$\cos y + \sin x$

Question 1 (ca. 12 marks)

(d) $g_x = a f_u + c f_v$
$g_y = b f_u + d f_v$ False

Decide whether the following statements are true or false, and justify your answers.

a) The function $f(x,y) = \dfrac{\sin(x)\cos(y)}{1+x^2+y^2}$, $(x,y) \in \mathbb{R}^2$ attains a global maximum and a global minimum.

(e) $x\sin y - y\cos x$ False
$F(x,y) = x\sin y - y\cos x$
$\int_C w = F(1,0) - F(0,1) \neq 0$

b) The function $g(x,y) = xy(x^2+2y^2-3)$, $(x,y) \in \mathbb{R}^2$ has at least 9 critical points.

c) Suppose you start at the point $(1,0)$ and move 0.5 units in the (x,y)-plane following (and continuously adjusting) the direction of steepest ascent of $h(x,y) = \dfrac{xy}{x^2+y^2}$. Afterwards you are closer to the y-axis than before.

(f) True

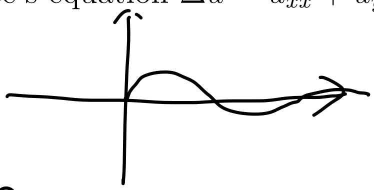
d) If C^2-functions $f(u,v)$ and $g(x,y)$ are related by $g(x,y) = f(ax+by, cx+dy)$ $= f(u,v)$ $(a,b,c,d \in \mathbb{R},\ ad-bc \neq 0)$ then $g_{xx}g_{yy} - g_{xy}^2 = (ad-bc)(f_{uu}f_{vv} - f_{uv}^2)$. ?

e) The line integral of $(\sin y + y\sin x)\,\mathrm{d}x + (x\cos y - \cos x)\,\mathrm{d}y$ along the quarter circle $x^2+y^2=1$, $x \geq 0$, $y \geq 0$ (in the mathematically positive direction) is zero. ?

$x = r\cos\varphi$ $\quad dx = -r\sin\varphi\, d\varphi$
$y = r\sin\varphi$ $\quad dy = r\cos\varphi\, d\varphi$
$\int_0^{\frac{\pi}{2}} (\sin(r\sin\varphi) + r\sin\varphi \sin(r\cos\varphi))(-r\sin\varphi)\, d\varphi$ $\quad \int_0^{\frac{\pi}{2}} r\cos\varphi\cos(r\sin\varphi) - \cos(r\cos\varphi)$

f) If P, Q are C^2-functions on $\mathbb{R}^2$ such that both $P\,\mathrm{d}x + Q\,\mathrm{d}y$ and $Q\,\mathrm{d}x - P\,\mathrm{d}y$ are exact, then P and Q solve Laplace's equation $\Delta u = u_{xx} + u_{yy} = 0$.

True
(a) $x = 0, k\pi, \ldots \quad f = 0$
$|f(x,y)| \leq \dfrac{|\sin x|}{1+x^2}$, $g(x) = \dfrac{\sin x}{1+x^2}$

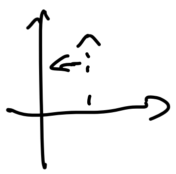


(b) $g_x(x,y) = y(x^2+2y^2-3) + 2x^2y = 3x^2y + 2y^3 - 3y = y(3x^2+2y^2-3) = 0$ True
$g_y(x,y) = x(x^2+2y^2-3) + 4xy^2 = 6xy^2 + x^3 - 3x = x(6y^2+x^2-3) = 0$
① $x = y = 0$ ② $x = 0, y = \pm\frac{\sqrt{3}}{\sqrt{2}}$ ③ $y = 0, x = \pm\sqrt{3}$ ④ $3x^2+2y^2=3$, $6y^2+x^2=3$ $\quad 2x^2-4y^2=0$, $x^2=2y^2$ $\quad 8y^2=3$, $y^2=\frac{3}{8}$ $y = \pm\frac{\sqrt{3}}{2\sqrt{2}}$

True
(c) $h_x(x,y) = \dfrac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2} = \dfrac{y^3 - x^2y}{(x^2+y^2)^2} = \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}$

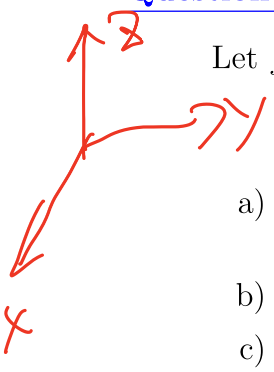
$h_y(x,y) = \dfrac{x(x^2+y^2) - 2xy^2}{(x^2+y^2)^2} = \dfrac{x^3 - xy^2}{(x^2+y^2)^2} = \dfrac{x(x^2-y^2)}{x^2+y^2}$

$h_x(1,0) = 0$
$h_y(1,0) = 1$



$y \nearrow$ $\quad h_x(x,y) < 0 \nearrow$
$h_y(x,y) > 0 \searrow$

Question 2 (ca. 9 marks)



$y = \frac{x^2}{\sqrt{2}}$

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 - 3x^2y + x^2 + 2y^2 - y.$$

a) Which symmetry property does f have? What can you conclude from this about the graph of f and the contours of f?

b) Determine the gradient $\nabla f(x,y)$ and the Hesse matrix $\mathbf{H}_f(x,y)$.

c) Determine all critical points of f and their types.
Hint: There are exactly 3 critical points.

d) Does f have a global extremum?

~~(d) $y = \frac{1}{3}$, $x \to \infty$, $f(x,y) \to \infty$~~

$\therefore$ no global maximum

yes, ~~global minimum~~

(d) no

$f(2,3) = -1 < -\frac{1}{8}$

(a) $f(-x) = f(x)$

$\therefore$ f is symmetric about ~~y axis~~ (y,z)-plane

the graph of f is symmetric about ~~y axis~~ (y,z)-plane

the contours of $f(x)$ ~~can be obtained by $f(-x)$~~ is symmetric about y axis

(b) $f_x(x,y) = 4x^3 - 6xy + 2x$

$f_y(x,y) = -3x^2 + 4y - 1$

$\therefore \nabla f(x,y) = \begin{pmatrix} 4x^3 - 6xy + 2x \\ -3x^2 + 4y - 1 \end{pmatrix}$

$\therefore f_{xx}(x,y) = 12x^2 - 6y + 2$, $f_{yy}(x,y) = 4$

$f_{xy}(x,y) = -6x$

$\therefore H_f(x,y) = \begin{pmatrix} 12x^2 - 6y + 2 & -6x \\ -6x & 4 \end{pmatrix}$

(c) $\begin{cases} f_x(x,y) = 4x^3 - 6xy + 2x = x(4x^2 - 6y + 2) = 0 \\ f_y(x,y) = -3x^2 + 4y - 1 = 0 \end{cases}$

$\Rightarrow \begin{cases} x = 0 \\ y = \frac{1}{4} \end{cases}$ or $\begin{cases} x = \pm 1 \\ y = 1 \end{cases}$

when $x = 0$, $y = \frac{1}{4}$, $\det(H_f(0, \frac{1}{4})) = \frac{1}{2} \cdot 4 - 0 = 2 > 0$

$f_{xx}(0, \frac{1}{4}) = \frac{1}{2} > 0$ $\therefore$ local minimal value

when $x = -1$, $y = 1$, $\det(H_f(-1,1)) = 32 - 36 = -4 < 0$

$\therefore$ saddle point

when $x = 1$, $y = 1$, $\det(H_f(1,1)) = 32 - 36 = -4 < 0$

$\therefore$ saddle point

Question 3 (ca. 9 marks)

Consider the surface S in $\mathbb{R}^3$ with equation

$$xz - y^2 + 2y + 2 = 0.$$ x can't be 0

a) Show that S is smooth.

b) Using the method of Lagrange multipliers, determine the point(s) on S that minimize(s) the distance from the origin $(0,0,0)$, and the corresponding distance d. (It need not be proved that at least one such point exists.)

c) Determine an equation for the tangent plane to S in the point $(2,-2,3)$.

d) The surface S is a (central) quadric. Determine its center and type.

(a) $z = f(x,y) = \dfrac{y^2-2y-2}{x}$

$\therefore f_x(x,y) = -\dfrac{y^2-2y-2}{x^2} = -\dfrac{(y-1)^2-1}{x^2}$

$f_y(x,y) = \dfrac{2y-2}{x}$

$\therefore |f'(x,y)| = \sqrt{\dfrac{(y-1)^4-2(y-1)^2+1+4x^2(y-1)^2}{x^4}} \neq 0$ for all (x,y)

$\therefore S$ is smooth

(b) $d(x,y,z) = x^2+y^2+z^2$

$y^2-2y+1=3$
$y = \pm\sqrt{3}+1$

$\nabla d(x) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$, $F(x,y,z) = xz - y^2 + 2y + 2 = 0 \Rightarrow \nabla F(x,y,z) = \begin{pmatrix} z \\ -2y+2 \\ x \end{pmatrix} \neq 0$

$\nabla d(x,y,z) = \lambda \nabla F(x,y,z)$, $\lambda \neq 0 \Rightarrow \begin{cases} 2x = \lambda z \\ 2y = \lambda(-2y+2) \\ 2z = \lambda x \end{cases} \Rightarrow \begin{cases} \lambda = -2 \\ x = \sqrt{2} \\ y = 2 \\ z = -\sqrt{2} \end{cases}$ or $\begin{cases} \lambda = -2 \\ x = -\sqrt{2} \\ y = 2 \\ z = \sqrt{2} \end{cases}$

$\therefore$ points: $(\sqrt{2}, 2, -\sqrt{2})$, $(-\sqrt{2}, 2, \sqrt{2})$,

or $\begin{cases} x = 0 \\ y = 1 \pm \sqrt{3} \\ z = 0 \end{cases}$

dist $= \sqrt{d(\sqrt{2},2,-\sqrt{2})} = \sqrt{d(-\sqrt{2},2,\sqrt{2})} = 2\sqrt{2}$

dist $= \sqrt{3}-1$

(c) $F_x(x,y,z) = z$, $F_y(x,y,z) = -2y+2$, $F_z(x,y,z) = x$

plane: $F_x(x,y,z)(x-x_0) + F_y(x,y,z)(y-y_0) + F_z(x,y,z)(z-z_0) = 0$

$\Rightarrow 3(x-2) + 6(y+2) + 2(z-3) = 0$

$\Rightarrow 3x + 6y + 2z = 0$

Question 4 (ca. 6 marks)

Evaluate the integral

$$\int_0^1 \frac{t^{1010}+t-2}{\ln t}\,\mathrm{d}t\,.$$

Hint: Consider $F(x)=\displaystyle\int_0^1 \frac{t^x+t-2}{\ln t}\,\mathrm{d}t$, $x\in(-1,\infty)$. Show first that F is well-defined and can be differentiated under the integral sign.

$$F(x)=\int_0^1 \frac{t^x+t-2}{\ln t}\,dt$$

$$F'(x)=\int_0^1 \frac{t^x\ln t}{\ln t}\,dt$$

$$=\int_0^1 t^x\,dt$$

$$=\frac{t^{x+1}}{x+1}\Big|_0^1=\frac{1}{x+1}$$

$$\therefore F(x)=\ln(x+1)+c$$

$$\therefore F(1010)=\ln(1011)+\ln 2$$

$$=\ln(2022)$$

$$F(0)=\int_0^1 \frac{t-1}{\ln t}\,dt=0$$

$$F(1)=\int_0^1 \frac{2(t-1)}{\ln t}\,dt=2F(0)$$

$$\therefore\quad 2c=\ln 2+c$$

$$c=\ln 2$$

$A = \int_D \sqrt{1+|\nabla f(x,y)|^2}\, d^2(x,y)$

Question 5 (ca. 6 marks)

a) Let K be the solid in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

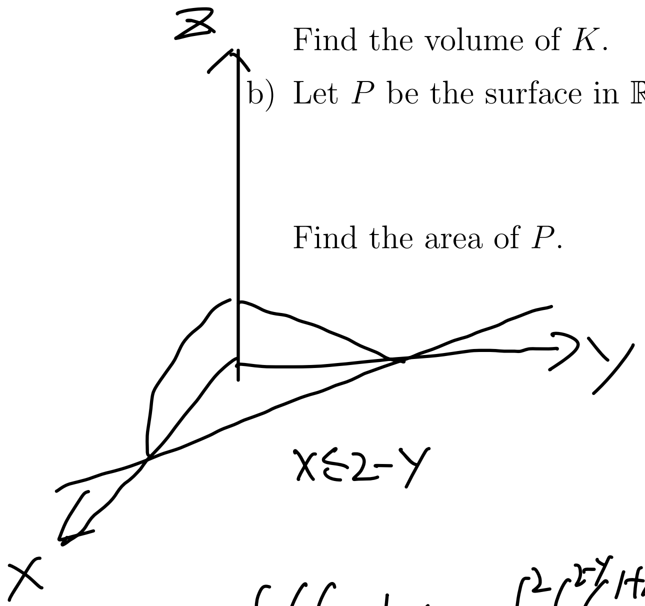
$$x \geq 0, \quad y \geq 0, \quad x+y \leq 2, \quad 0 \leq z \leq 1+x^2+2y.$$

Find the volume of K.

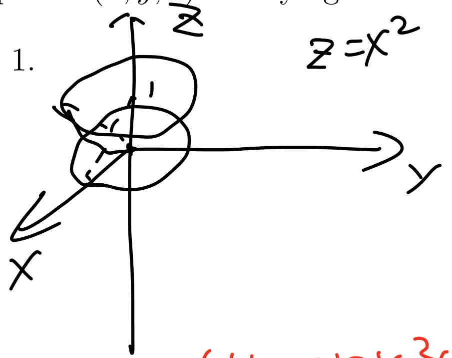
b) Let P be the surface in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$z = x^2+y^2 \leq 1.$$

Find the area of P.



$x \leq 2-y$



$z = x^2$

(a) $V = \iiint_E dV = \int_0^2 \int_0^{2-y} \int_0^{1+x^2+2y} dz\, dx\, dy$

$= \int_0^2 \int_0^{2-y} 1+x^2+2y \, dx\, dy$

$= \int_0^2 \frac{14}{3} + 3y - \frac{8}{3}y^2 - \frac{y^3}{3}\, dy$

$= \frac{62}{9}$

$\cancel{(b) \int_{\partial R} = 2\pi r \cdot h = 2\pi \cdot 1 \cdot 2 = 4\pi}$

$f(x,y) = x^2+y^2$

$S = \int_D \sqrt{1+|\nabla f(x,y)|^2}\, d^2(x,y)$

$= \int_D \sqrt{1+4x^2+4y^2}\, d^2(x,y)$

$= \int_0^1 \int_0^{2\pi} r\sqrt{1+4r^2}\, d\theta\, dr$

$= 2\pi \int_0^1 r\sqrt{1+4r^2}\, dr$

$= 2\pi \cdot \frac{1}{12}(1+4r^2)^{\frac{3}{2}}\Big|_0^1$

$= \frac{\pi}{6}(5\sqrt{5}-1)$

$\frac{\pi}{6}(5\sqrt{5}-1)$

$\int_0^2 \int_0^{2-y} 1+x^2+2y\, dx\, dy$

$\int_0^2 \frac{14}{3} - y - \frac{y^3}{3}\, dy$

$= \frac{14}{3}y - \frac{y^2}{2} - \frac{y^4}{12}\Big|_0^2$

$= 6$