Final Examination

Question 1 (ca. 9 marks)

Decide whether the following statements are true or false, and justify your answers briefly.

a) The surface in $\mathbb{R}^3$ with equation $xyz = x + y + z$ is smooth.

b) The function $f(x, y) = (2019 + x^4 + y^4)\mathrm{e}^{-x^2-y^2}$, $(x, y) \in \mathbb{R}^2$, has a global maximum but no global minimum.

c) If $\mathbf{x}^*$ solves the optimization problem "minimize $f(\mathbf{x})$ subject to $g_1(\mathbf{x}) = g_2(\mathbf{x}) = 0$" (where $f, g_1, g_2 \colon \mathbb{R}^3 \to \mathbb{R}$ are C^1-functions) and $\nabla g_1(\mathbf{x}^*)$, $\nabla g_2(\mathbf{x}^*)$ are both nonzero, then there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f(\mathbf{x}^*) = \lambda_1 \nabla g_1(\mathbf{x}^*) + \lambda_2 \nabla g_2(\mathbf{x}^*)$.

d) In $\mathbb{R}^2$ the line integral $\int_\gamma (ax + by)\,\mathrm{d}x + (cx + dy)\,\mathrm{d}y$ $(a, b, c, d, \in \mathbb{R})$ is independent of path if and only if $b = c$.

$x^2+y^2-2x^2$

e) Let $D = \mathbb{R}^2 \setminus S$, where $S = \{(x, 0); -1 \le x \le 1\}$ is the line segment connecting $(-1, 0)$ and $(1, 0)$. Then every vector field $F = (f_1, f_2)$ on D satisfying $(f_1)_y = (f_2)_x$ is a gradient field.

f) Let $D = \mathbb{R}^3 \setminus S$, where now S denotes the line segment connecting $(-1, 0, 0)$ to $(1, 0, 0)$. Then every vector field $F = (f_1, f_2, f_3)$ on D satisfying $\operatorname{curl}(F) \equiv \mathbf{0}$ is a gradient field.

True

(1) $F(x,y,z) = xyz - x - y - z = 0$, $\nabla F(x,y,z) = \begin{pmatrix} yz-1 \\ xz-1 \\ xy-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$yz = xz$

$(y-x)z = 0$

① $z = 0$ ✗

② $x = y = 1 = z$ not in set

∴ no point where $\nabla F(x,y,z)$ vanish

∴ surface is smooth

(b) ∵ $f(0,0) = 2019$, $\lim_{\substack{x\to\infty \\ y\to\infty}} f(x,y) = \dfrac{2019+x^4+y^4}{e^{x^2+y^2}} = 0$

∴ $f(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$ ∴ It has global maximum

no global minimum

(c) ~~true~~ false

(d) ~~assume $a=b=c=d=1$ false~~ true

(e) ?

(f) ?

Question 2 (ca. 7 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := xy(x+y-1).$$

a) Determine all critical points of f and their types.

b) Does f have a global extremum?

c) A polynomial $\ell(x,y) = ax+by+c$ with $(a,b) \neq (0,0)$ may be called a *line polynomial*, since $\ell(x,y) = 0$ defines a line L in $\mathbb{R}^2$. Show that the product $g = \ell_1\ell_2\ell_3$ of 3 line polynomials whose corresponding lines L_1, L_2, L_3 are in general position (i.e., determine a triangle in $\mathbb{R}^2$) must have at least 4 critical points.

(a) $\begin{cases} f_x(x,y) = y(x+y-1) + xy = 2xy + y^2 - y = 0 \\ f_y(x,y) = x(x+y-1) + xy = 2xy + x^2 - x = 0 \end{cases}$

$\Rightarrow \begin{cases} x=1 \\ y=0 \end{cases}$ or $\begin{cases} x=0 \\ y=1 \end{cases}$ or $\begin{cases} x=0 \\ y=0 \end{cases}$ or $\begin{cases} x=\frac{1}{3} \\ y=\frac{1}{3} \end{cases}$

$f_{xx}(x,y) = 2y$, $f_{yy}(x,y) = 2x$, $f_{xy} = 2x+2y-1$

when $(x,y)=(1,0)$ $4xy-(2x+2y-1)^2 = -1 < 0$ saddle point

when $(x,y)=(0,1)$ $4xy-(2x+2y-1)^2 = -1 < 0$ saddle point

when $(x,y)=(0,0)$ $4xy-(2x+2y-1)^2 = -1 < 0$ saddle point

when $(x,y)=(\frac{1}{3},\frac{1}{3})$ $4xy-(2x+2y-1)^2 = \frac{1}{3} > 0$, $f_{xx} = \frac{2}{3} > 0$ local minimum

(b) $\lim\limits_{\substack{x\to\infty \\ y\to\infty}} f(x,y) \to \infty$

$f(\frac{1}{3},\frac{1}{3}) = \frac{1}{9}\cdot(-\frac{1}{3}) = -\frac{1}{27}$

but $f(-1,-1) = -3 < -\frac{1}{27}$ $\therefore$ no globlal extremum

(c) $g(x,y) = (a_1x+b_1y+c_1)(a_2x+b_2y+c_2)(a_3x+b_3y+c_3)$

$= a_1a_2a_3x^3 + \ldots$

Question 3 (ca. 10 marks)

Consider the quadric surface S in $\mathbb{R}^3$ with equation

$$xy - xz - yz = 3.$$

a) Determine the type of S.

b) Show that there is a point on S that minimizes the distance to the origin $(0, 0, 0)$.

c) Determine all points on S with the property in b) and their distance from $(0, 0, 0)$.

d) Show that the point $\mathbf{q} = (-3, 1, 3)$ is on S, and determine an equation $a_1x + a_2y + a_3z = b$ for the tangent plane of S in $\mathbf{q}$.

$$y^2+y^2+y^2=3$$
$$3y^2=3$$
$$y^2=1$$
$$y=\pm 1$$

$$F_x = y-z$$
$$F_y = x-z$$
$$F_z = -x-y$$

$$-2x-\cancel{6}-6y+\cancel{6}+2z-6$$

$$-2(x+3)-6(y-1)+2(z-3)=0$$
$$\Rightarrow -2x-6y+2z-6=0$$
$$\Rightarrow x+3y-z=-3$$

$$xy-(x+y)(x+y)=3$$
$$-x^2-xy-y^2-3=0$$
$$x^2+xy+y^2+3=0$$
$$\therefore x=y,\quad z=-y$$

$$z=x+y$$

$$2x-\lambda_1 y+\lambda_1 z=0$$
$$\lambda_1 x-2y-\lambda_1 z=0$$
$$\lambda_1 x+\lambda_1 y+2z=0$$

$$(2+\lambda_1)(x-y)=0$$
$$(\lambda_1+2)(y+z)=0$$
$$(\lambda_1+2)(x+z)=0$$

$$F(x,y,z)=x^2+y^2+z^2$$

$$\begin{pmatrix}2x\\2y\\2z\end{pmatrix}=\lambda_1\begin{pmatrix}y-z\\x-z\\-x-y\end{pmatrix}\Rightarrow\{\lambda_1=-2 \text{ 无解} \Rightarrow \begin{cases}x=1\\y=1\\z=-1\end{cases} \text{ or } \begin{cases}x=-1\\y=-1\\z=1\end{cases}$$

(c) $d_1=\sqrt{3}=d_2$

Question 4 (ca. 7 marks)

Consider the function $F\colon [0,\infty) \to \mathbb{R}$ defined by

$$F(x) = \int_0^\infty \frac{\mathrm{e}^{-x(1+t^2)}}{1+t^2}\,\mathrm{d}t\,.$$

a) Show that F is continuous, $F'(x)$ exists for $x > 0$ and can be obtained by differentation under the integral sign, and $\lim_{x\to\infty} F(x) = 0$.

b) Show that

$$F'(x) = -\frac{\mathrm{e}^{-x}}{\sqrt{x}}\,I \quad \text{with} \quad I = \int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s\,.$$

c) Show that

$$F(x) = F(0) - \int_0^x \frac{\mathrm{e}^{-t}}{\sqrt{t}}\,I\,\mathrm{d}t = \frac{\pi}{2} - 2I\int_0^{\sqrt{x}} \mathrm{e}^{-s^2}\,\mathrm{d}s\,.$$

d) Letting $x \to \infty$, conclude from a) and c) that $I = \frac{1}{2}\sqrt{\pi}$.

$$F'(x) = \int_0^\infty -e^{-x(1+t^2)}\,dt$$
$$= \int_0^\infty -e^{-x-xt^2}\,dt$$
$$= -e^{-x}\cdot\int_0^\infty e^{-xt^2}\,dt$$

Question 5 (ca. 7 marks)

Consider the solid K in $\mathbb{R}^3$ defined by

$$K = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 \leq z \leq 1 + x\}.$$

a) Show that K is compact (i.e., closed and bounded).

b) Determine the volume $\operatorname{vol}(K)$.

Hint: Show that the projection of K onto the x-y plane is a disk, and use polar coordinates relative to the center of this disk.

c) Explain how to compute the surface area $\operatorname{vol}_2(\partial K)$.

Note: An explicit figure for $\operatorname{vol}_2(\partial K)$ is not required, but you should simplify the task of computing the surface area as far as possible.

$x^2+y^2 \leq 1+x \Rightarrow$

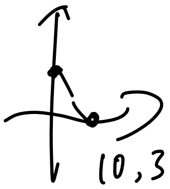
Question 6 (ca. 4 marks)

Let Δ be the (solid) triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(0,1)$. Evaluate the integral

$y = -3x + 4$

$$\int_\Delta \frac{3x+y}{x+3y}\,\mathrm{d}^2(x,y).$$

Hint: Use the obvious change of variables and the fact that the integral over the interior of Δ has the same value.



$(0,3)$

$u = 3x+y$

$v = x+3y$

$(0,0)$

$\Rightarrow (3,1)\ (1,3)$

$x = v - 3y$

$u = 3v - 8y$

$y = \frac{3}{8}v - \frac{1}{8}u$

$x = v - \frac{9}{8}v + \frac{3}{8}u$

$= -\frac{1}{8}v + \frac{3}{8}u$

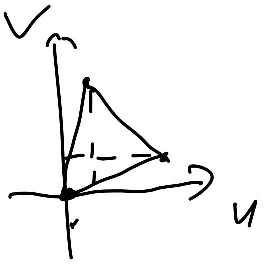
$$\therefore \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right| = \left|\begin{matrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8}\end{matrix}\right| = \frac{9-1}{64} = \frac{1}{8}$$

$$\therefore \iint_D \frac{3x+y}{x+3y}\,d^2(x,y) = \iint_S \frac{u}{v}\cdot\frac{1}{8}\,du\,dv$$

$$= \int_0^1\int_{\frac{v}{3}}^{3v} \frac{u}{v}\cdot\frac{1}{8}\,du\,dv + \int_1^3\int_{\frac{v}{3}}^{4-v} \frac{u}{8v}\,du\,dv$$

$$= \int_0^1 \frac{5v}{9}\,dv + \int_1^3 \frac{1}{v} - \frac{1}{2} + \frac{v}{18}\,dv$$

$$= \ln 3 - \frac{1}{2}$$



$v = -u + 4$

$u = 4 - v$

$\frac{16 - 8v + v^2 - \frac{v^2}{9}}{16v}$ $\quad \frac{8v^2}{9} \cdot \frac{1}{16v}$

$\frac{9v^2 - \frac{1}{9}v^2}{16v}$

$\frac{80v^2}{144v}$

$v = 3u - x$

$v = \frac{1}{3}u + \frac{8}{3}y$

$\frac{1}{v} - \frac{1}{2} + \frac{v}{18}$

$\ln v - \frac{v}{2} + \frac{v^2}{36}\Big|_1^3$

$\ln 3 - \frac{3}{2} + \frac{1}{4} + \frac{1}{2} - \frac{1}{36}$

$\ln 3 - \frac{3}{4} + \frac{1}{4}$