question	1	2	3	4	5	6	7	8	9
score									

MATH 213 Final Exam Sample
Fall 2022

NAME: ______________________

Instructor: M. Zhang

Please answer all *ten* questions.

- Show all work for full credit.
- The number of points for each question is noted.
- You may have 6 single-sided pages of cheat sheets (or equivalent); *no* other informational aids are permitted.
- No calculators (or equivalent) are permitted.
- You may not consult or communicate with anyone except the proctors during the Exam.
- NOTE: failure to abide by the above requirements will result in an Exam score of *zero*.

Good luck!

Answer:

1. Logical statement

   (a) Use logical equivalences to prove the following statements.

      (i) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

      (ii) $\neg(p \to q) \to \neg q$ is a tautology.

      (iii) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

   (b) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \Rightarrow n \text{ is even}).$$

(i) $\neg(p \oplus q) \Leftrightarrow \neg((\neg p \wedge q) \vee (p \wedge \neg q))$

$\Leftrightarrow \neg(\neg p \wedge q) \wedge (\neg(p \wedge \neg q))$

$\Leftrightarrow (p \vee \neg q) \wedge (\neg p \vee q)$

$\Leftrightarrow (p \vee \neg q \wedge \neg p) \vee (p \vee \neg q \wedge q) \Leftrightarrow p \leftrightarrow q$

(ii) $\neg(p \to q) \to \neg q \Leftrightarrow (p \to q) \vee (\neg q) \Leftrightarrow \neg p \vee q \vee \neg q \Leftrightarrow 1$

(iii) $(p \to q) \to ((\neg(r \to p) \vee (r \to q))$

$\Leftrightarrow \neg(p \to q) \vee (\neg(r \to p)) \vee (r \to q)$

$\Leftrightarrow (p \wedge \neg q) \vee (r \wedge \neg p) \vee (\neg r \vee q)$

$\Leftrightarrow (p \wedge \neg q) \vee q \vee (r \wedge \neg p) \vee (\neg r)$

$= (p \vee q) \vee (\neg p \vee \neg r) \Leftrightarrow p \vee q \vee \neg p \vee \neg r \Leftrightarrow 1$

(b) $\exists n \in \mathbb{N}\ (n^3 + 6n + 5$ is odd $\Rightarrow n$ is odd$)$

$(p \vee \neg q \wedge m \qquad \neg q \wedge m \vee p$

R is uncountable

2. (a) If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?
   (b) Give an example of two uncountable sets A and B such that the difference $A - B$ is
   - finite, $A = R \cup \{i\}$, $B = R$
   - countably infinite, $A = R \cup \{i, 2i, \ldots\}$, $B = R$
   - uncountable. $A = C$, $B = R$, $A - B = \{x \mid x = a + bi, b \neq 0\}$ uncountable

(a) Yes

Assume $A - B = A'$ is countable

$\therefore A = B + A' = B + A'$

let B has a one-to-one and onto function

to map to $\{1, 3, 5, 7, \ldots\}$

and A' map to $\{0, 2, 4, 6, 8, \ldots\}$

the A map to $\{0, 1, 2, 3, \ldots\}$

$\Rightarrow$ A is countable

contradiction

3. Let $f_1 : \mathbf{R} \to \mathbf{R}^+$ and $f_2 : \mathbf{R} \to \mathbf{R}^+$. Let $g : \mathbf{R} \to \mathbf{R}$, and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

   (a) Prove or disprove that $f_1(x)/f_2(x)$ is $\Theta(1)$.

   (b) Prove or disprove that $f_1(f_2(x))$ is $\Theta(g(g(x)))$. disprove

(a) $\lim_{x\to\infty} \frac{|f_1(x)|}{|g(x)|} = 1$

$\lim_{x\to\infty} \frac{|f_2(x)|}{|g(x)|} = 1$

$\therefore \lim_{x\to\infty} \frac{|f_1(x)|}{|f_2(x)|} = 1$

$\therefore \frac{f_1(x)}{f_2(x)} = \Theta(1)$

(b) $\frac{1}{x} + 5$

$-\frac{1}{x} + 5$

when $x \to \infty$

$f_1(5) = \frac{26}{5}$

$g(g(x)) = 0$

0

4. Let $n \in \mathbb{N}$, $n > 0$. Show

we have $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$

$$\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2} \cdot \binom{2n+2}{n+1}.$$

$$C_{2n}^{n+1} + C_{2n}^{n} = \frac{1}{2} C_{2n+2}^{n+1}$$

$$\frac{1}{2} C_{2n+2}^{n+1} = \frac{1}{2}\left(C_{2n+1}^{n+1} + C_{2n+1}^{n}\right)$$

$$= \frac{1}{2}\left(C_{2n}^{n+1} + C_{2n}^{n} + C_{2n}^{n} + C_{2n}^{n-1}\right)$$

$$= C_{2n}^{n} + C_{2n}^{n+1}$$

5. Let a, b, and c be integers. Suppose m is an integer greater than 1 and $ac \equiv bc \pmod{m}$. Prove $a \equiv b \pmod{m/\gcd(c,m)}$.

$$\frac{ac-bc}{m} = k$$

$$\frac{(a-b)\gcd(c,m)}{m} = k$$

$\because \gcd(c,m) = sc + tm$, $s, t \in \mathbb{Z}$

$$\therefore \frac{(a-b)\gcd(c,m)}{m}$$

$$= \frac{(a-b)(sc+tm)}{m}$$

$$= s\left(\frac{(a-b)c}{m}\right) + (a-b)t$$

$= sk + (a-b)t$ which is also integer

6. (a) Solve the recurrence equation $t_n = 2t_{n-1} + n + 2^n$ subject to the initial condition $t_0 = 0$.

   (b) Determine the Θ-class of the function $t(n)$ determined in (a).

(a) $t_n = 2t_{n-1} + n + 2^n$

$F(n) = n + 2^n$

Characteristic Equation: $r - 2 = 0 \Rightarrow r = 2$

$\therefore t_n^{(h)} = \alpha_1 \cdot 2^n$

let $a_n^{(p)} = C_1 + C_2 \cdot n + C_3 \cdot n \cdot 2^n \Rightarrow$ $C_2 =$ $C_3 = 1$

$$C_1 + C_2 \cdot n + C_3 \cdot n \cdot 2^n = 2C_1 + 2nC_2 - 2C_2 + C_3(n-1) \cdot 2^n + n + 2^n$$

$$= 2(C_1 - C_2) + (2C_2 + 1)n + C_3 \cdot n \cdot 2^n + \cancel{(1 - C_3)2^n}$$

$n \cdot 2^n$

$C_3 = 1$

$C_2 = -1$

$a_n^{(p)} = -2 - n + n \cdot 2^n$

$\therefore t_n = t_n^{(h)} + t_n^{(p)}$

$= \alpha_1 \cdot 2^n - 2 - n + n \cdot 2^n$

$\therefore t_0 = \alpha_1 - 2 = 0 \quad \therefore \alpha_1 = 2$

$\therefore t_n = (2 + n) \cdot (2^n - 1)$

(b) $t_n = 2^{n+1} + n \cdot 2^n - 2 - n$

$\because \lim_{n \to \infty} \frac{|t_n|}{n \cdot 2^n} = \lim_{n \to \infty} \left(\frac{2}{n} + 1 - \frac{2}{n \cdot 2^n} - \frac{1}{2^n} \right)$

$= 1$

$\therefore \Theta(t_n) = n \cdot 2^n$

① : $t_n = 2t_{n-1} + n + 2^n \leftarrow t_n^{(h)} = t_{n_1}$
② $t_n = 2t_{n-1} + n \leftarrow t_n^{(h)} = t_{n_2}$
③ $t_n = 2t_{n-1} + 2^n \leftarrow t_n^{(h)} = t_{n_3}$
$\Rightarrow t_{n_1} = c_1 t_{n_2} + c_2 t_{n_3} + c_3$

7. Let E_1 and E_2 be equivalence relations on some set A.

   (a) Is $E_1 \cup E_2$ an equivalence relation on A?

   (b) Is $E_1 \cap E_2$ an equivalence relation on A?

(a) No

E_1 : $\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

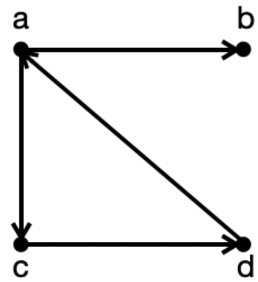
E_2 $\{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

$(1,2) \in E_1 \cup E_2$, $(2,3) \in E_1 \cup E_2$
but $(1,3) \notin E_1 \cup E_2$

(b) Yes

Reflexive : $(a,a) \in E_1$, $(a,a) \in E_2$ $\therefore (a,a) \in E_1 \cap E_2$
Symmetric : if $(a,b) \in E_1 \cap E_2$ $\therefore (b,a) \in E_1$, $(b,a) \in E_2$ $\therefore (b,a) \in E_1 \cap E_2$
Transitive : similar

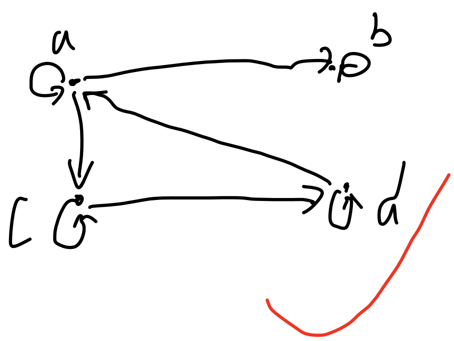
8. Consider relation R represented by the following graph.



(a) Indicate whether relation R satisfies the properties or not, respectively

- Reflexive not as $(a,a), (b,b), (c,c), (d,d) \notin R$
- Symmetric not as $(a,b) \in R, (b,a) \notin R$
- Antisymmetric Yes, as $(a,b), (a,c), (c,d), (d,a) \in R$, $(b,a), \ldots \notin R$
- Transitive not as $(a,c), (c,d) \in R, (a,d) \notin R$

(b) Draw the reflexive closure of relation R.



9. Consider a relation R defined on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$. Consider any of these functions $f : \mathbf{Z}^+$ to $\mathbf{Z}^+$ and $g : \mathbf{Z}^+$ to $\mathbf{Z}^+$, $(f, g) \in R$ if and only if $f(n)$ is $O(g(n))$.

   (a) Is R reflexive? Explain your answer.

   (b) Is R transitive? Explain your answer.

   (c) Prove or disprove R is an equivalence relation.

   (d) Prove or disprove R is a partial ordering.

(a) Yes as $f(n)$ is $O(f(n))$ for any $f(n)$

(b) Yes $(f,g) \in R \Rightarrow f(n)$ is $O(g(n)) \Rightarrow |f(n)| \leq c_1 |g(n)|$

$(g,h) \in R \Rightarrow g(n)$ is $O(h(n)) \Rightarrow |g(n)| \leq c_2 |h(n)|$

$\therefore |f(n)| \leq c_1 c_2 |h(n)|$

$\therefore f(n)$ is $O(h(n))$

$\therefore (f,h) \in R$

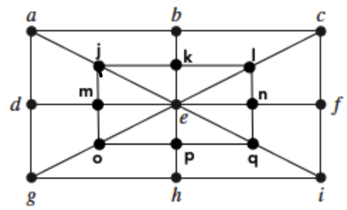
(c) No R is not symmetric $(f,g) \in R$, (g,f) may $\notin R$

(d) No as R is not antisymmetric

$f(n) = 2n$, $g(n) = 4n$

$(f,g) \in R$ $(g,f) \in R$, $f \neq g$

10. Consider the following graph:



(a) Does the graph contain a Hamilton cycle?

(b) Does the graph contain an Euler cycle?

(c) Show that the graph is not bipartite.

(d) Show that if a single vertex is removed, then the graph becomes bipartite and admits a perfect matching.

(a) Yes: a, b, c, l, k, j, e, n, f, i, q, p, h, g, o, m, d

(b) No, as a, c, g, i, d, b, f, h has odd degree

(c)

11. (Bonus) Show that $\log_2 3$ is an irrational number. Recall that an irrational number is a real number x that cannot be written as the ratio of two integers.

assume $\log_2 3$ is a rational number

$\therefore \log_2 3 = \frac{b}{a}$, $a, b \in \mathbb{Z}$

$\therefore \frac{\ln 3}{\ln 2} = \frac{b}{a}$

$\therefore a\ln 3 = b\ln 2$

$\therefore 3^a = 2^b$

3^a is always odd number

2^b is always even number

$\therefore$ contradiction

12. (Bonus) Prove that $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$ for all nonnegative integers n and k, where f_i denotes the ith Fibonacci number.

Assume $P(n)$: $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$

when $n = 0$

$f_n = 0$, $f_{n+1} = 1$

$\therefore f_k f_n + f_{k+1} f_{n+1} = f_{k+1}$

$f_{n+k+1} = f_{k+1}$

$\therefore$ when $n = 0$, $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$

when $n = 1$

$f_n = 1$, $f_{n+1} = 1$

$\therefore f_k f_n + f_{k+1} f_{n+1} = f_k + f_{k+1}$

$f_{n+k+1} = f_{k+2}$

$\because f_k + f_{k+1} = f_{k+2}$

$\therefore f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$

$\therefore$ when $n = 1$, $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$

when $n > 1$

$f_k f_{n-1} + f_{k+1} f_n = f_{n+k}$

$f_k f_{n-2} + f_{k+1} f_{n-1} = f_{n+k-1}$

$\therefore f_k f_n + f_{k+1} f_{n+1} = f_{n+k-1}$