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Multiplicative calculus and its applications

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Abstract

Two operations, differentiation and integration, are basic in calculus and analysis. In fact, they are the infinitesimal versions of the subtraction and addition operations on numbers, respectively. In the period from 1967 till 1970 Michael Grossman and Robert Katz gave definitions of a new kind of derivative and integral, moving the roles of subtraction and addition to division and multiplication, and thus established a new calculus, called multiplicative calculus. In the present paper our aim is to bring up this calculus to the attention of researchers and demonstrate its usefulness.

Keywords: Calculus; Derivative; Integral; Limit; Semigroup; Differential equation; Calculus of variations

1. Introduction

Differential and integral calculus, the most applicable mathematical theory, was created independently by Isaac Newton and Gottfried Wilhelm Leibnitz in the second half of the 17th century. Later Leonard Euler redirect calculus by giving a central place to the concept of function, and thus founded analysis.

Two operations, differentiation and integration, are basic in calculus and analysis. In fact, they are the infinitesimal versions of the subtraction and addition operations on numbers, respectively.

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In the period from 1967 till 1970 Michael Grossman and Robert Katz gave definitions of a new kind of derivative and integral, moving the roles of subtraction and addition to division and multiplication, and thus established a new calculus, called multiplicative calculus. Sometimes, it is called an alternative or non-Newtonian calculus as well. Unfortunately, multiplicative calculus is not so popular as the calculus of Newton and Leibnitz although it perfectly answers to all conditions expected from a theory that can be called a calculus. We, the authors of this paper, think that the gap is insufficient advertising of multiplicative calculus. We can account only two related papers [1,2].

Multiplicative calculus has a relatively restrictive area of applications than the calculus of Newton and Leibnitz. Indeed, it covers only positive functions. Therefore, one can ask whether it is reasonable to develop a new tool with a restrictive scope while a well-developed tool with a wider scope has already been created. The answer is similar to why do mathematicians use a polar coordinate system while there is a rectangular coordinate system, well-describing the points on a plane. We think that multiplicative calculus can especially be useful as a mathematical tool for economics and finance because of the interpretation given to multiplicative derivative below in Section 2.

In the present paper our aim is to bring up multiplicative calculus to the attention of researchers in the branch of analysis and demonstrate its usefulness. Sections 2 and 3 contain known results in the main. They are used in Section 4 to demonstrate some applications of multiplicative calculus.

2. Multiplicative derivatives

For motivation, assume that depositing \$$a$ in a bank account one gets \$$b$ after one year. Then the initial amount changes b/a times. How many times it changes monthly? For this, assume that the change for a month is p times. Then for one year the total amount becomes $b = ap^{12}$. Now we can compute p as $p = (b/a)^{1/12}$. Assuming that deposits change daily, at each hour, at each minute, at each second, etc. and the function, expressing its value at different time moments is f, we find the formula

$$\lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}, \tag{1}$$

showing how many times the amount $f(x)$ changes at the time moment x. Comparing (1) with the definition of the derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \tag{2}$$

we observe that the difference $f(x+h)-f(x)$ in (2) is replaced by the ratio $f(x+h)/f(x)$ in (1) and the division by h is replaced by the raising to the reciprocal power $1/h$.

The limit (1) is called the *multiplicative derivative* or, briefly, **derivative of f at x* and it is denoted by $f^*(x)$. If $f^*(x)$ exists for all x from some open set $A \subseteq \mathbb{R}$, where $\mathbb{R}$ denotes the real line, then the function $f^*: A \to \mathbb{R}$ is well defined. The function f^* itself is called the **derivative* of $f: A \to \mathbb{R}$ for which the symbol d^*f/dt can also be used. The *derivative of f^* is called the *second *derivative* of f and it is denoted by f^{**}. In a similar way the *nth *derivative* of f can be defined, which is denoted by $f^{*(n)}$ for $n = 0, 1, \ldots$, with $f^{*(0)} = f$.

If f is a positive function on A and its derivative at x exists, then one can calculate

$$
\begin{aligned}
f^*(x) &= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}\\
&= \lim_{h\to 0}\left(1+\frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}\cdot\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}}\\
&= e^{\frac{f'(x)}{f(x)}} = e^{(\ln\circ f)'(x)},
\end{aligned}
$$

where $(\ln\circ f)(x) = \ln f(x)$. If, additionally, the second derivative of f at x exists, then by an easy substitution we obtain

$$
f^{**}(x) = e^{(\ln\circ f^*)'(x)} = e^{(\ln\circ f)''(x)}.
$$

Here $(\ln\circ f)''(x)$ exists because $f''(x)$ exists. Repeating this procedure n times, we conclude that if f is a positive function and its nth derivative at x exists, then $f^{*(n)}(x)$ exists and

$$
f^{*(n)}(x) = e^{(\ln\circ f)^{(n)}(x)}, \quad n = 0, 1, \ldots. \tag{3}
$$

Note that the formula (3) includes the case $n = 0$ as well because

$$
f(x) = e^{(\ln\circ f)(x)}.
$$

Based on this, the function $f: A \to \mathbb{R}$ is said to be **differentiable* at x or on A if it is positive and differentiable, respectively, at x or on A.

Can we express $f^{(n)}$ in terms of $f^{*(n)}$? A formula, similar to Newton's binomial formula, can be derived. For this, note that by (3), we have

$$
\left(\ln\circ f^{*(n)}\right)(x) = (\ln\circ f)^{(n)}(x) = \left((\ln\circ f)^{(k)}\right)^{(n-k)}(x) = \left(\ln\circ f^{*(k)}\right)^{(n-k)}(x).
$$

Hence, using

$$
f'(x) = f(x)\left(\ln\circ f^*\right)(x),
$$

we calculate

$$
f''(x) = f'(x)\left(\ln\circ f^*\right)(x) + f(x)\left(\ln\circ f^{**}\right)(x)
$$

for f'', and

$$
f'''(x) = f''(x)\left(\ln\circ f^*\right)(x) + 2f'(x)\left(\ln\circ f^{**}\right)(x) + f(x)\left(\ln\circ f^{***}\right)(x)
$$

for f'''. Repeating this procedure n times, we derive the formula

$$
f^{(n)}(x) = \sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-k-1)!}f^{(k)}(x)\left(\ln\circ f^{*(n-k)}\right)(x). \tag{4}
$$

For the constant function $f(x) = c > 0$ on the interval (a, b), where $a < b$, we have

$$
f^*(x) = e^{(\ln c)'} = e^0 = 1, \quad x \in (a, b).
$$

Conversely, if $f^*(x) = 1$ for every $x \in (a, b)$, then from

$$
f^*(x) = e^{(\ln\circ f)'(x)} = 1,
$$

one can easily deduce that $f(x) = \mathrm{const} > 0$, $x \in (a, b)$. Thus, a function is a positive constant on an open interval if and only if its *derivative on this interval is identically 1. Recall that in a similar condition, involving derivative, the neutral element 0 in addition appears instead of the neutral element 1 of multiplication.

Here are some rules of *differentiation:

(a) $(cf)^*(x) = f^*(x)$,
(b) $(fg)^*(x) = f^*(x)g^*(x)$,
(c) $(f/g)^*(x) = f^*(x)/g^*(x)$,
(d) $(f^h)^*(x) = f^*(x)^{h(x)} \cdot f(x)^{h'(x)}$,
(e) $(f \circ h)^*(x) = f^*(h(x))^{h'(x)}$,

where c is a positive constant, f and g are *differentiable, h is differentiable and in part (e) $f \circ h$ is defined. For example, the rule (b) can be proved as follows:

$$\begin{aligned}(fg)^*(x) &= e^{(\ln\circ(fg))'(x)} = e^{(\ln\circ f)'(x)+(\ln\circ g)'(x)}\\ &= e^{(\ln\circ f)'(x)} \cdot e^{(\ln\circ g)'(x)} = f^*(x)g^*(x).\end{aligned}$$

At the same time, the respective rule for sum (and for difference as well) is complicated

$$(f+g)^*(x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}} \cdot g^*(x)^{\frac{g(x)}{f(x)+g(x)}}.$$

Let us formulate some useful results of differential calculus in terms of *derivative. They can be proved by application of the respective results of Newtonian calculus to the function $\ln\circ f$.

Theorem 1 *(Multiplicative Mean Value Theorem). If the function f is continuous on $[a,b]$ and *differentiable on (a,b), then there exists $a<c<b$ such that*

$$\frac{f(b)}{f(a)} = f^*(c)^{b-a}.$$

Corollary 1 *(Multiplicative Tests for Monotonicity). Let the function $f:(a,b)\to\mathbb{R}$ be *differentiable.*

(a) *If $f^*(x) > 1$ for every $x \in (a,b)$, then f is strictly increasing.*
(b) *If $f^*(x) < 1$ for every $x \in (a,b)$, then f is strictly decreasing.*
(c) *If $f^*(x) \geqslant 1$ for every $x \in (a,b)$, then f is increasing.*
(d) *If $f^*(x) \leqslant 1$ for every $x \in (a,b)$, then f is decreasing.*

Corollary 2 *(Multiplicative Tests for Local Extremum). Let the function $f:(a,b)\to\mathbb{R}$ be twice *differentiable.*

(a) *If f takes its local extremum at $c \in (a,b)$, then $f^*(c) = 1$.*
(b) *If $f^*(c) = 1$ and $f^{**}(c) > 1$, then f has a local minimum at c.*
(c) *If $f^*(c) = 1$ and $f^{**}(c) < 1$, then f has a local maximum at c.*

Theorem 2 *(Multiplicative Taylor's Theorem for One Variable). Let A be an open interval and let $f:A\to\mathbb{R}$ be $n+1$ times *differentiable on A. Then for any $x, x+h \in A$, there exists a number $\theta \in (0,1)$ such that*

$$f(x+h) = \prod_{m=0}^{n} \big(f^{*(m)}(x)\big)^{\frac{h^m}{m!}} \cdot \big(f^{*(n+1)}(x+\theta h)\big)^{\frac{h^{n+1}}{(n+1)!}}.$$

The above results can be extended to functions of several variables as well. For simplicity, consider the function $f(x, y)$ of two variables defined on some open subset of $\mathbb{R}^2$ $(= \mathbb{R} \times \mathbb{R})$. We can define partial *derivative of f in x, considering y as fixed, which is denoted by f_x^* or $\partial^* f/\partial x$. In a similar way, the partial *derivative of f in y can be defined, which is denoted by f_y^* or $\partial^* f/\partial y$. One can go on and define higher-order partial *derivatives of f for which the respective *notations are used.

Two results, generalizing the property (e) of *differentiation and Theorem 2, are as follows. They can also be proved by application of the respective results of Newtonian calculus to the function $\ln \circ f$.

Theorem 3 *(Multiplicative Chain Rule). Let f be a function of two variables y and z with continuous partial *derivatives. If y and z are differentiable functions on (a, b) such that $f(y(x), z(x))$ is defined for every $x \in (a, b)$, then*

$$\frac{d^* f(y(x), z(x))}{dx} = f_y^*\big(y(x), z(x)\big)^{y'(x)} f_z^*\big(y(x), z(x)\big)^{z'(x)}.$$

Theorem 4 *(Multiplicative Taylor's Theorem for Two Variables). Let A be an open subset of $\mathbb{R}^2$. Assume that the function $f : A \to \mathbb{R}$ has all partial *derivatives of order $n + 1$ on A. Then for every $(x, y), (x + h, y + k) \in A$ so that the line segment connecting these two points belongs to A, there exists a number $\theta \in (0, 1)$, such that*

$$f(x+h, y+k) = \prod_{m=0}^{n} \prod_{i=0}^{m} f^{*(m)}_{x^i y^{m-i}}(x, y)^{\frac{h^i k^{m-i}}{i!(m-i)!}} \prod_{i=0}^{n+1} f^{*(n+1)}_{x^i y^{n+1-i}}(x + \theta h, y + \theta k)^{\frac{h^i k^{n+1-i}}{i!(n+1-i)!}}.$$

3. Multiplicative integrals

Now let us define an analog of Riemann integral in multiplicative calculus. Let f be a positive bounded function on $[a, b]$, where $-\infty < a < b < \infty$. Consider the partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[a, b]$. Take the numbers $\xi_1, \ldots, \xi_n$ associated with the partition $\mathcal{P}$. The first step in the definition of proper Riemann integral of f on $[a, b]$ is the formation of the integral sum

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

To define the multiplicative integral of f on $[a, b]$ we will replace the sum by product and the product by raising to power

$$P(f, \mathcal{P}) = \prod_{i=1}^{n} f(\xi_i)^{(x_i - x_{i-1})}. \tag{5}$$

The function f is said to be *integrable in the multiplicative sense* or **integrable* if there exists a number P having the property: for every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[a, b]$ such that $|P(f, \mathcal{P}) - P| < \varepsilon$ for every refinement $\mathcal{P}$ of $\mathcal{P}_\varepsilon$ independently on selection of the numbers associated with the partition $\mathcal{P}$. The symbol

$$\int_a^b f(x)^{dx},$$

reflecting the feature of the product in (5), is used for the number P and it is called the *multiplicative integral* or **integral* of f on $[a,b]$. It is reasonable to let

$$\int_a^a f(x)^{dx} = 1 \quad \text{and} \quad \int_b^a f(x)^{dx} = \left(\int_a^b f(x)^{dx}\right)^{-1}.$$

It is easily seen that if f is positive and Riemann integrable on $[a,b]$, then it is *integrable on $[a,b]$ and

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln\circ f)(x)\,dx}. \tag{6}$$

Indeed, since the Riemann integral of $\ln\circ f$ on $[a,b]$ exists, the continuity of the exponential function and

$$P(f,\mathcal{P}) = e^{\sum_{i=1}^n (x_i - x_{i-1})(\ln\circ f)(\xi_i)} = e^{S(\ln\circ f,\mathcal{P})}$$

imply the above statement. Conversely, one can show that if f is Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx}. \tag{7}$$

Some rules of *integration are as follows:

(a) $$\int_a^b \left(f(x)^p\right)^{dx} = \left(\int_a^b f(x)^{dx}\right)^p, \quad p \in \mathbb{R},$$

(b) $$\int_a^b \left(f(x)g(x)\right)^{dx} = \int_a^b f(x)^{dx} \cdot \int_a^b g(x)^{dx},$$

(c) $$\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}},$$

(d) $$\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \cdot \int_c^b f(x)^{dx}, \quad a \leqslant c \leqslant b,$$

where f and g are *integrable on $[a,b]$. For example, the rule (b) follows from

$$\int_a^b \left(f(x)g(x)\right)^{dx} = e^{\int_a^b ((\ln\circ f)(x) + (\ln\circ g)(x))\,dx} = \int_a^b f(x)^{dx} \cdot \int_a^b g(x)^{dx}.$$

Theorem 5 *(Fundamental Theorem of Multiplicative Calculus). The following statements hold*:

(a) *Let* $f:[a,b]\to\mathbb{R}$ *be* **differentiable and let* f^* *be* **integrable. Then*

$$\int_a^b f^*(x)\,dx=\frac{f(b)}{f(a)}.$$

(b) *Let* $f:[a,b]\to\mathbb{R}$ *be* **integrable and let*

$$F(x)=\int_a^x f(t)^{dt},\quad a\leqslant x\leqslant b.$$

If f *is continuous at* $c\in[a,b]$, *then* F *is* **differentiable at* c *and* $F^*(c)=f(c)$.

Proof. Part (a) follows from

$$\int_a^b f^*(x)^{dx}=\int_a^b\left(e^{(\ln\circ f)'(x)}\right)^{dx}=e^{\int_a^b(\ln\circ f)'(x)\,dx}=e^{(\ln\circ f)(b)-(\ln\circ f)(a)}=\frac{f(b)}{f(a)}.$$

For part (b), write $F(x)=e^{\int_a^x(\ln\circ f)(t)\,dt}$, $a\leqslant x\leqslant b$. Then F is *differentiable at the point c of continuity of f. Furthermore,

$$F^*(c)=e^{(\ln\circ F)'(c)}=e^{\frac{F'(c)}{F(c)}}=e^{\frac{F(c)\cdot(\ln\circ f)(c)}{F(c)}}=e^{(\ln\circ f)(c)}=f(c)$$

completing the proof of part (b). □

Theorem 6 *(Multiplicative Integration by Parts). Let* $f:[a,b]\to\mathbb{R}$ *be* **differentiable, let* $g:[a,b]\to\mathbb{R}$ *be differentiable so the* f^g *is* **integrable. Then*

$$\int_a^b\left(f^*(x)^{g(x)}\right)^{dx}=\frac{f(b)^{g(b)}}{f(a)^{g(a)}}\cdot\frac{1}{\int_a^b(f(x)^{g'(x)})^{dx}}.$$

Proof. Use the property (d) of *differentiation and Theorem 5(a). □

4. Applications

4.1. Support to Newtonian calculus

Some facts of Newtonian calculus can be proved easily through multiplicative calculus. For example, the function

$$f(x)=\begin{cases}e^{-\frac{1}{x^2}} & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

called Cauchy's function, is infinitely many times differentiable on $\mathbb{R}$ (although it is not analytic). The proof of this fact by tools of Newtonian calculus is time consuming and, therefore, many authors avoid its proof. Using multiplicative calculus this can be proved relatively easy. Indeed, a verification of this statement is easy at every $x\neq 0$. It is more difficult to verify it at $x=0$.

Multiplicative calculus helps us to prove that $f^{(n)}(0) = 0$ for every $n = 0, 1, \ldots$. For this, let us consider $x > 0$ and show that

$$f^{*(n)}(x) = e^{\frac{(-1)^{n+1}(n+1)!}{x^{n+2}}}, \quad n = 0, 1, \ldots. \tag{8}$$

The formula (8) is true for $n = 0$. Assume that it is true for n and calculate straightforwardly the $(n+1)$st *derivative of f,

$$f^{*(n+1)}(x) = \lim_{h\to 0} e^{(\frac{(-1)^{n+1}(n+1)!}{(x+h)^{n+2}} - \frac{(-1)^{n+1}(n+1)!}{x^{n+2}})\cdot\frac{1}{h}}.$$

By the binomial formula,

$$\begin{aligned}
\ln f^{*(n+1)}(x) &= \lim_{h\to 0} \frac{(-1)^{n+2}(n+1)!\cdot((x+h)^{n+2} - x^{n+2})}{hx^{n+2}(x+h)^{n+2}} \\
&= \lim_{h\to 0} \frac{(-1)^{n+2}(n+1)!\cdot((n+2)hx^{n+1} + \cdots + h^{n+2})}{hx^{n+2}(x+h)^{n+2}} \\
&= \lim_{h\to 0} \frac{(-1)^{n+2}(n+1)!\cdot((n+2)x^{n+1} + \cdots + h^{n+1})}{x^{n+2}(x+h)^{n+2}} \\
&= \frac{(-1)^{n+2}(n+2)!\cdot x^{n+1}}{x^{2n+4}} = \frac{(-1)^{n+2}(n+2)!}{x^{n+3}}.
\end{aligned}$$

Hence,

$$f^{*(n+1)}(x) = e^{\frac{(-1)^{n+2}(n+2)!}{x^{n+3}}}.$$

By induction, (8) holds for every $n = 0, 1, \ldots$.

Next, we use the formula (4) and obtain

$$f^{(n)}(x) = \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}(n-1)!(n-k+1)!\cdot f^{(k)}(x)}{k!(n-k-1)!\cdot x^{n-k+2}}, \quad n = 1, 2, \ldots.$$

Multiple application of this formula yields

$$\frac{f^{(n)}(x)}{x} = f(x)\sum_{m=4}^{N_n} \frac{M_{n,m}}{x^m}, \quad n = 1, 2, \ldots,$$

where $M_{n,m}$ and N_n are integers and $N_n \geqslant 4$. We need not the exact values of these integers since by multiple application of L'Hopital's rule,

$$\lim_{x\to 0+} \frac{f(x)}{x^m} = \lim_{x\to 0+} \frac{e^{-\frac{1}{x^2}}}{x^m} = \lim_{x\to 0+} \frac{1/x^m}{e^{\frac{1}{x^2}}} = \lim_{z\to\infty} \frac{z^{\frac{m}{2}}}{e^z} = 0.$$

Thus,

$$\lim_{x\to 0+} \frac{f^{(n)}(x)}{x} = 0, \quad n = 0, 1, \ldots,$$

implying $f^{(n+1)}(0+) = 0$ whenever $f^{(n)}(0+) = 0$. Since $f(0) = 0$, by induction, we conclude that $f^{(n)}(0+) = 0$ for every $n = 0, 1, \ldots$. Now let $x < 0$. Since f is an even function, we easily deduce $f^{(n)}(0-) = 0$ for every $n = 0, 1, \ldots$. Thus, $f^{(n)}(0) = 0$ for every $n = 0, 1, \ldots$.

4.2. Semigroups of linear operators

Consider the linear differential equation

$$y'(x) = a(x)y(x), \quad x > 0, \qquad y(0) = y_0 > 0.$$

We can write it in the form $y'(x)/y(x) = a(x)$ or $e^{(\ln\circ y)'(x)} = e^{a(x)}$, assuming that its solution is positive. Thus,

$$y^*(x) = e^{a(x)}, \quad x > 0, \qquad y(0) = y_0,$$

and we can express the solution of the above differential equation in terms of *integral as

$$y(x) = y_0 \int_0^x \left(e^{a(t)}\right)^{dt}, \quad x \geqslant 0.$$

Thus, *integration is closely related to the representation theory of linear differential equations and can be used (after respective generalization) for construction of semigroups of linear bounded operators.

4.3. Multiplicative spaces

The concepts of *derivative and *integral are based on the ordinary limit operation. We can define a multiplicative limit or *limit as well. Given $x \in \mathbb{R}^+ = (0, \infty)$, we let the multiplicative absolute value of x be a number $|x|^*$ such that

$$|x|^* = \begin{cases} x & \text{if } x \geqslant 1, \\ 1/x & \text{if } x < 1. \end{cases}$$

This allows to define the multiplicative distance $d^*(x, y)$ between $x, y \in \mathbb{R}^+$ as

$$d^*(x, y) = \left|\frac{x}{y}\right|^*.$$

The following properties of multiplicative distance are obvious:

(a) $\forall x, y \in \mathbb{R}^+$, $d^*(x, y) \geqslant 1$,
(b) $d^*(x, y) = 1$ if and only if $x = y$,
(c) $\forall x, y \in \mathbb{R}^+$, $d^*(x, y) = d^*(y, x)$,
(d) (**triangle inequality*) $\forall x, y, z \in \mathbb{R}^+$, $d^*(x, z) \leqslant d^*(x, y)d^*(y, z)$.

On the base of this, one can define multiplicative metric spaces as alternative to the ordinary metric spaces. In particular, $\mathbb{R}^+$ is a multiplicative metric space and a sequence $\{x_n\}$ in $\mathbb{R}^+$, converges to $x \in \mathbb{R}^+$ in the multiplicative sense if for every $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d^*(x_n, x) < \varepsilon$ for every $n > N$. In fact, the convergence in $\mathbb{R}^+$ in both multiplicative and ordinary senses are equivalent. But they may be different in more general cases.

Another multiplicative metric space can be defined on the base of the collection $\mathbb{M}_n^+$ of positive $(n \times n)$-matrices. An $(n \times n)$-matrix A is said to be positive if $\mathbf{x}^{\mathrm{T}} A\mathbf{x} > 0$ for every n-vector $\mathbf{x}$, where $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$. If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of a positive $(n \times n)$-matrix A, then they are positive numbers. One can define a multiplicative norm of A as

$$\|A\|^* = \prod_{i=1}^n |\lambda_i|^*,$$

and the multiplicative distance between A and B as $d^*(A, B) = \|AB^{-1}\|^*$.

4.4. Multiplicative differential equations

It is natural to call the equation

$$y^*(x) = f\big(x, y(x)\big), \tag{9}$$

containing the *derivative of y, as a *multiplicative differential equation*. This equation has sense if f is a positive function defined on some subset G of $\mathbb{R} \times \mathbb{R}^+$. To give a theorem on existence and uniqueness of its solution, satisfying the condition

$$y(x_0) = y_0, \tag{10}$$

we need the analog of the Lipschitz condition in multiplicative case. Theorem 1 tells us that such a condition for f should have the form

$$\forall (x, y), (x, z) \in G, \quad \left|\frac{f(x, y)}{f(x, z)}\right|^* \leqslant L^{|y-z|}, \tag{11}$$

where $L > 1$ is a constant. In particular, if f has a partial *derivative f_y^* in its second variable and f_y^* is bounded in the multiplicative sense, i.e., there exists $M > 1$ such that

$$\forall (x, y) \in G, \quad \big|f_y^*(x, y)\big|^* \leqslant M,$$

then $f(x, y)$ satisfies the multiplicative Lipschitz condition.

Theorem 7 *(Multiplicative Differential Equation). Let f be a continuous function on the bounded open region G in $\mathbb{R} \times \mathbb{R}^+$ to (a, b), where $0 < a < b < \infty$. Assume that f satisfies the multiplicative analog of the Lipschitz condition given in* (11)*. Take $(x_0, y_0) \in G$. Then there exists $\varepsilon > 0$ such that Eq.* (9) *has a unique solution $y : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}^+$ satisfying the condition* (10)*.*

Proof. The multiplicative differential equation (9) can be transformed to the following ordinary differential equation

$$y'(x) = y(x) \ln f\big(x, y(x)\big). \tag{12}$$

Let us show that the function $F(x, y) = y \ln f(x, y)$, $(x, y) \in G$, satisfies the ordinary Lipschitz condition in y,

$$\begin{aligned}
\big|F(x, y) - F(x, z)\big| &\leqslant \big|y \ln f(x, y) - y \ln f(x, z)\big| + \big|y \ln f(x, z) - z \ln f(x, z)\big| \\
&\leqslant \left|y \ln \frac{f(x, y)}{f(x, z)}\right| + \big|\ln f(x, z)\big| \cdot |y - z| \\
&= y \ln \left|\frac{f(x, y)}{f(x, z)}\right|^* + \big|\ln f(x, z)\big| \cdot |y - z| \\
&\leqslant \big(y \ln L + \big|\ln f(x, z)\big|\big) \cdot |y - z|.
\end{aligned}$$

Here $\ln \circ f$ is bounded since f is bounded in the multiplicative sense. Also, y ranges in a bounded set. Hence, F satisfies the ordinary Lipschitz condition in y. We conclude that Eq. (12) has a unique solution satisfying the condition (10). This implies the conclusion of the theorem. □

4.5. Multiplicative calculus of variations

Consider the problem of minimizing the functional

$$J(y)=\int_a^b f\big(y(x),y'(x)\big)^{dx} \tag{13}$$

over all continuously differentiable functions $y(x)$ on $[a,b]$ with fixed end points $y(a)=y_1$ and $y(b)=y_2$. By the formula (6), this problem is equivalent to the minimization of

$$J_0(y)=\ln J(y)=\int_a^b \ln f\big(y(x),y'(x)\big)\,dx,$$

for which the methods of calculus of variations can be used. Nevertheless, we are insisting to use multiplicative methods. At first, let us consider the following multiplicative analog of the fundamental lemma of calculus of variations.

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is a positive continuous function so that*

$$\int_a^b \big(f(x)^{h(x)}\big)^{dx}=1$$

for every infinitely many times differentiable function $h:[a,b]\to\mathbb{R}$, then $f(x)=1$ for every $a\leqslant x\leqslant b$.

Proof. From

$$1=\int_a^b \big(f(x)^{h(x)}\big)^{dx}=e^{\int_a^b h(x)\ln f(x)\,dx},$$

we obtain

$$\int_a^b h(x)\ln f(x)\,dx=0.$$

By the fundamental lemma of calculus of variations, $\ln f(x)=0$ or $f(x)=1$ for every $a\leqslant x\leqslant b$. □

Turning back to the functional (13), assume that $f(y,y')$ has the second-order continuous partial *derivatives in y and y'. Let $h(x)$ be arbitrary continuously differentiable function on $[a,b]$ satisfying $h(a)=h(b)=0$ and let $\varepsilon\in\mathbb{R}$. Then

$$1\leqslant\frac{J(y+\varepsilon h)}{J(y)}=\frac{\int_a^b f(y(x)+\varepsilon h(x),y'(x)+\varepsilon h'(x))^{dx}}{\int_a^b f(y(x),y'(x))^{dx}}$$
$$=\int_a^b\left(\frac{f(y(x)+\varepsilon h(x),y'(x)+\varepsilon h'(x))}{f(y(x),y'(x))}\right)^{dx}$$

$$= o^*(\varepsilon) \int_a^b \bigl(f_y^*\bigl(y(x), y'(x)\bigr)^{\varepsilon h(x)} f_{y'}^*\bigl(y(x), y'(x)\bigr)^{\varepsilon h'(x)}\bigr)^{dx}$$

$$= o^*(\varepsilon) \left(\int_a^b \bigl(f_y^*\bigl(y(x), y'(x)\bigr)^{h(x)} f_{y'}^*\bigl(y(x), y'(x)\bigr)^{h'(x)}\bigr)^{dx} \right)^{\varepsilon},$$

where $o^*(\varepsilon)^{1/\varepsilon} \to 1$ if $\varepsilon \to 0$. Raising both sides of the obtained inequality to the power $1/\varepsilon$ and consequently, moving ε to $0+$ and $0-$, one can obtain

$$\int_a^b \bigl(f_y^*\bigl(y(x), y'(x)\bigr)^{h(x)} f_{y'}^*\bigl(y(x), y'(x)\bigr)^{h'(x)}\bigr)^{dx} = 1$$

or

$$\int_a^b \bigl(f_y^*\bigl(y(x), y'(x)\bigr)^{h(x)}\bigr)^{dx} \int_a^b \bigl(f_{y'}^*\bigl(y(x), y'(x)\bigr)^{h'(x)}\bigr)^{dx} = 1.$$

By multiplicative integration by parts formula,

$$\frac{\int_a^b (f_y^*(y(x), y'(x))^{h(x)})^{dx}}{\int_a^b (\frac{d^*}{dx} f_{y'}^*(y(x), y'(x))^{h(x)})^{dx}} = 1$$

or

$$\int_a^b \left(\left(\frac{f_y^*(y(x), y'(x))}{\frac{d^*}{dx} f_{y'}^*(y(x), y'(x))} \right)^{h(x)} \right)^{dx} = 1.$$

By Lemma 1,

$$\frac{f_y^*(y(x), y'(x))}{\frac{d^*}{dx} f_{y'}^*(y(x), y'(x))} = 1$$

or

$$f_y^*\bigl(y(x), y'(x)\bigr) = \frac{d^*}{dx} f_{y'}^*\bigl(y(x), y'(x)\bigr). \tag{14}$$

This is an analog of the Euler–Lagrange equation in the multiplicative case.

As an example, consider a problem of minimizing

$$J(y) = \int_0^1 \bigl(y'(x)^2\bigr)^{dx}$$

over all continuously differentiable functions y on $[0, 1]$ satisfying $y(0) = \lambda$ and $y(1) = \mu$. Here $f(y, y') = (y')^2$. To apply the formula (14), we calculate

$$f_y^*(y, y') = 1, \qquad f_{y'}^*(y, y') = e^{\frac{2}{y'}} \quad \text{and} \quad \frac{d^*}{dx} f_{y'}^*(y, y') = e^{\frac{d}{dx}(\frac{2}{y'})} = e^{-\frac{2y''}{(y')^2}}.$$

By (14), we deduce $2y'' + (y')^2 = 0$. This is a Riccati equation in y' and, hence,

$$y'(x) = \frac{2c_1}{2 + c_1 x},$$

implying

$$y(x) = 2c_1 \ln(2 + c_1 x) + c_2. \tag{15}$$

Here the constants c_1 and c_2 are determined from the system of equations

$$\begin{cases} 2c_1 \ln 2 + c_2 = \lambda, \\ 2c_1 \ln(2 + c_1) + c_2 = \mu. \end{cases} \tag{16}$$

In particular, the case $\lambda = \mu$ implies $c_1 = 0$ and $c_2 = \lambda$, i.e., the function at which (14) takes its minimum is the constant function $y(x) = \lambda$ on [0, 1]. Otherwise, it is the function (15) with c_1 and c_2 defined from (16).

5. Concluding remark

Assume φ is a bijective function of one variable. Define †derivative and †integral of the function f by

$$f^\dagger(x) = \varphi\big(\varphi^{-1} \circ f\big)'(x) \quad \text{and} \quad \int_a^b f(x)\, d^\dagger x = \varphi\left(\int_a^b \big(\varphi^{-1} \circ f\big)(x)\, dx\right),$$

where we assume that the range of f is a subset of the range of φ. On the base of these definitions one can develop †calculus. In fact, there are infinitely many calculi. The multiplicative calculus is one of them. We refer to Grossman and Katz [1] for other interesting calculi.

References

[1] M. Grossman, R. Katz, Non-Newtonian Calculus, Lee Press, Pigeon Cove, MA, 1972.
[2] D. Stanley, A multiplicative calculus, Primus IX (4) (1999) 310–326.