Name: ____________________ Student ID: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

$t=0$ $\quad \frac{d^2y}{dt^2}$ $\quad T=\frac{dV}{ds}$

1. Which of the following ODE's has distinct solutions $y_1, y_2\colon [-1,1] \to \mathbb{R}$ satisfying $y_1(0) = y_2(0)$?

   ☒ $y' = \sqrt{y^2+1}$ ☑ $t^2y'' = y$ ☒ $y' = \sqrt{|t|}\,y$ ☑ $(y')^3 = y$ ☒ $y' = t\,|y|$

   $y=0$

2. The ODE $(y^2+y)\,\mathrm{d}x - x\,\mathrm{d}y$ has the integrating factor

   ☐ 0 ☐ x^{-1} ☐ y^{-1} ☐ x^{-2} ☑ y^{-2}

3. For the solution $y(t)$ of the IVP $y' = y^4 - 4y^2,\ y(4.29) = 1$ the limit $\lim_{t\to+\infty} y(t)$ equals

   ☐ $-\infty$ ☐ -2 ☑ 0 ☐ 2 ☐ $+\infty$

   $y^2(y^2-4)$

4. For the solution $y(t)$ of the IVP $y' = y\ln t,\ y(1) = 1$ the value $y(\mathrm{e})$ is equal to

   ☐ e^{-2} ☐ e^{-1} ☐ 1 ☑ e ☐ e^2

   $e\cdot e^{t(\ln t-1)}$ $\quad e^{\int \ln t\,dt}$

5. For the solution $y\colon (0,\infty) \to \mathbb{R}$ of the IVP $t^2y'' - t\,y' + y = 1,\ y(1) = y'(1) = 0$ the value $y(\mathrm{e})$ is equal to

   $y=-t+t\ln t+1$

   ☐ $\ln 4$ ☑ 1 ☐ -1 ☐ $1+\ln 4$ ☐ $-1+\ln 4$

   $t=e^x$

6. The power series $z + \frac{1}{2}z^2 + \frac{1}{4}z^4 + \frac{1}{8}z^8 + \frac{1}{16}z^{16} + \cdots$ has radius of convergence

   ☐ 0 ☑ 1 ☐ $\sqrt{2}$ ☐ 2 ☐ ∞

   $\sum_{n=0}^{\infty} \frac{1}{2^n} z^{2^n}$ $\quad \lim_{n\to\infty} \frac{b(n+1)}{b(n)}$

7. The largest integer s such that $f_s(x) = \sum_{n=1}^{\infty} \frac{\cos(n^s x)}{n^3}$ is differentiable on $\mathbb{R}$ is equal to

   ☐ 0 ☑ 1 ☒ 2 ☐ 3 ☐ 4

   $f_s'(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2x)}{n}$ for $s=2$

8. For which choice of $f_n(x)$ does the function series $\sum_{n=1}^{\infty} f_n$ converge uniformly on $\mathbb{R}$?

   ☒ $f_n(x) = \sqrt[n]{x^2}/n^4$ ☒ $f_n(x) = n/(x^4+n^2)$ ☒ $f_n(x) = \sqrt{x^2+n}/n^4$
   ☑ $f_n(x) = n/(x^2+n^4)$ ☒ $f_n(x) = (-1)^n\ln(x^2+n)/n^4$

   $\frac{(-1)^n\ln(x^2+n)}{n^4}$

9. If $y(t)$ solves $y' = \dfrac{y+2t}{y+t}$ then $z(t) = y(t)/t$ solves

   ☑ $z' = \dfrac{2-z^2}{t(z+1)}$ ☐ $z' = \dfrac{z^2-2}{t^2(z+1)}$ ☐ $z' = \dfrac{z+2}{z+1}$ ☐ $z' = \dfrac{z+2}{t(z+1)}$

   ☐ $z' = \dfrac{2-z^2}{z+1}$

$\frac{x^{2/n}}{n^4}$ $\quad x^2 + \frac{x}{16} + \frac{x^{2/3}}{81}$

$\frac{1}{1+x^4} + \frac{2}{4+x^4} + \dots + \frac{n}{n^2+x^4}$

Continued on the back side

$y''+y=e$

$\phi_1(t)=-1+\int_0^t -1+t\,dt$
$=-1-t+\frac{t^2}{2}$ $\quad -1+\int_0^t -1 \quad +\frac{t^2}{2}\Big|_0^t$

10. The sequence $\phi_0, \phi_1, \phi_2, \dots$ of Picard-Lindelöf iterates for the IVP $\quad y' = y + t,\ y(0) = -1 \quad$ has $\phi_2(t)$ equal to

☐ $\frac{1}{6}t^3$ ☐ $-1-t-\frac{1}{2}t^2$ ☑ $-1-t+\frac{1}{6}t^3$ ☐ $-1-t-\frac{1}{2}t^2+\frac{1}{6}t^3$
☐ $-1-t$

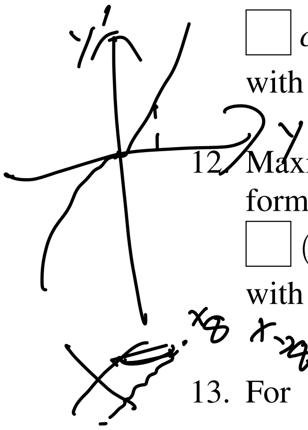
$5y^4+1$

$e^{(a+bi)x}$ $\quad C_1e^{ax}\sin bx$
$=e^{ax}\cdot e^{bxi}=e^{ax}(\cos bx + i\sin bx)$
$C_2e^{ax}\cos bx$

11. $y'' + y = \cos t \quad$ has a particular solution $y_p(t)$ of the form

☐ ct ☐ $c\cos t$ ☐ $c\sin t$ ☐ $ct\cos t$ ☑ $ct\sin t$

with a constant c.



$1-0\cdot t-1\cdot\frac{t^2}{2!}-2\cdot\frac{t^3}{3!}$ $\quad -(t-1)e^t=$ $\quad -\sum_{n=0}^{\infty}\frac{t^n}{n!}\cdot(n-1)=$

12. Maximal solutions of $\quad y' = y^5 + y \quad$ satisfying $y(0) = 1$ are defined on an interval of the form

$y'(0)=2>0$ $\quad \sum_{n=0}^{\infty}\frac{t^n}{n!}(1-n)=\sum_{n=0}^{\infty}\left(\frac{t^n}{n!}-\frac{t^n}{(n-1)!}\right)$

$y(y^4+1)$
$y=0$

☐ (a,b) ☐ $[a,b]$ ☐ $(a,+\infty)$ ☑ $(-\infty,b)$ ☐ $(-\infty,+\infty)$

with $a,b \in \mathbb{R}$.

$At=\begin{pmatrix}0 & -t\\ t & 2t\end{pmatrix}$ $\quad e^{\begin{pmatrix}0&-1\\1&2\end{pmatrix}t}$ $\quad =e^t-te^t$
$=(1-t)e^t$

13. For $\quad \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \quad$ and $t \in \mathbb{R}$, the matrix $e^{\mathbf{A}t}$ is equal to

☑ $\begin{pmatrix} (1-t)e^t & -t\,e^t \\ t\,e^t & (1+t)e^t \end{pmatrix}$ ☒ $\begin{pmatrix} 1 & e^t \\ e^{-t} & e^{2t} \end{pmatrix}$ ☒ $\begin{pmatrix} (1+t)e^t & t\,e^t \\ -t\,e^t & (1-t)e^t \end{pmatrix}$

☐ $\begin{pmatrix} (1-t)e^t & t\,e^t \\ -t\,e^t & (1+t)e^t \end{pmatrix}$ ☒ $\begin{pmatrix} (1+t)e^t & -t\,e^t \\ t\,e^t & (1-t)e^t \end{pmatrix}$

14. The matrix norm of $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ (subordinate to the Euclidean length on $\mathbb{R}^2$) is contained in the interval

☐ $[1,2]$ ☐ $(2,3]$ ☑ $(3,4]$ ☐ $(4,5]$ ☐ $(5,6]$

15. A map $T\colon M \to M$ satisfying $|T(x) - T(y)| \leq \frac{2021}{2022}|x - y|$ for all $x, y \in M$ must have a fixed point if M is equal to

☐ $(0,+\infty)$ ☐ $[0,1)$ ☑ $[0,+\infty)$ ☐ $(0,1)$ ☐ $\mathbb{Q}$

All intervals are in $\mathbb{R}$, and $\mathbb{Q}$ is the field of rational numbers.

16. This midterm exam was

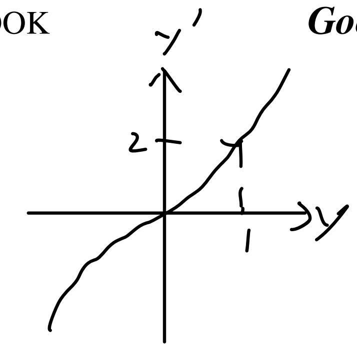
☐ too easy ☐ too difficult ☑ too long ☐ too short ☐ just appropriate

Time allowed: 50 min CLOSED BOOK ***Good luck!***

$\begin{pmatrix}2 & -3\\ 3 & 2\end{pmatrix}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$

$\begin{pmatrix}2\cos t-3\sin t\\ 3\cos t+2\sin t\end{pmatrix}$

$R=\sqrt{4c^2-12cs+9s^2+9c^2+12cs+4s^2}$
$=\sqrt{13}$



when $t\to\infty$, $y'>0$, $y\to\infty$, $y'\to\infty$
$\Rightarrow$ x有右界b $\Rightarrow$ 定义域可拓到 $,b)$
when $t\to-\infty$, $y'>0$, $y\to c$
$\Rightarrow$ x 左界为 $-\infty$ $\Rightarrow$ 定义域拓到 $(-\infty,$

$f = y' \Rightarrow f'$连续 $\Rightarrow f$满足 locally

$\sum_{n=0}^{\infty} \frac{1}{2^n} 2^{2^n} \Leftrightarrow \sum_{k=2^n}^{\infty} \frac{1}{2^k} 2^k = \sum_{k=2^n}^{\infty} a(k) 2^k$

$\therefore \lim_{k\to\infty} \frac{a(k+1)}{a_k} = \frac{2^{k+1}}{2^k} = 2$

由于从2^n转到k，使k不是逐一递增，有跳项

$\therefore \frac{1}{R} = \lim_{k\to\infty} \sqrt[k]{\frac{a(k+1)}{a_k}} = \lim_{n\to\infty} \sqrt[2^n]{2} = \lim_{n\to\infty} 2^{\frac{1}{2^n}} = 1 \Rightarrow R = 1$

$y' = \frac{2}{t+1} y - 1$ $\qquad -2t-1 \qquad \frac{2}{t+1}(-t)(t+1)$

$-2t-1$

$e^{\int \frac{2}{t+1} dt} = e^{2\ln(t+1)}$

$= (t+1)^2$

$(t+1)^{-2}$

$-2\ln(t+1)$

$c(t+1)^2 +$

-2

$(t+1)^2 \int -1 \, e^{\int -\frac{2}{t+1} dt} dt$

$(t+1)^2 \int \frac{-1}{(t+1)^2} dt$

$y(t) = -t^2 - 2t - 1 + t + 1$

$(t+1)^2 \cdot \frac{1}{t+1} = (t+1)$

$= -t^2 - t$

$c(t+1)^2 + t + 1 \qquad -(t+1)^2 + t + 1$

$y(0) = c + 1 = 0 \qquad c = -1$

$(x^4y^2 - y)dx + (x^2y^4 - x)dy = 0 \qquad \frac{1}{(xy)^2}$

$\frac{dM}{dy} = 2x^4y - 1 \;,\; \frac{dN}{dx} = 2y^4x - 1$

$$\therefore \frac{dM - dN}{M} = \frac{2xy(x^3-y^3)}{x^4y^2-y}$$

$y''' + y^{(n)}$ $\boxed{x_1}$ $\boxed{x_2} = \boxed{x_3}$ $\overset{1-j}{\boxed{x_4}}$ $\overset{1+j}{\boxed{x_5}}$

$c_1 e^{x_1 t}$

$c_2 t e^{x_2 t}$ $c_3 e^{x_2 t}$

$c_4 e^{\boxed{x_5} t}$ $\boxed{\varepsilon}$ $\cdot$ sin m

$c \in R$ x_1 m

$c e^{x_1 t}$ $m=1$ $te^{x_1 t}, e^{x_1 t}$, $m=2$

$m=3$: $t^2 e^{x_1 t}, te^{x_1 t}, e^{x_1 t}$