Math 286 Differential Equations Plus (Prof. Honold) **Midterm Exam** 2021/11/10

Name: ____________________ Student No.: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2\colon (0,2) \to \mathbb{R}$ satisfying $y_1(1) = y_2(1)$?

   ☒ $y' = \sqrt{t}\,|y|$ ☑ $y'' = yy'$ ☒ $y' = t\ln y$ ☒ $y' = y\ln t$ ☒ $yy' = 0$

   $y>0$

2. The ODE $3x\,\mathrm{d}x - (y - 3x^2/y)\,\mathrm{d}y$ has the integrating factor

   ☐ 0 ☐ x ☐ y ☐ x^2 ☑ y^2

$t\ln y = t$

$\frac{dt}{dy} = \frac{t}{y}$

3. For the solution $y(t)$ of the IVP $y' = y^4 - 1,\ y(2021) = 0$ the limit $\lim_{t\to+\infty} y(t)$ equals

   ☐ $-\infty$ ☑ -1 ☐ 0 ☐ 1 ☐ $+\infty$

4. For the solution $y(t)$ of the IVP $y' = e^{t-2y},\ y(0) = 0$ the value $y(1)$ is contained in

   ☐ $[0, \frac{1}{2}]$ ☑ $[\frac{1}{2}, 1]$ ☐ $[1, \frac{3}{2}]$ ☐ $[\frac{3}{2}, 2]$ ☐ $[2, \frac{5}{2}]$

5. For the solution $y\colon [0,\infty) \to \mathbb{R}$ of the IVP $(t+1)(y'+1) = 2y,\ y(0) = 0$ the value $y(1)$ is equal to

   ☑ -2 ☐ -4 ☐ -6 ☐ -8 ☐ -10

6. The power series $z + z^2 + z^4 + z^8 + z^{16} + \cdots$ has radius of convergence

   ☐ 0 ☐ $\frac{1}{2}$ ☑ 1 ☐ 2 ☐ ∞

7. The smallest integer a such that $f_a(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^a}$ is differentiable on $\mathbb{R}$ is equal to

   ☐ 1 ☐ 2 ☑ 3 ☐ 4 ☐ 5

8. For which choice of $f_n(x)$ does the function sequence (f_n) converge uniformly on $\mathbb{R}$?

   ☒ $f_n(x) = e^{-nx^2}$ ☑ $f_n(x) = x/(1+nx^2)$ ☒ $f_n(x) = 1/(1+nx^2)$
   ☒ $f_n(x) = x/(1+n^2)$ ☒ $f_n(x) = e^{-n^2x}$

9. If $y(t)$ solves $y' = 1 + y/t - y^2/t^2$ then $z = 1/\big(y(t) - t\big)$ solves

   ☐ $z' = z/t - 1/t^2$ ☐ $z' = tz + t^2$ ☐ $z' = tz - t^2$ ☐ $z' = t^2z + t$
   ☑ $z' = z/t + 1/t^2$

10. The sequence $\phi_0, \phi_1, \phi_2, \ldots$ of Picard-Lindelöf iterates for the IVP $y' = 2y + 2,\ y(0) = 2$ has $\phi_2(t)$ equal to

    ☐ $2t + 6t^2$ ☐ $2t + 5t^2$ ☑ $2 + 6t + 6t^2$ ☐ $2t + 4t^2$ ☐ $1 + 4t + 4t^2$

$\phi_1 = 2 + \int_0^t 4+2\, dt = 2+6t$

Continued on the back side

$\phi_2 = 2 + \int_0^t 6+12t\, dt = 2 + 6t + 6t^2$

$y'' = c_1e^t + (c_1+2c_2)e^t + (c_1+2c_2)te^t + 2c_2te^t + c_2t^2e^t = (2c_1+2c_2)e^t + (c_1+4c_2)te^t + c_2t^2e^t$

11. $y'' - 3y' + 2y = 2 + t\,\mathrm{e}^t$ has a particular solution $y_p(t)$ of the form

- ☐ $c_0 + c_1\mathrm{e}^t + c_2t\,\mathrm{e}^t$
- ☐ $c_0 + c_1t\,\mathrm{e}^t + c_2t^2\mathrm{e}^t$
- ☐ $c_0 + c_1t^2\mathrm{e}^t$
- ☐ $c_0t + c_1t\,\mathrm{e}^t + c_2t^2\mathrm{e}^t$
- ☐ $c_0t + c_1t^2\mathrm{e}^t$

$y' = c_0t + c_1e^t + c_1te^t + 2c_2te^t + c_2t^2e^t = c_0t + c_1e^t + (c_1+2c_2)te^t + c_2t^2e^t$

$c_0 + t(c_1+c_2t)e^t$

12. Maximal solutions of $y' = y^3 + 1$ satisfying $y(0) = 0$ are defined on an interval of the form

- ☐ (a,b)
- ☐ $[a,b]$
- ☐ $(a,+\infty)$
- ☐ $(-\infty,b)$
- ☐ $(-\infty,+\infty)$

with $a,b \in \mathbb{R}$.

$c_0(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}, \quad c_1(t) = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$

13. For $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$ and $t \in \mathbb{R}$, the matrix $\mathrm{e}^{\mathbf{A}t}$ is equal to

$\begin{pmatrix} e^{-t} & e^{2t} - e^{-t} \\ 0 & e^{2t} \end{pmatrix}$

- ☐ $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^{2t} - \mathrm{e}^{-t} \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$
- ☐ $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$
- ☐ $\begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$
- ☐ $\begin{pmatrix} -\mathrm{e}^{t} & \mathrm{e}^{t} \\ 0 & 2\mathrm{e}^{t} \end{pmatrix}$
- ☐ $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-t} - \mathrm{e}^{2t} \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & 0 \\ 0 & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & e^{2t} - e^{-t} \\ 0 & \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} \end{pmatrix}$

14. The matrix norm of $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ (subordinate to the Euclidean length on $\mathbb{R}^2$) is equal to

- ☐ 0
- ☐ 1
- ☐ $\sqrt{2}$
- ☐ 2
- ☐ $1+\sqrt{2}$

15. A contraction $T: M \to M$, $M \subseteq \mathbb{R}^2$, has a fixed point if M is

- ☐ infinite
- ☐ connected
- ☐ open
- ☐ closed
- ☐ bounded

Time allowed: 50 min | CLOSED BOOK | ***Good luck!***