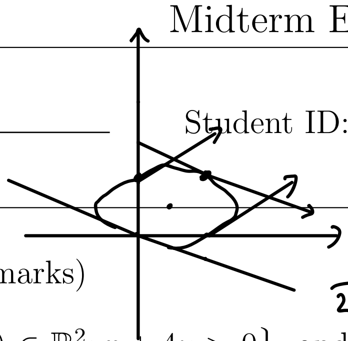
Math 241 Calculus III.
Prof. Honold

Midterm Exam 2

Nov 16, 2021
20:00–21:30

Name: ______________ Student ID: ______________ Major: ______

$\frac{4}{-4+8+16}$ $\frac{}{25}$



$\frac{-4}{16}$ $1+8+4$ $\frac{5}{25}$ $\frac{5}{25}$ $\frac{20}{25}$ $\frac{4}{5}$, $\frac{12}{-2}$ $y=\frac{1}{5}$ $\frac{5}{2}$ $x=-2$

Question 1 (ca. 10 marks)

Let $D=\{(x,y)\in\mathbb{R}^2; x+4y>0\}$, and consider the function $f\colon D\to\mathbb{R}$ defined by $y>-\frac{1}{4}x$

$$f(x,y)=\frac{x^2+4y^2}{x+4y}.$$

$y^4-8y^3+16y^2+4y^2$

(c) for (1,1), the ~~direction~~ unit vector of the origin is $(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$

$x=y^2-4y$ $\frac{}{y^2}$ x^2+16x^2 $\frac{17x^2}{9x}$

$f_x(x,y)=\frac{x^2+8xy-4y^2}{(x+4y)^2}$

(a) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y),\qquad \lim_{\substack{(x,y)\to(0,0)\\ x>0,\,y>0}} f(x,y),\qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

$f_y(x,y)=\frac{8y(x+4y)-4x^2-16y^2}{(x+4y)^2}$ $=\frac{1}{5}$ $y^2-8y+16\geq 0$ $\frac{5x^2}{5x}$

$=\frac{-4x^2+8xy+16y^2}{(x+4y)^2}$ $=\frac{1}{5}$

(including the possibility $+\infty$), or show that the limit does not exist.

(b) Derive equations for the contours of f and describe their shapes as accurately as possible.

(c) Determine the slope of the graph G_f at $(1,1)$ in the direction of the origin, and the maximal slope/direction of G_f at $(1,1)$.

$\therefore D_u f(1,1)=-\frac{\sqrt{2}}{10}-\frac{2\sqrt{2}}{5}=-\frac{\sqrt{2}}{2}$

$f_x(1,0)=1, f_y(1,0)=-4 \therefore \binom{1}{0}+R\binom{4}{1}$
$f_x(0,1)=-\frac{1}{4}, f_y(0,1)=1 \therefore \binom{0}{1}+R\binom{4}{1}$

Assume arbitrary unit direction vector $u=(\cos\varphi,\sin\varphi)$

(d) The 1-contour of f passes through the points $(1,0)$, $(0,1)$, and $(1,1)$. Determine equations for the tangent lines of the 1-contour in these points, and sketch the contour together with the tangent lines and the boundary of D (unit length at least 3 cm!).

$f_x(1,1)=\frac{1}{5}, f_y(1,1)=\frac{4}{5} \therefore \binom{1}{1}+R\binom{4}{-1}$

$D_u f(1,1)=-\frac{\sqrt{2}}{10}\cos\varphi-\frac{2\sqrt{2}}{5}\sin\varphi$

$\leq\sqrt{\frac{17}{50}}=\frac{\sqrt{34}}{10}$

$\cos\varphi=-\frac{2\sqrt{2}}{5}, \sin\varphi=-\frac{\sqrt{2}}{10}$

in the direction $(-\frac{2\sqrt{2}}{5},-\frac{\sqrt{2}}{10})$

(a) $\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{\substack{r\to 0\\ \theta\in[0,2\pi)}} f(r\cos\theta,r\sin\theta)=\lim_{\substack{r\to 0\\ \theta\in[0,2\pi)}} r\frac{1+3\sin^2\theta}{\cos\theta+4\sin\theta}=0$

$r>0$, $\cos\theta+4\sin\theta>0$

$\lim_{\substack{(x,y)\to(0,0)\\ x>0,y>0}} \cancel{f(x,y)=0}$ 该怎么做 doesn't exist

$\therefore$ ~~$\lim_{x\to\infty} f(x,2x)=\lim_{x\to\infty} f(x,x)=\lim_{x\to\infty}(\frac{17}{4}x-x)=\lim_{x\to\infty}\frac{8}{7}x=\infty$~~

$\frac{1}{50}+\frac{8}{25}$

$\therefore \lim_{|(x,y)|\to\infty} f(x,y)$ ~~doesn't exist~~ $=+\infty$

(b) Assume $\frac{x^2+4y^2}{x+4y}=z \Rightarrow x^2-zx+4y^2-4yz=0$

$\Rightarrow (x-\frac{z}{2})^2+(2y-z)^2=\frac{5}{4}z^2$

$\Rightarrow \frac{(x-\frac{z}{2})^2}{\frac{5}{4}z^2}+\frac{(y-\frac{z}{2})^2}{\frac{5}{16}z^2}=1$

$\therefore$ it is a ellipsoid with center point $(\frac{z}{2},\frac{z}{2})$, major axis length $a=\frac{\sqrt{5}}{2}|z|$

(0,0) is removed

minor axis length $b=\frac{\sqrt{5}}{4}|z|$

$72-36$ 18 12 36 18 9 2

-10 -7 $\begin{pmatrix}3\\6\end{pmatrix}\times\begin{pmatrix}0\\6\end{pmatrix}$ $\begin{matrix}36\\-36\end{matrix}$ $\begin{pmatrix}0\\0\\12\end{pmatrix}$ 24

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Question 1 (ca. 10 marks)

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$$f(x,y) = \frac{x^2 + 4y^2}{x + 4y}.$$

a) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{\substack{(x,y)\to(0,0)\\ x>0,\, y>0}} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibility $+\infty$), or show that the limit does not exist.

b) Derive equations for the contours of f and describe their shapes as accurately as possible.

c) Determine the slope of the graph G_f at $(1,1)$ in the direction of the origin, and the maximal slope/direction of G_f at $(1,1)$.

d) The 1-contour of f passes through the points $(1,0)$, $(0,1)$, and $(1,1)$. Determine equations for the tangent lines of the 1-contour in these points, and sketch the contour together with the tangent lines and the boundary of D (unit length at least 3 cm!).

(a) $\lim_{(x,y)\to(0,0)} f(x,y) =$

$\frac{-8}{27}$ $\frac{-10}{27}$ $\frac{-7}{27}$ $\frac{27}{\sqrt{213}}$ $\frac{\sqrt{213}}{27}$ $64+100+49$ 213 71

$\frac{216}{9 \cdot 18 \cdot 3}$ $\frac{2}{27}$ $1-8t^3$ $16t^3$

$\frac{2t^2}{(2t^2+1)^2}$ $\frac{4t(2t^2+1)^2-2(2t^2+1)\cdot 4t \cdot 2t^2}{(2t^2+1)^3}$

Question 2 (ca. 10 marks)

Consider the space curve C parametrized by

$$\mathbf{r}(t) = \left(3t+1, 3t^2, 2t^3\right), \quad t \in \mathbb{R}.$$

$\frac{1}{(2t^2+1)^3} \cdot \sqrt{64t^2+4-48t^2+144t^4+1-16t^3+64t^6}$

$t>0$ $t=0$ $6t^2+3$

$\int_0^t \sqrt{9+36t^2+36t^4}\,dt = \int_0^t 3(2t^2+1)\,dt = 2t^3+3t \Big|_0^t = 2t^3+3t = 5$ $\therefore t=1$

a) Is C contained in a plane? $k(1)=\frac{2}{27}$

b) Let E be the plane in $\mathbb{R}^3$ with equation $2x+4y=1$. Determine the (unique) point P on C that minimizes the distance to E, and the distance $d=d(P,E)$. *Hint:* What can you say about the tangent line of C in P?

$\therefore r(1)=(3,6,6)$, $r''(1)=(0,6,12)$
$r(1)=(4,3,2)$

c) Let $Q=(x,y,z)$ be the (unique) point on C with $z>0$ and such that the arc from $(1,0,0)$ to Q has length 5. Determine the osculating plane (in parametric form), center and radius of the osculating circle, and the TNB frame of C in Q. $\frac{27}{2}$

$\therefore \begin{pmatrix}4\\3\\2\end{pmatrix} + R_1\begin{pmatrix}3\\6\\6\end{pmatrix} + R_2\begin{pmatrix}0\\6\\12\end{pmatrix}$

$r''(t)=(0,6,12t)$

(a) $r'(t)=(3,6t,6t^2)$ $\therefore T(t)=\frac{r'(t)}{|r'(t)|}=\frac{1}{(2t^2+1)}\cdot(1,2t,2t^2)$

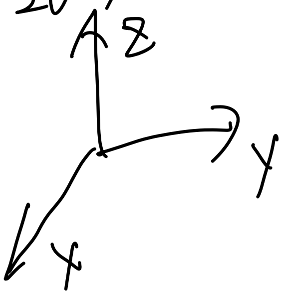
$r''(t)=(0,6,12t)$ $\therefore N(t)=\frac{T'(t)}{|T'(t)|}=\lambda\left(-\frac{8t}{(2t^2+1)^3}, \frac{2-12t^2}{(2t^2+1)^3}, \frac{1-8t^3}{(2t^2+1)^3}\right)$

$\begin{pmatrix}3t+1\\3t^2\\2t^3\end{pmatrix}+R_1\begin{pmatrix}3\\6t\\6t^2\end{pmatrix}+R_2\begin{pmatrix}0\\6\\12t\end{pmatrix}$

$T(1)=\left(\frac{1}{9},\frac{2}{9},\frac{2}{9}\right)$, $N(1)=\left(\frac{-8}{\sqrt{213}},\frac{-10}{\sqrt{213}},\frac{-7}{\sqrt{213}}\right)$
$B(1)=$

(a,b,c) $(2t^2c, 2tc, c)$

$3a+6tb+6t^2c=0$
$6b+12tc=0$ $=R\begin{pmatrix}2t^2\\2t\\1\end{pmatrix}$ 法向量变



$3a-12t^2c+6t^2c=0$ no

$a=2t^2c$

(b) Assume $P:(3t+1, 3t^2, 2t^3)$ $\vec{B}=(2,4,0)$
plane $E: \begin{pmatrix}\frac{1}{2}\\0\\0\end{pmatrix}+R_1\begin{pmatrix}2\\-1\\0\end{pmatrix}+R_2\begin{pmatrix}2\\-1\\1\end{pmatrix}$

point A on E: $\begin{pmatrix}\frac{1}{2}+2R_1+2R_2\\-R_1-R_2\\R_2\end{pmatrix}$

$r'(t)\cdot\vec{B}=0 \Rightarrow 6+24t=0$ $t=-\frac{1}{4}$

$\therefore P\left(\frac{1}{4}, \frac{3}{16}, -\frac{1}{32}\right)$, $PE: \begin{pmatrix}\frac{1}{4}\\\frac{3}{16}\\-\frac{1}{32}\end{pmatrix}+R\begin{pmatrix}2\\4\\0\end{pmatrix}$

$\left(\frac{1}{4}+2R\right)\cdot 2+\left(\frac{3}{16}+4R\right)\cdot 4=1 \Rightarrow R=-\frac{1}{80}$ $\therefore PE: \begin{pmatrix}\frac{9}{40}\\\frac{1}{80}\\-\frac{1}{32}\end{pmatrix}$

$\frac{1}{2}+4R+\frac{3}{4}+16R=1$ $d=$

$20R+\frac{5}{4}=1$ $-\frac{1}{4}$

$\frac{1}{8}\cdot\left(\frac{9}{5}\ \frac{11}{10}\ -\frac{5}{4}\right)$

$\frac{1}{}$ 81 $\frac{121}{}$ $+\frac{1}{}$

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$$f(x,y) \;=\; \frac{x^2+4y^2}{x+4y}.$$

$\le \frac{x}{x+4y}x + \frac{4y}{x+4y}y < x+y \to 0$

a) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{\substack{(x,y)\to(0,0)\\ x>0,\, y>0}} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

$\ge \frac{x^2+y^2}{\sqrt{17}\sqrt{x^2+y^2}}$

(including the possibility $+\infty$), or show that the limit does not exist.

b) Derive equations for the contours of f and describe their shapes as accurately as possible.

c) Determine the slope of the graph G_f at $(1,1)$ in the direction of the origin, and the maximal slope/direction of G_f at $(1,1)$.

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(a)

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Consider the space curve C parametrized by

$$\mathbf{r}(t) = \left(3t+1, 3t^2, 2t^3\right), \quad t \in \mathbb{R}.$$

a) Is C contained in a plane?

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c) Let $Q = (x, y, z)$ be the (unique) point on C with $z > 0$ and such that the arc from $(1, 0, 0)$ to Q has length 5. Determine the osculating plane (in parametric form), center and radius of the osculating circle, and the TNB frame of C in Q.

$\sqrt{1+2t^2}^2$

(a) $r'(t) = (3, 6t, 6t^2) = 3(1, 2t, 2t^2)$

$r''(t) = 3(0, 2, 4t)$

$\therefore$ osculating plane at $r(t)$: $\begin{pmatrix}3t+1\\3t^2\\2t^3\end{pmatrix} + R_1\begin{pmatrix}1\\2t\\2t^2\end{pmatrix} + R_2\begin{pmatrix}0\\2\\4t\end{pmatrix}$ change

no

(b) the normol vector of plane $2x+4y=1$: $\vec{B} = (2, 4, 0)$

when the distance $\downarrow$ $r'(t) \perp \vec{B} \Rightarrow 2 + 8t = 0 \Rightarrow t = -\frac{1}{4}$

$\therefore P = r(-\frac{1}{4}) = (\frac{1}{4}, \frac{3}{16}, -\frac{1}{32})$

assume the line through p with $\vec{B}$ as direction:

$\begin{pmatrix}\frac{1}{4}\\\frac{3}{16}\\-\frac{1}{32}\end{pmatrix} + c\begin{pmatrix}1\\2\\0\end{pmatrix}, c \in R$ $\therefore$ point $\begin{pmatrix}\frac{1}{4}+c\\\frac{3}{16}+2c\\-\frac{1}{32}\end{pmatrix}$ $\therefore \frac{1}{2} + 2c + \frac{3}{4} + 8c = 1$

$\Rightarrow 10c = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow c = -\frac{1}{40}$

$\therefore$ distance $= \left|-\frac{1}{40}\begin{pmatrix}1\\2\end{pmatrix}\right| = \frac{\sqrt{5}}{40}$

(c) $\int_0^t 3\sqrt{1+4t^2+4t^4}\,dt = \int_0^t 3(2t^2+1)\,dt = \int_0^t 6t^2+3\,dt = 2t^3+3t\Big|_0^t = 2t^3+3t = 5, t=1$

$\therefore Q = r(1) = (4, 3, 2)$, $r'(1) = 3(1, 2, 2)$, $r''(1) = 6(0, 1, 2)$

$\therefore$ osculating plane: $\begin{pmatrix}4\\3\\2\end{pmatrix} + c_1\begin{pmatrix}1\\2\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\\2\end{pmatrix}$

$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{1+2t^2}(1, 2t, 2t^2)$, $T(1) = \frac{1}{3}(1, 2, 2)$

$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{1}{1+2t^2}(-2t, 1-2t^2, 2t)$, $N(1) = \frac{1}{3}(-2, -1, 2)$

$\therefore B(1) = T(1) \times N(1) = \frac{1}{9}\begin{pmatrix}6\\-6\\3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$

$\therefore k(t) = \frac{2}{27}$ $\therefore r = \frac{27}{2}$ center: $\begin{pmatrix}4\\3\\2\end{pmatrix} + \frac{\cancel{27}^{9}}{2} \cdot \frac{1}{\cancel{3}}\begin{pmatrix}-2\\-1\\2\end{pmatrix} = \begin{pmatrix}-5\\-\frac{3}{2}\\11\end{pmatrix}$

$\frac{3}{16}$

$$\frac{1}{8}\cdot\left(\frac{1296+484+25}{400}\right)$$

$$\begin{array}{r} 81 \\ 16 \\ \hline 486 \\ 81 \\ \hline 1296 \end{array}$$

$$\frac{1}{8}\cdot\sqrt{\frac{\cancel{2005}\ 401}{\cancel{400}\ 80}} = \frac{\sqrt{32080}}{640}$$