TD 25 - DL

Ex 1 :

5)

$$\frac{1}{1+\ln(1+x)} = 1 - \ln(1+x) + \ln^2(1+x) + o(\ln^2(1+x)) = 1 - \left(x - \frac{x^2}{2} + o(x^2)\right) + x^2 + o(x^2)$$

$$= 1 - x + \frac{3}{2}x^2 + o(x^2)$$

$$+ (x^2 + o(x^2))\left(1 - x + \frac{3}{2}x^2 + o(x^2)\right)$$

$$= x^2 + o(x^2)$$

$$\frac{x^2 + x - \sin x}{1+\ln(1+x)} \sim x^2$$

$$\frac{x^2 + x - \sin x}{\ln(1+x)} \sim \frac{x^2}{x} = x$$

$$= x + o(x)$$

$$\underbrace{(x^2 + x - \sin x)}_{DL_3} \times \underbrace{\frac{1}{\ln(1+x)}}_{DL_0}$$

$$= \left(x^2 + \frac{x^3}{6} + o(x^3)\right) \times \frac{1}{x - \frac{x^2}{2} + o(x^2)}$$

$$= \left(x^2 + \frac{x^3}{6} + o(x^3)\right) \times \frac{1}{x\left(1 - \frac{x}{2} + o(x)\right)}$$

$$= \left(x + \frac{x^2}{6} + o(x^2)\right) \times \frac{1}{1 - \frac{x}{2} + o(x)} \qquad \frac{1}{1 - \frac{x}{2} + o(x)} = 1 + \frac{x}{2} + o(x)$$

$$= \left(x + \frac{x^2}{6} + o(x^2)\right)\left(1 + \frac{x}{2} + o(x)\right)$$

$$= x + \frac{x^2}{2} + o(x^2) + \frac{x^2}{6} + o(x^2) + o(x^2)$$

$$= x + \frac{2}{3}x^2 + o(x^2)$$

1) $e^x = e^a(x-a) + \frac{e^a}{2}(x-a)^2 + \underset{x \to a}{o}((x-a))$

2) $f : x \mapsto x\sqrt{x} = x^{3/2}$

$$
\begin{aligned}
(a+h)^{3/2} &= \left(a\left(1+\frac{h}{a}\right)\right)^{3/2} \\
&= a^{3/2}\left(1+\frac{h}{a}\right)^{3/2} \\
&= a^{3/2}\left(1 + \frac{3}{2}\frac{h}{a} + \frac{3}{2\times 2}\times\left(\frac{3}{2}-1\right)\left(\frac{h}{a}\right)^2 + o\left(\left(\frac{h}{a}\right)^2\right)\right) \\
&= a^{3/2}\left(1 + \frac{3}{2}\frac{h}{a} + \frac{3}{8}\frac{h^2}{a^2} + o(h^2)\right) \\
&= a^{3/2} + \frac{3}{2}\sqrt{a}\,h + \frac{3}{8\sqrt{a}}h^2 + o(h^2)
\end{aligned}
$$

$\boxed{\text{ou}}$ TY :

$\mathcal{C}^2$ aux voisinages de a

Rq : $x^{3/2} = \underset{0}{\underset{\parallel}{a_0}} + \underset{0}{\underset{\parallel}{a_1}}x + a_2 x^2 + o(x^2)$

$x^{3/2} = a_2 x^2 + o(x^2)$

$\underbrace{\frac{x^{3/2}}{x^2}}_{\xrightarrow[x\to 0]{} +\infty} = a_2 + o(1)$ Absurde

Donc f n'a pas de DL en 0 à l'OD2

3)
$$
\begin{aligned}
\ln\left(\frac{1+x}{1-x+x^2}\right) &= \ln(1+x) - \ln(1-x+x^2) \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) - \left[(-x+x^2) - (-x+x^2)^2\frac{1}{2} + (-x+x^2)^3\frac{1}{3} + \underbrace{o((-x+x^2)^3)}_{o(x^3)}\right) \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) + x - x^2 + \frac{1}{2}(x^2 - 2x^3) - \frac{1}{3}x^3 + o(x^3) \\
&= 2x + x^2\left(-\frac{1}{2} + \frac{1}{2} - 1\right) + x^3\left(\frac{1}{3} - 1 - \frac{1}{3}\right) + o(x^3)
\end{aligned}
$$

erreur calcul

$= 2x - x^2 - \frac{1}{3}x^3 + o(x^3)$

4) $$\frac{1}{(x+3)(1-x)} = \frac{1}{3\left(1+\frac{x}{3}\right)} \times \frac{1}{(1-x)}$$

$$= \frac{1}{3}\left[1 - \frac{x}{3} + \frac{x^2}{9} - \frac{x^3}{27} + o(x^3)\right)\left(1 + x + x^2 + x^3 + o(x^3)\right)$$

$$= \frac{1}{3}\left[1 + \frac{2}{3}x + x^2\left(1 - \frac{1}{3} + \frac{1}{9}\right) + x^3\left(-\frac{1}{27} + 1 - \frac{1}{3}\right) + o(x^3)\right]$$

$$= \frac{1}{3} + \frac{2}{9}x + \frac{7}{27}x^2 + \frac{17}{81}x^3 + o(x^3)$$

6) $$\frac{x \ln x}{x^2 - 1}$$

On pose $h = x - 1$

$$f(1+h) = \frac{(h+1)\ln(h+1)}{(h+1)^2 - 1} = \frac{(h+1)\ln(h+1)}{h(h+2)} = \frac{1}{h}\,\frac{(h+1)\ln(h+1)}{(h+2)}$$

$$= \frac{1}{h}\,\overset{DL_2}{\frac{1}{(h+2)}} \times \underbrace{\overset{DL_3}{(h+1)\ln(h+1)}}_{\sim h}$$

$$\frac{1}{h+2} = \frac{1}{2}\,\frac{1}{1+\frac{h}{2}} = \frac{1}{2}\left[\frac{h}{2} - \left(\frac{h}{2}\right)^2 + o(h^2)\right]$$

$$(h+1)\ln(h+1) = (1+h)\left[h - \frac{h^2}{2} + \frac{h^3}{3} + o(h^3)\right]$$

$$= h + \frac{h^2}{2} + h^3\left(\frac{1}{3} - \frac{1}{2}\right) + o(h^3)$$

$$= h + \frac{h^2}{2} - \frac{1}{6}h^3 + o(h^3)$$

$$\frac{1}{h}\,\frac{1}{(h+2)} \times (h+1)\ln(h+1) = \frac{1}{h} \times \frac{1}{2}\left[\frac{h}{2} - \left(\frac{h}{2}\right)^2 + o(h^2)\right] \times \left(h + \frac{h^2}{2} - \frac{1}{6}h^3 + o(h^3)\right)$$

$$= \frac{1}{2}\left[\frac{h}{2} - \frac{h^2}{4} + o(h^2)\right] \times \left[1 + \frac{h}{2} - \frac{1}{6}h^2 + o(h^2)\right]$$

$$f(1+h) = \frac{1}{2} - \frac{1}{12}h^2 + o(h^2)$$

$$f(x) = \frac{1}{2} - \frac{1}{12}(x-1)^2 + \underset{x \to 1}{o}\left((x-1)^2\right)$$

7) $\sqrt{1-x} - \sqrt{1+x}$

$$\sqrt{1+x} = (1+x)^{1/2}$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{2\times 3}x^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{2\times 4}x^4 + o(x^4)$$

$$= \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{7} + \frac{x^3}{16} - \frac{5}{128}x^4 + o(x^4)$$

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{7} - \frac{x^3}{16} - \frac{5}{128}x^4 + o(x^4)$$

$$f(x) = -x^3 - \frac{x^3}{8} + o(x^4)$$

8) $\ln\left(\frac{1}{\cos x}\right) = -\ln(\cos(x))$

$$= -\ln\Big[1 \underbrace{- \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + o(x^4)}_{h}\Big]$$

$$= -\left[-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right] + \frac{1}{2}\left[-\frac{x^2}{2} + \frac{x^4}{24} + o(x^2)\right]^2 + o\Big(\underbrace{\left(-\frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!}\right)^2}_{=o(x^4)}\Big)$$

$$= \frac{x^2}{2} - \frac{x^4}{24} + o(x^4) + \frac{1}{6}x^4 + o(x^4)$$

$$= \frac{x^2}{2} + \frac{1}{12}x^4 + o(x^4)$$

9) $(\ln(1+x))^2 = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)\right]$

$$\times\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)\right]$$

$$= x^2 + x^3\left[-\frac{1}{2} - \frac{1}{2}\right] + x^4\left[\frac{1}{3} + \frac{1}{3} + \frac{1}{4}\right] + o(x^4)$$

$$= x^2 - x^3 + \frac{11}{12}x^4 + o(x^4)$$

Ex2:

1) $\dfrac{\underbrace{\sin(x)}_{\sim x}\,(\overbrace{\tan x - x}^{=o(x)})}{\underbrace{\ln(1+x)}_{\sim x}} \sim \dfrac{x^3}{3}$

$\tan'(x) = 1 + \tan^2(x) = 1 + x^2 + o(x^2)$

$\tan(x) = x + \dfrac{x^3}{3} + o(x^3)$

$\dfrac{\sin(x)(\tan x - x)}{\ln(1+x)} \xrightarrow[x \to 0]{} 0$

2) $\dfrac{a^x - b^x}{x}$

$a^x = e^{x \ln a} = 1 + x \ln a + o(x)$

$b^x = e^{x \ln b} = 1 + x \ln b + o(x)$

$a^x - b^x = 1 + x \ln a + o(x) - 1 - x \ln b + o(x)$

$\dfrac{a^x - b^x}{x} = \ln a - \ln b + o(1)$