Calculus 1

Exercises in preparation the exams

Indice

NOTA: both ln and log denote the logarithm in base e.

1 Two hours exams

Exam of the 17.01.2022

THEME 1

Exercise 1 [10 punti] Given the function

$$f(x) = \arctan\left(\frac{|x+1|}{x^2+4}\right),$$

(i) find the domain, study the sign, compute the limits at the extremes of the domain;
(ii) study the derivability of f sul suo domain, compute the first derivative, study the monotonicity intervals and points of absolute/relative maximum or minimum;
(iii) draw the graph.

Exercise 2 [7 punti] Determine the solutions in $\mathbb{C}$ of the equation

$$\left(\frac{z}{i}\right)^3 = -8.$$

Exercise 3 [7 punti]
Study, by utilizing sviluppi of Mac Laurin applicati alla sequence

$$a_n = \frac{1}{n} - \sin\left(\frac{1}{n}\right) - \alpha \log\left(1 + \frac{1}{n^3}\right),$$

the convergence of the series $\displaystyle\sum_{n=1}^{\infty} n^2\, a_n$ for every $\alpha \in \mathbb{R}$.

Exercise 4 [8 punti]
By making use the definizione (and the metodo of sostituzione), compute the integral generalizzato

$$\int_0^{+\infty} \frac{\arctan t}{(1+t^2)(\arctan^2 t + 8\arctan t + 17)}\, dt.$$

: Discutere, for all values of the parameter $\alpha \in \mathbb{R}$, the convergence of the generalized integral

$$\int_0^{+\infty} \frac{\arctan t}{(1+t^2)^{2\alpha}(\arctan^2 t + 8\arctan t + 17)}\, dt.$$

NB: con log si indica the logarithm in base e.
Tempo a disposizione: 2 ore.

Exam of the 07.02.2022

THEME 1

Exercise 1 [10 punti] Given the function

$$f(x) = \log\left(|x| - x^2 + 2\right),$$

(i) determine the domain; determine the simmetria and the sign; compute the limits and asymptotes at the extremes of the domain;
(ii) study the derivability and calcolarne the first derivative ; study the monotonicity intervals individuando the points of maximum and of minimum both relative and absolute ;
(iii) draw the graph.

Exercise 2 [7 punti] Determine the insieme A of the numeri complessi $z \in \mathbb{C}$ tali che

$$\frac{|z + i\mathcal{I}m(z)|^2}{|z|^2 + \mathcal{R}e(z)^2} \geq 1$$

and disegnarlo in the Gauss plane .

Exercise 3 [7 punti]
Study the convergence of the series

$$\sum_{n=1}^{\infty} n \left\{ \alpha \sinh\left(\frac{1}{n^2}\right) + \log\left[\cosh\left(\frac{1}{n}\right)\right]\right\}$$

as $\alpha \in \mathbb{R}$.

Exercise 4 [8 punti]
By making use the integration by parts, compute

$$\int \arctan\left(\frac{2}{x}\right) dx.$$

. Study the convergence of the integral improprio

$$\int_0^{+\infty} \arctan\left(\frac{x^3+1}{x^\alpha}\right) dx$$

as $\alpha > 0$.

Tempo a disposizione: 2 ore.

Exam of the 01.07.2022

THEME 1

Exercise 1 [9 punti] Consider the function

$$f(x) = |x-2|\, e^{\frac{1}{(x-2)^2}}.$$

(i) determine the domain of f and the sign of f;
(ii) compute the main limits of f;
(iii) compute the derivative of f, discuss the monotonicity of f and determine the infimum and the supremum of f and relative and absolute minimum and maximum points;

(iv) compute asymptotes of $f^{(*)}$;
(v) draw a qualitative graph of f.

$^{(*)}$ this question it is 1 point .

Exercise 2 [8 punti] Determine in algebraic form the solutions in $\mathbb{C}$ of the equation

$$z^4 + (-2 - 2i)z^2 + 4i = 0.$$

Exercise 3 [7 punti]
(i) Determine, as $\alpha \in \mathbb{R}$, the limit

$$\lim_{x \to 0^+} \frac{(1+x)^{\alpha x} - 1}{x^2}.$$

Exercise 4 [8 punti] (i) Compute the following indefinite integral

$$\int \frac{\sqrt{t}}{1+t}\, dt.$$

(ii) Discutere the convergence of the generalized integral

$$\int_0^{+\infty} \frac{\sqrt{t}}{1+t^\alpha}\, dt$$

as $\alpha \in \mathbb{R}$.

NB: con log si indica the logarithm in base e.

Tempo a disposizione: 2 ore.

Exam of the 12.09.2022

THEME 1

Exercise 1 [8 punti] Given the function

$$f(x) = \arctan\left(\frac{1}{\sin x}\right),$$

(i) find the domain, study the periodicity and the simmetria, calcolarne the sign, compute the limits at the extremes of the domain;
(ii) study the derivability of f sul suo domain, compute the first derivative, find the monotonicity intervals and the points of minimum and of maximum, both relative and absolute , and infimum and superiore;
(iii) draw the graph of f.

Exercise 2 [8 punti] Find the solutions $z \in \mathbb{C}$ of the inequality

$$\left|\frac{z-i}{z-1}\right| \geq 1$$

and le si segni on Gauss plane .

Exercise 3 [8 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\sin x - \alpha x + \frac{1}{6}\alpha x^3}{\arctan(x^2+4x^3)}$$

for every $\alpha \in \mathbb{R}$.

Exercise 4 [8 punti] (a) Compute the integral definito:

$$\int_{\log 4}^{\log 6} \frac{e^x}{(e^x-2)(e^x-1)}\, dx$$

(b) Al variare of $\alpha \in \mathbb{R}$ Study the convergence of

$$\int_{\log 4}^{+\infty} \frac{e^x}{(e^x-2)^\alpha(e^x-1)}\, dx.$$

NB: con log si inca the logarithm in base e.

Tempo a sposizione: 2 ore.

2 1h30m exams

Exam of the 06.07.2020 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = |(x+3)\log(x+3)|, \qquad x \in D =]-3, +\infty[.$$

(i) Compute

$$\lim_{x\to -3^+} f(x), \quad \lim_{x\to +\infty} f(x).$$

(ii) Compute the first derivative of the function f, study the monotonicity intervals and draw the graph of f.

Exercise 2 [6 punti] Find the solutions of the equation

$$z^3 = 8i$$

in algebraic and exponential form (or trigonometric), and draw them on the Gauss plane .

Exercise 3 [6 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(1+n^2)\log n}{n^4}.$$

Exercise 4 [6 punti] Compute

$$\int_0^{+\infty} e^{-\sqrt{2x}}\, dx.$$

Exercise 5 [6 punti] Compute the limit

$$\lim_{x\to+\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x+2} - \sqrt[6]{x^2-1}\right)^2.$$

Tempo a sposizione: 1 ore and 30 minuti.

Exam of the 14.09.2020 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = \arctan\left(\frac{x+1}{x-1}\right), \qquad x \in (1, \infty).$$

(i) Inviduarne the asymptotes.
(ii) If ne determini the monotonicity .

Exercise 2 [6 punti] Consider the complex number $z = \sqrt{3} - i$.
(i) Scriverlo in exponential form .
(ii) Compute the real part of z^6.

Exercise 3 [6 punti] Establish the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}.$$

Exercise 4 [6 punti] Compute the limit

$$\lim_{x \to 0^+} \frac{\log(1 + \sinh x) - \sin x}{x^2}.$$

Exercise 5 [6 punti] Consider the generalized integral

$$\int_1^{\infty} \log\left(\frac{x^\alpha}{x^\alpha + 1}\right) dx.$$

(i) Compute the integral for $\alpha = 2$.
(ii) Establish for which $\alpha \in [0, \infty)$ it converges.
Tempo a sposizione: 1 ore and 30 minuti.

Exam of the 18.01.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \arctan\left(\frac{x}{x^2 + x + 1}\right);$$

(i) inviduarne the domain, stuarne the sign, compute the limits at the extremes of the domain;
(ii) calcolarne the first derivative, study the monotonicity intervals inviduando the punti estremanti;
(iii) draw the graph of f.

Exercise 2 [8 punti] Find in $\mathbb{C}$ the solutions of the equation

$$z^4 + (-1 + i)z^2 - i = 0.$$

Suggerimento: sostituire $w = z^2$.

Exercise 3 [8 punti]
(i) Compute

$$\lim_{n \to \infty} \frac{n^{2n}}{(n+1)^{2n}}.$$

(ii) Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}.$$

Exercise 4 [8 punti] Per $\alpha \in \mathbb{R}$, si consideri

$$f_\alpha(x) = \frac{1}{\sinh x + x^\alpha}.$$

(a) Study as $\alpha \in \mathbb{R}$ the convergence

$$\int_0^{\log 2} f_\alpha(x)\, dx.$$

(b) Compute

$$\int_0^{\log 2} f_0(x)\, dx.$$

Tempo a sposizione: 1 ore and 30 minuti.

Exam of the 08.02.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \sqrt{\frac{|x|}{x^2+1}}.$$

(i) Determine the domain of f, study the sign and the simmetria of f and compute limits and asymptotes at the extremes of the domain;
(ii) Study the derivability of f and compute the first derivative, study the monotonicity intervals inviduando the points of maximum/ absolute minimum/relativo;
(iii) draw the graph of f.

Exercise 2 [8 punti] Find the complex solutions of the equation

$$\frac{8}{z^3} = \frac{1+i}{1-i},$$

in algebraic and exponential form (or trigonometric), and draw them on the Gauss plane .

Exercise 3 [8 punti]
(i) Compute

$$\int \log(t+1)\, dt.$$

(ii) Dedurre the value of

$$\int_0^1 \frac{\log(\sqrt{x}+1)}{\sqrt{x}}\, dx.$$

Exercise 4 [8 punti]
(i) Individuare as $\alpha \in \mathbb{R}$ the order diinfinitesimal l of

$$n\left(\cos(1/n) - 1\right) + \frac{\alpha}{n}$$

(ii) Study as $\alpha \in \mathbb{R}$ the convergence of

$$\sum_{n=1}^{+\infty} \left| n\left(\cos(1/n) - 1\right) + \frac{\alpha}{n} \right|.$$

Tempo a disposizione: 1 ore and 30 minuti.

Exam of the 05.07.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \log\left(1 + \sqrt{1 - x^2}\right).$$

(i) Determine the domain of f, study the sign and the simmetria of f and compute the limits at the extremes of the domain;
(ii) Study the derivability of f and compute the first derivative, study the monotonicity intervals and find the points of absolute /relative maximum and minimum ;
(iii) draw the graph of f.
Exercise 2 [8 punti] Find the complex solutions of the equation

$$\mathrm{Im}(z^2) + |z|^2\mathrm{Re}\left(\frac{1}{z}\right) = 0,$$

and draw them on the Gauss plane .
Exercise 3 [8 punti]
Sia

$$f_\alpha(x) := \frac{\arctan x}{1 + x^{2\alpha}}.$$

(i) Compute

$$\int f_1(x)\,dx = \int \arctan x \left(\frac{1}{1+x^2}\right)\,dx.$$

(ii) Study as $\alpha \in [0, \infty)$ the convergence of

$$\int_1^{+\infty} f_\alpha(x)\,dx.$$

Exercise 4 [8 punti]
(i) Compute as $\alpha \in \mathbb{R}$ the limit

$$\lim_{n\to\infty} \frac{2\log[\cos(1/n)] + \alpha[\sin(1/n)]^2}{(1/n)^2}.$$

(ii) Dedurre the comportamento of the series

$$\sum_{n=1}^{\infty} \left\{2\log[\cos(1/n)] + [\sin(1/n)]^2\right\}.$$

Tempo a disposizione: 1 ore and 30 minuti.

Exam of the 13.09.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \frac{|\sin x|}{1 - 2\cos x} \quad .$$

(i) Find the domain; study the periodicity , the sign and the simmetria of f;
(ii) study the derivability and calcolarne the first derivative ; study the monotonicity intervals and find the points of maximum/ absolute minimum/relativo;
(iii) draw the graph.

Exercise 2 [8 punti] Find the solutions $z \in \mathbb{C}$ of the inequality

$$\left| \frac{z+1}{z} \right| \geq 1$$

and draw them on the Gauss plane .

Exercise 3 [8 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} n^{\alpha} \left(\frac{1}{n} - \sin \frac{1}{n} \right)$$

as $\alpha \in \mathbb{R}$.

Exercise 4 [8 punti]
Compute the integral

$$\int_{-1}^{0} \frac{x}{x^2 + 2x + 2} \, dx.$$

Tempo a disposizione: 1 ore and 30 minuti.

3 2h30m Exams

Traccia 1

1) Sia

$$f(x) = \frac{|x-1|-2}{x^2+1}, \quad x \in [-2, 4].$$

Studiarne the sign, the derivability, the monotonicity and the massimi and minimi locali and absolute .

2) Compute

$$\lim_{x \to 0} \frac{\sin x^3}{x(1-\cos x)+x^4}.$$

3) Compute le radici terze of $-27i$.

4) (a) Compute

$$\int_0^{\frac{\pi^2}{4}} \sqrt{x} \sin \sqrt{x} dx.$$

(b*) Determine for which $\alpha \in \mathbb{R}$ the following integral converges:

$$\int_0^{\sqrt{\frac{\pi}{2}}} x^\alpha \sin \sqrt{x} dx.$$

5*) Discutere, for all values of the parameter $\alpha \in \mathbb{R}$, the convergence of the series

$$\sum_{n=1}^{\infty} \log \left(n\left(e^{\frac{1}{n}} - 1\right) - \frac{\alpha}{n} \right).$$

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Tempo a disposizione: 2 ore and 30 minuti. **Si consiglia disvolgere for primi the esercizi senza the asterisco.**

Ogni affermazione deve essere adeguatamente giustificata.

Traccia 2

1) Sia

$$f(x) = \log(|x-1|+1) - \log x, \quad x \in]0,2].$$

Simplify it and study the derivability, the monotonicity and the massimi and minimi locali and absolute .

2) Compute

$$\lim_{x\to 0} \frac{1-\cos x^2}{x(x-\sin x) + x^3 \sin^2 x.}$$

3) Simplify the expression

$$\frac{\overline{(1+i)}^2}{(1-i)^2 \left(\frac{-1}{i} + \sqrt{3}\right)}$$

writing the result in algebraic form and in trigonometric form .

4) (a) Compute

$$\int_{\log \pi}^{2\log \pi} e^{2x} \cos e^x \, dx.$$

(b*) Study the convergence of the generalized integral

$$\int_{-\infty}^{\log \pi} e^{\alpha x} \cos e^x dx$$

as $\alpha \in \mathbb{R}$.

5*) Sia

$$f(x) = \int_{\sqrt{\frac{\pi}{2}}}^{x} \sin(t^2) \, dt.$$

(a) Computethe Taylor expansion of f diorder 2 at the point $x_0 = \sqrt{\frac{\pi}{2}}$ $\big($letting the value of $f(\sqrt{\frac{\pi}{2}})$ as known$\big)$;

(b) study the monotonicity and the convexity and the concavity of f in the interval $[-1, 2]$;

(c) compute , as $\lambda \in \mathbb{R}$, the numero of the solutions of the equation $f(x) = \lambda$ contained in the interval $[-1, 2]$.

Tempo a disposizione: 2 ore and 30 minuti. **Si consiglia disvolgere for primi the esercizi senza the asterisco.**

Ogni affermazione deve essere adeguatamente giustificata.

Traccia 3

1) Sia

$$f(x) = \arctan|x^2 - 1|, \quad x \in [-1, 2].$$

Studiarne the derivability, the monotonicity and the massimi and minimi locali and absolute .

2) Compute

$$\lim_{x\to 0} \frac{(1-\cosh x)^2}{\sin x(x - \arctan x) + x^3 \sinh^2 x.}$$

3) Solve the equation

$$(z^2 + 2i)(z^3 + 8) = 0$$

writing the result in algebraic form and in trigonometric form .

4) (a) Compute

$$\int_{\log 3}^{1} \frac{e^x}{e^{2x} - 3e^x + 2}\, dx.$$

(b*) Study the convergence of the generalized integral

$$\int_{\log 2}^{+\infty} \frac{e^x}{(e^{2x} - 3e^x + 2)^\alpha}\, dx$$

as $\alpha \in \mathbb{R}$.

5*) Sia

$$f(x) = \begin{cases} \alpha(\arctan\sin x + 1) & \text{for } x \le 0 \\ e^{x+1} & \text{for } x > 0. \end{cases}$$

(a) Determine all the $\alpha \in \mathbb{R}$ such that the graph of f admits a tangent line in $(0, f(0))$ and compute it for tali α;

(b) discuss , as $\lambda \in \mathbb{R}$, the numero of the solutions of the equation $f(x) = e + \lambda x$ contained in the interval $[0, 2]$.

Tempo a disposizione: 2 ore and 30 minuti. **Si consiglia disvolgere for primi the esercizi senza the asterisco.**

Ogni affermazione deve essere adeguatamente giustificata.

Traccia 4

1) [6 punti] Study the absolute convergence and the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(\sin x)^n}{n}$$

for all values of the parameter $x \in [0, 2\pi[$.

2) [4 punti] Compute

$$\lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{\arctan x}.$$

3) [4 punti] Solve the inequality

$$\left| e^{i \operatorname{Re} z} (\bar{z} - i) \right| \leq 1$$

and draw the solutions in the Gauss plane .

4) [4 + 4 punti]

(a) Compute

$$\int_0^2 \sqrt{4 - x^2}\, dx \qquad \text{(eseguire a sostituzione iperbolica).}$$

(b*) Study the convergence of the generalized integral

$$\int_0^2 (4 - x^2)^\alpha \, dx$$

as $\alpha \in \mathbb{R}$.

5*) [7 punti] Consider the function

$$f(x) = |1 - x| \, e^{\arctan(4/x)}.$$

1) Determine the domain, compute the main limits of f and determine the asymptotes.
2) Compute f' nei punti where is possibile and determine the monotonicity intervals and the points of extreme of f.
3) Draw a graph of f.

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Tempo a disposizione: 2 ore and 30 minuti. **Si consiglia disvolgere for primi the esercizi senza the asterisco.**

Ogni affermazione deve essere adeguatamente giustificata.

4 Other exams

Limits .
1) Compute the limits

$$\lim_{x\to 2}\frac{\sqrt{x}-2}{x-2},\qquad \lim_{x\to-\infty} x\log\frac{1-x}{3-x},\qquad \lim\left(\frac{x^2}{x^2-2}\right)^{x^2}.$$

2) Compute the limits

$$\lim_{n\to\infty}\frac{2^n-n!\sin\frac{1}{n}-n}{2^{n-1}+(n-1)!},\qquad \lim_{n\to\infty}\frac{e^n-n\sin n}{n!-2^n}.$$

Series.
1) Determine the character of the series

$$\sum_{n=1}^{\infty}2^{\frac{1}{n}},\qquad \sum_{n=1}^{\infty}\frac{1-\cos n}{n^2},\qquad \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+n-1}$$

2) Study the convergence and the absolute convergence for all values of the parameter $x\in\mathbb{R}$ of the series

$$\sum_{n=1}^{\infty}\left(1-\cos\frac{x^n}{n}\right),\qquad \sum_{n=1}^{\infty}\arctan\frac{x^n}{n}.$$

Funzioni.
1) Discutere the derivability and compute le derivate prime and seconde of the functions

$$f_1(x)=\log\frac{1}{\cos x},\qquad f_2(x)=\log|\sin x-\frac{1}{2}|$$

in the loro domain.
2) Verificare the identità

$$\arcsin x=\arctan\frac{x}{\sqrt{1-x^2}},\qquad x\in]-1,1[.$$

3) Study the monotonicity and determine the points of maximum and minimum relative and absolute di

$$f_1(x)=\sin x-x\cos x,\qquad f_2(x)=\sqrt{x}-\sqrt{x-1}\ (\text{for } x\in[0,1]),\qquad \arctan\left|x-\frac{1}{x}\right|\ (\text{for } x\neq 0).$$

4) Study the numero disolutions of the equation

$$x-\log|x|=\alpha\qquad (x\neq 0)$$

as $\alpha\in\mathbb{R}$.

Integrali.
1) Computethe Taylor expansion of order 3 con centro $x_0=1$ of the functions

$$F_1(x)=\int_1^x\frac{e^t}{t}\,dt\qquad F_2(x)=\int_1^x\frac{\log t}{t}\,dt$$

and dire if in the interval $[1,2]$ they are invertibili.

2) Compute the integral i

$$\int x\log^2 x\,dx,\quad \int_0^1 \frac{x^2-4}{x^2+5x+4}\,dx,\quad \int \frac{x}{\cos^2 x}\,dx,\quad \int_1^{+\infty} \frac{dx}{x^2+x},\quad \int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx.$$

3) Study the convergence degli integral i

$$\int_0^{+\infty} \frac{x^\alpha}{\sqrt{e^x-1}}\,dx,\quad \int_0^{+\infty} \frac{dx}{x^\alpha+x^2}$$

as $\alpha \in \mathbb{R}$.

5 Exams of the last 4 years (see solutions at the end)

Exam of the 23.01.2017

THEME 1

Exercise 1 [6 punti] Compute the integral

$$\int_{\log 3}^{2} \frac{e^x}{e^{2x}-4}\, dx$$

Exercise 2 [6 punti] Solve the inequality

$$\left|2z^2 - 2\bar{z}^2\right| < 3$$

and draw the solutions on Gauss plane .

Exercise 3 [6 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} n^2\Big(\cos\frac{1}{n} - 1 + \sin\frac{1}{2n^\alpha}\Big)$$

for all values of the parameter $\alpha > 0$.

Exercise 4 [8 punti] Consider the function

$$f(x) := \arcsin\frac{|x|-4}{x^2+2}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D;
ii) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute and compute the main limits of f';
iii) draw a qualitative graph of f.

Exercise 5 [6 punti] Study the convergence of the generalized integral

$$\int_0^{+\infty} \frac{\arctan x\ |\arctan(x-1)|}{|1-x^2|^\alpha\ (\sinh\sqrt{x})^\beta}\, dx$$

as $\alpha, \beta \in \mathbb{R}$.

Exercise . Sia I a interval chiuso and limitato and sia $f : I \to \mathbb{R}$ a function continuous and tale che $f(x) \in I$ for every $x \in I$. Dimostrare that esiste almeno a $x \in I$ tale che $f(x) = x$.

THEME 2

Exercise 1 [6 punti] Compute the integral

$$\int_0^1 \frac{e^x}{e^{2x}+4e^x+5}\, dx$$

Exercise 2 [6 punti] Solve the inequality

$$\left|4\bar{z}^2 - 4z^2\right| < 5$$

and draw the solutions on Gauss plane .

Exercise 3 [6 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} n^2\big(2 - e^{1/2n^\alpha} - \cos(1/n)\big)$$

for all values of the parameter $\alpha > 0$.

Exercise 4 [8 punti] Consider the function

$$f(x) := \arcsin \frac{4-|x|}{1+2x^2}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D;
ii) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute and compute the main limits of f';
iii) draw a qualitative graph of f.

Exercise 5 [6 punti] Study the convergence of the generalized integral

$$\int_0^{+\infty} \frac{|\arctan(x-2)|\ \arctan x}{|x^2-4|^\alpha\ (\sinh \sqrt[3]{x})^\beta}\, dx$$

as $\alpha, \beta \in \mathbb{R}$.

THEME 3

Exercise 1 [6 punti] Compute the integral

$$\int_{\log 4}^{3} \frac{e^x}{e^{2x}-9}\, dx$$

Exercise 2 [6 punti] Solve the inequality

$$\left|3z^2 - 3\bar{z}^2\right| < 2$$

and draw the solutions on Gauss plane .

Exercise 3 [6 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} n^2\big(\cosh(1/n^\alpha) + \cos(1/n) - 2\big)$$

for all values of the parameter $\alpha > 0$.

Exercise 4 [8 punti] Consider the function

$$f(x) := \arcsin \frac{|x|-4}{2x^2+3}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D;
ii) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute and compute the main limits of f';
iii) draw a qualitative graph of f.

Exercise 5 [6 punti] Study the convergence of the generalized integral

$$\int_0^{+\infty} \frac{|\arctan(3-x)|\ \arctan x}{|9-x^2|^\alpha\ (\cosh\sqrt{x}-1)^\beta}\,dx$$

as $\alpha, \beta \in \mathbb{R}$.

THEME 4

Exercise 1 [6 punti] Compute the integral

$$\int_0^1 \frac{e^x}{e^{2x}-4e^x+5}\,dx$$

Exercise 2 [6 punti] Solve the inequality

$$\left|9\bar{z}^2 - 9z^2\right| < 2$$

and draw the solutions on Gauss plane .

Exercise 3 [6 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} n^2\big(e^{1/n^2} - \tan 1/n^\alpha - 1\big)$$

for all values of the parameter $\alpha > 0$.

Exercise 4 [8 punti] Consider the function

$$f(x) := \arcsin\frac{4-4|x|}{5x^2+3}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D;
ii) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute and compute the main limits of f';
iii) draw a qualitative graph of f.

Exercise 5 [6 punti] Study the convergence of the generalized integral

$$\int_0^{+\infty} \frac{\arctan x\ |\arctan(1-2x)|}{|1-4x^2|^\alpha\ (\cosh x-1)^\beta}\,dx$$

as $\alpha, \beta \in \mathbb{R}$.

Exam of the 13.02.2017

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) := \log\left|x^2 - 2x - 3\right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine the points of extreme relative and absolute of f;
(iii) compute f'' and study the concavity and the convexity of f;
(iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{n^n}{n!}.$$

Exercise 3 [4 punti] Given

$$f(z) = \frac{2 + iz}{iz + 1},$$

determine the domain and determine all the $z \in \mathbb{C}$ tali che $f(z) = z$. Express tutte the solutions in algebraic form.

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\arctan x - \sin x + x^{\frac{10}{3}} \log x}{x^\alpha (1 - \cos^2 x)}$$

as $\alpha > 0$.

Exercise 5 [8 punti] Study the convergence of the generalized integral

$$\int_2^{+\infty} \frac{1}{x^\alpha \sqrt{x-2}}\, dx$$

as $\alpha \in \mathbb{R}$ and calcolarlo for $\alpha = 1$.

THEME 2

Exercise 1 [8 punti] Consider the function

$$f(x) := \log\left|x^2 + x - 6\right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine relative and absolute extreme points of f;
(iii) compute f'' and study the concavity and the convexity of f;
(iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} \left(\frac{2}{3}\right)^n \frac{n^n}{n!}.$$

Exercise 3 [4 punti] Given

$$f(z) = \frac{-1-2iz}{iz-1},$$

determine the domain and determine all the $z \in \mathbb{C}$ tali che $f(z) = 2z$. Express tutte the solutions in algebraic form.

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\arctan x - \sinh x + x^{\frac{11}{2}} \log x}{x^\alpha(1-\cosh^2 x)}$$

as $\alpha > 0$.

Exercise 5 [8 punti] Study the convergence of the generalized integral

$$\int_3^{+\infty} \frac{1}{x^\alpha \sqrt{x-3}}\, dx$$

as $\alpha \in \mathbb{R}$ and calcolarlo for $\alpha = 1$.

THEME 3

Exercise 1 [8 punti] Consider the function

$$f(x) := \log\left|x^2 - 2x - 8\right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine the points of extreme relative and absolute of f;
(iii) compute f'' and study the concavity and the convexity of f;
(iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} \frac{1}{3^n} \frac{n^n}{n!}.$$

Exercise 3 [4 punti] Given

$$f(z) = \frac{-2+3iz}{2iz-3},$$

determine the domain and determine all the $z \in \mathbb{C}$ tali che $f(z) = -z$. Express tutte the solutions in algebraic form.

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{x^{\frac{9}{2}} \log x - \tan x + \sin x}{x^\alpha(1-\cosh^2 x)}$$

as $\alpha > 0$.

Exercise 5 [8 punti] Study the convergence of the generalized integral

$$\int_4^{+\infty} \frac{1}{x^\alpha \sqrt{x-4}}\, dx$$

as $\alpha \in \mathbb{R}$ and calcolarlo for $\alpha = 1$.

THEME 4

Exercise 1 [8 punti] Consider the function

$$f(x) := \log \left|x^2 + 3x - 4\right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine the points of extreme relative and absolute of f;
(iii) compute f'' and study the concavity and the convexity of f;
(iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{n=1}^{+\infty} \left(\frac{2}{7}\right)^n \frac{n^n}{n!}.$$

Exercise 3 [4 punti] Given

$$f(z) = \frac{1-4iz}{iz+4},$$

determine the domain and determine all the $z \in \mathbb{C}$ tali che $f(z) = z$. Express tutte the solutions in algebraic form.

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\sinh x - \tan x - x^{\frac{15}{4}} \log x}{x^\alpha (1-\cos^2 x)}$$

as $\alpha > 0$.

Exercise 5 [8 punti] Study the convergence of the generalized integral

$$\int_5^{+\infty} \frac{1}{x^\alpha \sqrt{x-5}}\, dx$$

as $\alpha \in \mathbb{R}$ and calcolarlo for $\alpha = 1$.

Exam of the 10.07.2017

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) := \log \left| e^{2x} - 4 \right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine the points of extreme relative and absolute of f;
iii) compute f'' and study the concavity and the convexity of f;
iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Draw in the Gauss plane the insieme

$$S := \left\{ z \in \mathbb{C} : \operatorname{Re} \frac{z-1}{z-i} \geq 0,\ |z+1-i| \leq 1 \right\}.$$

Exercise 3 [5 punti] Compute the integral

$$\int e^{2x} \arctan(3e^{x}) dx.$$

Exercise 4 [7 punti] Compute the limit

$$\lim_{x \to 0^+} \frac{\arctan \sin x - \sinh x}{x^{\alpha}(1 - \cos^2 x)}$$

for all values of the parameter $\alpha > 0$.

Exercise 5 [7 punti] Study the convergence semplice and assoluta di

$$\sum_{n=2}^{+\infty} \frac{(1-e^{a})^n}{n + \sqrt{n}}$$

as $a \in \mathbb{R}$.

THEME 2

Exercise 1 [8 punti] Consider the function

$$f(x) := \log \left| e^{-3x} - 9 \right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine the points of extreme relative and absolute of f;
iii) compute f'' and study the concavity and the convexity of f;
iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Draw in the Gauss plane the insieme

$$S := \left\{ z \in \mathbb{C} : \operatorname{Re} \frac{z+1}{z-i} > 0,\ |z-1-i| \leq 1 \right\}.$$

Exercise 3 [5 punti] Compute the integral

$$\int e^{2x} \arctan(2e^x) dx.$$

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\sin \arctan x - \sinh x}{x^\alpha (1 - \cosh^2 x)}$$

for all values of the parameter $\alpha > 0$.

Exercise 5 [7 punti] Study the convergence semplice and assoluta of

$$\sum_{n=2}^{+\infty} \frac{(1-2^a)^n}{n + \log n}$$

as $a \in \mathbb{R}$.

Exam of the 18.09.2017

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) := \frac{3x}{\log |2x|}.$$

i) Determine the domain D and study le simmetries and the sign of f; determine the limits of f at the extremes of D, the prolungabilità of f and the asymptotes;
ii) study the derivability, compute the derivative and its main limits , study the monotonicity e determine the points of extreme relative and absolute of f;
iii) compute f'' and study the concavity and the convexity of f;
iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Given the polynomial

$$z^4 + z^3 + 8thez + 8i$$

determine prima a Root Test intera and le other roots , writing them in algebraic form.

Exercise 3 [5 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{3x}{n}\right)^{n^2}$$

as $x \in \mathbb{R}$.

Exercise 4 [7 punti] Compute, for all values of the real parameter α, the limit

$$\lim_{x\to 0^+} \frac{\cosh(\alpha x) - e^{x^2} + x \log(\cos x)}{x - \sin x + e^{-1/x^2}}.$$

Exercise 5 [7 punti] Study the convergence of the generalized integral

$$\int_0^{+\infty} xe^{ax}(2+\cos x)\,dx$$

as $a \in \mathbb{R}$. Compute

$$\int_0^{+\infty} xe^{-x}\cos x\,dx$$

(sugg.: compute preliminarly a primitive of $e^{-x}\cos x$).

THEME 2

Exercise 1 [8 punti] Consider the function

$$f(x) := \frac{2x}{\log|3x|}.$$

i) Determine the domain D and study le simmetries and the sign of f; determine the limits of f at the extremes of D, the prolungabilità of f and the asymptotes;
ii) study the derivability, compute the derivative and its main limits , study the monotonicity e determine the points of extreme relative and absolute of f;
iii) compute f'' and study the concavity and the convexity of f;
iv) draw a qualitative graph of f.

Exercise 2 [5 punti] Given the polynomial

$$z^4 - z^3 - 27thez + 27i$$

determine prima a Root Test intera and le other roots , writing them in algebraic form.

Exercise 3 [5 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty}\left(1-\frac{2x}{n}\right)^{n^2}$$

as $x \in \mathbb{R}$.

Exercise 4 [7 punti] Compute, for all values of the real parameter α, the limit

$$\lim_{x\to 0^+} \frac{\cos x - e^{\alpha x^2} + x\log(\cosh x)}{x - \sinh x + e^{-1/x^2}}.$$

Exercise 5 [7 punti] Study the convergence of the generalized integral

$$\int_0^{+\infty} xe^{ax}(2-\sin x)\,dx$$

as $a \in \mathbb{R}$. Compute

$$\int_0^{+\infty} xe^{-x}\sin x\,dx$$

(sugg.: compute preliminarly a primitive of $e^{-x}\sin x$).

Exam of the 29.01.2018

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) := \log \frac{|x^2 - 5|}{x+1}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ;the study of the second derivative may be skipped
iii) draw a qualitative graph of f.

Exercise 2 [6 punti] Consider the sequence

$$a_n = \frac{(-1)^n e^{2n} \sin \frac{1}{n}}{(n-1)!}, \quad n \in \mathbb{N}, n \geq 2.$$

a) Compute $\lim_{n\to\infty} a_n$;
b) study the absolute convergence and the convergence semplice of the series $\sum_{n=2}^{\infty} a_n$.

Exercise 3 [5 punti] Sia $f(z) = z^2 + \bar{z}|z|$. Solve the equation

$$zf(z) = |z|^3 - 8i,$$

writing the solutions in algebraic form and disegnandole in the Gauss plane .

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to+\infty} \frac{\log(x+3) - \log(x+1) - \sin\frac{2}{x}}{\cos\sin\frac{1}{2x} - e^{\frac{\alpha}{x^2}} - e^{-x}}$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [8 punti] a) Study the convergence of the generalized integral

$$\int_{\sqrt{2}}^{+\infty} \frac{1}{x^\alpha \sqrt{x^2-2}}\, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 1$.

Exercise . Sia $x_0 \in \mathbb{R}$ and define the sequence $\{a_n : n \in \mathbb{N}\}$ ponendo

$$a_0 = x_0 \text{ e, for every } n \geq 1,\ a_{n+1} = \sin a_n.$$

a) prove that a_n is definitively monotonic for $n \to +\infty$;
b) prove that $\lim_{n\to+\infty} a_n = 0$.

THEME 2

Exercise 1 [6 punti] Consider the function

$$f(x) := \log \frac{|x^2-3|}{x+1}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ;the study of the second derivative may be skipped;
iii) draw a qualitative graph of f.

Exercise 2 [6 punti] Consider the sequence

$$a_n = \frac{(-1)^n e^{3n} \sinh \frac{1}{n}}{(n-1)!}, \quad n \in \mathbb{N}, n \geq 2.$$

a) Compute $\lim_{n\to\infty} a_n$;
b) study the absolute convergence and the convergence semplice of the series $\sum_{n=2}^{\infty} a_n$.

Exercise 3 [5 punti] Sia $f(z) = -z^2 + \bar{z}|z|$. Solve the equation

$$zf(z) = |z|^3 - 8i,$$

writing the solutions in algebraic form and disegnandole in the Gauss plane .

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to+\infty} \frac{\log(x+1) - \log(x+2) + \sinh \frac{1}{x}}{\cosh \sin \frac{1}{x} - e^{\frac{\alpha}{x^2}} - e^{-2x}}$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [8 punti] a) Study the convergence of the generalized integral

$$\int_{\frac{1}{2}}^{+\infty} \frac{1}{x^\alpha \sqrt{4x^2-1}}\, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 1$.

THEME 3

Exercise 1 [6 punti] Consider the function

$$f(x) := \log \frac{|x^2-4|}{x-1}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ;the study of the second derivative may be skipped ;
iii) draw a qualitative graph of f.

Exercise 2 [6 punti] Consider the sequence

$$a_n = \frac{(-1)^n e^{\frac{n}{2}} \arctan \frac{1}{n}}{(n-1)!}, \quad n \in \mathbb{N}, n \geq 2.$$

a) Compute $\lim_{n\to\infty} a_n$;
b) study the absolute convergence and the convergence semplice of the series $\sum_{n=2}^{\infty} a_n$.

Exercise 3 [5 punti] Sia $f(z) = z^2 + \bar{z}|z|$. Solve the equation

$$zf(z) = |z|^3 + 27i,$$

writing the solutions in algebraic form and disegnandole in the Gauss plane .

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to+\infty} \frac{\log(x-2) - \log(x-1) + \arctan \frac{1}{x}}{\cos \sinh \frac{2}{x} - \cos \frac{\alpha}{x} - e^{-\frac{x}{2}}}$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [8 punti] a) Study the convergence of the generalized integral

$$\int_2^{+\infty} \frac{1}{x^\alpha \sqrt{x^2-4}} \, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 1$.

THEME 4

Exercise 1 [6 punti] Consider the function

$$f(x) := \log \frac{|x^2-6|}{x+1}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; the study of the second derivative may be skipped ;
iii) draw a qualitative graph of f.

Exercise 2 [6 punti] Consider the sequence

$$a_n = \frac{(-1)^n e^{\frac{n}{3}} \tan \frac{1}{n}}{(n-1)!}, \quad n \in \mathbb{N}, n \geq 2.$$

a) Compute $\lim_{n\to\infty} a_n$;
b) study the absolute convergence and the convergence semplice of the series $\sum_{n=2}^{\infty} a_n$.

Exercise 3 [5 punti] Sia $f(z) = -z^2 + \bar{z}|z|$. Solve the equation

$$zf(z) = |z|^3 + 27i,$$

writing the solutions in algebraic form and disegnandole in the Gauss plane .

Exercise 4 [7 punti] Compute the limit

$$\lim_{x\to+\infty}\frac{\log(x+3)-\log(x+1)-\tan\frac{2}{x}}{\cosh\sinh\frac{3}{x}-\cosh\frac{\alpha}{x}-e^{-3x}}$$

as $\alpha\in\mathbb{R}$.

Exercise 5 [8 punti] a) Study the convergence of the generalized integral

$$\int_{\frac{1}{3}}^{+\infty}\frac{1}{x^{\alpha}\sqrt{9x^2-1}}\,dx$$

as $\alpha\in\mathbb{R}$;
b) calcolarlo for $\alpha=1$.

Exam of the 16.02.2018

THEME 1

Exercise 1 [7 punti] Consider the function

$$f(x)=\begin{cases} e^{x-\frac{1}{|x-2|}} & \text{for } x\neq 2\\ 0 & \text{for } x=2.\end{cases}$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) si dica if f is continuous in the whole $\mathbb{R}$.
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; compute the main limits of f'; in particolare si dica if f is differentiable in the whole $\mathbb{R}$; the study of the second derivative can be skipped ;
iv) draw a qualitative graph of f.

Exercise 2 [6 punti] Study as $x\in\mathbb{R}$ the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty}\frac{(2x-1)^n}{(2n+3)^2}.$$

Exercise 3 [6 punti] Solve the equation

$$z^2\overline{z}+z\overline{z}^2=4\,\mathrm{Im}(iz)$$

and draw the solutions on Gauss plane .

Exercise 4 [6 punti]
Compute the limit

$$\lim_{x\to0}\frac{(4\cos x-\alpha)^2-4x^4}{x^4\sin^2 x}$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [7 punti] a) Study the convergence of the generalized integral

$$\int_0^{\frac{\pi^2}{3}} x^\alpha \sin(\sqrt{3x})\, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = \frac{1}{2}$.

THEME 2

Exercise 1 [7 punti] Consider the function

$$f(x) = \begin{cases} e^{-x-\frac{1}{|x+2|}} & \text{for } x \neq -2 \\ 0 & \text{for } x = -2. \end{cases}$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) si dica if f is continuous in the whole $\mathbb{R}$.
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; compute the main limits of f'; in particolare si dica if f is differentiable in the whole $\mathbb{R}$; the study of the second derivative may be skipped ;
iv) draw a qualitative graph of f.

Exercise 2 [6 punti] Study as $x \in \mathbb{R}$ the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{(3n+2)^2}.$$

Exercise 3 [6 punti] Solve the equation

$$-\mathrm{Im}(z^2\overline{z} - z\overline{z}^2) = 8i(z - \overline{z})$$

and draw the solutions on Gauss plane .

Exercise 4 [6 punti]
Compute the limit

$$\lim_{x\to 0} \frac{4(\cosh x - \alpha)^2 - x^4}{x^4 \arctan^2 x}$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [7 punti] a) Study the convergence of the generalized integral

$$\int_0^{\frac{\pi^3}{2}} x^{\alpha-1} \sin(\sqrt[3]{2x})\, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 1$.

THEME 3

Exercise 1 [7 punti] Consider the function

$$f(x) = \begin{cases} e^{x-\frac{1}{|x-3|}} & \text{for } x \neq 3 \\ 0 & \text{for } x = 3 \end{cases}$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) si dica if f is continuous in the whole $\mathbb{R}$.
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; compute the main limits of f'; in particolare si dica if f is differentiable in the whole $\mathbb{R}$; the study of the second derivative may be skipped ;
iv) draw a qualitative graph of f.

Exercise 2 [6 punti] Study as $x \in \mathbb{R}$ the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty} \frac{(3x+1)^n}{(2n+5)^2}.$$

Exercise 3 [6 punti] Solve the equation

$$z\bar{z}^2 - z^2\bar{z} = 2i\,\mathrm{Im}(\bar{z} - z)$$

and draw the solutions on Gauss plane .

Exercise 4 [6 punti]
Compute the limit

$$\lim_{x\to 0} \frac{(e^{x^2} - 2\alpha)^2 - x^4}{x^4 \sinh^2 x}$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [7 punti] a) Study the convergence of the generalized integral

$$\int_0^{\frac{\pi^2}{8}} x^{1-\alpha} \sin(\sqrt{2x})\, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = \frac{1}{2}$.

THEME 4

Exercise 1 [7 punti] Consider the function

$$f(x) = \begin{cases} e^{-x-\frac{1}{|x+3|}} & \text{for } x \neq -3 \\ 0 & \text{for } x = -3 \end{cases}$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;

ii) si dica if f is continuous in the whole $\mathbb{R}$.
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; compute the main limits of f'; in particolare si dica if f is differentiable in the whole $\mathbb{R}$; the study of the second derivative may be skipped ;
iv) draw a qualitative graph of f.

Exercise 2 [6 punti] Study as $x \in \mathbb{R}$ the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{(3n+5)^2}.$$

Exercise 3 [6 punti] Solve the equation

$$\operatorname{Im}\left(\bar{z}^2 z - z^2 \bar{z}\right) = 4 \operatorname{Re}(iz)$$

and draw the solutions on Gauss plane .

Exercise 4 [6 punti]
Compute the limit

$$\lim_{x\to 0} \frac{2(3\alpha - e^{x^2})^2 - 2x^4}{x^4 \tan^2 x}$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [7 punti] a) Study the convergence of the generalized integral

$$\int_0^{\frac{\pi^3}{24}} x^\alpha \sin(\sqrt[3]{3x})\, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 0$.

Exam of the 9.07.2018

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = \log\left|2 - 3e^{3x}\right|.$$

i) Si determini the domain D and study the sign of f;
ii) si determininio the limits of f at the extremes of D and the asymptotes;
iii) find the derivative and study the monotonicity of f, determinandone the points of extreme relative and absolute ; the study of the second derivative may be skipped ;
iv) draw a qualitative graph of f.

Exercise 2 [6 punti] Solve the inequality

$$|z|^2 \operatorname{Re}\left(\frac{1}{z}\right) \le \operatorname{Im}\left(\bar{z}^2\right)$$

rappresentandone the solutions on Gauss plane .

Exercise 3 [6 punti] Compute the limit

$$\lim_{x\to+\infty} \frac{\left(\log(1+x) - \log x - \frac{\alpha}{x}\right)^2}{\left(1-\cos\frac{1}{x}\right)^2 + e^{-x}}$$

as $\alpha \in \mathbb{R}$.

Exercise 4 [6 punti] Study as $\alpha \in \mathbb{R}$ the convergence of the series

$$\sum_{n=1}^{\infty} n \arctan\left(\frac{2^{\alpha n}}{n}\right).$$

Exercise 5 [8 punti] a) Compute a primitive di

$$f(x) = \frac{x^2}{(x^2+1)(x^2+2)}$$

(sugg.: cercare a decomposizione of the integrand of the tipo $\frac{A}{x^2+1} + \frac{B}{x^2+2}$).
b) Study the convergence of the generalized integral

$$\int_0^{+\infty} \log\frac{x^\alpha+2}{x^\alpha+1}\,dx.$$

as $\alpha > 0$.
c) Compute the integral for $\alpha = 2$.

THEME 2

Exercise 1 [6 punti] Consider the function

$$f(x) = \log\left|2e^{2x} - 3\right|.$$

i) Si determini the domain D and study the sign of f;
ii) si determininio the limits of f at the extremes of D and the asymptotes;
iii) find the derivative and study the monotonicity of f, determinandone the points of extreme relative and absolute ; the study of the second derivative may be skipped;
iv) draw a qualitative graph of f.

Exercise 2 [6 punti] Solve the inequality

$$\operatorname{Im}\left(\frac{1}{z}\right) \geq \frac{\operatorname{Im}\left(z^2 - \bar{z}^2\right)}{|z|^2}$$

rappresentandone the solutions on Gauss plane .

Exercise 3 [6 punti] Compute the limit

$$\lim_{x\to+\infty} \frac{\left(\cosh\frac{1}{x} - 1\right)^2 - e^{-x}}{\left(\log(2+x) - \log x + \frac{2\alpha}{x}\right)^2}$$

as $\alpha \in \mathbb{R}$.

Exercise 4 [6 punti] Study as $\alpha \in \mathbb{R}$ the convergence of the series

$$\sum_{n=1}^{\infty} n^2 \arctan\left(\frac{4^{\alpha n}}{n^2}\right).$$

Exercise 5 [8 punti] a) Compute a primitive di

$$f(x) = \frac{x^2}{(x^2+4)(x^2+1)}$$

(sugg.: cercare a decomposizione of the integrand of the tipo $\frac{A}{x^2+1} + \frac{B}{x^2+4}$).
b) Study the convergence of the generalized integral

$$\int_0^{+\infty} \log \frac{x^\alpha + 1}{x^\alpha + 4}\, dx.$$

as $\alpha > 0$.
c) Compute the integral for $\alpha = 2$.

Exam of the 17.09.2018

THEME 1

Exercise 1 [7 punti] Consider the function

$$f(x) := \begin{cases} e^{-\frac{2}{|x|}}(2|x|-3) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

i) Determine the domain D, le simmetries and study the sign of f;
ii) determine the limits of f at the extremes of D and the asymptotes;
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; the study of the second derivative may be skipped ;
iv) study the continuity and ( ) the derivability of f (in particolare in $x = 0$);
v) draw a qualitative graph of f.

Exercise 2 [6 punti] Sia

$$P_\lambda(z) = \lambda - 4thez + 2iz^2 + z^3.$$

Find $\lambda_0 \in \mathbb{C}$ in modo che $z = -2i$ sia a zero of P_{λ_0}. Solve the equation

$$P_{\lambda_0}(z) = 0$$

and express the solutions in algebraic form.

Exercise 3 [6 punti] Discutere for all values of the real parameter α the convergence of the series

$$\sum_{n=2}^{\infty} \frac{\log(n + \sin n)}{n^{\frac{\alpha}{2}} + 2}$$

Exercise 4 [6 punti] Compute as $\alpha \in \mathbb{R}^+$ the limit

$$\lim_{x\to 0^+} \frac{x - \sinh x - x^\alpha}{\cos x - 1 + x^{\frac{7}{3}} \log x}.$$

Exercise 5 [7 punti] Given the integral

$$\int_0^{\frac{1}{\sqrt{2}}} x^{\frac{\alpha}{2}} \arcsin 2x^2 \, dx,$$

a) study the convergence as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 2$.

THEME 2

Exercise 1 [7 punti] Consider the function

$$f(x) := \begin{cases} e^{-\frac{1}{|x|}} \big(2 - 3|x|\big) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

i) Find the domain D, le simmetries and study the sign of f;
ii) determine the limits of f at the extremes of D and the asymptotes;
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; the study of the second derivative may be skipped ;
iv) study the continuity and ( ) the derivability of f (in particolare in $x = 0$);
v) draw a qualitative graph of f.

Exercise 2 [6 punti] Sia

$$P_\lambda(z) = \lambda + 2iz + 3iz^2 + z^3.$$

Determine $\lambda_0 \in \mathbb{C}$ in modo che $z = -3i$ sia a zero of P_{λ_0}. Solve the equation

$$P_{\lambda_0}(z) = 0$$

and express the solutions in algebraic form.

Exercise 3 [6 punti] Discutere for all values of the real parameter α the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\log(n + \cos n)}{n^{2\alpha} + 1}$$

Exercise 4 [6 punti] Compute as $\alpha \in \mathbb{R}^+$ the limit

$$\lim_{x\to 0^+} \frac{\sin x - x - x^\alpha}{\cosh x - 1 + x^{\frac{5}{2}} \log x}.$$

Exercise 5 [7 punti] Given the integral

$$\int_0^{\sqrt{2}} x^{2\alpha} \arcsin \frac{x^2}{2} \, dx,$$

a) study the convergence as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = \frac{1}{2}$.

Exam of the 21.01.2019

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = e^{\frac{|x^2-16|}{x+3}}, \qquad x \in D =]-\infty, -3[.$$

i) determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{e^{2x} - 1 - \sin(2x)}{\sinh^2 x + x^{\frac{9}{2}}}.$$

Exercise 3 [4 punti] Solve the equation

$$iz^2 + (1+2i)z + 1 = 0$$

in $z \in \mathbb{C}$, writing the solutions in algebraic form.

Exercise 4 [5+3+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f(t) := \frac{\log\left(1 + \frac{t}{2}\right)}{t^{2\alpha}}.$$

i) Compute $\int_1^2 f(t)\,dt$ con $\alpha = 1$.

ii) Sia $F(x) := \int_2^x f(t)\,dt$ con $\alpha = \frac{1}{2}$. Scrivere the Taylor formula of the second order for F centrata in $x = 2$.

iii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f(t)\,dt$.

Exercise 5 [7 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(\log\alpha)^n}{1+\sqrt{2n}}$$

as $\alpha \in]0, +\infty[$.

Exercise Determine all the values of $a \in \mathbb{R}$ such that the function $f(x) = e^x - ax^3$ sia convex in the whole $\mathbb{R}$.

THEME 2

Exercise 1 [6 punti] Consider the function

$$f(x) = e^{\frac{|x^2-4|}{x+1}}, \qquad x \in D =]-\infty, -1[.$$

i) determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\sinh(3x) - \log(1+3x)}{\sin^2 x + x^{\frac{11}{2}}}.$$

Exercise 3 [4 punti] Solve the equation

$$iz^2 + (-1-2i)z + 1 = 0$$

in $z \in \mathbb{C}$, writing the solutions in algebraic form.

Exercise 4 [5+3+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f(t) := \frac{\log\left(1+\frac{t}{4}\right)}{t^{\frac{\alpha}{2}}}.$$

i) Compute $\int_1^4 f(t)\,dt$ con $\alpha = 4$.

ii) Sia $F(x) := \int_4^x f(t)\,dt$ con $\alpha = 2$. Scrivere the Taylor formula of the second order for F centrata in $x = 4$.

iii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f(t)\,dt$.

Exercise 5 [7 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(\tan\alpha)^n}{\sqrt{2n}-1}$$

as $\alpha \in]-\pi/2, +\pi/2[$.

Exercise Determine all the values of $a \in \mathbb{R}$ such that the function $f(x) = e^x - ax^3$ sia convex in the whole $\mathbb{R}$.

THEME 3

Exercise 1 [6 punti] Consider the function

$$f(x) = e^{\frac{|x^2-3|}{x-1}}, \qquad x \in D =]-\infty, 1[.$$

i) determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{e^{3x} - 1 - \sinh(3x)}{\log^2(1+x) + x^{2\pi}}.$$

Exercise 3 [4 punti] Solve the equation

$$iz^2 + (1 - 2i)z - 1 = 0$$

in $z \in \mathbb{C}$, writing the solutions in algebraic form.

Exercise 4 [5+3+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f(t) := \frac{\log(1 + 2t)}{t^{\alpha - 1}}.$$

i) Compute $\int_1^{\frac{3}{2}} f(t)\,dt$ con $\alpha = 3$.

ii) Sia $F(x) := \int_3^x f(t)\,dt$ con $\alpha = 2$. Scrivere the Taylor formula of the second order for F centrata in $x = 3$.

iii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f(t)\,dt$.

Exercise 5 [7 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=2}^{+\infty} \frac{(1 + \log\alpha)^n}{\sqrt{n} - 1}$$

as $\alpha \in]0, +\infty[$.

Exercise Determine all the values of $a \in \mathbb{R}$ such that the function $f(x) = e^x - ax^3$ sia convex in the whole $\mathbb{R}$.

THEME 4

Exercise 1 [6 punti] Consider the function

$$f(x) = e^{\frac{|x^2 - 5|}{x - 2}}, \qquad x \in D =]-\infty, 2[.$$

i) determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Compute the limit

$$\lim_{x \to 0^+} \frac{\sinh(2x) - \log(1 + 2x)}{\arctan(x^2) + x^{2e}}.$$

Exercise 3 [4 punti] Solve the equation

$$iz^2 + (-1 + 2i)z - 1 = 0$$

in $z \in \mathbb{C}$, writing the solutions in algebraic form.

Exercise 4 [5+3+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f(t) := \frac{\log\left(1 + \frac{t}{3}\right)}{t^{\alpha + 1}}.$$

i) Compute $\int_1^3 f(t)\,dt$ con $\alpha = 1$.

ii) Sia $F(x) := \int_3^x f(t)\,dt$ con $\alpha = 0$. Scrivere the Taylor formula of the second order for F centrata in $x = 3$.

iii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f(t)\,dt$.

Exercise 5 [7 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(\tan 2\alpha)^n}{1+\sqrt{n}}$$

as $\alpha \in]-\pi/4, +\pi/4[$.

Exercise Determine all the values of $a \in \mathbb{R}$ such that the function $f(x) = e^x - ax^3$ sia convex in the whole $\mathbb{R}$.

Exam of the 11.02.2019

THEME 1

Exercise 1 [6 punti] Sia

$$f(x) = |(x+3)\log(x+3)|, \qquad x \in D =]-3, +\infty[.$$

(i) Determine i limits of f at the extremes of D and the asymptotes; study the prolungabilità for continuity in $x = -3$;
(ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(1+n^2)\sin n}{n^4}$$

Exercise 3 [4 punti] Solve the inequality

$$\frac{1}{2} \le \frac{\big(\mathrm{Re}\,(\bar z + i) - 1\big)^2}{4} + \frac{\big(\mathrm{Im}\,(\bar z + i) - 1\big)^2}{4} \le 1$$

and draw the solutions on Gauss plane .

Exercise 4 [5 punti] Compute

$$\int_0^{+\infty} e^{-\sqrt{2x}}\,dx.$$

Exercise 5 [3+3 punti] Sia

$$f_\alpha(x) = \frac{e^{-\sqrt{2x}} - 1}{x^{\alpha-1}}.$$

(a) study the convergence of the integral

$$\int_0^{+\infty} f_\alpha(x)\,dx$$

as $\alpha \in \mathbb{R}$.
(b) Per $\alpha = 2$, sia $F(x) = \int_1^{\cos x} f_\alpha(t)\,dt$: si calcoli $F'(\pi/3)$.

Exercise 6 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\cosh(\alpha\,x) - \cosh\left(e^{2x}-1\right)}{x^3}$$

for all values of the parameter $\alpha > 0$.

Exercise Compute

$$\lim_{x\to+\infty} \int_x^{x+e^{-x}} e^t \arctan t\,dt.$$

THEME 2

Exercise 1 [6 punti] Sia

$$f(x) = |(x+2)\log(x+2)|, \qquad x \in D =]-2, +\infty[.$$

(i) Determine i limits of f at the extremes of D and the asymptotes; study the prolungabilità for continuity in $x = -2$;
(ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Study the convergence of the series

$$\sum_{n=2}^{\infty} \frac{n^3 \sin n}{1-n^5}$$

Exercise 3 [4 punti] Solve the inequality

$$\frac{1}{3} \le \frac{\left(\mathrm{Re}\,(\bar{z}+2i)-1\right)^2}{9} + \frac{\left(\mathrm{Im}\,(\bar{z}+2i)-1\right)^2}{9} \le 1$$

and draw the solutions on Gauss plane .

Exercise 4 [5 punti] Compute

$$\int_0^{+\infty} e^{-\sqrt{3x}}\,dx.$$

Exercise 5 [3+3 punti] Sia

$$f_\alpha(x) = \frac{e^{-\sqrt{3x}}-1}{x^{2\alpha+1}}.$$

(a) study the convergence of the integral

$$\int_0^{+\infty} f_\alpha(x)\,dx$$

as $\alpha \in \mathbb{R}$.
(b) Per $\alpha = 0$, sia $F(x) = \int_1^{\sin x} f_\alpha(t)\,dt$: si calcoli $F'(\pi/6)$.

Exercise 6 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\cos(\alpha\,x) - \cos\log\left(1+5x\right)}{x^3}$$

for all values of the parameter $\alpha > 0$.

Exercise Compute

$$\lim_{x\to+\infty}\int_x^{x+e^{-x}} e^t \arctan t\, dt.$$

THEME 3

Exercise 1 [6 punti] Sia

$$f(x) = |(x+1)\log(x+1)|, \qquad x \in D =]-1, +\infty[.$$

(i) Determine i limits of f at the extremes of D and the asymptotes; study the prolungabilità for continuity in $x = -1$;
(ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Study the convergence of the series

$$\sum_{n=2}^{\infty} \frac{n^2 \sin(n^2)}{1-n^5}$$

Exercise 3 [4 punti] Solve the inequality

$$\frac{1}{2} \le \frac{\big(\mathrm{Re}\,(\bar z - i) - 1\big)^2}{9} + \frac{\big(\mathrm{Im}\,(\bar z - i) - 1\big)^2}{9} \le 1$$

and draw the solutions on Gauss plane .

Exercise 4 [5 punti] Compute

$$\int_0^{+\infty} e^{-\sqrt{x/2}}\, dx.$$

Exercise 5 [3+3 punti] Sia

$$f_\alpha(x) = \frac{e^{-\sqrt{x/2}} - 1}{x^{\alpha-3}}.$$

(a) study the convergence of the integral

$$\int_0^{+\infty} f_\alpha(x)\, dx$$

as $\alpha \in \mathbb{R}$.
(b) Per $\alpha = 4$, sia $F(x) = \int_1^{\sinh x} f_\alpha(t)\, dt$: si calcoli $F'(\log 3)$.

Exercise 6 [7 punti] Compute the limit

$$\lim_{x\to0^+} \frac{\cos(\alpha\, x) - \cos\log\big(1+2x\big)}{x^3}$$

for all values of the parameter $\alpha > 0$.

Exercise Compute

$$\lim_{x\to+\infty}\int_x^{x+e^{-x}} e^t \arctan t\, dt.$$

THEME 4

Exercise 1 [6 punti] Sia

$$f(x) = |(x+4)\log(x+4)|, \qquad x \in D =]-4, +\infty[.$$

(i) Determine i limits of f at the extremes of D and the asymptotes; study the prolungabilità for continuity in $x = -4$;
(ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute and draw the graph.

Exercise 2 [4 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2-n^2)\sin(n^2)}{n^5}$$

Exercise 3 [4 punti] Solve the inequality

$$\frac{1}{3} \leq \frac{\left(\mathrm{Re}\,(\bar{z}-2i)-1\right)^2}{4} + \frac{\left(\mathrm{Im}\,(\bar{z}-2i)-1\right)^2}{4} \leq 1$$

and draw the solutions on Gauss plane .

Exercise 4 [5 punti] Compute

$$\int_0^{+\infty} e^{-\sqrt{x/3}}\, dx.$$

Exercise 5 [3+3 punti] Sia

$$f_\alpha(x) = \frac{e^{-\sqrt{x/3}} - 1}{x^{2\alpha-1}}.$$

(a) study the convergence of the integral

$$\int_0^{+\infty} f_\alpha(x)\, dx$$

as $\alpha \in \mathbb{R}$.
(b) Per $\alpha = 1$, sia $F(x) = \int_1^{\arctan x} f_\alpha(t)\, dt$: si calcoli $F'(\sqrt{3})$.

Exercise 6 [7 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\cosh(\alpha\, x) - \cosh\left(1-e^{3x}\right)}{x^3}$$

for all values of the parameter $\alpha > 0$.

Exercise Compute

$$\lim_{x\to+\infty} \int_x^{x+e^{-x}} e^t \arctan t\, dt.$$

Exam of the 8.07.2019

THEME 1

Exercise 1 [6 punti] Sia

$$f(x) = e^{\frac{2}{|2+\log x|}}.$$

a) Determine the domain D of f; determine the limits of f at the extremes of D and study the prolungabilità for continuity di f in $x = 0$;
b) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute ;
c) draw a qualitative graph of f.

Exercise 2 [4 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{1-2\sqrt{n}}.$$

Exercise 3 [4 punti] Solve the equation

$$\frac{z}{\bar{z}} = -\frac{(\operatorname{Im} z)^2}{|iz^2|}$$

and draw the solutions on Gauss plane .

Exercise 4 [5+3+4 punti] a) Compute a primitive of the function

$$e^x \log(1+2e^x).$$

Per $\alpha \in \mathbb{R}$, define $f_\alpha(x) = e^{\alpha x} \log(1+2e^x)$:
b) study the convergence of the generalized integral

$$\int_0^{+\infty} f_\alpha(x)\,dx$$

as $\alpha \in \mathbb{R}$;
c) find the Taylor expansion diorder 2 centered in $x_0 = 1$ of the function

$$F(x) = \int_1^x f_0(t)\,dt.$$

Exercise 5 [6 punti] Compute the limit

$$\lim_{x\to+\infty} x^\alpha \left(\sqrt[8]{x^2-2} - \sqrt[4]{x+1}\right)$$

for all values of the parameter $\alpha > 0$.

THEME 2

Exercise 1 [6 punti] Sia

$$f(x) = e^{\frac{1}{|3+\log x|}}.$$

a) Determine the domain D of f; determine the limits of f at the extremes of D and study the prolungabilità for continuity di f in $x = 0$;
b) study the derivability, compute the derivative and study the monotonicity of f; determine the points

of extreme relative and absolute ;
c) draw a qualitative graph of f.

Exercise 2 [4 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty}(1-\sqrt{n})\sinh\frac{1}{n^2}.$$

Exercise 3 [4 punti] Solve the equation

$$\frac{z}{\bar{z}}=\frac{(\operatorname{Re} z)^2}{|iz^2|}$$

and draw the solutions on Gauss plane .

Exercise 4 [5+3+4 punti] a) Compute a primitive of the function

$$e^x\log(1+3e^x).$$

Per $\alpha\in\mathbb{R}$, define $f_\alpha(x)=e^{\alpha x}\log(1+3e^x)$:
b) study the convergence of the generalized integral

$$\int_0^{+\infty} f_\alpha(x)\,dx$$

as $\alpha\in\mathbb{R}$;
c) find the Taylor expansion diorder 2 centered in $x_0=2$ of the function

$$F(x)=\int_2^x f_0(t)\,dt.$$

Exercise 5 [6 punti] Compute the limit

$$\lim_{x\to+\infty} x^\alpha\left(\sqrt[3]{x+2}-\sqrt[6]{x^2-1}\right)$$

for all values of the parameter $\alpha>0$.

Exam of the 17.09.2019

THEME 1

Exercise 1 [7 punti] Sia

$$f(x)=\log\left|e^{3x}-2\right|.$$

a) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and determine the asymptotes;
b) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute ;
c) draw a qualitative graph of f.

Exercise 2 [5 punti] Compute the limit

$$\lim_{x\to0^+}\frac{e^{x-2x^2}-1-x}{\sinh x^2+x^{7/3}\log x}.$$

Exercise 3 [4 punti] Solve the inequality

$$\operatorname{Re} z \leq \operatorname{Re}\left(\frac{3}{z}\right)$$

and draw the solutions on Gauss plane .

Exercise 4 [6+3 punti] a) Compute the indefinite integral

$$\int \left(\tan\frac{x}{2}\right)^3 dx \qquad \text{(sugg.: eseguire the sostituzione } \tan\frac{x}{2} = u\text{).}$$

b) study the convergence of the generalized integral

$$\int_0^{\frac{\pi}{6}} \frac{\tan x}{x^{\alpha+2}} dx$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [4+3 punti] (i) Si dimostri that the sequence

$$a_n = \log(n+1) - \log\sqrt{n^2 + \alpha n + 4}$$

is infinitesimal for $n \to \infty$ (for every α) and for $\alpha = 2$ compute the order ;
(ii) study the convergence of the series

$$\sum_{n=2}^{\infty} a_n$$

as $\alpha \in \mathbb{R}$.

THEME 2

Exercise 1 [7 punti] Sia

$$f(x) = \log\left|e^{2x} - 3\right|.$$

a) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and determine the asymptotes;
b) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute ;
c) draw a qualitative graph of f.

Exercise 2 [5 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{e^{x-3x^2} - 1 - x}{\sin x^2 + x^{5/2}\log x}.$$

Exercise 3 [4 punti] Solve the inequality

$$\operatorname{Re} z \leq \operatorname{Re}\left(\frac{4}{z}\right)$$

and draw the solutions on Gauss plane .

Exercise 4 [6+3 punti] a) Compute the indefinite integral

$$\int (\tan 2x)^3\, dx \qquad \text{(sugg.: eseguire the sostituzione } \tan 2x = u).$$

b) study the convergence of the generalized integral

$$\int_0^{\frac{\pi}{6}} \frac{\tan x}{x^{2\alpha-1}} dx$$

as $\alpha \in \mathbb{R}$.

Exercise 5 [4+3 punti] (i) Si dimostri that the sequence

$$a_n = \log(n+1) - \log\sqrt{n^2 + \alpha n + 3}$$

is infinitesimal for $n \to \infty$ (for every α) and for $\alpha = 2$ compute the order ;
(ii) study the convergence of the series

$$\sum_{n=2}^{\infty} a_n$$

as $\alpha \in \mathbb{R}$.

Exam of the 20.01.2020

THEME 1

Exercise 1 [7 punti] Consider the function

$$f(x) = \sin\left(2\arctan(|x|^3)\right)$$

i) determine the domain D, the sign, simmetries , i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ; the study of the second derivative may be skipped .

iii) draw the qualitative graph .

Exercise 2 [6 punti] Compute the limit

$$\lim_{x\to 0^+} \left(1 + \sin x\right)^{x^a}$$

as $a \in \mathbb{R}$, usando the form "exp{log . . . }".

Exercise 3 [4 punti] Trovare the zeros in $\mathbb{C}$ di

$$(z^3 + 5)(z^2 + z + 1) = 0.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f_\alpha(t) := \frac{e^{2t} + 2e^t}{(e^t - 1)^\alpha} \ .$$

i) Compute a primitive of f_α con $\alpha = 1$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f_\alpha(t)\,dt$.

Exercise 5 [6 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(3\sin x)^n n}{n^2 + \sqrt{n}}$$

as $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Exercise Sia $\{a_n\}$ a sequence tale che $a_n > 0$ and $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$ for every $n \in \mathbb{N}$. Si dimostri che $\sum_{n=1}^{\infty} a_n$ diverges.

Tempo a disposizione: 2 ore and 45 minuti.

THEME 2

Exercise 1 [7 punti] Consider the function

$$f(x) = 1 - \sin\left(2\arctan(|x|^3)\right)$$

i) determine the domain D, the sign, simmetries , i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ; the study of the second derivative may be skipped .

iii) draw the qualitative graph .

Exercise 2 [6 punti] Compute the limit

$$\lim_{x\to 0^+} \left(1 - \sinh x\right)^{x^a}$$

as $a \in \mathbb{R}$, usando the form "$\exp\{\log \dots\}$".

Exercise 3 [4 punti] Trovare the zeros in $\mathbb{C}$ di

$$(z^2 - z + 1)(z^3 + 4) = 0.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f_\alpha(t) := \frac{e^{2t} - 3e^t}{(e^t - 1)^\alpha}\ .$$

i) Compute a primitive of f_α con $\alpha = 1$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f_\alpha(t)\,dt$.

Exercise 5 [6 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(4\cos x)^n n}{n^2 + 1}$$

as $x \in [0, \pi]$.

Exercise Sia $\{a_n\}$ a sequence tale che $a_n > 0$ and $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$ for every $n \in \mathbb{N}$. Si dimostri che $\sum_{n=1}^{\infty} a_n$ diverges.

Tempo a disposizione: 2 ore and 45 minuti.

THEME 3

Exercise 1 [7 punti] Consider the function

$$f(x) = \sin\left(2\arctan(|x|^5)\right)$$

i) determine the domain D, the sign, simmetries , i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ; the study of the second derivative may be skipped .

iii) draw the qualitative graph .

Exercise 2 [6 punti] Compute the limit

$$\lim_{x\to 0^+} \left(1-\sin x\right)^{x^a}$$

as $a \in \mathbb{R}$, usando the form "$\exp\{\log\ldots\}$".

Exercise 3 [4 punti] Trovare the zeros in $\mathbb{C}$ di

$$(z^3+3)(z^2+z+2) = 0.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f_\alpha(t) := \frac{e^{2t}-2e^t}{(e^t-1)^\alpha}\ .$$

i) Compute a primitive of f_α con $\alpha = 1$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f_\alpha(t)\,dt$.

Exercise 5 [6 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(4\sin x)^n n}{n^2+2\sqrt{n}}$$

as $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Exercise Sia $\{a_n\}$ a sequence tale che $a_n > 0$ and $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$ for every $n \in \mathbb{N}$. Si dimostri che $\sum_{n=1}^{\infty} a_n$ diverges.

Tempo a disposizione: 2 ore and 45 minuti.

THEME 4

Exercise 1 [7 punti] Consider the function

$$f(x) = 1-\sin\left(2\arctan(|x|^5)\right)$$

i) determine the domain D, the sign, simmetries , i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ; the study of the second derivative may be skipped.

iii) draw the qualitative graph .

Exercise 2 [6 punti] Compute the limit

$$\lim_{x\to 0^+} \left(1+\sinh x\right)^{x^a}$$

as $a \in \mathbb{R}$, usando the form "exp{log . . . }".

Exercise 3 [4 punti] Trovare the zeros in $\mathbb{C}$ di

$$(z^2 - z + 2)(z^3 + 2) = 0.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f_\alpha(t) := \frac{e^{2t} + 3e^t}{(e^t - 1)^\alpha} \ .$$

i) Compute a primitive of f_α con $\alpha = 1$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f_\alpha(t)\,dt$.

Exercise 5 [6 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(3\cos x)^n n}{n^2+2}$$

as $x \in [0, \pi]$.

Exercise Sia $\{a_n\}$ a sequence tale che $a_n > 0$ and $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$ for every $n \in \mathbb{N}$. Si dimostri che $\sum_{n=1}^{\infty} a_n$ diverges.

Tempo a disposizione: 2 ore and 45 minuti.

Exam of the 10.02.2020

THEME 1

Exercise 1 [7 punti] Consider the function

$$f(x) = \exp\left\{\left|\frac{x}{x+1}\right|\right\}.$$

i) Find the domain D, i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ;

iii) draw the qualitative graph .

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{k=1}^{\infty} 3^k \frac{k!}{k^k}.$$

Exercise 3 [5 punti] Solve in $\mathbb{C}$ nella form preferita (algebrica, esponenziale, trigonometrica):

$$z^3 = \frac{1}{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i}.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ and

$$f_\alpha(t) := \frac{e^{-2/t}}{3t^\alpha} .$$

i) Compute a primitive of f_α con $\alpha = 3$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^{+\infty} f_\alpha(t)\, dt$.

Exercise 5 [6 punti] Compute the following limit

$$\lim_{x \to 0^+} \frac{\sin(x - x^3) - \log(1 + \sinh x) + \alpha x^2}{x^3}$$

as $\alpha \in \mathbb{R}$.

Exercise Sia $\alpha \in [0, +\infty[$ and si definisca

$$F_\alpha(x) := \int_0^x t^\alpha e^{-t^2}\ dt,\ \ x \geqslant 0.$$

Establish for which values of α one has che F_α is concave sull'interval $[1, +\infty[$. There are values $\alpha > 0$ so that F_α sia concave su $[0, +\infty[$?

Tempo a disposizione: 2 ore and 45 minuti.

THEME 2

Exercise 1 [7 punti] Consider the function

$$f(x) = \exp\left\{ \left| \frac{x+1}{x} \right| \right\}.$$

i) Find the domain D, i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ;

iii) draw the qualitative graph .

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{k=1}^{\infty} 4^k \frac{k!}{k^k}.$$

Exercise 3 [5 punti] Solve in $\mathbb{C}$ nella form preferita (algebrica, esponenziale, trigonometrica):

$$z^3 = \frac{1}{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ and

$$f_\alpha(t) := \frac{2e^{-3/t}}{t^\alpha} .$$

i) Compute a primitive of f_α con $\alpha = 3$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^{+\infty} f_\alpha(t)\,dt$.

Exercise 5 [6 punti] Compute the following limit

$$\lim_{x\to 0^+} \frac{\sinh(x - x^3) - \log(1 + \sin x) + \alpha x^2}{x^3}$$

as $\alpha \in \mathbb{R}$.

Exercise Sia $\alpha \in [0, +\infty[$ and si definisca

$$F_\alpha(x) := \int_0^x t^\alpha e^{-t^2}\,dt, \; x \geqslant 0.$$

Establish for which values of α one has che F_α is concave sull'interval $[1, +\infty[$. There are values $\alpha > 0$ so that F_α sia concave su $[0, +\infty[$? Tempo a disposizione: 2 ore and 45 minuti.

THEME 3

Exercise 1 [7 punti] Consider the function

$$f(x) = \exp\left\{ \left| \frac{x}{x-1} \right| \right\}.$$

i) Find the domain D, i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ;

iii) draw the qualitative graph .

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{k=1}^{\infty} 5^k \frac{k!}{k^k}.$$

Exercise 3 [5 punti] Solve in $\mathbb{C}$ nella form preferita (algebrica, esponenziale, trigonometrica):

$$z^3 = \frac{1}{-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i}.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ and

$$f_\alpha(t) := \frac{3e^{-2/t}}{t^\alpha} \ .$$

i) Compute a primitive of f_α con $\alpha = 3$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^{+\infty} f_\alpha(t)\, dt$.

Exercise 5 [6 punti] Compute the following limit

$$\lim_{x \to 0^+} \frac{\sin(x + x^3) - \log(1 + \sinh x) + \alpha x^2}{x^3}$$

as $\alpha \in \mathbb{R}$.

Exercise Sia $\alpha \in [0, +\infty[$ and si definisca

$$F_\alpha(x) := \int_0^x t^\alpha e^{-t^2}\, dt, \ x \geqslant 0.$$

Establish for which values of α one has che F_α is concave sull'interval $[1, +\infty[$. There are values $\alpha > 0$ so that F_α sia concave su $[0, +\infty[$?

Tempo a disposizione: 2 ore and 45 minuti.

THEME 4

Exercise 1 [7 punti] Consider the function

$$f(x) = \exp\left\{ \left| \frac{x-1}{x} \right| \right\}.$$

i) Find the domain D, i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ;

iii) draw the qualitative graph .

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{k=1}^{\infty} 6^k \frac{k!}{k^k}.$$

Exercise 3 [5 punti] Solve in $\mathbb{C}$ nella form preferita (algebrica, esponenziale, trigonometrica):

$$z^3 = \frac{1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i}.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ and

$$f_\alpha(t) := \frac{e^{-3/t}}{2t^\alpha} \ .$$

i) Compute a primitive of f_α con $\alpha = 3$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^{+\infty} f_\alpha(t)\,dt$.

Exercise 5 [6 punti] Compute the following limit

$$\lim_{x \to 0^+} \frac{\sinh(x + x^3) - \log(1 + \sin x) + \alpha x^2}{x^3}$$

as $\alpha \in \mathbb{R}$.

Exercise Sia $\alpha \in [0, +\infty[$ and si definisca

$$F_\alpha(x) := \int_0^x t^\alpha e^{-t^2}\ dt, \ x \geqslant 0.$$

Establish for which values of α one has che F_α is concave sull'interval $[1, +\infty[$. There are values $\alpha > 0$ so that F_α sia concave su $[0, +\infty[$?

Tempo a disposizione: 2 ore and 45 minuti.

Exam of the 06.07.2020 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = |(x+3)\log(x+3)|, \qquad x \in D =]-3, +\infty[.$$

(i) Compute

$$\lim_{x \to -3^+} f(x), \quad \lim_{x \to +\infty} f(x).$$

(ii) Compute the first derivative of the function f, study the monotonicity intervals and draw the graph of f.

Exercise 2 [6 punti] Find the solutions of the equation

$$z^3 = 8i$$

in algebraic and exponential form (or trigonometric), and draw them on the Gauss plane .

Exercise 3 [6 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(1+n^2)\log n}{n^4}.$$

Exercise 4 [6 punti] Compute

$$\int_0^{+\infty} e^{-\sqrt{2x}}\, dx.$$

Exercise 5 [6 punti] Compute the limit

$$\lim_{x\to+\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x+2} - \sqrt[6]{x^2-1}\right)^2.$$

Tempo a disposizione: 1 ore and 30 minuti.

Exam of the 14.09.2020 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = \arctan\left(\frac{x+1}{x-1}\right), \qquad x \in (1,\infty).$$

(i) Individuarne the asymptotes.
(ii) If ne determini the monotonicity .

Exercise 2 [6 punti] Consider the complex number $z = \sqrt{3} - i$.
(i) Scriverlo in exponential form .
(ii) Compute the real part of z^6.

Exercise 3 [6 punti] Establish the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}.$$

Exercise 4 [6 punti] Compute the limit

$$\lim_{x\to 0^+} \frac{\log(1+\sinh x) - \sin x}{x^2}.$$

Exercise 5 [6 punti] Consider the generalized integral

$$\int_1^{\infty} \log\left(\frac{x^\alpha}{x^\alpha+1}\right) dx.$$

(i) Compute the integral for $\alpha = 2$.
(ii) Establish for which $\alpha \in [0,\infty)$ it converges.

Tempo a disposizione: 1 ore and 30 minuti.

6 Selected execises from former years exams

Functions .
1) (20.02.2013) Given the function

$$f(x) = x\left|3 + \frac{1}{\log(2x)}\right|,$$

(a) determine the domain, calcolarne the limits at the extremes and determine asymptotes;
(b) study the prolungabilità at the extremes of the domain and the derivability;
(c) compute f' and determine the monotonicity intervals and the points of extreme (maximum and minimum) relative and absolute of f;
(d) compute the main limits of f';
(e) draw a qualitative graph of f (the study of the concavity and of the convexity is not required).

2) (3.02.2014) Consider the function

$$f(x) = \arctan\left(\frac{2x}{\log|x| - 1}\right).$$

1) Determine the domain and discuss the simmetria and the sign of f.

2) Compute the main limits of f, determine the asymptotes and discuss brevemente the continuity.

3) Compute f' and determine the monotonicity intervals and the points of extreme of f.

4) Compute the main limits of f' and study the derivability of f in $x = 0$.

5) Draw a graph of f the study of the second derivative may be skipped.

3) (26.01.2015) Consider the function

$$f(x) = |x+1|\, e^{\frac{-1}{|x+3|}}.$$

(a) Determine the domain D of f; determine the limits of f at the extremes of D and the asymptotes; study the continuity and the prolungamenti for continuity;
(b) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute and compute the main limits of f';
(c) draw a qualitative graph of f.

Series
1) (16.09.2013) Discutere, for all values of the parameter $\alpha \in \mathbb{R}$, the convergence of the series

$$\sum_{n=1}^{\infty} \log\left(n\left(\mathrm{e}^{\frac{1}{n}} - 1\right) - \frac{\alpha}{n}\right).$$

2) (26.01.2015) Determine all the $x \in \mathbb{R}$ such that the series

$$\sum_{n=2}^{\infty} \frac{\log n}{n-1}(x-2)^n$$

converga, resp. converga assolutamente.

3) (25.01.2016) Determine all the $x \in \mathbb{R}$ such that the series

$$\sum_{n=5}^{+\infty} \frac{\big(\log(x-3)\big)^n}{n-1}$$

converga, resp. converga assolutamente.

Limits

1) (20.02.2013) Compute the limit

$$\lim_{x\to 0^+} \frac{x^{7/2}\,\log^2 x - 1 + \sin x^2 + \cos\left(1 - e^{\sqrt{2}\,x}\right)}{\sinh x - x^\alpha}$$

for all values of the parameter $\alpha > 0$.

2) (3.02.2014) Compute the limit

$$\lim_{x\to 0^+} \frac{\sinh x^\alpha - \cos(\sqrt{x})\log(1+\sin x)}{\log\cos 2x + x^3 \log x}$$

for all values of the parameter $\alpha > 0$.

3) (20.02.2015) (a) Compute the order diinfinitesimal l di

$$e^{x-x^2} - \cos(\alpha x) - \sin x$$

for $x \to 0$ as $\alpha \in \mathbb{R}$;

(b) compute the limit

$$\lim_{x\to 0} \frac{e^{x-x^2} - \cos(\alpha x) - \sin x}{\sinh x - \log(1+\sin x)}$$

as $\alpha \in \mathbb{R}$.

Esercizi sui numeri complessi

1) (7.02.2012) Solve the equation

$$i\,\mathrm{Re} z + z^2 = |z|^2 - 1$$

and draw the solutions on Gauss plane .

2) (23.02.2012) Write in algebraic form the zeros of the polynomial

$$\big(z^2 + iz + 2\big)\big(z^3 - 8i\big).$$

3) (18.09.2012) Express in algebraic form the solutions of the equation

$$z^6 - iz^3 + 2 = 0$$

and rappresentarle on Gauss plane .

4) (5.02.2013) Compute tutte the solutions $z \in \mathbb{C}$ of the equation

$$\left(\frac{2z+1}{2z-1}\right)^3 = 1,$$

scriverle in algebraic form and rappresentarle in the Gauss plane .

5) (15.07.2013) Compute tutte the solutions $z \in \mathbb{C}$ of the equation

$$z^5 = -16\overline{z}$$

writing them prima in trigonometric form /esponenziale and in algebraic form; draw them infine on Gauss plane .

6) (15-07.2014) Express in trigonometric form the solutions of the equation

$$\frac{z^4}{z^4+1} = 1 - \frac{i}{\sqrt{3}}, \qquad z \in \mathbb{C}$$

and draw them in the Gauss plane .

7) (20.02.2015) Si risolva the inequality

$$\mathrm{Re}\Big((z+i)^2\Big) \leq \mathrm{Im}\Big(i(\bar{z}-2i)^2\Big) \tag{1}$$

and se ne disegni the insieme of the solutions in the Gauss plane .

8) (16.07.2015) Si risolva the equation

$$\Big(\frac{1}{18} - \frac{i\sqrt{3}}{18}\Big)\bar{z}^2 = 1,$$

disegnandone the solutions in the Gauss plane .

9) (15.02.2016) Determine tutte the solutions of the 'equation

$$\bar{z}^2 = 2thez, \qquad z \in \mathbb{C},$$

writing them in algebraic form and rappresentandole on Gauss plane .

10) (11.07.2016) Solve in the Gauss plane the equation

$$2\bar{z}^3 = 3i,$$

rappresentandone the solutions in algebraic form.

Integrali

1) (7.02.2012) Compute the integral

$$\int_0^8 e^{\sqrt[3]{x}}\, dx.$$

2) (23.02.2012) Given the function

$$f(x) = \frac{2e^x+1}{e^{2x}+2e^x+2},$$

(a) compute a primitiva;

(b) si provi that the generalized integral $\int_0^{+\infty} f(x)\, dx$ and converging and lo si calcoli.

3) (17.07.2012) (a) Compute the order diinfinito for $x \to 3$ of the function

$$g(x) = \frac{x}{9-x^2};$$

b) dire for which $\alpha \geq 0$ converges the integral

$$I = \int_0^3 \frac{x}{\left(9-x^2\right)^{\alpha}} dx;$$

c) calcolarlo for $\alpha = \frac{1}{2}$.
4) (5.02.2013) Compute the integral
$$\int_{\log 8}^{+\infty} \frac{\sqrt{e^x+1}}{e^x-3}\,dx.$$
5) (20.02.2013) Compute the integral
$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^4} \sin \frac{1}{x}\,dx$$
6) (15.07.2013) a) Study the convergence of the generalized integral
$$\int_0^{+\infty} \frac{e^{2\alpha x}-1}{e^{2x}+1} dx$$
as $\alpha \in \mathbb{R}$.
b) Compute the integral for $\alpha = 1/2$.
7) (3.02.2014) Study the convergence of the integral
$$\int_0^{+\infty} \frac{\arctan x}{(x+2)^{\frac{\alpha-1}{2}}(4+x)^{2\alpha}}\,dx$$
as $\alpha \in \mathbb{R}$ and calcolarlo for $\alpha = 1$.
8) (15.07.2014) Trovare for which $\alpha \in \mathbb{R}$ converges the integral
$$\int_0^{+\infty} \frac{1}{x^\alpha\,(3+2\sqrt{x}+x)} dx$$
and calcolarlo for $\alpha = 1/2$.
9) (12.09.2014) Determine the $\alpha \in \mathbb{R}$ for the quali the integral
$$\int_0^4 \frac{\sqrt{x}}{(4-x)^\alpha} dx$$
converges and calcolarlo for $\alpha = 1/2$.
10) (25.01.2016) Compute the integral
$$\int_0^{1/2} \big(\arcsin 2x\big)^2 dx$$
11) (11.07.2016) Establish for which $\alpha \in \mathbb{R}$ the following integral is converging
$$\int_0^{\pi/8} \frac{\sin 2x}{|\log(\cos 2x)|^\alpha \cos 2x}\,dx$$
and calcolarne the value for $\alpha = 1/2$.

7 Other exercises

FUNZIONI

Exercise. Determine, as $\lambda > 1$ the numero disolutions of the equation

$$\lambda^x = x^\lambda.$$

Soluzione. The equation (that ha the soluzione λ) is equivalente a

$$x \log \lambda = \lambda \log x.$$

Posto $f(x) = x \log \lambda$, $g(x) = \lambda \log x$, one has $f'(x) = \log \lambda$, $g'(x) = \frac{\lambda}{x}$ and hence le two functions they are tangenti if

$$\begin{cases} x \log \lambda = \lambda \log x \\ \log \lambda = \frac{\lambda}{x}. \end{cases}$$

Si ricava $\lambda = \lambda \log x$ that is, $x = e$ and hence $\log \lambda = \frac{\lambda}{e}$ from which $\lambda = e$. The function $\log x$ is tangent alla retta $y = \frac{\log \lambda}{\lambda} x$ if $\lambda = e$. the coefficiente angolare of the retta ha a maximum for $\lambda = e$ and hence confrontando the graph of $\log x$ con quello of the retta $y = \frac{\log \lambda}{\lambda} x$ si ottengono always two solutions $\forall \lambda > 1$. Per $\lambda = e$ one has a sola soluzione.

Exercise Given the function $f : \mathbb{R} \to \mathbb{R}$ defined da

$$f(x) = 3x^4 + 4(2a - a^2)x^3 - 12a^3x^2 + a^6,$$

dove $a > 0$ is a parameter fissato. Determine
a) i points of maximum and minimum of f and the values of f in tali punti;
$x = -2a, a^2$ points of minimum ; $x = 0$ point of maximum; $f(-2a) = -16a^4 - 16a^5 + a^6$, $f(a^2) = -a^8 - 4a^7 + a^6$ $f(0) = a^6$.

b) i values of a so that the equation $f = 0$ ha 2 zeros positivi;
Basta imporre $f(a^2) < 0$ that implica $a > -2 + \sqrt{5}$;

c) i values of a so that the equation $f = 0$ ha non piu' dia zero negative ;
Basta imporre $f(-2a) \geq 0$ that implica $a \geq 8 + \sqrt{80}$.

d) i values of $a \geq 0$ so that f is convex.
$a = 0$

Exercise. Determine, as $\lambda \in \mathbb{R}$ the numero disolutions of the equation

$$\frac{1}{10(1 + x^2)} + \left|1 - \sqrt{|x|}\right| = \lambda.$$

Soluzione. Studio the function $f(x) = \frac{1}{10(1+x^2)} + \left|1 - \sqrt{|x|}\right|$ is pari and hence basta studiarla for $x \geq 0$. One has $f(0) = \frac{11}{10}$, $\lim_{x \to +\infty} = +\infty$ e

$$f'(x) = \begin{cases} \frac{-2x}{10(1+x^2)^2} - \frac{1}{2\sqrt{x}}, & 0 < x < 1 \\ \frac{-2x}{10(1+x^2)^2} + \frac{1}{2\sqrt{x}}, & x > 1. \end{cases}$$

f is decreasing in $(0, 1)$ and increasing in $(1, +\infty$, hence $x = 0$ is point direlative maximum and $x = 1$ is point of absolute minimum. The graph is porta alle solutions

$$\begin{cases} \lambda < \frac{1}{20} & \text{ness a soluzione} \\ \lambda = \frac{1}{20} & \text{2 solutions} \\ \frac{1}{20} < \lambda < \frac{11}{10} & \text{4 solutions} \\ \lambda = \frac{11}{10} & \text{3 solutions} \\ \lambda > \frac{11}{10} & \text{2 solutions .} \end{cases}$$

Exercise Given the function

$$f(x) = 4x^3 - 4ax^2 + a^2x - 1,$$

determine for which values of $a > 0$

a) $f(x)$ ha esattamente three zeros;

b) tali zeros they are all positivi.

Soluzione.

a)

$$f'(x) = 12x^2 - 8ax + a^2 = 0 \iff x = \frac{a}{2},\ x = \frac{a}{6}.$$

$x = \frac{a}{2}$ is point of maximum and $x = \frac{a}{6}$ is point of minimum. In order to find three zeros one must assume $f(\frac{a}{2}) < 0 < f(\frac{a}{6})$, that is verified if and only if $a > \frac{3}{\sqrt[3]{2}}$.

b) Since $f(0) = -1 < 0$ for $0 < x < \frac{a}{6}$ there is a zero, as well as there is a zero in $(\frac{a}{6}, \frac{a}{2}$ and, finally a zero $x > \frac{a}{2}$, $\lim_{x\to+\infty} f(x) = +\infty$.

Exercise. Given the function

$$f_a(x) = x^a - ax^2, \quad a > 0,$$

compute $\sup\{f_a(x),\, x \ge 0\}$ and $\inf\{f_a(x),\, x \ge 0\}$, specifying if they are maximum o minimum .

Soluzione.

$$a > 2 \ \Rightarrow\ \lim_{x\to+\infty} f_a(x) = +\infty = \sup\{f_a(x),\, x \ge 0\};$$

$$a \le 2 \ \Rightarrow\ \lim_{x\to+\infty} f_a(x) = -\infty = \inf\{f_a(x),\, x \ge 0\}.$$

One has $f_a'(x) = ax^{a-1} - 2ax = 0 \iff x = 0,\ 2^{\frac{1}{a-2}}$ if $a \ne 2$. $2^{\frac{1}{a-2}}$ is of minimum if $a > 2$, of maximum if $a < 2$. Therefore

$$a > 2 \ \Rightarrow\ \min\{f_a(x),\, x \ge 0\} = 2^{\frac{a}{a-2}} - a2^{\frac{2}{a-2}};$$

$$a < 2 \ \Rightarrow\ \max\{f_a(x),\, x \ge 0\} = 2^{\frac{a}{a-2}} - a2^{\frac{2}{a-2}};$$

$$a = 2 \ \Rightarrow\ \max\{f_2(x),\, x \ge 0\} = 0.$$

Exercise. Study, as $\lambda \in \mathbb{R}$, the numero disolutions of the equation

$$-e^x + e^4|x-1| = \lambda.$$

Sol. Studio the function

$$f(x) = -e^x + e^4|x-1|.$$

Dom$f = \mathbb{R}$. $\lim_{x\to+\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = +\infty$.

$$f'(x) = \begin{cases} -e^x + e^4, & \text{for } x > 1 \\ -e^x - e^4, & \text{for } x < 1. \end{cases}$$

C'ànd a point dimax in $(4, 2e^4)$ and a point dimin. (angoloso) in $(1, -e)$. Therefore

$$\begin{cases} \lambda > 2e^4,\ \lambda < -e \quad 1 \text{ sol.}, \\ -e < \lambda < 2e^4 \quad 3 \text{ sol.}, \\ \lambda = -e, 2e^4 \quad 2 \text{ sol.}. \end{cases}$$

Exercise. Sia

$$f(x) = \ln(x+4) + \frac{x+8}{x+4}.$$

- Compute the intervals diconcavity and diconvexity of f sul suo domain.
- Individuare the maximum interval A contenente -3 dove f risulti invertible .
- Sia g the function inversa of the f ristretta su A. Compute $g'(f(-3))$.

SOL. Dom$f = \{x > -4\}$. $f'(x) = x/(4+x)^2$, $f''(x) = (4-x)/(4+x)^3$. One has $f''(x) > 0$ for $-4 < x < 4$ and ivi the function is convex, for $x > 4$ concave. The maximal neighbourhood of -3 in cui f ànd monotonic (decreasing) and hence invertible is $-4 < x < 0$. One has $f(-3) = 5$, and

$$g'(f(-3)) = \frac{1}{f'(-3)} = -\frac{1}{3}.$$

FUNZIONI INTEGRALI

Exercise. Study the convexity and concavity of the function

$$F(x) = \int_2^x g(\sin t)\, dt, \quad x \in \mathbb{R},$$

dove g is a function differentiable in $\mathbb{R}$ and tale che $g'(x) < 0$.

Soluzione One has

$$F'(x) = g(\sin x),\ \text{e } F''(x) = g'(\sin x)\cos x,$$

from which

$$F''(x) > 0 \iff \cos x < 0 \iff \frac{\pi}{2} + 2K\pi < x < \frac{3\pi}{2} + 2K\pi$$

for K intero; in the union of tali intervals F is convex, and in the complementare is concave.

Exercise. Study the function

$$F(x) = \int_0^x \frac{(t+1)(3-t)}{\arctan(1+t^2)}\, dt,$$

specifying, in particolare, the intervals dicrescenza and decrescenza.
Compute $\lim_{x\to+\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$ and tracciare a qualitative graph .

Soluzione. One has $F'(x) = \frac{(x+1)(3-x)}{\arctan(1+x^2)}$ and $F'(x) > 0 \iff -1 < x < 3$. $\lim_{x\to+\infty} F'(x) = -\infty$ and $\lim_{x\to-\infty} F'(x) = -\infty$, from which si ricava $F'(x) < -1$ for $|x| > M$ and hence $\lim_{x\to+\infty} F(x) = -\infty$ and $\lim_{x\to-\infty} F(x) = +\infty$.

LIMITI

Exercise Compute the following limits (the third one as $\alpha \in \mathbb{R}$),

$$\lim_{x\to 3} \frac{x^6 - 3^6}{x^8 - 3^8} = 1/12, \qquad \lim_{n\to+\infty} \left(1 + \frac{1}{n^n}\right)^{n!} = 1,$$

$$\lim_{x\to 0^+} \frac{\sqrt{x^\alpha + x} + \sqrt{x}}{}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x^\alpha + x} + \sqrt{x}}{}$$

$$= \begin{cases} +\infty & \text{if } \alpha < 2/3 \\ 1 & \text{if } \alpha = 2/3 \\ 0 & \text{if } \alpha > 2/3 \end{cases}$$

8 Solutions of Theme 1 of the last 4 years exams

Exam of the 23.01.2017

THEME 1

Exercise 1 Compute the integral

$$\int_{\log(3)}^{2} \frac{e^x}{e^{2x} - 4}\, dx$$

Solution. One has

$$\begin{aligned}
\int_{\log(3)}^{2} \frac{e^x}{e^{2x} - 4}\, dx &= (\text{setting } e^x = t, \text{ so that } dx = dt/t) \int_3^{e^2} \frac{1}{t^2 - 4}\, dt \\
&= -\frac{1}{4} \int_3^{e^2} \frac{1}{t+2} - \frac{1}{t-2}\, dt = \frac{1}{4} \ln \frac{|t-2|}{|t+2|}\Big|_3^{e^2} \\
&= \frac{1}{4}\left[\log 5 \frac{e^4 - 2}{e^4 + 2}\right] = \frac{1}{2}\left(\operatorname{settanh}\frac{3}{2} - \operatorname{settanh}\frac{e^2}{2}\right).
\end{aligned}$$

Exercise 2 Solve the inequality

$$\left|2\bar{z}^2 - 2z^2\right| < 3$$

and draw the solutions on Gauss plane .

Solution. One has

$$2\bar{z}^2 - 2z^2 = 4(\bar{z} - z)(\bar{z} + z) = (\text{setting } z = x + iy)\ 8ixy,$$

from which

$$\left|2\bar{z}^2 - 2z^2\right| = 8|xy|.$$

The soluzione is hence

$$\{z = x + iy \in \mathbb{C} : |xy| < \frac{3}{8}\},$$

rappresentata in the picture 1.

grafici/2017-I-T1c.pdf

Figura 1: Solutions of exercise 2 (Theme 1).

Exercise 3 Study the convergence of the series

$$\sum_{n=1}^{+\infty} n^2\big(\cos(1/n) - 1 + \sin(1/2n^\alpha)\big)$$

for all values of the parameter $\alpha > 0$.

Solution. Dagli sviluppi of Mac Laurin of $\cos x$ and of $\sin x$ one has, for $n \to +\infty$,

$$\cos(1/n) - 1 + \sin(1/2n) = -\frac{1}{2n^2} + \frac{1}{4! \cdot n^4} + o\left(\frac{1}{n^4}\right) + \frac{1}{2n^\alpha} - \frac{1}{3!8n^{3\alpha}} + \frac{1}{5!32n^{5\alpha}} + o\left(\frac{1}{n^{5\alpha}}\right),$$

so that the general term of the series, for $n \to +\infty$, is asymptotic to

$$\begin{cases} (\text{if } \alpha < 2) & \frac{n^2}{2n^\alpha} \\ (\text{if } \alpha = 2) & 1/24n^2 \\ (\text{if } \alpha > 2) & -1/2 \end{cases}$$

and hence ha sign definitively constant for $n \to +\infty$. If $\alpha \neq 2$ the general term of the series is not infinitesimal l and hence the series diverges (a $-\infty$). Per $\alpha = 2$ the series converges.

Exercise 4 Consider the function

$$f(x) := \arcsin \frac{|x| - 4}{2 + x^2}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D;
ii) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute and compute the main limits of f';
iii) draw a qualitative graph of f.

Solution. (i) The function is pari. $D = \{x \in \mathbb{R} : -1 \leq \frac{|x|-4}{2+x^2} \leq 1\}$. The inequality $\frac{|x|-4}{2+x^2} \leq 1$ equivale a $|x| - 6 - x^2 \leq 0$, that is always verificata, while $\frac{|x|-4}{2+x^2} \geq -1$ equivale a $x^2 + |x| - 2 \geq 0$, that is verificata for $x \leq -1$ and $x \geq 1$. Therefore $D = [1, +\infty[\ (\cup] - \infty, -1])$. D'ora in assumeremo always $x \geq 0$. The function is continuous in D, $f(1) = \arcsin(-1) = -\pi/2$ and $\lim_{x\to+\infty} f(x) = \arcsin 0 = 0$, horizontal asymptote. The sign of f is dato dal sign of the argument of the arcoseno, so that $f(x) \geq 0$ if and only if $x - 4 \geq 0$ and hence $x \geq 4$.
(ii) In D si posthey are applicare le regole diderivazione if the argument of the arcoseno is diverso da ± 1, that is, for $x > 1$. Per tali x one has

$$f'(x) = \frac{x^2 + 2 - 2x(x-4)}{(2+x^2)^2} \frac{1}{\sqrt{1 - \left(\frac{x-4}{2+x^2}\right)^2}} = \frac{-x^2 + 8x + 2}{(1+2x^2)\sqrt{2x^2 + x - 3}},$$

from which one deduces that $f'(x) \leq 0$ if and only if $-x^2+8x+2 \leq 0$, for $x > 1$, that is, for $1 < x < 4+3\sqrt{2}$, that therefore is the point di absolute maximum, while $x = 1$ is the point of absolute minimum. One has

$$\lim_{x\to 1^+} f'(x) = +\infty,$$

so that the graph of f, rappresentato nella figura 2, ha tangent verticale in $(1, \pi/2)$.

Exercise 5 Study the convergence of the generalized integral

$$\int_0^{+\infty} \frac{|\arctan(x-1)| \arctan x}{|1-x^2|^\alpha\ (\sinh\sqrt{x})^\beta}\, dx$$

as $\alpha, \beta \in \mathbb{R}$.

Solution. The integranda $f(x)$ is continuous in $]0,1[\cup]1,+\infty[$, so that bisogna study the convergence of the integral separatamente for $x \to 0^+$, for $x \to 1$ and for $x \to +\infty$.
Per $x \to 0$,

$$f(x) \sim \frac{x \arctan 1}{x^{\beta/2}} = \arctan 1 \frac{1}{x^{\frac{\beta}{2}-1}},$$

and hence the integral converges in 0 if and only if $\beta < 4$.
Per $x \to 1$,

$$f(x) \sim \frac{\arctan 1\, |x-1|}{|x-1|^\alpha |x+1|^\alpha (\sinh\sqrt{2})^\beta} = \frac{\arctan 1}{2^\alpha (\sinh\sqrt{2})^\beta} \frac{1}{|x-1|^{\alpha-1}},$$

and hence the integral converges in 1 if and only if $\alpha < 2$.
Per $x \to +\infty$, if $\beta > 0$

$$f(x) \leq \frac{\pi^2}{4\,(\sinh\sqrt{x})^\beta} \leq \frac{\pi^2}{2^{(2-\beta)}\, e^{(\beta\sqrt{x})}}.$$

Quest'ultima expression is $o(1/x^2)$ for $x \to +\infty$ and hence converges.
If $\beta = 0$,

$$f(x) \sim \pi^2/4x^{2\alpha},$$

grafici/2017-I-T1f.pdf

Figura 2: the graph of f (Theme 1).

hence converges if $\alpha > 1/2$. If $\beta < 0$,

$$f(x) \sim \pi^2 e^{-\beta/2}/2^{2-\beta} > 1/x$$

for $x \to +\infty$ and hence the integral diverges. In sintesi, the integral converges if $\alpha < 2$ and $0 < \beta < 4$ o if $\beta = 0$ and $1/2 < \alpha < 2$.

Exercise . Sia I a interval chiuso and limitato and sia $f : I \to \mathbb{R}$ a function continuous and tale che $f(x) \in I$ for every $x \in I$. Dimostrare that esiste almeno a $x \in I$ tale che $f(x) = x$.

Solution. Consideriamo the function $g(x) = f(x) - x$, that vogliamo dimostrare that si annulla in almeno a point of $I := [a, b]$. If $g(a), g(b) \neq 0$ allora necessariamente $g(a) > 0$ and $g(b) < 0$, so that for the teorema degli zeros esiste $\bar{x} \in]a, b[$ tale che $g(\bar{x}) = 0$.

Exam of the 13.02.2017

THEME 1

Exercise 1 Consider the function

$$f(x) := \log \left| x^2 - 2x - 3 \right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine the points of

extreme relative and absolute of f;
(iii) compute f'' and study the concavity and the convexity of f;
(iv) draw a qualitative graph of f.

Soluzione. i) Clearly $D = \{x \in \mathbb{R} \ : \ x^2 - 2x - 3 \neq 0\} = \mathbb{R} \setminus \{-1, 3\}$. Per the sign abbiamo

$$f(x) \geq 0, \quad \Longleftrightarrow \quad |x^2 - 2x - 3| \geq 1, \quad \Longleftrightarrow \quad x^2 - 2x - 3 \leq -1, \ \vee \ x^2 - 2x - 3 \geq +1.$$

Abbiamo che $x^2 - 2x - 2 \leq 0$ if and only if $x_0 := 1 - \sqrt{3} \leq x \leq 1 + \sqrt{3} =: x_1$ and $x^2 - 2x - 4 \geq 0$ if and only if $x \leq 1 - \sqrt{5} =: x_2$ oppure $x \geq 1 + \sqrt{5} =: x_3$. Therefore $f(x) \leq 0$ if and only if x appartiene ad uno of the two intervals $[x_2, x_0]$ and $[x_1, x_3]$. As for i limits, one has:
ànd chiaro che $x^2 + 3x - 4 \longrightarrow +\infty$ for $x \longrightarrow \pm\infty$, cosicché $\lim_{x \to \pm\infty} f(x) = +\infty$. Yet non ci they are asymptotes poiché, for $x \to \pm\infty$,

$$\frac{f(x)}{x} = \frac{\log(x^2 - 2x - 3)}{x} \sim \frac{\log x^2}{x} = \frac{2\log|x|}{x} \longrightarrow 0,$$

essendo $\log|x| = o(x)$ for $x \to \pm\infty$. Per $x \longrightarrow -1, 3$ one has always che $|x^2 + 3x - 4| \longrightarrow 0+$ hence in ogni case $f(x) \longrightarrow -\infty$ so that one has the asymptotes verticali $x = -1, 3$.
ii) The function is superposition dicontinuous functions ove definite, hence is continuous on tutto the proprio domain. Furthermore, is superposition didifferentiable functions, eccetto quando $x^2 + 3x - 4 = 0$, that for ò they are punti that non appartengono al domain of f: si conclude che f is differentiable in the proprio domain. Since $(\log|y|)' = \frac{1}{y}$ one has immediately che

$$f'(x) = \frac{2x - 2}{x^2 - 2x - 3}.$$

Let us study the sign of f'. the sign of the denominator is positive for $x < -1$ oppure $x > 3$. the numerator is positive for $x > 1$. We deduce the tabella following:

	$-\infty \quad -1$	$-1 \quad 1$	$1 \quad 3$	$3 \quad +\infty$
sgn $(2x-2)$	$-$	$-$	$+$	$+$
sgn (x^2-2x-3)	$+$	$-$	$-$	$+$
sgn f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

I punti $x = -1, 3$ non appartengono al domain, while $x = 1$ is a localmaximum stretto. There are no nánd massimi nánd minimi globali essendo f il limitata sia inferiormente that superiormente.
iii) Clearly f' is differentiable ove defined in quanto rational function: we have che

$$f''(x) = \frac{-2x^2 + 4x - 10}{(x^2 - 2x - 3)^2}.$$

Therefore $f'' \geq 0$ if and only if $2x^2 - 4x - 10 \leq 0$, that is, mai. One concludes that $f'' < 0$ ovunque (where defined) so that the function is concave in ciascuno degli intervals that compongono the suo domain.
iv) the graph of f is rappresentato figura 3.

Exercise 2 Study the convergence of the series

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{n^n}{n!}.$$

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Figura 3: the graph of f (Theme 1).

Soluzione. The series is clearly a termini with constant sign . one can applicare the criterio asymptotic of the rapporto. Detto a_n the general term, one has

$$\frac{a_{n+1}}{a_n} = \frac{1}{2}\frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \frac{1}{2}\frac{(n+1)^n}{n^n} = \frac{1}{2}\left(1+\frac{1}{n}\right)^n \longrightarrow \frac{1}{2}e > 1.$$

Hence the series diverges.

Exercise 3 Given

$$f(z) = \frac{2+iz}{iz+1},$$

determine the domain and determine all the $z \in \mathbb{C}$ tali che $f(z) = z$. Express tutte the solutions in algebraic form.

Soluzione. Perché the fraction sia defined occorre che $iz + 1 \neq 0$, that is, che $z \neq \frac{-1}{i} = \frac{i}{-i \cdot i} = i$. Ora, for $z \neq i$,

$$f(z) = z \quad \iff \quad 2 + iz = z(iz+1) \quad \iff \quad iz^2 + (1-i)z - 2 = 0.$$

Thisis un'equation disecondo grado a coefficienti complessi, and the formula risolutiva tradizionale funziona allo stesso modo (pur diintendere the root come radice complessa). One has hence

$$\begin{aligned} z_{1,2} &= \frac{i-1+\sqrt{1-1-2i+8i}}{2i} = \frac{i-1+\sqrt{6i}}{2i} = \frac{i-1\pm\sqrt{6}e^{i\frac{\pi}{4}}}{2i} \\ &= \frac{1}{2} + \frac{i}{2} \pm \frac{\frac{\sqrt{12}}{2}(1+i)}{2i} = \frac{1\pm\sqrt{3}}{2} + \frac{1\mp\sqrt{3}}{2}i \end{aligned}$$

Exercise 4 Compute the limit

$$\lim_{x\to 0^+} \frac{\arctan x - \sin x + x^{\frac{10}{3}}\log x}{x^\alpha(1-\cos^2 x)}$$

as $\alpha > 0$.

Soluzione. Osservato che, in virtù of the notable limit $\lim_{x\to 0+} x^\gamma \log x = 0$ for every $\gamma \in \mathbb{R}$, one has subito that si tratta of a indeterminate form of the tipo $\frac{0}{0}$. Determiniamo the termini principali col metodo degli sviluppi asintotici. Abbiamo che, for $x \to 0^+$,

$$N(x) := x - \frac{x^3}{3} + o(x^3) - \big(x - \frac{x^3}{6} + o(x^3)\big) + x^{\frac{10}{3}} \log x = -\frac{x^3}{6} + o(x^3) + x^{\frac{10}{3}} \log x.$$

Let us observe that $x^{\frac{10}{3}} \log x = o(x^3)$ for $x \to 0^+$: infatti

$$\frac{x^{\frac{10}{3}} \log x}{x^3} = x^{\frac{10}{3}-3} \log x \to 0, \quad \text{essendo } \frac{10}{3} - 3 > 0,$$

always in virtù of the notable limit sopra richiamato. Therefore $N(x) = \frac{x^3}{6} + o(x^3)$ for $x \to 0^+$. Quanto al denominator , one can osservare preliminarly che

$$(1 - \cos^2 x) = (1 - \cos x)(1 + \cos x) \sim \frac{x^2}{2} \cdot 2 = x^2,$$

so that $D(x) := x^\alpha (1 - \cos^2 x) \sim x^\alpha \cdot x^2 = x^{\alpha+2}$ for $x \to 0^+$. In conclusione, for $x \to 0^+$ one has

$$\frac{N(x)}{D(x)} \sim \frac{\frac{x^3}{6}}{x^{\alpha+2}} \longrightarrow \begin{cases} 0, & 1-\alpha > 0, \iff \alpha < 1, \\ -\frac{1}{6}, & 1-\alpha = 0, \iff \alpha = 1, \\ -\infty, & 1-\alpha < 0, \iff \alpha > 1. \end{cases}$$

Exercise 5 Study the convergence of the generalized integral

$$\int_2^{+\infty} \frac{1}{x^\alpha \sqrt{x-2}}\, dx$$

as $\alpha \in \mathbb{R}$ and calcolarlo for $\alpha = 1$.

Soluzione. Sia $f_\alpha(x) := \frac{1}{x^\alpha \sqrt{x-2}}$ the function integranda. Notiamo that it is continuous in $]2, +\infty[$ and hence the integral is generalizzato sia in $x = 2$ that for $x \to +\infty$. Avendo clearly f_α also sign constant , andiamo a study the comportamento asymptotic at the extremes of the integration interval . Per $x \to +\infty$ we have che

$$f_\alpha(x) \sim \frac{1}{x^\alpha \sqrt{x}} = \frac{1}{x^{\alpha+1/2}},$$

so that $\int^{+\infty} f_\alpha(x)\, dx < +\infty$ if and only if $\int^{+\infty} \frac{1}{x^{\alpha+1/2}}\, dx < +\infty$ that is, if and only if $\alpha + 1/2 > 1$, ovvero $\alpha > 1/2$. Per $x \to 2+$ one has che

$$f_\alpha(x) \sim \frac{1}{2^\alpha \sqrt{x-2}},$$

that is integrabile in $x = 2+$. In conclusione, f_α is integrabile in senso generalizzato in $[2, +\infty[$ if and only if $\alpha > 1/2$.

Calcoliamo the integral in the case $\alpha = 1$. Siccome is generalizzato we have che

$$\int_2^{+\infty} \frac{1}{x\sqrt{x-2}}\, dx = \lim_{a\to 2+,\ b\to+\infty} \int_a^b \frac{1}{x\sqrt{x-2}}\, dx.$$

Sostituendo $x-2=y^2$ $(y>0)$, one has

$$\int \frac{1}{x\sqrt{x-2}}\,dx = \int \frac{2y}{(y^2+2)y}\,dy = \frac{1}{2}\int \frac{2}{\left(\frac{y}{\sqrt{2}}\right)^2+1}\,dy.$$

Sostituendo ancora $y/\sqrt{2}=t$, one has

$$\int \frac{1}{\left(\frac{y}{\sqrt{2}}\right)^2+1}\,dy = \int \frac{1}{t^2+1}\,dt = \arctan t + c = \arctan\frac{y}{\sqrt{2}} + c.$$

Therefore,

$$\int_2^{+\infty} \frac{1}{x\sqrt{x-2}}\,dx = \lim_{a\to 2+,\ b\to+\infty}\left(\arctan\sqrt{\frac{b-2}{\sqrt{2}}} - \arctan\sqrt{\frac{a-2}{\sqrt{2}}}\right) = \frac{\pi}{\sqrt{2}}.$$

Exam of the 10.07.2017

THEME 1

Exercise 1 Consider the function

$$f(x) := \log\left|e^{2x}-4\right|.$$

i) Determine the domain D and study the sign of f; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivata, study the monotonicity and determine the points of extreme relative and absolute of f;
iii) compute f'' and study the concavity and the convexity of f;
iv) draw a qualitative graph of f.
Solution. i) the domain is

$$D = \{x : e^{2x} \neq 4\} = \mathbb{R}\setminus\{\log 2\}.$$

SI ha $f(x)\geq 0$ if and only if $|e^{2x}-4|\geq 1$, that is, if and only if $e^{2x}\geq 5$ oppure $e^{2x}\leq 3$, hence

$$f\left(\frac{\log 5}{2}\right) = f\left(\frac{\log 3}{2}\right) = 0 \quad \text{and } f(x)>0 \text{ if and only if } x > \frac{\log 5}{2} \text{ oppure } x < \frac{\log 3}{2}.$$

One has moreover

$$\lim_{x\to-\infty} f(x) = \log 4, \quad \lim_{x\to\log 2} f(x) = -\infty, \quad \lim_{x\to+\infty} f(x) = +\infty.$$

As for the oblique asymptote for $x\to+\infty$, one has

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\log\left(e^{2x}-4\right)}{x} = 2, \quad \lim_{x\to+\infty}(f(x)-2x) = \lim_{x\to+\infty}\log\left(e^{2x}-4\right)-2x = \lim_{x\to+\infty}\log\frac{e^{2x}-4}{e^{2x}} = 0.$$

Therefore $y=2x$ is oblique asymptote for $x\to+\infty$, $y=2\log 2$ is horizontal asymptote for $x\to-\infty$ and in $x=\log 2$ one has a vertical asymptote.
ii) f is differentiable in the whole D, where one has

$$f'(x) = \frac{2e^{2x}}{e^{2x}-4}.$$

f is therefore strictly decreasing for $x < \log 2$ and strictly increasing for $x > \log 2$. Non risultano hence points of extreme.
iii) Un calcolo direttogives

$$f''(x) = \frac{-16e^{2x}}{(e^{2x}-4)^2},$$

so that f is concave in $]-\infty, \log 2[$ and in $]\log 2, +\infty[$.
iv) the graph is in the picture **??**.

Exercise 2 Draw in the Gauss plane the insieme

$$S := \left\{ z \in \mathbb{C} : \operatorname{Re}\frac{z-1}{z-i} \geq 0,\ |z+1-i| \leq 1 \right\}.$$

Solution. Si tratta in first luogo didetermine the real part of $\frac{z-1}{z+i}$. One has, setting $z = x+iy$,

$$\operatorname{Re}\frac{x-1+iy}{x+i(y-1)} = \operatorname{Re}\frac{(x-1+iy)(x-i(y-1))}{x^2+(y-1)^2} = \frac{x(x-1)+y(y-1)}{x^2+(y-1)^2}.$$

One has therefore

$$\begin{aligned} S &= \{(x,y) \in \mathbb{R}^2 : x^2 - x + y^2 - y \geq 0, (x+1)^2 + (y-1)^2 \leq 1\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \geq \frac{1}{2}, (x+1)^2 + (y-1)^2 \leq 1 \right\}, \end{aligned}$$

that is, the parte esterna al cerchio centered in $(\frac{1}{2}, \frac{1}{2})$ and raggio $\frac{1}{\sqrt{2}}$ and interna al cerchio centered in $(-1,1)$ and raggio 1, rappresentata in the picture 5.

Figura 5: Solutions of exercise 2 (Theme 1).

Exercise 3 Compute the integral

$$\int e^{2x} \arctan(3e^x) dx.$$

Solution. Eseguendo the sostituzione $x = \log t$ one has

$$\begin{aligned} \int e^{2x} \arctan(3e^x) dx &= \int t \arctan(3t)\, dt = \frac{t^2}{2}\arctan 3t - \frac{3}{2}\int \frac{t^2}{1+9t^2}\, dt \\ &= \frac{t^2}{2}\arctan 3t - \frac{3}{2}\Big[\frac{1}{9}\int \frac{1+9t^2}{1+9t^2}\, dt - \frac{1}{9}\int \frac{1}{1+(3t)^2}\, dt\Big] \\ &= \frac{t^2}{2}\arctan 3t - \frac{t}{6} + \frac{\arctan 3t}{18} + c \\ &= \frac{e^{2x}}{2}\arctan 3e^x - \frac{e^x}{6} + \frac{\arctan 3e^x}{18} + c, \quad c \in \mathbb{R}. \end{aligned}$$

Exercise 4 Compute the limit

$$\lim_{x \to 0^+} \frac{\arctan \sin x - \sinh x}{x^\alpha (1 - \cos^2 x)}$$

for all values of the parameter $\alpha > 0$.
Solution. Da $\arctan y = y - \frac{y^3}{3} + o(y^3)$ for $y \to 0$ si deduce, for $x \to 0$,

$$\arctan \sin x = \sin x - \frac{\sin^3 x}{3} + o(x^3) = x - \frac{x^3}{6} - \frac{x^3}{3} + o(x^3) = x - \frac{x^3}{2} + o(x^3),$$

so that, for $x \to 0$,

$$\arctan \sin x - \sinh x = x - \frac{x^3}{2} - \left(x + \frac{x^3}{6}\right) + o(x^3) = -\frac{2x^3}{3} + o(x^3).$$

Therefore one has

$$\lim_{x\to 0^+} \frac{\arctan \sin x - \sinh x}{x^\alpha(1-\cos^2 x)} = \lim_{x\to 0^+} \frac{-2x^3/3 + o(x^3)}{x^{\alpha+2} + o(x^{2+\alpha})} = \begin{cases} 0 & \text{for } \alpha < 1 \\ -\frac{2}{3} & \text{for } \alpha = 1 \\ -\infty & \text{for } \alpha > 1. \end{cases}$$

Exercise 5 Study the convergence semplice and assoluta di

$$\sum_{n=2}^{+\infty} \frac{(1-e^a)^n}{n+\sqrt{n}}$$

as $a \in \mathbb{R}$.
Solution. Per the absolute convergence one may usare the Root Test, which gives

$$\lim_{n\to\infty} \sqrt[n]{\frac{|1-e^a|^n}{n+\sqrt{n}}} = |1-e^a|.$$

The series absolutely converges (and hence semplicemente) if $|1-e^a| < 1$ and diverges assolutamente and does not converge semplicemente if $|1-e^a| > 1$, in quanto the general term is not infinitesimal l. Per $|1-e^a| = 1$ the Root Test nongives informazioni. Risolvendo le disequazioni one deduces that the series absolutely converges for $a < \log 2$ and does not converge for $a > \log 2$. Per $a = \log 2$ the series diventa

$$\sum_{n=2}^{+\infty} \frac{(-1)^n}{n+\sqrt{n}}.$$

Per asintoticità con the series armonica $\sum 1/n$ this series does not converge assolutamente. Furthermore, it simply converges for the criterio diLeibniz, essendo the general term a sign alterno and – in value absolute – infinitesimal l and decreasing.

Exam of the 18.09.2017

THEME 1

Exercise 1 Consider the function

$$f(x) := \frac{3x}{\log|2x|}.$$

i) Determine the domain D and study le simmetries and the sign of f; determine the limits of f at the extremes of D, the prolungabilità of f and the asymptotes;
ii) study the derivability, compute the derivative and its main limits , study the monotonicity e determine the points of extreme relative and absolute of f;
iii) compute f'' and study the concavity and the convexity of f;
iv) draw a qualitative graph of f.
Solution. i) the domain is $D = \{x : x \neq 0, \log|2x| \neq 0\} = \{x : x \neq 0, x \neq \pm\frac{1}{2}\}$. The function is visibilmente dispari, so that the study in $[0, +\infty[$. Per $x > 0$, $f(x) > 0$ if and only if $x > \frac{1}{2}$. One has

$$\begin{aligned}
&\lim_{x\to 0^+} f(x) = 0 \quad \text{(so that } f \text{ is prolungabile con continuity in } x = 0)\\
&\lim_{x\to \frac{1}{2}^-} f(x) = -\infty\\
&\lim_{x\to \frac{1}{2}^+} f(x) = +\infty\\
&\lim_{x\to +\infty} f(x) = +\infty\\
&\lim_{x\to +\infty} \frac{f(x)}{x} = 0 \quad \text{(so that non c' is oblique asymptote for } x \to +\infty).
\end{aligned}$$

ii) Per $x > 0, x \neq \frac{1}{2}$ one has

$$f'(x) = \frac{3\log 2x - 3}{\log^2 2x}.$$

Essendo f prolungabile con continuity in $x = 0$, vediamo if the prolungamento of f is differentiable in 0. A tale scopo calcoliamo

$$\lim_{x\to 0^+} f'(x) = 0,$$

so that the prolungata of f is differentiable also in $x = 0$, con derivative nulla. The sign of f' dipende solo dal sign of $\log 2x - 1$, that is positive if and only if $x > e/2$. Therefore $e/2$ is a point of strict local minimum . There are no extremes absolute .
iii) Per $x > 0, x \neq \frac{1}{2}$ one has

$$f''(x) = 3\frac{\frac{\log^2 2x}{x} - 2(\log 2x - 1)\frac{\log 2x}{x}}{\log^4 2x} = 3\frac{2 - \log 2x}{x\log^3 2x},$$

that one has > 0 if and only if $\frac{1}{2} < x < \frac{e^2}{2}$, that is, f is convex in the interval $]\frac{1}{2}, \frac{e^2}{2}[$ and concave negli intervals $]0, \frac{1}{2}[$ and $[\frac{1}{2}, +\infty[$.
iv) the graph of f is riportato nella figura 6.

Figura 6: the graph of f (Theme 1).

Exercise 2 Given the polynomial

$$z^4 + z^3 + 8thez + 8i$$

determine first a integer root and then le other roots , writing them in algebraic form.
Solution. Per tentativi, a radice intera is $z = -1$: infatti

$$(-1)^4 + (-1)^3 - 8i + 8i = 0.$$

Eseguendo the divisione dipolinomi, oppure, più semplicemente, raccogliendo z^3 nei primi two addendi and $8i$ negli ultimi two , one has

$$z^4 + z^3 + 8thez + 8i = (z+1)\big(z^3 + 8i\big),$$

so that le restanti three radici they are le radici cubiche of $-8i = 8e^{\frac{3}{2}\pi i}$, that is, they are

$$2e^{i\frac{\pi}{2}} = 2i,\ 2e^{(\frac{1}{2}+\frac{2}{3})\pi i} = 2e^{\frac{7}{6}\pi i} = -\sqrt{3} - i,\ 2e^{(\frac{1}{2}+\frac{4}{3})\pi i} = 2e^{\frac{11}{6}\pi i} = \sqrt{3} - i.$$

Exercise 3 Study the convergence of the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{3x}{n}\right)^{n^2}$$

as $x \in \mathbb{R}$.

Solution. The series is a termini definitively positivi for every $x \in \mathbb{R}$. the Root Testgives

$$\lim_{n\to+\infty} \sqrt[n]{\left(1 + \frac{3x}{n}\right)^{n^2}} = \lim_{n\to+\infty} \left(1 + \frac{3x}{n}\right)^{n} = e^{3x}.$$

The series therefore converges for every $x < 0$ and diverges for every $x > 0$. Per $x = 0$ the Root Test nongives informazioni, ma for tale x the series ha for general term 1 and hence diverges.

Exercise 4 Compute, for all values of the real parameter α, the limit

$$\lim_{x\to 0^+} \frac{\cosh(\alpha x) - e^{x^2} + x\log(\cos x)}{x - \sin x + e^{-1/x^2}}.$$

Solution. the numerator, for $x \to 0$, si sviluppa come

$$\begin{aligned}
\cosh \alpha x &= 1 + \frac{1}{2}\alpha^2 x^2 + \frac{1}{24}\alpha^4 x^4 + o(x^4) = 1 + \frac{1}{2}\alpha^2 x^2 + o(x^3)\\
-e^{x^2} &= -1 - x^2 - \frac{1}{2}x^4 + o(x^4) = -1 - x^2 + o(x^3)\\
x \log \cos x &= x \log\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right) = x\Big(\frac{-x^2}{2} + o(x^2)\Big) = \frac{-x^3}{2} + o(x^3),
\end{aligned}$$

so that

$$\cosh(\alpha x) - e^{x^2} + x\log(\cos x) = x^2\Big(\frac{\alpha^2}{2} - 1\Big) - \frac{x^3}{2} + o(x^3) = \begin{cases} x^2\Big(\frac{\alpha^2}{2} - 1\Big) + o(x^2) & \text{if } \alpha \neq \pm\sqrt{2}\\ -\frac{x^3}{2} + o(x^3) & \text{if } \alpha = \pm\sqrt{2}. \end{cases}$$

the denominator , for $x \to 0$, si sviluppa come

$$x - \sin x + e^{-1/x^2} = \frac{x^3}{6} + o(x^3),$$

in quanto $e^{-1/x^2} = o(x^\beta)$ for every β reale. The limit hence vale

$$\lim_{x\to 0^+} \frac{\cosh(\alpha x) - e^{x^2} + x\log(\cos x)}{x - \sin x + e^{-1/x^2}} = \lim_{x\to 0^+} \frac{x^2\left(\frac{\alpha^2}{2} - 1\right) - \frac{x^3}{2} + o(x^3)}{\frac{x^3}{6} + o(x^3)}$$

$$= \lim_{x\to 0^+} \begin{cases} \frac{x^2\left(\frac{\alpha^2}{2}-1\right)+o(x^2)}{\frac{x^3}{6}+o(x^3)} & \text{if } \alpha \neq \pm\sqrt{2} \\ \frac{-\frac{x^3}{2}+o(x^3)}{\frac{x^3}{6}+o(x^3)} & \text{if } \alpha = \pm\sqrt{2}. \end{cases}$$

$$= \begin{cases} +\infty & \text{if } |\alpha| < \sqrt{2} \\ -\infty & \text{if } |\alpha| > \sqrt{2} \\ -3 & \text{if } \alpha = \pm\sqrt{2}. \end{cases}$$

Exercise 5 Study the convergence of the generalized integral

$$\int_0^{+\infty} xe^{ax}(2 + \cos x)\, dx$$

as $a \in \mathbb{R}$. Compute

$$\int_0^{+\infty} xe^{-x}\cos x\, dx$$

(sugg.: compute preliminarly a primitive of $e^{-x}\cos x$).
Solution. Una primitive of the integrand può essere calcolata for every a, so that the discussione of the convergence può essere fatta sia direttamente dalla definizione, sia mediante criteri of convergence . Usando the comparison criterion one has, for $a \geq 0$,

$$xe^{ax}(2 + \cos x) \geq x \text{ for every } x \geq 0$$

and hence the integral diverges. Per $a < 0$ the asymptotic comparison dà, ad esempio,

$$xe^{ax}(2 + \cos x) = o(e^{ax/2}),$$

perché

$$\lim_{x\to+\infty} \frac{xe^{ax}(2 + \cos x)}{e^{ax/2}} \leq \lim_{x\to+\infty} \frac{3xe^{ax}}{e^{ax/2}} = \lim_{x\to+\infty} 3xe^{\frac{ax}{2}} = 0.$$

Siccome $\int_0^{+\infty} e^{ax/2}\, dx < +\infty$, the integral converges.
Per the primitiva, calcoliamo preliminarly

$$\int e^{-x}\cos x\, dx = -e^{-x}\cos x - \int e^{-x}\sin x\, dx$$
$$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\, dx,$$

so that

$$\int e^{-x}\cos x\, dx = \frac{e^{-x}}{2}(\sin x - \cos x) + c.$$

Ora integriamo by parts prendendo x come factor finito and $e^{-x}\cos x$ come differential factor. Risulta

$$\int xe^{-x}\cos x\,dx = x\frac{e^{-x}}{2}(\sin x-\cos x) - \int \frac{e^{-x}}{2}(\sin x-\cos x)\,dx.$$

Calcoliamo separatamente

$$\begin{aligned}\int e^{-x}\sin x\,dx &= -e^{-x}\sin x + \int e^{-x}\cos x\,dx\\ &= -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx,\end{aligned}$$

so that

$$\int e^{-x}\sin x\,dx = -\frac{e^{-x}}{2}(\sin x+\cos x)+c.$$

In definitiva,

$$\begin{aligned}\int_0^{+\infty} xe^{-x}\cos x\,dx &= \lim_{b\to+\infty}[-x\frac{e^{-x}}{2}(\sin x-\cos x)\Big|_0^b + \frac{1}{4}e^{-x}(\sin x+\cos x)\Big|_0^b\\ &\quad + \frac{1}{4}e^{-x}(\sin x-\cos x)\Big|_0^b]\\ &= 0.\end{aligned}$$

(NB. Non is strano that the result sia nullo: the integrand non ha sign constant .)

Exam of the 29.01.2018

THEME 1

Exercise 1 Consider the function

$$f(x) := \log\frac{|x^2-5|}{x+1}.$$

i) Determine the domain D of f, its simmetries and study the sign; determine the limits of f at the extremes of D and the asymptotes;
ii) study the derivability, compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ;
iii) draw a qualitative graph of f.
Solution. i) Da $\frac{|x^2-5|}{x+1}>0$ segue che $D=\{x>-1, x\neq\sqrt{5}\}$. There are no simmetries evidenti. $f(x)\leq 0$ if and only if

$$x>-1$$

e

$$|x^2-5|\leq x+1 \Leftrightarrow -x-1\leq x^2-5\leq x+1 \Leftrightarrow \begin{cases} x^2+x-4\geq 0\\ x^2-x-6\leq 0,\end{cases}$$

that is, if and only if $\frac{-1+\sqrt{17}}{2}\leq x\leq 3$.

One has moreover

$$\lim_{x\to-1^+} f(x) = +\infty$$
$$\lim_{x\to\sqrt{5}} f(x) = -\infty$$
$$\lim_{x\to+\infty} f(x) = +\infty.$$

Siccome

$$\lim_{x\to+\infty} \frac{f(x)}{x} = 0,$$

non ci they are asymptotes obliqui, monly one two asymptotes verticali (oltre ad a asintoto orrizzontale "così alto that non si vede" (cit.))
ii) Le regole diderivazione posthey are essere applicate in the whole D, perché the point in which the argument of the modulo si annulla non appartiene al domain. Siccome $f(x) = \log|x^2 - 5| - \log(x+1)$ and ricordando che $\frac{d}{dx}\log|g(x)| = \frac{g'(x)}{g(x)}$ dove $g(x) \neq 0$, one has for every $x \in D$

$$f'(x) = \frac{2x}{x^2-5} - \frac{1}{x+1} = \frac{x^2+2x+5}{(x^2-5)(x+1)}.$$

Siccome the polynomial al numeratore is always positive, $f'(x) > 0$, and hence f is increasing, if and only if $x > \sqrt{5}$. There are no points of extreme.
iii) the graph of f is in the picture 7.

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Figura 7: the graph of f (Theme 1).

Exercise 2 Consider the sequence

$$a_n = \frac{(-1)^n e^{2n} \sin\frac{1}{n}}{(n-1)!}, \quad n \in \mathbb{N}, n \geq 2.$$

a) Compute $\lim_{n\to\infty} a_n$;
b) study the absolute convergence and the convergence semplice of the series $\sum_{n=2}^{\infty} a_n$.
Solution. a) Siccome $a_n \sim \frac{(e^2)^n}{n!}$ for $n \to \infty$, $\lim_{n\to\infty} a_n = 0$ (ricordando a limit fondamentale).

b) the criterion of the asymptotic comparison and the criterio of the rapporto danno

$$\lim_{n\to\infty} \frac{e^{2(n+1)}}{(n+1)!} \frac{n!}{e^{2n}} = \lim_{n\to\infty} \frac{e^2}{n+1} = 0,$$

so that the series absolutely converges and hence converges.
the fatto che $a_n \to 0$ si poteva also dedurre direttamente dalla convergence of the series .

NOTA: applicando the criterio diLeibniz one may dedurre direttamente the convergence of the series . Risulta che $|a_n|$ is decreasing if and only if $e^2 \leq n$, the that is vero for every $n > 2$ (the dimostrazione richiede a po' dilavoro). Resta comunque da verificare the absolute convergence . Siccome in questo case is vera, the uso of the criterio diLeibniz is of the tutto inutile.

Exercise 3 Sia $f(z) = z^2 + \bar{z}|z|$. Solve the equation

$$zf(z) = |z|^3 - 8i,$$

writing the solutions in algebraic form and disegnandole in the Gauss plane .
Solution. The equation is

$$z^3 + z\bar{z}|z| = |z|^3 - 8i.$$

Siccome $z\bar{z}|z| = |z|^2|z| = |z|^3$, the equation diventa

$$z^3 = -8i.$$

Le three radici cubiche of $-8i = 8e^{i3\pi/2}$ they are date da

$$2e^{i\frac{\pi}{2}} = 2i,\ 2e^{i\frac{7\pi}{6}} = -\sqrt{3} - i,\ 2e^{i\frac{11\pi}{6}} = \sqrt{3} - i,$$

rappresentate in the picture 8.

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Figura 8: Solutions of exercise 3 (Theme 1).

Exercise 4 Compute the limit

$$\lim_{x\to+\infty} \frac{\log(x+3) - \log(x+1) - \sin\frac{2}{x}}{\cos\sin\frac{1}{2x} - e^{\frac{\alpha}{x^2}} - e^{-x}}$$

as $\alpha \in \mathbb{R}$.
Solution. the numerator:

$$\begin{aligned}
\log(x+3) - \log(x+1) - \sin\frac{2}{x} &= \log x + \log\Big(1+\frac{3}{x}\Big) - \log x - \log\Big(1+\frac{1}{x}\Big) - \sin\frac{2}{x} \\
&= \log\Big(1+\frac{3}{x}\Big) - \log\Big(1+\frac{1}{x}\Big) - \sin\frac{2}{x} \\
&= \frac{3}{x} - \frac{9}{2x^2} - \frac{1}{x} + \frac{1}{2x^2} - \frac{2}{x} + o\Big(\frac{2}{x^2}\Big) \\
&= -\frac{4}{x^2} + o\Big(\frac{1}{x^2}\Big)
\end{aligned}$$

for $x \to +\infty$. the denominator (ricordando che $e^{-x} = o(1/x^\alpha)$ for $x \to +\infty$ for every α):

$$\begin{aligned}
\cos\sin\frac{1}{x} - e^{\frac{\alpha}{x^2}} - e^{-x} &= 1 - \frac{1}{2}\sin^2\frac{1}{2x} + \frac{1}{24}\sin^4\frac{1}{2x} - \Big(1+\frac{\alpha}{x^2}+\frac{\alpha^2}{2x^4}\Big) + o\Big(\frac{1}{x^4}\Big) \\
&= -\frac{1}{2}\Big(\frac{1}{2x} - \frac{1}{6(2x)^3}\Big)^2 + \frac{1}{24(2x)^4} - \frac{\alpha}{x^2} - \frac{\alpha^2}{2x^4} + o\Big(\frac{1}{x^4}\Big) \\
&= \Big(-\frac{1}{8} - \alpha\Big)\frac{1}{x^2} + \Big(\frac{1}{96} + \frac{1}{24\cdot 2^4} - \frac{\alpha^2}{2}\Big)\frac{1}{x^4} + o\Big(\frac{1}{x^4}\Big)
\end{aligned}$$

$$= \begin{cases} -\Big(\frac{1}{8}+\alpha\Big)\frac{1}{x^2} + o\Big(\frac{1}{x^2}\Big) & \text{if } \alpha \neq -\frac{1}{2} \\ \frac{1}{192}\frac{1}{x^4} + o\Big(\frac{1}{x^4}\Big) & \text{if } \alpha = -\frac{1}{2} \end{cases}$$

for $x \to +\infty$. Di conseguenza,

$$\lim_{x\to+\infty} \frac{\log(x+3) - \log(x+1) - \sin\frac{2}{x}}{\cosh\sin\frac{1}{2x} - e^{\frac{\alpha}{x^2}} - e^{-x}} = \lim_{x\to+\infty} \begin{cases} \dfrac{\frac{-4}{x^2}+o\Big(\frac{1}{x^2}\Big)}{-\Big(\frac{1}{8}+\alpha\Big)\frac{1}{x^2}+o\Big(\frac{1}{x^2}\Big)} = \frac{32}{1+8\alpha} & \text{if } \alpha \neq -\frac{1}{8} \\ \dfrac{\frac{-4}{x^2}+o\Big(\frac{1}{x^2}\Big)}{\frac{1}{192}\frac{1}{x^4}+o\Big(\frac{1}{x^4}\Big)} = -\infty & \text{if } \alpha = \frac{1}{8}. \end{cases}$$

NOTA: the numerator poteva also essere scritto come

$$\begin{aligned}
\log(x+3) - \log(x+1) - \sin\frac{2}{x} &= \log\frac{x+3}{x+1} - \sin\frac{2}{x} = \log\Big(1+\frac{2}{x+1}\Big) - \sin\frac{2}{x} \\
&= \frac{2}{x+1} - \frac{1}{2}\Big(\frac{2}{x+1}\Big)^2 - \frac{2}{x} + o\Big(\frac{2}{x^2}\Big) \\
&= -2\frac{2x+1}{x(x+1)^2} + o\Big(\frac{2}{x^2}\Big) \\
&= -2\frac{1+2x}{x(x+1)^2} + o\Big(\frac{2}{x^2}\Big) \sim -\frac{4}{x^2}
\end{aligned}$$

for $x \to +\infty$. The maggior parte degli studenti that ha svolto the calcolo in questo modo ha tralasciato the termine diorder 2 nello sviluppo of the logarithm.

Exercise 5 a) Study the convergence of the generalized integral

$$\int_{\sqrt{2}}^{+\infty} \frac{1}{x^\alpha\sqrt{x^2-2}}\,dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 1$.
Solution. a) The integranda $f(x)$ is continuous in $]\sqrt{2}, +\infty[$, so that si deve controllare the convergence sia for $x \to \sqrt{2}^+$ that for $x \to +\infty$. Per $x \to \sqrt{2}+$,

$$f(x) \sim \frac{1}{\sqrt{x - \sqrt{2}}},$$

so that the integral converges for every α. Per $x \to +\infty$,

$$f(x) \sim \frac{1}{x^{\alpha+1}},$$

so that the integral converges if and only if $\alpha > 0$.
b) Con the sostituzione $x = \sqrt{2}\cosh t$, one has (for $t > 0$)

$$\int_{\sqrt{2}}^{+\infty} \frac{1}{x\sqrt{x^2-2}}\,dx = \int_0^{+\infty} \frac{\sqrt{2}\sinh t}{2\cosh t \sinh t}\,dt = \sqrt{2}\int_0^{+\infty} \frac{e^t}{1+e^{2t}}\,dt = \sqrt{2}\arctan e^t\Big|_0^{+\infty} = \sqrt{2}\Big(\frac{\pi}{2} - \frac{\pi}{4}\Big) = \frac{\sqrt{2}\pi}{4}.$$

In alternativa, con the sostituzione $y = \sqrt{x^2-2}$, seguita dalla sostituzione $z = y/\sqrt{2}$, one gets ,

$$\int_2^{+\infty} \frac{1}{x\sqrt{x^2-4}}\,dx = \int_0^{+\infty} \frac{1}{y^2+4}\,dy = \frac{1}{\sqrt{2}}\int_0^{+\infty} \frac{1}{z^2+1}\,dz = \frac{1}{\sqrt{2}}\arctan z\Big|_0^{+\infty} = \frac{1}{\sqrt{2}}\frac{\pi}{2}.$$

Un third modo di compute the integral is the following:

$$\int_{\sqrt{2}}^{+\infty} \frac{1}{x\sqrt{x^2-2}}\,dx = \int_{\sqrt{2}}^{+\infty} \frac{dx}{x^2\sqrt{1-2/x^2}} = \frac{\sqrt{2}}{2}\int_1^{+\infty} \frac{dt}{t^2\sqrt{1-1/t^2}} = \frac{1}{\sqrt{2}}\arcsin\frac{1}{t}\Big|_1^{+\infty} = \frac{1}{\sqrt{2}}\frac{\pi}{2}.$$

Exercise . Sia $x_0 \in \mathbb{R}$ and si definisca the sequence $\{a_n : n \in \mathbb{N}\}$ ponendo

$$a_0 = x_0 \text{ e, for every } n \geq 1,\ a_{n+1} = \sin a_n.$$

a) prove that a_n is definitively monotonic for $n \to +\infty$;
b) prove that $\lim_{n\to+\infty} a_n = 0$.
Solution. a) Per $n \geq 1$ one has $|a_n| = |\sin(a_{n-1})| \leq 1$. If $a_1 \in [0,1]$, allora da $\sin x \leq x$ $\forall x \geq 0$ si ricava $a_{n+1} = \sin a_n \leq a_n$ and hence the sequence is definitively decreasing. If instead $a_1 \in [-1,0]$ one gets the sequence is definitively increasing.
b) In ogni case the sequence ha a limit $\ell \in [-1,1]$. If for assurdo fosse $\ell \neq 0$ si avrebbe, essendo the function seno continuous,

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{|\sin \ell|}{|\ell|} < 1,$$

the that implicherebbe the convergence of the series $\sum_{n=0}^{\infty} |a_n|$, the that a sua volta implicherebbe che a_n converges a 0, cosicché $0 = \ell \neq 0$. Hence $\ell = 0$. In alternativa, always for the continuity di sin,

$$\ell = \lim a_{n+1} = \lim \sin a_n = \sin \ell$$

that ha $\ell = 0$ come unica soluzione.

Exam of the 16.02.2018

THEME 1

Exercise 1 Consider the function

$$f(x) = \begin{cases} e^{x-\frac{1}{|x-2|}} & \text{for } x \neq 2 \\ 0 & \text{for } x = 2. \end{cases}$$

i) Determine the domain D of f, its simmetries and study the sign; In order to determine i limits of f at the extremes of D and the asymptotes;
ii) si dica if f is continuous in the whole $\mathbb{R}$.
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; compute the main limits of f'; in particolare si dica if f is differentiable in the whole $\mathbb{R}$; the study of the second derivative may be skipped
iv) draw a qualitative graph of f.
Solution. i)DOMINIO: $|x-2| \neq 0 \iff x \neq 2$, hence $D = \mathbb{R}\backslash\{2\} \cup \{2\} = \mathbb{R}$
LIMITI:

$$\lim_{x\to 2} f(x) = e^2 \cdot e^{-\infty} = e^2 \cdot 0 = 0 \qquad \lim_{x\to+\infty} f(x) = e^{+\infty} = +\infty \qquad \lim_{x\to-\infty} f(x) = e^{-\infty} = 0$$

ASINTOTI

$$\lim_{x\to+\infty} f(x)/x = \lim_{x\to+\infty} \frac{e^x}{x} = +\infty$$

hence non ci they are asymptotes obliqui.
ii) CONTINUITY: The function is continuous in $\mathbb{R}\backslash\{2\}2$ perchè superposition dicontinue. È continuous also for $x = 2$ since $\lim_{x\to 2} = 0 = f(2)$. Hence f is continuous.
iii) if $x > 2$ one has

$$f'(x) = \left(e^{x-\frac{1}{x-2}}\right)\left(1 + \frac{1}{(x-2)^2}\right);$$

if $x < 2$ one has

$$f'(x) = \left(e^{x+\frac{1}{x-2}}\right)\left(1 - \frac{1}{(x-2)^2}\right).$$

Hence $f'(x) \geq 0$ if

$$\begin{cases} x > 2 \\ 1 + \frac{1}{(x-2)^2} \geq 0 \end{cases} \bigcup \begin{cases} x < 2 \\ 1 - \frac{1}{(x-2)^2} \geq 0 \end{cases}$$

that is, if

$$\begin{aligned} x \in]2,+\infty[&\bigcup \left(]-\infty,2[\bigcap\{x:\ (x-2)^2 \geq 1\}\right) \\ =]2,+\infty[&\bigcup \left(]-\infty,2[\bigcap\{x:\ (x-2) \leq -1 \text{ oppure } (x-2) \geq 1\}\right) \end{aligned}$$

that is, if

$$x \in]2,+\infty[\bigcup]-\infty,1].$$

Furthermore,, since

$$\lim_{x\to 2-} f'(x) = \lim_{x\to 2-} \left(e^{x+\frac{1}{x-2}}\right)\left(1 - \frac{1}{(x-2)^2}\right) = -e^2 \lim_{x\to 2-} \frac{e^{\frac{1}{x-2}}}{(x-2)^2} = 0,$$

$$\lim_{x\to 2+} f'(x) = \lim_{x\to 2+} \left(e^{x-\frac{1}{x-2}}\right)\left(1+\frac{1}{(x-2)^2}\right) = -e^2 \lim_{x\to 2+} \frac{e^{-\frac{1}{x-2}}}{(x-2)^2} = 0$$

one has che f is differentiable in $x = 2$ and $f'(2) = 0$. Concludendo, f is differentiable on tutto the domain $D = \mathbb{R}$, anzi is diclasse C^1.

Dalthe study of the monotonicity f ha a relative maximum in $x = 1$ and a absolute minimum in $x = 0$.

iv) the graph is in the picture 9.

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Figura 9: the graph of f (Theme 1).

Exercise 2 Study as $x \in \mathbb{R}$ the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{(2n+3)^2}.$$

Solution. Let us study the absolute convergence con the criterio of the rapporto

$$\lim_{n\to\infty} \frac{|2x-1|^{n+1}}{(2n+5)^2} \frac{(2n+3)^2}{|2x-1|^n} = |2x-1| \lim_{n\to\infty} \frac{(2n+3)^2}{(2n+5)^2} = |2x-1|$$

o, alternativamente, con Root Test

$$\lim_{n\to\infty} \sqrt[n]{\frac{|2x-1|^n}{(2n+3)^2}} = \lim_{n\to\infty} |2x-1| \sqrt[n]{\frac{1}{(2n+3)^2}} = |2x-1|.$$

Therefore the series absolutely converges – and hence converges– for $0 < x < 1$ and diverges assolutamente and does not converge (perché the general term is not infinitesimal l) for $x < 0$ and for $x > 1$. Per $x = 0$

and $x = 1$ the Root Test and of the rapporto non danno informazioni. Per $x = 0$, $x = 1$ the series diventa

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2x-1)^2}, \quad \sum_{n=1}^{\infty} \frac{1}{(2x-1)^2},$$

rispettivamente, and hence absolutely converges, and hence semplicemente, for asymptotic comparison con the series converging

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Per $0 \leq x < 1/2$ the convergence semplice one may also dedurre dal criterio diLeibniz.

Exercise 3 Solve the equation

$$z^2\overline{z} + z\overline{z}^2 = 4\,\mathrm{Im}(iz)$$

and diventasegnarne the solutions on Gauss plane .
Solution. Poniamo $z = \rho(\cos 0a + i \sin 0a)$. The equation diventa

$$2\rho^3 \cos 0a = 4\rho \cos 0a.$$

Hence, $\rho = 0$, that is, $z = 0$, oppure

$$\rho^2 \cos 0a = 2 \cos 0a,$$

vale a diventare $\rho^2 = 2$, o $z = \pm\rho i$, $\rho > 0$. Concludendo, l' insieme of the solutions on Gauss plane is the unione of the retta verticale for the origine and the circolo diraggio $\sqrt{2}$, rappresentati in the picture 10.

grafici/2018-II-T1c-eps-converted-to.pdf

Figura 10: Solutions of exercise 3 (Theme 1).

Exercise 4
a) Compute the limit

$$\lim_{x\to 0} \frac{(4\cos x - \alpha)^2 - 4x^4}{x^4 \sin^2 x}$$

as $\alpha \in \mathbb{R}$.
Solution. the denominator is asymptotic to x^6 for $x \to 0$. the numerator: one has, for $x \to 0$,

$$\begin{aligned}(4\cos x - \alpha)^2 - 4x^4 &= \big(4 - \alpha - 2x^2 + \frac{x^4}{6} + o(x^4)\big)^2 - 4x^4 \\ &= \begin{cases} 4 - \alpha + o(1) & \text{for } \alpha \neq 4 \\ 4x^4 - \frac{2x^6}{3} - 4x^4 + o(x^6) & \text{for } \alpha = 4 \end{cases} \\ &= \begin{cases} 4 - \alpha + o(1) & \text{for } \alpha \neq 4 \\ -\frac{2x^6}{3} + o(x^6) & \text{for } \alpha = 4. \end{cases}\end{aligned}$$

One has therefore

$$\lim_{x\to 0} \frac{4(\cos x - \alpha)^2 - x^4}{x^4 \sin^2 x} = \begin{cases} +\infty & \text{for } \alpha \neq 4 \\ -\frac{2}{3} & \text{for } \alpha = 4. \end{cases}$$

Exercise 5 a) Study the convergence of the generalized integral

$$\int_0^{\frac{\pi^2}{3}} x^\alpha \sin(\sqrt{3x})\, dx$$

as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = \frac{1}{2}$.
Solution. a) The integranda $g(x)$ is continuous in the integration interval, possibly except at the first extreme. Per $x \to 0^+$ one has

$$g(x) \sim \sqrt{3}\, x^{\alpha + \frac{1}{2}}.$$

The integral is converging if and only if the exponent is greater than -1, that is, if and only if $\alpha > -\frac{3}{2}$.
b) One has, con the sostituzione $3x = t^2$, which gives $dx = \frac{2}{3} t\, dt$,

$$\begin{aligned}\int_0^{\frac{\pi^2}{3}} x^{\frac{1}{2}} \sin\left(\sqrt{3x}\right) dx &= \frac{2}{3\sqrt{3}} \int_0^\pi t^2 \sin t\, dt \\ \text{(by parts)} &= \frac{2}{3\sqrt{3}} \Big(- t^2 \cos t \Big|_0^\pi + 2 \int_0^\pi t \cos t\, dt \Big) \\ \text{(by parts)} &= \frac{2}{3\sqrt{3}} \pi^2 + \frac{2}{3\sqrt{3}} \Big(2t \sin t \Big|_0^\pi - 2 \int_0^\pi \sin t\, dt \Big) \\ &= \frac{2}{3\sqrt{3}} (\pi^2 - 4).\end{aligned}$$

Exam of the 9.07.2018

THEME 1

Exercise 1 Consider the function

$$f(x) = \log\left|2 - 3e^{3x}\right|.$$

i) Si determini the domain D and study the sign of f;
ii) si determininio the limits of f at the extremes of D and the asymptotes;

iii) find the derivative and study the monotonicity of f, determinandone the points of extreme relative and absolute ; the study of the second derivative may be skipped;
iv) si diventasegni a qualitative graph of f.
Solution. i) the domain of f is dato dalla condizione $3e^{3x} \neq 2$, that is,

$$D = \{x \in \mathbb{R} : x \neq \frac{\log \frac{2}{3}}{3}\}.$$

the sign of f is positive if and only if $|2-3e^{3x}| > 1$. Elevando al quadrato one gets the inequality equivalente

$$9e^{6x} - 12e^{3x} + 3 > 0.$$

setting $e^{3x} = y$ and diventavidendo for 3, one gets the inequality $3y^2 - 4y + 1 > 0$, that ha for solutions $y < 1/3$, $y > 1$. Therefore $f(x) \geq 0$ if and only if

$$x \leq \frac{-\log 3}{3} \quad \text{oppure} \quad x \geq 0.$$

In alternativa: if $2 - 3e^{3x} \geq 0$, one has:

$$|2 - 3e^{3x}| > 1 \iff 2 - 3e^{3x} > 1 \iff e^{3x} < \frac{1}{3} \iff x < \frac{1}{3}\log(\frac{1}{3}) = -\frac{1}{3}\log 3.$$

If instead $2 - 3e^{3x} < 0$:

$$|2 - 3e^{3x}| > 1 \iff 3e^{3x} - 2 > 1 \iff e^{3x} > 1 \iff x > \frac{1}{3}\log(1) = 0.$$

Therefore $f(x) \geq 0$ if and only if

$$x \leq \frac{-\log 3}{3} \quad \text{oppure} \quad x \geq 0.$$

ii) One has

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \log\left(2 - 3e^{3x}\right) = \log 2,$$

perché $\lim_{x\to-\infty} e^{3x} = 0$, hence the retta $y = \log 2$ is a horizontal asymptote. Furthermore,

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \log\left(3e^{3x} - 2\right) = +\infty,$$

e

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\log\left(3e^{3x} - 2\right)}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{3x + \log\left(3 - 2e^{-3x}\right)}{x} = 3,$$
$$\lim_{x\to+\infty} f(x) - 3x = \lim_{x\to+\infty} \log\left(3 - 2e^{-3x}\right) = \log 3.$$

Therefore the retta $y = 3x + \log 3$ is oblique asymptote for $x \to +\infty$.
Finally ,

$$\lim_{x\to\frac{\log\frac{2}{3}}{3}} f(x) = \lim_{y\to 0^+} \log y = -\infty,$$

$x = \frac{\log\frac{2}{3}}{3}$ is a vertical asymptote.
(iii) Le regole diderivazione posthey are essere applicate in the whole D, perché the point in which the

Figura 11: the graph of f (Theme 1).

argument of the modulo si annulla non appartiene al domain. Ricordando che $\frac{d}{dx}\log|g(x)| = \frac{g'(x)}{g(x)}$ dove $g(x) \neq 0$, one has for every $x \in D$

$$f'(x) = \frac{9e^{3x}}{3e^{3x}-2}.$$

Siccome the numerator is always positive, $f'(x) > 0$, and hence f is increasing, if and only if $x > \frac{\log\frac{2}{3}}{3}$. There are no points of extreme.
(iv) the graph is in the picture 11.

Exercise 2 Solve the inequality

$$|z|^2 \operatorname{Re}\left(\frac{1}{z}\right) \leq \operatorname{Im}\left(\bar{z}^2\right)$$

rappresentandone the solutions on Gauss plane .
Solution. Notiamo prima that bisogna avere $z \neq 0$. Poniamo $z = x + iy$. Siccome, for $z \neq 0$,

$$\operatorname{Re}\left(\frac{1}{z}\right) = \operatorname{Re}\left(\frac{\bar{z}}{z\bar{z}}\right) = \operatorname{Re}\left(\frac{x-iy}{|z|^2}\right) = \frac{x}{|z|^2},$$

the inequality, for $z \neq 0$, is equivalente a

$$x \leq \operatorname{Im}\left((x-iy)^2\right) = \operatorname{Im}\left(x^2 - y^2 - 2ixy\right) = -2xy,$$

that a sua volta is equivalente a

$$x(1+2y) \leq 0,\ x^2 + y^2 \neq 0,$$

that ha for solutions the insieme

$$\left(\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \leq -\frac{1}{2}\} \cup \{(x,y) \in \mathbb{R}^2 : x \leq 0, y \geq -\frac{1}{2}\}\right) \setminus \{(0,0)\}.$$

Solutions they are in the picture 12. **NB**: $z = 0$ is da togliere!

Figura 12: Solutions of exercise 2 (Theme 1).

Exercise 3 Compute the limit

$$\lim_{x\to+\infty} \frac{\left(\log(1+x) - \log x - \frac{\alpha}{x}\right)^2}{\left(1-\cos\frac{1}{x}\right)^2 + e^{-x}}$$

as $\alpha \in \mathbb{R}$.
Solution. Per $x \to +\infty$ one has

$$\log(1+x) - \log x - \frac{\alpha}{x} = \log x + \log\left(1+\frac{1}{x}\right) - \log x - \frac{\alpha}{x} = \frac{1}{x} - \frac{1}{2x^2} - \frac{\alpha}{x} + o\left(\frac{1}{x^2}\right) = \begin{cases} \frac{1-\alpha}{x} + o\left(\frac{1}{x}\right) & \text{for } \alpha \neq 1 \\ -\frac{1}{2x^2} + o\left(\frac{1}{x^2}\right) & \text{for } \alpha = 1. \end{cases}$$

One has therefore, for $x \to +\infty$,

$$\left(\log(1+x) - \log x - \frac{\alpha}{x}\right)^2 = \begin{cases} \frac{(1-\alpha)^2}{x^2} + o\left(\frac{1}{x^2}\right) & \text{for } \alpha \neq 1 \\ \frac{1}{4x^4} + o\left(\frac{1}{x^4}\right) & \text{for } \alpha = 1. \end{cases}$$

Per the denominator one has

$$\left(1 - \cos\frac{1}{x}\right)^2 + e^{-x} = \left(\frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right)^2 + e^{-x} = \frac{1}{4x^4} + o\left(\frac{1}{x^4}\right),$$

poiché $e^{-x} = o\left(\frac{1}{x^n}\right)$ for $x \to +\infty$ qualunque sia $n > 0$. Therefore one has

$$\lim_{x\to+\infty} \frac{\left(\log(1+x) - \log x - \frac{\alpha}{x}\right)^2}{\left(1 - \cos\frac{1}{x}\right)^2 + e^{-x}} = \begin{cases} +\infty & \text{for } \alpha \neq 1 \\ 1 & \text{for } \alpha = 1. \end{cases}$$

Exercise 4 Study as $\alpha \in \mathbb{R}$ the convergence of the series

$$\sum_{n=1}^{\infty} n \arctan\left(\frac{2^{\alpha n}}{n}\right).$$

Solution. The series is a termini positivi. Osserviamo innanzitutto that for $\alpha > 0$ the general term is not infinitesimal l, in quanto $\lim_{n\to\infty} 2^{\alpha n}/n = +\infty$, so that $\lim_{n\to\infty} \arctan\left(\frac{2^{\alpha n}}{n}\right) = \pi/2$, and hence

$$\lim_{n\to\infty} n \arctan\left(\frac{2^{\alpha n}}{n}\right) = +\infty.$$

Therefore for $\alpha > 0$ the series diverges. Per $\alpha \leq 0$ one can usare the criterion of the asymptotic comparison, that dice that the series ha lo stesso character of the series

$$\sum_{n=1}^{\infty} n\frac{2^{\alpha n}}{n} = \sum_{n=1}^{\infty} 2^{\alpha n}.$$

Quest'ultima is the series geometrica diragione 2^α, that converges if and only if $2^\alpha < 1$, hence if and only if $\alpha < 0$.

Exercise 5 a) Compute a primitive di

$$f(x) = \frac{x^2}{(x^2+1)(x^2+2)}$$

(sugg.: cercare a decomposizione of the integrand of the tipo $\frac{A}{x^2+1} + \frac{B}{x^2+2}$).
b) Study the convergence of the generalized integral

$$\int_0^{+\infty} \log\frac{x^\alpha + 2}{x^\alpha + 1}\, dx.$$

as $\alpha > 0$.
c) Compute the integral for $\alpha = 2$.
Solution. a) One has

$$\frac{x^2}{(x^2+1)(x^2+2)} = \frac{A}{x^2+1} + \frac{B}{x^2+2} = \frac{x^2(A+B) + 2A + B}{(x^2+1)(x^2+2)},$$

from which

$$A+B=1,\ 2A+B=0, \quad \text{that is, } A=-1, B=2.$$

Therefore

$$\int f(x)\,dx = \int \Big(\frac{-1}{x^2+1}+\frac{2}{x^2+2}\Big)\,dx = -\arctan x + \int \frac{1}{\big(\frac{x}{\sqrt{2}}\big)^2+1}dx$$

$$= -\arctan x + \sqrt{2}\int \frac{1}{t^2+1}\,dt = -\arctan x + \sqrt{2}\arctan\frac{x}{\sqrt{2}} + k, k \in \mathbb{R}.$$

b) The integrand is continuo in $[0,+\infty[$, so that bisogna controllare the convergence of the integral solo for $x \to +\infty$. Siccome the integrand is positive, usiamo the criterion of the asymptotic comparison . One has

$$\log\frac{x^\alpha+2}{x^\alpha+1} = \log\Big(1+\frac{2}{x^\alpha}\Big) - \log\Big(1+\frac{1}{x^\alpha}\Big) = \frac{1}{x^\alpha} + o\Big(\frac{1}{x^\alpha}\Big)$$

for $x \to +\infty$. Therefore the integral converges if and only if $\alpha > 1$.
c) Integrando by parts one has

$$\int_0^c \log\frac{x^2+2}{x^2+1}\,dx = x\log\frac{x^2+2}{x^2+1}\Big|_0^c - \int_0^c x\frac{x^2+1}{x^2+2}\frac{2x(x^2+1)-2x(x^2+2)}{(x^2+1)^2}\,dx$$

$$= c\log\frac{c^2+2}{c^2+1} - \int_0^c \frac{-2x^2}{(x^2+2)(x^2+1)}\,dx = \quad \text{[tenendo conto of the calcolo fatto in a)]}$$

$$= c\log\frac{c^2+2}{c^2+1} + 2\Big(-\arctan c + \sqrt{2}\arctan\frac{c}{\sqrt{2}}\Big).$$

Therefore

$$\int_0^{+\infty} \log\frac{x^2+2}{x^2+1}\,dx = \lim_{c\to+\infty}\left(c\log\frac{c^2+2}{c^2+1} + 2\Big(-\arctan c+\sqrt{2}\arctan\frac{c}{\sqrt{2}}\Big)\right) = \pi(\sqrt{2}-1),$$

in quanto

$$\lim_{c\to+\infty} c\log\frac{c^2+2}{c^2+1} = \lim_{c\to+\infty} c\left(\frac{1}{c^2}+o\Big(\frac{1}{c^2}\Big)\right) = 0.$$

Exam of the 17.09.2018

THEME 1

Exercise 1 Consider the function

$$f(x) := \begin{cases} e^{-\frac{2}{|x|}}\big(2|x|-3\big) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

i) Determine the domain D, le simmetries and study the sign of f;
ii) In order to determine i limits of f at the extremes of D and the asymptotes;
iii) compute the derivative and study the monotonicity of f and determine the points of extreme relative and absolute ; the study of the second derivative may be skipped ;
iv) study the continuity and ( ) the derivability of f (in particolare in $x = 0$);

v) draw a qualitative graph of f.
Solution. i) $D = \mathbb{R}$, ovviamente and the function is pari. One has

$$f(x) \geq 0 \text{ if and only if } |x| \geq \frac{3}{2} \text{ oppure } x = 0.$$

D'ora in study f for $x \geq 0$.
ii) One has

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^{-\frac{2}{x}}(2x-3) = +\infty.$$

Per the calcolo of the asintoto one has

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} e^{-\frac{2}{x}}\frac{2x-3}{x} = 2$$

e

$$\lim_{x\to+\infty} f(x) - 2x = \lim_{x\to+\infty} \Big(2x\big(e^{-\frac{2}{x}} - 1\big) - 3e^{-\frac{2}{x}}\Big) = \lim_{x\to+\infty} \left(2x\Big(-\frac{2}{x} + o\big(\frac{1}{x}\big)\Big) - 3e^{-\frac{2}{x}}\right) = -7,$$

so that the retta $y = 2x - 7$ is oblique asymptote for $x \to +\infty$.
iii) Per $x > 0$ si posthey are applicare le regole diderivazione, dato that one has $f(x) = e^{-\frac{2}{x}}(2x-3)$. Therefore

$$f'(x) = 2e^{-\frac{2}{x}} + \frac{2e^{-\frac{2}{x}}}{x^2}(2x-3) = \frac{2e^{-\frac{2}{x}}}{x^2}(x^2+2x-3).$$

One has therefore che $f'(x) \geq 0$ if and only if $x^2 + 2x - 3 \geq 0$, that is, (for $x > 0$) if and only if $x \geq 1$. Therefore $x = 1$ is the point of absolute minimum, and is a minimum stretto, while $x = 0$ is a point of relative maximum stretto, in quanto $f(x) < 0 = f(0)$ for $0 < |x| < \frac{3}{2}$ (mostrato in (i)).
iv) The function is continuous in $]0, +\infty[$ in quanto superposition difunctions elementari. Per study the continuity in 0 bisogna compute

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} e^{-\frac{2}{x}}(2x-3) = -3\lim_{x\to0^+} e^{-\frac{2}{x}} = 0 = f(0).$$

Therefore f is continuous also in $x = 0$. Per study the derivability in $x = 0$ one may compute the limit

$$\lim_{x\to0^+} f'(x) = \lim_{x\to0^+} \frac{2e^{-\frac{2}{x}}}{x^2}(x^2+2x-3) = -3\lim_{x\to0^+} \frac{2e^{-\frac{2}{x}}}{x^2} = 0$$

for the noto confronto tra esponenziali and potenze. Therefore f is differentiable also in $x = 0$ (and the derivative is continuous also in $x = 0$).
v) the graph of f is in the picture 13.

Figura 13: the graph of f (Theme 1).

Exercise 2 Sia

$$P_\lambda(z) = \lambda - 4thez + 2iz^2 + z^3.$$

Find $\lambda_0 \in \mathbb{C}$ in modo che $z = -2i$ sia a zero of P_{λ_0}. Solve the equation

$$P_{\lambda_0}(z) = 0$$

and express the solutions in algebraic form.
Solution. $P_\lambda(-2i) = \lambda - 8 - 8i + 8i$, from which $P_\lambda(-2i) = 0$ if and only if $\lambda = 8$. the polynomial dicui trovare the zeros is hence $P_{\lambda_0}(z) = 8 - 4thez + 2iz^2 + z^3$. Siccome $z = -2i$ is a zero of P, P is divisible for $z + 2i$ and one has, in particolare,

$$P_{\lambda_0}(z) = (z + 2i)(z^2 - 4i).$$

Le altre solutions of the equation $P_{\lambda_0}(z) = 0$ they are therefore le two radici quadrate of $4i = 4e^{i\frac{\pi}{2}}$, that is, they are

$$\pm 2e^{i\frac{\pi}{4}} = \pm\sqrt{2}(1 + i).$$

Exercise 3 Discutere for all values of the real parameter α the convergence of the series

$$\sum_{n=2}^{\infty} \frac{\log(n + \sin n)}{n^{\frac{\alpha}{2}} + 2}$$

Solution. The series is a termini positivi and one may hence usare the criterion of the asymptotic comparison . One has

$$\log\big(n + \sin n\big) \sim \log n \text{ for } n \to \infty,$$

perché

$$\lim_{n\to\infty} \frac{\log\big(n + \sin n\big)}{\log n} = \lim_{n\to\infty} \frac{\log n + \log\Big(1 + \frac{\sin n}{n}\Big)}{\log n} = \lim_{n\to\infty} \frac{\log n + \frac{\sin n}{n} + o\Big(\frac{1}{n}\Big)}{\log n} = 1.$$

Furthermore,

$$\frac{1}{n^{\frac{\alpha}{2}} + 2} \sim \frac{1}{n^{\frac{\alpha}{2}}} \text{ for } n \to \infty.$$

The series converges therefore if and only if converges the series

$$\sum_{n=1}^{\infty} \frac{\log n}{n^{\frac{\alpha}{2}}}.$$

Quest'ultima converges if and only if $\frac{\alpha}{2} > 1$, that is, if and only if $\alpha > 2$. Infatti, if $\frac{\alpha}{2} \le 1$, the general term of the series is $\ge \frac{1}{n}$ and hence the series diverges. If instead $\frac{\alpha}{2} > 1$ and scelgo $1 < \beta < \frac{\alpha}{2}$, allora, for $n \to \infty$,

$$\frac{\log n}{n^{\frac{\alpha}{2}}} = o\Big(\frac{1}{n^\beta}\Big),$$

dal limit fondamentale

$$\lim_{n\to\infty} \frac{\log n}{n^\gamma} = 0 \text{ for every } \gamma > 0$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^\beta}$ converges.

Exercise 4 Compute as $\alpha \in \mathbb{R}^+$ the limit

$$\lim_{x\to 0^+} \frac{x - \sinh x - x^\alpha}{\cos x - 1 + x^{\frac{7}{3}} \log x}.$$

Solution. One has, for $x \to 0^+$,

$$x - \sinh x - x^\alpha = -\frac{x^3}{6} + o(x^3) - x^\alpha \sim \begin{cases} -x^\alpha & \text{if } \alpha < 3 \\ -\frac{7}{6}x^3 & \text{if } \alpha = 3 \\ -\frac{x^3}{6} & \text{if } \alpha > 3 \end{cases}$$

$$\cos x - 1 + x^{\frac{7}{3}} \log x = -\frac{x^2}{2} + o(x^2) + x^{\frac{7}{3}} \log x = -\frac{x^2}{2} + o(x^2) \sim -\frac{x^2}{2}$$

in quanto

$$\lim_{x\to 0+} \frac{x^{\frac{7}{3}}\log x}{x^2} = \lim_{x\to 0+} \sqrt[3]{x}\log x = 0.$$

Therefore ,

$$\lim_{x\to 0^+} \frac{x-\sinh x - x^\alpha}{\cos x - 1 + x^{\frac{7}{3}}\log x} = \begin{cases} +\infty & \text{if } \alpha < 2 \\ 2 & \text{if } \alpha = 2 \\ 0 & \text{if } \alpha > 2. \end{cases}$$

Exercise 5 Given the integral

$$\int_0^{\frac{1}{\sqrt{2}}} x^{\frac{\alpha}{2}} \arcsin 2x^2\, dx,$$

a) study the convergence as $\alpha \in \mathbb{R}$;
b) calcolarlo for $\alpha = 2$.
Solution. a) The integrand $g(x) = x^{\frac{\alpha}{2}} \arcsin 2x^2$ is positive, so that one may usare the criterion of the asymptotic comparison . One has, for $x \to 0^+$,

$$g(x) \sim 2x^{\frac{\alpha}{2}+2},$$

so that the integral converges if and only if $\frac{\alpha}{2} + 2 > -1$, that is, if and only if $\alpha > -6$.
b) One has

$$\int_0^{\frac{1}{\sqrt{2}}} x \arcsin 2x^2\, dx = \text{(by parts)}\ \frac{x^2}{2}\arcsin 2x^2\Big|_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{2}\frac{4x}{\sqrt{1-4x^4}}\, dx$$
$$= \frac{\pi}{8} - \int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-4x^4}}\, dx = \frac{\pi}{8} + \frac{\sqrt{1-4x^4}}{4}\Big|_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{8} - \frac{1}{4}.$$

Exam of the 21.01.2019

THEME 1

Exercise 1 Consider the function

$$f(x) = e^{\frac{|x^2-16|}{x+3}}, \qquad x \in D =]-\infty, -3[.$$

i) Determine the limits of f at the extremes of D and the asymptotes; study the prolungabilità for continuity in $x = -3$;
ii) study the derivability, calcolarne the derivata, study the monotonicity and determine the points of extreme relative and absolute .

Solution.
i) Let us observe that

$$\lim_{x\to -\infty} \frac{|x^2-16|}{x+3} = -\infty, \qquad \lim_{x\to -3^-} \frac{|x^2-16|}{x+3} = -\infty$$

hence con a change of variable

$$\lim_{x\to -\infty} f(x) = \lim_{y\to -\infty} e^y = 0, \qquad \lim_{x\to -3^-} f(x) = \lim_{y\to -\infty} e^y = 0.$$

In particolare f ha a horizontal asymptote ($y = 0$) for $x \to -\infty$. Furthermore, f può essere prolungata come function continuous da sinistra in -3 setting $f(-3) = 0$.

ii) Calcoliamo, for $x \neq -4$,

$$
\begin{aligned}
f'(x) &= e^{\frac{|x^2-16|}{x+3}} \frac{d}{dx} \frac{|x^2-16|}{x+3} = e^{\frac{|x^2-16|}{x+3}} \frac{\operatorname{sgn}(x^2-16) 2x(x+3) - |x^2-16|}{(x+3)^2} \\
&= e^{\frac{|x^2-16|}{x+3}} \operatorname{sgn}(x^2-16) \frac{2x(x+3) - (x^2-16)}{(x+3)^2} \\
&= e^{\frac{|x^2-16|}{x+3}} \operatorname{sgn}(x^2-16) \frac{x^2+6x+16}{(x+3)^2},
\end{aligned}
$$

dove "sgn" indica the function sign.

Osservando che $e^{\frac{|x^2-16|}{x+3}} > 0$ and $(x+3)^2 > 0$ for every $x \in D$, vogliamo valutare the sign of

$$
(x^2 + 6x + 16)\operatorname{sgn}(x^2 - 16)
$$

Calcolando the discriminante of $x^2 + 6x + 16$, $\Delta = 36 - 64 < 0$ one gets $x^2 + 6x + 16 > 0$ for every x. Furthermore,

$$
\operatorname{sgn}(x^2-16) > 0 \quad \Leftrightarrow \quad x^2 - 16 > 0 \quad \Leftrightarrow \quad |x| > 4 \quad \Leftrightarrow \quad x < -4 \text{ o } x > 4.
$$

Since ci interessano solo the values of $x \in D$, ovvero $x < -3$, we get $\operatorname{sgn}(x^2 - 16) > 0$ for $x < -4$ and $\operatorname{sgn}(x^2-16) < 0$ for $-4 < x < -3$. Ne one has

$$
f'(x) > 0 \text{ (and hence } f \text{ increasing) for } x < -4, \quad f'(x) < 0 \text{ (and hence } f \text{ decreasing) for } x \in]-4, -3[,
$$

from which segue che -4 ánd a point absolute maximum and for the teorema diFermat, non posthey are esservi altri points of extreme.

Finally $x = -4$ is the unique point in cui f one has non differentiable (is a angular point) perché

$$
\lim_{x \to -4^+} f'(x) = -8 = - \lim_{x \to -4^-} f'(x).
$$

the graph of f is in the picture 14.

Exercise 2 Compute the limit

$$
\lim_{x \to 0^+} \frac{e^{2x} - 1 - \sin(2x)}{\sinh^2 x + x^{\frac{9}{2}}}.
$$

Solution. By making use the sviluppo diTaylor $e^y = 1 + y + \frac{y^2}{2} + o(y^2)$, $\sin y = y + o(y^2)$ con $y = 2x$ we get

$$
e^{2x} = 1 + 2x + 2x^2 + o(x^2), \quad \sin 2x = 2x + o(x^2), \quad \text{for } x \to 0
$$

and therefore the numerator può essere scritto come

$$
e^{2x} - 1 - \sin 2x = 2x^2 + o(x^2) \quad \text{for } x \to 0
$$

Scrivendo $\sinh x = x + o(x)$ we have $\sinh^2 x = (x + o(x))^2 = x^2 + o(x^2)$ for $x \to 0$. Furthermore,, essendo $\frac{9}{2} > 2$, it is $x^{\frac{9}{2}} = o(x^2)$ for $x \to 0$. Ne segue

$$
\sinh^2 x + x^{\frac{9}{2}} = x^2 + o(x^2).
$$

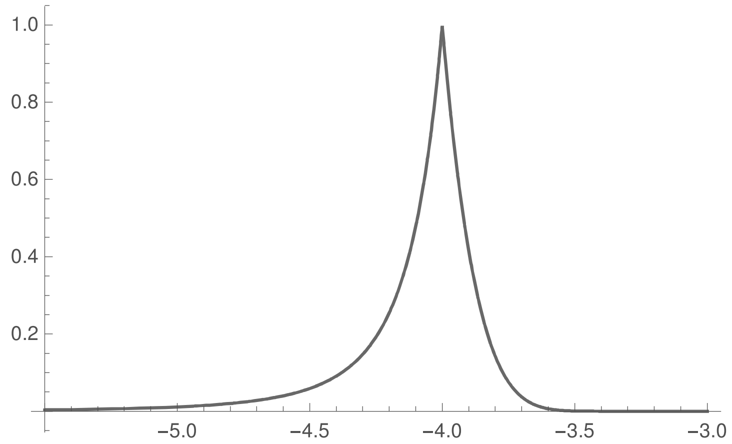

Figura 14: the graph of f (Theme 1).

Da cui

$$\lim_{x\to 0} \frac{e^{2x} - 1 - \sin 2x}{\sinh^2 x + x^{\frac{9}{2}}} = \lim_{x\to 0} \frac{2x^2 + o(x^2)}{x^2 + o(x^2)} = 2.$$

Exercise 3 Solve the equation

$$iz^2 + (1 + 2i)z + 1 = 0$$

in $z \in \mathbb{C}$, writing the solutions in algebraic form.

Solution. Vale

$$z = \frac{-1 - 2i + \sqrt{(1 + 2i)^2 - 4i}}{2i} = \frac{-1 - 2i + \sqrt{-3}}{2i},$$

dove $\sqrt{-3}$ denota le two radici complesse of -3, that they are $\pm i\sqrt{3}$ (while $\sqrt{3}$ denota the radice quadrata positiva of 3). Questo one may verificare scrivendo le radici nella form ρe^{i0a}, richiedendo che

$$3 = 3e^{i0} = \left(\rho e^{i0a}\right)^2 = \rho^2 e^{2i0a},$$

from which $\rho = \sqrt{3}$ and $0a = k\pi$ for $k \in \mathbb{Z}$. Abbiamo hence that le two radici they are

$$z_\pm = \frac{-1 - 2i \pm i\sqrt{3}}{2i} = -1 \pm \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Exercise 4
Siano $\alpha \in \mathbb{R}$ fissato and

$$f(t) := \frac{\log\left(1 + \frac{t}{2}\right)}{t^{2\alpha}}.$$

i) Compute $\int_1^2 f(t)\,dt$ con $\alpha = 1$.

ii) Sia $F(x) := \int_2^x f(t)\,dt$ con $\alpha = \frac{1}{2}$. Scrivere the formula of Taylor of the second order for F centrata in $x = 2$.

iii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f(t)\,dt$.

Solution. i) Integriamo by parts and we get

$$\int_1^2 \frac{\log\left(1+\frac{t}{2}\right)}{t^2}dt = -\frac{\log\left(1+\frac{t}{2}\right)}{t}\Big|_1^2 + \int_1^2 \frac{1}{t(2+t)}dt = -\frac{\log 2}{2} + \log\frac{3}{2} + \int_1^2 \frac{1}{t(2+t)}dt$$

To compute the second integral usiamo the metodo of the fratti semplici: poniamo

$$\frac{1}{t(2+t)} = \frac{A}{t} + \frac{B}{2+t} = \frac{2A+At+Bt}{t(2+t)},$$

from which $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. In conclusione

$$\int_1^2 \frac{1}{t(2+t)}dt = \frac{1}{2}\int_1^2 \frac{1}{t}dt - \frac{1}{2}\int_1^2 \frac{1}{2+t}dt = \frac{1}{2}\big(\log t - \log(2+t)\big)\Big|_1^2 = \frac{1}{2}\log\frac{3}{2}.$$

In conclusione

$$\int_1^2 \frac{\log\left(1+\frac{t}{2}\right)}{t^2}dt = -\frac{\log 2}{2} + \log\frac{3}{2} + \frac{1}{2}\log\frac{3}{2} = \log\frac{3\sqrt{3}}{4}.$$

ii) the polynomial diTaylor is

$$T_F^{2,2}(x) = F(2) + F'(2)(x-2) + \frac{F''(2)}{2}(x-2)^2,$$

therefore devo compute

$$F(0) = 0$$

$$F'(x) = \frac{d}{dx}\int_1^x f(t)dt = f(x) = \frac{\log\left(1+\frac{x}{2}\right)}{x} \quad \Rightarrow \quad F'(2) = \frac{\log 2}{2},$$

e

$$F''(x) = f'(x) = \frac{\frac{1}{2+x}x - \log(1+\frac{x}{2})}{x^2} \quad \Rightarrow \quad F''(2) = \frac{1}{8} - \frac{\log 2}{4}.$$

Ne segue

$$f(x) = \frac{\log 2}{2}(x-2) + \frac{1}{2}\left(\frac{1}{8} - \frac{\log 2}{4}\right)(x-2)^2 + o(x-2)^2$$

for $x \to 2$.

iii) Let us observe that for $\alpha \leq 0$ the function f is palesemente continuous and limitata su $[0,1]$, so that the integral esiste finito. Per $\alpha > 0$ dobbiamo valutare the comportamento asymptotic of $f(t)$ for $t \to 0^+$, essendo comunque f continuous and limitata on ogni interval $[\delta, 1]$ for every $0 < \delta < 1$. Abbiamo

$$f(t) = \frac{\log\left(1+\frac{t}{2}\right)}{t^{2\alpha}} = \frac{\frac{t}{2} + o(t)}{t^{2\alpha}} \sim \frac{1}{2t^{2\alpha-1}}, \quad \text{for } t \to 0^+.$$

Per the criterion of the asymptotic comparison it is hence

$$\int_0^1 f(t)dt \text{ converges } \Leftrightarrow \int_0^\delta \frac{1}{2t^{2\alpha-1}}dt \text{ converges for some } \delta > 0 \Leftrightarrow 2\alpha - 1 < 1.$$

Hence the integral converges if and only if $\alpha < 1$.

Exercise 5 Study the convergence semplice and assoluta of the series

$$\sum_{n=0}^{+\infty} \frac{(\log\alpha)^n}{1+\sqrt{2n}}$$

as $\alpha \in]0, +\infty[$.

Solution. Let us study the convergence of the series

$$\sum_{n=0}^{+\infty} \frac{y^n}{1+\sqrt{2n}}.$$

Per $|y| < 1$ the series absolutely converges . Questo può essere easily provato usando the Root Test, essendo

$$\lim_{n\to\infty} \left(\frac{|y|^n}{1+\sqrt{2n}}\right)^{\frac{1}{n}} = |y| < 1$$

oppure osservando che $n|y|^n \to 0$ for $|y| < 1$, hence $|y|^n \leq \frac{1}{n}$ definitively for $n \to \infty$ and visto that the series

$$\sum_{n=0}^{+\infty} \frac{1}{n(1+\sqrt{2n})}$$

converges ($\frac{1}{n(1+\sqrt{2n})} \sim \frac{1}{n^{\frac{3}{2}}}$) possiamo concludere usando the teorema of the confronto.

Per $|y| > 1$ the general term of the series diverges, hence the series non può convergere.

Per $y = 1$ the series diverges for asymptotic comparison con the series $\sum_{n=0}^{+\infty} \frac{1}{\sqrt{2n}}$.

Finally , for $y = -1$ the series converges for the criterio diLeibniz, essendo the modulo of the general term of the series decreasing a 0. Yet , for the case precedente, the series does not converge assolutamente.

Sostituendo $\log \alpha = y$ we get that the series originale absolutely converges if and only if $-1 < \log \alpha < 1$, ovvero if and only if $\frac{1}{e} < \alpha < e$, simply converges if and only if $-1 \leq \log \alpha < 1$, ovvero if and only if $\frac{1}{e} \leq \alpha < e$ and diverges in all the altri casi, ovvero $0 < \alpha < \frac{1}{e}$ and $\alpha \geq e$.

Exercise Determine all the values of $a \in \mathbb{R}$ such that the function $f(x) = e^x - ax^3$ sia convex in the whole $\mathbb{R}$.

Solution. Da $f''(x) = e^x - 6ax$, one has che f is convex if and only if

(A) $f''(x) = e^x - 6ax \geq 0$ for every $x \in \mathbb{R}$

Ora, if $a < 0$,

$$\lim_{x\to-\infty} f''(x) = \lim_{x\to-\infty} e^x - 66ax \geq 0 = -\infty$$

and hence **(A)** is not verified.

If $a = 0$ instead **(A)** is verified.

If $a > 0$ study the function $g(x) := f''(x) = e^x - 6ax$. One has $g'(x) = e^x - 6a \geq 0 \iff x \geq \log(6a)$. Hence g ha a absolute minimum in $x = \log(6a)$. Therefore **(A)** is verificata if and only if $g(\log(6a)) = 6a - 6a\log(6a) \geq 0$, that is, if and only if $1 - 1\log(6a) \geq 0$, hence if and only if $a \leq \frac{e}{6}$.

In conclusione f is convex if and only if $a \leq \frac{e}{6}$.

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Exam of the 11.02.2019

THEME 1

Exercise 1. Sia

$$f(x) = |(x+3)\log(x+3)|, \qquad x \in D =]-3, +\infty[.$$

(i) Determine i limits of f at the extremes of D and the asymptotes; study the prolungabilità for continuity in $x = -3$;

(ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points diextreme relative and absolute and draw the graph.

Solution.
(i) Con the change of variable $y = x + 3$ we get

$$\lim_{x \to -3^+} f(x) = \lim_{y \to 0^+} |y \log y| = 0.$$

Questo in particolare implica che f one may prolungare for continuity in $x = -3$ setting $f(-3) = 0$.
Clearly vale

$$\lim_{x \to \infty} |x + 3| = \infty, \quad \lim_{x \to \infty} |\log(x + 3)| = \infty \quad \Rightarrow \quad \lim_{x \to \infty} f(x) = \infty.$$

D'altronde

$$\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{|x + 3|}{x} |\log(x + 3)| = 1 \cdot \lim_{x \to \infty} |\log(x + 3)| = \infty,$$

hence the function non ha a oblique asymptote for $x \to \infty$.

(ii) Let us observe that in the domain D the function $(x + 3)\log(x + 3)$ si annullonly one for $x + 3 = 1$, ovvero $x = -2$. Hence in $D \setminus \{-2\}$ the function f is differentiable in quanto prodotto and superposition didifferentiable functions, and si calcola

$$f'(x) = \operatorname{sgn}((x + 3)\log(x + 3))((x + 3)\log(x + 3))' = \operatorname{sgn}((x + 3)\log(x + 3))(\log(x + 3) + 1),$$

ovvero

$$\begin{aligned} f'(x) &= -(\log(x + 3) + 1) \text{ for } -3 < x < -2 \\ f'(x) &= \log(x + 3) + 1 \text{ for } x > -2. \end{aligned}$$

Si vede easily that $f'(x) > 0$ for every $x > -2$, hence f is strictly monotonic increasing for $x > -2$.
Per $-3 < x < -2$ vale

$$f'(x) > 0 \Leftrightarrow \log(x + 3) < -1 \Leftrightarrow x + 3 < \frac{1}{e} \Leftrightarrow x < -3 + \frac{1}{e}.$$

Con analoghi calcoli one has hence che

$$f'(x) > 0 \text{ for } -3 < x < -3 + \frac{1}{e}, \quad f'(x) = 0 \text{ for } x = -3 + \frac{1}{e}, \quad f'(x) < 0 \text{ for } -3 + \frac{1}{e} < x < -2.$$

Ne segue che f is strictly monotonic increasing for $-3 < x < -3 + \frac{1}{e}$ and strictly monotonic decreasing for $-3 + \frac{1}{e} < x < -2$.
Therefore $-3 + \frac{1}{e}$ is a point of localmaximum , while -2 is a point of absolute minimum (infatti $f(x) \geq 0$ for every $x \in D$ and $f(-2) = 0$).
Si può easily osservare che

$$\lim_{x \to -2^-} f'(x) = -1, \quad \lim_{x \to -2^+} f'(x) = 1$$

and questo (for a teorema eventualmente visto a lezione) implica che f is not differentiable for $x = -2$.
the graph of f is in the picture 23.

Exercise 2. Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(1 + n^2)\sin n}{n^4}$$

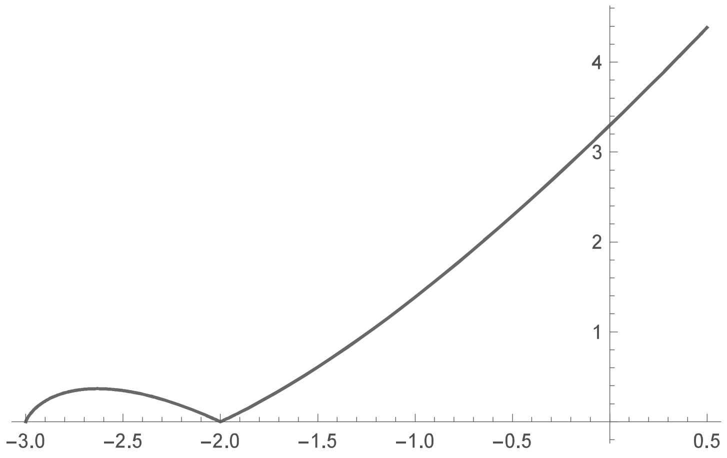


Figura 15: the graph of f (Theme 1).

Solution. Let us observe that $|\sin n| \leq 1$ for every n, and hence

$$\sum_{n=1}^{\infty} \left| \frac{(1+n^2)\sin n}{n^4} \right| = \sum_{n=1}^{\infty} |\sin n| \left| \frac{1+n^2}{n^4} \right| \leq \sum_{n=1}^{\infty} \frac{1+n^2}{n^4}.$$

Since abbiamo

$$\frac{1+n^2}{n^4} \sim \frac{n^2}{n^4} = \frac{1}{n^2} \quad \text{for } n \to \infty,$$

o (equivalentemente) scrivendo

$$\frac{1+n^2}{n^4} = \frac{n^2(1+o(1/n^2))}{n^4} = \frac{1+o(1)}{n^2},$$

for the criterio of convergence asymptotic deduce that the series a termini positivi

$$\sum_{n=1}^{\infty} \frac{1+n^2}{n^4}$$

converges, and hence for the principio of the confronto the series originale absolutely converges .

Exercise 3 [4 punti] Solve the inequality

$$\frac{1}{2} \leq \frac{\big(\mathrm{Re}\,(\bar{z}+i)-1\big)^2}{4} + \frac{\big(\mathrm{Im}\,(\bar{z}+i)-1\big)^2}{4} \leq 1$$

and draw the solutions on Gauss plane .

Solution. Scriviamo in algebraic form $z = x + iy$, $\bar{z} = x - iy$. Hence

$$\mathrm{Re}\,(\bar{z}+i) = \mathrm{Re}\,(x - iy + i) = x, \quad \mathrm{Im}\,(\bar{z}+i) = \mathrm{Im}\,(x - iy + i) = 1 - y.$$

The inequality può essere therefore riscritta come

$$\frac{1}{2} \leq \frac{\big(x-1\big)^2}{4} + \frac{\big(-y\big)^2}{4} \leq 1,$$

ovvero

$$2 \leq (x-1)^2 + y^2 \leq 4.$$

Ricordando che $(x-x_0)^2 + (y-y_0)^2 = r^2$ is the equation of a circonferenza diraggio r centrata in (x_0, y_0), we get that the inequality determina the corona circolare compresa tra le circonferenze diraggi $\sqrt{2}$ and 2 and centrate in $(1, 0)$.

the disign of the solutions is in the picture 16.

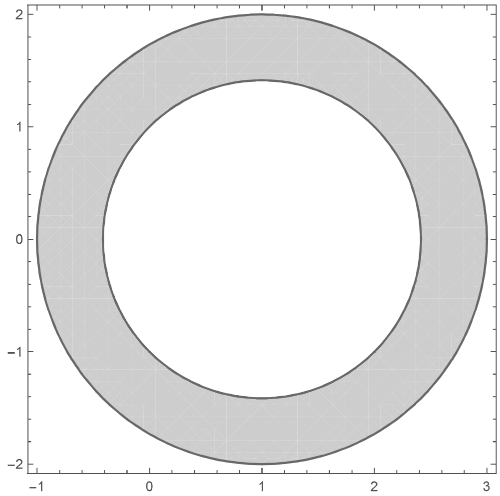


Figura 16: The soluzione of the exercise 3 (Theme 1).

Exercise 4. Compute

$$\int_0^{+\infty} e^{-\sqrt{2x}}\, dx.$$

Solution. By making use the change of variable $\sqrt{2x} = y$, from which $x = \frac{y^2}{2}$ and $dx = ydy$, we get

$$\int_0^{+\infty} e^{-\sqrt{2x}}\, dx = \int_0^{+\infty} e^{-y} y\, dy.$$

Integrando by parts one has

$$\int_0^{+\infty} e^{-y} y\, dy = \left[-e^{-y} y\right]_0^{+\infty} + \int_0^{+\infty} e^{-y}\, dy = 0 + \left[-e^{-y}\right]_0^{+\infty} = 0 - (-1) = 1.$$

Exercise 5. Sia

$$f_\alpha(x) = \frac{e^{-\sqrt{2x}} - 1}{x^{\alpha-1}}.$$

(a) study the convergence of the integral

$$\int_0^{+\infty} f_\alpha(x)\, dx$$

as $\alpha \in \mathbb{R}$.

(b) Per $\alpha = 2$, sia $F(x) = \int_1^{\cos x} f_\alpha(t)\, dt$: si calcoli $F'(\pi/3)$.

Solution. (a) Let us observe that the function f_α is continuous for $0 < x < +\infty$. Consideriamo

$$\int_0^1 f_\alpha(x)\,dx. \tag{2}$$

Essendo $e^{-\sqrt{2x}} = 1 - \sqrt{2x} + o(\sqrt{x})$ for $x \to 0$, abbiamo

$$f_\alpha(x) = \frac{-\sqrt{2x} + o(\sqrt{x})}{x^{\alpha-1}} = \frac{-\sqrt{2} + o(1)}{x^{\alpha-\frac{3}{2}}} \sim \frac{-\sqrt{2}}{x^{\alpha-\frac{3}{2}}},$$

hence, for the criterio of convergence asymptotic, the integral in (2) converges if and only if

$$\int_0^1 \frac{-\sqrt{2}}{x^{\alpha-\frac{3}{2}}}\,dx$$

converges, ovvero (portando $-\sqrt{2}$ fuori dall'integral) if and only if $\alpha - \frac{3}{2} < 1$, hence if and only if $\alpha < \frac{5}{2}$.

Let us study ora

$$\int_1^{+\infty} f_\alpha(x)\,dx. \tag{3}$$

Since $e^{-\sqrt{2x}} \to 0$ for $x \to \infty$ abbiamo

$$f_\alpha(x) \sim \frac{-1}{x^{\alpha-1}}$$

and for the criterio asymptotic of convergence , the integral in (3) converges if and only if

$$\int_1^{+\infty} \frac{-1}{x^{\alpha-1}}\,dx.$$

converges, ovvero if and only if $\alpha - 1 > 1$, hence if and only if $\alpha > 2$.

Therefore the integral originale converges if and only if $2 < \alpha < \frac{5}{2}$.

(b) Scriviamo

$$G(y) = \int_1^y f_2(t)\,dt = \int_1^y \frac{e^{-\sqrt{2t}} - 1}{t}\,dt.$$

Per the teorema fondamentale of the calcolo vale

$$G'(y) = f_2(y) = \frac{e^{-\sqrt{2y}} - 1}{y}.$$

Abbiamo $F(x) = G(\cos x)$. Per the chain rule , hence

$$F'(\pi/3) = G'(\cos(\pi/3))(-\sin(\pi/3)) = -\frac{\sqrt{3}}{2}G'(1/2) = -\frac{\sqrt{3}}{2}\frac{e^{-1} - 1}{1/2} = -\sqrt{3}(1 - 1/e).$$

Exercise 6 Compute the limit

$$\lim_{x\to 0^+} \frac{\cosh(\alpha\, x) - \cosh\left(e^{2x} - 1\right)}{x^3}$$

for all values of the parameter $\alpha > 0$.

Solution. Ricordiamo che $\cosh y = 1 + \frac{y^2}{2} + o(y^2)$, ed $e^y = 1 + y + o(y)$ for $y \to 0$, hence possiamo espandere the numerator come

$$\begin{aligned}
\text{Num} &= 1 + \frac{\alpha^2 x^2}{2} + o(x^2) - \cosh\left(2x + o(x)\right) \\
&= 1 + \frac{\alpha^2 x^2}{2} + o(x^2) - \left[1 + \frac{(2x + o(x))^2}{2} + o((x + o(x))^2)\right] \\
&= 1 + \frac{\alpha^2 x^2}{2} + o(x^2) - \left[1 + 2x^2 + o(x^2)\right] \\
&= \frac{(\alpha^2 - 4)x^2}{2} + o(x^2).
\end{aligned}$$

Therefore

$$\lim_{x\to 0^+} \frac{\cosh(\alpha\, x) - \cosh\left(e^{2x} - 1\right)}{x^3} = \lim_{x\to 0^+} \frac{\frac{\alpha^2 - 4}{2} + o(1)}{x} = \begin{cases} -\infty & \text{for } 0 < \alpha < 2 \\ +\infty & \text{for } \alpha > 2. \end{cases}$$

the case $\alpha = 2$ one has più difficile perchánd is not possibile compute $\lim_{x\to 0^+} \frac{o(1)}{x}$. Dobbiamo therefore ottenere un'expansion of the numerator all'order successivo (the third). This volta scriviamo $\cosh y = 1 + \frac{y^2}{2} + o(y^3)$, ed $e^y = 1 + y + y^2 + o(y^2)$ for $y \to 0$. In particolare

$$\begin{aligned}
\cosh(e^{2x} - 1) &= \cosh(2x + 2x^2 + o(x)^2) \\
&= 1 + \frac{(2x + 2x^2 + o(x)^2)^2}{2} + o((2x + 2x^2 + o(x)^2)^3) \\
&= 1 + \frac{4x^2 + 8x^3 + o(x^3)}{2} + o(x^3) \\
&= 1 + 2x^2 + 4x^3 + o(x^3).
\end{aligned}$$

Per $\alpha = 2$ we have $\cosh(\alpha x) = 1 + 2x^2 + o(x^3)$, hence

$$\text{Num} = 1 + 2x^2 + o(x^3) - (1 + 2x^2 + 4x^3 + o(x^3)) = -4x^3 + o(x^3)$$

Therefore

$$\lim_{x\to 0^+} \frac{\cosh(2x) - \cosh\left(e^{2x} - 1\right)}{x^3} = \lim_{x\to 0^+} \frac{-4x^3 + o(x^3)}{x^3} = -4.$$

Exercise . Compute

$$\lim_{x\to +\infty} \int_x^{x+e^{-x}} e^t \arctan t \, dt.$$

Solution. Essendo the integrand continuous in a neighbourhood of $+\infty$ (in realtà in the whole $\mathbb{R}$), for the mean value theorem esiste $t_x \in [x, x + e^{-x}]$ tale che

$$\int_x^{x+e^{-x}} e^t \arctan t \, dt = e^{-x}\, e^{t_x} \arctan t_x$$

and hence, siccome the integrand is increasing,

$$e^{-x} e^x \arctan x \le e^{-x}\, e^{t_x} \arctan t_x \le e^{-x} e^{x+e^{-x}} \arctan\left(x + e^{-x}\right),$$

that is,

$$\arctan x \le \int_x^{x+e^{-x}} e^t \arctan t \, dt \le e^{e^{-x}} \frac{\pi}{2}.$$

Siccome

$$\lim_{x\to+\infty} e^{e^{-x}} = 1,$$

applicando the teorema of the Carabinieri one gets

$$\lim_{x\to+\infty} \int_x^{x+e^{-x}} e^t \arctan t\, dt = \frac{\pi}{2}.$$

Exam of the 8.07.2019

THEME 1

Exercise 1 [6 punti] Sia

$$f(x) = e^{\frac{2}{|2+\log x|}}.$$

a) Determine the domain D of f; determine i limits of f at the extremes of D and study the prolungabilità for continuity di f in $x = 0$;
b) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme ;
c) draw a qualitative graph of f.

Solution (a) Essendo the domain of e^x tutto $\mathbb{R}$, and the domain of $\log x$ all the $x > 0$, the domain of f is determinato dalle two condizioni:

$$x > 0, \quad 2 + \log x \neq 0.$$

The seconda relazione equivale a $x \neq e^{-2}$, hence

$$D = \{x > 0 : x \neq e^{-2}\}.$$

Con three cambi of variabile one gets

$$\lim_{x\to0^+} f(x) = \lim_{y\to-\infty} e^{\frac{2}{|2+y|}} = \lim_{s\to+\infty} e^{\frac{2}{s}} = \lim_{t\to0^+} e^t = 1.$$

Hence f può essere estesa for continuity in 0 setting $f(0) = 1$.
Furthermore,

$$\lim_{x\to e^{-2}} f(x) = \lim_{y\to0^+} e^{\frac{2}{y}} = \lim_{s\to+\infty} e^s = +\infty.$$

Hence f non può essere estesa for continuity in e^{-2}.
(b) The function is differentiable at all points of its domain, essendo superposition didifferentiable functions (the function $|\cdot|$ is not differentiable only at 0, ma $2 + \log x$ is zero only at e^{-2}, which doesn't belong to the domain.) The derivata, calcolata con the chain rule is :

$$f'(x) = e^{\frac{2}{|2+\log x|}} \left(-\frac{2}{|2+\log x|^2}\right) \frac{2+\log x}{|2+\log x|} \frac{1}{x} = -\frac{2e^{\frac{2}{|2+\log x|}}}{x|2+\log x|^3}(2+\log x).$$

Notice that the fraction of the right-hand side is always positiva in the domain D, from which:

$$f'(x) > 0 \Leftrightarrow 2 + \log x < 0 \Leftrightarrow 0 < x < e^{-2},$$

$$f'(x) < 0 \Leftrightarrow 2 + \log x > 0 \Leftrightarrow x > e^{-2},$$

ed $f'(x) \neq 0$ for every $x \in D$. In particular, f non ha punti critical .

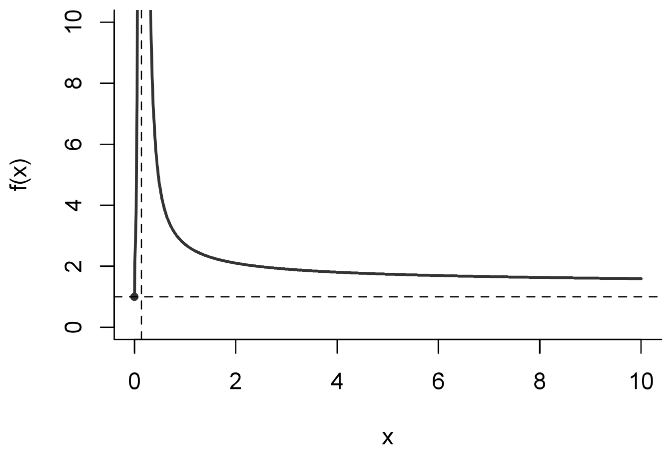


Figura 17: the graph of f (Theme 1).

the graph of f is in the picture 17.

Exercise 2 [4 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{1 - 2\sqrt{n}}.$$

Solution Per $n \to \infty$ we have $\sin \frac{1}{n} \sim \frac{1}{n}$ and $1 - 2\sqrt{n} \sim -2\sqrt{n}$. Therefore the general term of the series is asymptotic to $\frac{-1}{2n^{\frac{3}{2}}}$. Since $\frac{3}{2} > 1$, the series

$$\sum_{n=1}^{\infty} \frac{-1}{2n^{\frac{3}{2}}}$$

converges, and for the principio of convergence asymptotic, also the prima series converges.

Exercise 3 [4 punti] Solve the equation

$$\frac{z}{\bar{z}} = -\frac{(\operatorname{Im} z)^2}{|iz^2|}$$

and draw the solutions on Gauss plane .

Solution Let us observe that the equation is ben definitonly one for $z \neq 0$. Therefore, assumendo $z \neq 0$ possiamo semplificare moltiplicando a sinistra for $\frac{z}{z}$, ottenendo

$$\frac{z}{\bar{z}} = \frac{z^2}{|z|^2} = -\frac{(\operatorname{Im} z)^2}{|z|^2},$$

where we have also usato che $|iz^2| = |z^2| = |z|^2$. Moltiplicando for $|z|^2$ we get

$$z^2 = -\text{Im z}^2.$$

Scrivendo $z = x + iy$ one gets $z^2 = x^2 - y^2 + 2ixy$, $-(\mathrm{Im} z)^2 = -y^2$, and we get hence

$$x^2 - y^2 + 2ixy = -y^2. \tag{4}$$

Quest'equation può essere soddisfattonly one if $2ixy = 0$. Questo implica $xy = 0$, ovvero $x = 0$ o $y = 0$ (non entrambi perché $z \neq 0$). In the case $x = 0$, $y \neq 0$, we have a soluzione of (4). In the case $y = 0$ and $x \neq 0$ instead (4) is not soddisfatta. Therefore the solutions they are

$$\{z = x + iy \in \mathbb{C} : x = 0, y \neq 0\},$$

ovvero the asse immaginario privato of the origine, come one may vedere in the picture 18.

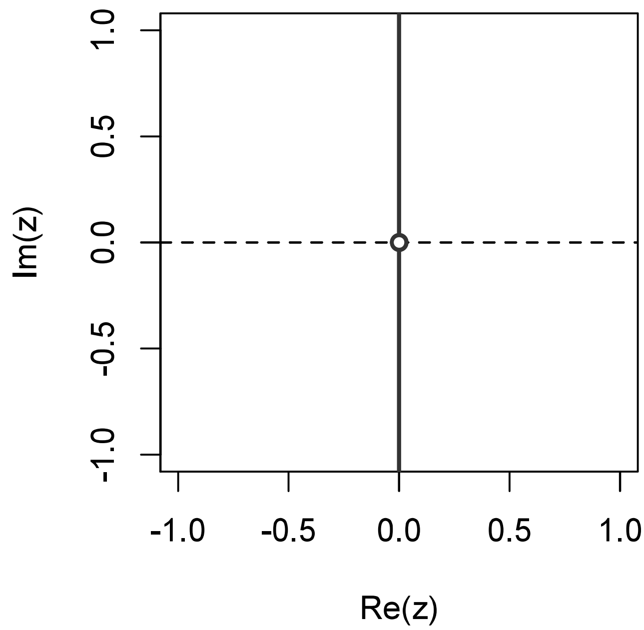


Figura 18: The insieme of the solutions of the exercise 3 (Theme 1).

Exercise 4 [5+3+4 punti] a) Compute a primitive of the function

$$e^x \log(1 + 2e^x).$$

Per $\alpha \in \mathbb{R}$, define $f_\alpha(x) = e^{\alpha x} \log(1 + 2e^x)$:
b) study the convergence of the generalized integral

$$\int_0^{+\infty} f_\alpha(x)\, dx$$

as $\alpha \in \mathbb{R}$;
c) find the Taylor expansion of order 2 centered in $x_0 = 1$ of the function

$$F(x) = \int_1^x f_0(t)\,dt.$$

Solution a) Con the sostituzione $y = e^x$, $dy = e^x dx$ and un'integration by parts one gets

$$\int e^x \log(1+2e^x)dx = \int \log(1+2y)dy = y\log(1+2y) - \int \frac{2y}{1+2y}dy.$$

Ora, for ridurre the numerator of the integrand a destra scriviamo

$$\frac{2y}{1+2y} = 1 - \frac{1}{1+2y},$$

Hence

$$\int \frac{2y}{1+2y}dy = \int \left(1 - \frac{1}{1+2y}\right) dy = y - \frac{\log(1+2y)}{2}.$$

Aggiungendo ai termini precedenti and sostituendo $y = e^x$ one gets

$$\int e^x \log(1+2e^x)dx = e^x \log(1+2e^x) - e^x + \frac{\log(1+2e^x)}{2} + c.$$

b) Per ogni $\alpha \in \mathbb{R}$ the function f_α is continuous in $[0, +\infty)$, hence for study the convergence of the suo integral, study the comportamento of f_α for $x \to \infty$. Per $\alpha \geq 0$ we have che

$$\lim_{x\to\infty} f_\alpha(x) = +\infty,$$

so that the integral $\int_0^{+\infty} f_\alpha(x)dx$ diverges.

Per $\alpha < 0$, we have $f_\alpha(x) = O(x^{-2})$ for $x \to \infty$, and for the criterion of the asymptotic comparison, the integral converges.

c) Abbiamo

$$F(x) = F(1) + F'(1)(x-1) + \frac{F''(1)}{2}(x-1)^2 + o(|x-1|^2) \quad \text{for } x \to 1.$$

Abbiamo

$$F(1) = 0, \quad F'(1) = f_0(1) = \log(1+2e), \quad f_0'(x) = \frac{2e^x}{1+2e^x}, \quad f_0'(1) = \frac{2e}{1+2e},$$

hence

$$F(x) = \log(1+2e)(x-1) + \frac{e}{1+2e}(x-1)^2 + o(|x-1|^2) \quad \text{for } x \to 1.$$

Exercise 5 [6 punti] Compute the limit

$$\lim_{x\to+\infty} x^\alpha \left(\sqrt[8]{x^2-2} - \sqrt[4]{x+1}\right)$$

for all values of the parameter $\alpha > 0$.

Solution. Let us utilize the expansion $(1+y)^\alpha = 1 + \alpha y + o(y)$ for $y \to 0$ and scriviamo

$$\sqrt[8]{x^2-2} = (x^2-2)^{\frac{1}{8}} = x^{\frac{1}{4}}\left(1 - \frac{2}{x^2}\right)^{\frac{1}{8}} = x^{\frac{1}{4}}\left(1 - \frac{1}{4x^2} + o\left(\frac{1}{x^2}\right)\right), \quad \text{for } x \to \infty,$$

$$\sqrt[4]{x-1} = (x-1)^{\frac{1}{4}} = x^{\frac{1}{4}}\left(1-\frac{1}{x}\right)^{\frac{1}{4}} = x^{\frac{1}{4}}\left(1-\frac{1}{4x}+o\left(\frac{1}{x}\right)\right), \quad \text{for } x \to \infty.$$

Sottraendo we get

$$x^\alpha\left(\sqrt[8]{x^2-2}-\sqrt[4]{x+1}\right) = x^\alpha \cdot x^{\frac{1}{4}}\left(-\frac{1}{4x}+o\left(\frac{1}{x}\right)\right) = -\frac{x^{\alpha-\frac{3}{4}}}{4}(1+o(1)) \quad \text{for } x \to \infty.$$

Therefore

$$\lim_{x\to+\infty} x^\alpha\left(\sqrt[8]{x^2-2}-\sqrt[4]{x+1}\right) = \lim_{x\to+\infty}\left(-\frac{x^{\alpha-\frac{3}{4}}}{4}(1+o(1))\right) = \begin{cases} 0 & \text{for } \alpha < \frac{3}{4} \\ -\frac{1}{4} & \text{for } \alpha = \frac{3}{4} \\ -\infty & \text{for } \alpha > \frac{3}{4}. \end{cases}$$

Exam of the 17.09.2019

Theme 1

Exercise 1. Sia
$$f(x) = \log\left|e^{3x}-2\right|.$$

a) Determine the domain D and study the sign of f; determine i limits of f at the extremes of D and determine the asymptotes;
b) study the derivability, compute the derivative and study the monotonicity of f; determine the points of extreme relative and absolute ;
c) draw a qualitative graph of f.

Solution. a) Clearly $D = \{x \in \mathbb{R} : |e^{3x}-2| > 0\} = \{x \in \mathbb{R} : e^{3x}-2 \neq 0\} = \mathbb{R}\backslash\{\frac{\log 2}{3}\}$. Segno:

$$f(x) \geqslant 0 \iff |e^{3x}-2| \geqslant 1 \iff e^{3x}-2 \leqslant -1 \text{ and } e^{3x}-2 \geqslant 1 \iff x \leqslant 0, \text{ and } x \geqslant \frac{\log 3}{3}.$$

When $=$ one has also the zeros of f. Limits and asymptotes: we have to study the function for $x \longrightarrow \pm\infty, \frac{\log 2}{3}$. Easily one has $f(-\infty) = \log 2$, hence $y = \log 2$ is horizontal asymptote at $-\infty$. A $+\infty$ easily $f(+\infty) = +\infty$. Cerchiamo a oblique asymptote $y = mx + q$. As for m, we have che

$$m = \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\log(e^{3x}-2)}{x} = \lim_{x\to+\infty}\frac{\log(e^{3x}\cdot 1_x)}{x} = \lim_{x\to+\infty}\frac{3x+0_x}{x} = 3.$$

As for q, we have

$$q = \lim_{x\to+\infty}(f(x)-3x) = \lim_{x\to+\infty}\left(\log(e^{3x}-2)-\log e^{3x}\right) = \lim_{x\to+\infty}\log\frac{e^{3x}-2}{e^{3x}} = \log 1 = 0.$$

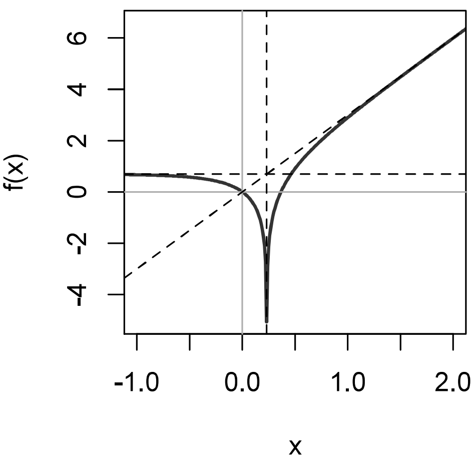
Conclusione: $y = 3x$ is oblique asymptote at $+\infty$. Finally ,

$$\lim_{x\to\frac{\log 2}{3}}\log|e^{3x}-2| = \log 0+ = -\infty,$$

from which $x = \frac{\log 2}{3}$ is vertical asymptote.
b) Clearly f is continuous sul proprio domain essendo superposition dicontinuous functions ove definite. È also differentiable poichánd the unique point in which one may not applicare the chain rule is x t.c. $e^{3x}-2=0$, that is, $x = \frac{\log 2}{3}$, that for ò non appartiene al domain of f. The derivative is

$$f'(x) = \frac{1}{|e^{3x}-2|}\operatorname{sgn}(e^{3x}-2)\cdot 3e^{3x} = \frac{3e^{3x}}{e^{3x}-2}.$$

Da this segue che
$$f'(x) \geqslant 0, \quad \Longleftrightarrow \quad e^{3x} - 2 > 0, \quad \Longleftrightarrow \quad x > \frac{\log 2}{3}.$$
One concludes that $f \searrow$ su $]-\infty, \frac{\log 2}{3}[$ while $f \nearrow$ su $]\frac{\log 2}{3}, +\infty[$. There are no , diconseguenza nánd minimi nánd massimi (diqualsiasi natura).
c) Grafico.



Exercise 2. Compute the limit
$$\lim_{x \to 0^+} \frac{e^{x-2x^2} - 1 - x}{\sinh x^2 + x^{7/3} \log x}.$$
Solution. Since $\lim_{x\to 0+} x^\alpha \log x = 0$, si vede easily that the limit si presenta come a form of the tipo 0/0. Let us study the ordine diinfinitesimal l dinumeratore and denominator . Since
$$e^t = 1 + t + o(t) = 1 + t + \frac{t^2}{2} + o(t^2),$$
abbiamo
$$N = 1 + (x - 2x^2) + o(x - x^2) - 1 - x = -2x^2 + o(x) = o(x),$$
insufficiente for at comportamento preciso,
$$N = 1 + (x - 2x^2) + \frac{(x - 2x^2)^2}{2} + o((x - 2x^2)^2) - 1 - x = -2x^2 + \frac{x^2}{2} + o(x^2) \sim -\frac{3}{2}x^2 \text{ for } x \to 0^+.$$
Per the denominator is sufficiente ricordare che $\sinh t = t + o(t)$ so that
$$D = x^2 + o(x^2) + x^{7/3} \log x = x^2 + o(x^2) \text{ for } x \to 0^+,$$
essendo $x^{7/3} \log x = o(x^2)$ poiché $\frac{x^{7/3} \log x}{x^2} = x^{1/3} \log x \longrightarrow 0$ for $x \longrightarrow 0+$. Hence
$$\frac{N}{D} \sim \frac{-\frac{3}{2}x^2}{x^2} \longrightarrow -\frac{3}{2}.$$

Exercise 3. Solve the inequality

$$\operatorname{Re} z \leq \operatorname{Re}\left(\frac{3}{z}\right)$$

and draw the solutions on Gauss plane .

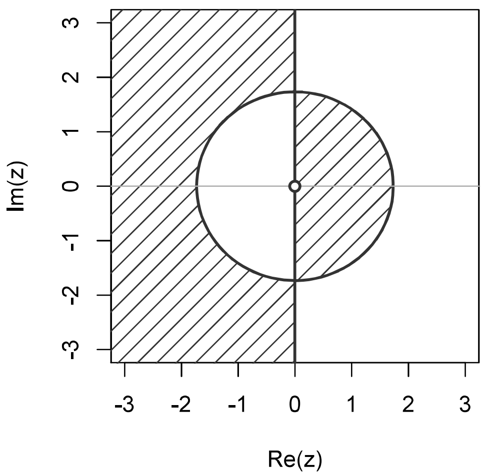
Solution. Sia $z = x + iy$ con $x, y \in \mathbb{R}$. Then $\operatorname{Re} z = x$ while essendo $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$,

$$\operatorname{Re}\frac{3}{z} = \frac{3x}{x^2+y^2}.$$

Therefore, $z \neq 0$ verifica the inequality if and only if

$$x \leqslant \frac{3x}{x^2+y^2} \iff \begin{cases} x > 0, \quad 1 \leqslant \frac{3}{x^2+y^2}, & \iff \quad x^2+y^2 \leq 3, \\ x = 0, \quad \forall y \in \mathbb{R}\backslash\{0\}, & \\ x < 0, \quad 1 \geqslant \frac{3}{x^2+y^2}, & \iff \quad x^2+y^2 \geq 3. \end{cases}$$

Figura:



Exercise 4. a) Compute the indefinite integral

$$\int \left(\tan\frac{x}{2}\right)^3 dx \qquad \text{(sugg.: eseguire the sostituzione } \tan\frac{x}{2} = u\text{).}$$

b) study the convergence of the generalized integral

$$\int_0^{\frac{\pi}{6}} \frac{\tan x}{x^{\alpha+2}} dx$$

as $\alpha \in \mathbb{R}$.

Solution. a) Seguendo the hint $u = \tan x/2$, $x = 2\arctan u$ from which $dx = \frac{2}{1+u^2}$, therefore

$$\begin{aligned}\int \left(\tan \tfrac{x}{2}\right)^3\ dx &= \int \tfrac{2u^3}{1+u^2}\ du = 2\int \tfrac{u(u^2+1-1)}{1+u^2}\ du = \int 2u - \tfrac{2u}{1+u^2}\ du = u^2 - \log(1+u^2) \\ &= \left(\tan \tfrac{x}{2}\right)^2 - \log\left(1 + \left(\tan \tfrac{x}{2}\right)^2\right).\end{aligned}$$

b) Sia $f(x) = \frac{\tan x}{x^{\alpha+2}}$. Certamente $f \in C(]0, \frac{\pi}{6}])$ for every α and is continuous anche in $x = 0$ (hence integrabile sicuramente) for $\alpha + 2 \leqslant 0$, that is, for $\alpha \leqslant -2$. Per $\alpha > -2$ we have at generalized integral in $x = 0$. Since $\tan x = x + o(x) = x1_x$ for $x \longrightarrow 0$,

$$f(x) \sim \frac{x}{x^{\alpha+2}} = \frac{1}{x^{\alpha+1}} \sim \frac{1}{x^{\alpha+1}} \ \text{ for } x \to 0^+,$$

integrabile in 0 if and only if $\alpha + 1 < 1$, that is, $\alpha < 0$ for asymptotic comparison . Morale: the generalized integral esiste finito if and solo if $\alpha < 0$.

Exercise 5. (i) Si dimostri that the sequence

$$a_n = \log(n+1) - \log\sqrt{n^2 + \alpha n + 4}$$

is infinitesimal for $n \to \infty$ (for every α) and for $\alpha = 2$ compute the order ;
(ii) study the convergence of the series

$$\sum_{n=2}^{\infty} a_n$$

as $\alpha \in \mathbb{R}$.

Solution. i) Let us observe that

$$a_n = \log \frac{n+1}{\sqrt{n^2 + \alpha n + 4}} \sim \log \frac{n}{n} \longrightarrow 0.$$

In order to find the order diinfinitesimal l occorre essere più precisi. Notiamo che, by dividing numerator and denominator by n, and usando le proprietà of the logarithms

$$a_n = \log\left(1 + \frac{1}{n}\right) - \frac{1}{2}\log\left(1 + \frac{\alpha}{n} + \frac{4}{n^2}\right).$$

Since $\log(1+t) = t + o(t) = t - \frac{t^2}{2} + o(t^2)$,

$$\begin{aligned}a_n &= \tfrac{1}{n} - \tfrac{1}{2n^2} + o\left(\tfrac{1}{n^2}\right) - \tfrac{1}{2}\left(\tfrac{\alpha}{n} + \tfrac{4}{n^2} - \frac{\left(\frac{\alpha}{n} + \frac{4}{n^2}\right)^2}{2} + o\left(\tfrac{1}{n^2}\right)\right) \\ &= \tfrac{2-\alpha}{2n} - \tfrac{5-\alpha^2}{2n^2} + o\left(\tfrac{1}{n^2}\right).\end{aligned}$$

In particolare, if $\alpha = 2$ one gets $a_n \sim -\frac{1}{2n^2}$.
ii) Per quanto visto al point i),

$$a_n \sim \begin{cases} \frac{2-\alpha}{2n} \equiv \frac{C}{n}, & \alpha \neq 2, \\ -\frac{1}{2n^2} \equiv \frac{C}{n^2}, & \alpha = 2, \end{cases}$$

from which si conclude che $\sum_n a_n$ converges if and only if $\alpha = 2$ in virtù of the criterion of the asymptotic comparison .

Exam of the 20.01.2020

THEME 1

Exercise 1 [7 punti] Consider the function

$$f(x) = \sin\left(2\arctan(|x|^3)\right)$$

i) determine the domain D, the sign, simmetries , i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ; the study of the second derivative may be skipped

iii) draw the qualitative graph .

Solution. i) Clearly $D =]-\infty, +\infty[$. Clearly f is pari, hence basta limitarsi althe study su $[0, +\infty[$. Since $2\arctan|x|^3 \in [0, \pi[$, f is always positiva and moreover $f = 0$ sse $x = 0$. Limits : there is only one interesting limit, $\lim_{x\to+\infty} f(x) = \sin\pi = 0$, from which the retta $y = 0$ is horizontal asymptote at $+\infty$.

ii) Essendo f superposition didifferentiable functions, eccetto for $x = 0$, one has

$$f'(x) = \cos\left(2\arctan|x|^3\right)\frac{6x^2\text{sgn } x}{1+x^6}, \quad \forall x \neq 0.$$

Per $x = 0$ chiaramente f is continuous and siccome

$$\lim_{x\to 0} f'(x) = 0,$$

for the test diderivability it follows that $\exists f'(0) = 0$. Per the monotonicity, study the sign of f': for $x > 0$,

$$f'(x) \geqslant 0, \iff \cos\left(2\arctan x^3\right) \geqslant 0, \iff 2\arctan x^3 \leqslant \frac{\pi}{2}, \iff \arctan x^3 \leqslant \frac{\pi}{4}, \iff x^3 \leqslant 1,$$

that is, for $x \leqslant 1$. Hence f is increasing su $[0, 1]$ and decreasing su $[1, +\infty[$. Si deduce easily the monotonicity su D and che $x = 0$ is point of minimum globale while $x = \pm 1$ they are massimi globali.

Exercise 2 [6 punti] Compute the limit

$$\lim_{x\to 0+}\left(1+\sin x\right)^{x^a}$$

as $a \in \mathbb{R}$, usando the form "exp{log . . . }".

Solution. Per $x \longrightarrow 0+$, $1 + \sin x \longrightarrow 1$ while

$$x^a \longrightarrow \begin{cases} 0, & \text{if } a > 0, \\ 1, & \text{if } a = 0, \\ +\infty, & \text{if } a < 0. \end{cases}$$

Since $1^0 = 1$ and $1^1 = 1$ si deduce that the limit it is 1 for every $a \geqslant 0$. Per $a < 0$, $1^{+\infty}$ is indeterminate form. Since

$$(1+\sin x)^{x^a} = e^{x^a \log(1+\sin x)},$$

grafici/1app1920_disigntema1.pdf

ricordato che $\log(1+t) = t1_t$ and che $\sin x = x1_x$ abbiamo

$$(1+\sin x)^{x^a} = e^{x^a \sin x \cdot 1_x} = e^{x^{a+1}1_x} \longrightarrow \begin{cases} e^0 = 1, & \text{if } -1 < a < 0, \\ e^1 = e, & \text{if } a = -1, \\ e^{+\infty} = +\infty, & \text{if } a < -1. \end{cases}$$

Exercise 3 [4 punti] Trovare the zeros in $\mathbb{C}$ di

$$(z^3+5)(z^2+z+1) = 0.$$

Solution. Clearly

$$(z^3+5)(z^2+z+1) = 0, \quad \iff \quad z^3 = -5, \ \vee \ z^2+z+1 = 0.$$

In the first case , si tratta di compute le radici terze of -5. Premesso che $-5 = 5u(\pi)$ ($u(0a) = \cos 0a + i \sin 0a$), for the formula diDe Moivre, $z = \rho u(0a)$ is t.c.

$$z^3 = -5, \quad \iff \quad \begin{cases} \rho^3 = 5, \\ \\ 0a = \frac{\pi}{3} + k\frac{2\pi}{3}, \ k = 0,1,2, \end{cases} \quad \iff \quad z = \sqrt[3]{5}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \ -\sqrt[3]{5}, \ \sqrt[3]{5}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right).$$

In the second case,

$$z_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ fissato and

$$f_\alpha(t) := \frac{e^{2t} + 2e^t}{(e^t - 1)^\alpha}.$$

i) Compute a primitive of f_α con $\alpha = 1$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^1 f_\alpha(t)\, dt$.

Solution. i) Abbiamo che

$$\int \frac{e^{2t}+2e^t}{e^t-1}\ dt \overset{u=e^t,\ t=\log u,\ dt=du/u}{=} \int \frac{u^2+2u}{(u-1)} \frac{du}{u} = \int \frac{u+2}{u-1}\ du = \int \left(1 + \frac{3}{u-1}\right)\ du$$

$$= u + 3\log|u-1| = e^t + 3\log|e^t - 1|.$$

ii) Considerato che $f_\alpha \in C(]0,1])$, the integral $\int_0^1 f_\alpha(t)\ dt$ is generalizzato in 0. Essendo $f_\alpha \geqslant 0$ su $]0,1]$, possiamo applicare the test of the asymptotic comparison for stabilire the convergence of the integral. Notiamo che

$$f_\alpha(t) = \frac{3_t}{(e^t - 1)^\alpha} = \frac{3_t}{(t1_t)^\alpha} \sim_{0+} \frac{3}{t^\alpha},$$

so that esiste $\int_0^1 f_\alpha$ sse esiste $\int_0^1 \frac{1}{t^\alpha}\ dt$, sse $\alpha < 1$ come ben noto.

Exercise 5 [6 punti] Study the convergence semplice and assoluta of the series

$$\sum_{n=1}^{+\infty} \frac{(3\sin x)^n n}{n^2 + \sqrt{n}}$$

as $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Solution. Let us study the absolute convergence, that is, the convergence of the series

$$\sum_n |a_n| = \sum_n \frac{n 3^n |\sin x|^n}{n^2 + \sqrt{n}}.$$

A tal fine, let us apply the Root Test: essendo

$$|a_n|^{1/n} = \frac{n^{1/n} 3|\sin x|}{n^{2/n} 1_n} \longrightarrow 3|\sin x|,\ \forall x \in [-\pi/2, \pi/2],$$

(ricordiamo che $n^{1/n} \longrightarrow 1$) we have che:

- if $3|\sin x| < 1$ (that is, $|\sin x| < \frac{1}{3}$ ovvero, essendo $x \in [-\pi/2, \pi/2]$, sse $x \in]-\arcsin 1/3, \arcsin 1/3[$), the series absolutely converges (hence also semplicemente);
- if $3|\sin x| > 1$ (that is, for $[-\pi/2, \pi/2] \setminus [-\arcsin 1/3, \arcsin 1/3]$), the series diverges assolutamente and poichánd the test dice in questo case che $|a_n| \longrightarrow +\infty$, the condizione necessaria of convergence is not verificata, so that the series does not converge nemmeno semplicemente.

Rimangono the casi $\sin x = \pm\frac{1}{3}$, nei quali the test precedente fallisce. Per $\sin x = 1/3$, the series diventa

$$\sum_n \frac{n}{n^2 + \sqrt{n}} \sim \sum_n \frac{1}{n},\ \text{divergente.}$$

Since the terms have constant sign, convergence semplice and assoluta coincidono (hence there is no kind of convergence). Finally , for $\sin x = -1/3$,

$$\sum_n (-1)^n \frac{n}{n^2 + \sqrt{n}},$$

that is a series a termini disign alternato. The absolute convergence ritorna al case precedente (hence is esclusa). Per the convergence semplice possiamo applicare the test diLeibniz purché

$$\frac{n}{n^2 + \sqrt{n}} \searrow 0.$$

The convergence at 0 is evidente. Per the monotonicity possiamo procedere direttamente oppure introdurre the function ausiliaria $f(x) := \frac{x}{x^2+\sqrt{x}}$ and osservare che

$$f'(x) = \frac{x^2 + \sqrt{x} - x(2x + \frac{1}{2\sqrt{x}})}{(x^2 + \sqrt{x})^2} = \frac{-x^2 + \frac{\sqrt{x}}{2}}{(x^2 + \sqrt{x})^2}.$$

Siccome $f' \leqslant 0$ sse $-x^2 + \sqrt{x}/2 \leqslant 0$ ovvero $x^{3/2} \geqslant \frac{1}{2}$, in particolare for $n \geqslant 1$ one has $f(n) \searrow$, from which the conclusione: the series simply converges (ma non assolutamente) for the test diLeibniz.

Exercise Sia $\{a_n\}$ a sequence tale che $a_n > 0$ and $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$ for every $n \in \mathbb{N}$. Si dimostri che $\sum_{n=1}^{\infty} a_n$ diverges.

Solution. Dall'ipotesi segue che $(n+1)a_{n+1} \geqslant na_n$, cioé (na_n) is increasing: allora $na_n \geqslant a_1 > 0$, from which $a_n \geqslant \frac{a_1}{n}$ for every $n \geqslant 1$. Ma allora, the series diverges for confronto con the series armonica.

Tempo a disposizione: 2 ore and 45 minuti.

Exam of the 10.02.2020

THEME 1

Exercise 1 [7 punti] Consider the function

$$f(x) = \exp\left\{\left|\frac{x}{x+1}\right|\right\}.$$

i) Find the domain D, i limits at the extremes of D and the asymptotes;

ii) study the derivability, calcolarne the derivata, study the monotonicity, determine the points of extreme relative and absolute ;

iii) draw the qualitative graph .

Solution. i) Clearly $D = \mathbb{R} \setminus \{-1\}$. I limits at the extremes of D they are

$$\lim_{x\to\pm\infty} f(x) = \lim_{t\to 1} e^t = e, \quad \lim_{x\to -1} f(x) = \lim_{t\to+\infty} e^t = +\infty.$$

Therefore, f ha a horizontal asymptote of equation $y = e$ for $x \to \pm\infty$, and a vertical asymptote of equation $x = -1$ for $x \to -1$.

ii) f is composta da differentiable functions tranne where the modulo si annulla, that is, f is sicuramente differentiable in ogni $x \in D \setminus \{0\} = \mathbb{R} \setminus \{0, -1\}$. the point $x = 0$ viene studiato a parte. Distinguiamo tra the case in cui $\frac{x}{x+1} > 0$, that is, $x > 0$ oppure $x < -1$, and the case in cui $\frac{x}{x+1} < 0$, that is, $-1 < x < 0$.

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- if $x \in]-\infty, -1[\cup]0, +\infty[$

$$f(x) = \exp\left\{\frac{x}{x+1}\right\}$$

$$f'(x) = \exp\left\{\frac{x}{x+1}\right\}\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{1}{(x+1)^2}\exp\left\{\frac{x}{x+1}\right\},$$

that is strictly positiva, Thereforerefore f is increasing su $]-\infty, -1[\cup]0, +\infty[$.

- If $x \in]-1, 0[$ one has

$$f(x) = \exp\left\{-\frac{x}{x+1}\right\}$$

$$f'(x) = \exp\left\{-\frac{x}{x+1}\right\}\frac{d}{dx}\left(-\frac{x}{x+1}\right) = -\frac{1}{(x+1)^2}\exp\left\{-\frac{x}{x+1}\right\},$$

that is strictly negativa, Thereforerefore f is decreasing su $]-1, 0[$.

Si vede che $\lim_{x\to 0^+} f'(x) = 1e^0 = 1$, while $\lim_{x\to 0^-} f'(x) = -1e^0 = -1$. Therefore f is not differentiable in $x = 0$, that is a angular point . Essendo D a union of intervals aperti, f può avere extremes locali solo where Therefore derivative si annulla and in points of non derivability. Come osservato sopra, $f'(x) \neq 0$, and the unique extreme si trova in $x = 0$, dove f ha the suo absolute minimum con $f(0) = 1$.

iii) Grafico:

Exercise 2 [5 punti] Study the convergence of the series

$$\sum_{k=1}^{\infty} 3^k \frac{k!}{k^k}.$$

Solution. The series has positive terms . let us apply the criterio of the rapporto asymptotic . One has

$$\frac{a_{k+1}}{a_k} = \frac{3^{k+1}(k+1)!}{(k+1)^{k+1}}\frac{k^k}{3^k k!} = \frac{3(k+1)k^k}{(k+1)(k+1)^k} = \frac{3}{(1+\frac{1}{k})^k} \to \frac{3}{e} \text{ for } k \to \infty.$$

Essendo $\frac{3}{e} > 1$, the series diverges for the criterio of the rapporto asymptotic .

Exercise 3 [5 punti] Solve in $\mathbb{C}$ nella form preferita (algebrica, esponenziale, trigonometrica):

$$z^3 = \frac{1}{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}.$$

Solution. Essendo $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = e^{\frac{5\pi}{4}i}$, the equation da risolvere diventa

$$z^3 = \frac{1}{e^{\frac{5\pi}{4}i}} = e^{-\frac{5\pi}{4}i} = e^{\frac{3\pi}{4}i}.$$

Per the teorema diDe Moivre the solutions they are

$$z_0 = e^{\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad z_1 = e^{(\frac{\pi}{4}+\frac{2\pi}{3})i} = e^{\frac{11\pi}{12}i} = e^{-\frac{\pi}{12}i}, \quad z_2 = e^{(\frac{\pi}{4}+\frac{4\pi}{3})i} = e^{\frac{19\pi}{12}i} = e^{-\frac{5\pi}{12}i}$$

Applicando le formule dibisezione, one has

$$\cos\left(-\frac{\pi}{12}\right) = \sqrt{\frac{1+\cos\left(-\frac{\pi}{6}\right)}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}},$$

$$\sin\left(-\frac{\pi}{12}\right) = -\sqrt{\frac{1-\cos\left(-\frac{\pi}{6}\right)}{2}} = -\sqrt{\frac{1-\sqrt{3}/2}{2}},$$

from which anche

$$\cos\left(-\frac{5\pi}{12}\right) = -\sqrt{\frac{1+\sqrt{3}/2}{2}},$$

$$\sin\left(-\frac{5\pi}{12}\right) = -\sqrt{\frac{1-\sqrt{3}/2}{2}},$$

cosicché

$$z_1 = \sqrt{\frac{1+\sqrt{3}/2}{2}} - \sqrt{\frac{1-\sqrt{3}/2}{2}}i \qquad z_2 = -\sqrt{\frac{1+\sqrt{3}/2}{2}} - \sqrt{\frac{1-\sqrt{3}/2}{2}}i.$$

Exercise 4 [4+3 punti] Siano $\alpha \in \mathbb{R}$ and

$$f_\alpha(t) := \frac{e^{-2/t}}{3t^\alpha} .$$

i) Compute a primitive of f_α con $\alpha = 3$.

ii) Determine for which $\alpha \in \mathbb{R}$ esiste finito $\int_0^{+\infty} f_\alpha(t)\, dt$.

Solution. i) Con the sostituzione $y = -2/t$ one has $t = -2/y$, $dt = \frac{2}{y^2}dy$, and hence

$$\int f_3(t)dt = \int \frac{e^{-2/t}}{3t^3} = \int \frac{e^y}{3}\frac{-y^3}{8}\frac{2}{y^2}dy = -\frac{1}{12}\int ye^y dy.$$

Integrando by parts, one gets

$$\int f_3(t)dt = -\frac{1}{12}\int ye^y dy = -\frac{1}{12}\left(ye^y - \int e^y dy\right) = \frac{1}{12}(1-y)e^y = \frac{1}{12}\left(1+\frac{2}{t}\right)e^{-2/t}.$$

ii) f_α is continuous su $(0,\infty)$. Per qualsiasi $\alpha \in \mathbb{R}$ one has (for the gerarchia degli infiniti) $\lim_{x\to 0^+} \frac{e^{-2/t}}{3t^\alpha} = 0$. Therefore , the function f_α può essere prolungata for continuity in $t = 0$, so that is, always integrabile in $[0, c]$, for qualsiasi $c > 0$. Per $t \to +\infty$, da $\frac{2}{t} \to 0$ one gets $e^{-2/t} \sim 1$ so that

$$f_\alpha(t) \sim \frac{1}{3t^\alpha},$$

and essendo f_α a sign constant , in virtù of the test of the asymptotic comparison, the integral esiste if and only if $\alpha > 1$.

Exercise 5 [6 punti] Compute the following limit

$$\lim_{x\to 0^+} \frac{\sin(x-x^3)-\log(1+\sinh x)+\alpha x^2}{x^3}$$

as $\alpha \in \mathbb{R}$.

Solution. the limit is a indeterminate form $0/0$. Analizziamo the numerator. Ricordando che (for $t \to 0$)

$$\sin t = t+o(t) = t-\frac{t^3}{6}+o(t^3), \quad \log(1+t) = t+o(t) = t-\frac{t^2}{2}+\frac{t^3}{3}+o(t^3), \quad \sinh t = t+o(t) = t+\frac{t^3}{6}+o(t^3),$$

si vede che (for $x \to 0^+$)

$$\begin{aligned}\text{Numerator } &= (x-x^3)-\tfrac{(x-x^3)^3}{6}+o((x-x^3)^3)\\ &-\left(x+\tfrac{x^3}{6}+o(x^3)-\tfrac{1}{2}\left(x+\tfrac{x^3}{6}+o(x^3)\right)^2+\tfrac{1}{3}\left(x+\tfrac{x^3}{6}+o(x^3)\right)^3+o\left(\left(x+\tfrac{x^3}{6}+o(x^3)\right)^3\right)\right)+\alpha x^2\\ &= \left(\alpha+\tfrac{1}{2}\right)x^2+\left(-1-\tfrac{1}{6}-\tfrac{1}{6}-\tfrac{1}{3}\right)x^3+o(x^3) = \left(\alpha+\tfrac{1}{2}\right)x^2-\tfrac{5}{3}x^3+o(x^3) \sim \left(\alpha+\tfrac{1}{2}\right)x^2-\tfrac{5}{3}x^3.\end{aligned}$$

Si conclude allora che

$$\lim_{x\to 0^+}\frac{\text{Numerator}}{x^3} = \lim_{x\to 0^+}\left(\frac{\alpha+\frac{1}{2}}{x}-\frac{5}{3}\right) = \begin{cases}\infty, & \alpha > -\frac{1}{2},\\ -\infty, & \alpha < -\frac{1}{2},\\ -\frac{5}{3}, & \alpha = -\frac{1}{2}.\end{cases}$$

Exercise Sia $\alpha \in [0,+\infty[$ and define

$$F_\alpha(x) := \int_0^x t^\alpha e^{-t^2}\,dt, \ x \geqslant 0.$$

Establish for which values of α one has che F_α is concave sull'interval $[1,+\infty[$. There are values $\alpha > 0$ so that F_α sia concave su $[0,+\infty[$?

Solution. The function F_α is a function integral of $f_\alpha(t) := t^\alpha e^{-t^2}$. Essendo this ben defined and continuous su $[0,+\infty[$ (si ricorda $\alpha \geqslant 0$), anche F_α is ben defined , continuous and differentiable (for the teorema fondamentale of the calcolo) e

$$F_\alpha'(x) = f_\alpha(x) = x^\alpha e^{-x^2}.$$

Da this,

$$F_\alpha''(x) = e^{-x^2}\left(\alpha x^{\alpha-1}+x^\alpha(-2x)\right) = x^{\alpha-1}e^{-x^2}\left(\alpha-2x^2\right).$$

Siccome F_α is twice differentiable, for a noto result

$$F_\alpha \text{ concave su } [1,+\infty[, \quad\iff\quad F_\alpha''(x) \leqslant 0, \ \forall x \in [1,+\infty[.$$

Essendo

$$F_\alpha''(x) \leqslant 0, \quad\iff\quad \alpha-2x^2 \leqslant 0, \quad\overset{x\geqslant 0}{\iff}\quad x \geqslant \sqrt{\frac{\alpha}{2}},$$

F_α is concave su $[1,+\infty[$ if and only if $\sqrt{\frac{\alpha}{2}} \leqslant 1$, that is, $\alpha \leqslant 2$. Lo stesso calcolo mostra che, for every $\alpha > 0$ one has $F_\alpha''(x) > 0$ for every $x \in [0,\sqrt{\frac{\alpha}{2}}[$, so that F_α non può essere concave su $[0,+\infty[$ for alcun value of $\alpha > 0$. Per $\alpha = 0$, one has che

$$F_0''(x) = -2xe^{-x^2} < 0 \quad \forall x > 0,$$

hence F_0 is concave su $[0,+\infty[$.

Tempo a disposizione: 2 ore and 45 minuti.

Exam of the 06.07.2020 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = |(x+3)\log(x+3)|, \qquad x \in D =]-3, +\infty[.$$

(i) Compute

$$\lim_{x\to -3^+} f(x), \quad \lim_{x\to +\infty} f(x).$$

Solution.

$$\lim_{x\to -3^+} |(x+3)\log(x+3)| = \lim_{x\to -3^+} -(x+3)\log(x+3) = \lim_{x\to -3^+} -\frac{\log(x+3)}{\frac{1}{x+3}} \overset{\text{(De the Hôpital)}}{=}$$
$$= \lim_{x\to -3^+} x+3 = 0$$

$$\lim_{x\to +\infty} |(x+3)\log(x+3)| = \lim_{x\to +\infty} (x+3) \lim_{x\to +\infty} \log(x+3) = +\infty$$

(ii) Compute the first derivative of the function f, study the monotonicity intervals and draw the graph of f.

Solution. Per ogni x tale che $f(x) \neq 0$, that is,, for every $x \in D\backslash\{-2\}$,

$$f'(x) = \operatorname{sgn}\Big((x+3)\log(x+3)\Big)\left(\log(x+3)+1\right)$$

$$\begin{gathered} f'(x) \geq 0 \iff \\ x \in \Big\{x \in D,\ (x+3)\log(x+3) > 0,\ \log(x+3)+1 \geq 0\Big\} \bigcup \\ \bigcup \Big\{x \in D,\ (x+3)\log(x+3) < 0,\ \log(x+3)+1 \leq 0\Big\} \\ \iff \\ x \in \Big\{x > -2 \quad x \geq -3 + \tfrac{1}{e}\Big\} \bigcup \\ \bigcup \Big\{-3 < x < -2,\ x \leq -3 + \tfrac{1}{e}\Big\} = \\ \big]-3, -3 + \tfrac{1}{e}\big[\ \bigcup\ \big]-2, +\infty\big[\end{gathered}$$

Therefore f is monotonic increasing in each of the $\big]-3, -3+\frac{1}{e}\big[$ and $]-2, +\infty[$, while is monotonic decreasing in $-3+\frac{1}{e}, -2[$. Therefore the function ha a localmaximum at the point $x = -3+\frac{1}{e}$, where it is $f\left(-3+\frac{1}{e}\right) = \frac{1}{e}$, and a minimum locale at the point $x = -2$, where it is $f(-2) = 0$. From the theorem of the right and left limit of the dertivate one gets

$$f'_+(-2) = \lim_{x\to -2+} f'(x) = 1 \qquad f'_-(-2) = \lim_{x\to -2-} f'(x) = -1.$$

Hence $x = -2$ is a angular point con tangent sinistra of equation $y = -x-2$ and tangent destra of equation $y = x+2$.

Exercise 2 [6 punti] Find the solutions of the equation

$$z^3 = 8i$$

in algebraic and exponential form (or trigonometric), and draw them on the Gauss plane .

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Solution. Let us begin by writing $8i$ in trigonometric form :

$$8i = 8e^{i\frac{\pi}{2}}$$

Therefore $8i$ ha modulo $\rho = 8$ and argument $0a = \frac{\pi}{2}$. Solve l' equation means trovare le third roots of $8i$, that noi sappiamo essere in numero of three . Let us call them z_0, z_1, z_2. One has

$$z_0 = \rho^{\frac{1}{3}} e^{i\frac{0a}{3}} = 2e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$z_1 = \rho^{\frac{1}{3}} e^{i\left(\frac{0a}{3}+\frac{2\pi}{3}\right)} = 2e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = 2e^{i\frac{5\pi}{6}} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$$

$$z_2 = \rho^{\frac{1}{3}} e^{i\left(\frac{0a}{3}+\frac{4\pi}{3}\right)} = 2e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)} = 2e^{i\frac{3\pi}{2}} = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$$

Come sapevamo già dalla teoria, the solutions on the Gauss plane are the verteces dian equilateral triangle inscribed in a circle diradius 2. More precisely, one of the three verteces si trova in $(0, -2)$ and an edge is a subset of the line $y = 1$.

Exercise 3 [6 punti] Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(1+n^2)\log n}{n^4}.$$

Solutione.

Si tratta of a series a termini positivi. *Proviamo* ad applicare the criterio of the rapporto:

$$\lim_{n\to\infty} \frac{\frac{(1+(n+1)^2)\log(n+1)}{(n+1)^4}}{\frac{(1+n^2)\log n}{n^4}} = \lim_{n\to\infty} \frac{n^4}{(n+1)^4} \frac{(2+n^2+2n)}{(1+n^2)} \frac{\log(n+1)}{\log n} = \lim_{n\to\infty} \frac{\log(n+1)}{\log n} =$$

$$= \lim_{n\to\infty} \frac{\log(n(1+1/n)}{\log n} = \lim_{n\to\infty} \frac{\log n + \log(1+1/n)}{\log n} = 1$$

Purtroppo siamo in the *case in which the criterio of the rapporto nongives alc a informazione.*

Tentiamo allora the strada of the *confronto* (asymptotic).

the factor $\dfrac{(1+n^2)}{n^4}$ is asymptotic to $\dfrac{1}{n^2}$, that fornirebbe a series converging. there is the factor $\log n$, that peggiora the situazione. Però noi sappiamo che, for $x \to \infty$ $\log x = o(x^{\alpha})$ for qualsiasi $\alpha > 0$: infatti

$$\lim_{x\to\infty} \frac{\log x}{x^{\alpha}} \overset{\text{(De the Hôpital)}}{=} \lim_{x\to\infty} (-\frac{1}{x} x^{-\alpha+1}) = \lim_{x\to\infty} x^{-\alpha} = 0.$$

Therefore, scegliendo ad esempio $\alpha = 1/2$, one has che

$$\frac{(1+n^2)\log n}{n^4} = o\frac{(1+n^2)n^{\frac{1}{2}}}{n^4} = o\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is converging, for the criterio of the rapporto asymptotic si conclude that also the series data is converging .

Osservazione. Si sarebbe potuto scegliere a qualsiasi $\alpha \in]0,1[$al posto of $\alpha = \frac{1}{2}$. Invece the $\alpha \geq 1$ sarebbero stati inservibili, in quanto the series $\sum_{n=1}^{\infty} \frac{1}{n^{2-\alpha}}$ is divergente for $\alpha \geq 1$.

Exercise 4 [6 punti] Compute

$$\int_0^{+\infty} e^{-\sqrt{2x}}\, dx.$$

Solution. Per ogni $r > 0$, calcoliamo l' integral $\int_0^r e^{-\sqrt{2x}}\, dx$. Con the sostituzione $y(x) = \sqrt{2x}$, that is, $x(y) = \frac{y^2}{2}$ one gets

$$\int_0^r e^{-\sqrt{2x}}\, dx = \int_0^{\frac{r^2}{2}} e^{-y} \frac{d\left(\frac{y^2}{2}\right)}{dy}\, dy = \int_0^{\frac{r^2}{2}} e^{-y} y dy \overset{\text{(by parts)}}{=} [-e^{-y}y]_0^{\frac{r^2}{2}} + \int_0^{\frac{r^2}{2}} e^{-y} dy = -\frac{r^2 e^{-\frac{r^2}{2}}}{2} - e^{-\frac{r^2}{2}} + 1.$$

Hence

$$\int_0^{+\infty} e^{-\sqrt{2x}}\, dx = \lim_{r\to+\infty} \int_0^r e^{-\sqrt{2x}}\, dx = \lim_{r\to+\infty} \left(-\frac{r^2 e^{-\frac{r^2}{2}}}{2} - e^{-\frac{r^2}{2}} + 1 \right) = 1$$

(perché $\lim_{r\to+\infty} \frac{r^2 e^{-\frac{r^2}{2}}}{2} = \lim_{r\to+\infty} \frac{r^2}{2e^{\frac{r^2}{2}}} = 0$)

Exercise 5 [6 punti] Compute the limit

$$\lim_{x\to+\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x+2} - \sqrt[6]{x^2-1} \right)^2.$$

Solution.

$$\lim_{x\to+\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x+2} - \sqrt[6]{x^2-1} \right)^2 = \lim_{x\to+\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x}\sqrt[3]{1+\frac{2}{x}} - \sqrt[3]{x}\sqrt[6]{1-\frac{1}{x^2}} \right)^2 =$$

$$= \lim_{x\to+\infty} x^{\frac{5}{3}} \left(\sqrt[3]{1+\frac{2}{x}} - \sqrt[6]{1-\frac{1}{x^2}} \right)^2 = \lim_{x\to+\infty} x^{\frac{5}{3}} \left(\sqrt[3]{1+\frac{2}{x}} - \sqrt[6]{1-\frac{1}{x^2}} \right)^2.$$

By making use lo sviluppo diTaylor

$$(1+y)^\alpha = 1 + \alpha y + o(y) \qquad y \to 0$$

(valido study ogni $\alpha \in \mathbb{R}$), one gets

$$\lim_{x\to+\infty} x^{\frac{4}{3}} \left(\sqrt[3]{x+2} - \sqrt[6]{x^2-1} \right)^2 = \lim_{x\to+\infty} x^{\frac{5}{3}} \left(1 + \frac{2}{3x} + o\left(\frac{1}{x}\right) - 1 + \frac{1}{6x^2} + o\left(\frac{1}{x^2}\right) \right)^2 =$$

$$= \lim_{x\to+\infty} x^{\frac{5}{3}} \left(\frac{2}{3x} + o\left(\frac{1}{x}\right) \right)^2 = \lim_{x\to+\infty} x^{\frac{5}{3}} \left(\frac{2}{3x} \right)^2 \overset{\text{(P.S.I.)}}{=} \lim_{x\to+\infty} \frac{4}{9} x^{-\frac{1}{3}} = 0$$

Tempo a disposizione: 1 ore and 30 minuti.

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Exam of the 14.09.2020 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [6 punti] Consider the function

$$f(x) = \arctan\left(\frac{x+1}{x-1}\right), \qquad x \in (1, +\infty).$$

(i) Individuarne the asymptotes.
(ii) If ne determini the monotonicity .

Solution.

(i) the function is defined and continuous in the whole the domain $(1, +\infty)$, therefore the asymptotes riguardano solo $x \to 1+$ and $x \to +\infty$. Da

$$\lim_{x\to 1+} \arctan\left(\frac{x+1}{x-1}\right) \underset{y=\frac{x+1}{x-1}}{=} \lim_{y\to+\infty} \arctan y = \frac{\pi}{2}$$

$$\lim_{x\to +\infty} \arctan\left(\frac{x+1}{x-1}\right) \underset{y=\frac{x+1}{x-1}}{=} \lim_{y\to 1} \arctan y = \frac{\pi}{4}$$

one gets the function ha a horizontal asymptote study $y \to +\infty$ of equation $y = \frac{\pi}{4}$.

(ii) Calcoliamo the derivative of f:

$$\frac{df}{dx}(x) = \frac{1}{1 + \dfrac{(x+1)^2}{(x-1)^2}} \cdot \frac{-2}{(x-1)^2} = -\frac{2}{(x-1)^2 + (x+1)^2}.$$

Therefore $\frac{df}{dx}(x) < 0$ study ogni $x \in (1, +\infty)$, from which segue that the function is strictly decreasing in the domain $(1, +\infty)$.

Exercise 2 [6 punti] Consider the complex number $z = \sqrt{3} - i$.
(i) Scriverlo in exponential form .
(ii) Compute the real part of z^6.

Solution.

(i) One has $\rho := |z| = \sqrt{3+1} = 2$, from which

$$z = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 2e^{-i\frac{\pi}{6}}$$

(ii)

$$Re(z^6) = Re(2^6 e^{-i\pi}) = -64 \ \ (= z^6)$$

Exercise 3 [6 punti] Establish the convergence semplice and assoluta of the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}.$$

Solution. (i) In virtù of the criterio diLeibniz the series simply converges :

- ha segni alterni;
- $\frac{n}{n^2+1}$ is decreasing infatti,

$$\frac{n_1}{n_1^2+1} \geq \frac{n_2}{n_2^2+1} \iff n_2(n_1^2+1) \leq n_1(n_2^2+1) \iff (n_2-n_1)(1-n_2n_1) \leq 0 \iff n_2 \geq n_1,$$

(the ultimo passaggio dovuto al fatto che $(1-n_2n_1) \leq 0$); oppure si calcola the derivative

$$\left(\frac{x}{x^2+1}\right)' = \frac{-x^2+1}{(x^2+1)^2} \leq 0 \iff |x| \geq 1 \quad \text{if} \quad x \geq 1$$

- one has $\lim\limits_{n\to+\infty}(-1)^n \dfrac{n}{n^2+1} = 0.$

(ii) The series does not converge assolutamente, perchànd the series

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{n}{n^2+1}\right| = \sum_{n=1}^{\infty}\frac{n}{n^2+1}$$

ànd asintotica alla series armonica.

Exercise 4 [6 punti] Compute the limit

$$\lim_{x\to 0^+}\frac{\log(1+\sinh x)-\sin x}{x^2}.$$

Solution. da

$$\log(1+\sinh x)-\sin x = \sinh x - \frac{(\sinh x)^2}{2} + o((\sinh x)^2) - \sin x = \frac{x^3}{3} + o(x^3) - \frac{(x)^2}{2} + o(x^2)$$

one has

$$\lim_{x\to 0^+}\frac{\log(1+\sinh x)-\sin x}{x^2} = \lim_{x\to 0^+}\frac{-\frac{(x)^2}{2}+o(x^2)}{x^2} = -\frac{1}{2}.$$

Oppure si applica De the Hôpital twice.

Exercise 5 [6 punti] Consider the generalized integral

$$\int_1^{\infty} \log\left(\frac{x^\alpha}{x^\alpha+1}\right) dx.$$

(i) Compute the integral for $\alpha = 2$.
(ii) Establish study quali $\alpha \in [0,\infty)$ it converges.

Solution.

(i)

$$\lim_{k\to+\infty}\int_1^k \log\left(\frac{x^2}{x^2+1}\right)\,dx = \lim_{k\to+\infty}\left(\left[x\log\left(\frac{x^2}{x^2+1}\right)\right]_1^k - \int_1^k \frac{2}{(1+x^2)}\,dx\right) =$$
$$\lim_{k\to+\infty}\left(k\log\left(\frac{k^2}{k^2+1}\right) - \log\left(\frac{1}{2}\right) - 2\arctan k + 2\arctan 1\right)$$
$$\lim_{k\to+\infty}\left(k\log\left(1-\frac{1}{k^2+1}\right) - \log\left(\frac{1}{2}\right) - 2\arctan k + 2\arctan 1\right)$$
$$= \log 2 - \pi + \frac{\pi}{2} = \log 2 - \frac{\pi}{2}$$

(ii)

$$\log\left(\frac{x^\alpha}{x^\alpha+1}\right) = \log\left(1-\frac{1}{x^\alpha+1}\right) = -\frac{1}{x^\alpha+1} + o\left(\frac{1}{x^\alpha+1}\right)$$

study asymptotic comparison con $-\frac{1}{x^\alpha}$ converges if and only if $\alpha > 1$.

NB: con log si indica the logarithm in base e.

Tempo a disposizione: 1 ore and 30 minuti.

Exam of the 18.01.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \arctan\left(\frac{x}{x^2+x+1}\right);$$

(i) find the domain, study the sign, compute the limits at the extremes of the domain;
(ii) calcolarne the first derivative, study the monotonicity intervals and find the punti estremanti;
(iii) draw the graph of f.

Solution. (i). Iniziamo dalthe study of the domain. The denominator $x^2+x+1>0$ is always strictly positive, in quanto the determinante $\Delta = -3$ is negative . Considerando also that the domain of the function arctan is tutto $\mathbb{R}$, we get $D = \mathbb{R}$.
Let us study the sign of f: since $x^2+x+1>0$ study ogni $x\in\mathbb{R}$, abbiamo

$$f(x) = \arctan\left(\frac{x}{x^2+x+1}\right) \geq 0 \iff \frac{x}{x^2+x+1} \geq 0 \iff x \geq 0$$

e

$$f(x) = 0 \iff x = 0.$$

Let us study the limits at the extremes of the domain; abbiamo

$$\lim_{x\to+\infty}\arctan\left(\frac{x}{x^2+x+1}\right) = \arctan\left(\lim_{x\to+\infty}\left(\frac{x}{x^2+x+1}\right)\right) = 0$$

$$\lim_{x\to-\infty}\arctan\left(\frac{x}{x^2+x+1}\right) = \arctan\left(\lim_{x\to-\infty}\left(\frac{x}{x^2+x+1}\right)\right) = 0.$$

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Hence $y = 0$ is horizontal asymptote a $+\infty$ and a $-\infty$.

(ii). The derivative of f is

$$f'(x) = \frac{1}{1 + \left(\frac{x}{x^2+x+1}\right)^2} \frac{-2x^2 - x + x^2 + x + 1}{(x^2+x+1)^2} = \frac{1}{\left(1 + \left(\frac{x}{x^2+x+1}\right)^2\right)(x^2+x+1)^2}(-x^2+1).$$

Hence

$$f'(x) \geq 0 \iff 1 - x^2 \geq 0 \iff x \in [-1, 1]$$

e

$$f'(x) = 0 \iff 1 - x^2 = 0 \iff x \in \{-1, 1\}.$$

We deduce che f is cresente in the interval $[-1, 1]$, decreasing in $]-\infty, -1]$ and in $[1, +\infty[$, hence $x = -1$ is point of minimum globale while $x = 1$ is point of global maximum.

(iii). the graph of the function is sketched in the picture .

Exercise 2 [8 punti] Find in $\mathbb{C}$ the solutions of the equation

$$z^4 + (-1+i)z^2 - i = 0.$$

Suggerimento: sostituire $w = z^2$.

Solution. Con the sostituzione $w = z^2$ one gets

$$w^2 + (-1+i)w - i = 0$$

whose solutions are

$$w_{1,2} = \frac{1 - i + \sqrt{2i}}{2} = \frac{1 - i \pm (1+i)}{2} = \{1, \, -i\}.$$

Therefore, the solutions they are 4 and coincidono con the union of the solutions of $z^2 = 1$ and $z^2 = -i$, it is a dire

$$z_1 = 1, \quad z_2 = -1, \quad z_3 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \quad z_4 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

the solutions **Exercise 3 [8 punti]**

(i) Compute

$$\lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n}}.$$

(ii) Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}.$$

Solution. (i).

$$\lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n}} = \lim_{n\to\infty} \frac{1}{(1+1/n)^{2n}} = \left(\lim_{n\to\infty} \frac{1}{(1+1/n)^n}\right)^2 = e^{-2}.$$

(ii). Since the series has positive terms, let us apply the criterio of the rapporto asymptotic and the result of (i), ottenendo

$$\begin{aligned}\lim_{n\to\infty}\frac{(2n+2)!n^{2n}}{(2n)!(n+1)^{2n+2}} &= \lim_{n\to\infty}\frac{(2n+2)(2n+1)n^{2n}}{(n+1)^2(n+1)^{2n}} = \lim_{n\to\infty}\frac{(4n^2+6n+2)n^{2n}}{(n^2+2n+1)(n+1)^{2n}}\\ &= 4\lim_{n\to\infty}\frac{n^{2n}}{(n+1)^{2n}} = 4e^{-2}.\end{aligned}$$

Vale $4e^{-2} < 1$; the series is converging .

Exercise 4 [8 punti] study $\alpha \in \mathbb{R}$, si consideri

$$f_\alpha(x) = \frac{1}{\sinh x + x^\alpha}.$$

(a) Study as $\alpha \in \mathbb{R}$ the convergence

$$\int_0^{\log 2} f_\alpha(x)\,dx.$$

(b) Compute

$$\int_0^{\log 2} f_0(x)\,dx.$$

Solution. (a). Si tratta of an integrand a values positivi hence possiamo sfruttare the criterion of the asymptotic comparison . Da

$$f_\alpha(x) = \frac{1}{\sinh x + x^\alpha} = \frac{1}{x + o(x) + x^\alpha}$$

we get that, study $x \to 0$ the function is asintotica a $\frac{1}{x}$ if $\alpha > 1$, a $\frac{2}{x}$ if $\alpha = 1$ and a $\frac{1}{x^\alpha}$ if $\alpha < 1$. Therefore the integral converges $\iff \alpha < 1$.

(b) Con the sostituzione $t = e^x$ (that is, $x = \log t$) one gets

$$\begin{aligned}\int_0^{\log 2} f_0(x)\,dx &= \int_0^{\log 2}\frac{1}{\sinh x + 1}dx = \int_0^{\log 2}\frac{2}{e^x - e^{-x} + 2}dx = \int_1^2 \frac{2}{(t - t^{-1} + 2)t}dt\\ &= \int_1^2 \frac{2}{t^2 + 2t - 1}dt.\end{aligned}$$

Le radici of $t^2 + 2t - 1 = 0$ they are $-1 \pm \sqrt{2}$, hence

$$\frac{1}{t^2+2t-1} = \frac{A}{t+1-\sqrt{2}} + \frac{B}{t+1+\sqrt{2}}$$

study suitable $A, B \in \mathbb{R}$. One has $1 = A(t+1+\sqrt{2}) + B(t+1-\sqrt{2})$, from which

$$\begin{cases} A + B = 0 \\ A(1+\sqrt{2}) + B(1-\sqrt{2}) = 1 \end{cases}$$

hence

$$\begin{cases} A + B = 0 \\ -B(1+\sqrt{2}) + B(1-\sqrt{2}) = 1 \end{cases}$$

and therefore $A = \frac{\sqrt{2}}{4}$, $B = -\frac{\sqrt{2}}{4}$. We deduce

$$\begin{aligned}
\int_0^{\log 2} f_0(x)\,dx &= \left(\frac{\sqrt{2}}{2}\int_1^2 \frac{1}{t+1-\sqrt{2}}dt - \frac{\sqrt{2}}{2}\int_1^2 \frac{1}{t+1+\sqrt{2}}dt\right) \\
&= \left[\frac{\sqrt{2}}{2}\log|t+1-\sqrt{2}| - \frac{\sqrt{2}}{2}\log|t+1+\sqrt{2}|\right]_1^2 \\
&= \frac{\sqrt{2}}{2}\log\left(\frac{3-\sqrt{2}}{3+\sqrt{2}}\right) - \frac{\sqrt{2}}{2}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right).
\end{aligned}$$

Tempo a disposizione: 1 ore and 30 minuti.

Exam of the 08.02.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \sqrt{\frac{|x|}{x^2+1}}.$$

(i) Determine the domain of f, study the sign and the simmetria of f and compute limits and asymptotes at the extremes of the domain;
(ii) Study the derivability of f and compute the first derivative, study the monotonicity intervals and find the points of maximum/ absolute minimum/relativo;
(iii) draw the graph of f.

Solution. (i). Iniziamo dal domain. The denominator $x^2+1>0$ is always strictly positive . the numerator $|x|$ is always maggiorearctangent uguale dizero. Considerando that the domain of the function Root Test is $[0,\infty)$, we get $D=\mathbb{R}$.
Let us study the sign and le simmetries of f. The function is pari: $f(x)=f(-x), \forall x\in\mathbb{R}$. Furthermore, it ha always values non negativi:

$$f(x) = \sqrt{\frac{|x|}{x^2+1}} \geq 0 \iff x \in \mathbb{R},$$

e

$$f(x) = 0 \iff x = 0.$$

Let us study the limits at the extremes of the domain; abbiamo:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Hence $y=0$ is horizontal asymptote a $+\infty$ and a $-\infty$.

(ii) Let us study the derivability of f. One has che $f \in C^{(1)}(\mathbb{R}\setminus\{0\})$ in quanto superposition difunctions $C^{(1)}(\mathbb{R})$, esclusa the function $g(x)=|x|$ that stonly one in $C^{(1)}(\mathbb{R}\setminus\{0\})$. study ogni $x\neq 0$ one has

$$f'(x) = \frac{1}{2\sqrt{\frac{|x|}{x^2+1}}}\frac{\operatorname{sgn}(x)(x^2+1)-|x|2x}{(x+1)^2}.$$

Hence,

$$\{x>0 \text{ e } f'(x)>0\} \iff (x^2+1)-2x^2>0 \iff 1-x^2>0 \iff x\in]0,1[$$

e

$$\{x > 0 \text{ and } f'(x) = 0\} \iff x = 1$$

study simmetria, one has

$$\{x < 0 \text{ and } f'(x) > 0\} \iff x \in]-\infty, -1[$$

e

$$\{x < 0 \text{ and } f'(x) = 0\} \iff x = -1.$$

Furthermore,, study the teorema of the limit of the derivata,

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0^+} \frac{1-x^2}{2(x^2+1)^2\sqrt{\frac{|x|}{x^2+1}}} = \lim_{x\to 0^+} \frac{1}{2\sqrt{|x|}} = +\infty$$

$$\lim_{x\to 0-} f'(x) = \lim_{x\to 0^-} \frac{(-1)(1-x^2)}{2(x^2+1)^2\sqrt{\frac{|x|}{x^2+1}}} = \lim_{x\to 0^-} \frac{-1}{2\sqrt{|x|}} = -\infty$$

therefore, the function is not differentiable in $x = 0$, where ha a cuspid.

Dalla precedente analisi and dalla continuity of the function one has that the function is increasing in each of the two intervals $[0, 1]$ and $]-\infty, -1]$ and is decreasing in each of the two intervals $[-1, 0]$ and $[1, +\infty[$.

Furthermore, vi is a maximum (resp. minimum) globale in $x = 1$ (resp. $x = -1$).

(iii). the graph of the function is sketched in the picture .

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Figura 19: the graph of f.

Exercise 2 [8 punti] Find the complex solutions of the equation

$$\frac{8}{z^3} = \frac{1+i}{1-i},$$

in algebraic and exponential form (or trigonometric), and draw them on the Gauss plane .

Solution. Da
$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{2i}{2} = i$$
we get
$$z^3 = \frac{8}{i} = -8i.$$
Solutions of equation they are le (three) radici terze of $-8i = 8e^{i\frac{3}{2}\pi}$, that is,
$$z_1 = 2e^{i\frac{1}{2}\pi} = 2i, \quad z_2 = 2e^{i\frac{7}{6}\pi} = -\sqrt{3} - i, \quad z_3 = 2e^{i\frac{11}{6}\pi} = \sqrt{3} - i$$
and they are disegnate nella figura following

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Figura 20: Solutions of exercise 2.

Exercise 3 [8 punti]
(i) Compute
$$\int \log(t+1)\, dt.$$
(ii) Dedurre the value of
$$\int_0^1 \frac{\log(\sqrt{x}+1)}{\sqrt{x}}\, dx.$$

Solution.(i) study parti:

$$\int \log(t+1)\,dt = \log(t+1)t - \int \frac{t}{t+1}dt$$
$$= t\log(t+1) - \int \left(1 - \frac{1}{t+1}\right)dt$$
$$= t\log(t+1) - t + \log|t+1| + c,$$

con $c \in \mathbb{R}$.

(ii) Utilizzando the sostituzione $t = \sqrt{x}$,

$$\int_0^1 \frac{\log(\sqrt{x}+1)}{\sqrt{x}}\,dx = \lim_{c\to 0+}\int_c^1 \frac{\log(\sqrt{x}+1)}{\sqrt{x}}\,dx = \lim_{c\to 0+}\int_{\sqrt{c}}^1 \frac{\log(t+1)}{t}2t\,dt$$
$$= \lim_{c\to 0+} 2\int_{\sqrt{c}}^1 \log(t+1)\,dt$$
$$= \lim_{c\to 0+} 2\left[t\log(t+1) - t + \log(t+1)\right]_{\sqrt{c}}^1$$
$$= 2(2\log 2 - 1)$$

Exercise 4 [8 punti]
(i) Individuare as $\alpha \in \mathbb{R}$ the order diinfinitesimal l of

$$n\left(\cos(1/n) - 1\right) + \frac{\alpha}{n}$$

(ii) Study as $\alpha \in \mathbb{R}$ the convergence of

$$\sum_{n=1}^{+\infty}\left|n\left(\cos(1/n)-1\right) + \frac{\alpha}{n}\right|.$$

Solution. (i)

$$n\left(\cos(1/n)-1\right) + \frac{\alpha}{n} = n\left(-\frac{1}{2n^2} + \frac{1}{24n^4} + o(\frac{1}{n^4})\right) + \frac{\alpha}{n} = \frac{-1/2+\alpha}{n} + \frac{1}{24n^3} + o\left(\frac{1}{n^3}\right)$$

ànd diorder 1 study ogni $\alpha \neq 1/2$ and diorder 3 study $\alpha = 1/2$.

(ii) The terms of this series have constant sign . Da quanto visto at the previous point , the general term of the series verifica le following asintoticità

$$a_n \sim \begin{cases} \frac{-1/2+\alpha}{n} & \text{if } \alpha \neq 1/2 \\ \frac{1}{24n^3} & \text{if } \alpha = 1/2. \end{cases}$$

Applicando the teorema of the asymptotic comparison con the series armonica generalizzata, we get that the series converges if $\alpha = 1/2$ and diverges if $\alpha \neq 1/2$.
Tempo a disposizione: 1 ore and 30 minuti.

Exam of the 05.07.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \log\left(1 + \sqrt{1 - x^2}\right).$$

(i) Determine the domain of f, study the sign and the simmetria of f and compute the limits at the extremes of the domain;
(ii) Study the derivability of f and compute the first derivative, study the monotonicity intervals and find the points of maximum/ absolute minimum/relativo;
(iii) draw the graph of f.
Solution. (i). In order to determine the domain bisogna imporre that the radicando sia nonnegativo and the argument of the logarithm sia positive . The disuguaglianza $1 - x^2 \geq 0$ ha come solutions $x \in [-1, 1]$. study questi values of x, is ovvio that the argument of the logarithm sia positive . Therefore

$$\text{dom}(f)\text{=[-1,1]}.$$

To single out the simmetries , let us observe that it is

$$f(-x) = \log\left(1 + \sqrt{1 - (-x)^2}\right) = f(x);$$

the function is pari.
Let us study the sign of the function: $f(x) \geq 0$ equivale a

$$1 + \sqrt{1 - x^2} \geq 1 \qquad \text{that is,} \qquad \sqrt{1 - x^2} \geq 0.$$

Since $\sqrt{\ldots}$ is sicuramente nonnegativo, deduce that the function is always nonnegativa and si annullonly one in $x = \pm 1$ that they are therefore points of absolute minimum (con $f(\pm 1) = 0$).
From the theorem on the algebra of continuous functions and the theorem on superposition of dicontinuopus functions, $f \in C^0(dom(f))$. We henc

$$\lim_{x \to 1^-} f(x) = f(1) = 0$$

and analogamente, for simmetria, $\lim_{x \to -1^+} f(x) = 0$.

(ii). In $(-1, 1)$, for the teorema sull'algebra of the derivate and quello sulla derivative of the function composta, we get that the f is differentiable . The derivability in ± 1 va studiata separatamente. Abbiamo

$$f'(x) = \frac{1}{1 + \sqrt{1 - x^2}} \cdot \frac{-x}{\sqrt{1 - x^2}}.$$

Since it is $\lim_{x \to 1^-} f'(x) = -\infty$ (and for simmetria $\lim_{x \to -1^+} f'(x) = +\infty$), concludiamo che f is not differentiable in $x = \pm 1$. Furthermore,, the intervals dicrescenza they are determinati da $f' \geq 0$ that is, $x \leq 0$. Therefore che

- f is increasing in $[-1, 0]$
- f is decreasing in $[0, -1]$
- $x = 0$ is the unique point diabsolute maximum
- $x = \pm 1$ they are points of absolute minimum (già lo sapevamo).

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Figura 21: graph of the exercise 1

grafici/3app2021_grafico2.pdf

Figura 22: graph of the exercise 2

(iii). Si veda the graph in the picture 1.

Exercise 2 [8 punti] Find the complex solutions of the equation

$$\mathrm{Im}(z^2) + |z|^2 \mathrm{Re}\left(\frac{1}{z}\right) = 0,$$

and draw them on the Gauss plane .
Solution. Innanzitutto notiamo that the equation ha senso solo for $z \neq 0$. Per tali values of z risolviamo the equation usando the algebraic form of the numeri complessi: $z = x + iy$ con $x, y \in \mathbb{R}$. Abbiamo

$$z^2 = (x^2 - y^2) + 2ixy, \qquad |z|^2 = x^2 + y^2, \qquad \frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

The equation iniziale diventa

$$2xy + x = 0 \qquad \text{that is,} \qquad x(2y + 1) = 0$$

that ha solutions

$$x = 0 \qquad \text{e} \qquad y = -\frac{1}{2}$$

that formano le two rette (for $z \neq 0$) in the graph in Figura 2.

Exercise 3 [8 punti]
Sia

$$f_\alpha(x) := \frac{\arctan x}{1 + x^{2\alpha}}.$$

(i) Compute

$$\int f_1(x)\, dx = \int \arctan x \left(\frac{1}{1 + x^2}\right) dx.$$

(ii) Study as $\alpha \in [0, \infty)$ the convergence of

$$\int_1^{+\infty} f_\alpha(x)\, dx.$$

Solution. (i). By making use the sostituzione $\arctan x = t$ (ricordarsi: $(\arctan x)' = \frac{1}{1+x^2}$) we get

$$\int \arctan x \left(\frac{1}{1 + x^2}\right) dx = \int t\, dt = \frac{t^2}{2} + c = \frac{\arctan^2 x}{2} + c, \qquad c \in \mathbb{R}.$$

(ii). Osserviamo $f \in C^0([1, +\infty))$ (e $f > 0$ su $[1, +\infty)$); hence the integral is improprio solo for $x \to +\infty$. Let us study the asintoticità of f_α for $x \to +\infty$:

$$f_\alpha(x) \sim \frac{\pi}{2} \cdot \frac{1}{1 + x^{2\alpha}} \sim \frac{\pi}{2} \cdot \frac{1}{x^{2\alpha}} \qquad \text{for } x \to +\infty.$$

Applicando the criterion of the asymptotic comparison for the integral the impropri (and ricordando che $\int_1^{+\infty} x^a dx$ converges if and only if $a < -1$) we get that the integral dipartenza is converging if and only if $\alpha > 1/2$.

Exercise 4 [8 punti]
(i) Compute as $\alpha \in \mathbb{R}$ the limit

$$\lim_{n\to\infty} \frac{2\log[\cos(1/n)] + \alpha[\sin(1/n)]^2}{(1/n)^2}.$$

(ii) Dedurre the comportamento of the series

$$\sum_{n=1}^{\infty} \left\{2\log[\cos(1/n)] + [\sin(1/n)]^2\right\}.$$

Solution. (*i*). By making use the sviluppi diMc Laurin of $\cos x$ and of $\log(1+x)$, for $n \to +\infty$ abbiamo

$$\begin{aligned}
\log[\cos(1/n)] &= \log\left[1 + \left(-\frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^3}\right)\right)\right] \\
&= -\frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^3}\right) + +o\left(-\frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^3}\right)\right) \\
&= -\frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right).
\end{aligned}$$

Furthermore,, usando lo sviluppo diMc Laurin of $\sin x$, for $n \to +\infty$ abbiamo

$$[\sin(1/n)]^2 = \left[\frac{1}{n} + o\left(\frac{1}{n^2}\right)\right]^2 = \frac{1}{n^2} + o\left(\frac{1}{n^3}\right).$$

Deduciamo that the numerator verifica

$$\text{num.} = (\alpha - 1)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right);$$

confollowingmente vale

$$\lim_{n\to\infty} \frac{2\log[\cos(1/n)] + \alpha[\sin(1/n)]^2}{(1/n)^2} = \alpha - 1 \qquad \forall\alpha \in \mathbb{R}.$$

(*ii*). Let us observe that the point precedente con $\alpha = 1$ dà

$$\lim_{n\to+\infty} \frac{2\log[\cos(1/n)] + [\sin(1/n)]^2}{(1/n)^2} = 0$$

that is,

$$2\log[\cos(1/n)] + [\sin(1/n)]^2 = o[(1/n)^2] \qquad \text{for } n \to +\infty.$$

We deduce in particolare that the termine of the nostra series is definitively positive . Furthermore,, applicando the criterion of the asymptotic comparison and ricordando che $\sum(1/n)^2$ is converging, we get that the series is converging .

Tempo a disposizione: 1 ore and 30 minuti.

Exam of the 13.09.2021 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [8 punti] Consider the function

$$f(x) = \frac{|\sin x|}{1 - 2\cos x} \quad .$$

(i) Find the domain; study the periodicity , the sign and the simmetria of f;
(ii) study the derivability and calcolarne the first derivative ; study the monotonicity intervals and find the points of maximum/ absolute minimum/relativo;
(iii) draw the graph.

Solution. (i). The function is defined for every $x \in \mathbb{R}$ tale che

$$1 - 2\cos x \neq 0 \iff \cos x \neq \frac{1}{2} \iff x \in \mathbb{R} \setminus \left\{ \pm\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z} \right\}$$

Clearly the function is periodica con periodo 2π. Furthermore,

$$f(x) = \frac{|\sin x|}{1 - 2\cos x} = \frac{|\sin(-x)|}{1 - 2\cos(-x)} = f(-x)$$

therefore the function is pari, that is, the suo graphis simmetrico rispetto all' asse of the ordinate.

Limito the study al domain $[-\pi, \pi] \setminus \{\pm\frac{\pi}{3}\}$; calcolo the limit at the extremes

$$\lim_{x \to \pi/3-} f(x) = -\infty, \qquad \lim_{x \to \pi/3+} f(x) = +\infty.$$

(ii). Per ogni point of the domain tale che $|\sin x| \neq 0$, cioé $x \neq k\pi$, $k \in \mathbb{Z}$, one has

$$f'(x) = \frac{\cos x \frac{|\sin x|}{\sin x}(1 - 2\cos x) - 2\sin x |\sin x|}{(1 - 2\cos x)^2} = \frac{|\sin x|}{(1 - 2\cos x)^2} \left(\frac{\cos x}{\sin x}(1 - 2\cos x) - 2\sin x \right)$$

$$f'(x) \geq 0 \iff \frac{1}{\sin x} \left(\cos x - 2\cos^2 x + 2\sin^2 x \right) = \frac{1}{\sin x} (\cos x - 2) \geq 0$$

In $]0, \pi[\setminus \pi/3$ one has $\sin x > 0$ and $\cos x - 2 < 0$, hence $f'(x) < 0$, therefore le restrictions to the intervals $]0, \pi/3[,]\pi/3, \pi[$ they are strictly decreasing. By symmetry, le restrictions to the intervals $] - \pi/3, 0[,] - \pi, -\pi/3[$ they are strictly crescenti. e, the function ha a minimum locale in π, with value $f(\pi) = 0$ and hence in ogni point $\pi + 2k\pi$, $k \in \mathbb{Z}$. One has moreover

$$\lim_{x \to 0-} f'(x) = 1 \quad \lim_{x \to 0+} f'(x) = -1 \quad \lim_{x \to \pi-} f'(x) = -\frac{1}{3} \lim_{x \to -\pi+} f'(x) = \frac{1}{3};$$

hence the funziona presenta punti angolosi in $k\pi$, $\forall k \in \mathbb{Z}$.
(iii). Vedi figura.

Figura 23: the graph of f.

Exercise 2 [8 punti] Find the solutions $z \in \mathbb{C}$ of the inequality

$$\left| \frac{z + 1}{z} \right| \geq 1$$

and draw them on the Gauss plane .

Solution. Innanzitutto Let us observe that the campo diesistenza of the disuguaglianza is dato da $|z| \neq 0$ that is, da $z \neq 0$. Forniamo two metodi disoluzione.

1. Eleviamo al quadrato entrambi the membri:

$$\frac{|z+1|^2}{|z|^2} \geq 1 \iff (x+1)^2 + y^2 \geq x^2 + y^2 \iff 1 + 2x \geq 0 \iff x \geq -1/2$$

2.
$$\left|\frac{z+1}{z}\right| \geq 1 \iff \left|1 + \frac{x - iy}{x^2 + y^2}\right| \geq 1 \iff \left|x^2 + y^2 + x - iy\right| \geq x^2 + y^2$$

if and only if

$$x^4 + y^4 + x^2 + 2x^2y^2 + 2x^3 + 2xy^2 + y^2 \geq x^4 + y^4 + 2x^2y^2$$

if and only if

$$x^2 + 2x^3 + 2xy^2 + y^2 \geq 0 \iff (2x+1)(x^2+y^2) \geq 0 \iff 2x + 1 \geq 0 \iff x \geq -1/2.$$

Solutions they are the numeri complessi $z = x + iy$ (con $x, y \in \mathbb{R}$) tali che: $z \neq 0$ and $x \geq -1/2$.

Exercise 3 [8 punti]
Study the convergence of the series

$$\sum_{n=1}^{\infty} n^{\alpha} \left(\frac{1}{n} - \sin\frac{1}{n}\right)$$

as $\alpha \in \mathbb{R}$.
Solution. By making use lo sviluppo diMcLaurin of $\sin x$ one gets is asintotica alla series

$$\sum_{n=1}^{\infty} \frac{1}{6} n^{\alpha-3},$$

in particolare is a series a termini positivi. Possiamo therefore applicare the criterion of the asymptotic comparison and dedurre that it is converging if and only if $\alpha - 3 < -1$, that is, $\alpha < 2$, and is divergenteper $\alpha \geq 2$.

Exercise 4 [8 punti]
Compute the integral

$$\int_{-1}^{0} \frac{x}{x^2 + 2x + 2}\, dx.$$

Solution. Abbiamo

$$\begin{aligned}
\int_{-1}^{0} \frac{x}{x^2 + 2x + 2}\, dx &= \frac{1}{2} \int_{-1}^{0} \frac{2x + 2 - 2}{x^2 + 2x + 2}\, dx \\
&= \frac{1}{2} \int_{-1}^{0} \frac{2x + 2}{x^2 + 2x + 2}\, dx - \int_{-1}^{0} \frac{1}{x^2 + 2x + 2}\, dx \\
&= \frac{1}{2} \log(x^2 + 2x + 2)\Big|_{-1}^{0} - \int_{-1}^{0} \frac{1}{(x+1)^2 + 1}\, dx \\
&= \frac{1}{2} \log(x^2 + 2x + 2)\Big|_{-1}^{0} - \arctan(x+1)\Big|_{-1}^{0} \\
&= \frac{1}{2} \log(2) - \frac{\pi}{4}.
\end{aligned}$$

Exam of the 17.01.2022 - Modalità telematica (causa COVID)

THEME 1

Exercise 1 [10 punti] Given the function

$$f(x) = \arctan\left(\frac{|x+1|}{x^2+4}\right),$$

(i) find the domain:

$$Domain = \mathbb{R};$$

study the sign:

$$f(x) \geq 0 \iff \frac{|x+1|}{x^2+4} \geq 0 \qquad \text{Hence } f(x) \geq 0 \ \forall x \in \mathbb{R}. \text{ Furthermore, } \quad f(x) = 0 \iff x = -1;$$

compute the limits at the extremes of the domain:

$$\lim_{x\to\pm\infty} \arctan\left(\frac{|x+1|}{x^2+4}\right) \underset{y=\frac{|x+1|}{x^2+4}}{=} \lim_{y\to 0} \arctan y = 0,$$

hence $y = 0$ is horizontal asymptote a $-\infty$ and a $+\infty$.

(ii) Study the derivability of f sul suo domain, compute the first derivative :
Let us study f separatamente nelle regioni
$x > -1 \iff |x+1| = x+1$ and
$x < -1 \iff |x+1| = -(x+1)$, so that one has:

$$f(x) = \arctan\left(\mp\frac{(x+1)}{x^2+4}\right) \qquad x \lessgtr -1,$$

and hence

$$f'(x) = \frac{\mp\frac{x^2+4-2x(x+1)}{(x^2+4)^2}}{1+\frac{(x+1)^2}{(x^2+4)^2}} = \pm\frac{x^2+2x-4}{(x^2+4)^2+(x+1)^2} \qquad \text{if } \lessgtr -1,$$

$$\lim_{x\to -1-} f'(x) = -\frac{1}{5} \qquad \lim_{x\to -1+} f'(x) = \frac{1}{5} \implies f'_-(-1) = -\frac{1}{5}, \quad f'_+(-1) = \frac{1}{5}.$$

Hence the function is differentiable for every $x \in \mathbb{R}\backslash\{-1\}$ while in $x = -1$ vi is a angular point .

Study the intervals dimonotonicity :

$$\begin{cases} f'(x) \geq 0 \\ x < -1 \end{cases} \iff \begin{cases} x^2+2x-4 \geq 0 \\ x < -1 \end{cases} \iff x \leq -1-\sqrt{5}$$

Hence the function is strictly increasing in $]-\infty, -1-\sqrt{5}[$ and strictly decreasing in $]-1-\sqrt{5}, -1[$. Furthermore,

$$\begin{cases} f'(x) \geq 0 \\ x > -1 \end{cases} \iff \begin{cases} x^2+2x-4 \leq 0 \\ x > -1 \end{cases} \iff -1 < x \leq -1+\sqrt{5}.$$

Hence the function is strictly increasing in $]-1, -1+\sqrt{5}[$ and strictly decreasing in $]-1+\sqrt{5}, +\infty[$. Finally ,

$$f'(-1+\sqrt{5}) = f'(-1-\sqrt{5}) = 0$$

grafici/2201_funzione1.pdf

Figura 24: Grafico of the function.

and the punti $-1-\sqrt{5}$, $-1+\sqrt{5}$ they are direlative maximum . Da $f(-1)=0$, the point $x=-1$ is point of absolute minimum.

(iii) draw the graph of f.

Exercise 2 [7 punti] Determine the solutions in $\mathbb{C}$ of the equation

$$\left(\frac{z}{i}\right)^3 = -8 \iff z^3 = 8i = 8\left(\cos\left(\frac{1}{2}\pi\right) + i\sin\left(\frac{1}{2}\pi\right)\right)$$

Dobbiamo that is, trovare le radici terze of $8i$, that is,, con the formula diDe Moivre,

$$z_0 = 2\left(\cos\left(\frac{1}{6}\pi\right) + i\sin\left(\frac{1}{6}\pi\right)\right) = \sqrt{3}+i$$

$$z_1 = 2\left(\cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right)\right) = -\sqrt{3}+i$$

$$z_2 = 2\left(\cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right)\right) = -2i.$$

(Stanno sui verteces of a triangolo equilatero inscritto in the cerchio diraggio 2 con a vertex in $= -2i$)

Exercise 3 [7 punti]

(i) Mediante suitable sviluppi diTaylor, determine, as $\alpha \in \mathbb{R}$, a sviluppo of the sequence

$$a_n = \frac{1}{n} - \sin\left(\frac{1}{n}\right) - \alpha\log\left(1+\frac{1}{n^3}\right) \qquad \text{for } n \to +\infty.$$

One has

$$a_n = \frac{1}{n} - \sin\left(\frac{1}{n}\right) - \alpha\log\left(1+\frac{1}{n^3}\right) = \frac{1}{n} - \frac{1}{n} + \frac{1}{6n^3} - \frac{1}{120n^5} + o\left(\frac{1}{n^5}\right) - \alpha\frac{1}{n^3} + \alpha\frac{1}{2n^6} + o\left(\frac{\alpha}{n^6}\right) =$$

$$= \frac{1-6\alpha}{6n^3} - \frac{1}{120n^5} + o\left(\frac{1}{n^5}\right)$$

(ii) Study the convergence of the series

$$\sum_{n=1}^{\infty} n^2\, a_n.$$

Da

$$\sum_{n=1}^{\infty} n^2\, a_n = \sum_{n=1}^{\infty} \left(\frac{1-6\alpha}{6n} - \frac{1}{120n^3} + o\left(\frac{1}{n^3}\right)\right)$$

segue che, for $\alpha = \frac{1}{6}$, the termine generico of the series is always negative for n sufficientemente grande and is asymptotic to $\frac{1}{n^3}$. Therefore the series converges. If instead $\alpha \neq \frac{1}{6}$, the termine generico of the series is with constant sign for n sufficientemente grande and is asymptotic to $\frac{1}{n}$. Therefore the series diverges for $\alpha \neq \frac{1}{6}$.

Exercise 4 [8 punti]

(i) By making use the definizione, compute the integral generalizzato

$$\int_0^{+\infty} \frac{\arctan t}{(1+t^2)(\arctan^2 t + 8\arctan t + 17)}\, dt;$$

Consideriamo the sostituzione $y = \arctan t$, that implica $dy = \frac{1}{1+t^2}dt$:

$$\lim_{c\to\infty} \int_0^c \frac{\arctan t}{(1+t^2)(\arctan^2 t + 8\arctan t + 17)}\, dt = \lim_{c\to+\infty} \int_0^{\arctan c} \frac{y}{y^2+8y+17}\, dy =$$

$$= \lim_{c\to+\infty} \left[\frac{1}{2}\log((y+4)^2+1) - 4\arctan(y+4)\right]_0^{\arctan c}$$

$$= \frac{1}{2}\log((\pi/2+4)^2+1) - 4\arctan(\pi/2+1) - \frac{1}{2}\log 17 + 4\arctan(4)$$

where we have usato:

$$\int \frac{y}{y^2+8y+17}dy = \int \frac{y}{y^2+8y+16+1}dy = \frac{1}{2}\int \frac{2(y+4)}{(y+4)^2+1}dy - 4\int \frac{1}{(y+4)^2+1}dy =$$

$$= \frac{1}{2}\log((y+4)^2+1) - 4\arctan(y+4) + c, \qquad c \in \mathbb{R}$$

discuss the convergence of the generalized integral

$$\int_0^{+\infty} \frac{\arctan t}{(1+t^2)^{2\alpha}(\arctan^2 t + 8\arctan t + 17)}\, dt;$$

for every $\alpha \in \mathbb{R}$.

The integrand , for $t \to +\infty$, is asymptotic to

$$\frac{C}{t^{4\alpha}}$$

for a suitable constant $C > 0$, hence the integral converges for

$$4\alpha > 1 \iff \alpha > \frac{1}{4}.$$

Exam of the 07.02.2022

Exercise 1 [10 punti] Given the function

$$f(x) = \log\left(|x| - x^2 + 2\right),$$

(i) determine the domain; determine the simmetria and the sign.

$$x \in \text{Domain} \iff |x| - x^2 + 2 > 0 \iff |x|^2 - |x| - 2 < 0 \ (\text{da } x^2 = |x|^2)$$

inequality that is risolta da

$$|x| \in \left]-1, 2\right[\iff |x| \in [0, 2[\iff x \in \left]-2, 2\right[$$

Hence Domain$= \left]-2, 2\right[$
The function is chiaramente pari.
The function is continuous perchánd composta dicontinuopus functions .

In alternativa si sarebbe potuto also argomentare come segue.

$$f(x) = \begin{cases} \log\left(x - x^2 + 2\right) & \forall x \geq 0 \\ \log\left(-x - x^2 + 2\right) & \forall x < 0 \end{cases}$$

Si osserva che f is pari, and si limita the study a $x \geq 0$. Hence $x \in$ Domain and $x \geq 0$ if and only if $x - x^2 + 2 > 0$ and $x \geq 0$, that is, $x \in [0, 2[$. Since f is pari, one has

$$\text{Domain} =]-2, 2[$$

.

Furthermore,

$$\begin{cases} f(x) \geq 0 \\ x \geq 0 \end{cases} \iff \begin{cases} x - x^2 + 2 \geq 1 \\ x \geq 0 \end{cases} \iff x \in \left[0, \frac{1+\sqrt{5}}{2}\right].$$

By symmetry, si conclude che

$$f(x) \geq 0 \iff x \in \left[-\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$$

Compute the limits and asymptotes at the extremes of the domain:

$$\lim_{x\to 2} f(x) = -\infty \quad \text{that for simmetria implica} \quad \lim_{x\to -2} f(x) = -\infty$$

così in 2 and -2 ci they are two asymptotes verticali.
(ii) study the derivability and calcolarne the first derivative ; study the monotonicity intervals individuando the points of maximum and of minimum sia relatvi that absolute :

$$\begin{cases} x > 0 \\ f'(x) = \dfrac{1-2x}{x - x^2 + 2} > 0 \end{cases} \iff x \in \left]0, \frac{1}{2}\right[.$$

Furthermore, $f'(x) = 0, \ x > 0$ if and only if $x = \frac{1}{2}$ Since f is continuous in its domain, if we deduce che f is strictly increasing in $\left[0, \frac{1}{2}\right]$, strictly decreasing in $\left[\frac{1}{2}, 2\right[$, and ha a point direlative maximum in $x = \frac{1}{2}$.

By symmetry, one has also che f is strictly decreasing in $\left[-\frac{1}{2},0\right]$, strictly increasing in $\left]-2,-\frac{1}{2}\right[$, and ha a point direlative maximum in $x=-\frac{1}{2}$.

In particolare $x=\frac{1}{2},-\frac{1}{2}$ they are points of absolute maximum .
Per $x=0$ (the function is continuous): $f(0)=\log 2$. $x=0$ is hence point direlative minimum (ma non absolute perché f tende a $-\infty$ at the extremes).
One has moreover $\lim_x \to 0^+ f'(x)=1/2=f'_+(0)$, that for simmetria implica $f'_-(0)=-1/2$, Hence 0 is a angular point .

(iii) draw the graph.

Exercise 2 [7 punti] Determine the insieme A of the numeri complessi $z\in\mathbb{C}$ tali che

$$\frac{|z+i\mathcal{I}m(z)|^2}{|z|^2+\mathcal{R}e(z)^2}\geq 1$$

and disegnarlo in the Gauss plane .

If scriviamo $z=x+iy$, the inequality diventa

$$\frac{|x+2yi|^2}{2x^2+y^2}=\frac{x^2+4y^2}{2x^2+y^2}\geq 1$$

the numerator is 0 if and solo if $(x,y)=(0,0)$. Negli altri punti is positive, therefore for $(x,y)\neq(0,0)$ the inequality is equivalente a

$$x^2+4y^2\geq 2x^2+y^2 \iff$$
$$3y^2-x^2=(\sqrt{3}y-x)(\sqrt{3}y+x)\geq 0 \iff$$
$$x+iy\in\left\{x+iy,\ \ y\leq x/\sqrt{3},\ \ y\leq -x/\sqrt{3}\right\}\bigcup\left\{x+iy,\ \ y\geq x/\sqrt{3},\ \ y\geq -x/\sqrt{3}\right\},\quad (x,y)\neq(0,0).$$

Exercise 3 [7 punti]
Study the convergence of the series

$$\sum_{n=1}^{\infty} n\left\{\alpha\sinh\left(\frac{1}{n^2}\right)+\log\left[\cosh\left(\frac{1}{n}\right)\right]\right\}$$

as $\alpha\in\mathbb{R}$.

Da

$$n\left\{\alpha\sinh\left(\frac{1}{n^2}\right)+\log\left[\cosh\left(\frac{1}{n}\right)\right]\right\}=$$
$$=n\left\{\alpha\frac{1}{n^2}+\alpha\cdot o\left(\frac{1}{n^4}\right)+\log\left[1+\frac{1}{2n^2}+\frac{1}{24n^4}+o\left(\frac{1}{n^5}\right)\right]\right\}=$$
$$=n\left\{\alpha\frac{1}{n^2}+\alpha\cdot o\left(\frac{1}{n^4}\right)+\frac{1}{2n^2}-\frac{1}{12n^4}+o\left(\frac{1}{n^4}\right)\right\}=$$
$$=(2\alpha+1)\frac{1}{2n}-\frac{1}{12n^3}+o\left(\frac{1}{n^3}\right)$$

deduce that it is a series a sign definitively constant e, applicando the criterion of the asymptotic comparison, che converges if and only if $2\alpha + 1 = 0$, i.e. $\alpha = -1/2$.

Exercise 4 [8 punti]
By making use the integration by parts, compute

$$\int \arctan\left(\frac{2}{x}\right)\, dx.$$

Abbiamo

$$\begin{aligned}
\int \arctan\left(\frac{2}{x}\right)\, dx &= \int \arctan\left(\frac{2}{x}\right)\, dx \\
&= x \arctan\left(\frac{2}{x}\right) + \int \frac{2}{x(1+4/x^2)} dx \\
&= x \arctan\left(\frac{2}{x}\right) + \int \frac{2x}{x^2+4} dx \\
&= x \arctan\left(\frac{2}{x}\right) + \log(x^2+4) + c, \quad c \in \mathbb{R}.
\end{aligned}$$

. Study the convergence of the integral improprio

$$\int_0^{+\infty} \arctan\left(\frac{x^3+1}{x^\alpha}\right)\, dx$$

as $\alpha > 0$.

Let us observe that the function integrand is always $C^{(0)}((0,+\infty))$ and nonnegativa. All'extreme $x = 0$ the integrand tende a $\pi/2$ hence

$$\int_0^1 \arctan\left(\frac{x^3+1}{x^\alpha}\right)\, dx$$

ànd integrable for ogni $\alpha > 0$. Let us study the integrabilità of

$$\int_1^{+\infty} \arctan\left(\frac{x^3+1}{x^\alpha}\right)\, dx$$

If $\alpha \leq 3$ the argument of the arctangent is always > 1 so that $\arctan\left(\frac{x^3+1}{x^\alpha}\right) > \pi/4$. It follows that the integral diverges. If $\alpha > 3$, the argument of arctangent tende a zero for $x \to +\infty$, and the integrand is asymptotic to $1/x^{\alpha-3}$. Hence the ultimo integral converges if and only if $\alpha - 3 > 1$, that is, $\alpha > 4$.

In conclusion

$$\int_0^{+\infty} \arctan\left(\frac{x^3+1}{x^\alpha}\right)\, dx$$

converges if and only if $\alpha > 4$.

Exam of the 01.07.2022

Exercise 1 [9 punti] Consider the function

$$f(x) = |x-2|\, e^{\frac{1}{(x-2)^2}}.$$

(i) determine the domain of f and the sign of f;

$$Domain = \mathbb{R}\backslash\{2\}$$

e

$$f(x) > 0$$

for every $x \in Domain$, perché prodotto didue functions positive.
(ii) compute the main limits of f;

$$\lim_{x\to\pm\infty} |x-2|\, e^{\frac{1}{(x-2)^2}} = +\infty \cdot 1 = +\infty$$

$$\lim_{x\to 2\pm} |x-2|\, e^{\frac{1}{(x-2)^2}} \underset{y=\frac{1}{(x-2)^2}}{=} \lim_{y\to+\infty} y^{-\frac{1}{2}}\, e^{y} = +\infty$$

(iii) compute the derivative of f, discuss the monotonicity of f and determine the infimum and the supremum of f and relative and absolute minimum and maximum points; Per ogni $x > 2$

$$\frac{df}{dx}(x) = e^{\frac{1}{(x-2)^2}} - 2(x-2)e^{\frac{1}{(x-2)^2}}\frac{1}{(x-2)^3} = e^{\frac{1}{(x-2)^2}}\left(1 - \frac{2}{(x-2)^2}\right) = e^{\frac{1}{(x-2)^2}} \cdot \frac{x^2-4x+2}{(x-2)^2}$$

Analogamente, Per ogni $x < 2$

$$\frac{df}{dx}(x) = -\left(e^{\frac{1}{(x-2)^2}} - 2(x-2)e^{\frac{1}{(x-2)^2}}\frac{1}{(x-2)^3}\right) = -e^{\frac{1}{(x-2)^2}}\left(1 - \frac{2}{(x-2)^2}\right) = -e^{\frac{1}{(x-2)^2}} \cdot \frac{x^2-4x+2}{(x-2)^2}$$

Hence, poiché $x^2 - 4x + 2 > 0$ if and only if $x > 2 + \sqrt{2}$ o $x < 2 - \sqrt{2}$, one has che $\frac{df}{dx}(x) > 0$ if and only if $x > 2 + \sqrt{2}$ o $2 - \sqrt{2} < x < 2$, while $\frac{df}{dx}(2+\sqrt{2}) = \frac{df}{dx}(2-\sqrt{2}) = 0$.

Furthermore, $f(2+\sqrt{2}) = f(2-\sqrt{2}) = \sqrt{2}$.

Hence the function is strictly increasing in $[2-\sqrt{2}, 2[$ and in $[2+\sqrt{2}, +\infty[$, is strictly decreasing in $]-\infty, 2-\sqrt{2}[$ and in $]2, 2+\sqrt{2}]$, cosicchánd in $2+\sqrt{2}$ and in $2-\sqrt{2}$ it ha two minimi relative that they are also absolute . Furthermore, the function is il limitata superiormente.
(iv) compute asymptotes of f;

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{(x-2)\, e^{\frac{1}{(x-2)^2}}}{x} = 1 \cdot 1 = 1$$

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{(2-x)\, e^{\frac{1}{(x-2)^2}}}{x} = -1 \cdot 1 = -1$$

$$\lim_{x\to+\infty} (f(x) - x) = (x-2)\, e^{\frac{1}{(x-2)^2}} - x =$$

Utilizzo lo sviluppo of e^y for $y \to 0$, con $y = \frac{1}{(x-2)^2}$

$$= \lim_{x\to+\infty} \left(x - 2 + \frac{x-2}{(x-2)^2} + o\left(\frac{1}{(x-2)}\right) - x\right) = -2$$

Analogamente

$$\lim_{x\to-\infty} (f(x) + x) = (2-x)\, e^{\frac{1}{(x-2)^2}} - x = \lim_{x\to+\infty} \left(2 - x + \frac{2-x}{(x-2)^2} + o\left(\frac{1}{(x-2)}\right) + x\right) = 2$$

In conclusion, for $x \to +\infty$, one has l' asintoto $y = -2 + x$ and for $x \to -\infty$, one has l' asintoto $y = 2 - x$
(v) draw a qualitative graph of f.

Exercise 2 [8 punti] Determine in algebraic form the solutions in $\mathbb{C}$ of the equation

$$z^4 + (-2 - 2i)z^2 + 4i = 0.$$

Pongo $w := z^2$. The equation for w is

$$w^2 + (-2 - 2i)w + 4i = 0$$

whose solutions are

$$w_1 = 1 + i + r_1, \;\; w_2 = 1 + i + r_2$$

dove r_1, r_2 they are le radici quadrate of $(1+i)^2 - 4i = 1 - 2i - 1 = -2i$. That is, $r_1 = -1+i$ and $r_2 = 1-i$, from which $w_1 = 2i, w_2 = 2$ so that the solutions si trovano unendo the solutions $z^2 = 2i$ a quelle of $z^2 = 2$. Ne segue (con the solito de Moivre) che the solutions they are

$$z_1 = 1 + i, z_2 = -1 - i, z_3 = \sqrt{2}, \;\; z_4 = -\sqrt{2}$$

Exercise 3 [7 punti]
(i) Determine, as $\alpha \in \mathbb{R}$, the limit

$$\lim_{x \to 0^+} \frac{(1+x)^{\alpha x} - 1}{x^2}.$$

Utilizzando the principio of sostituzione in the prodotto/quoziente of limits con functions asintotiche, osservando che, for $x \to 0^+$, $e^{\alpha x \log(1+x)} - 1 \sim \alpha x \log(1+x) \sim \alpha x^2$, one gets

$$\lim_{x \to 0^+} \frac{(1+x)^{\alpha x} - 1}{x^2} = \lim_{x \to 0^+} \frac{e^{\log(1+x)^{\alpha x}} - 1}{x^2} = \lim_{x \to 0^+} \frac{\alpha x \log(1+x)}{x} = \lim_{x \to 0^+} \frac{\alpha x^2}{x} = \alpha.$$

Exercise 4 [8 punti] (i) Compute the following indefinite integral

$$\int \frac{\sqrt{t}}{1+t}\, dt.$$

Pongo $y := \sqrt{t}$, so that $dy = \frac{1}{2}(\sqrt{t})^{-1}dt = \frac{1}{2}y^{-1}dt$, that is, $dt = 2ydy$. One has hence

$$\int \frac{\sqrt{t}}{1+t}\, dt = 2\int \frac{y^2}{1+y^2}dy = 2\left[\int \frac{1+y^2}{1+y^2}dy[- \int \frac{1}{1+y^2}dy\right] =$$
$$2\,(y - \arctan(y)) + c = 2\left(\sqrt{t} - \arctan(\sqrt{t})\right) + c, \qquad \forall c \in \mathbb{R}.$$

(ii) Discutere the convergence of the generalized integral

$$\int_0^{+\infty} \frac{\sqrt{t}}{1+t^\alpha}\, dt$$

as $\alpha \in \mathbb{R}$.

Per $t \to 0+$, if $\alpha \geq 0$ the function integrand is continuous nello 0. If instead $\alpha < 0$ the function is prolungabile for continuità, uguale a 0 nello 0. Hence non ci they are problemi diintegrabilità in a right neighbourhood of 0.

Per $t \to +\infty$, if $\alpha < 0$, $\frac{\sqrt{t}}{1+t^\alpha} \sim \sqrt{t}$ that is not integrable for $t \to +\infty$. If $\alpha = 0$ $\frac{\sqrt{t}}{1+t^\alpha} \sim \frac{\sqrt{t}}{2}$ che, similmente, is not integrable for $t \to +\infty$. If $\alpha > 0$, $\frac{\sqrt{t}}{1+t^\alpha} \sim \frac{1}{t^{\alpha-\frac{1}{2}}}$, that for $t \to +\infty$ is integrabile if and only if $\alpha - \frac{1}{2} > 1$ cioé, if and only if $\alpha > \frac{3}{2}$.