# TD 28 - Matrices

## Ex 1 :

1)

$$M \begin{array}{c} \\ i \\ \end{array}\!\!\begin{array}{c} j \\ \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \end{array} = \begin{pmatrix} \boxed{0} & \begin{matrix} m_{1i} \\ \vdots \\ m_{ni} \end{matrix} & \boxed{0} \end{pmatrix} = \begin{pmatrix} \boxed{0} & C_i(A) & \boxed{0} \end{pmatrix}$$

2)

$$\begin{array}{c} \\ i \end{array}\!\!\begin{array}{c} j \\ \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \end{array} M = \begin{pmatrix} \boxed{0} \\ L_j(M) \\ \boxed{0} \end{pmatrix} \begin{array}{c} \\ i \\ \\ \end{array}$$

$$(ME_{ij})_{k\ell} = \sum_{p=1}^{n} M_{kp} (E_{ij})_{p\ell} = \sum_{p=1}^{n} M_{kp}\, \delta_{ip}\, \delta_{j\ell} = M_{ki}\, \delta_{j\ell}$$

$$= \begin{cases} 0 & \text{si } j \neq \ell \text{ donc } \forall \ell \neq j \;\; C_j \cdot (ME_{ij}) = 0 \\ M_{ki} & \text{si } j = \ell \text{ donc } C_j(ME_{ij}) = C_i(M) \end{cases}$$

3) dans le cours

## Ex 2 :

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{3}{4} & 1 & -\frac{1}{4} \\ -1 & -1 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} -1 & 1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Voir TD - Moi

Ex 3:

1) $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} n & \cdots & n \\ \vdots & & \vdots \\ n & \cdots & n \end{pmatrix}$

$J^2 = nJ$

$J^3 = J^2 \times J$

$= nJ \times J$

$= nJ^2$

$= n \times n J$

$= n^2 J$

$\forall\ p \geqslant 1 \quad J^p = n^{p-1} J$

$M = \begin{pmatrix} a & & b \\ & \ddots & \\ b & & a \end{pmatrix} = bJ + (a-b)I_n$

$M^p = \left(bJ + (a-b)I_n\right)^p$

$bJ$ et $(a-b)I_n$ commutent

donc :

$M^p = \sum_{k=0}^{p} \binom{p}{k} b^k J^k \left((a-b)I_n\right)^{p-k}$

$= \sum_{k=0}^{p} \binom{p}{k} b^k (a-b)^{p-k} J^k$

$= \sum_{k=1}^{p} \binom{p}{k} b^k (a-b)^{p-k} n^{k-1} J + (a-b)^p I_n$

$= \frac{1}{n}\left(\sum_{k=1}^{p} \binom{p}{k} (nb)^k (a-b)^{p-k}\right) J + (a-b)^p I_n$

$= \frac{1}{n}\left((nb+a-b)^p - (a-b)^p\right) J + (a-b)^p I_n$

$= \alpha_p J + \beta_p I_n$

$= \begin{pmatrix} \alpha_p+\beta_p & & \\ \alpha_p & \ddots & \\ & & \alpha \end{pmatrix}$

$rg\begin{pmatrix} a & & b \\ & \ddots & \\ b & & a \end{pmatrix} = rg\begin{pmatrix} a & b-a & \cdots & b-a \\ b & a-b & & 0 \\ \vdots & 0 & \ddots & \\ b & 0 & \cdots\ 0 & a-b \end{pmatrix}$

$C_2 \leftarrow C_2 - C_1$

$C_3 \leftarrow C_3 - C_1$

$C_n \leftarrow C_n - C_1$

Si $a=b$, alors $\operatorname{rg} M = \operatorname{rg} \begin{pmatrix} a & & & \\ \vdots & & 0 & \\ a & & & \end{pmatrix} = \begin{cases} 1 & \text{si } a \neq 0 \\ 0 & \text{sinon} \end{cases}$

Si $a \neq b$, alors $\operatorname{rg} M = \operatorname{rg} \begin{pmatrix} a & -1 & \cdots & -1 \\ b & 1 & & 0 \\ \vdots & & \ddots & \\ b & 0 & & 1 \end{pmatrix} = \operatorname{rg} \begin{pmatrix} * & -1 & \cdots & -1 \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$, avec $* = a-(n-1)b$

$\operatorname{rg} M = \begin{cases} n-1 & \text{si } a-(n-1)b = 0 \\ n & \text{sinon} \end{cases}$

CCl : $A \in GL_n(\mathbb{K}) \iff \begin{cases} a \neq b \\ a \neq (n-1)b \end{cases}$

Calcul de $A^{-1}$ à faire

$B$ est inversible car triangulaire supérieure avec tous ses coeff diagonaux non nuls.

3) $\underset{\substack{\| \\ (e_1,\dots,e_n)}}{\operatorname{Mat}_{\mathcal{B}}} f = \begin{pmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix} \iff \begin{cases} f(e_1) = e_1 \\ f(e_2) = e_1 + e_2 \\ f(e_3) = e_2 + e_3 \\ \vdots \\ f(e_n) = e_{n-1} + e_n \end{cases}$

$f^{-1}(e_1) = e_1$

$f^{-1}(e_2) = e_2 - e_1$ car $f(e_2 - e_1) = e_2$

$f^{-1}(e_3) = e_1 - e_2 + e_3$ car $f(e_3 - e_2 + e_1) = e_3$

$f^{-1}(e_4) = e_1 + e_2 - e_3 + e_4$ car $f(e_4 - e_3 + e_2 - e_1) = e_4$

$\operatorname{Mat}_{\mathcal{B}} f^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 & \\ 0 & 1 & -1 & 1 & \\ \vdots & 0 & 1 & -1 & \\ & \vdots & 0 & 1 & \ddots \\ & & \vdots & 0 & \\ 0 & 0 & 0 & 0 & \end{pmatrix} = \left( \sigma_{1 \leq j} \, (-1)^{i+j} \right)_{1 \leq i,j \leq n}$

Ex 4 :

1) Soit $(P,Q) \in \mathbb{R}_2[X]^2$ et $\lambda \in \mathbb{R}$

$$f(\lambda P + Q) = \dots$$
$$= \lambda f(P) + f(Q)$$

De plus, soit $f \in \mathbb{R}_2[X]$

Mq $f(P) \in \mathbb{R}_2[X]$

$$f(P) = \underbrace{2(X+1)P}_{\deg \leqslant 3} - \underbrace{(X-1)^2 P'}_{\deg \leqslant 3} \in \mathbb{R}_3[X]$$

On pose $P = aX^2 + bX + C$

Le coeff de $f(P)$ devant $X^3$ vaut $2a - 2a = 0$

Donc $f(P) \in \mathbb{R}_2[X]$

cce : $f \in \mathcal{L}(\mathbb{R}_2[X])$

2) $f(1) = 2X + 2$

$f(X) = 2(X+1)X - (X-1)^2 = X^2 + 4X - 1$

$f(X^2) = 6X^2 - 2X$

$$\mathrm{Mat}_{\mathcal{B}}\, f = \begin{pmatrix} 2 & -1 & 0 \\ 2 & 4 & -2 \\ 0 & 1 & 6 \end{pmatrix}$$

3) $f \in GL_n(\mathbb{R}_2[X]) \iff f$ bij

$\iff f$ inj car $f$ endo de $\underbrace{\mathbb{R}_2[X]}_{\text{dim finie}}$

$\iff A \in GL_3(\mathbb{K})$

$\iff \mathrm{rg}\, A = 3$

$\iff \mathrm{Ker}\, A = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$$\text{rg} A = \text{rg}\begin{pmatrix} 2 & -1 & 0 \\ 2 & 4 & -2 \\ 0 & 1 & 6 \end{pmatrix}$$

$$= \text{rg}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 4 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \text{rg}\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 3 & 16 \end{pmatrix} = 3$$

Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M_{31}(\mathbb{R})$

$$X \in \text{Ker} A \iff \begin{cases} 2x - y = 0 \\ 2x + 4y - 2z = 0 \\ x - 6z = 0 \end{cases}$$

$$\iff \begin{cases} y = 2x \\ 2x + 8x - \frac{1}{3}x = 0 \\ z = \frac{1}{6}x \end{cases}$$

$$\iff x = y = z = 0$$

Soit $P \in \text{Ker} f$

On pose $P = aX^2 + bX + c$

$$f(P) = 2(X+1)P - (X-1)^2 P'$$

$$= 2(X+1)(aX^2+bX+c) - (X-1)^2(2aX+b)$$

$$= (2a + 2b - b + 4a)X^2$$

$$+ (2c + 2b + 2b - 2a)X$$

$$+ 2c - b$$

$f(P) = 0$

$$\text{donc} \begin{cases} 6a + b = 0 \\ -2a + 4b + 2c = 0 \\ -b + 2c = 0 \end{cases}$$

$$f(P) = a\, f(X^2) + b\, f(X) + c\, f(1)$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Ex 5:

$$f : M_{31}(\mathbb{R}) \longrightarrow M_{31}(\mathbb{R})$$
$$X \mapsto AX$$

1) $$rg(A - \lambda I_3) = rg \begin{pmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$\begin{array}{ccc} C_1 & C_2 + C_1 & C_3 + (\lambda - 2) C_1 \end{array}$$
$$= rg \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1-\lambda & 0 \\ -\lambda & 1-\lambda & \lambda + \lambda - 2 \end{pmatrix}$$

Si $\lambda = 1$ alors $rg(A - \lambda I_3) = rg \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = 1$

Sinon :

$$rg(A - \lambda I_3) = rg \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1-\lambda & 0 \\ -\lambda & 1-\lambda & 2(\lambda - 1) \end{pmatrix} = 3 \quad \text{car } 1 - \lambda \neq 0$$

$Ker(A - \lambda I_3) = \{0\}$ si $\lambda \neq 1$

$$Ker(A - \lambda I_3) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M_{31}(\mathbb{R}) : (A - I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ 0 \\ x + y \end{pmatrix}$$

$$Ker(A - I_3) = \left\{ \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}, (x, y) \in \mathbb{R}^2 \right\} = Vect\left( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$$

2) Idée 1 :

* $\mathcal{B} = 3$ et $\mathcal{B}$ liberté

Idée 2 :

$$\mathrm{Mat}_{\mathcal{B}_{can}}(\mathcal{B}) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = P$$

$$\mathrm{rg}\, P = \mathrm{rg} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \mathrm{rg} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = 3$$

$P \in GL_3(\mathbb{R})$

donc $\mathcal{B}$ est une base de $\mathbb{R}^3$

3) $f : \mathbb{R}^3 \to \mathbb{R}^3$
$(x,y,z) \mapsto (2x+y-z,\ y,\ x+y)$

$f((1,0,1)) = (1,0,1) = e_1$

$f((-1,1,0)) = (-1,1,0) = e_2$

$f((1,1,1)) = (2,1,2) = e_1 + e_3$

Donc :

$$\underset{\substack{\| \\ (e_1,e_2,e_3)}}{\mathrm{Mat}_{\mathcal{B}}} f = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4) On a : $D = P^{-1}AP$ où $P = P_{\mathcal{B}_{can}, \mathcal{B}}$

Donc $A = PDP^{-1}$

puis : $A^n = PD^nP^{-1}$

$$D^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Mq $D^n = \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

On a :

$$D^n = \left( I_3 + E_{13} \right)^n$$

$I_3 E_{13} = E_{13} I_3$

$$= \sum_{k=0}^{n} \binom{n}{k} E_{13}^k$$

$E_{12}^2 = E_{13} E_{13} = 0$

$$= I_3 + n E_{13}$$

$P = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ et $P^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$$P E_{13} P^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A^n = P \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} P^{-1}$$

$$= P \left( I_3 + n E_{13} \right) P^{-1}$$

$$= I_3 + n P E_{13} P^{-1}$$

$$= \begin{pmatrix} n+1 & n & -n \\ 0 & 1 & 0 \\ n & n & -n+1 \end{pmatrix}$$

Vérifier pour $\begin{cases} n = 0 \\ n = 1 \end{cases}$

Ex 6 :

1)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & 10 & -4 \\ -8 & 16 & -6 \end{pmatrix}$$

2) On pose $P = X^2 + aX + b$

$P(A) = A^2 + aA + bI_3$

$P(A) = 0 \iff a = -3$ et $b = 2$

4) Il existe $(Q_n, R_n) \in \mathbb{R}[X]^2$

$$X^n = Q_n(X^2 - 3X + 2) + \underbrace{R_n}_{\deg \leq 1}$$

$a_n X + b_n$

On évalue en $-1$ et en $-2$

On a : $1 = a_n + b_n$

$2^n = 2a_n + b_n$

puis $a_n = 2^n - 1$

$b_n = 2 - 2^n$

$X^n = Q_n(X^3 - 3X + 2) + (2^n - 1)X - 2^n + 2$

OK pour $\begin{cases} n = 0 \\ n = 1 \end{cases}$

5) $A^n = Q_n(A)(A^2 - 3A + 2I_3) + (2^n - 1)A + (2 - 2^n)I_3$

$= (2^n - 1)A + (2 - 2^n)I_3$

$\forall n \in \mathbb{N} :$ $\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = A^n \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}$

$$= \begin{pmatrix} 2(2^n - 1) + 2 + 2^n & 2^n - 1 & 1 - 2^n \\ 0 & 2^n - 1 + 2 - 2^n & 0 \\ 2^n - 1 & 2^n - 1 & 2 - 2^n \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}$$

3) $A^2 - 3A + 2I_n = 0$

$$2I_n = 3A - A^2$$

$$= A(3I_3 - A)$$

Ccl : $A \in GL_3(\mathbb{R})$ et $A^{-1} = \frac{1}{2}(3I_3 - A)$

Ex 7 :

1) Soit $(\lambda_0, \dots, \lambda_{p-1}) \in \mathbb{K}^p$ tq $\sum_{k=0}^{p-1} \lambda_k f^k(x) = 0$

Mq $\lambda_0 = \dots = \lambda_{p-1} = 0$

$$f^{p-1}\left(\sum_{k=0}^{p-1} \lambda_k f^k(x)\right) = 0$$

$$= \sum_{k=0}^{p-1} \lambda_k f^{p+k-1}(x) = \lambda_0 \underbrace{f^{p-1}(x)}_{\neq 0_E}$$

$$\sum_{k=1}^{p-1} \lambda_k f^k(x) = 0$$

$$f^{p-2}\left(\sum_{k=1}^{p-1} \lambda_k f^k(x)\right) = \sum_{k=1}^{p-1} \lambda_k f^{p+k-2}(x) = \lambda_1 \underbrace{f^{p-1}(x)}_{\neq 0_E}$$

$\boxed{\text{ou:}}$ Soit $(\lambda_0, \lambda_1, \dots, \lambda_{p-1}) \in \mathbb{K}^p$ tq $\sum_{k=0}^{p-1} \lambda_k f^k(x) = 0$

Mq $\lambda_0 = \dots = \lambda_{p-1} = 0$

Supp $(\lambda_0, \dots, \lambda_{p-1}) \neq (0, \dots, 0)$

Soit $i_0 = \text{Min}\{k \in [\![0; p-1]\!] \text{ tq } \lambda_k \neq 0\}$

On a : $\sum_{k=0}^{p-1} \lambda_k f^k(x) = \sum_{k=i_0}^{p-1} \lambda_k f^k(x) = 0$

$$f^{p-1-i_0}\left(\sum_{k=i_0}^{p-1} \lambda_k f^k(x)\right) = 0$$

$$\sum_{k=i_0}^{p-1} \lambda_k f^{k+p-1-i_0}(x) = 0$$

$$\lambda_{i_0} \underbrace{f^{p-1}(x)}_{\neq 0_E} = 0$$

Donc $\lambda_{i_0} = 0$ ABSURDE car $\lambda_{i_0} \neq 0$

donc $(\lambda_0, \dots, \lambda_{p-1}) = (0, \dots, 0)$

2) Supp $f^n = 0$ et $f^{n-1} \neq 0$

Il existe donc $x_0 \in E$ : $f^{n-1}(x_0) \neq 0$

D'après 1, $(x_0, f(x_0), \dots, f^{n-1}(x_0))$ est libre

Comme $\#(x_0, f(x_0), \dots, f^{n-1}(x_0)) = n = \dim E$

Donc $B = (x_0, f(x_0), \dots, f^{n-1}(x_0))$ est une base de $E$.

Déterminons $\mathrm{Mat}_B f$ ?

$$\mathrm{Mat}_B f = \begin{pmatrix} 0 & 0 & & & 0 \\ 1 & 0 & & 0 & \vdots \\ 0 & 1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A = (\delta_{i+1,j})_{1 \leq i,j \leq n}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ & \vdots & 0 & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$ perdu

$$A = \mathrm{Mat}_{(e_1,\dots,e_n)} f \iff \begin{cases} f(e_1) = 0 \\ f(e_2) = e_1 \\ f(e_3) = e_2 \\ \vdots \\ f(e_n) = e_{n-1} \end{cases}$$

On prends $B' = (f^{n-1}(x_0), f^{n-2}(x_0), \dots, f(x_0), x_0)$

On a :

$$\mathrm{Mat}_{B'} f = (\delta_{i+1,j})_{1 \leq i,j \leq n} = A$$

Mq $f^n = 0$ et $f^{n-1} \neq 0$

$\mathrm{Mat}_B f^n = A^n$

$\mathrm{Mat}_{B'} f^{n-1} = A^{n-1}$

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ & \vdots & 0 & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ 0 & \cdots & & & 0 \end{pmatrix} = (\delta_{i+2,j})_{1 \leq i,j \leq n}$$

Par récurrence :

$$A^p = (\sigma_{i+p,j})_{1 \le i,j \le n}$$

donc $A^n = 0$

$$A^{n-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & 0 \\ & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}$$

et $\mathrm{Mat}_{\mathcal{B}} f^{n-1} = A^{n-1}$