HKN ECE 310 Review Session

09/24/2023

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Logistics

This session is not recorded.

Slides can be found on the HKN website (Google ‘HKN UIUC’, it will be the first link that pops up.

Extra office hours will be held on **9/27 at 6 pm** in the RSO Office (next to the Daily Byte). They will end when your exam begins.

Overview

Topics

- CT and DT Signals
- LSI and LSIC Systems
- Impulse Response
- Z-transforms
- Stability, Causality, etc
- Multiple Systems

CT and DT Signals

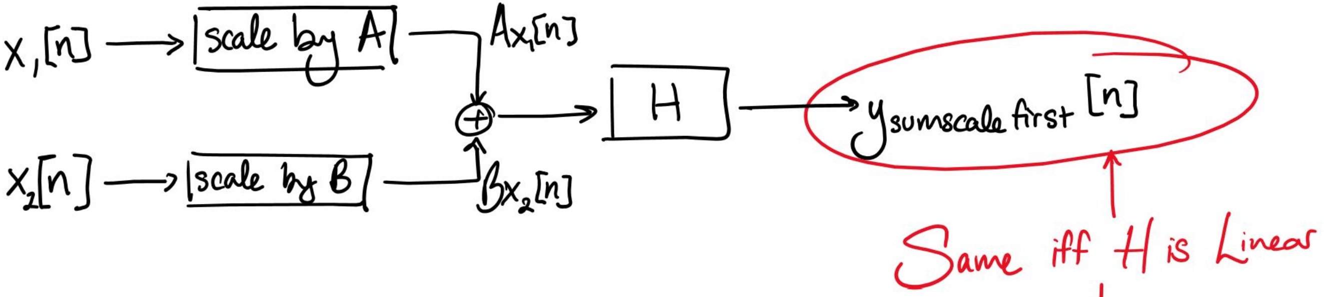
- Discrete-Time (DT) Signals are really just Continuous-Time (CT) Signals, but sampled
  - Example: your phone battery level
- $x[n]$ represents DT signals, while it is written as $x(t)$ for CT
- Frequency can also be either Discrete or Continuous - You'll learn how to deal with this later.

LSI and LSIC Systems aka LTI and LTIC Systems

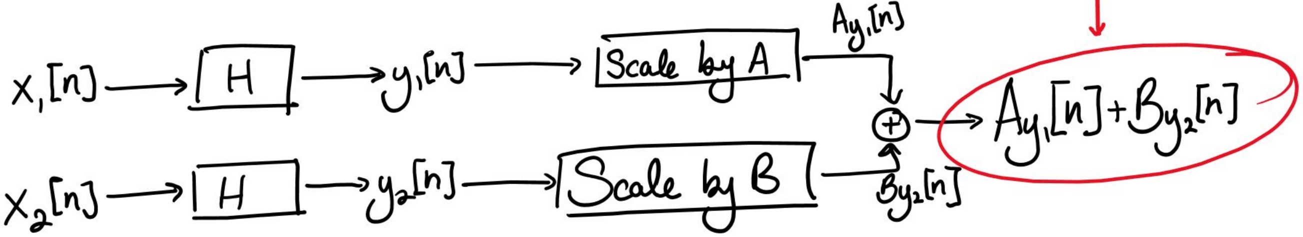
- Given a system H which produces output y[n] on input x[n], we can check for three properties:
  - **Linearity**: Saying system H is linear is equivalent to saying:
    - Given $x_1[n] \rightarrow y_1[n]$, and given $x_2[n] \rightarrow y_2[n]$, then $ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$
    - Any scaling and summing of original inputs is equivalently reflected in the outputs
  - **Shift Invariance**: Saying H is **SI** (shift invariant) or **TI** (time-invariant) is equivalent to saying:
    - Given $x[n] \rightarrow y[n]$, then $x[n-k] \rightarrow y[n-k]$
    - Any delay in the original input is equivalently reflected in the output
  - **Causality**: output cannot depend on future input values.
    - No “official” formal definition in this class.

Pictographic Representation of Linearity

Apply and Feed



Feed and Apply



Pictographic Representation of Shift Invariance

Apply and Feed

$$x[n] \longrightarrow \boxed{\text{Delay by } k} \xrightarrow{x[n-k]} \boxed{H} \longrightarrow y_{delayfirst}[n]$$

Same iff H is TI

Feed and Apply

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] \longrightarrow \boxed{\text{Delay by } k} \longrightarrow y[n-k]$$

General Approach on Proving L/TI

1. Feed and apply
    a. Define input/output pairs to the system
        i. $x[n] \rightarrow y[n]$
    b. Apply transformation to the outputs
2. Apply and feed
    a. Define a new function applying whatever variant
        i. Testing time invariance: $g[n] = x[n-k]$
        ii. Testing linearity: $g[n] = ax_1[n] + bx_2[n]$
        iii. (Not strictly necessary, but it's easy to get lost...)
    b. Feed this into the system determine output $y_g[n]$
3. Compare the results from steps 1 & 2.

Examples: Proving LTI via a system's equations

- Prove, or disprove, that $y[n] = \dfrac{x[2n]}{n^2}$ is linear.
- Prove, or disprove, that $y[n] = \dfrac{x[n-\boldsymbol{\alpha}]}{x[n-\boldsymbol{\beta}]}$ is time invariant.
- Prove, or disprove, that $y[n] = x[n] + x[2023]$ is time invariant.

Causality of Systems via their Equations

- Suppose we have some equation describing some system
  - $y[n] = x[n-4] + y[n-1] + n^2 x[n]$
- "Current outputs cannot depend on future inputs"
- A more formal definition (for those who want):
  - The maximum index of y (call it ny), must always be greater or equal to the maximum index of x (call it nx) at any point in time

Justifying Causality of Systems via their Equations

- For every system, determine whether it is causal or not and justify.

1. $y[n] = x[n] + y[n-1] + x[n-200] + x[\lfloor \frac{n}{2} \rfloor]$
2. $y[n] = x[n] + x[n-1] + x[n-2] + \ldots$
3. $y[n] = \cos(2023n)$
4. $y[n] = x[|n|]$
5. $y[n] = |x[n]|$
6. $y[n] - y[n-3] = x[n+2]$

BIBO Stability of Systems via their Equations

- Given a system H, the system is **BIBO stable** if and only if every bounded input $x[n]$ produces a bounded output $y[n]$.
    - That is, if $|x[n]| < \boldsymbol{\beta}$ for some $\boldsymbol{\beta}$, then $|y[n]| < \boldsymbol{\alpha}$ for some $\boldsymbol{\alpha}$.
    - “Bounded output” must be well-defined output
- We can check for BIBO stability even if the system isn’t LTI
    - Proof: FA21 got tested on it…

Justifying BIBO Stability of Systems via their Equations

Determine whether the following sequences are BIBO stable or not. If not, provide a bounded input that causes the input to be unbounded.

1. $y[n] = 20x[n]$
2. $y[n] = \dfrac{(½)^n x[n]}{x[310]}$
3. $y[n] = \dfrac{1}{n^2} + \dfrac{x[n]}{1-(x[n]-5)^2}$
4. $y[n] = x[n] + x[n-1] + y[n-1]$
5. $y[n] = x[n](-1)^n - 2y[n-1]$

Impulse Response

Let $x[n]$ be the input to an LSI system with impulse response $h[n]$. Then, the system output $y[n]$ is given by

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]h[n-m]$$

Equivalently, $y[n] = x[n] * h[n]$.

Three ways of determining the output:

1. Graphical method
2. Flipping a sequence + sequential multiplication and adding
3. ~~Plug it into scipy~~

Which of the following make sense?

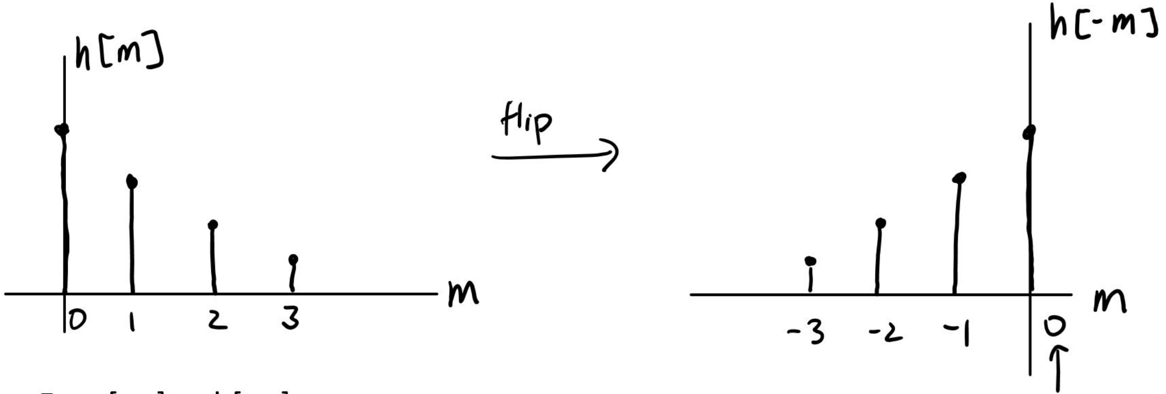
Let x[n] be some arbitrary signal. Let h[n] be the impulse response of some linear system.

1. x[n] is linear.
2. x[n] is shift-invariant.
3. x[n] is causal.
4. x[n] is BIBO-stable.
5. x[n] is unbounded.

6. h[n] is linear.
7. h[n] is shift-invariant.
8. h[n] is causal.
9. h[n] is BIBO-stable.
10. h[n] is unbounded.

Graphical Method

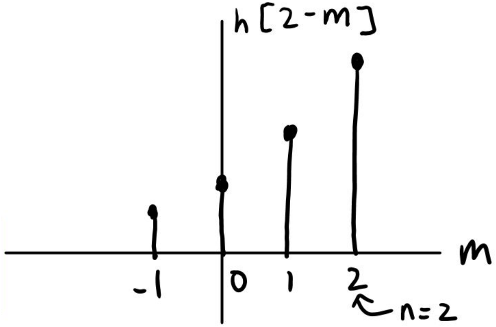
1. Flip either x[n] or h[n] and find as a function of m rather than n

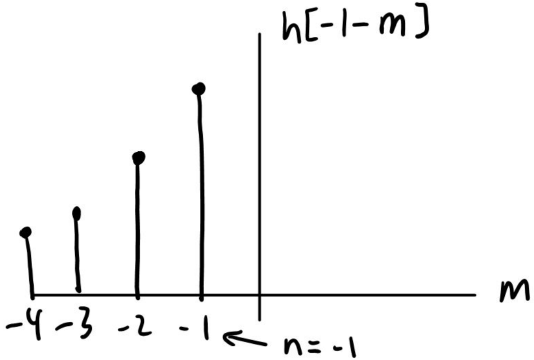


This m = 0 point is equivalent to the n value during convolution

For x[-m] or h[-m]:

- n > 0 → shift right
- n < 0 → shift left

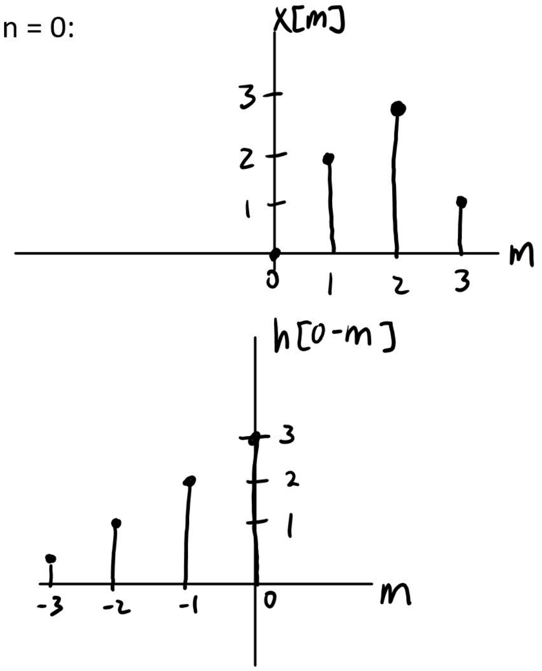




Graphical Method (cont)

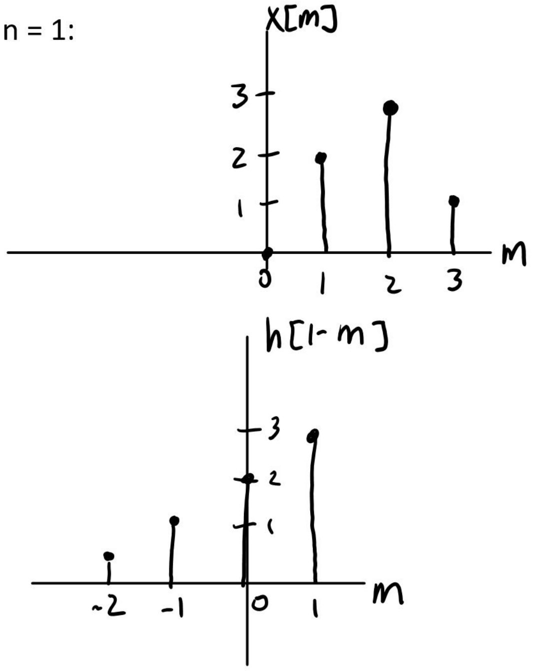
2. Pass the flipped signal, x[n-m] or h[n-m] from left to right and find the convolution result $y[n] = \sum_{m=-\infty}^{+\infty} x[m]h[n-m]$ or $y[n] = \sum_{m=-\infty}^{+\infty} h[m]x[n-m]$ as follows:

n = 0:



y[0] = 0 * 3 = 0

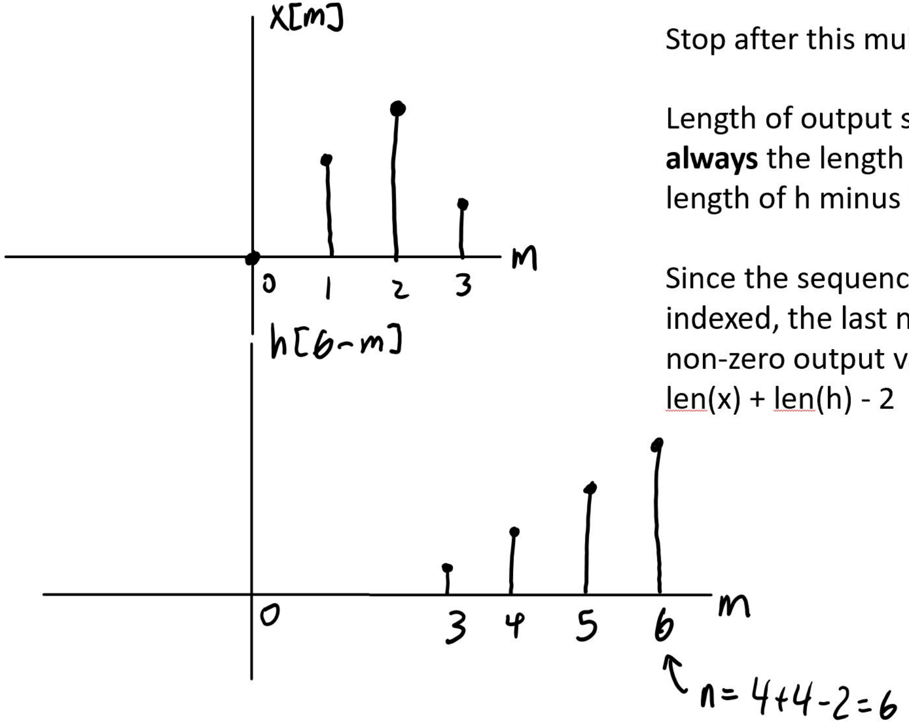
n = 1:



y[1] = 0 * 2 + 2 * 3 = 6

Graphical Method (cont)

3. Continue until you have found a number of points equal to the length of x + the length of h minus 1



Stop after this multiplication!

Length of output sequence is **always** the length of x + the length of h minus 1!

Since the sequences are zero-indexed, the last n that gives a non-zero output value should be len(x) + len(h) - 2

Sequence Method

Given two sequences x[n] and h[n]:

- x[n] = [-1, 0, 1, 2]
- h[n] = [2, 3, 1, 0]

1. First flip one of the two sequences, and obtain both as a function of m

- x[m] = [-1, 0, 1, 2]
- h[-m] = [0, 1, 3, 2]

Sequence Method (cont)

2. Similar to the graphical method, shift the flipped sequence from left-to-right, multiplying overlapping elements and adding the products together for each n

n = 0:

x[m] = [-1, 0, 1, 2]

h[0-m] = [0, 1, 3, 2]

y[0] = -1 * 2 = -2

n = 1:

x[m] = [-1, 0, 1, 2]

h[1-m] = [0, 1, 3, 2]

y[1] = -1 * 3 + 0 * 2 = -3

And so on…

Sequence Method (cont)

3. Continue until you have found a number of points equal to the length of x + the length of h minus 1

n = 6:

x[m] = [-1, 0, 1, 2]

h[6-m] = [0, 0, 0, 0, 1, 3, 2]

can be omitted
(for a metaphor to
the graphical method)

y[6] = 2 * 0 = 0

Stop after this multiplication!

Length of output sequence is **always** the length of x + the length of h minus 1!

Since the sequences are zero-indexed, the last n that gives a non-zero output value should be len(x) + len(h) - 2

Matrix Method

Turn one of your signals to a column vector, the other to a matrix. Multiply -> Profit

x[n] = [-1, 0, 1, 2]
h[n] = [2, 3, 1, 0]

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 7 \\ 7 \\ 2 \end{bmatrix}$$

Impulse response and causality

Suppose we have an LTI system with impulse response $h[n]$.

- The system is causal (the system is LTIC) implies: $h[n] = 0$ for $n < 0$.
  - Signals for which $x[n] = 0$ for $n < 0$ are called "causal signals" for this reason.
- The system is non-causal implies $h[n] \neq 0$ for $n < 0$

Z-transforms

$$y[z] = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$$

- Typically perform inverse z-transform by inspection or by partial fraction decomposition
- Important properties:
  - Linearity
  - Multiplication
  - Delay property
  - Use properties whenever you can to go fast on the test!
- A Z-transform is uniquely determined by TWO things:
  - Expression for transform, F(z)
  - Region of Convergence, range of |z|
    - Continuous region
    - Does not include poles
- Convolution theorem: $x[n] * h[n] \leftrightarrow$ X(z)H(z)

Z-transforms: Example

$$x[n] = 5^n u[n-2] + (0.2)^n \delta[n+1]$$

$$x[n] = 5^2(5^{n-2})u[n-2] + (0.2)^n \delta[n+1]$$

$$Z \downarrow$$

$$X(z) = \frac{5^2 z^{-2}}{1-5z^{-1}} + 5z$$

Convolution Theorem in a Picture

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI Sys.} \\ H\end{matrix}} \longrightarrow y[n] = x[n] * h[n]$$

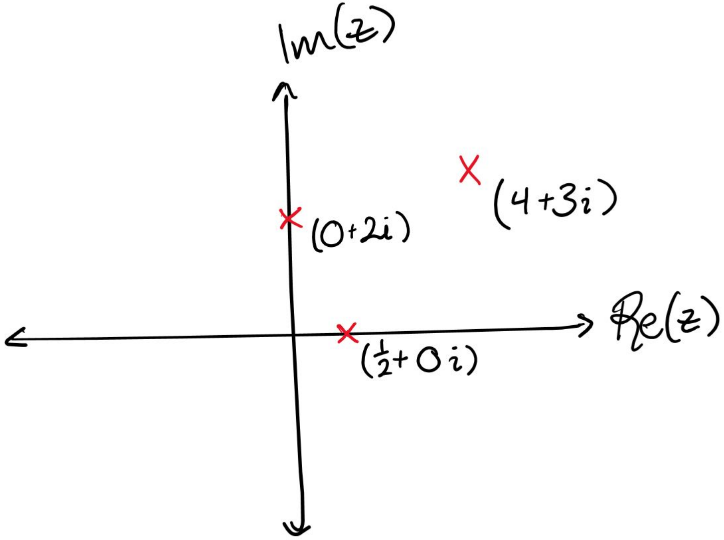
$$\Big\Updownarrow \; \mathcal{Z}$$

$$X(z) \longrightarrow \boxed{\begin{matrix}\text{LTI Sys.} \\ H\end{matrix}} \longrightarrow Y(z) = X(z)H(z)$$

Concept check: Possible ROCs

Below is a pole plot of z-domain function $F(z)$.

(a) What are all possible ROC's?



ROCs and Sequences – General Patterns

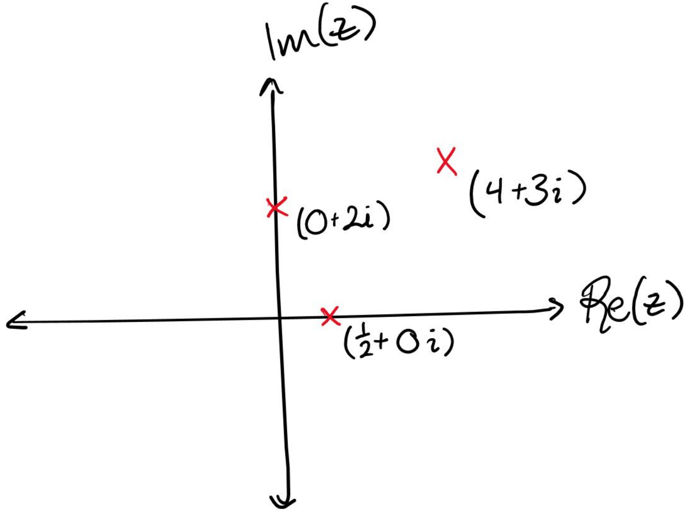
Suppose we have $x[n] \leftrightarrow X(z)$.

- Right sided sequence ($x[n] = 0$ for $n < k$ for some k):
  - Example: $x[n] = u[n-4]3^{n-4}$
  - ROC: $|z| > r$ for some r
  - Hint: Causal impulse responses fall under this category.
- Left sided sequence ($x[n] = 0$ for $n > k$ for some k):
  - Example: $x[n] = u[-(n+1)]4^{n+1}$
  - ROC: $|z| < r$
  - Hint: Anticausal impulse responses fall under this category.
- Finite sequence (both right-sided and left-sided)
  - ROC: all $|z|$ except potentially $|z|=0$ and $|z| = \text{infinity}$
- Infinite sequence in both directions (neither left-sided nor right-sided)
  - ROC: $a < |z| < b$ for some a, b

Concept check: Sequences and ROCs

Below is the same pole plot of z-domain function $F(z)$.

(a) Which ROC's result in a double sided sequence? Left sided? Right sided?



Example: Z-transform to sequence

Given

$$H(z) = \frac{z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$

and that $H(z)$ represents the transfer function for a causal system, find the impulse response.

Relationship between LTI systems and Transfer Function

A system H is LTI if and only if...

1. Its transfer function H(z), the z-transform of the impulse response, exists.
2. H(z) is equal to the ratio of the z-transform of the output to the z-transform of the input.

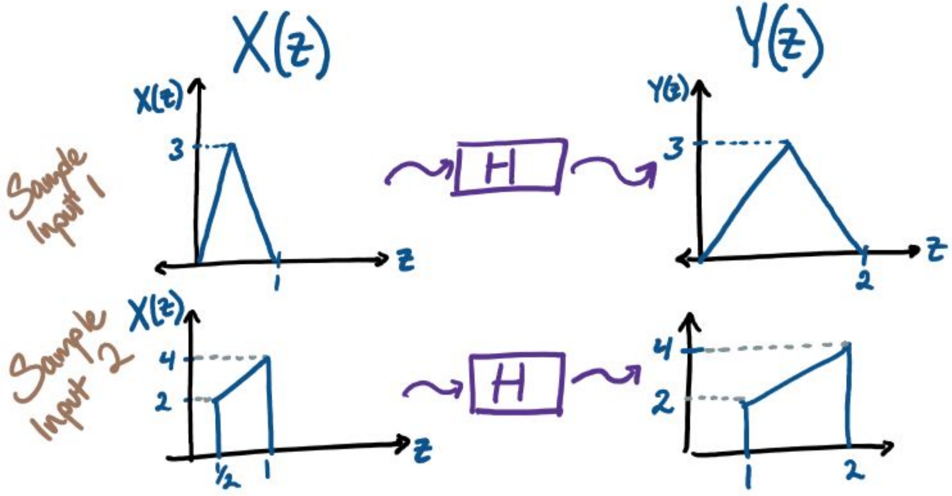
$$H(z) = \frac{Y_a(z)}{X_a(z)} = \frac{Y_b(z)}{X_b(z)} =$$

- This ratio is the same for every input/output pair.

Checking for LTI in the Z-domain

We are given a system H. We send two different inputs, $x_1[n] \leftrightarrow X_1(z)$and $x_2[n] \leftrightarrow X_2(z)$ and obtain outputs $y_1[n]$and $y_2[n]$. Is it possible that this system is LTI, or is it impossible?

$$Y(z) = X\left(\frac{z}{2}\right)$$



[a] Possibly LTI

[b] Absolutely not LTI

Conceptual Check

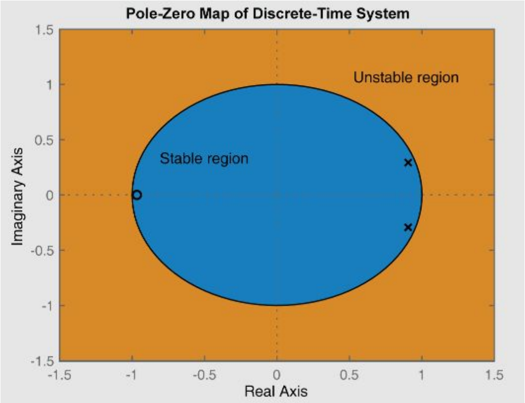
Suppose we know a system is LTI. Suppose that some finite-length input $x_a[n] \leftrightarrow X_a(z)$ has output $y_a[n] \leftrightarrow Y_a(z)$.

- True/False: If we know that for some z_0, $Y_a(z_0) = 0$, but $X_a(z_0) \neq 0$, then we know that $Y_{\text{any input}}(z_0) = 0$.

- True/False: Consider some value z_0. Knowing $X_a(z_0) = 0$ does not give us information about the value of $Y_a(z_0)$.

Marginal Stability

- A LTIC system is **marginally stable** if none of its poles have a radius greater than 1, and if it has one or more distinct poles with radius 1.
- For a system in general, it is marginally stable if the ROC borders |z| = 1 but doesn't contain it.
- In this class, marginally stable = **unstable**.



Source: https://in.mathworks.com/help/ident/ref/lti.pzmap.html

Stability and Causality of LTI systems

Given an LSI system with impulse response $h[n] \leftrightarrow H(z)$:

- H is BIBO stable if and only if the ROC of the z-transform $H(z)$ includes the unit circle.
- H is causal implies that the ROC of $H(z)$ has form |z| > r for some r.

BIBO Stability of LTI systems

- Checking for BIBO stability of LTI systems

1. Absolute summability of impulse response
    a. i.e. $\sum_{n=-\infty}^{+\infty}|h[n]| < \infty$
2. Location of poles and ROC of the transfer function
    a. ROC must include unit circle (poles + causality)
3. System definition (proof by counterexample)
    a. Find a bounded input that produces an unbounded output

Example

$$H(z) = \frac{1 - 2z^{-2}}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

- Find the LCCDE for the system.
- Is the system BIBO stable?

A semi-mechanical way to find BIBO unstable pairs

1 Finding Bounded output and unbounded output pairs

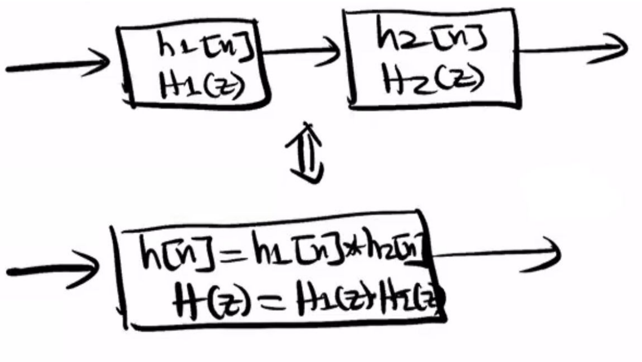
1. Rewrite the denominator in terms of $(1 - kz^{-1})$ factors exclusively
2. If $\exists k$ where $|k| > 1$, then $\delta[n]$ makes it unbounded
3. If $\exists k$ where $k = 1$, then $u[n]$ makes it unbounded
4. If $\exists k$ where $|k| = 1$ but $k \neq 1$, then $k = e^{j\omega n}$. Then $\cos(\omega n)u[n]$ makes it unbounded

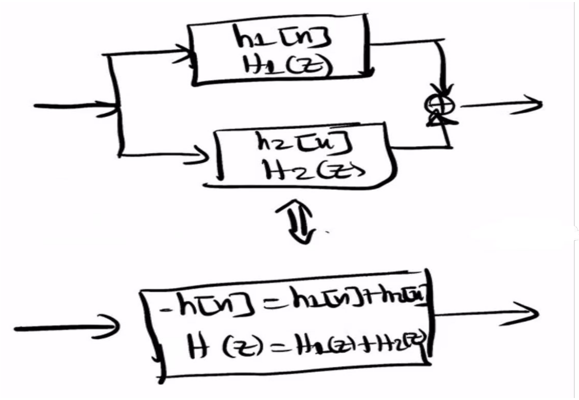
Conceptual Question

- If a system is BIBO stable then it must be causal. T/F

Multiple Systems

- We can combine systems in **serial** or **parallel** connections.





DTFT

- Given a discrete-time signal, we can find the DTFT using the following:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$X(\omega) = X(z)|_{z=e^{j\omega}}$ if X(z)'s ROC includes the unit circle.

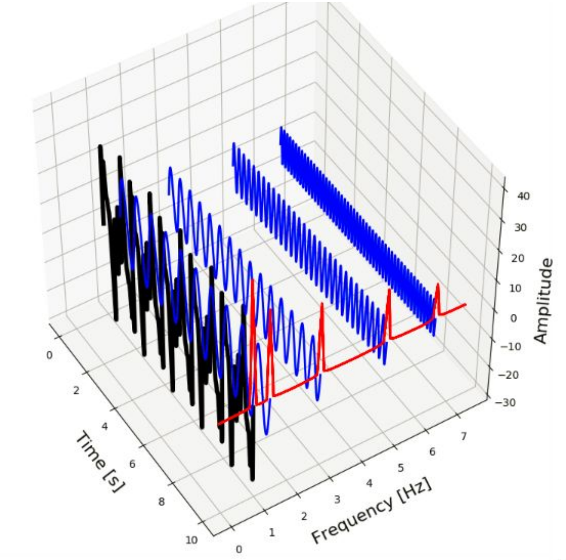
(i.e. can be thought of a slice of the z-transform of x[n] around the unit circle)

- Periodic w/ period $T = 2\pi$

DTFT

What does the DTFT tell us?

- Frequency content of a discrete signal
- Magnitude and phase response of a system

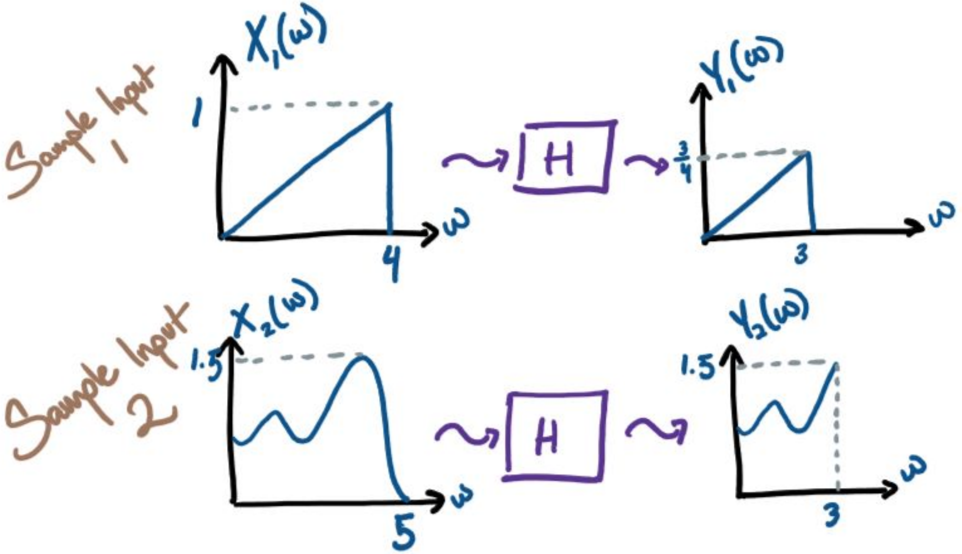


Conceptual Question

- The DTFT of a sequence always exists as long as its z-transform exists. T/F
- If the z-transform of the function does not include $|z|=1$, then the DTFT of the sequence does not exist. T/F

Concept Check: LTI and DTFT

- Once again, we are given a system H. We feed $x_1[n] \leftrightarrow X_1(\omega)$, and we get $y_1[n] \leftrightarrow Y_1(\omega)$. We do this with input $x_2[n] \leftrightarrow X_2(\omega)$, , and we obtain $y_2[n] \leftrightarrow Y_2(\omega)$.



[a] Possibly LTI

[b] Absolutely not LTI

Finding DTFT

- What is the DTFT of the following systems?
  - $x_1[n] = \ (\frac{1}{2})^n u[-n]$
  - $x_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$
  - $x_3[n] = \sum_{m=-\infty}^{+\infty} (\frac{1}{2})^{n-m+1} \text{sinc}\left(\frac{m}{2}\right) u[n-m]$

DTFT Question

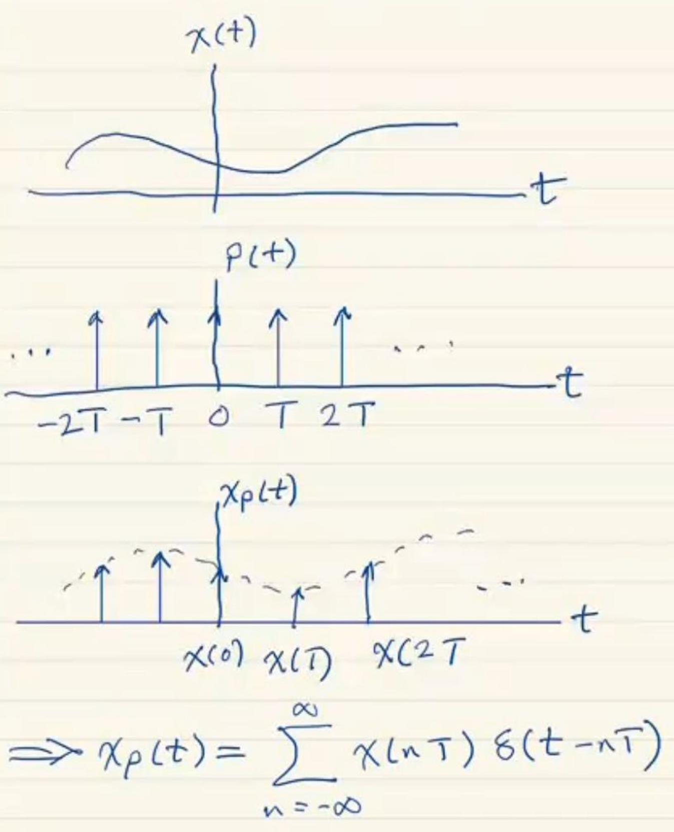
$$\{x[n]\}_{n=-1}^{2} = \{1, j, 0, -j\}$$

$$X_d(0) = ?$$

$$X_d(\pi) = ?$$

Sampling

- We can multiply a function by an impulse train to **sample** it.



Bonus: ECE 310 MT1 any speedrun tactics

- Time is a concern
- PFE: change of variable / direct PFE
- Polynomial division: Google "synthetic division"
  - Useful if the denominator is factorable into single-root terms
- Tables: Use the properties and table pairs whenever you can!
  - Z-transform pairs/properties, DTFT pairs/properties: learn them by heart, or find a place on your reference sheet to put them.

Thank you!

Good luck on Midterm 1!

Let us know if we were helpful: