1. CTFT: Continuous-time Fourier transform:

$x(t) \xleftrightarrow{CTFT} X_c(\Omega)$

$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(\Omega)e^{j\Omega t}d\Omega$

$X_c(\Omega)=\int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$

Property: $x(t-t_0)\leftrightarrow e^{-j\omega t_0}X(j\omega)$

$e^{j\omega_0 t}x(t)\leftrightarrow X(j(\omega-\omega_0))$

$x^*(t)\leftrightarrow X^*(-j\omega)$ | $x(t)*y(t)\leftrightarrow X(j\omega)Y(j\omega)$

$x(-t)\leftrightarrow X(-j\omega)$ | $x(t)y(t)\leftrightarrow \frac{1}{2\pi}X(j\omega)*Y(j\omega)$

$x(at)\leftrightarrow \frac{1}{|a|}X(\frac{j\omega}{a})$ | $\frac{dx(t)}{dt}\leftrightarrow j\omega X(j\omega)$

$\int_{-\infty}^{t}x(t)dt\leftrightarrow \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$ | $tx(t)\leftrightarrow j\frac{dX(j\omega)}{d\omega}$

x(t) is real also apply for DTFT $\begin{cases} X(j\omega)=X^*(-j\omega) \\ Re\{X(j\omega)\}=Re\{X(-j\omega)\} \\ Im\{X(j\omega)\}=-Im\{X(-j\omega)\} \\ |X(j\omega)|=|X(-j\omega)| \\ \measuredangle X(j\omega)=-\measuredangle X(-j\omega) \end{cases}$ | x(t) real, even $\Leftrightarrow$ X(jw) real even | x(t) real, odd $\Leftrightarrow$ X(jw) purely imaginary, odd | also apply for DTFT

Parseval's Relation for Aperiodic Signals:

$\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2d\omega$

Transform Pairs: $\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}\leftrightarrow 2\pi\sum_{k=-\infty}^{\infty}a_k\delta(\omega-k\omega_0)$

$e^{j\omega_0 t}\leftrightarrow 2\pi\delta(\omega-\omega_0)$ | $\cos\omega_0 t\leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

$x(t)=1\leftrightarrow 2\pi\delta(\omega)$ | $\sin\omega_0 t\leftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

$\delta(t)\leftrightarrow 1$ | $u(t)\leftrightarrow \frac{1}{j\omega}+\pi\delta(\omega)$

$\delta(t-t_0)\leftrightarrow e^{-j\omega t_0}$ | $\sum_{n=-\infty}^{\infty}\delta(t-nT)\leftrightarrow \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2\pi k}{T})$

$x(t)\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}\leftrightarrow \frac{2\sin\omega T_1}{\omega}$ | $\frac{\sin Wt}{\pi t}\leftrightarrow X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$

$e^{-at}u(t), Re\{a\}>0\leftrightarrow\frac{1}{a+j\omega}$ | $te^{-at}u(t), Re\{a\}>0\leftrightarrow\frac{1}{(a+j\omega)^2}$

$x(t)$ or $x[n]$ real $\Leftrightarrow X(j\omega)=X^*(-j\omega)$ ☆

3. Sinusoidal response of LSI system: for fixed ω_0

$\{e^{j\omega_0 n}\}_n\rightarrow\boxed{H_d(\omega)}\rightarrow\{H_d(\omega_0)e^{j\omega_0 n}\}_n$

if $\{h[n]\}$ is real-valued then

$\{\cos(\omega_0 n+\phi)\}_n\rightarrow\boxed{H_d(\omega)}\rightarrow\{|H_d(\omega_0)|\cos(\omega_0 n+\phi+\measuredangle H_d(\omega_0))\}_n$

求magnitude/phase response example:

$H(\omega)=2-e^{-2j\omega}+2e^{-j4\omega}\Rightarrow H(\omega)=2e^{-2j2\omega}(e^{j2\omega}+e^{-j2\omega})-e^{-j2\omega}$

$=4\cos(2\omega)e^{-j2\omega}-e^{-j2\omega}$

$=e^{-j2\omega}(4\cos(2\omega)-1)$

$\therefore |H(\omega)|=|4\cos(2\omega)-1|$

$\measuredangle H(\omega)=-2\omega+\measuredangle(4\cos(2\omega)-1)$

只画$[-\pi,\pi]$即可

$\measuredangle H(\omega)$ range 限制在$(-\pi,\pi]$

2. DTFT: Discrete-time Fourier transform:

$x[n]\xleftrightarrow{DTFT}X_d(\omega)$

$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_d(\omega)e^{j\omega n}d\omega$

$X_d(\omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$

Property: $X_d(\omega)=X(z)|_{z=e^{j\omega}}$ if ROC_x includes unit circle

2π-periodic: $X_d(\omega)=X_d(\omega+2k\pi)$ for any $k\in\mathbb{Z}$

if $\{x[n]\}$ is real-valued, then $X_d(-\omega)=X_d^*(\omega)$

or $|X_d(-\omega)|=|X_d(\omega)|$, $\measuredangle X_d(-\omega)=-\measuredangle X_d(\omega)$

if $y[n]=x[n]w[n]$, then $Y_d(\omega)=(X_d*W_d)(\omega)$

$=\frac{1}{2\pi}\int_{2\pi}X_d(\theta)W_d(\omega-\theta)d\theta$

Parseval's relation: $\sum_n|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X_d(\omega)|^2d\omega$

$x[n-n_0]\leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$ | $e^{j\omega_0 n}x[n]\leftrightarrow X(e^{j(\omega-\omega_0)})$

$x^*[n]\leftrightarrow X^*(e^{-j\omega})$ | $x[-n]\leftrightarrow X(e^{-j\omega})\Rightarrow X(-\omega)$

$x_{(k)}[n]=\begin{cases}x[\frac{n}{k}] & \text{if } n=mk, m\in\mathbb{Z}\\ 0 & \text{if } n\neq mk\end{cases}\Leftrightarrow X(e^{jk\omega})$ 等效 $X(k\omega)$

$x[n]*y[n]\leftrightarrow X(e^{j\omega})Y(e^{j\omega})$ | ~~x[n]y[n] ⟺ (1/2π)∫2π X(e…)~~

$x[n]y[n]\leftrightarrow\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

$nx[n]\leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$

Transform Pairs: only consider $(-\pi,\pi]$

$\delta[n]\leftrightarrow 1$ | $u[n]\leftrightarrow\frac{1}{1-e^{-j\omega}}+\pi\delta(\omega)$

$a^n u[n]\leftrightarrow\frac{1}{1-ae^{-j\omega}}, |a|<1$ | $e^{j\omega_0 n}\leftrightarrow 2\pi\delta(\omega-\omega_0)$

$\cos(\omega_0 n)\leftrightarrow\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ | $\sin(\omega_0 n)\leftrightarrow -j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

$rect(\frac{n-k}{L})\leftrightarrow\frac{\sin(\frac{L\omega}{2})}{\sin(\frac{\omega}{2})}e^{-j\omega k}$ | $sinc(Ln)\leftrightarrow\frac{\pi}{L}rect(\frac{\omega}{2L})$

$sinc^2(Ln)\leftrightarrow\frac{\pi}{L}\Delta(\frac{\omega}{2L})$ | $1\leftrightarrow 2\pi\delta(\omega)$

$(n+1)a^n u[n], |a|<1\leftrightarrow\frac{1}{(1-ae^{-j\omega})^2}$

$rect(t)=\begin{cases}1, & |t|<\frac{1}{2}\\ 0, & |t|>\frac{1}{2}\end{cases}$

$\delta(a(x-x_0))=\frac{1}{|a|}\delta(x-x_0)$

只有BIBO stable 才满足 $H_d(e^{j\omega})=H(z)|_{z=e^{j\omega}}$

否则只能用DTFT pair得 $H_d(e^{j\omega})$

Ex. $H_d(\omega)=\omega e^{j\pi\cos\omega}$, $x[n]=3+e^{j\frac{\pi}{3}n}+\sin(\frac{\pi}{2}n+\frac{\pi}{4})$

求$y[n]$:

$x[n]=3e^{j0\cdot n}+e^{j\frac{\pi}{3}n}+\frac{e^{j(\frac{\pi}{2}n+\frac{\pi}{4})}-e^{-j(\frac{\pi}{2}n+\frac{\pi}{4})}}{2j}$

$H_d(0)=0$, $H_d(\frac{\pi}{3})=\frac{\pi}{3}e^{j\frac{\pi}{2}}$, $H_d(\frac{\pi}{2})=\frac{\pi}{2}$, $H_d(-\frac{\pi}{2})=-\frac{\pi}{2}$

$\therefore y[n]=0\cdot 3e^{j0\cdot n}+\frac{\pi}{3}e^{j\frac{\pi}{2}}\cdot e^{j\frac{\pi}{3}n}+\frac{\pi}{2}\cdot\frac{e^{j(\frac{\pi}{2}n+\frac{\pi}{4})}+e^{-j(\frac{\pi}{2}n+\frac{\pi}{4})}}{2j}$

$=\frac{\pi}{3}e^{j(\frac{\pi}{2}+\frac{\pi}{3}n)}-\frac{j\pi}{2}\cos(\frac{\pi}{4}+\frac{\pi}{2}n)$

Ex. 求$H(\omega)$的magnitude/phase response, $H(\omega)=1-e^{-j\omega}+e^{-2j\omega}-e^{-3j\omega}$

$H(\omega)=e^{-1.5j\omega}(2j\sin(1.5\omega)-2j\sin(0.5\omega))$

$\therefore |H(\omega)|=|2\sin(1.5\omega)-2\sin(0.5\omega)|$

$\measuredangle H(\omega)=-1.5\omega+\measuredangle(2j\sin(1.5\omega)-2j\sin(0.5\omega))$

$=\begin{cases}-1.5\omega+\frac{\pi}{2}, & \sin(1.5\omega)\geq\sin(0.5\omega)\\ -1.5\omega-\frac{\pi}{2}, & \sin(1.5\omega)<\sin(0.5\omega)\end{cases}$

4. ADC/DAC:

描述频域而非时域

For $X_c(\Omega)$, 最右非0 $\Omega_0=2\pi B \Rightarrow$ bandwidth $B=\frac{\Omega_0}{2\pi}$

To avoid aliasing: $2\pi B < 2\pi f_s - 2\pi B \Rightarrow$ Nyquist rate $f_s > 2B$

ADC: 先得$X_c(\Omega)$, 再从其图得$X_d(w)$

$x(t) \to \underset{T}{\boxed{ADC}} \to x[n]=x(nT)$ $\left\{\begin{array}{l} w=\Omega T \\ \text{值乘}\frac{1}{T} \\ \text{有}2\pi\text{周期}\end{array}\right.$

$X_d(w)=\frac{1}{T}\sum_{k\in\mathbb{Z}} X_c\left(\frac{w+2k\pi}{T}\right)$

若有aliasing, 先画图再叠加计算

DAC: 先得$Y_c(\Omega)$, 再从其图得$Y[n]$

$y[n] \to \underset{T,p(t)}{\boxed{DAC}} \to y(t)$ $\left\{\begin{array}{l} \Omega=\frac{w}{T} \\ \text{值乘}T\end{array}\right.$

$y(t)=\sum_{n\in\mathbb{Z}} y[n]p(t-nT)$

$Y_c(\Omega)=Y_d(\Omega T)P_c(\Omega)$

$P_c^{(ideal)}(\Omega)=\begin{cases}T & \text{if } |\Omega|<\frac{\pi}{T}\\ 0 & \text{else}\end{cases}$

$p^{(ZOH)}(t)=\begin{cases}1, & 0\le t<T\\ 0, & \text{else}\end{cases}$ $P_c^{(ZOH)}(\Omega)=\frac{e^{-j\frac{\Omega T}{2}}\left(2\sin\left(\frac{\Omega T}{2}\right)\right)}{\Omega}$

< ideal reconstruction filter is non-causal as its time domain has form sinc

5. DFT: x(t)得x[n]再得X[k]

$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi kn}{N}}$

$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi kn}{N}}$

$X[k]=X_d(w)|_{w=\frac{2\pi k}{N}}$

$W_N=e^{-j\frac{2\pi}{N}}$

$W_N^{kn}=e^{-j\frac{2\pi kn}{N}}$

circular shift: for a fixed m, if $y[n]=x[\langle n-m\rangle_N]$, then we have $Y[k]=X[k]e^{-j\frac{2\pi km}{N}}=X[k]W_N^{km}$

if $y[n]=x[\langle -n\rangle_N]$ we have $Y[k]=X[-k]$

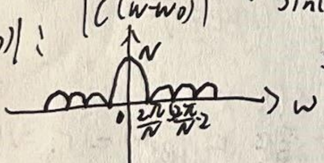
if $y[n]=z[n]W_N^{-mn}$ then $Y[k]=Z[\langle k-m\rangle_N]$

For $x[n]=A\cos(w_0 n), 0\le n\le N-1$

DTFT: $X(w)=\frac{A}{2}C(w-w_0)+\frac{A}{2}C(w+w_0)$

where $C(w)=e^{-j\frac{w}{2}(N-1)}\left(\frac{\sin(\frac{N}{2}w)}{\sin(\frac{1}{2}w)}\right)$

$|C(w-w_0)|$:



zeros: $w-w_0=\frac{2\pi}{N}k, k\neq 0$

Circular modulation

$x[n]\cos\left(\left(\frac{2\pi}{N}\right)k_0 n\right) \Leftrightarrow \frac{1}{2}X[\langle k+k_0\rangle_N]+\frac{1}{2}X[\langle k-k_0\rangle_N]$

If the signal is symmetric, then its DFT's magnitude has even symmetric, Its DFT's phase has odd symmetric

DFT of DFT is original signal multiplied by N

zero-padding can improve the resolution

each peak in DFT has height $\frac{AN}{2}$

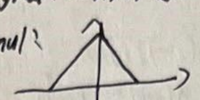
Upsampling: 压缩 contracts DTFT 需LPF 删除copy

$y[n]=\begin{cases}x[\frac{n}{U}], & n=\pm U, 2U, \dots\\ 0, & \text{else}\end{cases}$

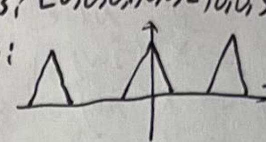
$Y_d(w)=\sum_{n=-\infty}^{\infty}y[n]e^{-jwn}=\sum_{n=-\infty}^{\infty}x[n]e^{-jwUn}$ $Y_d(w)=X_d(Uw)$

Signal: [0,1,2,3], Upsampled by 3: [0,0,0,1,0,0,2,0,0,3,0,0]

original:



Upsampled:



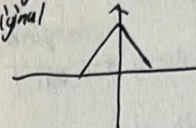
Downsampling: 延伸 expands DTFT, 可导致aliasing

$x_D[n]=x[Dn]$

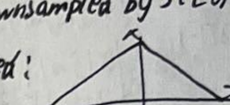
$Y_d(w)=\frac{1}{D}\sum_{k=0}^{D-1}X_d\left(\frac{w-2\pi k}{D}\right)$ ☆

Signal: [0,1,2,3,4,5,6,7,8,9], Downsampled by 3: [0,3,6,9]

original



Downsampled:



~~$w_{original} D = w_D$~~

注意、$H_d(w)$正负，大小

continuous-time frequencies: Ω

discrete-time frequencies: w

spacing between the DFT: 相邻k的Δw: $\frac{2\pi}{N}$

$x_a(t)\to\boxed{A/D}\to\boxed{H_d(w)}\to\boxed{D/A}\to y_a(t)$

if no alisaing in $H_d(w)$ 保持过滤的w the system is LTI

eigenfunction of transfer functions: Az^n

eigenfunction of frequency responses: $Ae^{jw_0 n}$

$\cos(w_0 n)$ is not eigenfunction of frequency responses

Full lobe separation: $w_2-w_1>\frac{4\pi}{N}$

so increase the length x[n] can distinguish frequencies that are closer to one another (not by zero-padding)

Half lobe separation: $w_2-w_1>\frac{2\pi}{N}$

Rectangular window $r[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$

Hamming window $h[n]=\begin{cases}0.54-0.46\cos\left(\frac{2\pi n}{N-1}\right) & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$