TD13 - Limites, équivalents et suites

Ex 1 :

1) Supp $\lim\limits_{+\infty} f = \ell$, $\ell \in \mathbb{R}$

Soit $x \in \mathbb{R}$

$\forall n \in \mathbb{N} \quad f(x) = \underbrace{f(x + nT)}_{\to \ell}$

Donc $f(x) = \ell$

2) Supp $\lim\limits_{+\infty} f = +\infty$

Soit $x \in \mathbb{R}$

$\forall n \in \mathbb{N} \quad f(x) = \underbrace{f(x + nT)}_{\to +\infty}$

Donc $f(x) = +\infty$ Absurde

3) $f \nearrow$ donc f a une limite

Ex 3 :

1) $f : x \mapsto \dfrac{\cos x - 1}{\sqrt{\tan x}}$

$D_f = \{x \in \mathbb{R},\ \tan x > 0\} = \bigcup\limits_{k \in \mathbb{Z}} \left]k\pi, \frac{\pi}{2} + k\pi\right[$

$$\frac{\cos x - 1}{\sqrt{\tan x}} \sim \frac{-\frac{x^2}{2}}{\sqrt{x}} = -\frac{1}{2} x^{3/2}$$

2) $f : x \mapsto \ln(1+e^x) - \ln 2$

$D_f = \{ x \in \mathbb{R} : 1 + e^x > 0 \} = \mathbb{R}$

$\ln(1+e^x) - \ln 2 = \ln\left(\frac{1+e^x}{2}\right)$

$\ln(1+h) \underset{h \to 0}{\sim} h$

$\ln(y) \underset{y \to 0}{\sim} y - 1$ chgmt de variable

$\ln\left(\frac{1+e^x}{2}\right) \sim \frac{1+e^x}{2} - 1 = \frac{e^x - 1}{2}$

$\ln\left(\frac{1+e^x}{2}\right) = \ln\left(1 + \frac{e^x - 1}{2}\right) \sim \frac{e^x - 1}{2} \sim \frac{x}{2}$

3) $f : x \mapsto \dfrac{1 - \sin\left(\frac{\pi(1-x)}{2}\right)}{x - \sqrt{x^2+2x}}$

$D_f = \left\{ x \in \mathbb{R} \text{ tq } \begin{cases} x^2 + 2x \geq 0 \\ x \neq \sqrt{x^2+2x} \end{cases} \right\}$

Soit $x \in \mathbb{R}$

. $x^2 + 2x - x(x+2)$

donc $x^2 + 2x \geq 0 \iff x \in]-\infty ; -2] \cup [0 ; +\infty[$

. $x = \sqrt{x^2+2x} \iff \begin{cases} x^2 = x^2 + 2x \\ x \geq 0 \end{cases}$

$\iff x = 0$

Donc $D_f =]-\infty ; -2] \cup]0 ; +\infty[$

$1 - \sin\left(\frac{\pi(1-x)}{2}\right) = 1 - \sin\left(\frac{\pi}{2} + \frac{x\pi}{2}\right) = 1 - \cos\left(\frac{\pi x}{2}\right) \sim \frac{1}{2}\left(\frac{\pi x}{2}\right)^2 = \frac{\pi^2}{8} x^2$

$\underset{= o\sqrt{x}}{x} - \underbrace{\sqrt{x^2+2x}}_{\underset{0}{\sim} \sqrt{2x}} \underset{0}{\sim} \sqrt{2x}$

Donc $f(x) \underset{0}{\sim} \dfrac{\frac{\pi^2}{8} x^2}{-\sqrt{2x}} = -\dfrac{\pi^2}{8\sqrt{2}} x^{3/2}$

Rq : $x - \sqrt{x^2+2x} \underset{\infty}{\sim} \,?$ Mettre en facteur les termes dominants

$$x\left(1-\sqrt{1+\frac{2}{x}}\right) \sim x \times \left(\frac{-1}{2}\right) \times \frac{2}{x} = -1$$

4) $f : x \longmapsto \dfrac{\tan^2 x}{1+\frac{1}{x^2}}$

$$D_f = \{x \in D_{\tan} \quad x \neq 0\} = D_{\tan} \setminus \{0\} = \left(\bigcup_{k\in\mathbb{Z}} \left]-\frac{\pi}{2}+k\pi \,;\, \frac{\pi}{2}+k\pi\right[\right) \setminus \{0\}$$

$$\frac{\tan^2 x}{1+\frac{1}{x^2}} \underset{0}{\sim} \frac{x^2}{\frac{1}{x^2}} = x^4 \xrightarrow[x\to 0]{} 0$$

5) $f : x \longmapsto \dfrac{\sqrt[3]{x^3-2}}{\sqrt{x^2+x}}$

$$D_f = \{x \in \mathbb{R} : \underbrace{x^2+x}_{x(x+1)} > 0\} =]-\infty ; -1[\cup]0 ; +\infty[$$

$$\sqrt[3]{x^3-2} \underset{0}{\sim} \sqrt[3]{-x}$$

$$\sqrt{x^2+x} \underset{0}{\sim} \sqrt{x}$$

Donc $f(x) \underset{0}{\sim} \dfrac{\sqrt[3]{-x}}{\sqrt{x}} = \dfrac{-x^{\frac{1}{3}}}{x^{\frac{1}{2}}} = -x^{\frac{1}{6}} \xrightarrow[x \to 0^{>}]{} -\infty$

6) $f : x \longmapsto \sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}} - \sqrt{\frac{1}{x}}$

$D_f = \mathbb{R}^{+*}$

$$\frac{1}{x}+\sqrt{\frac{1}{x}} \underset{0}{\sim} \frac{1}{x}$$

$$\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}} \sim \sqrt{\frac{1}{x}}$$

$$\frac{1}{x} + \underbrace{\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}_{\sim \sqrt{\frac{1}{x}} = o\left(\frac{1}{x}\right)} \sim \frac{1}{x}$$

$$\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}} \sim \sqrt{\frac{1}{x}}$$

$$f(x) = \sqrt{\frac{1}{x}}\left(\sqrt{1+\underbrace{x\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}_{\underset{0}{\sim}\sqrt{x}}} - 1\right)$$

$$\sim \sqrt{\frac{1}{x}} \times \frac{1}{2}\sqrt{x} = \frac{1}{2}$$

7) $f : x \longmapsto \sqrt[3]{\underbrace{\frac{1}{x^3}+\frac{1}{x}+1}_{\sqrt{\frac{1}{x^3}} - \frac{1}{x}}} - \sqrt{\underbrace{1+\frac{1}{x^2}}_{\sim \frac{1}{|x|}}}$

$Df = \mathbb{R}^*$

$$f(x) \underset{\substack{x \to 0 \\ <}}{=} \frac{1}{x} + o\left(\frac{1}{x}\right) + \frac{1}{x} + o\left(\frac{1}{x}\right) \sim \frac{2}{x}$$

Soit $x > 0$

$$f(x) = \frac{1}{x^3}\left(\sqrt[3]{1+x^2+x^3} - \sqrt{1+x^2}\right)$$

$$= \frac{1}{x^3}\left(\underbrace{\sqrt[3]{1+x^2+x^3}}_{\sim \frac{1}{3}x^2} + \underbrace{1 - \sqrt{1+x^2}}_{\sim -\frac{1}{2}x^2}\right) \sim \frac{-x}{6}$$