Question 1 (ca. 14 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = y^2 - 2$ satisfying $y(0) = 1$ and $y(1) = 2$.

b) The maximal solution of the initial value problem $y' = y^2 \cos t$, $y(0) = 1$ exists at time $t = 2$.

$y' = y^2$ $\quad -\frac{1}{x+c}$ $\quad c=-1$ $\quad -\frac{1}{x-1}$

c) Every solution $y(t)$ of $t^2y'' - ty' + y = 0$, $t > 0$, satisfies $\lim_{t\downarrow 0} y(t) = 0$.

$\tan x$ $\quad \frac{1}{\cos x}$ $\quad x=\frac{\pi}{2}$ $\quad -\frac{\pi}{2}$ $\quad \frac{\pi}{2}$

d) The initial value problem $(\cos x)y'' + (\sin x)y' + y = 0$, $y(0) = y'(0) = 1$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ which is defined at $x = 1$.

e) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^2 = \mathbf{I}_n$ (the $n \times n$ identity matrix) then $e^{\mathbf{A}t} = (\cosh t)\mathbf{I}_n + (\sinh t)\mathbf{A}$.

$x^2+2x+2=0$ $\quad (x+1)^2=-1$ $\quad x=\pm i-1$

f) Every solution $\mathbf{y}(t)$ of the system $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \mathbf{y}$ satisfies $\lim_{t\to\infty} \mathbf{y}(t) = (0,0)^{\mathsf{T}}$.

g) There exist real numbers $b_1, b_2, b_3, \dots$ such that $x - x^2 = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$ for all $x \in [0,1]$.

$k(k-1)-k+1=0$ $\quad t$
k^2-2k+1 $\quad k=1$
$(k-1)^2=0$

Question 2 (ca. 7 marks)

Consider the differential equation

$$(x-1)^2y'' + 2(x^2-1)y' - 4y = 0. \tag{DE}$$

a) Show that $x_0 = 1$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(1, \infty)$.
Hint: It turns out that $r_1 - r_2 \in \mathbb{Z}$, but the more complicated machinery developed in the lecture/textbook for this case is not needed.

c) Using the result of b), discuss the general solution of (DE) on $(-\infty, 1)$ and on $\mathbb{R}$.

Question 3 (ca. 5 marks)

Consider the ODE

$$y' = t^3 + \frac{2}{t}y - \frac{1}{t}y^2, \qquad t > 0. \tag{R}$$

a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = t^r$.

b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear ODE, and explain the precise correspondence between solutions of (R) and solutions of the linear ODE.

c) Solve the linear ODE in b) and use the result to determine the general solution of (R).

Question 4 (ca. 9 marks)

Consider $\mathbf{A} = \begin{pmatrix} 1 & -2 & -2 \\ -4 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{y}(0) = (0, 0, 0)^{\mathsf{T}}$.

Question 5 (ca. 6 marks)

Consider the differential equation

$$(3xy + 2y^2)\,\mathrm{d}x + (3x^2 + 6xy + 3y^2)\,\mathrm{d}y = 0. \tag{DF}$$

a) Show that $(0, 0)$ is the only singular point of (DF).

b) Transform (DF) into an exact equation and determine the general solution in implicit form.

c) Is every point of $\mathbb{R}^2$ on a unique integral curve of (DF) ?

Question 6 (ca. 7 marks)

Determine all real solutions $y(t)$ of

$$2\,y^{(5)} - y''' + y'' = 1 + t - 2\sin t.$$

Solutions

1 a) False. There is the constant solution $y(t) \equiv \sqrt{2}$. By the Intermediate Value Theorem, a solution satisfying $y(0) = 1$, $y(1) = 2$ would attain the value $\sqrt{2}$ at some $t_0 \in (0,1)$, contradicting the Uniqueness Theorem. $\boxed{2}$

b) False. Solving this separable ODE in the standard way, we obtain $\mathrm{d}y/y^2 = \cos t\,\mathrm{d}t$, $-1/y = \sin t + C$, $y = -\frac{1}{\sin t + C}$ as general solution. The initial condition $y(0) = 1$ gives $C = -1$, so that $y(t) = \frac{1}{1-\sin t}$, $t \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$. At $t = 2$ this solution is not defined. $\boxed{2}$

c) True. This is an Euler equation with parameters $\alpha = -1$, $\beta = 1$, indicial equation $r^2 + (\alpha - 1)r + \beta = r^2 - 2r + 1 = (r-1)^2$ and general solution $y(t) = c_1 t + c_2 t \ln t$ on $(0, \infty)$. Since $\lim_{t\downarrow 0} t = \lim_{t\downarrow 0} t \ln t = 0$, it follows that $\lim_{t\downarrow 0} y(t) = 0$. $\boxed{2}$

d) True. The explicit form of this homogeneous linear 2nd-order ODE is

$$y'' + \frac{\sin x}{\cos x}\, y' + \frac{1}{\cos x}\, y = 0.$$

$x_0 = 0$ is an ordinary point and the coefficient functions $p(x) = \frac{\sin x}{\cos x}$, $q(x) = \frac{1}{\cos x}$ are analytic in the disk $|z| < \pi/2$, because the complex cosine function $z \mapsto \cos z = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}\right)$ has the same zeros $z = (2k+1)\pi/2$, $k \in \mathbb{Z}$, as the real cosine function. For this note that $\cos z = 0$ is equivalent to $\mathrm{e}^{2\mathrm{i}z} = -1$ and, writing $z = x + \mathrm{i}y$, in turn to $\mathrm{e}^{-2y+2\mathrm{i}x} = \mathrm{e}^{-2y}\left(\cos(2x) + \mathrm{i}\sin(2x)\right) = -1$. This implies $\sin(2x) = 0$, i.e., $x = m\pi/2$ with $m \in \mathbb{Z}$ and $\cos(2x) = (-1)^m$, which in turn forces that $m = 2k+1$ is odd and $y = 0$. Thus $z = x = (2k+1)\pi/2$.

According to the lecture this guarantees that every associated IVP has a power series solution $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $\geq \pi/2$ and hence is defined at $x = 1$. $\boxed{2}$

e) True. $\mathbf{\Phi}(t) = (\cosh t)\mathbf{I}_n + (\sinh t)\mathbf{A}$ satisfies

$$\begin{aligned}
\mathbf{\Phi}'(t) &= (\sinh t)\mathbf{I}_n + (\cosh t)\mathbf{A},\\
\mathbf{A}\mathbf{\Phi}(t) &= (\cosh t)\mathbf{A} + (\sinh t)\mathbf{A}^2 = (\cosh t)\mathbf{A} + (\sinh t)\mathbf{I}_n = \mathbf{\Phi}'(t),\\
\mathbf{\Phi}(0) &= (\cosh 0)\mathbf{I}_n + (\sinh 0)\mathbf{A} = \mathbf{I}_n.
\end{aligned}$$

These properties characterize the matrix exponential function of $\mathbf{A}$ uniquely, so that we must have $\mathbf{\Phi}(t) = \mathrm{e}^{\mathbf{A}t}$. $\boxed{2}$

f) True. Denoting the matrix in question by $\mathbf{A}$, we have $\chi_{\mathbf{A}}(X) = X^2 + 2X + 2 = (X + 1 - \mathrm{i})(X + 1 + \mathrm{i})$, so that the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1 - \mathrm{i}$, $\lambda_2 = -1 + \mathrm{i}$. If $\mathbf{v}_1, \mathbf{v}_2$ are corresponding eigenvectors, every solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ has the form

$$\mathbf{y}(t) = c_1 \mathrm{e}^{(-1+\mathrm{i})t}\mathbf{v}_1 + c_2 \mathrm{e}^{(-1-\mathrm{i})t}\mathbf{v}_2 = \mathrm{e}^{-t}\left(c_1 \mathrm{e}^{\mathrm{i}t}\mathbf{v}_1 + c_2 \mathrm{e}^{-\mathrm{i}t}\mathbf{v}_2\right)$$

for some constants c_1, c_2 and clearly satisfies $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$.

Alternatively, the system is asymptotically stable since $\mathrm{trace}(\mathbf{A}) = -2 < 0$ and $\det(\mathbf{A}) = 2 > 0$. So there exists $\delta > 0$ such that every solution $\mathbf{y}(t)$ with $|\mathbf{y}(0)| < \delta$ satisfies $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$. An arbitrary nonzero solution $\mathbf{y}(t)$ can be scaled by the constant $c = \frac{\delta}{2\mathbf{y}(0)}$ to obtain $|c\,\mathbf{y}(0)| < \delta$. Since $t \mapsto c\,\mathbf{y}(t)$ is a solution as well, it follows that $\lim_{t\to\infty} c\,\mathbf{y}(t) = \mathbf{0}$ and hence $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$. $\boxed{2}$

g) True. The function $f(x) = x - x^2$, $x \in [0,1]$, can be extended to an odd 2-periodic function on $\mathbb{R}$, since $f(0) = f(1) = 0$. The extension is piecewise C^1 (in fact even C^1) and hence represented by its Fourier series everywhere. Since $L = 2$ and the extension is odd, its Fourier series is a pure sine series $\sum_{k=1}^{\infty} b_k \sin(k\pi x)$. $\boxed{+2}$

$$\sum_{1} = 12 + 2$$

2 a) The explicit form of (DE) is

$$\begin{aligned} &y'' + \frac{2(x+1)}{x-1}\,y' - \frac{4}{(x-1)^2}\,y = 0 \\ \iff\quad &y'' + \left(\frac{4}{x-1} + 2\right) y' - \frac{4}{(x-1)^2}\,y = 0 \end{aligned}$$

One sees that the coefficient of y' has a pole of order 1 at $x_0 = 1$, and the coefficient of y has a pole of order 2 at $x_0 = 1$. This implies that $x_0 = 1$ is a regular singular point. $\boxed{1}$

b) From a) we have, using the notation in the lecture, that $p_0 = 4$, $p_1 = 2$, $q_0 = -4$ and all other coefficients p_i, q_i are zero. $\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + 3r - 4 = (r-1)(r+4) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 1$ are $r_1 = 1$, $r_2 = -4$. We have $r_1 - r_2 = 5 \in \mathbb{Z}$, but we will show that nevertheless two fundamental solutions y_1, y_2 of the form

$$y_1(x) = (x-1)\sum_{n=0}^{\infty} a_n (x-1)^n, \quad y_2(x) = (x-1)^{-4}\sum_{n=0}^{\infty} b_n (x-1)^n \qquad \boxed{1}$$

exist. For this it will be convenient to use the abbreviation $t = x - 1$, which turns (DE) into

$$t^2 y'' + 2t(t+2)y' - 4y = t^2 y'' + (2t^2 + 4t)y' - 4y = 0.$$

First we determine $y_1(x)$. We have

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n t^{n+1}, \\ t^2 y_1''(x) &= \sum_{n=0}^{\infty} (n+1)n a_n t^{n+1}, \\ t^2 y_1'(x) &= \sum_{n=0}^{\infty} (n+1) a_n t^{n+2} = \sum_{n=1}^{\infty} n a_{n-1} t^{n+1}, \\ t y_1'(x) &= \sum_{n=0}^{\infty} (n+1) a_n t^{n+1}. \end{aligned}$$

Substituting these into (DE) gives, setting $a_{-1} = 0$,

$$\sum_{n=0}^{\infty} \left[(n+1)n a_n + 2n a_{n-1} + 4(n+1)a_n - 4a_n\right] t^{n+1} = \sum_{n=0}^{\infty} \left[(n+5)n a_n + 2n a_{n-1}\right] t^{n+1} = 0.$$

Equating coefficients gives the recurrence relation

$$a_n = -\frac{2}{n+5}a_{n-1} \quad \text{for } n = 1, 2, \dots.$$

The coefficient a_0 can be chosen freely. Using the normalization $a_0 = 1$ we obtain

$$\begin{aligned} y_1(x) &= (x-1) - \frac{2}{6}(x-1)^2 + \frac{2^2}{6\cdot 7}(x-1)^3 - \frac{2^3}{6\cdot 7\cdot 8}(x-1)^4 \pm \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{6\cdot 7\cdot 8\cdots(n+5)}(x-1)^{n+1}. \end{aligned} \qquad \boxed{2}$$

For the determination of $y_2(x)$ we repeat the computation with exponents decreased by 5. The result is

$$\sum_{n=0}^{\infty} \left[(n-4)(n-5)b_n + 2(n-5)b_{n-1} + 4(n-4)b_n - 4b_n\right] t^{n-4} =$$
$$= \sum_{n=0}^{\infty} \left[n(n-5)b_n + 2(n-5)b_{n-1}\right] t^{n-4} = 0.$$

Here we obtain the recurrence relation

$$b_n = -\frac{2}{n}\, b_{n-1} \quad \text{for } n = 1, 2, 3, 4, 6, \dots$$

and, setting $b_0 = 1$, $b_5 = 0$

$$y_2(x) = (x-1)^{-4} - \frac{2}{1!}(x-1)^{-3} + \frac{2^2}{2!}(x-1)^{-2} - \frac{2^3}{3!}(x-1)^{-1} + \frac{2^4}{4!}, \qquad \boxed{1\tfrac{1}{2}}$$

which is a finite sum!

Note: If instead we stipulate the recurrence relation $b_n = (-2/n)b_{n-1}$ for all n, then we obtain

$$\widetilde{y}_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}(x-1)^{n-4} = (x-1)^{-4}\mathrm{e}^{-2(x-1)} = y_2(x) - \frac{2^5}{5!}\, y_1(x).$$

Alternative solution: We use the general recurrence relation for the functions $a_n(r)$, viz. $a_0(r) = 1$ and

$$\begin{aligned} a_n(r) &= -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_{n-1}(r) \\ &= -\frac{1}{(r+n-1)(r+n+4)} \left[(r+n-1)2\right] a_{n-1}(r) = -\frac{2a_{n-1}(r)}{r+n+4}. \end{aligned}$$

Thus the coefficients $a_n(1)$ of $y_1(x)$ satisfy the recurrence relation $a_n(1) = -\frac{2\,a_{n-1}(1)}{n+5}$ (the same as for a_n above) and the coefficients $a_n(-4)$ of $y_1(x)$ satisfy the recurrence relation $a_n(-4) = -\frac{2\,a_{n-1}(-4)}{n}$ (the same as for b_n above, except that it holds also for

$n = 5$). The rest of the computation remains the same and leads to $y_1(x)$ and the nonterminating series solution $\widetilde{y}_2(x)$.

The general (real) solution on $(1, \infty)$ is then $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$, $c_1, c_2 \in \mathbb{R}$. $\boxed{\frac{1}{2}}$
That solutions are defined on the whole of $(1, \infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C} \setminus \{1\}$, but follows also readily from the easily established fact that the radius of convergence of both power series is ∞.

c) The solution on $(-\infty, 1)$ is exactly the same (i.e., the formulas for $y_1(x)$, $y_2(x)$ remain unchanged), since according to the general theory we have to replace $(x-1)^r$ by $|x-1|^r$. But this leaves $y_2(x)$ unchanged and changes only the sign of $y_1(x)$. The linear span of $y_1(x)$, $y_2(x)$ remains the same. $\boxed{1}$

Since $y_1(x)$ is analytic everywhere and $\lim_{x\to 1} y_2(x) = \infty$, the solution on $\mathbb{R}$ is $y(x) = c_1\, y_1(x)$, $c_1 \in \mathbb{R}$. $\boxed{+1}$

$$\sum_2 = 7 + 1$$

3 a) Substituting $y_1(t) = t^r$ into the ODE gives

$$r\, t^{r-1} = t^3 + 2\, t^{r-1} - t^{2r-1},$$

so that we can take $r = 2$ and $y_1(t) = t^2$. $\boxed{1}$

b) $y = t^2 + 1/z \Longrightarrow y' = 2t - z'/z^2$
Substituting this into (R) gives

$$\begin{aligned} 2t - \frac{z'}{z^2} &= t^3 + \frac{2}{t}\left(t^2 + \frac{1}{z}\right) - \frac{1}{t}\left(t^2 + \frac{1}{z}\right)^2 \\ &= t^3 + 2t + \frac{2}{tz} - t^3 - \frac{2t}{z} - \frac{1}{tz^2} \\ \Longleftrightarrow \quad z' &= \left(2t - \frac{2}{t}\right) z + \frac{1}{t}. \end{aligned}$$

This is of the form $z' = a(t)z + b(t)$, hence first-order (inhomogeneous) linear. $\boxed{2}$
Since $y = t^2 + 1/z$ is equivalent to $z = 1/(y - t^2)$ and $y(t) \neq t^2$ for all t in the domain of y (by the Uniqueness Theorem), this defines a one-to-one correspondence between solutions of (R) different from y_1 and solutions of $z' = \left(2t - \frac{2}{t}\right) z + 1/t$. $\boxed{\frac{1}{2}}$

c) The general solution of $z' = \left(2t - \frac{2}{t}\right) z$ is

$$z(t) = c \exp\left(\int 2t - \frac{2}{t}\, \mathrm{d}t\right) = c \exp\left(t^2 - 2\ln t\right) = c\, \mathrm{e}^{t^2}/t^2, \quad c \in \mathbb{R} \qquad \boxed{1}$$

Variation of parameters then yields a particular solution z_p of $z' = \left(2t - \frac{2}{t}\right) z + 1/t$:

$$z_p(t) = \mathrm{e}^{t^2}/t^2 \int t^2 \mathrm{e}^{-t^2} (1/t)\, \mathrm{d}t = \mathrm{e}^{t^2}/t^2 \left(-\tfrac{1}{2}\mathrm{e}^{-t^2}\right) = -\frac{1}{2t^2}. \qquad \boxed{1}$$

$\Longrightarrow$ The general solution of $z' = \left(2t - \frac{2}{t}\right) z + 1/t$ is

$$z(t) = -\frac{1}{2t^2} + c\,\mathrm{e}^{t^2}/t^2 = \frac{2c\,\mathrm{e}^{t^2} - 1}{2t^2}, \quad c \in \mathbb{R}.$$

$\Longrightarrow$ The general solution of (R) is

$$y_c(t) = t^2 + \frac{2t^2}{2c\,\mathrm{e}^{t^2} - 1} = t^2\,\frac{2c\,\mathrm{e}^{t^2} + 1}{2c\,\mathrm{e}^{t^2} - 1}, \quad c \in \mathbb{R} \cup \{\infty\},$$ $\boxed{1\frac{1}{2}}$

or, slightly simplified,

$$y_c(t) = t^2\,\frac{c\,\mathrm{e}^{t^2} + 1}{c\,\mathrm{e}^{t^2} - 1}, \quad c \in \mathbb{R} \cup \{\infty\}.$$

For $c = \infty$ we obtain the special solution $y(t) = t^2$, which would otherwise be missing.

The (maximal) solutions $y_\infty(t) = t^2$, $y_0(t) = -t^2$, and $y_c(t)$ for $c < 0$ are defined on $(0, \infty)$. For $c \geq 1$ the same is true, since then $c\,\mathrm{e}^{t^2} > 1$ for all $t > 0$.

"Solutions" $y_c(t)$ with $0 < c < 1$ do, strictly speaking, not form solutions of their own (because their maximal domain is not an interval) but give rise to two maximal solutions ("branches") $y_c^{(1)}(t)$, $y_c^{(2)}(t)$ with domains $\left(0, \sqrt{-\ln c}\right)$ and $\left(\sqrt{-\ln c}, \infty\right)$, respectively. $\boxed{+1}$

$$\sum_3 = 7 + 1$$

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4 The characteristic polynomial of $\mathbf{A}$ is

$$\chi_{\mathbf{A}}(X) = \begin{vmatrix} X-1 & 2 & 2 \\ 4 & X+1 & -2 \\ 0 & 0 & X+3 \end{vmatrix} = (X+3)\,[(X-1)(X+1) - 4 \cdot 2] = (X+3)(X^2-9)$$
$$= (X-3)(X+3)^2.$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$ with algebraic multiplicity 1 and $\lambda_2 = -3$ with algebraic multiplicity 2. $\boxed{2}$

$\underline{\lambda_1 = 3:}$

$$\mathbf{A} - 3\,\mathbf{I} = \begin{pmatrix} -2 & -2 & -2 \\ -4 & -4 & 2 \\ 0 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} -2 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda_1 = 3$ is generated by $\mathbf{v}_1 = (1, -1, 0)^\mathsf{T}$.

$\underline{\lambda_2 = -3:}$

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 4 & -2 & -2 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda_2 = -3$ is generated by $\mathbf{v}_2 = (1, 2, 0)^\mathsf{T}$ and $\mathbf{v}_3 = (1, 0, 2)^\mathsf{T}$.

A fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is therefore

$$\mathbf{y}_1(t) = \mathrm{e}^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = \mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3(t) = \mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$ $\boxed{3}$

$\mathbf{A}$ is invertible, since 0 is not an eigenvalue of $\mathbf{A}$.
$\Longrightarrow$ The system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ has the constant solution $\mathbf{y}(t) \equiv -\mathbf{A}^{-1}\mathbf{b}$, which is found by solving the system $\mathbf{A}\mathbf{x} = -\mathbf{b}$.

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ -4 & -1 & 2 & 0 \\ 0 & 0 & -3 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & -9 & -6 & 0 \\ 0 & 0 & -3 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$\Longrightarrow x_3 = 1$, $x_2 = -2x_3/3 = -2/3$, $x_1 = 2x_2 + 2x_3 = 2/3$
$\Longrightarrow$ The constant solution is $\mathbf{y}(t) \equiv \left(\frac{2}{3}, -\frac{2}{3}, 1\right)^{\mathsf{T}}$. $\boxed{1}$
$\Longrightarrow$ The general (real) solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ is

$$\mathbf{y}(t) = c_1\,\mathrm{e}^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2\,\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_3\,\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix}, \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

Fitting the initial conditions gives for c_1, c_2, c_3 the linear system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -\frac{2}{3} \\ -1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & -\frac{2}{3} \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{array}\right)$$

$\Longrightarrow c_3 = -1/2$, $c_2 = -c_3/3 = 1/6$, $c_1 = -2/3 - c_2 - c_3 = -1/3$
$\Longrightarrow$ The solution of the IVP is

$$\begin{aligned} \mathbf{y}(t) &= -\tfrac{1}{3}\,\mathrm{e}^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \tfrac{1}{6}\,\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \tfrac{1}{2}\,\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} - \frac{1}{3}\mathrm{e}^{3t} - \frac{1}{3}\mathrm{e}^{-3t} \\ -\frac{2}{3} + \frac{1}{3}\mathrm{e}^{3t} + \frac{1}{3}\mathrm{e}^{-3t} \\ 1 - \mathrm{e}^{-3t} \end{pmatrix}. \end{aligned} \qquad \boxed{2}$$

$$\sum_4 = 8$$

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5 a) $M(x,y) = 3xy + 2y^2 = y(3x+2y)$, $N(x,y) = 3x^2 + 6xy + 3y^2 = 3(x+y)^2$ have no common zero except $(0,0)$. $\Longrightarrow (0,0)$ is the only singular point. $\boxed{1}$

b) We have

$$M_y - N_x = 3x + 4y - (6x + 6y) = -3x - 2y = M(-1/y).$$

Thus $(M_y - N_x)/M$ depends only on y, and there is an integrating factor of the form $g(y)$.

The integrability condition $(gM)_y = (gN)_x$ then becomes $g'M + gM_y = gN_x$, i.e.,

$$g' = \frac{g(N_x - M_y)}{M} = \frac{g}{y}.$$

The solution of this ODE is $g(y) = cy$, so that we can take $g(y) = y$. $\boxed{2}$
$\Longrightarrow$ On $\mathbb{R}^2 \setminus x$-axis the ODE $(3xy + 2y^2)\,\mathrm{d}x + (3x^2 + 6xy + 3y^2)\,\mathrm{d}y = 0$ is equivalent to the exact ODE

$$\left(3xy^2 + 2y^3\right)\mathrm{d}x + \left(3x^2y + 6xy^2 + 3y^3\right)\mathrm{d}y = 0. \qquad \boxed{\tfrac{1}{2}}$$

An antiderivative f of the corresponding exact differential is determined in the usual way by "partial integration" with respect to x, say.

$$\begin{aligned} f(x,y) &= \tfrac{3}{2}\,x^2y^2 + 2\,xy^3 + g(y),\\ f_y(x,y) &= 3\,x^2y + 6\,xy^2 + g'(y) \overset{!}{=} 3x^2y + 6xy^2 + 3y^3 \end{aligned}$$

$\Longrightarrow g'(y) = 3\,y^3 \Longrightarrow g(y) = \frac{3}{4}\,y^4 + C \Longrightarrow f(x,y) = \frac{3}{2}\,x^2y^2 + 2\,xy^3 + \frac{3}{4}\,y^4 + C$
The general implicit solution of the exact ODE is then given by (in slightly simplified form and with a different C)

$$6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = C, \quad C \in \mathbb{R}. \qquad \boxed{2}$$

Solutions with $C < 0$ don't exist and for $C = 0$ the x-axis is obtained, since $6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = y^2(6x^2 + 8xy + 3y^2)$ and the quadratic has discriminant $8^2 - 4\cdot 6\cdot 3 = -8 < 0$.

Since the x-axis (equivalently, the function $y(x) \equiv 0$) is a solution of (DF), multiplication by y hasn't introduced any new solution, and $6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = C$, $C \geq 0$ solves (DF) as well. $\boxed{\tfrac{1}{2}}$

c) Yes. This is implicit in the preceding discussion. Intersection points of integral curves must be singular, so that the only candidate for such a point is the origin. But the corresponding contour of f, the 0-contour, consists of a single integral curve, viz. the x-axis. $\boxed{+1}$

$$\sum_5 = 6 + 1$$

6 The characteristic polynomial is

$$\begin{aligned} a(X) &= 2X^5 - X^3 + X^2 = X^2(2X^3 - X + 1)\\ &= X^2(X+1)(2X^2 - 2X + 1) \qquad \text{(since -1 is a root)}\\ &= 2X^2(X+1)(X^2 - X + \tfrac{1}{2})\\ &= 2X^2(X+1)\left(X - \frac{1+\mathrm{i}}{2}\right)\left(X - \frac{1-\mathrm{i}}{2}\right) \end{aligned}$$

with zeros $\lambda_1 = 0$ of multiplicity 2, $\lambda_2 = -1$, $\lambda_3 = (1+\mathrm{i})/2$, $\lambda_4 = (1-\mathrm{i})/2$.

$\Longrightarrow$ A complex fundamental system of solutions of the associated homogeneous equation is 1, t, e^{-t}, $\mathrm{e}^{(1+\mathrm{i})t/2}$, $\mathrm{e}^{(1-\mathrm{i})t/2}$, and the corresponding real fundamental system is

$$1, \quad t, \quad \mathrm{e}^{-t}, \quad \mathrm{e}^{t/2}\cos(t/2), \quad \mathrm{e}^{t/2}\sin(t/2). \qquad \boxed{2}$$

In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the two systems $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = 1 + t$, $b_2(t) = \mathrm{e}^{\mathrm{i}t}$. Superposition and extraction of the real part then yields the particular solution $y_p(t) = y_1(t) - 2\,\mathrm{Im}\,y_2(t)$. $\boxed{1}$

(1) Since $\mu = 0$ is a zero of multiplicity 2 of $a(X)$, the proper Ansatz in this case is $y_1(t) = c_0t^2 + c_1t^3$. Substituting it into the ODE we get

$$2\cdot 0 - 6\,c_1 + 2\,c_0 + 6\,c_1t = 1 + t.$$

$\Longrightarrow c_1 = 1/6$, $c_0 = 1$, and $y_1(t) = t^2 + t^3/6$. $\boxed{1\tfrac{1}{2}}$

(2) Since $a(\mathrm{D})\mathrm{e}^{\mathrm{i}t} = a(\mathrm{i})\mathrm{e}^{\mathrm{i}t}$ and $a(\mathrm{i}) \neq 0$, we can take $y_2(t) = \frac{1}{a(\mathrm{i})}\mathrm{e}^{\mathrm{i}t}$. Since $a(\mathrm{i}) = 2\mathrm{i}^5 - \mathrm{i}^3 + \mathrm{i}^2 = 3\mathrm{i} - 1$, we obtain

$$\begin{aligned} y_2(t) &= \frac{1}{-1+3\mathrm{i}}\,\mathrm{e}^{\mathrm{i}t} = \frac{-1-3\mathrm{i}}{10}\,(\cos t + \mathrm{i}\sin t)\,, \\ \operatorname{Im} y_2(t) &= -\frac{3}{10}\cos t - \frac{1}{10}\sin t. \end{aligned} \qquad \boxed{1\tfrac{1}{2}}$$

It follows that a particular real solution of the inhomogeneous equation is $y_p(t) = t^2 + \frac{1}{6}\,t^3 + \frac{3}{5}\cos t + \frac{1}{5}\sin t$ and that the general real solution of the inhomogeneous equation is

$$y(t) = t^2 + \frac{1}{6}\,t^3 + \frac{3}{5}\cos t + \frac{1}{5}\sin t + c_1 + c_2\,t + c_3\,\mathrm{e}^{-t} + c_4\,\mathrm{e}^{t/2}\cos(t/2) + c_5\,\mathrm{e}^{t/2}\sin(t/2),$$

$$c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}. \qquad \boxed{1}$$

$$\sum_{6} = 7$$

$$\sum_{\text{Final Exam}} = 47 + 5$$
