University of Illinois
ECE 310 **Profs. Do, Snyder, Moustakides**

Midterm Exam 1

7:00-9:00pm, Wednesday, September 27, 2023

Name: ____________________________________

Section: 9:00 AM 12:00 PM 3:00 PM

NetID: _____________

Score: _____________

Problem	Pts.	Score
1	12	
2	12	
3	12	
4	5	
5	9	
6	20	
7	20	
8	10	
Total	100	

Instructions

- You may not use any books, calculators, or notes other than one <u>handwritten</u> two-sided sheet of 8.5" x 11" paper.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.

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(12 Pts.)

1. Select **True** or **False** to each of the following statements:

   (a) Any causal discrete-time system must also be time-invariant. **True/<u>False</u>**

   (b) The ROC of a given z-transform cannot contain any poles or zeros. **True/<u>False</u>**

   (c) Two BIBO stable systems connected in parallel always form a BIBO stable system. **<u>True</u>/False**

   (d) For a discrete-time system with impulse response $h[n]$, the system output $y[n]$ to any input signal $x[n]$ is always given by $y[n] = x[n] * h[n]$. **True/<u>False</u>**

   (e) Two LTI systems given by impulse responses $h_1[n]$ and $h_2[n]$ are connected in series in some order, i.e. $h_1[n]$ or $h_2[n]$ may come first. If we pass an input signal $x[n]$ to this system, we will receive the same output signal $y[n]$ for either ordering of $h_1[n]$ and $h_2[n]$. **<u>True</u>/False**

   (f) A BIBO stable LTI system with transfer function $H(z) = \frac{1}{1-3z^{-1}}$ must be non-causal. **<u>True</u>/False**

(12 Pts.)

2. For each of the systems with input $x[n]$ and output $y[n]$ shown in the table, indicate by "**yes**" or "**no**" whether the properties indicated apply to the system. You will only be graded on your answers in the boxes and not on any work you show.

	Linear	Time-invariant	Causal	Stable
$y[n] = \log(\lvert n\rvert + 1)x[n]$	**Yes**	**No**	**Yes**	**No**
$y[n] = x[n] * u[n+1]$	**Yes**	**Yes**	**No**	**No**
$y[n] = x[n] + 3$	**No**	**Yes**	**Yes**	**Yes**

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(12 Pts.)

3. For each of the following parts, calculate the convolution $y[n] = x[n] * h[n]$.

   (a) $\{x[n]\}_{n=0}^{4} = \{\underset{\uparrow}{1}, 2, 3, 2, 1\}, \quad \{h[n]\}_{n=0}^{1} = \{\underset{\uparrow}{-1}, 1\}.$

$$
\begin{aligned}
y &= \mathbf{H}x \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$\boxed{y[n] = \{\underset{\uparrow}{-1}, -1, -1, 1, 1, 1\}}$$

   (b) $x[n] = \left(\frac{3}{4}\right)^n u[n], \quad h[n] = 2u[n] - u[n-1] - u[n-2]$

$$
\begin{aligned}
y[n] &= \left(\frac{3}{4}\right)^n u[n] * (2u[n] - u[n-1] - u[n-2]) \\
&= \left(\frac{3}{4}\right)^n u[n] * (2\delta[n] + \delta[n-1]) \\
&= 2\left(\frac{3}{4}\right)^n u[n] + \left(\frac{3}{4}\right)^{n-1} u[n-1]
\end{aligned}
$$

$$\boxed{y[n] = 2\left(\frac{3}{4}\right)^n u[n] + \left(\frac{3}{4}\right)^{n-1} u[n-1]}$$

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(5 Pts.)

4. Suppose we have an LTI system described by impulse response $h[n]$. We know that the system response to input $x[n] = u[n]$ is given by $y[n] = \delta[n] + \delta[n-1]$. Which of the following sequences correctly expresses $h[n]$? **(Circle one)**

   (a) $h[n] = \{\underset{\uparrow}{1}, 0, 1\}$

   (b) $\boxed{h[n] = \{\underset{\uparrow}{1}, 0, -1\}}$

   (c) $h[n] = \{\underset{\uparrow}{1}, 1\}$

   (d) $h[n] = \{\underset{\uparrow}{1}, -1\}$

(9 Pts.)

5. Calculate the z-transform $X(z)$ of $x[n] = nu[n+1]$.

$$\begin{aligned}
nu[n+1] &= (n+1)u[n+1] - u[n+1] \\
u[n+1] &\overset{\mathcal{Z}}{\mapsto} \frac{z}{1-z^{-1}}, 1 < |z| < \infty \quad \text{(time shift property)} \\
nu[n] &\overset{\mathcal{Z}}{\mapsto} \frac{z^{-1}}{(1-z^{-1})^2}, \ |z| > 1 \quad \text{(differentiation property)} \\
(n+1)u[n+1] &\overset{\mathcal{Z}}{\mapsto} \frac{1}{(1-z^{-1})^2}, \ |z| > 1 \quad \text{(differentiation + time shift)}
\end{aligned}$$

$$\boxed{X(z) = \frac{1}{(1-z^{-1})^2} - \frac{z}{1-z^{-1}}, \ 1 < |z| < \infty}$$

Alternatively, let $y[n] = u[n+1]$. We have

$$Y(z) = \frac{z}{1-z^{-1}}.$$

Since $x[n] = ny[n]$, we have

$$X(z) = -z\frac{dY(z)}{dz} = -z\frac{d}{dz}\left(\frac{z}{1-z^{-1}}\right) = \boxed{\frac{-z(1-2z^{-1})}{(1-z^{-1})^2}}$$

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(20 Pts.)

6. We are given the following transfer function $H(z)$ of an LTI system

$$H(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

(a) Determine all possible ROCs and the corresponding impulse response $h[n]$ for each ROC.

$$\begin{aligned}
\frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} &= \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - 2z^{-1}} \\
1 - z^{-1} &= A_1(1 - 2z^{-1}) + A_2(1 - \frac{1}{2}z^{-1}) \\
z = \frac{1}{2} \implies A_1 &= \frac{1}{3} \\
z = 2 \implies A_2 &= \frac{2}{3}.
\end{aligned}$$

	First term	Second term	ROC	$h[n]$
Case 1	Right-sided	Right-sided	$\|z\| > 2$	$\frac{1}{3}(\frac{1}{2})^n u[n] + \frac{2}{3}(2)^n u[n]$
Case 2	Left-sided	Left-sided	$\|z\| < \frac{1}{2}$	$-\frac{1}{3}(\frac{1}{2})^n u[-n-1] - \frac{2}{3}(2)^n u[-n-1]$
Case 3	Right-sided	Left-sided	$\frac{1}{2} < \|z\| < 2$	$\frac{1}{3}(\frac{1}{2})^n u[n] - \frac{2}{3}(2)^n u[-n-1]$

Note: When the first term is left-sided and the second term is right-sided, the ROC is null (empty set). Thus, the z-transform sum of the corresponding $h[n]$ converges for no values of z and we do not require it in the above answer table.

(b) Determine the corresponding difference equation with transfer function $H(z)$.

$$\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} \\
Y(z)(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1}) &= X(z)(1 - z^{-1}) \\
Y(z)(1 - \frac{5}{2}z^{-1} + z^{-1}) &= X(z)(1 - z^{-1})
\end{aligned}$$

$$\boxed{y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1]}$$

(20 Pts.)

7. Consider a **causal** LTI system with the following transfer function:

$$H(z) = \frac{1 - 3z^{-1} + 2z^{-2}}{1 + \frac{3}{4}z^{-1} - \frac{1}{4}z^{-2}}.$$

(a) Determine the poles, zeros, and ROC of the system.

$$H(z) = \frac{(1 - 2z^{-1})(1 - z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 + z^{-1})}$$

Poles: $z = \frac{1}{4}, z = -1$

Zeros: $z = 1, z = 2$

ROC: $|z| > 1$

(b) Calculate the system response $y[n]$ to input signal $x[n] = u[n]$.

$$\begin{aligned}
Y(z) &= X(z)H(z) \\
&= \left(\frac{1}{1 - z^{-1}}\right)\left(\frac{(1 - 2z^{-1})(1 - z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 + z^{-1})}\right) \\
&= \frac{1 - 2z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 + z^{-1})} \\
&= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 + z^{-1}} \\
1 - 2z^{-1} &= A_1(1 + z^{-1}) + A_2(1 - \frac{1}{4}z^{-1}) \\
z = \frac{1}{4} &\implies A_1 = -\frac{7}{5} \\
z = -1 &\implies A_2 = \frac{12}{5}.
\end{aligned}$$

$$\boxed{-\frac{7}{5}\left(\frac{1}{4}\right)^n u[n] + \frac{12}{5}(-1)^n u[n]}$$

(c) Is the system described by $H(z)$ BIBO stable? If the system is not BIBO stable, give an example of a bounded input that will produce an unbounded output.

The given system is **not BIBO stable** because the ROC does not contain the unit-circle. The bounded input signal $x[n] = \cos(\pi n)u[n] = (-1)^n u[n]$ will produce an unbounded output.

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(10 Pts.)

8. Suppose we have an LTI system described by

$$y[n] - \frac{2}{3}y[n-1] - \frac{8}{9}y[n-2] = 3x[n] + \alpha x[n-1],$$

where α is a finite, real-valued constant and there exists a transfer function $H(z)$ for the system.

(a) Assuming the system is **causal**, for what value(s) of α is this system BIBO stable?

$$\begin{aligned} Y(z)\left(1 - \frac{2}{3}z^{-1} - \frac{8}{9}z^{-2}\right) &= X(z)(3 + \alpha z^{-1}) \\ \frac{Y(z)}{X(z)} = H(z) &= \frac{3 + \alpha z^{-1}}{1 - \frac{2}{3}z^{-1} - \frac{8}{9}z^{-2}} \\ &= \frac{3 + \alpha z^{-1}}{(1 - \frac{4}{3}z^{-1})(1 + \frac{2}{3}z^{-1})}, |z| > \frac{4}{3} \end{aligned}$$

If $\alpha = \mathbf{-4}$, we will have a zero at $z = \frac{4}{3}$ that will cancel the pole at $\frac{4}{3}$ that makes the system originally unstable.

(b) Assuming the system is **non-causal** and **two-sided**, for what value(s) of α is this **non-causal** system BIBO stable?

If the system is two-sided and thus non-causal, we must have the ROC is $\frac{2}{3} < |z| < \frac{4}{3}$. This system is already BIBO stable since its ROC contains the unit-circle. Thus, the system is stable **for all values of** α.

For the other possible two-sided version of the system, the ROC is empty and thus the transfer function $H(z)$ would be undefined for all values of z.