共轭

一、complex number

rectangular form: $x=a+bj$ (complex conjugate $x^*=a-bj$)

polar form: $x=Re^{j\theta} \Rightarrow xy=RSe^{j(\theta+\phi)}$; $\frac{x}{y}=\frac{R}{S}e^{j(\theta-\phi)}$

magnitude: $|x|=\sqrt{a^2+b^2}=R$

phase: $\angle x=\theta=\begin{cases}\arctan(\frac{b}{a}), & a\geq 0\\ \arctan(\frac{b}{a})+\pi, & a<0\end{cases}$

$=\begin{cases} j, & a=0, b>0 \\ -j, & a=0, b<0\end{cases} = \begin{cases}\arctan(\frac{b}{a}), & R>0\\ \arctan(\frac{b}{a})+\pi, & R<0\end{cases}$

Euler's identity:

$x=R(\cos\theta+j\sin\theta)$

$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{2j}$

common discrete signal representation

Kronecker delta/unit impulse: $\delta[n]=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$

unit step: $u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$. Sinusoids: $x[n]=A\sin(w_0n+\theta)$

exponentials: $x[n]=\beta a^n$

二、discrete-time systems, T是映射

$\Rightarrow x[n]\xrightarrow{T} y[n]$ 或 $y[n]=T(x[n])$

$x[n]=\{0,1,2\}$

$x[n]=\{0,\underset{\uparrow}{1},2\}$

$x[n]=\{0,\underline{1},2\}$

① linear: $T(ax_1[n]+bx_2[n])=aT(x_1[n])+bT(x_2[n])$

② time-invariant: $y[n-n_0]=T(x[n-n_0])$

③ causal: system output not depends on future samples

④ BIBO stable: for any $x[n]$ that $|x[n]|<\beta$ 对 $\forall n$ 成立, 有 $|T(x[n])|<\alpha$ 输入有界,输出有界 | 或 $H(z)$ 之收敛域包含 unit circle

或 $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$

前提 LTI 系统

1、impulse response: $h[n]=T(\delta[n])$

$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$

ROC shape of $H(z)$	Causal	Right-sided	left-sided	Condition for BIBO
$\lvert z\rvert>P_{max}$	✓	✓	×	$P_{max}<1$
$\lvert z\rvert<P_{min}$	×	×	✓	$P_{min}>1$
$P_{max}<\lvert z\rvert<\infty$	×	✓	×	$P_{max}<1$
$a<\lvert z\rvert<b$	×	✓	✓	$\begin{cases}P_{max}=a<1 \text{ for } h_r[n]\\ P_{min}=b>1 \text{ for } h_l[n]\end{cases}$

	impulse response $h[n]$	Transfer function $H(z)$
Parallel	$h_1[n]+h_2[n]$	$H_1(z)+H_2(z)$
series	$h_1[n]*h_2[n]$	$H_1(z)H_2(z)$

$\text{rect}(t)=\begin{cases}1, & |t|<\frac{1}{2}\\ 0, & |t|>\frac{1}{2}\end{cases}$

$\delta(a(x-x_0))$

$=\frac{1}{|a|}\delta(x-x_0)$

$\Delta(t)=(1-|t|), |t|\leq 1$

2、convolution:

$x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$

$\begin{cases} h_1[n]*h_2[n]=h_2[n]*h_1[n] \\ (x[n]*h_1[n])*h_2[n]=x[n]*(h_1[n]*h_2[n]) \\ x[n]*\delta[n]=x[n] \\ x[n]*(h_1[n]+h_2[n])=x[n]*h_1[n]+x[n]*h_2[n]\end{cases}$

if $x[n]$ starts at n_s, ends at n_e

$h[n]$ starts at m_s, ends at m_e

then $y[n]=x[n]*h[n]$ sarts at n_s+m_s, ends at n_e+m_e

3、LCCDE: linear constant-coefficient difference equations

$y[n]=\sum_{i=1}^{K}b_iy[n-i]+\sum_{j=0}^{M-1}c_jx[n-j]$

① $K>0$: IIR: infinite impulse response

② $K=0$: FIR: finite impulse response

4、Block diagrams

① delay block: $x[n-k]\rightarrow\boxed{z^{-1}}\rightarrow x[n-k-1]$

② coefficient/gain block: $x[n]\rightarrow \overset{c}{\triangleright} \rightarrow cx[n]$

③ adder block: $x[n]\rightarrow \oplus \rightarrow x[n]+z[n]$ ($z[n]$)

Parseval's Relation:

$\int_{-\infty}^{\infty}|x(t)|^2dt$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(jw)|^2dw$

5、找 bounded input 和 unbounded output pairs

① rewrite 分母 in terms of $1-kz^{-1}$ factors

② if $\exists k$ where $|k|>1$, then $\delta[n]$ makes it unbounded

if $\exists k$ where $k=1$, then $u[n]$ makes it unbounded

if $\exists k$ where $|k|=1$ but $k\neq 1$, then $k=e^{jwn}$ then $\cos(wn)u[n]$ makes it unbounded

三、z-transform:

$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$, $x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$

pairs:

$\delta[n]\xmapsto{z}1$, ROC: all z, $u[n]\xmapsto{z}\frac{1}{1-z^{-1}}$, ROC: $|z|>1$

$a^nu[n]\xmapsto{z}\frac{1}{1-az^{-1}}$, ROC: $|z|>|a|$ $-a^nu[-(n+1)]\xmapsto{z}\frac{1}{1-az^{-1}}$, ROC: $|z|<|a|$

$na^nu[n]\xmapsto{z}\frac{az^{-1}}{(1-az^{-1})^2}$, ROC: $|z|>|a|$ $-na^nu[-(n+1)]\xmapsto{z}\frac{az^{-1}}{(1-az^{-1})^2}$, $|z|<|a|$

$\cos(w_0n)u[n]\xmapsto{z}\frac{1-\cos(w_0)\cdot z^{-1}}{1-2\cos(w_0)\cdot z^{-1}+z^{-2}}$ ROC: $|z|>1$

$\sin(w_0n)u[n]\xmapsto{z}\frac{\sin(w_0)\cdot z^{-1}}{1-2\cos(w_0)\cdot z^{-1}+z^{-2}}$ ROC: $|z|>1$

$a^n\cos(w_0n)u[n]\xmapsto{z}\frac{1-a\cos(w_0)z^{-1}}{1-2a\cos(w_0)z^{-1}+a^2z^{-2}}$, ROC: $|z|>|a|$

$a^n\sin(w_0n)u[n]\xmapsto{z}\frac{a\sin(w_0)\cdot z^{-1}}{1-2a\cos(w_0)z^{-1}+a^2z^{-2}}$, ROC: $|z|>|a|$

$X(z)=0$: zeros $X(z)\rightarrow\infty$ poles

property:

$x[n-k]\xmapsto{z}z^{-k}X(z)$ ROC: R_x except $z=\infty$ or $z=0$

$ax_1[n]+bx_2[n]\xmapsto{z}aX_1(z)+bX_2(z)$ ROC: at least $R_{x_1}\cap R_{x_2}$

$x_1[n]*x_2[n]\xmapsto{z}X_1(z)X_2(z)$ ROC: at least $R_{x_1}\cap R_{x_2}$

$nx[n]\xmapsto{z}-z\frac{dX(z)}{dz}$ ROC: R_x

$x^*[n]\xmapsto{z}X^*(z^*)$ ROC: R_x

$x[-n]\xmapsto{z}X(z^{-1})$ ROC: $\frac{1}{R_x}$

$a^nx[n]\xmapsto{z}X(\frac{z}{a})$ ROC: $|a|R_x$

$Re\{x[n]\}\xmapsto{z}\frac{1}{2}[X(z)+X^*(z^*)]$ ROC: at least R_x

$Im\{x[n]\}\xmapsto{z}\frac{1}{2j}[X(z)-X^*(z^*)]$ ROC: at lesast R_x

6. LCCDEs: $y[n]+\sum_{k=1}^{N}a_ky[n-k]=\sum_{k=0}^{M}b_kx[n-k]$ N个pole M个zeros

有feedback项吗 —No→ FIR: finite-length impulse response

↓Yes

有at least one pole没有与zero抵消吗 —No→ FIR; —Yes→ IIR: infinite-length impulse response

e.g. $H(z)=1+z^{-1}+z^{-2}$, 分子幂绝对值大, FIR

分母 z 的幂的绝对值比分子大, 就是 IIR $H(z)=\frac{1}{1-az^{-1}}$

7. marginal stability 边缘稳定: ROC $|z|>1$ 或 $|z|<1$

四、1、CTFT: Continunous time Fourier transform:

$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\Omega)e^{j\Omega t}d\Omega$

$X_c(\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt$

Pairs:

$\sum_{k=-\infty}^{\infty}a_ke^{jkw_0t}\leftrightarrow 2\pi\sum_{k=-\infty}^{\infty}a_k\delta(w-kw_0)$

$e^{jw_0t}\leftrightarrow 2\pi\delta(w-w_0)$

$x(t)=1\leftrightarrow 2\pi\delta(w)$

$\delta(t)\leftrightarrow 1$

$\delta(t-t_0)\leftrightarrow e^{-jwt_0}$

$x(t)\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}\leftrightarrow\frac{2\sin(wT_1)}{w}$

$e^{-at}u(t), Re\{a\}>0\leftrightarrow\frac{1}{a+jw}$

$\cos(w_0t)\leftrightarrow\pi[\delta(w-w_0)+\delta(w+w_0)]$

$\sin(w_0t)\leftrightarrow\frac{\pi}{j}[\delta(w-w_0)-\delta(w+w_0)]$

$u(t)\leftrightarrow\frac{1}{jw}+\pi\delta(w)$

$\sum_{n=-\infty}^{\infty}\delta(t-nT)\leftrightarrow\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(w-\frac{2\pi k}{T})$

$\frac{\sin wt}{\pi t}\leftrightarrow X(jw)=\begin{cases}1, & |w|<W\\ 0, & |w|>W\end{cases}$

$te^{-at}u(t), Re\{a\}>0\leftrightarrow\frac{1}{(a+jw)^2}$

if $x(t)$ or $x[n]$ real $\Leftrightarrow X(jw)=X^*(-jw)$

Property:

$x(t-t_0)\leftrightarrow e^{-jwt_0}X(jw)$

$x^*(t)\leftrightarrow X^*(-jw)$

$x(-t)\leftrightarrow X(-jw)$

$x(at)\leftrightarrow\frac{1}{|a|}X(\frac{jw}{a})$

$\int_{-\infty}^{t}x(t)dt\leftrightarrow\frac{1}{jw}X(jw)+\pi X(0)\delta(w)$

$e^{jw_0t}x(t)\leftrightarrow X(j(w-w_0))$

$x(t)*y(t)\leftrightarrow X(jw)Y(jw)$

$x(t)y(t)\leftrightarrow\frac{1}{2\pi}X(jw)*Y(jw)$

$\frac{dx(t)}{dt}\leftrightarrow jwX(jw)$

$tx(t)\leftrightarrow j\frac{d}{dw}X(jw)$

$x(t)$ is real $\begin{cases}X(jw)=X^*(-jw)\\ Re\{X(jw)\}=Re\{X(-jw)\}\\ Im\{X(jw)\}=-Im\{X(-jw)\}\\ |X(jw)|=|X(-jw)|\\ \angle X(jw)=-\angle X(-jw)\end{cases}$

$x(t)$ real even $\Leftrightarrow X(jw)$ real even

$x(t)$ real odd $\Leftrightarrow X(jw)$ purely imaginary, odd

2. DTFT: has 2π period

$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_d(w)e^{jwn}dw$, $X_d(w)=\sum_{n=-\infty}^{\infty}x[n]e^{-jwn}$

Pairs:

$\delta[n]\leftrightarrow 1$ | $u[n]\leftrightarrow\frac{1}{1-e^{-jw}}+\pi\delta(w)$

$a^nu[n]\leftrightarrow\frac{1}{1-ae^{-jw}}, 0<|a|<1$ | $e^{jw_0n}\leftrightarrow 2\pi\delta(w-w_0)$

$\cos(w_0n)\leftrightarrow\pi[\delta(w-w_0)+\delta(w+w_0)]$ | $\sin(w_0n)\leftrightarrow -j\pi[\delta(w-w_0)-\delta(w+w_0)]$

$rect(\frac{n-k}{L})\leftrightarrow\frac{\sin(\frac{Lw}{2})}{\sin(\frac{w}{2})}e^{-jwk}$ | $sinc(Ln)\leftrightarrow\frac{\pi}{L}rect(\frac{w}{2L})$

$sinc^2(Ln)\leftrightarrow\frac{\pi}{L}\Delta(\frac{w}{2L})$ | $1\leftrightarrow 2\pi\delta(w)$

$(n+1)a^nu[n], |a|<1\leftrightarrow\frac{1}{(1-ae^{-jw})^2}$

Property:

只有BIBO stable才满足 $H_d(e^{jw})=H(z)|_{z=e^{jw}}$, 否则只能用DTFT pair得 $H_d(w)$

$X_d(w)=X(z)|_{z=e^{jw}}$ if ROC_x includes unit circle

if $\{x[n]\}$ is real-valued, then $X_d(-w)=X_d^*(w)$ ~ hermitian property

$|X_d(-w)|=|X_d(w)|$, $\angle X_d(-w)=-\angle X_d(w)$

if $y[n]=x[n]w[n]$, then $Y_d(w)=(X_d * W_d)(w)$
$=\frac{1}{2\pi}\int_{2\pi}X_d(\theta)W_d(w-\theta)d\theta$

Parseval's relation: $\sum_n|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X_d(w)|^2dw$

3. Sinusoidal response of LSI system: for fixed w_0

$\{e^{jw_0n}\}_n\rightarrow\boxed{H_d(w)}\rightarrow\{H_d(w_0)e^{jw_0n}\}_n$

if $\{h[n]\}$ is real-valued then

$\{\cos(w_0n+\phi)\}_n\rightarrow\boxed{H_d(w)}\rightarrow\{|H_d(w_0)|\cos(w_0n+\phi+\angle H_d(w_0))\}_n$

2. DTFT property cont.:

$x[n-n_0]\leftrightarrow e^{-jwn_0}X(e^{jw})$ | $e^{jw_0n}x[n]\leftrightarrow X(e^{j(w-w_0)})$

$x^*[n]\leftrightarrow X^*(e^{-jw})$ | $x[-n]\leftrightarrow X(e^{-jw})$

$x_{(k)}[n]=\begin{cases}x[\frac{n}{k}], & \text{if } n=mk, m\in\mathbb{Z}\\ 0, & \text{if } n\neq mk\end{cases}$

$x[n]*y[n]\leftrightarrow X(e^{jw})Y(e^{jw})$

$x[n]y[n]\leftrightarrow\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(w-\theta)})d\theta$

$nx[n]\leftrightarrow j\frac{dX(e^{jw})}{dw}$

3. Frequency response cont.:

impulse response $h[n]\overset{Z}{\leftrightarrow}H(z)$ transfer function
$h[n]\overset{F}{\leftrightarrow}H(w)$ frequency response

① complex exponential signals ⇒ eigenfunctions of LTI systems

$y[n]=x[n]*h[n]$, $x[n]=Az^n$, $A,z\in\mathbb{C}\Rightarrow y[n]=H(z)\cdot A\cdot z^n$

$y[n]=x[n]*h[n]$, $x[n]=Ae^{jw_0n}$, $A\in\mathbb{C}, w_0\in\mathbb{R}\Rightarrow y[n]=H(w_0)\cdot Ae^{jw_0n}$

② 问一个system是否real-valued: if $H(-w)=H^*(w)\Rightarrow$ real value
实际在问h[n]是不是real-valued

③ magnitude response: $|H(w)|=\sqrt{H(w)\cdot H^*(w)}$

decibel scale: $|H(w)|_{dB}=20\log_{10}|H(w)|$

phase response: $\angle H(w)=\tan^{-1}(\frac{Im\{H(w)\}}{Re\{H(w)\}})$

4. group delay: $\tau_{gd}=-\frac{dH(w)}{dw}$

uniform group delay: τ_{gd}不变 ⇒ 经过~system后图片在x,y轴上 slightly shifted

non-uniform group delay: τ_{gd}会变

5. ideal sampling:

① A/D: $x[n]=x(nT), -\infty<n<+\infty$

sample frequency: $f_s=\frac{1}{T}$

$X_d(w)=\frac{1}{T}\sum_{n=-\infty}^{\infty}X_a(\frac{w-2n\pi}{T})$, $w=\Omega T$

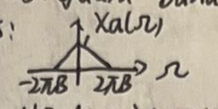
② bandlimited signal: $x(t)$ with CTFT $X_a(\Omega)$: $X_a(\Omega)=0$ for $|\Omega|>2\pi B$, B为the max linear frequency present in a bandlimited signal

Nyquist criterion: $f_s>2B$ Nyquist rate: $f_{nyquist}=2B$

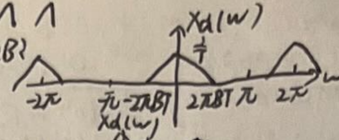
$f<2B$: aliasing

$f>2B$: guard band

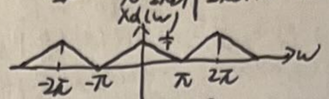
原本:



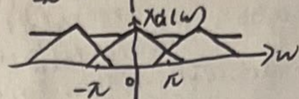
f>2B:



at the Nyquist rate:



below the Nyquist rate:



③ recover bandlimit signal $x(t)$ from digital $x[n]$

process → interpolation D/A

$y(t)=\sum_{n\in\mathbb{Z}}y[n]p(t-nT)$ $Y_c(\Omega)=Y_d(\Omega T)P_c(\Omega)$

For ideal D/A

$y(t)=\sum_{n=-\infty}^{\infty}y[n]sinc(\frac{\pi(t-nT)}{T})$, $P_c(\Omega)=\begin{cases}T, & \text{if } |\Omega|<\frac{\pi}{T}\\ 0, & \text{else}\end{cases}$

For ZOH D/A

$p(t)=\begin{cases}1, & 0\le t<T\\ 0, & \text{else}\end{cases}$ $P_c(\Omega)=\frac{e^{-j\frac{\Omega T}{2}}(2\sin(\frac{\Omega T}{2}))}{\Omega}$

ideal reconstruction filter is noncausal as its time domain has form sinc

6. DFT: discrete ~~time~~ fourier transform

$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi kn}{N}}$, $X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi kn}{N}}$

$X[k]=X_d(w)|_{w=\frac{2\pi k}{N}}$, $W_N=e^{-j\frac{2\pi}{N}}$, $W_N^{kn}=e^{-j\frac{2\pi kn}{N}}$

circular shift: for a fixed m,

if $y[n]=x[\langle n-m\rangle_N]$, then we have

$Y[k]=X[k]e^{-j\frac{2\pi km}{N}}=X[k]W_N^{kn}$

if $y[n]=x[\langle -n\rangle_N]$, then we have

$Y[k]=X[-k]$

if $y[n]=z[n]W_N^{-mn}$, then we have

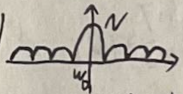
$Y[k]=Z[\langle k-m\rangle_N]$

For $x[n]=A\cos(w_0n)$, $0\le n\le N-1$

DTFT: $X(w)=\frac{A}{2}C(w-w_0)+\frac{A}{2}C(w+w_0)$

where $C(w)=e^{-\frac{jw}{2}(N-1)}(\frac{\sin(\frac{N}{2}w)}{\sin(\frac{1}{2}w)})$

$|C(w-w_0)|$



peak: $\frac{A}{2}N$

zeros: $w-w_0=\frac{2\pi}{N}\cdot k$, $k\neq 0$

circular modulation:

$x[n]\cos((\frac{2\pi}{N})k_0n)\leftrightarrow\frac{1}{2}X[\langle k+k_0\rangle_N]+\frac{1}{2}X[\langle k-k_0\rangle_N]$

if the signal is symmetric, then its DFT's magnitude has even symmetric, its DFT's phase has odd symmetric.

DFT of DFT is original signal multiplied by N

zero-padding can improve the resolution

circular convolution:

$x[n]\circledast h[n]=\sum_{m=0}^{N-1}x[m]h[\langle n-m\rangle_N]=\sum_{m=0}^{N-1}h[m]x[\langle n-m\rangle_N]$

$y[n]=x[n]\circledast h[n]\overset{DFT}{\longleftrightarrow}Y[k]=X[k]H[k]$

circular convolution 只适用于两个长度一样的signal 使用,不一样长的在短的signal后补0

To guarantee that linear and circular convolutions coincide (或: $\{x_n\}*\{h[n]\}=DFT^{-1}\{DFT\{x[n]\}\cdot DFT\{h[n]\}\}$): 只用zero-padding 到 $N=N_1+N_2-1$

7. Spectral analysis：在1个finite-length离散信号可拆成多个sin()

① the frequency of components may be too ~~●~~ close to separate ⇒ 用更大的 N, zero-padding; full-lobe separation: $w_2-w_1>\frac{4\pi}{N}$

half-lobe separation: $w_2-w_1>\frac{2\pi}{N}$

[eg]. 用8kHz采样, $F_1=82Hz$, $F_2=87Hz$

$w_1=\Omega_1 T=\frac{2\pi\cdot 82}{8000}$, $w_2=\Omega_2 T=\frac{2\pi\cdot 87}{8000}$

用full-lobe separation: $w_2-w_1>\frac{4\pi}{N}$ 解 N

② the amplitudes of components may vary greatly

[eg]. $s_1[n]=\cos(0.5\pi n)+\frac{1}{10}\cos(0.625\pi n)$, $0\le n<63$

$\Rightarrow s_1[n]=(u[n]-u[n-64])\cdot(\cos(0.5\pi n)+\frac{1}{10}\cos(0.625\pi n))$, $-\infty<n<\infty$

用window相乘 $x[n]w[n]\overset{F}{\longleftrightarrow}\frac{1}{2\pi}X(w)*W(w)$

又 $\cos(w_0 n)\overset{F}{\longleftrightarrow}\pi(\delta(w-w_0)+\delta(w+w_0))\rightarrow S_1(w)$ 会变成window function的平移和缩放的组合

rectangular window: $r[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{else}\end{cases}$

hamming window: $h[n]=\begin{cases}\frac{25}{46}-\frac{21}{46}\cos(\frac{2\pi n}{N-1}) & 0\le n\le N-1\\ 0 & \text{else}\end{cases}$

better side lobe attenuation

→ a lowpass FIR filter design the Hamming window have a wider transition band and smaller passband ripples

实际上,所有window function的tradeoff: the main lobe width v.s. side lobe attenuation (main lobe窄, side lobe就高)

8. FFT: Fast fourier transform

① 直接计算DFT: $X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\cdot\frac{2\pi k}{N}n}$, $0\le k\le N-1$

$O(N^2)$: 求k时一次,对于每个k都要求一次

② decimation-in-time (DIT) FFT

twiddle factor $W=e^{-j\cdot\frac{2\pi}{N}}$; $W_N^{kn}=e^{-j\cdot\frac{2\pi k}{N}n}$

③ FFT: 只对circular convolution (即DFT起作用) → 考虑linear convolution, 让其与circular convolution等效 ← zero padding

④ algorithm: $x[n]$: length-N; $h[n]$: length-L; $y[n]=x[n]*h[n]$

zero-padding: 让 $x[n]$, $h[n]$ 都为 length: $N+L-1$ $O(1)$

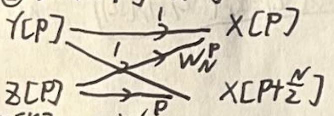
用FFT算 $x[n]$, $h[n]$ 的DFT, $X_{zp}[k]$, $H_{zp}[k]$ $O(M\log N)$

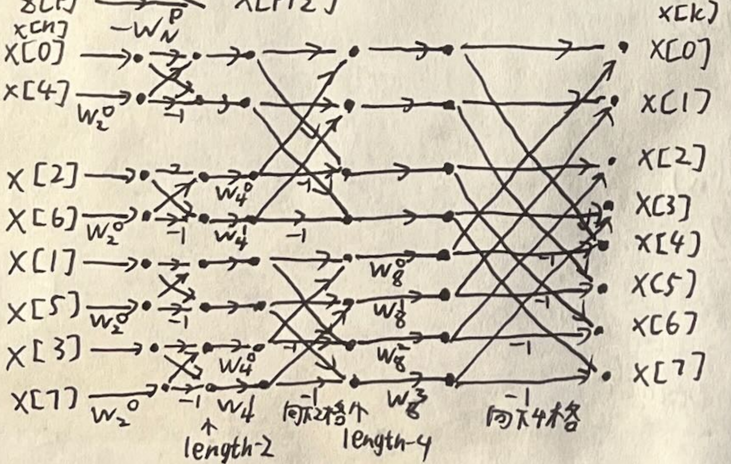
$Y[k]=X_{zp}[k]\cdot H_{zp}[k]$ 对应项相乘 $O(N)$

用FFT算inverse DFT of $Y[k]\rightarrow y[n]$ $O(N\cdot\log N)$

随着 N 越来越大, fast convolution algorithm更有优势

⑤ butterfly diagram (time version)



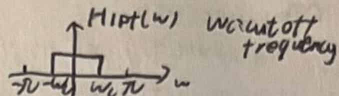


五、filter design → LTI frequency-selective filter

1. 几种基础filter (ideal)

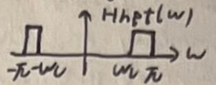
① low-pass filter:

$H_{lpf}(w)=\begin{cases}1 & |w|\le w_c\\ 0 & w_c<|w|\le\pi\end{cases}$



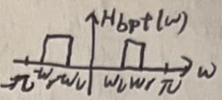
② high-pass filter:

$H_{hpf}(w)=\begin{cases}0 & |w|\le w_c\\ 1 & w_c<|w|\le\pi\end{cases}$



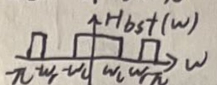
③ band-pass filter:

$H_{bpf}(w)=\begin{cases}1 & w_L\le|w|<w_H\\ 0 & \text{else}\end{cases}$



④ band-stop filter:

$H_{bsf}(w)=\begin{cases}0 & w_L\le|w|<w_H\\ 1 & \text{else}\end{cases}$



2. issue with ideal filter

① impulse response is infinite long

② system is non-causal

③ system is not BIBO stable

3. digital filter 术语

① passband: 频率通过的范围, passband ripple: the amount of variation in the gain of a passband (ideal为0dB, 是与0dB的差), 可以用 δp 表示passband ripple (单位为dB) 且 dB: $20\log_{10}(|H(w)|)$

② stopband: 频率不通过 (attenuate / remove) 的范围, stopband attenuation: the highest gain achieved by a given stopband.

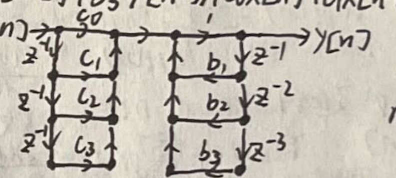
③ transition band: 在passband和stopband之间的频率范围, transition bandwidth: transition band的宽度

我们想要small passband ripple, strong stopband attenuation和narrow transition bandwidth

trade off: stopband attenuation和transition bandwidth against length or complexity of the digital filter.

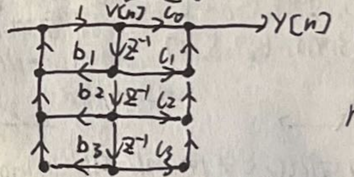
4. block diagram: 根据LCCDE来画: $y[n]=b_1y[n-1]+b_2y[n-2]+b_3y[n-3]+c_0x[n]+c_1x[n-1]+c_2x[n-2]+c_3x[n-3]$

①



addition: 6
multiplication: 7
register/delay: 6

②



addition: 6
multiplication: 7
register/delay: 3

5. linear-phase FIR filters:

① FIR filter: only input terms ($c_i x[n-i]$项)

通过选取filter前 N 项来逼近之前提到的issue

$h_{lpf}[n]=\frac{\sin(w_c n)}{\pi n}$, $0\le n\le N-1$, 再考虑group delay:

$\tau_g(w)=-\frac{d\angle H(w)}{dw}$

uniform group delay: 在time-domain上所有element shifted uniformly

linear phase response 对应 uniform group delay

为此,平移filter: $h_{lpf}[n]=\frac{\sin(w_c(n-\alpha))}{\pi(n-\alpha)}$, $0\le n\le N-1$, $\alpha=\frac{N-1}{2}$, 效果关于 α 对称, stopband attenuation更好.

② linear-phase filter types：根据even/odd symmetry和长度even/odd划分

I. even symmetry, odd length N：$h_1[n]=h_1[N-n-1]$

II. even symmetry, even length：

III. odd symmetry, odd length：$h_3[n]=-h_3[N-n-1]$

IV. odd symmetry, even length

symmetry	length	$H_d(0)$	$H_d(\pi)$	possible ideal filter type
even	odd	\	\	LP, HP, BP, BS
even	even	\	0	LP, BP
odd	odd	0	0	BP
odd	even	0	\	BP, HP

④ strict linear phase：$\angle H(w)$ 是与w一次线性关系，无截距，无jump

⑤ window method：用来approximate any ideal frequency response $D(w)$

I. take the inverse DTFT of $D(w)$ 得 ~~…~~ $d[n]$ (ideal，无限长)

II. shift $d[n]$ by $\alpha=\frac{N-1}{2}$，得 $g[n]=d[n-\alpha]$

III. ~~to~~ window function $w[n]$ 来得 $h[n]=g[n]\cdot w[n]$

tradeoff：transition bandwidth 和 stopband attenuation of FIR filter

window种类：

rectangular window：$w[n]=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{else}\end{cases}$ narrowest transition band, weakest stopband attenuation

hamming window：$w[n]=\begin{cases}\frac{25}{46}-\frac{21}{46}\cos\left(\frac{2\pi n}{N-1}\right), & 0\le n\le N-1\\ 0, & \text{else}\end{cases}$ widest transition band, best stopband attenuation

(triangle)

Bartlett window $w[n]=\begin{cases}\frac{2n}{N-1}, & 0\le n\le\frac{N-1}{2}\\ 2-\frac{2n}{N-1}, & \frac{N-1}{2}\le n\le N-1\\ 0 & \text{else}\end{cases}$

Hann window $w[n]=\begin{cases}0.5-0.5\cos\left(\frac{2\pi n}{N-1}\right) & 0\le n\le N-1\\ 0 & \text{else}\end{cases}$

Blackman window $w[n]=\begin{cases}0.42-0.5\cos\left(\frac{2\pi n}{N-1}\right)+0.08\left(\frac{4\pi n}{N-1}\right) & 0\le n\le N-1\\ 0 & \text{else}\end{cases}$

[eg]. lowpass filter：$D_{lpf}(w)=\begin{cases}1 & |w|\le w_c\\ 0 & |w|>w_c\end{cases}$

先 invert DTFT：$d_{lpf}[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_{lpf}(w)\,dw=\frac{\sin(w_c n)}{\pi n}$

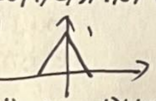
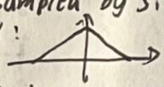
再平移并乘上window func：$h_{lpf}[n]=\frac{\sin(w_c(n-\alpha))}{\pi(n-\alpha)}w[n]$

5. Downsampling：又延伸 expands DTFT，可导致alisaing

① $x_D[n]=x[Dn]$ $\quad Y_d(w)=\frac{1}{D}\sum_{k=0}^{D-1}X_d\left(\frac{w-2\pi k}{D}\right)$

相当于坐标乘D，因为有 $k=0\sim k=D-1$，保障2π周期

signal：[0,1,2,3,4,5,6,7,8,9]，Downsampled by 3：[0,3,6,9]

original  Downsampled：

② anti-aliasing filter：先用anti-aliasing filter再downsampling

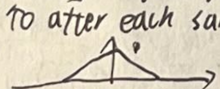
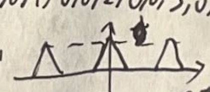
$x[n]\rightarrow$ [anti-alising filter $w_c=\frac{\pi}{D}$] $\rightarrow$ [downsampling $\downarrow D$] $\rightarrow y[n]$ $\quad H_D(w)=\begin{cases}1 & |w|\le\frac{\pi}{D}\\ 0 & |w|>\frac{\pi}{D}\end{cases}$

6. upsampling：压缩DTFT，需LPF多余copy

① $y[n]=\begin{cases}x\left[\frac{n}{U}\right], & n=\pm U,\pm 2U\\ 0, & \text{else}\end{cases}$ $\quad Y_d(w)=\sum_{n=-\infty}^{\infty}y[n]e^{-jwn}=\sum_{n=-\infty}^{\infty}x[n]e^{-jwnU}=X_d(Uw)$

signal：[0,1,2,3] Upsampled by 3：[0,0,0,1,0,0,2,0,0,3,0,0]

insert U-1 0 after each sample

original  Upsampled：

② interpolation filter：low-pass filter $H_u(w)=\begin{cases}U, & |w|\le\frac{\pi}{U}\\ 0, & |w|>\frac{\pi}{U}\end{cases}$

多个倍数U是为了ensure the original samples from $x[n]$ 不会被rescaled

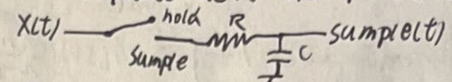
$x[n]\rightarrow$ [upsampling $\uparrow U$] $\rightarrow$ [interpolation filter $h_u[n]\leftrightarrow H_U(w)$ $w_c=\frac{\pi}{U}$] $\rightarrow z[n]$

③ rate conservation by a non-integer factor：cascading interpolator 和 decimator factor $R=\frac{U}{D}$：先upsample by factor of U；再用low-pass filter ($w_c=\min\{\frac{\pi}{U},\frac{\pi}{D}\}$)；最后downsample by factor of D.

$x[n]\rightarrow$ [upsampling $\uparrow U$] $\rightarrow$ [interpolation/anti-aliasing filter low-pass, $w_c=\min\{\frac{\pi}{U},\frac{\pi}{D}\}$ (相当于两filter乘在一起)] $\rightarrow$ [downsampling $\downarrow D$] $\rightarrow y[n]$

7. Practical sampling and reconstruction of analog signals：

① practical A/D conversion：先sampling：~~x_{sample}~~ $x_{sample}(t)=x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT)$，再convert成discrete time sequence $x[n]$ ⇒ I. sample and hold 用capacitor实现



quantization：round measurements to a pre-defined set of value

② pratical D/A conversion：

I. ideal D/A：$x(t)=\sum_{n=-\infty}^{\infty}x[n]\,\text{sinc}\left(\frac{\pi(t-nT)}{T}\right)$ ~~问题~~ ~~non-causa~~

问题non-causal, not stable

II. zero-order hold DAC：$x_r(t)=\sum_{n=-\infty}^{\infty}x[n]g(t-nT)$

$y_{ZOH}(t)=\begin{cases}1 & 0\le t\le T\\ 0 & \text{else}\end{cases}\rightarrow G_{ZOH}(\Omega)=T\cdot e^{-j\frac{\Omega T}{2}}\cdot\text{sinc}\left(\frac{\Omega T}{2}\right)$

问题：outer regions of the central copy are pushed down by $G_{ZOH}(\Omega)$

⇒ 用reconstruction (compensation) filter：$H_r(\Omega)$

$H_r(\Omega)=\begin{cases}\frac{1}{\text{sinc}\left(\frac{\Omega T}{2}\right)} & |\Omega|\le\frac{\pi}{T}\\ 0 & \text{else}\end{cases}$

bandwidth for $H_r(\Omega)$：$\frac{2(\pi-w)}{T}$

$x[n]\rightarrow$ [reassociate in time $x_{sample}(nT)=x[n]$] $\xrightarrow{X_s(\Omega)}$ [zero-order hold $X_r(\Omega)=X_s(\Omega)G_{ZOH}(\Omega)$] $\xrightarrow{X_r(\Omega)}$ [Reconstruction filter $X_a(\Omega)=X_r(\Omega)H_r(\Omega)$] $\xrightarrow{X_a(\Omega)} x(t)$