Name: ______________ Student ID: ______________ Major: ______

Question 1 (ca. 11 marks)

$\frac{11}{8}$

Consider the function $f\colon D \to \mathbb{R}$ defined by

$$f(x,y) = \frac{3x}{x^2+xy+y^2}.$$

$\leq \frac{3x}{x^2+y^2} = \frac{3}{x+\frac{y^2}{x}}$

Here $D \subseteq \mathbb{R}^2$ is the maximum possible domain for f.

a) Determine D.

b) Which obvious symmetry property does f have? What can you conclude from this about the graph and the contours of f?

c) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{\substack{(x,y)\to(0,0)\\ x>0,\, y>0}} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibilities $\pm\infty$), or show that the limit does not exist.

$x^2+xy+y^2 \geq \frac{1}{2}(x^2+y^2)$

d) Show that f has no critical point (i.e., no point at which ∇f vanishes).

e) Sketch the 1-contour of f as accurately as possible. Your drawing should include the points with a horizontal or vertical tangent.

f) Determine the slope of the graph G_f at $(1,-1)$ in the direction of the origin (NW), and the maximal slope/direction of G_f at $(1,-1)$.

(a) $\because f(x,y) = \frac{3x}{x^2+xy+y^2}$ $\therefore x^2+xy+y^2 = x^2+xy+\frac{y^2}{4}+\frac{3y^2}{4} = (x+\frac{y}{2})^2+\frac{3y^2}{4} \geq 0$ only $x=0, y=0$ equals

$\therefore D: \{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0)\}$

(b) $\because f(-x,-y) = -\frac{3x}{x^2+xy+y^2} = -f(x,y)$

$\therefore$ the symmetry is $f(-x,-y) = -f(x,y)$, f is symmetric about the origin

~~the contours of f is symmetric about the origin~~

−k-contour of f can be obtained from the k-contour by reflection at the origin of $\mathbb{R}^2$

(c) (i) $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{3x}{3x^2} = \lim_{x\to 0}\frac{1}{x} = \infty$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,\sqrt{x}) = \lim_{x\to 0}\frac{3}{x+\sqrt{x}+1} = 3$ $\therefore$ doesn't exist

(ii) $\lim_{\substack{(x,y)\to(0,0)\\ x>0,y>0}} f(x,y) = \lim_{\substack{x\to 0\\ x>0}} f(x,x) = \lim_{x\to 0}\frac{3x}{3x^2} = \lim_{x\to 0}\frac{1}{x} = \infty$

$\lim_{\substack{(x,y)\to(0,0)\\ x>0,y>0}} f(x,y) = \lim_{\substack{x\to 0\\ x>0}} f(x,\sqrt{x}) = \lim_{x\to 0}\frac{3}{x+\sqrt{x}+1} = 3$ $\therefore$ doesn't exist

(iii) $\lim_{|(x,y)|\to\infty}|f(x,y)| = \lim_{|(x,y)|\to\infty}\frac{3|x|}{x^2+xy+y^2} \leq \lim_{|(x,y)|\to\infty}\frac{6|x|}{x^2+y^2} \leq \lim_{|(x,y)|\to\infty}\frac{6}{\sqrt{x^2+y^2}} = 0$

Math 241 Calculus III.
Prof. Honold

Midterm Exam 2

April 21, 2022
18:00–19:30

Name: ______________ Student ID: ______________ Major: ______

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$x>0$

$x^2 = 4+2y+y^2$

$3x = x^2+xy+y^2$

$x=1$ $\quad y^2+y-2=0$

$y=1$ $\quad (y+2)(y-1)$

$y=-2$

$3x = x^2-x+1$

$y=-1$

x^2-4x+1

$(x-2)^2=3$ $\quad x=\pm\sqrt{3}+2$

(d) $f_x(x,y) = \dfrac{3(x^2+xy+y^2)-3x\cdot(2x+y)}{(x^2+xy+y^2)^2} = \dfrac{3y^2-3x^2}{(x^2+xy+y^2)^2} = 0 \Rightarrow y = \pm x \neq 0$

$f_y(x,y) = -\dfrac{3x^2+6xy}{(x^2+xy+y^2)^2} = 0 \Rightarrow y = -\frac{1}{2}x \neq 0$

$\therefore \{y\in\mathbb{R}\setminus\{0\} \mid y=-\frac{1}{2}x\} \cap \{y\in\mathbb{R}\setminus\{0\} \mid y=\pm x\} = 0$

$\therefore f$ has no critical point.

(e) $f(x,y)=1 \Rightarrow 3x = x^2+xy+y^2$

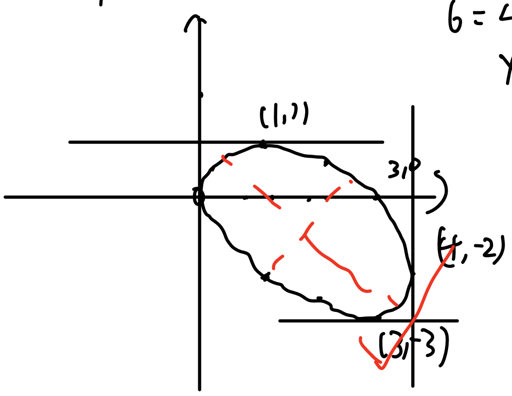
$6 = 4+2y+y^2$

$y^2+2y+1=3$

$y=\pm\sqrt{3}-1$

$y=0$

$x(x-3)=0$



$\frac{y}{2}+x-\frac{3}{2}=0$

(f) the unit direction vector $u=(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$

$\therefore f_u(1,-1) = (f_x(1,-1)\ f_y(1,-1))\begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$

$= (0\quad 3)\begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} = \frac{3\sqrt{2}}{2}$

$\therefore$ the slope is $\frac{3\sqrt{2}}{2}$

$f_x(1,-1)=0,\ f_y(1,-1)=3$

$u=(\cos\varphi,\sin\varphi)$

$f_u(1,-1) = 0\cdot\cos\varphi + 3\sin\varphi \le 3,\ \varphi=\frac{\pi}{2}$

$\therefore$ maximal slope: 3

direction vector $(0,1)$

$y=-2x+3$

$y=-1$

Question 2 (ca. 7 marks)

Consider the differentiable map $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x,y) = \left(2xy - 2x, x^2 - y^2 + 2y\right).$$

a) For $(x,y) \in \mathbb{R}^2$ compute the Jacobi matrix $\mathbf{J}_G(x,y)$.

b) In which points $(x,y) \in \mathbb{R}^2$ is G conformal?

c) Determine the G-image $S = G(Q)$ of the unit square $Q = [0,1]^2$, and graph Q and the region S on paper (unit length at least 2 cm).
Hint: It suffices to determine the G-images of the vertices and edges of Q. The edges are segments of the coordinate lines with equations $x = 0$, $x = 1$, $y = 0$, $y = 1$. You should indicate the correspondence between edges and their G-images by using the same color (or line style such as "dashed", "dotted", etc.).

d) Is the figure obtained in c) compatible with the result in b)? Justify your answer!

(a) $\because G(x,y) = (2xy-2x, x^2-y^2+2y)$

$\therefore \dfrac{\partial G_1(x,y)}{\partial x} = 2y-2, \quad \dfrac{\partial G_1(x,y)}{\partial y} = 2x$

$\dfrac{\partial G_2(x,y)}{\partial x} = 2x, \quad \dfrac{\partial G_2(x,y)}{\partial y} = 2-2y$

$\therefore J_G(x,y) = \begin{pmatrix} \frac{\partial G_1(x,y)}{\partial x} & \frac{\partial G_1(x,y)}{\partial y} \\ \frac{\partial G_2(x,y)}{\partial x} & \frac{\partial G_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 2y-2 & 2x \\ 2x & 2-2y \end{pmatrix}$

(b) $J_G(x,y)^T = \begin{pmatrix} 2y-2 & 2x \\ 2x & 2-2y \end{pmatrix}$

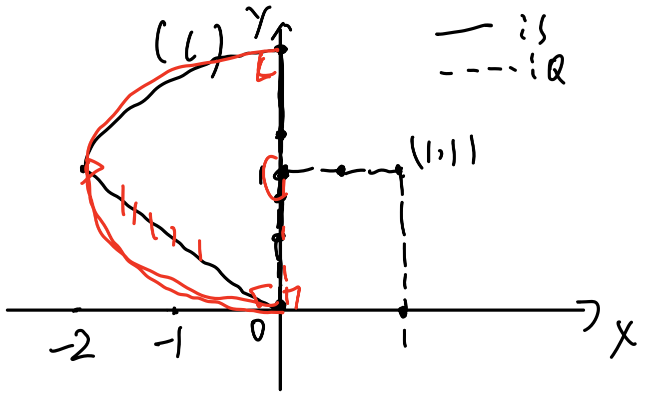
when G conformal $J_G(x,y)^T J_G(x,y) = \lambda I_2 \Rightarrow \begin{pmatrix} 2y-2 & 2x \\ 2x & 2-2y \end{pmatrix}\begin{pmatrix} 2y-2 & 2x \\ 2x & 2-2y \end{pmatrix}$

$= \begin{pmatrix} (2y-2)^2+4x^2 & 0 \\ 0 & 4x^2+(2y-2)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore (2y-2)^2+4x^2 \neq 0 \quad \therefore \cancel{x^2+(y-1)^2 = \tfrac{1}{4}}$ $\quad (x,y) \neq (0,1)$

$\therefore$ for points on circle whose center at $(0,1)$, $r = \frac{1}{2}$ are conformal

(c)



— : S
- - - : Q

$G(0,0) = (0,0)$
$G(0,1) = (0,1)$
$G(1,0) = (-2,1)$
$G(1,1) = (0,2)$
$G(0,\frac{1}{2}) = (0,\frac{3}{4})$

~~(d) yes as $(0,\frac{1}{2}),(\frac{1}{2},1)$ should be conformal~~

$G(0,\frac{1}{2}) = (0,\frac{3}{4})$

$G(\frac{1}{2},1) = (0,\frac{5}{4})$

$0, \cancel{-\frac{1}{4}+1}+2$

$y = -\frac{x^2}{4}+2 \qquad y' = -\frac{x}{2}$

$0\,0,\ 1,0 \qquad -1$

$(x,1)$

$G(x,0) = (-2x, x^2)$

$y = \frac{x^2}{4} \qquad y = \frac{x}{2}$

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Question 3 (ca. 4 marks)

The total resistance R produced by two conductors with resistances R_1 and R_2 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

a) Find the linear approximation of $R = R(R_1, R_2)$ near (R_1, R_2).

b) Suppose $R_1 = 10\,\Omega$, $R_2 = 5\,\Omega$ with a possible tolerance of $\pm 0.5\,\Omega$. Using the Mean Value Theorem, state (tight) rigorous upper and lower bounds for the true value of R.

Note: In b) it is not necessary to determine explicit figures.

(a) $\because \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ $\qquad \Delta R$

$\therefore R = \frac{R_1 R_2}{R_1 + R_2}$

$\therefore \frac{\partial R}{\partial R_1} = \frac{R_2(R_1+R_2) - R_1R_2}{(R_1+R_2)^2} = \frac{R_2^2}{(R_1+R_2)^2}$, $\frac{\partial R}{\partial R_2} = \frac{R_1(R_1+R_2) - R_1R_2}{(R_1+R_2)^2} = \frac{R_1^2}{(R_1+R_2)^2}$

$\therefore J_R(R_1,R_2) = \begin{pmatrix} \frac{R_2^2}{(R_1+R_2)^2} & \frac{R_1^2}{(R_1+R_2)^2} \end{pmatrix}$

$\therefore$ linear approximation: $R = R(R_1+R_2) + J_R(R_1,R_2)\begin{pmatrix} \Delta R_1 \\ \Delta R_2 \end{pmatrix}$

$= \frac{R_1R_2}{R_1+R_2} + \frac{R_2^2 \Delta R_1}{(R_1+R_2)^2} + \frac{R_1^2 \Delta R_2}{(R_1+R_2)^2}$

$\Delta R \approx$

(b) $\Delta R = R - R(R_1+R_2)$

$= \frac{R_2^2 \Delta R_1}{(R_1+R_2)^2} + \frac{R_1^2 \Delta R_2}{(R_1+R_2)^2}$

$\leq \frac{(5.5)^2 \cdot 0.5}{(9.5+4.5)^2} + \frac{(10.5)^{\cancel{2}} \cdot 0.5}{(9.5+4.5)^2} = \frac{281}{784} \approx 0.36$

$\because R = \frac{50}{15} = \frac{10}{3} \approx 3.33$ $\quad \therefore R \in [3.33-0.36, 3.33+0.36]$

$= [2.97, 3.69]$