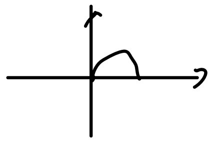

Question 1 (ca. 12 marks)

$y(0) = -\frac{2}{c-1} - 1 = \frac{1}{2}$ $\frac{2}{c-1} = -\frac{3}{2}$

Decide whether the following statements are true or false, and justify your answers.

(a) phase line

(b)

a) There exists a solution $y(t)$ of $y' = 2y - y^2$ satisfying $y(0) = y(1) = 1$. false

b) The maximal solution of the initial value problem $y' = y^2 - t$, $y(0) = \frac{1}{2}$ exists at time $t = 2021$. true

$y' = y^2$ $-\frac{1}{x+c}$

c) Every solution $y\colon (0, \infty) \to \mathbb{R}$ of $t^2 y'' + 3t\, y' + 2y = 0$ has infinitely many zeros. true

$y = t^k$ $k(k-1)+3k+2=0$ $k^2+2k+2=0$

Radius

d) The initial value problem $(x^2+4)y'' + (x+4)y' - 4y = 0$, $y(1) = y'(1) = 1$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ which is defined at $x = 3$. true

A有个根

e) Suppose $\mathbf{A} \in \mathbb{R}^{2\times 2}$ satisfies $\mathbf{A}^3 = \mathbf{I}$ (the 2×2 identity matrix), but $\mathbf{A} \neq \mathbf{I}$. Then every solution $\mathbf{y}(t)$ of the linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ must satisfy $\lim_{t\to\infty} \mathbf{y}(t) = (0,0)^{\mathsf{T}}$. true

$e^{at} = \cosh$ A <0 $x^2-(a+b)x+ad-bc=0$

反例

f) Suppose $f, g\colon (-1,1) \to \mathbb{R}$ are C^1-functions. Then the IVP $y' = f(t)g(y)$, $y(0) = 0$ has a solution $y(t)$ that is defined for all $t \in (-1,1)$. false

$y = \frac{1}{x+\frac{1}{2}} - 2$

$y' = -\frac{1}{(x+\frac{1}{2})^2}$

Question 2 (ca. 10 marks)

$p_0 = \lim_{x\to 0} x \cdot \frac{x(1-x)}{2x^2} = \frac{1}{2}$ exist finite

$q_0 = \lim_{x\to 0} x^2 \frac{-6}{2x^2} = -3$ exist

$x_0 = 0$ regular singular point

Consider the differential equation

$$2x^2 y'' + x(1-x)y' - 6\,y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 7 marks)

Consider the ODE

$$y' = y^2 + \frac{5}{t}y + \frac{5}{t^2}, \qquad t > 0. \tag{R}$$

a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = c\,t^r$ with constants c, r.

b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear ODE.

c) Using b), determine all maximal solutions of (R) and their domain.

Question 4 (ca. 6 marks)

For the matrix $\mathbf{A} = \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -10 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 \end{pmatrix}$ determine the general solution of the linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Question 5 (ca. 7 marks)

Consider the differential equation

$$x(3y^2 - 1)\,\mathrm{d}x + y\,\mathrm{d}y = 0. \tag{DF}$$

a) Determine the general solution of (DF) in implicit form.

b) Determine the maximal solution $y(x)$ satisfying $y(1) = \frac{1}{3}$ and its domain.
Hint: $\ln\left(\frac{3}{2}\right) \approx 0.4$

c) Is every point of $\mathbb{R}^2$ on a unique integral curve of (DF)?

Question 6 (ca. 8 marks)

a) Determine a real fundamental system of solutions of

$$y^{(5)} + 4\,y^{(4)} + 24\,y''' + 40\,y'' + 100\,y' = 0.$$

Hint: The characteristic polynomial is divisible by the square of a quadratic polynomial.

b) Determine the general real solution of

$$y^{(5)} + 4\,y^{(4)} + 24\,y''' + 40\,y'' + 100\,y' = 200\,t - \mathrm{e}^{-t}.$$

c) Find the Laplace transform $Y(s)$ of the solution of the ODE in b) with initial values $y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 0$.

Solutions

1 a) False. Since $y' = y(2-y)$ is positive if $0 < y < 2$, and any solution starting in the strip $0 < y < 2$ is confined to this strip (e.g., because the strip is bounded by the constant solutions $y(t) \equiv 0$ and $y(t) \equiv 2$), the solution with $y(0) = 1$ must be strictly increasing and hence satisfy $y(1) > 1$. $\boxed{2}$

Alternatively one can argue that this ODE is of the form $y' = ay^2 + by + c$ and the corresponding canonical form, viz. $z' = -z^2 + 1$, is the same as for the Logistic Equation (cf. our discussion in the lecture and H16 of HW2). Hence solutions starting between the two equilibrium solutions must be monotonically increasing.

b) True. For $t \geq 0$ we have $-t \leq y^2 - t \leq y^2$, so that the solution $\phi_1(t)$ of $y' = y^2 \wedge y(0) = \frac{1}{2}$ is an upper bound for $y(t)$ and the solution $\phi_2(t)$ of $y' = -t \wedge y(0) = \frac{1}{2}$ is a lower bound for $y(t)$, Solving the two auxiliary IVP's gives $\phi_1(t) = 1/(2-t)$, $\phi_2(t) = (1-t^2)/2$. Hence $y(t)$ is defined at least on $[0, 2)$. $\boxed{1}$

Since $y(1) \leq \phi_1(1) = 1$ and $y^2 - t \leq y^2 - 1$ for $t \geq 1$, the solution $\phi_3(t)$ of $y' = y^2 - 1 \wedge y(1) = 1$ is an upper bound for $y(t)$. Solving the auxilieary IVP gives $\phi_3(t) \equiv 1$. Since $\phi_2(t)$ and $\phi_3(t)$ are defined on $[1, \infty)$, the same must be true of $y(t)$. $\boxed{+1}$

c) True. This is an Euler equation with parameters $\alpha = 3$, $\beta = 2$, indicial equation $r^2 + (\alpha - 1)r + \beta = r^2 + 2r + 2 = (r + 1 + \mathrm{i})(r + 1 - \mathrm{i}) = 0$, complex fundamental system $t^{-1\pm\mathrm{i}}$, and general real solution $y(t) = t^{-1}\left(c_1 \cos \ln t + c_2 \sin \ln t\right)$ on $(0, \infty)$. $\boxed{1}$

Since $c_1 \cos x + c_2 \sin x = A \sin(x - \alpha)$ for some $A \geq 0$, $\alpha \in [0, 2\pi)$, we can write the solution in the form $y(t) = A\,t^{-1} \sin(\ln t - \alpha)$ and conclude that $t_k = \mathrm{e}^{k\pi+\alpha}$, $k = 0, 1, 2, \dots$, are solutions. $\boxed{1}$

d) True. The explicit form of this homogeneous linear 2nd-order ODE is

$$y'' + \frac{x+4}{x^2+4}\, y' - \frac{4}{x^2+4}\, y = 0.$$

$x_0 = 1$ is an ordinary point and the coefficient functions $p(x) = \frac{x+4}{x^2+4}$, $q(x) = -\frac{4}{x^2+4}$ are analytic (even rational) in the disk $|z - 1| < \sqrt{5}$ (the disk with center 1 that has the singularities $\pm 2\mathrm{i}$ of $p(x)$ and $q(x)$ on its boundary). Hence (referring to a theorem proved in the lecture) there exists a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ of the IVP with radius of convergence $\geq \sqrt{5}$. Since $\sqrt{5} > 2$, this solution is defined at $x = 3$. $\boxed{2}$

e) True. If $\mathbf{v} \in \mathbb{C}^2$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda \in \mathbb{C}$, we have $\mathbf{v} = \mathbf{A}^3\mathbf{v} = \lambda^3\mathbf{v}$, and hence $\lambda^3 = 1$. Thus $\lambda \in \left\{1, \frac{-1+\mathrm{i}\sqrt{3}}{2}, \frac{-1-\mathrm{i}\sqrt{3}}{2},\right\}$. If $\mathbf{A}$ has an eigenvalue $\neq 1$, the eigenvalues must be $\lambda_{1/2} = \frac{-1\pm\mathrm{i}\sqrt{3}}{2}$ (since complex eigenvalues of real matrices occur in conjugate pairs). Since their real part is $-\frac{1}{2} < 0$, the matrix $\mathbf{A}$ is asymptotically stable. If $\lambda_1 = \lambda_2 = 1$ then $(\mathbf{A} - \mathbf{I})^2 = \mathbf{0}$ (by the Cayley-Hamilton Theorem), which gives $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, $\mathbf{I} = \mathbf{A}^3 = 2\mathbf{A}^2 - \mathbf{A} = 3\mathbf{A} - 2\mathbf{I}$, and hence $\mathbf{A} = \mathbf{I}$; contradiction. $\boxed{+2}$

f) False. Separable ODE's $y' = f(t)g(y)$ may have maximal solutions with strictly smaller domain than $f(t)$. $\boxed{1}$

This can happen even for autonomous ODE's, and we can take $y' = 2(y+1)^2$, i.e., $f(t) = 1$, $g(y) = 2(y+1)^2$ as counterexample. The general solution of this ODE is $y(t) = \frac{1}{C-2t} - 1$, $C \in \mathbb{R}$, and $y(0) = 0$ gives $C = 1$. But $y(t) = \frac{1}{1-2t} - 1$ is not defined for $t \in \left[\frac{1}{2}, 1\right)$. $\boxed{+2}$

$$\sum_1 = 8 + 5$$

---

2 a) The explicit form of (DE) is

$$y'' + \frac{1-x}{2x}\,y' - \frac{3}{x^2}\,y = 0$$

Using the notation of the lecture/textbook, $p(x) = \frac{1-x}{2x} = \frac{1}{2}\,x^{-1} - \frac{1}{2}$ has a pole of order 1 at 0, and $q(x) = -\frac{3}{x^2}$ has a pole of order 2 at 0. This implies that 0 is a regular singular point of (DE). $\boxed{1}$

Alternatively, use that the limits defining p_0, q_0 below are finite.

b) From a) we have $p_0 = \lim_{x\to 0} x\,p(x) = 1/2$, $q_0 = \lim_{x\to 0} x^2\,q(x) = -3$.
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{1}{2}\,r - 3 = (r-2)(r+3/2) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = 2$, $r_2 = -3/2$. Since $r_1 - r_2 \notin \mathbb{Z}$, there exist two fundamental solutions y_1, y_2 of the form

$$y_1(x) = x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2},$$
$$y_2(x) = x^{-3/2} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n-3/2} \qquad \boxed{1}$$

with normalization $a_0 = b_0 = 1$.

First we determine the analytic solution $y_1(x)$. We have

$$\begin{aligned}
0 &= 2x^2\,y_1'' + x(1-x)y_1' - 6\,y_1 \\
&= 2x^2 \sum_{n=0}^{\infty}(n+2)(n+1)a_n x^n + (x - x^2)\sum_{n=0}^{\infty}(n+2)a_n x^{n+1} - 6\sum_{n=0}^{\infty} a_n x^{n+2} \\
&= \sum_{n=0}^{\infty} \left[2(n+2)(n+1) + n + 2 - 6\right] a_n x^{n+2} - \sum_{n=0}^{\infty}(n+2)a_n x^{n+3} \\
&= \sum_{n=0}^{\infty}(2n^2 + 7n)a_n x^{n+2} - \sum_{n=1}^{\infty}(n+1)a_{n-1}x^{n+2} \\
&= \sum_{n=1}^{\infty} \left[n(2n+7)a_n - (n+1)a_{n-1}\right] x^{n+2}.
\end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = \frac{n+1}{n(2n+7)} a_{n-1} \quad \text{for } n = 1, 2, 3, \dots. \qquad \boxed{1}$$

and with $a_0 = 1$ further $a_n = \frac{(n+1)!}{n!\,9\cdot 11\cdot 13\cdots(2n+7)} = \frac{n+1}{9\cdot 11\cdot 13\cdots(2n+7)}$.

$$\begin{aligned}\Longrightarrow y_1(x) &= \sum_{n=0}^{\infty} \frac{n+1}{9\cdot 11\cdot 13\cdots(2n+7)}\, x^{n+2} \qquad \boxed{1\tfrac{1}{2}}\\ &= x^2 + \frac{2}{9}\, x^3 + \frac{3}{9\cdot 11}\, x^4 + \frac{4}{9\cdot 11\cdot 13}\, x^5 + \frac{5}{9\cdot 11\cdot 13\cdot 15}\, x^7 + \cdots\end{aligned}$$

For the determination of $y_2(x)$ we repeat the process with exponents decreased by 3.5 :

$$\begin{aligned}0 &= 2x^2\, y_2'' + x(1-x)y_2' - 6\, y_2\\ &= 2x^2 \sum_{n=0}^{\infty} \left(n - \tfrac{3}{2}\right)\left(n - \tfrac{5}{2}\right) b_n x^{n-7/2} + (x - x^2) \sum_{n=0}^{\infty} \left(n - \tfrac{3}{2}\right) b_n x^{n-5/2} - 6 \sum_{n=0}^{\infty} b_n x^{n-3/2}\\ &= \sum_{n=0}^{\infty} \left[2\left(n - \tfrac{3}{2}\right)\left(n - \tfrac{5}{2}\right) + n - \tfrac{3}{2} - 6\right] b_n x^{n-3/2} - \sum_{n=0}^{\infty} \left(n - \tfrac{3}{2}\right) b_n x^{n-1/2}\\ &= \sum_{n=0}^{\infty} (2n^2 - 7n) b_n x^{n-3/2} - \sum_{n=1}^{\infty} \left(n - \tfrac{5}{2}\right) b_{n-1} x^{n-3/2}\\ &= \sum_{n=1}^{\infty} \left[n(2n-7)b_n - \left(n - \tfrac{5}{2}\right) b_{n-1}\right] x^{n-3/2}.\end{aligned}$$

Here we obtain the recurrence relation

$$b_n = \frac{n - 5/2}{n(2n-7)}\, b_{n-1} = \frac{2n-5}{2n(2n-7)}\, b_{n-1} \quad \text{for } n = 1, 2, 3, \dots, \qquad \boxed{1}$$

and with $b_0 = 1$ further $b_n = \frac{(-3)(-1)\cdots(2n-5)}{2\cdot 4\cdots 2n(-5)(-3)\cdots(2n-7)} = \frac{2n-5}{2\cdot 4\cdots 2n(-5)}$.

$$\begin{aligned}y_2(x) &= -\frac{1}{5} \sum_{n=0}^{\infty} \frac{2n-5}{2\cdot 4\cdots 2n}\, x^{n-3/2} \qquad \boxed{1\tfrac{1}{2}}\\ &= x^{-3/2} + \frac{3}{5\cdot 2}\, x^{-1/2} + \frac{1}{5\cdot 2\cdot 4}\, x^{1/2} - \frac{1}{5\cdot 2\cdot 4\cdot 6}\, x^{3/2} - \frac{3}{5\cdot 2\cdot 4\cdot 6\cdot 8}\, x^{5/2} - \frac{5}{5\cdot 2\cdot 4\cdot 6\cdot 8\cdot 10}\, x^{7/2} - \cdots\end{aligned}$$

Alternative solution: We use the general recurrence relation for the rational functions $a_n(r)$, viz. $a_0(r) = 1$ and

$$a_n(r) = -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_{n-1}(r) \quad \text{for } n \geq 1.$$

Since $F(r) = (r-2)(r+3/2)$, $p_1 = -1/2$, $p_2 = p_3 = \cdots = q_1 = q_2 = \cdots = 0$, we obtain

$$\begin{aligned}a_n(r) &= -\frac{1}{(r+n-2)(r+n+3/2)} \left[(r+n-1)(-1/2)\right] a_{n-1}(r)\\ &= \frac{r+n-1}{(r+n-2)(2r+2n+3)}\, a_{n-1}(r), \qquad n \geq 1.\end{aligned}$$

Thus the coefficients $a_n(2)$ of $y_1(x)$ satisfy the recurrence relation $a_n(2) = \frac{n+1}{n(2n+7)}\,a_{n-1}(2)$ (the same as for a_n above) and the coefficients $a_n(-3/2)$ of $y_2(x)$ satisfy the recurrence relation $a_n(-3/2) = \frac{n-5/2}{(n-7/2)2n}\,a_{n-1}(-3/2)$ (the same as for b_n above). The rest of the computation remains the same.

The general (real) solution on $(0,\infty)$ is then $y(x) = c_1\,y_1(x) + c_2\,y_2(x)$, $c_1, c_2 \in \mathbb{R}$. $\boxed{\frac{1}{2}}$

That solutions are defined on the whole of $(0,\infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C}\setminus\{0\}$, but follows also readily from the easily established fact that the radius of convergence of both power series is ∞. $\boxed{\frac{1}{2}}$

c) The solution on $(-\infty,0)$ is $y(x) = c_1\,y_1(x) + c_2\,y_2^-(x)$ with the same power series $y_1(x)$ as in b) and

$$y_2^-(x) = -\frac{1}{5\,|x|^{3/2}}\sum_{n=0}^{\infty}\frac{2n-5}{2\cdot 4\cdots 2n}\,x^n = -\frac{1}{5(-x)^{3/2}}\sum_{n=0}^{\infty}\frac{2n-5}{2\cdot 4\cdots 2n}\,x^n. \qquad \boxed{1}$$

(This is not the same as $y_2(-x)$, whose coefficients have an additional factor $(-1)^n$.)

Since $y_1(x)$ is analytic everywhere and $\lim_{x\downarrow 0} y_2(x) = \infty$, the general solution on $\mathbb{R}$ is $y(x) = c_1\,y_1(x)$, $c_1 \in \mathbb{R}$. $\boxed{1}$

$$\sum_2 = 10$$

---

3 a) Substituting $y_1(t) = c\,t^r$ into the ODE gives

$$cr\,t^{r-1} = c^2t^{2r} + 5c\,t^{r-1} + 5\,t^{-2},$$

which holds if $r=-1$ and $-c = c^2+5c+5$, i.e., $c^2+6c+5=0$, which has solutions $c\in\{-1,-5\}$. Thus we can take $y_1(t) = -t^{-1}$ or $y_1(t) = -5\,t^{-1}$. $\boxed{1}$

b) Taking $y_1(t) = -t^{-1}$ in a), the substitution becomes $y = -t^{-1} + 1/z$, $y' = 1/t^2 - z'/z^2$. Substituting this into (R) gives

$$\begin{aligned}
\frac{1}{t^2} - \frac{z'}{z^2} &= \left(-\frac{1}{t}+\frac{1}{z}\right)^2 + \frac{5}{t}\left(-\frac{1}{t}+\frac{1}{z}\right) + \frac{5}{t^2} = \frac{1}{t^2} - \frac{2}{tz} + \frac{1}{z^2} - \frac{5}{t^2} + \frac{5}{tz} + \frac{5}{t^2}\\
\iff\quad -\frac{z'}{z^2} &= \frac{3}{tz} + \frac{1}{z^2}\\
\iff\quad z' &= -\frac{3}{t}\,z - 1.
\end{aligned}$$

This is of the form $z' = a(t)z + b(t)$, hence first-order (inhomogeneous) linear. $\boxed{2}$

c) The general solution of $z' = (-3/t)z$ is

$$z(t) = c\exp\int -\frac{3}{t}\,\mathrm{d}t = \frac{c}{t^3}, \quad c\in\mathbb{R}. \qquad \boxed{1}$$

Variation of parameters then yields a particular solution z_p of $z' = (-3/t) - 1$:

$$z_p(t) = t^{-3}\int t^3(-1)\,\mathrm{d}t = -\frac{t}{4}. \qquad \boxed{1}$$

$\Longrightarrow$ The general solution of $z' = (-3/t)z - 1$ is

$$z(t) = -\frac{t}{4} + \frac{c}{t^3}, \quad c \in \mathbb{R}.$$

$\Longrightarrow$ The general solution of (R) is

$$y(t) = -\frac{1}{t} + \frac{1}{-t/4 + c/t^3} = -\frac{1}{t} + \frac{t^3}{c - t^4/4}, \quad c \in \mathbb{R} \cup \{\infty\}, \qquad \boxed{1}$$

where $c = \infty$ represent the solution y_1.

The maximal domain of $y(t)$ is $(0, \infty)$ for $c \in \{0, \infty\}$ and $c < 0$ ($c = 0$ corresponds to the 2nd solution $y(t) = -5\,t^{-1}$ discovered in a).) For $c > 0$ the expression for $y(t)$ defines two maximal solutions on $\bigl(0, \sqrt[4]{4c}\bigr)$ and $\bigl(\sqrt[4]{4c}, \infty\bigr)$. $\boxed{1}$

$$\sum_{3} = 7$$

---

4 The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}
\chi_{\mathbf{A}}(X) &= \begin{vmatrix} X+8 & 0 & -5 & 2 \\ -5 & X+1 & 4 & -1 \\ 10 & 0 & X-7 & 2 \\ 0 & 0 & -3 & X-2 \end{vmatrix} = (X+1) \begin{vmatrix} X+8 & -5 & 2 \\ 10 & X-7 & 2 \\ 0 & -3 & X-2 \end{vmatrix} \\
&= (X+1) \begin{vmatrix} X+8 & -5 & 2 \\ 2-X & X-2 & 0 \\ 0 & -3 & X-2 \end{vmatrix} = (X+1)(X-2) \begin{vmatrix} X+8 & -5 & 2 \\ -1 & 1 & 0 \\ 0 & -3 & X-2 \end{vmatrix} \\
&= (X+1)(X-2) \begin{vmatrix} X+3 & -5 & 2 \\ 0 & 1 & 0 \\ -3 & -3 & X-2 \end{vmatrix} = (X+1)(X-2) \begin{vmatrix} X+3 & 2 \\ -3 & X-2 \end{vmatrix} \\
&= (X+1)(X-2)\,[(X+3)(X-2) - (-3)2] \\
&= X(X+1)^2(X-2).
\end{aligned}$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$ with algebraic multiplicity 1, $\lambda_2 = -1$ with algebraic multiplicity 2, $\lambda_3 = 2$ with algebraic multiplicity 1. $\boxed{2}$

$\underline{\lambda_1 = 0:}$

$$\mathbf{A} - 0\,\mathbf{I} = \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -10 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -2 & 0 & 2 & 0 \\ -8 & 0 & 8 & 0 \end{pmatrix} \to \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The latter is in "permuted" echelon form with x_1 as a free variable, say. Setting $x_1 = 1$ gives $x_3 = 1$, $x_4 = -3/2$, $x_2 = -1/2$.

$\Longrightarrow$ The eigenspace corresponding to $\lambda_1 = 0$ is generated by $\mathbf{v}_1 = (2, -1, 2, -3)^{\mathsf{T}}$.

$\underline{\lambda_2 = -1:}$

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} -7 & 0 & 5 & -2 \\ 5 & 0 & -4 & 1 \\ -10 & 0 & 8 & -2 \\ 0 & 0 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & -3 & 0 \\ 5 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda_2 = -1$ is generated by $\mathbf{v}_2 = (0,1,0,0)^\mathsf{T}$ and $\mathbf{v}_3 = (-1,0,-1,1)^\mathsf{T}$.

$\underline{\lambda_3 = 2:}$

$$\mathbf{A} - 2\,\mathbf{I} = \begin{pmatrix} -10 & 0 & 5 & -2 \\ 5 & -3 & -4 & 1 \\ -10 & 0 & 5 & -2 \\ 0 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 5 & -3 & -4 & 1 \\ 0 & -6 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Setting $x_4 = 1$ gives $x_3 = 0$, $x_2 = 0$, $x_1 = -1/5$.
$\Longrightarrow$ The eigenspace corresponding to $\lambda_3 = 2$ is generated by $\mathbf{v}_4 = (-1,0,0,5)^\mathsf{T}$.

Since eigenvectors corresponding to different eigenvalues are linearly independent, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ form a basis of $\mathbb{R}^4$ (and $\mathbf{A}$ is diagonalizable).
$\Longrightarrow$ A fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is

$$\mathbf{y}_1(t) \equiv \begin{pmatrix} 2 \\ -1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{y}_2(t) = \mathrm{e}^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3(t) = \mathrm{e}^{-t} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_4(t) = \mathrm{e}^{2t} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 5 \end{pmatrix},$$

$\boxed{4}$

and the general (real) solution is $\mathbf{y}(t) = c_1\,\mathbf{y}_1(t) + c_2\,\mathbf{y}_2(t) + c_3\,\mathbf{y}_3(t) + c_4\,\mathbf{y}_4(t)$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

$$\sum_4 = 6$$

---

5 a) Dividing (DF) by $3y^2 - 1$ gives the exact (even separable) equation

$$x\,\mathrm{d}x + \frac{y}{3y^2 - 1}\,\mathrm{d}y = 0.$$

A function F with $\mathrm{d}F = x\,\mathrm{d}x + \frac{y}{3y^2-1}\,\mathrm{d}y$ is $F(x,y) = \frac{1}{2}\,x^2 + \frac{1}{6}\ln|3y^2 - 1|$, and hence the general solution of (DF) in implicit form is

$$\frac{1}{2}\,x^2 + \frac{1}{6}\ln\left|3y^2 - 1\right| = C, \quad C \in \mathbb{R}. \qquad \boxed{2}$$

This must be complemented by the horizontal lines $y = \pm 1/\sqrt{3}$, which have been lost when dividing by $3y^2 - 1$. Since $y = \text{const.}$ implies $\mathrm{d}y = 0$, these are indeed solutions (even explicit solutions $y(x) \equiv \pm 1/\sqrt{3}$). $\boxed{1}$

b) $y(1) = 1/3$ requires $C = \frac{1}{2} + \frac{1}{6}\ln\frac{2}{3}$. Then we solve the corresponding contour equation for y:

$$\begin{aligned}
\frac{1}{6}\ln\left|3y^2-1\right| &= \frac{1}{2}(1-x^2)+\frac{1}{6}\ln\frac{2}{3}\\
\ln\left|3y^2-1\right| &= 3(1-x^2)+\ln\frac{2}{3}\\
\left|3y^2-1\right| &= \frac{2}{3}\,\mathrm{e}^{3(1-x^2)}\\
3y^2-1 &= -\frac{2}{3}\,\mathrm{e}^{3(1-x^2)} && \left(\text{since } 3\,y(1)-1=-\tfrac{2}{3}\right)\\
y=y(x) &= \sqrt{\frac{1-\frac{2}{3}\,\mathrm{e}^{3(1-x^2)}}{3}} && \boxed{2}
\end{aligned}$$

The domain I of $y(x)$ is determined by

$$\begin{aligned}
& 1-\frac{2}{3}\,\mathrm{e}^{3(1-x^2)}\geq 0\\
\iff\quad & 3(1-x^2)\leq \ln\frac{3}{2}\\
\iff\quad & x^2\geq 1-\frac{1}{3}\ln\frac{3}{2}\approx 0.87
\end{aligned}$$

Since I must be an interval containing 1, we obtain $I=\left[\sqrt{0.87},\infty\right)$. $\boxed{1}$

More precisely $I=(a,\infty)$ with $a=\sqrt{1-\frac{1}{3}\ln\frac{3}{2}}\approx 0.929970410262577$. Since $y(x)$ is not differentiable at a, we exclude a from the domain.

c) No. The integral curves of (DF) are the contours of $F(x,y)=\frac{1}{2}\,x^2+\frac{1}{6}\ln|3y^2-1|$. From $\mathrm{d}F(x,y)=x\,\mathrm{d}x+\frac{y}{3y^2-1}\,\mathrm{d}y$ we get $F_x=x$, $F_y=\frac{y}{3y^2-1}$, $F_{xx}=1$, $F_{xy}=F_{yx}=0$, and $F_{yy}=-\frac{3y^2+1}{(3y^2-1)^2}$. Since $F_x(0,0)=F_y(0,0)=0$, $F_{xx}(0,0)=1$, $F_{yy}(0,0)=-1$, the origin $(0,0)$ is a saddle point of F and hence contained in two distinct integral curves. $\boxed{+2}$

$$\sum_5 = 6+2$$

6 a) The characteristic polynomial is

$$\begin{aligned}
a(X) &= X^5+4\,X^4+24\,X^3+40\,X^2+100\,X\\
&= X(X^4+4\,X^3+24\,X^2+40\,X+100)\\
&= X(X^2+2\,X+10)^2\\
&= X(X+1-3\mathrm{i})^2(X+1+3\mathrm{i})^2
\end{aligned}$$

with zeros $\lambda_1=0$ of multiplicity 1 and $\lambda_2=-1+3\mathrm{i}$, $\lambda_3=-1-3\mathrm{i}$ of multiplicity 2.
$\Longrightarrow$ A complex fundamental system of solutions is 1, $\mathrm{e}^{(-1+3\mathrm{i})t}$, $t\,\mathrm{e}^{(-1+3\mathrm{i})t}$, $\mathrm{e}^{(-1-3\mathrm{i})t}$, $t\,\mathrm{e}^{(-1-3\mathrm{i})t}$ and the corresponding real fundamental system is

$$1,\quad \mathrm{e}^{-t}\cos(3t),\quad \mathrm{e}^{-t}\sin(3t),\quad t\,\mathrm{e}^{-t}\cos(3t),\quad t\,\mathrm{e}^{-t}\sin(3t). \qquad \boxed{2}$$

b) In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the two systems $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = 200\,t$, $b_2(t) = \mathrm{e}^{-t}$. Superposition then yields the particular solution $y_p(t) = y_1(t) - y_2(t)$.

(1) Since $\mu = 0$ is a zero of multiplicity 1 of $a(X)$, the proper Ansatz in this case is $y_1(t) = c_0 t + c_1 t^2$. Substituting it into the ODE we get

$$40(2c_1) + 100(c_0 + 2c_1 t) = 200\,t.$$

$\Longrightarrow c_1 = 1$, $c_0 = -\frac{4}{5}$, and $y_1(t) = t^2 - \frac{4}{5}\,t$. $\boxed{1}$

(2) Since $\mu = -1$ is not a root of $a(X)$, we can take $y_2(t) = \frac{1}{a(-1)}\,\mathrm{e}^{-t} = -\frac{1}{81}\,\mathrm{e}^{-t}$. $\boxed{1}$

$\Longrightarrow y_p(t) = t^2 - \frac{4}{5}\,t + \frac{1}{81}\,\mathrm{e}^{-t}$ is a particular solution. $\boxed{1}$

The general real solution is then

$$\begin{aligned} y(t) = c_1 + c_2\,\mathrm{e}^{-t}\cos(3t) + c_3\,\mathrm{e}^{-t}\sin(3t) + c_4\,t\,\mathrm{e}^{-t}\cos(3t) + c_5\,t\,\mathrm{e}^{-t}\sin(3t) \\ + t^2 - \tfrac{4}{5}\,t + \tfrac{1}{81}\,\mathrm{e}^{-t} \qquad \text{with } c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}. \end{aligned} \quad \boxed{1}$$

c) The Laplace transform of the right-hand side of the ODE in b) is

$$200\,\mathcal{L}\{t\} - \mathcal{L}\left\{e^{-t}\right\} = \frac{200}{s^2} - \frac{1}{s+1} = \frac{-s^2 + 200\,s + 200}{s^2(s+1)}. \quad \boxed{1}$$

Using the formulas for the Laplace transform of the derivatives of $y(t)$ and the given initial conditions, this implies $s^5\,Y(s) + 4s^4\,Y(s) + 24s^3\,Y(s) + 40s^2\,Y(s) + 100s\,Y(s) = \frac{-s^2+200\,s+200}{s^2(s+1)}$, i.e.,

$$Y(s) = \frac{-s^2 + 200\,s + 200}{s^3(s+1)(s^2+2s+10)^2}. \quad \boxed{1}$$

$$\sum_{6} = 8$$

$$\sum_{\text{Final Exam}} = 45 + 7$$