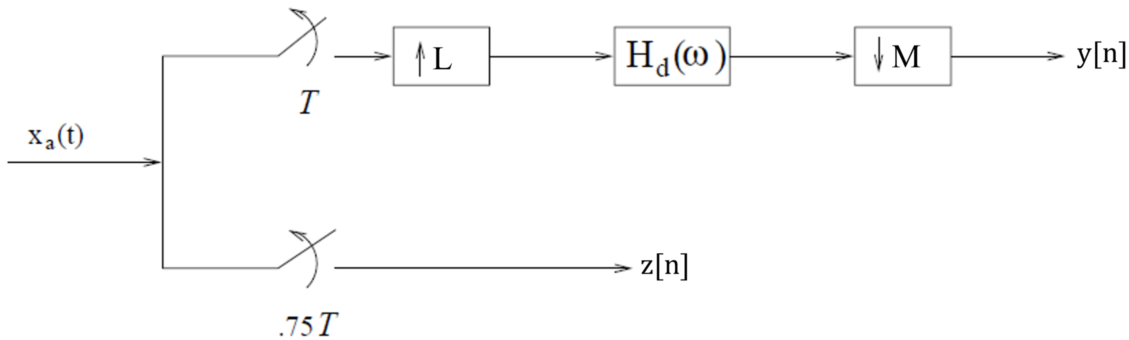
$$\boxed{z[n] = y[\langle n-3\rangle_5] = \{\underset{\uparrow}{-1}, 5, 9, -6, -19\}}$$

(c) What is the minimum N such that $x[n] \circledast_N h[n] = x[n] * h[n]$?

$$\boxed{N = 9}$$

(6 Pts.)

11. Consider the following system consisting of two synchronized ideal A/D converters. Assume that the input analog signal $x_a(t)$ is bandlimited to π/T. Complete the digital rate conversion subsystem by determining M, L, and $H_d(\omega)$ such that $y[n] = z[n]$.



The top branch should multiply the sampling period by $\frac{3}{4}$, which is equivalent to increasing the implicit sampling rate by $\frac{4}{3}$. Thus, we need

$$\boxed{L = 4, \quad M = 3}$$

Technically, any positive integer multiple of $L = 4k$ and $M = 3k$ could work with appropriate choice of $H_d(\omega)$, but we give the answer for $L = 4$, $M = 3$. The necessary $H_d(\omega)$ will be the stricter of the interpolation and anti-aliasing filter and have a gain of 4 to make the height of $Y_d(\omega) = Z_d(\omega)$:

$$\boxed{H_d(\omega) = \begin{cases} 4, & |\omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \le \pi \end{cases}}$$

(4 Pts.)

12. Let $\{x[n]\}_{n=0}^{N-1} = \cos\left(\frac{\pi}{6}n\right) + \cos\left(\frac{\pi}{4}n\right)$, $0 \leq n \leq N-1$ be a length-N discrete-time signal. For which of the following values of N will the resulting DFT $X[k]$ have only four non-zero values? (Please circle one choice.)

    (a) $N = 8$

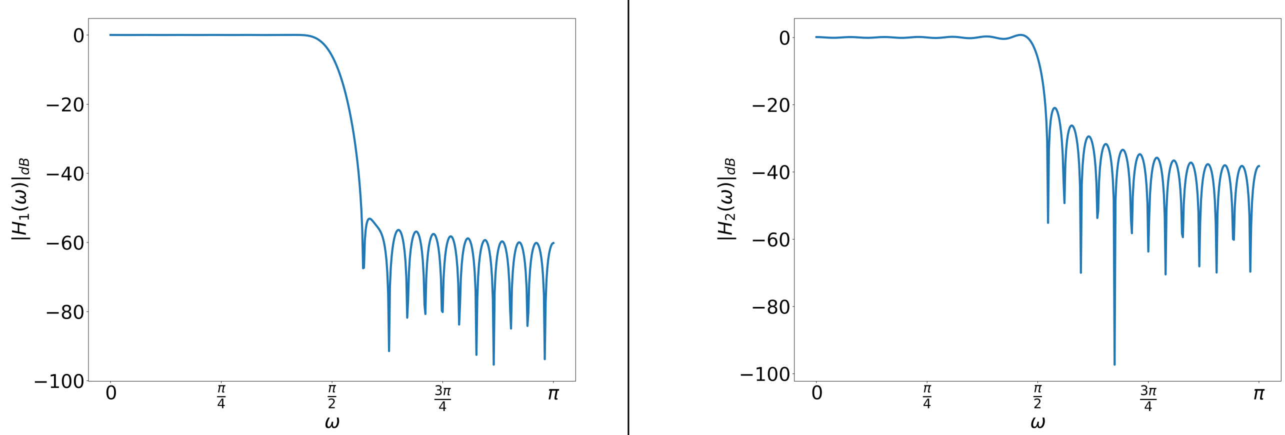
    (b) $N = 12$

    (c) $\boxed{N = 24}$

    (d) $N = 36$

(4 Pts.)

13. Suppose we use the window method for FIR filter design to create a length-51 low-pass filter with cutoff frequency $\omega_c = \frac{\pi}{2}$. We use a rectangular window and Hamming window to design two versions of this filter. The figure below depicts these two magnitude responses on a dB scale.



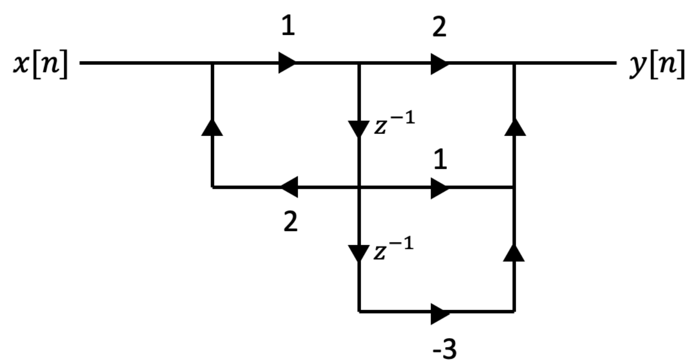
(a) $|H_1(\omega)|$ (b) $|H_2(\omega)|$

Circle the magnitude response corresponding to the filter designed with the **Hamming window** and **justify your reasoning**.

The frequency response on the left has a **wider transition band** and **stronger stopband attenuation**, which is consistent with the effects of the Hamming window compared to the rectangular window.

(6 Pts.)

14. Consider the causal LTI system shown below:



(a) Find the transfer function $H(z)$ for this system.

$$\boxed{H(z) = \frac{2 + z^{-1} - 3z^{-2}}{1 - 2z^{-1}}, \ \text{ROC} = |z| > 2}$$

(b) Determine the LCCDE relating $x[n]$ and $y[n]$ for the flow graph.

$$\boxed{y[n] = 2y[n-1] + 2x[n] + x[n-1] - 3x[n-2]}$$

(c) Is the system stable?

No

(9 Pts.)

15. An FIR filter is described by the below unit pulse response $h[n]$ where K is a real-valued constant.

$$h[n] = \{-\underset{\uparrow}{1}, K, 6, K, -1\}$$

(a) Determine $R(\omega)$ (in terms of K), α, and β such that the frequency response of this filter $H_d(\omega) = R(\omega)e^{j(-\alpha\omega+\beta)}$ of this filter where $R(\omega)$ is a real-valued function and α and β are real-valued constants.

$$\begin{aligned} H_d(\omega) &= -1 + Ke^{-j\omega} + 6e^{-j2\omega} + Ke^{-j3\omega} - e^{-j4\omega} \\ &= e^{-j2\omega}\left(6 + K(e^{j\omega} + e^{-j\omega}) - (e^{j2\omega} + e^{-j2\omega})\right) \\ &= (6 + 2K\cos(\omega) - 2\cos(2\omega))\, e^{-j2\omega} \end{aligned}$$

$R(\omega) = 6 + 2K\cos(\omega) - 2\cos(2\omega)$

$\alpha = 2$

$\beta = 0$

(b) For what value of K will $H_d(0) = 0$?

$$H_d(0) = 6 + 2K\cos(0) - 2\cos(0) = 0$$

$$\Longrightarrow \boxed{K = -2}$$

(c) For what value of K will $H_d(\pi) = 0$?

$$H_d(\pi) = 6 + 2K\cos(\pi) - 2\cos(2\pi) = 0$$

$$\Longrightarrow \boxed{K = 2}$$

(d) For what values of K will $H_d(\omega)$ have strictly **linear phase**?

The amplitude response reaches a minimum at $\omega = \pi$ for positive K since both cosine terms evaluate to -1. Thus, we need

$$6 - 2K - 2 \geq 0.$$

Similarly, if K is negative, the amplitude response reaches a minimum at $\omega = 0$. So we also need

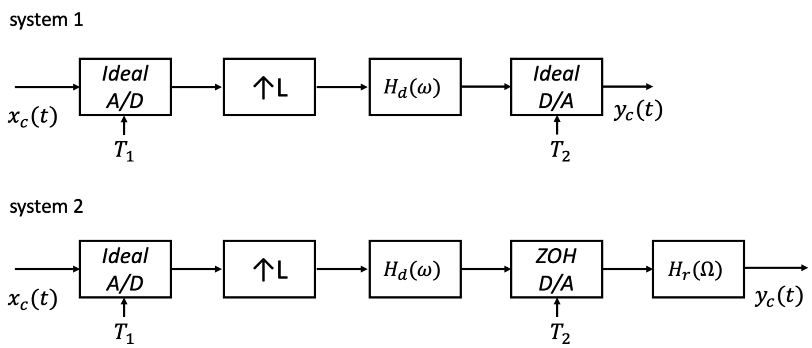
$$6 + 2K - 2 \geq 0.$$

Altogether, this implies we need

$$\boxed{-2 \leq K \leq 2}$$

(7 Pts.)

16. Consider the two systems in the following figure, where $X_c(\Omega)$ is bandlimited to $4 \cdot 10^3\pi$, T_1 is $\frac{1}{4\cdot 10^3}$ second, and $H_d(\omega)$ is an ideal LPF whose cut-off frequency is π/L.



(a) For System 1, express T_2 in terms of T_1 and L so that the signals $x_c(t)$ and $y_c(t)$ are identical.

$$\boxed{T_2 = \frac{T_1}{L}}$$

(b) In System 2, suppose the analog compensation filter $H_r(\Omega)$ has a transition band starting at $\Omega_1 = 4 \cdot 10^3\pi$ radian/second and ending at $\Omega_2 = 76 \cdot 10^3\pi$ radian/second. Determine the minimum L so that System 2 functions as System 1.

We must identify the minimum upsampling factor L such that the adjacent spectral copy, i.e. centered at $\frac{2\pi}{T_2}$, does not extend to the left of $\Omega = 76 \cdot 10^3\pi$. After upsampling, the central copy of the DTFT will have a maximum frequency of $\frac{\pi}{L}$. Thus, for sampling period $T_2 = \frac{T_1}{L}$, we need

$$\frac{2\pi - \frac{\pi}{L}}{T_2} \geq 76\pi \cdot 10^3$$

$$\frac{2\pi L - \pi}{T_1} \geq 76\pi \cdot 10^3$$

$$8\pi L \cdot 10^3 - 4\pi \cdot 10^3 \geq 76\pi \cdot 10^3$$

$$\boxed{L \geq 10}$$