$\frac{1}{\cos^2 y}$

Math 285 Differential Equations (Prof. Honold) **Midterm Exam** 2023/04/14

Name: ____________________ Student ID: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2 \colon \mathbb{R} \to \mathbb{R}$ satisfying $y_1(0) = y_2(0) = 1$?

   ?

   ■ $y' = y^{2/3}$ ☐ $y' = \sqrt{y+1}/y$ ☐ $y' = \tan y$ ☐ $t\,y' = y$ ☐ $y' = \ln|y|$

2. The ODE $\quad xy\,\mathrm{d}x + (1+x^2)\,\mathrm{d}y \quad$ has the integrating factor

   ☐ 0 ☐ 1 ☐ x ■ ✓ y ☐ xy

3. For the solution $y(t)$ of the IVP $\quad y' = y^3 - 4y^2,\ y(2023) = 1 \quad$ the limit $\lim\limits_{t\to+\infty} y(t)$ equals

   ☐ $-\infty$ ■ ✓ 0 $y^2(y-4)$ ☐ 2 ☐ 4 ☐ $+\infty$

4. For the solution $y(t)$ of the IVP $\quad y' = \frac{ty+1}{t^2+1},\ y(0) = 2 \quad$ the value $y(1)$ is equal to

   ☐ $\sqrt{2}$ ☐ 2 ☐ $1+\sqrt{2}$ ☐ 3 ■ ✓ $1+2\sqrt{2}$

5. For the solution $y(t)$ of the IVP $\quad y' = \left(y^2-3\right)/(ty),\quad y(1) = 2 \quad$ the value $y(2)$ is equal to

   ☐ $\sqrt{6}$ ■ ✓ $\sqrt{7}$ ☐ $\sqrt{8}$ ☐ 3 ☐ $\sqrt{10}$

   $y' = \frac{y^2-3}{ty} = \frac{y}{t} - \frac{3}{ty} = \frac{1}{t}\left(\frac{y^2-3}{y}\right)$ $\quad y' \cdot \frac{y}{y^2-3} = \frac{1}{t}$ $\quad \frac{2y}{y^2-3}$ $\quad \frac{dy}{dt} \cdot \frac{d(\ln y^2-3)}{dy} = \frac{2}{t}$

6. For the solution $y \colon (0,\infty) \to \mathbb{R}$ of the IVP $\quad 2t^2y'' - t\,y' - 2y = 0,\ \ y(1) = 0,\ y'(1) = 5$ the value $y(4)$ is equal to

   ☐ 5 ☐ 17 ☐ 29 ■ 31 ☐ 59

7. The power series $\displaystyle\sum_{n=1}^{\infty} z^{n!}$ (where $n! = 1\cdot 2\cdots n$) has radius of convergence

   ☐ 0 ☐ $1/\mathrm{e}$ ■ ✓ 1 ☐ e ☐ ∞

8. The smallest integer s such that $\displaystyle f_s(x) = \sum_{n=1}^{\infty} \frac{x\sin(nx)}{n^s+1}$ is differentiable on $\mathbb{R}$ is equal to

   ☐ 0 ☐ 1 ☐ 2 ■ ✓ 3 ☐ 4

   $\frac{d f_s(x)}{dx} = \sum_{n=1}^{\infty} \frac{\sin nx + nx\cos nx}{n^s+1}$

9. For which choice of $f_n(x)$ does the function sequence (f_n) converge uniformly on $[0,\infty)$?

   ?

   ☒ $n/(x+n)$ ■ $(x^2-x+n)/(x^2+n)$ ☒ $x/(x+n)$

   ☒ $(x+n)/(x+n^2)$ ☒ $(x+n)/(x^2+n)$

   $\sum n$

10. The family of curves $y = 1 + Cx^3$, $C \in \mathbb{R}$, solves the ODE

    ☐ $3x^2\,\mathrm{d}x - \mathrm{d}y = 0$ ☐ $3y\,\mathrm{d}x - x\,\mathrm{d}y = 0$ ☐ $3y\,\mathrm{d}x + x\,\mathrm{d}y = 0$

    ■ ✓ $3(y-1)\,\mathrm{d}x - x\,\mathrm{d}y = 0$ ☐ $(3x^2+1)\,\mathrm{d}x - x\,\mathrm{d}y = 0$

$(y-1) = cx^3$

$(y-1)x^{-3} = c$ $\quad -3x^{-4}(y-1)dx + x^{-3}dy = 0$

$3(y-1)dx - xdy = 0$

Continued on the back side

$\phi_1 = -1 + \int 1\, dt = -1+t$

$\phi_2 = -1 + \int (-1+t)^2 dt$

11. The sequence $\phi_0, \phi_1, \phi_2, \ldots$ of Picard-Lindelöf iterates for the IVP $y' = y^2 \wedge y(0) = -1$ has $\phi_2(t)$ equal to

$= -1 + \int (1-2t+t^2)\, dt = -1+t-t^2+\frac{t^3}{3}$

☐ $1+t+t^2+t^3$ ☑ $-1+t-t^2+\frac{1}{3}t^3$ ☐ $-1+t-t^2+t^3$

☐ $-1+t$ ☐ $1+t+t^2+\frac{1}{3}t^3$

12. $y''' - y' + 6 = \mathrm{e}^{-2t}$ has a particular solution $y_p(t)$ of the form

☐ $c_0 + c_1 t$ ☐ $c_0 t + c_1 t^2 \mathrm{e}^{-2t}$ ☐ $c_0 + c_1 t\, \mathrm{e}^{-2t}$ ☑ $c_0 t + c_1 \mathrm{e}^{-2t}$

☐ $c_0 + c_1 \mathrm{e}^{-2t}$

$x^3 - x = 0$ $\quad -6+e^{-2t}$

$x(x^2-1)=0$

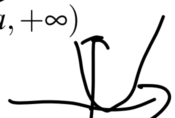
with constants $c_0, c_1 \in \mathbb{R}$. $\quad c_0 t + c_1 e^{-2t}$

$x_1 = 0,\ x_2 = -1,\ x_3 = 1$

13. Maximal solutions of $y' = y^2 - 2y + 1$ satisfying $y(0) = 0$ are defined on an interval of the form

☐ (a,b) ☐ $[a,b]$ ☑ $(a,+\infty)$ ☐ $(-\infty,b)$ ☐ $(-\infty,+\infty)$

with $a, b \in \mathbb{R}$.



$(y-1)^2$

$(a, +\infty)$

14. The matrix norm of $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -4 & 8 \end{pmatrix}$ (subordinate to the Euclidean length on $\mathbb{R}^2$) is contained in the interval

☐ $[1,3]$ ☐ $(3,5]$ ☐ $(5,7]$ ☐ $(7,9]$ ☑ $(9,11]$

15. For the matrix $\mathbf{A}$ in Question 14, the function $b_{21}(t)$ in $\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$ is equal to

☐ $-\frac{4}{9} + \frac{4}{9}\mathrm{e}^{9t}$ ☑ $\frac{4}{9} - \frac{4}{9}\mathrm{e}^{9t}$ ☐ $\frac{2}{9} - \frac{2}{9}\mathrm{e}^{9t}$ ☐ $\frac{2}{9} - \frac{4}{9}\mathrm{e}^{9t}$ ☐ $\frac{4}{9} - \frac{2}{9}\mathrm{e}^{9t}$

16. $(\mathrm{d}/\mathrm{d}e)e^t$ is equal to

☐ 1 ☐ e^{t-1} ☑ $t\,e^{t-1}$ ☐ e^t ☐ 0

Time allowed: 50 min 8+8 CLOSED BOOK ***Good luck!***

$x^2 - 9x = 0$ $\quad \begin{pmatrix} 1 & -2 \\ -4 & 8 \end{pmatrix}$

$x_1 = 0 \quad x_2 = 9$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{9} & \\ \frac{4}{9} - \frac{4}{9}e^{9t} & \end{pmatrix}$

$c_0 = a_1 + a_2 e^{9t} = 1$

$c_1 = a_3 + a_4 e^{9t} = -\frac{1}{9} + \frac{1}{9}e^{9t}$

$c_0(0) = a_1 + a_2 = 1 \quad a_1 = 1$

$c_0'(0) = 9a_2 = 0 \quad a_2 = 0$

$c_1(0) = a_3 + a_4 = 0 \quad a_3 = -\frac{1}{9}$

$c_1'(0) = 9a_4 = 1 \quad a_4 = \frac{1}{9}$

Notes

Notes have only been written for Group A. In those places where Group B differs from Group A, the difference is indicated briefly at the end of the note.

1 $y' = y^{2/3} = \sqrt[3]{y^2}$ is defined for all $(t,y) \in \mathbb{R}^2$ and behaves like $y' = \sqrt{|y|}$, which we have discussed in the lecture. The EUT doesn't apply, since the derivative of $y \mapsto y^{2/3}$ is unbounded near $y = 0$.
More precisely, there is the solution $y_1(t) = \frac{1}{27}t^3$ (obtained from the Ansatz $y(t) = ct^r$). Since $y_1'(0) = 0$,

$$y_2(t) = \begin{cases} \frac{1}{27}t^3 & \text{if } t \geq 0, \\ 0 & t < 0 \end{cases}$$

is also a solution. These solutions satisfy $y_1(3) = y_2(3) = 1$. Since $y' = y^{2/3}$ is automous, $t \mapsto y_1(t+3)$ and $t \mapsto y_2(t+3)$ are solutions as well, and have the required initial conditions.
The other 4 ODE's either satisfy the assumptions of the EUT globally ($y' = \tan y$ and $y' = \ln|y|$), or have no solutions with $y(0) = 1$ ($ty' = y$), or have non-uniqueness only at points that a solution with the given initial condition cannot reach ($y' = \sqrt{y+1}/y$).

2 Multiplying the ODE by y gives $xy^2\,\mathrm{d}x + (1+x^2)y\,\mathrm{d}y = 0$ which of the form $P\,\mathrm{d}x + Q\,\mathrm{d}y$ with $P_y = 2xy = Q_x$ and hence exact on $\mathbb{R}^2$. Answers B,C,D don't have this property. Answer A is also false: Zero is not considered as an integrating factor, since multiplication by zero renders the ODE useless.

3 The phase line can be used to answer this question. The ODE is of the form $y' = f(y)$ with $f(y) = y^3 - 4y^2 = y^2(y-4)$, which is negative in the intervall determined by adjacent zeros of f into which the starting value $y_0 = 1$ falls, viz. $(0,4)$. Hence $y(t)$ tends to the left end point of this interval for $t \to +\infty$.

4 This ODE is 1st-order linear with associated homogeneous ODE $y' = \frac{t}{t^2+1}y$. The solution of the latter is

$$y_h(t) = c\exp\left(\int \frac{t\,\mathrm{d}t}{t^2+1}\right) = c\exp\left(\frac{1}{2}\ln(t^2+1)\right) = c\sqrt{t^2+1}.$$

A particular solution of the inhomogeneous ODE is $y(t) = t$ (shame on you if you haven't found it!), and hence the general solution is $y(t) = t + c\sqrt{t^2+1}$, which has $y(0) = c$. In Group A the initial condition $y(0) = 2$ gives $y(1) = 1 + 2\sqrt{2}$, while in Group B $y(0) = \sqrt{2}$ gives $y(1) = 3$.

5 This is a separable ODE, which can be solved by the standard method (Group B comes first):

$$\begin{aligned}
\frac{y}{y^2-2}\,dy &= \frac{\mathrm{d}t}{t} \\
\int_2^y \frac{\eta}{\eta^2-2}\,\mathrm{d}\eta &= \int_1^t \frac{\mathrm{d}\tau}{\tau} \\
\left[\frac{1}{2}\ln\left(\eta^2-2\right)\right]_2^y &= [\ln\tau]_1^t \\
\frac{1}{2}\left(\ln\left(y^2-2\right) - \ln 2\right) &= \ln t \\
\ln\frac{y^2-2}{2} &= \ln\left(y^2-2\right) - \ln 2 = 2\ln t = \ln(t^2) \\
\frac{y^2-2}{2} &= t^2 \\
y &= \sqrt{2t^2+2}
\end{aligned}$$

$\Longrightarrow y(2) = \sqrt{10}$

In Group A the computation is

$$\begin{aligned}
\left[\frac{1}{2}\ln\left(\eta^2-3\right)\right]_2^y &= [\ln\tau]_1^t \\
\frac{1}{2}\left(\ln\left(y^2-3\right)-\ln 1\right) &= \ln t \\
\ln\left(y^2-3\right) &= 2\ln t = \ln(t^2) \\
y^2-3 &= t^2 \\
y &= \sqrt{t^2+3},
\end{aligned}$$

and $y(2)=\sqrt{7}$.

6 This Euler equation has a solution of the form $y(t)=t^k$, as argued in the lecture (or use Exercise H46 of Homework 8). Plugging this Ansatz into the ODE leads to $2k(k-1)-k-2=2k^2-3k-2=0=2(k-2)(k+1/2)=0$ with solutions $k=2$ and $k=1/2$. Hence the general (real) solution is

$$y(t)=c_1t^2+c_2\frac{1}{\sqrt{t}},\quad c_1,c_2\in\mathbb{R}.$$

The given initial conditions imply $c_1=2$, $c_2=-2$, $y(t)=2t^2-2/\sqrt{t}$, and $y(4)=31$.

7 The radius of convergence is 1, the same as for any power series with coefficients in $\{0,1\}$ that is not a polynomial; remember my remarks in the lecture.

8 For checking the differentiability of $f_s(x)$ one has to look at the series of derivatives, which is

$$\sum_{n=1}^{\infty}\frac{\sin(nx)+nx\cos(nx)}{n^s+1}=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^s+1}+x\sum_{n=1}^{\infty}\frac{n\cos(nx)}{n^s+1}.$$

For $s=3$ the two series on the right-hand side converge uniformly on $\mathbb{R}$ by the Weierstrass test, since $\sum_{n=1}\infty\frac{1}{n^3+1}$ and $\sum_{n=1}\infty\frac{n}{n^3+1}$ converge in $\mathbb{R}$. This implies the series of derivatives converges uniformly on all intervals of the form $[-R,R]$, $R>0$, which is sufficient to show that f_3 is differentiable on $\mathbb{R}$.
For $s=2$ the first series on the right-hand side converges still uniformly, but the second series doesn't since it behaves like $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$, which diverges at $x=0,\pm2\pi,\pm4\pi,\dots$. The factor x causes uniform convergence of the series of derivatives near $x=0$ but not at other multiples of 2π. Consequently, f_2 is not differentiable at $x=\pm2\pi,\pm4\pi,\dots$.

9 In (A) the point-wise limit is 1, but $\frac{n}{x+n}$ for fixed n can be made close to zero by choosing x large. Hence no uniform response to $\varepsilon<1$ can exist.
In (B) the point-wise limit is 1, and

$$\left|\frac{x^2-x+n}{x^2+n}-1\right|=\frac{x}{x^2+n}\leq\frac{x}{2x\sqrt{n}}=\frac{1}{2\sqrt{n}}\to 0\quad\text{for } n\to\infty,$$

showing uniform convergence.
In (C) the point-wise limit is 0, but $\frac{x}{x+n}$ for fixed n can be made close to 1 by choosing x large.
In (D) the point-wise limit is 0, but the same argument as in (C) applies.
In (E) the point-wise limit is 1, but

$$\left|\frac{x+n}{x^2+n}-1\right|=\frac{\left|x-x^2\right|}{x^2+n}.$$

For large x this is again close to 1 instead of zero.

10 Rewriting the equation as $(y-1)x^{-3}=C$, we see that the curves are the contours of $f(x,y)=(y-1)x^{-3}$ and hence satisfy the ODE

$$f_x\,\mathrm{d}x+f_y\,\mathrm{d}y=-3(y-1)x^{-4}\,\mathrm{d}x+x^{-3}\,\mathrm{d}y=0,$$

which expresses the orthogonality of the contours to the gradient ∇f. Multiplying by $-x^4$, this simplifies to $3(y-1)\,\mathrm{d}x-x\,\mathrm{d}y=0$.

11 $\phi_0(t)=-1$, $\phi_1(t)=-1+\int_0^t\phi_0(s)^2\,\mathrm{d}s=-1+\int_0^t\mathrm{d}s=-1+t$, $\phi_2(t)=-1+\int_0^t\phi_1(s)^2\,\mathrm{d}s=-1+\int_0^t(s-1)^2\,\mathrm{d}s=-1+\int_0^t(s^2-2s+1)\,\mathrm{d}s=-1+\left[s^3/3-s^2+s\right]_0^t=-1+t-t^2+t^3/3$.

12 This question contains a trap, viz. that it have characteristic polynomial $X^3-X+6=(X+2)(X^2-2X+3)$, which has $\mu=-2$ as root. But this is false, and the ODE in standard form is rather $y'''-y'=-6+e^{-2t}$ with characteristic polynomial X^3-X, which has $\mu=0$ but not $\mu=-2$ as a root, so that the correct Ansatz is $y=y_1+y_2$ with $y_1(t)=c_0t$, $y_2(t)=c_1\mathrm{e}^{-2t}$.

13 Since maximal solutions of IVPs are unique, the statement should have read “The maximal solution ...” rather than “Maximal solutions ...”.
Solutions $y=y(t)$ satisfy $\int_0^y\frac{\mathrm{d}\eta}{\eta^2-2\eta+1}=\int_0^y\frac{\mathrm{d}\eta}{(\eta-1)^2}=\int_0^t\mathrm{d}\tau=t$. We have $\lim_{y\uparrow 1}\int_0^y\frac{\mathrm{d}\eta}{(\eta-1)^2}=\lim_{y\uparrow 1}\left[-\frac{1}{\eta-1}\right]_0^y=\lim_{y\uparrow 1}\left(\frac{1}{1-y}-1\right)=+\infty$ and $a:=\lim_{y\downarrow-\infty}\int_0^y\frac{\mathrm{d}\eta}{(\eta-1)^2}=-\int_{-\infty}^0\frac{\mathrm{d}\eta}{(\eta-1)^2}\in\mathbb{R}$ (since this improper integral converges). This shows that the maximal solution is defined on $(a,+\infty)$.

14 $\|\mathbf{A}\|$ is equal to the square root of the largest eigenvalue of $\mathbf{A}^\mathsf{T}\mathbf{A}$, which in this case is $\left(\begin{smallmatrix}17&-34\\-34&68\end{smallmatrix}\right)$. The eigenvalues of this matrix are $\lambda_1=85$, $\lambda_2=0$, and hence the answer is $\sqrt{85}>9$ (more precisely, $\sqrt{85}\approx 9.22$).
A fast way to compute the eigenvalues of $\mathbf{A}^\mathsf{T}\mathbf{A}$ is the following: Since $\mathbf{A}^\mathsf{T}\mathbf{A}$ isn't invertible, one eigenvalue must be zero. Then the other eigenvalue must be equal to the trace of the matrix, which is 85.
Applying the same argument to $\mathbf{A}$ gives that its eigenvalues are 0 and 9. This implies $\|\mathbf{A}\|\geq 9$, since in general $\|\mathbf{A}\|\geq|\lambda|$ for any eigenvalue of $\mathbf{A}$. Thus all but two answers are excluded.
In Group B we have $\left(\begin{smallmatrix}17&34\\34&68\end{smallmatrix}\right)$, so that the answer is the same.

15 The last two answers can be excluded right away, because $\mathrm{e}^{\mathbf{A}0}$ is the 2×2 identity matrix, and hence $b_{21}(0)=0$.
The matrix $\mathbf{A}$ satisfies $\mathbf{A}^2=9\mathbf{A}$ (Cayley-Hamilton), and hence $\mathbf{A}^k=9^{k-1}\mathbf{A}$ for $k\geq 1$.

$$\Longrightarrow\ \mathrm{e}^{\mathbf{A}t}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sum_{k=1}^{\infty}\frac{9^{k-1}}{k!}A=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\mathrm{e}^{9t}-1}{9}\begin{pmatrix}1&-2\\-4&8\end{pmatrix}$$

Thus $b_{21}(t)=\frac{4}{9}-\frac{4}{9}\mathrm{e}^{9t}$.
Alternatively, use the method in Exercise H48 of Homework 8 to determine $\mathrm{e}^{\mathbf{A}t}$. A fundamental system of solutions of $(\mathrm{D}^2-9\mathrm{D})y=0$ is $\left\{1,\mathrm{e}^{9t}\right\}$, and the special fundamental system satisfying the initial conditions of H48 c) is determined from this as $c_0(t)=1$, $c_1(t)=\left(\mathrm{e}^{9t}-1\right)/9$. Thus $\mathrm{e}^{\mathbf{A}t}=\mathbf{I}_2+\frac{\mathrm{e}^{9t}-1}{9}\mathbf{A}$, the same as above.
In Group B the answer is $b_{21}(t)=-\frac{4}{9}+\frac{4}{9}\mathrm{e}^{9t}$.

16 Purportedly this was a favorite question of German mathematician ERNST WITT (1911–1991) when he examined Calculus students at Hamburg University.