TD 31 - Déterminants

Ex 1:

1)

2)
$$\det\begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix} = \begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{vmatrix}$$

$$= a(b-a)(c-b)(d-c)$$

3)
$$\begin{vmatrix} 1 & a+b & a^3+b^3 \\ 1 & b+c & b^3+c^3 \\ 1 & c+a & c^3+a^3 \end{vmatrix} = \begin{vmatrix} 1 & a+b & a^3+b^3 \\ 0 & c-a & c^3-a^3 \\ 0 & c-b & c^3-b^3 \end{vmatrix}$$ $L_2 \leftarrow L_2 - L_1$, $L_3 \leftarrow L_3 - L_1$

$$= \begin{vmatrix} c-a & c^3-a^3 \\ c-b & c^3-b^3 \end{vmatrix}$$

$$= (c-a)(c-b)\begin{vmatrix} 1 & c^2+ca+a^2 \\ 1 & c^2+bc+b^2 \end{vmatrix}$$

$$= (c-a)(c-b)\begin{vmatrix} 1 & c^2+ac+a^2 \\ 0 & bc-ac-a^2+b^2 \end{vmatrix}$$

$$= (c-a)(c-b)(b-c) \times (a+b+c)$$

4) $\begin{vmatrix} 1 & \cos\alpha & \cos(2\alpha) \\ 1 & \cos\beta & \cos(2\beta) \\ 1 & \cos\gamma & \cos(2\gamma) \end{vmatrix}$

$$= \begin{vmatrix} 1 & \cos\alpha & \cos(2\alpha) \\ 0 & \cos\beta - \cos\alpha & \cos(2\beta) - \cos(2\alpha) \\ 0 & \cos\gamma - \cos\alpha & \cos(2\gamma) - \cos(2\alpha) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\beta - \cos\alpha & \cos(2\beta) - \cos(2\alpha) \\ \cos\gamma - \cos\alpha & \cos(2\gamma) - \cos(2\alpha) \end{vmatrix}$$

$$= 2(\cos\beta - \cos\alpha)(\cos\gamma - \cos\alpha) \begin{vmatrix} 1 & \cos\beta + \cos\alpha \\ 1 & \cos\gamma + \cos\alpha \end{vmatrix}$$

$$= 2V(\cos\alpha, \cos\beta, \cos\gamma)$$

5) $\begin{vmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & \vdots \\ \vdots & & \ddots & & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 1 \\ & -1 & 0 \\ 0 & & \ddots & \\ & & & -1 \end{vmatrix} = (-1)^{n-1}$

$L_i \leftarrow L_i - L_1$

$\forall i \in [\![2, n]\!]$

6) $\begin{vmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{vmatrix} = (n-1)\begin{vmatrix} 1 & 0 & & 1 \\ \vdots & & \ddots & \\ 1 & 1 & & 0 \end{vmatrix}$

$c_1 \leftarrow \sum_{j=1}^{n} c_j$

$L_i \leftarrow L_i + L_1$

$$= (n-1)\begin{vmatrix} 1 & & 0 \\ 0 & \ddots & \\ \vdots & & \\ 0 & \cdots & 0 \quad 1 \end{vmatrix}$$

$$= (n-1)\begin{vmatrix} 1 & \cdots & & 1 \\ & -1 & 0 & \vdots \\ 0 & & \ddots & -1 \\ & & & -1 \end{vmatrix} = (n-1)(-1)^{n-1}$$

7) $\begin{vmatrix} 1 & & 1 \\ & 2 & \\ & & \ddots \\ 1 & & n \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 1 \\ 1 & 2 & 0 \\ & & \ddots \\ 1 & & n-1 \end{vmatrix} = (n-1)!$

Ex 2:

Dvpt / C_1

1) $D_n = \begin{vmatrix} 2\cos\theta & 1 & & 0 \\ 1 & \ddots & \ddots & \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 1 & \theta \end{vmatrix} = 2\cos\theta \overbrace{\begin{vmatrix} 2\cos\theta & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2\cos\theta \end{vmatrix}}^{D_{n-1}} - \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 2\cos\theta & 1 & 0 \\ 0 & 1 & \ddots & 1 \\ 0 & 0 & 1 & 2\cos\theta \end{vmatrix}$

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$$D_n = 2\cos\theta\, D_{n-1} - D_{n-2}$$

$$X^2 - 2\cos\theta\, X + X = 0$$

$$\Delta = (-2\cos\theta)^2 - 4$$
$$= 4\cos^2\theta - 4$$
$$= 4(\cos^2\theta - 1)$$

$\Delta \geq 0$	$\Delta < 0$
$4(\cos^2\theta - 1) \geq 0$	$4(\cos^2\theta - 1) < 0$
$\cos^2\theta - 1 \geq 0$	$-\sin^2\theta < 0$
$-\sin^2\theta \geq 0$	$\sin^2\theta > 0$
$\sin^2\theta \leq 0$	$\theta \in \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$
$\sin^2\theta = 0$	
$\theta = k\pi, k \in \mathbb{Z}$	

Si $\theta = k\pi, k \in \mathbb{Z}$, $\Delta = 0$

$$x_0 = \frac{2\cos\theta}{2} = \cos\theta$$

$$S_{k\pi} = \{ (\lambda n + \mu)\cos^n\theta , (\lambda, \mu) \in \mathbb{C}^2 \}$$

Si $\theta \in \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$, $\Delta \leq 0$

$$x_{1,2} = \frac{2\cos\theta \pm 2i\sqrt{1-\cos^2\theta}}{2}$$

$$= \cos\theta \pm i\sqrt{\sin^2\theta}$$

$$= \cos\theta \pm i|\sin\theta|$$

$$S_{x_{1,2}} = \{ (\cos\theta \pm i|\sin\theta|)^n (A\cos(n\theta) + B\sin(n\theta)), (A, B) \in \mathbb{R}^2 \}$$

Dvpt / L_n

2) $\begin{vmatrix} 1+x^2 & x & & 0 \\ x & 1+x^2 & \ddots & \\ 0 & \ddots & \ddots & x \\ 0 & 0 & x & 1+x^2 \end{vmatrix} = (1+x^2)\begin{vmatrix} 1+x^2 & x & & 0 \\ x & \ddots & \ddots & \\ & \ddots & \ddots & x \\ 0 & & x & 1+x^2 \end{vmatrix}$

$- x \begin{vmatrix} 1+x^2 & x & 0 & 0 \\ x & \ddots & \ddots & \vdots \\ 0 & \ddots & x & 0 \\ \vdots & x & 1+x^2 & 0 \\ 0 & 0 & x & x \end{vmatrix}$

$= (1+x^2) D_{n-1} - x^2 D_{n-2}$

$X^2 - (1+x^2)X + x^2 = (X-1)(X-x^2)$

Ex 3 :

1) Si $a = c$

$$\begin{vmatrix} b & & a \\ & \ddots & \\ a & & b \end{vmatrix} = \begin{vmatrix} b & a-b & \cdots & a-b \\ a & b-a & & 0 \\ \vdots & & \ddots & \\ a & 0 & & b-a \end{vmatrix} \qquad C_i \leftarrow C_i - C_1$$

$$= \begin{vmatrix} b+a(n-1) & 0 & \cdots & 0 \\ a & b-a & & \vdots \\ \vdots & & \ddots & 0 \\ a & 0 & & b-a \end{vmatrix} \qquad L_1 \leftarrow L_1 + \sum_{i=2}^{n} L_i$$

$$= \big(b + a(n-1)\big)(b-a)^{n-1}$$

$\boxed{\text{ou :}}$ $C_1 \leftarrow \sum C_j$

$$\begin{vmatrix} b & & a \\ & \ddots & \\ a & & b \end{vmatrix} = \begin{vmatrix} b+(n-1)a & a & \cdots & a \\ \vdots & b & \ddots & \vdots \\ & a & \ddots & a \\ b+(n-1)a & a & \cdots & b \end{vmatrix}$$

$$= b+(n-1)a \begin{vmatrix} 1 & a & \cdots & a \\ \vdots & b & \ddots & \vdots \\ & a & \ddots & a \\ 1 & a & \cdots & b \end{vmatrix}$$

$$= \big(b+(n-1)a\big) \begin{vmatrix} 1 & a & \cdots & a \\ 0 & b-a & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & b-a \end{vmatrix}_n \qquad L_i \leftarrow L_i - L_1 \quad \forall i \in [\![2, n]\!]$$

$$= \big(b + (n-1)a\big)(b-a)^{n-1}$$

2)

$$f : x \mapsto \begin{vmatrix} b+x & c+x & \cdots & c+x \\ a+x & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c+x \\ a+x & \cdots & a+x & b+x \end{vmatrix}$$

Soit $x \in \mathbb{R}$

$$f(x) = \begin{vmatrix} b+x & c+x & \cdots & c+x \\ a+x & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c+x \\ a+x & \cdots & a+x & b+x \end{vmatrix}$$

$C_i \leftarrow C_i - C_1$

$$= \begin{vmatrix} b+x & c-b & \cdots & \cdots & c-b \\ a+x & b-a & c-b & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c-b \\ a+x & 0 & \cdots & 0 & b-a \end{vmatrix}$$

$= \alpha x + \beta$ en développant / C_1

Evaluer en $-a$ et $-c$

$$f(-a) = \begin{vmatrix} b-a & c-a & \cdots & c-a \\ & \ddots & \ddots & \vdots \\ & 0 & \ddots & c-a \\ & & & b-a \end{vmatrix} = (b-a)^n$$

$$f(-c) = \begin{vmatrix} b-c & & & \\ a-c & \ddots & 0 & \\ \vdots & \ddots & \ddots & \\ a-c & \cdots & a-c & b-c \end{vmatrix} = (b-c)^n$$

$$\alpha = \frac{f(-a) - f(-c)}{-a+c} \qquad \beta = f(-a) + \alpha a$$

$$f(x) = \alpha(x+a) + f(-a)$$

$$= \frac{(b-a)^n - (b-c)^n}{-a+c}(x+a) + (b-a)^n$$

En particulier, $f(0) = \dfrac{(b-a)^n - (b-c)^n}{c-a}\, a + (b-a)^n$

3)

Soit $g : c \mapsto \begin{vmatrix} b & & c \\ & \ddots & \\ a & & b \end{vmatrix}$ $\quad a, b$ fixé

$$g(a) = (b-a)^{n-1}(b+(n-1)a)$$

Si $c \neq a$

$$g(c) = \frac{(b-a)^n \times c - a(b-c)^n}{c-a}$$

$$g(a+\underset{\neq 0}{h}) = \frac{(a+h)(b-a)^n - a(b-a-h)^n}{h}$$

$$\overset{b\neq a}{=} \frac{(a+h)(b-a)^n - a(b-a)^n\left(1 - \frac{h}{b-a}\right)^n}{h}$$

$$= \frac{(b-a)^n}{h}\left(a+h - a\left(1 - \frac{h}{b-a}\right)^n\right)$$

$$= \frac{(b-a)^n}{h}\left(\cancel{a} + h - a\left(\cancel{1} - \frac{nh}{b-a} + o(h)\right)\right)$$

$$= \frac{(b-a)^n}{h}\, h\left(1 + \frac{na}{b-a} + o(1)\right)$$

$$= (b-a)^n\left(1 + \frac{na}{b-a} + o(1)\right)$$

$$= (b-a)^{n-1}(b-a+na)$$

$$= (b-a)^{n-1}(b+(n-1)a)$$ Retrouve le truc plus haut

Ex 4:

1) $$\begin{vmatrix} a+b & ab & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & ab \\ 0 & & & 1 & a+b \end{vmatrix} = (a+b) \begin{vmatrix} a+b & ab & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & ab \\ 0 & & & 1 & a+b \end{vmatrix} - \begin{vmatrix} ab & 0 & \cdots & \cdots & 0 \\ a+b & \ddots & \ddots & & \vdots \\ 1 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & ab \\ 0 & & 1 & a+b & ab \end{vmatrix}$$

$$= (a+b) \begin{vmatrix} a+b & ab & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & ab \\ 0 & & & 1 & a+b \end{vmatrix} - ab \begin{vmatrix} a+b & ab & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & ab \\ 0 & & & 1 & a+b \end{vmatrix}$$

$$= (a+b)\, D_{n-1} - ab\, D_{n-2}$$

2) $$\begin{vmatrix} 1 & 1 & \cdots & \cdots & 1 \\ b_1 & a_1 & a_1 & \cdots & a_1 \\ b_2 & b_2 & a_2 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ b_n & \cdots & \cdots & b_n & a_n \end{vmatrix}$$