University of Illinois
ECE 310

Spring 2021
Profs. Moon, Katselis, and Shomorony

Midterm Exam 2

7:00-8:50 PM, Thursday, April 8, 2021

Please do not start the exam until the starting time.

- You may not use any books, electronic devices, or notes other than **one** handwritten two-sided sheet of 8.5" x 11" paper.
- You should solve the problems on blank sheets of paper, take pictures of your solutions, and upload them to Gradescope before the end of the exam time.

GOOD LUCK!

1. **(15 Pts.)** Answer **True** or **False** to each of the following statements:

   (a) By sampling a continuous-time signal $x_c(t) = \cos(17\pi t)$ with some sampling period T, it is possible to obtain a discrete time signal $x[n] = \cos(3\pi n/4)$. **True/False**

   (b) If the Nyquist sampling rate for a continuous-time signal $x_c(t)$ is F_s, then the Nyquist sampling rate for $y_c(t) = x_c(2t)$ is $2F_s$. **True/False**

   (c) If $\{x[n]\}_{n=0}^{5}$ is real-valued and $\{X[k]\}_{k=0}^{5}$ is its DFT, then $X[0]$ is real-valued. **True/False**

   (d) The value of $\int_3^\infty (t+1)\delta(t)dt$ is 0. **True/False**

   (e) Let $\{x[n]\}_{n=0}^{7} = \{1, -1, 6, 7, 9, -6, -7, 9\}$. Consider the corresponding 8-point DFT $\{X[k]\}_{k=0}^{7}$. Then, $X[0] = 0$. **True/False**

   Solution:

   (a) **True**,

   $$x[n] = x_c(nT) \implies T = \frac{3}{68}$$

   (b) **True**, Denoting max frequency in $x_c(t)$ as Ω_{max}, then the max frequency in $x_c(2t)$ will be $2\Omega_{max}$. Therefore, the Nyquist rate will be doubled for $y_c(t)$.

   (c) **True**, $X[0]$ is real-valued since it is sum of real valued numbers.

   $$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$
   $$X[0] = \sum_{n=0}^{7} x[n]$$

   (d) **True**,

   $$\int_3^\infty (t+1)\delta(t)dt = \int_3^\infty \delta(t)dt = 0$$

   (e) **False**.

   $$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$
   $$X[0] = \sum_{n=0}^{7} x[n]$$
   $$X[0] = 18$$

2. **(10 Pts.)** Let $X_d(\omega) = e^{-j4\omega}\sin(2\omega)$. Sketch the magnitude and phase of $X_d(\omega)$ (label your plots carefully).

   Solution:

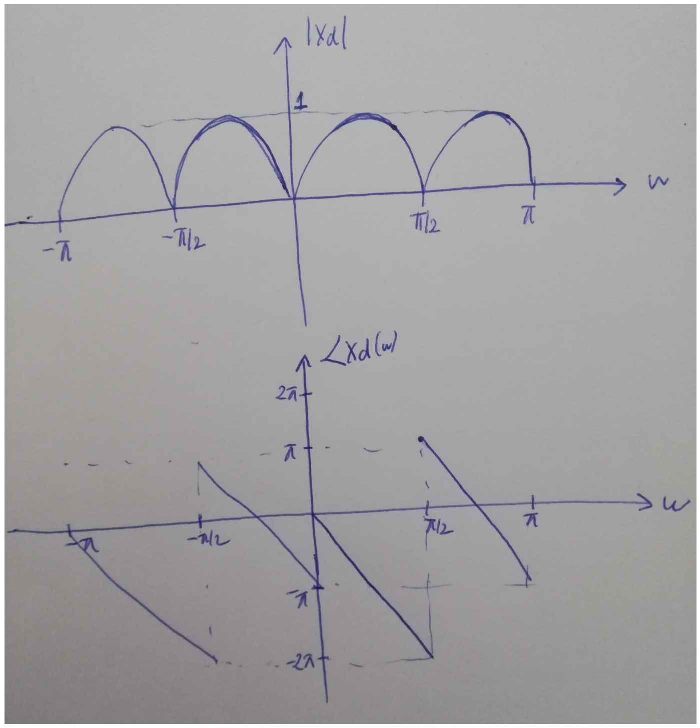


Figure 1: Magnitude and Phase of $X_d(\omega)$

3. **(9 Pts.)** The DTFT of $x[n] = 3\delta[n+1] - 3\delta[n-7]$ can be written as:

$$X_d(w) = Ae^{-jB\omega}\sin(C\omega)$$

Determine the (possibly complex) values of A, B, and C.

Solution:

$$\begin{aligned} X_d(\omega) &= 3e^{j\omega} - 3e^{-j7\omega} \\ &= 3e^{-j3\omega}(e^{j4\omega} - e^{-j4\omega}) \\ &= 3e^{-j3\omega}2jsin(4\omega) \\ &= 6je^{-j3\omega}sin(4\omega) \end{aligned}$$

Therefore $A = 6j$, $B = 3$ and $C = 4$.

4. **(8 Pts.)** Let $\{X[k]\}_{k=0}^{7}$ be the 8-point DFT of $x[n] = \{1, -2, 3, -4, 5, -6, 7, -8\}$. Determine the sequence $\{y[n]\}_{n=0}^{7}$ whose DFT is $Y[k] = e^{-j\left(\frac{6\pi}{8}k+\pi\right)}X[k], k = 0, 1, \ldots, 7$.

Solution:

$$Y[k] = -e^{-j\left(\frac{2\pi}{8}k3\right)}X[k] \iff y[n] = -x[< n-3 >_8]$$

Overall,

$$y[n] = \{6, -7, 8, -1, 2, -3, 4, -5\}$$

5. **(8 Pts.)** Let $\{X[k]\}_{k=0}^{6} = \{-2, -4j, 3, 5, 3, 4j, -2j\}$ be the 7-point DFT of a signal $\{x[n]\}_{n=0}^{6}$. Is $\{x[n]\}_{n=0}^{6}$ real? Justify your answer.

Solution: It is not real. We can observe this by simply checking $x[0]$, by the definition of inverse DFT:

$$x[0] = \frac{1}{N}\sum_{k=0}^{6} X[k]e^{j\frac{2\pi kn}{7}}\Bigg|_{n=0} = \frac{1}{N}\sum_{k=0}^{6} X[k].$$

Then it is clear to see that $x[0]$ is a complex value.

6. **(9 Pts.)** For each statement below, decide whether it best describes the CTFT, the DTFT or the DFT. You need not show your work and no partial credit will be given.

   (a) Suppose that a signal $x[n]$ satisfies $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ (i.e., it is absolutely summable). Then, this transform can be easily computed if the corresponding z-transform is known.

   (b) This transform can be applied to recorded data of finite length using a computer.

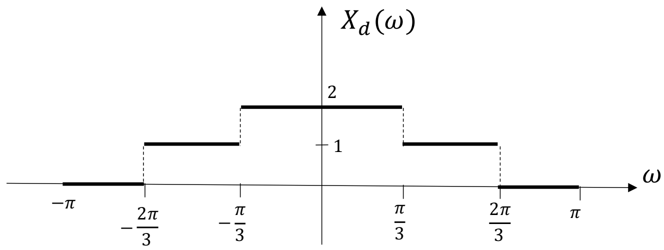
   (c) This transform is defined for infinitely many values of its frequency variable and it is typically **not** periodic as a function of its frequency variable.

   Solution: (a) DTFT, since we just need to replace z in the z-transform by $e^{j\omega}$.

   (b) DFT, DFT is suitable for a finite-length sequence, and a computer works based on digital signals.

   (c) CTFT, by definition CTFT is a integral of the continuous signal on the range $[-\infty, \infty]$, which involves infinitely many values. Also, unlike DTFT, it is not always periodic.

7. **(14 Pts.)** The DTFT of the signal $x[n] = A_1 \frac{\sin(\omega_0 n)}{\omega_0 n} + A_2 \frac{\sin(2\omega_0 n)}{2\omega_0 n}$ is as shown below.



Determine the constants A_1, A_2, and ω_0.

Solution: We can use the following DTFT pair for some $W > 0$:

$$\frac{\sin(Wn)}{\pi n} \iff \begin{cases} 1, & |w| \leq W, \\ 0, & W \leq |w| \leq \pi. \end{cases}$$

Therefore, let $x[n] = x_1[n] + x_2[n]$ such that $x_1[n] := A_1 \frac{\sin(\omega_0 n)}{\omega_0 n}$ and $x_2[n] := A_2 \frac{\sin(2\omega_0 n)}{2\omega_0 n}$, then the DTFT of $x[n]$ is

$$X_d(\omega) = X_{d,1}(\omega) + X_{d,2}(\omega)$$

where

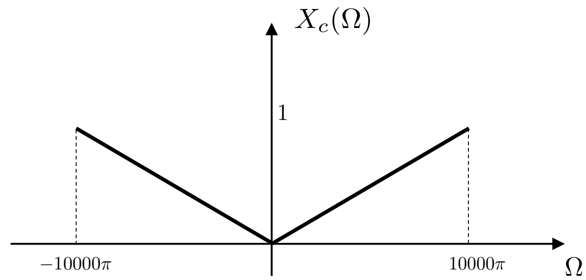
$$X_{d,1}(\omega) = \begin{cases} \dfrac{A_1 \pi}{w_0}, & |w| \leq w_0, \\ 0, & w_0 \leq |w| \leq \pi. \end{cases} \qquad X_{d,2}(\omega) = \begin{cases} \dfrac{A_2 \pi}{2w_0}, & |w| \leq 2w_0, \\ 0, & 2w_0 \leq |w| \leq \pi. \end{cases}$$

Then from the plot of $X_d(\omega)$ we observe that $w_0 = \frac{\pi}{3}$, also from the magnitude we have

$$\frac{A_1 \pi}{w_0} + \frac{A_2 \pi}{2w_0} = 2, \quad \frac{A_2 \pi}{2w_0} = 1.$$

By solving these we obtain $A_2 = 2w_0/\pi = 2/3$ and $A_1 = w_0/\pi = 1/3$.

8. **(15 Pts.)** The continuous-time signal $x_c(t)$ has the real-valued Fourier transform shown below. The signal $x_c(t)$ is sampled with a sampling period of T to produce the discrete-time signal $x[n] = x_c(nT)$.



(a) What is the Nyquist rate for the signal $x_c(t)$?

(b) Sketch the DTFT $X_d(\omega)$ of $x[n]$ for $-\pi < \omega < \pi$ for the sampling frequency $F_s = 1/T = 5$ kHz.

(c) Is the signal $x[n]$ real-valued? Justify your answer.

Solution: (a) The bandwidth of $x_c(t)$ is $\Omega_0 := 10000\pi$, then the Nyquist rate f is given as

$$\frac{\Omega_0}{f} = \pi \quad \rightarrow \quad f = 10000 \text{ Hz}$$

(b) Since $F_s < f$ is less than the Nyquist rate, there exists aliasing, where the digital frequency that corresponds to 10000π is $10000\pi/F_s = 2\pi$. In Figure 2, on left we show the $X_c\left(\frac{\Omega}{F_s}\right)$ and its neighboring copies $X_c\left(\frac{\Omega}{F_s} + 2\pi\right)$ and $X_c\left(\frac{\Omega}{F_s} - 2\pi\right)$. By summing these together, on right we plot $X_d(\omega)$.

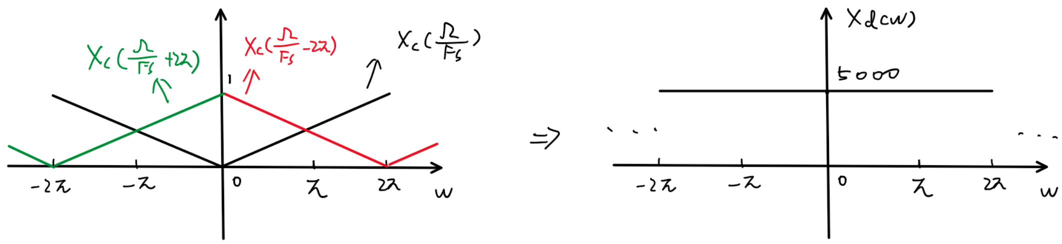
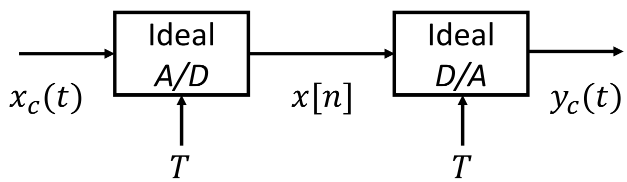


Figure 2: Problem 8. The sketch of $X_d(\omega)$.

(c) Yes, because the CTFT $X_c(\Omega)$ is real and symmetric, then $x_c(t)$ is real-valued, so does $x[n]$.

9. **(12 Pts.)**



Suppose the output of the D/A in the system above is found to be

$$y_c(t) = 2\cos(300\pi t)$$

when the sampling frequency is $F_s = 1/T = 400$ Hz.

(a) Determine $x[n]$.

(b) Assume $x_c(t)$ is bandlimited to 400 Hz. Determine the two different input signals $x_c(t) = x_1(t)$ and $x_c(t) = x_2(t)$ that could have produced the given output of the D/A.

Solution:

(a) Considering the relation between angular and normalized frequency $\omega = \Omega T$, the normalized frequency of $x[n]$ corresponds to:

$$\omega_0 = \frac{300\pi}{400} = \frac{3\pi}{4} \text{ (rad/sample)}.$$

Additionally, considering the scaling factor $T = \frac{1}{400}$ of the ideal lowpass filter included in the D/A converter and the sequence being a cosine (sum of deltas), $x[n]$ corresponds to:

$$x[n] = 2\cos\left(\frac{3\pi n}{4}\right).$$

(b) Using again the relation $\omega = \Omega T$, the angular frequency of $x_1(t)$ corresponds to:

$$\begin{aligned} \Omega_1 &= \frac{3\pi}{4} \cdot 400 = \; 300\pi \text{ (rad/s)} \\ &\Rightarrow x_1(t) = 2\cos(300\pi t) \end{aligned}$$

which satisfies the band limit condition. A second frequency satisfying it generates aliasing. Let $x_2[n]$ be:

$$\begin{aligned} x_2[n] &= 2\cos\left[\left(\frac{3\pi}{4} - 2\pi\right)n\right] \\ &= 2\cos\left(\frac{-5\pi n}{4}\right) \\ &= 2\cos\left(\frac{5\pi n}{4}\right). \end{aligned}$$

Using the frequency relation:

$$\begin{aligned} \Omega_2 &= \frac{5\pi}{4} \cdot 400 = \; 500\pi \text{ (rad/s)} \\ &\Rightarrow x_2(t) = 2\cos(500\pi t). \end{aligned}$$