

$$g) F = \{ (u_n) \in \mathbb{R}^{\mathbb{N}} \text{ tq } u \text{ arithmétique}\}$$

$$\rightarrow 0 \in F$$

$$\rightarrow \text{Soit } (u, v) \in F$$

$$\text{Par def il existe } (r, r') \in \mathbb{R}^2 \text{ tq pour tout } n \in \mathbb{N} \begin{cases} u_{n+1} = u_n + r \\ v_{n+1} = v_n + r' \end{cases}$$

$$\text{Soit } n \in \mathbb{N}$$

$$u_{n+1} + v_{n+1} = u_n + r + v_n + r'$$

$$= (u_n + v_n) + (r + r')$$

Donc $u + v$ est une suite arithmétique de raison $(r + r')$ et $(u_n + v_n)_{n \in \mathbb{N}} \in F$

$$\rightarrow \text{Soit } \lambda \in \mathbb{R} \text{ et } n \in \mathbb{N}$$

$$\lambda u_{n+1} = \lambda(u_n + r) = \lambda u_n + \lambda r$$

Donc $(\lambda u_n)_{n \in \mathbb{N}}$ est une suite arithmétique de raison (λr) et $(\lambda u_n)_{n \in \mathbb{N}} \in F$

Famille génératrice $\{(A_n + B)_{n \in \mathbb{N}}, (A, B) \in \mathbb{R}^2\}$

10) Non pas stable par +

$$\text{Contre ex } u = (2^n)_{n \in \mathbb{N}}, v = (3^n)_{n \in \mathbb{N}}$$

$$u + v = (2^n + 3^n)_{n \in \mathbb{N}}$$

$$\frac{u_1 + v_1}{u_0 + v_0} = \frac{5}{2} \neq \frac{u_2 + v_2}{u_1 + v_1} = \frac{13}{5}$$

$$\begin{aligned} M) |(\lambda f + g)(y) - (\lambda f + g)(z)| &\leq |\lambda(f(y) - f(z)) + g(y) - g(z)| \\ &\leq |\lambda| |f(y) - f(z)| + |g(y) - g(z)| \\ &\leq |\lambda| k |z - y| + k' |z - y| \\ &\underbrace{(\leq |\lambda| k + k') |z - y|}_{(1)} \end{aligned}$$

$$13) \text{ Que si } \beta = 0$$

Ex 2:

$$1) \rightarrow 0_{\mathbb{R}^{\mathbb{N}}} \in F \cap G$$

$\rightarrow F$ et G sont inclus dans $\mathbb{R}^{\mathbb{N}}$

\rightarrow Soit $(\lambda, \nu) \in \mathbb{R}$ et $(u, v) \in F^2$

Soit $n \in \mathbb{N}$

$$\begin{aligned} (\lambda u + \nu v)_{2n} &= \lambda u_{2n} + \nu v_{2n} \\ &= \lambda u_{2n} + \nu v_{2n} \in F \end{aligned}$$

Soit $(u, v) \in G^2$

$$(\lambda u + \nu v)_{2n} = \lambda u_{2n} + \nu v_{2n} = 0 \in G$$

2) M₁ F et G sont supplémentaires c.à.d $F \oplus G = E$

$$\text{M}_1 \quad F \cap G = \{0_E\}$$

$$F \cap G = \{(u_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \forall n \in \mathbb{N} \quad u_{2n} = u_{2n+1} = 0\} = \{0_E\}$$

$$\text{M}_2 \quad F \oplus G = \mathbb{R}^{\mathbb{N}}$$

$$\text{Soit } u \in \mathbb{R}^{\mathbb{N}}$$

$$\text{On pose } v \text{ la suite t.q } \forall n \in \mathbb{N} \quad v_{2n} = v_{2n+1} = u_{2n}$$

$$\text{et } w = u - v$$

Vérifions que $v \in F$ / $w \in G$

$$\text{M}_3 \quad w \in G$$

$$\text{Soit } n \in \mathbb{N}$$

$$w_{2n} = u_{2n} - v_{2n} = 0$$

$$\text{donc } w \in G$$

$$\therefore \text{Donc } F \oplus G = E$$

$$\begin{aligned} u &= v + w \quad / \quad w_{2n} = 0 \\ v_{2n} &= v_{2n+1} \\ u_{2n} &= v_{2n} + \cancel{w_{2n}} = v_{2n+1} \\ &\stackrel{=} 0 \end{aligned}$$

Autre:

Soit $\lambda \in \mathbb{R} \setminus \{1\}$

$$K_\lambda = \{u \in \mathbb{R}^N : u_{2n+1} = \lambda u_{2n}\}$$

$$\text{M}\mathcal{F} F \oplus K_\lambda = \mathbb{R}^N$$

Soit $u \in F \cap K_\lambda$

$$\forall n \in \mathbb{N} \quad u_{2n+1} = u_{2n} = \lambda u_{2n}$$

$$\underbrace{(1-\lambda)}_{\neq 0} u_{2n} = 0$$

$$\text{donc } \forall n \in \mathbb{N} \quad u_{2n+1} = u_{2n} = 0$$

$$\text{Donc } F + K_\lambda = F \oplus K_\lambda$$

$$\text{M}\mathcal{F} F \oplus K_\lambda \supset \mathbb{R}^N$$

Soit $u \in \mathbb{R}^N$

$$\text{On pose } w \text{ tq } \forall n \in \mathbb{N} \quad \begin{cases} w_{2n} = \frac{u_{2n+1} - u_{2n}}{\lambda - 1} \\ w_{2n+1} = \lambda w_{2n} \end{cases}$$

$$\text{et } v = u - w$$

$$\begin{aligned} u &= v + w \\ \overline{v}_{2n} &= \overline{w}_{2n+1} = \lambda w_{2n} \end{aligned}$$

$$\begin{aligned} u_{2n+1} - u_{2n} &= \overline{v}_{2n} - \overline{v}_{2n} + w_{2n+1} - w_{2n} \\ &= (\lambda - 1) w_{2n} \end{aligned}$$

$$w_{2n} = \frac{u_{2n+1} - u_{2n}}{\lambda - 1}$$

=

$$\begin{aligned} M_q & \subset K_\lambda \\ w \in F \\ u = w + v \end{aligned}$$

Soit $n \in \mathbb{N}$

$$w_{2n+1} = u_{2n+1} - w_{2n+1} = u_{2n+1} - \lambda \left(\frac{u_{2n+1} - u_{2n}}{\lambda - \lambda} \right)$$

$$= \dots$$

$$= \frac{\lambda u_{2n} - u_{2n+1}}{\lambda - \lambda}$$

$$w_{2n} = u_{2n} - w_{2n} = \dots = \dots$$

$$= w_{2n+1}$$

Donc $w \in F$

$$A = \underbrace{\{u \in \mathbb{R}^{\mathbb{N}} : \forall n \in \mathbb{N} \quad u_{2n+1} = u_{2n} + 3\}}_{\text{pas } u_n \text{ ev}}$$

Ex 3:

$$1) \rightarrow E \subset \mathcal{C}([0,1], \mathbb{R})$$

$$\rightarrow \mathcal{O}_{\mathcal{C}([0,1])} \subset E$$

$\rightarrow M_q$ stable par CL

\rightarrow Trouver supplémentaire.

$$F = \{f \in \mathcal{C}([0,1], \mathbb{R}) \text{ constante}\} \quad \text{Enlève valeur moyenne.}$$

$$M_q | E + F = E \oplus F$$

$$\rightarrow M_q | E \cap F \subset \{0\}$$

Soit $f \in E \cap F$, f constante et $\int_0^1 f(t) dt = 0$

Il existe $c \in \mathbb{R}$ tq $\forall t \in [0,1]$, $f(t) = c$

$$\text{et } \int_0^1 f = \int_0^1 c = c = 0$$

$$\text{et } E+F = E \oplus F$$

$\rightarrow M_E(E+F) \supseteq \mathcal{C}([0,1], \mathbb{R})$

Soit $g \in \mathcal{C}([0,1], \mathbb{R})$

$$\lceil g = f + h$$

$$\int_0^1 h(t) dt = 0$$

$$f = c \in \mathbb{R}$$

$$\int_0^1 g = \int_0^1 f = \int_0^1 c = c$$

$$\text{Donc } c = \int_0^1 g$$

$$\text{et } h = g - \int_0^1 g \quad \lceil$$

$$\text{On pose } f: [0,1] \longrightarrow \mathbb{R} \\ n \mapsto \int_0^1 g$$

$$\text{et } h: [0,1] \longrightarrow \mathbb{R} \\ n \mapsto g(n) - \int_0^1 g(t) dt$$

$$M_F \underset{h \in E}{\cap} \lceil$$

$$g = f + h \lceil$$

$$\int_0^1 h = \int_0^1 g - \int_0^1 \left(\int_0^1 g(t) dt \right) = \int_0^1 g - \int_0^1 g = 0$$

$$\text{Donc } h \in E$$

$$F \oplus E = \mathcal{C}([0,1], \mathbb{R})$$

Complément: $f_0 \in \mathcal{C}([0,1], \mathbb{R}) \setminus E$

$$M_F(E \oplus \mathbb{R} f_0) = \mathcal{C}([0,1], \mathbb{R})$$

$$M_E(E \cap \mathbb{R} f_0) \subset \{0\}$$

Sait $f \in ENRf_0$, $\int_0^1 f = 0$

Il existe $r \in \mathbb{R}$ $f = r f_0$

Dans $\int_0^1 r f_0 = 0 = r \int_0^1 f_0$ $\cancel{\text{est}} \neq 0_R$

donc $r = 0_R$ puis $f = 0_{\mathcal{C}([0,1], \mathbb{R})}$

car $ENRf_0 \subset \{0\}$

donc $E + \mathbb{R}f_0 = E \oplus \mathbb{R}f_0$

Sait $f \in \mathcal{C}([0,1], \mathbb{R})$

$\Gamma_f = \frac{f|_E}{g + h} \in \mathbb{R}f_0$

$$\int_0^1 g = 0$$

Il existe $r \in \mathbb{R}$ tq $h = r f_0$

donc $\int_0^1 f = \int_0^1 r f_0$

$$\int_0^1 f = r \int_0^1 f_0$$

$$r = \frac{\int_0^1 f}{\int_0^1 f_0} \quad \downarrow$$

On pose: $h: [0,1] \rightarrow \mathbb{R}$
 $x \mapsto f_0(x) - \frac{\int_0^x f}{\int_0^x f_0} \neq 0$ car $f_0 \notin E$

$$g: [0,1] \rightarrow \mathbb{R}$$

 $x \mapsto f(x) - h(x)$

Mq $f = g + h$ -

$$g \in E$$

$$h \in \mathbb{R}f_0$$

$$\int_0^1 g = \int_0^1 f - \int_0^1 \left(f_0 - \frac{\int_0^x f}{\int_0^x f_0} \right) = 0$$

Donc $g \in E$ puis $E \oplus \mathbb{R}f_0 = \mathcal{C}([0,1], \mathbb{R})$

$$2) E = \{ f \in C([0, 1], \mathbb{R}) \mid \int_0^1 f(t) dt = 0 \text{ et } f(0) = 0 \}$$

On pose $F = \left\{ \begin{array}{l} [0, 1] \rightarrow \mathbb{R} \\ x \mapsto ax + b \end{array}, (a, b) \in \mathbb{R}^2 \right\}$

$$\mathcal{M}_1 \cap E \oplus F = \mathcal{C}([0, 1], \mathbb{R})$$

$$\text{cad } \mathcal{M}_1 \cap E \cap F = \{ 0_{C([0, 1], \mathbb{R})} \}$$

Soit $f \in E \cap F$

$$\text{il existe } (a, b) \in \mathbb{R}^2 \text{ tq } f(x) = ax + b$$

$$\text{on a } f(0) = 0 \text{ donc } b = 0$$

$$\text{De plus } \int_0^1 f = 0$$

$$\text{or } \int_0^1 f(t) dt = \int_0^1 ax dt = 0$$

$$\text{donc } a = 0$$

$$\text{Donc } f = 0_{C([0, 1], \mathbb{R})}$$

$$\Gamma \vdash h = \underbrace{g + f}_{\in F}$$

$$\text{il existe } (a, b) \in \mathbb{R}^2 \text{ tq } f(x) = ax + b$$

$$\text{et on a: } \int_0^1 g(t) dt = 0 \quad \text{et } g(0) = 0$$

$$h(0) = g(0) + f(0) = b$$

$$\int_0^1 g = \int_0^1 h - \int_0^1 f = 0$$

$$\int_0^1 h = \int_0^1 f = \frac{a}{2} + b = \frac{a}{2} + h(0)$$

$$a = 2(\int_0^1 h - h(0)) \quad \square$$

On pose: $f: [0, 1] \rightarrow \mathbb{R}$
 $x \mapsto 2(\int_0^1 h - h(0))x + h(0)$

$$g: [0, 1] \rightarrow \mathbb{R}$$

$$x \mapsto h(x) - f(x)$$

\mathcal{M}_1 $h \in E$
 $g \in F \vee$
 $f = g + h \vee$

$$g(0) = h(0) - f(0)$$

$$= h(0) - h(0) = 0$$

$$\text{et } \int_0^1 f = \int_0^1 g - \int_0^1 h$$

$$= \int_0^1 g - \int_0^1 2t \left(\int_0^t h - h(0) \right) dt - \int_0^1 h(0)$$

$$= \dots$$

$$= 0$$

$$(f_1, f_2) \in (\mathcal{C}([0, 1], \mathbb{R}), \mathbb{E})^2$$

$$\text{tq } f_2 \in \mathbb{R} f_1$$

$$\text{Mq Vect}(f_1, f_2) \oplus E = \mathcal{C}([0, 1], \mathbb{R})$$

Bonus choisir f_1 et f_2

$$3) F = \ker \varphi \quad \varphi: \mathcal{C}([0, 1], \mathbb{R}) \rightarrow \mathbb{R}^3$$

$$f \mapsto \left(f(0), f'(0), \int_0^1 f \right)$$

$$\lim \quad \lim \quad \lim \Rightarrow \ker \varphi \Rightarrow \text{ev}$$

$$G = \left\{ \begin{array}{l} [0, 1] \rightarrow \mathbb{R} \\ t \mapsto at^2 + bt + c \end{array}, (a, b, c) \in \mathbb{R}^3 \right\}$$

$$= \text{Vect} \left(\begin{array}{l} [0, 1] \rightarrow \mathbb{R} \\ t \mapsto t^k \end{array} \right)_{k \in \llbracket 0, 2 \rrbracket}$$

$$\text{Mq } F \cap G \subset \{0\}$$

$$\text{Mq } \mathcal{C}([0, 1], \mathbb{R}) \subset F + G$$

$$\Gamma \vdash h = \underbrace{f}_{\epsilon F} + \underbrace{g}_{\epsilon G}$$

il existe $(a, b, c) \in \mathbb{R}^3$ tq $g: x \mapsto ax^2 + bx + c$

$$h(0) = f(0) + g(0) = c$$

$$h'(0) = f'(0) + g'(0) = b$$

$$\int_0^1 h = \int_0^1 f + \int_0^1 g = \int_0^1 g(t) dt = \left[\frac{1}{3} ax^3 + \frac{1}{2} bx^2 + cx \right]_0^1$$

$$= \frac{1}{3} a + \frac{1}{2} b + c$$

$$4) F = \{(x, y, z, t) \in \mathbb{R}^4 : x=y=z=t\}$$

Si $\mu \in ENF$, $\mu = (x, y, z, t) \in \mathbb{R}^4$

$$\text{alors on a: } \begin{cases} x+y+z+t=0 \\ x=y=z=t \end{cases} \quad \text{done } x=y=z=t=0$$

cad $\mu = (0, 0, 0, 0)$

cad $E+F = E \oplus F$

Soit $w \in \mathbb{R}^4$ tq $w = (x, y, z, t)$

$$\text{On pose } v = \left(\frac{x+y+z+t}{4}, \frac{x+y+z+t}{4}, \frac{x+y+z+t}{4}, \frac{x+y+z+t}{4} \right)$$

$$\text{et } w = v - v$$

Vérifions que $v \in F$, $v \in E$ et $w = v - v$

$$\text{On a } x'+y'+z'+t'=0 \quad \text{avec } v = \left(\underbrace{x}_{w'}, \underbrace{y}_{w'}, \underbrace{z}_{w'}, \underbrace{t}_{w'} \right)$$

done $v \in E$

$$5) F = \{(x, y, z) \in \mathbb{R}^3 : x+y=0\} \quad \text{Base } (-3, -1, 1)$$

M₁ $E+F = E \oplus F$

Soit $(x, y, z) \in \mathbb{R}^3$

$$\begin{cases} x+y+2z=0 \\ y+z=0 \\ x+y=0 \end{cases}$$

$$\text{done } \begin{cases} x=-y \\ z=-y \\ -4y=0 \end{cases}$$

done $(x, y, z) = (0, 0, 0)$

\rightarrow Soit $(x, y, z) \in \mathbb{R}^4$

$$\Gamma(x, y, z) = (x', y', z') + v \quad \text{avec } \underline{x'+y'=0}$$

$$(x, y, z) = \lambda(-3, -1, 1) + (x', y', z')$$

$$= (-3\lambda + x', -\lambda + y', \lambda + z')$$

$$x+y = -4\lambda$$

$$\text{donc } \lambda = -\frac{x+y}{4}$$

$$\text{On pose } v = -\frac{x+y}{4} (-3, -1, 1)$$

$$\text{et } w = (x, y, z) - v$$

Vérifions que $w \in F$, $v \in E$ et $v + w = (x, y, z)$

$$6) F = \{ u \in \mathbb{R}^{\mathbb{N}} \mid u_0 = 0 = u_1 \}$$

$$\text{My } \mathbb{R}^{\mathbb{N}} = F \oplus G$$

$$\text{on } G = \{(a_n + b)_{n \in \mathbb{N}}, (a, b) \in \mathbb{R}^2\}$$

. Soit $u \in F \cap G$

Pour def il existe $(a, b) \in \mathbb{R}^2$ tq $\forall n \in \mathbb{N} \quad u_n = a_n + b$

$$\text{Donc } \begin{cases} u_0 = b \\ u_1 = a + b \end{cases}$$

$$\text{Comme } u_0 = u_1 = 0, \quad a = b = 0$$

$$\text{donc } u = 0_{\mathbb{R}^{\mathbb{N}}}$$

Soit $u \in \mathbb{R}^{\mathbb{N}}$

$$\Gamma \quad u = \underbrace{v}_{\substack{\in F \\ v_0 = v_1 = 0}} + \underbrace{w}_{\substack{\in G \\ w_0 = w_1 = 0}}$$

$$\begin{aligned} u_0 &= v_0 + w_0 = b = 0 \\ u_1 &= v_1 + w_1 = a + b \end{aligned}$$

$$\text{On a } \begin{cases} b = u_0 \\ a = u_1 - u_0 \end{cases} \quad \square$$

$$\text{On pose } \forall n \in \mathbb{N} \quad w_n = ((u_n - u_0)n + u_0)$$

$$\text{et } v = u - w$$

$$\text{My } w \in G \vee$$

$$u \in F$$

$$w + v = u \quad \checkmark$$

$$v_0 = u_0 - w_0 = u_0 - u_0 = 0$$

$$v_1 = u_1 - w_1 = u_1 - ((u_1 - u_0)n + u_0) = 0$$

$$v \in F$$

$$\mathbb{R}^{\mathbb{N}} = E \oplus G$$

$$F = \{ f \in \mathbb{R}^{\mathbb{R}} : f(2) = f(13) = 0 \} \quad G = \{ f \in \mathbb{R}^{\mathbb{R}} \mid f \text{ affine} \}$$

$$F = \{ f \in \mathbb{R}^{\mathbb{R}} : f(2) = f(15) = f(-1) = 0 \} \quad G = \left\{ \begin{array}{l} R \rightarrow R \\ x \mapsto ax^2 + bx + c \end{array}, \quad (a, b, c) \in \mathbb{R}^3 \right\}$$

$$7) E = \{ f \in C^1(\mathbb{R}, \mathbb{R}) \mid f' + 2f = 0 \} = \left\{ \begin{array}{l} \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto Ce^{-2x} \end{array} , C \in \mathbb{R} \right\}$$

$F = \{ f \in C^1(\mathbb{R}, \mathbb{R}) : f(3) = 0 \}$ Degré de liberté infini - 1

$$E \cap F = O_{\mathbb{R}^{\mathbb{R}}}$$

$$\text{M} \text{y } E \oplus F = C^1(\mathbb{R}, \mathbb{R})$$

Soit $h \in C^1(\mathbb{R}, \mathbb{R})$

$$\begin{cases} h = g + f \\ g: x \mapsto Ce^{-2x} \end{cases} \quad f(3) = 0$$

$$\begin{aligned} h(3) &= Ce^{-6} \\ C &= h(3)e^6 \end{aligned}$$

$$\text{On pose } g: x \mapsto h(3)e^{6-2x}$$

$$f = h - g$$

$$\text{M} \text{y } g \in E, f \in F, h = f + g$$

$$f(3) = h(3) - g(3) = h(3) - h(3) = 0$$

$$\text{donc } f \in F$$

Ex 4:

1) \rightarrow avec $\lambda_1 = \lambda_2 = 0$
 \rightarrow deux vecteurs non colinéaires

2) Voir Photo dim 12 h 16

$$\lambda_1 = 0 \quad \lambda_2 = -2 \quad \lambda_3 = 2 \quad \lambda_4 = -1$$

$$2e_3 - e_1 - e_4 = 0$$

3) Voir Photo dim 12 h 21

4) Soit $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$

$$\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$$

$$\text{donc } \forall x \in \mathbb{R} \quad \lambda_1 \cos(x) + \lambda_2 \sin(x) + f = 0$$

$$\text{En particulier } \lambda_1 + \lambda_3 = 0 \quad (x=0) \quad \lambda_2 + \lambda_3 = 0 \quad (x=\frac{\pi}{2}) \quad -\lambda_1 + \lambda_3 = 0 \quad (x=\pi)$$

donc $\lambda_1 = \lambda_2 = \lambda_3 = 0 \Rightarrow$ famille libre

Ex 5:

$$\begin{aligned}
 1) E &= \{(x, y, z, t) \in \mathbb{R}^4 : x+y+z+t=0\} \\
 &= \{(x, y, z, -x-y-z), (x, y, z) \in \mathbb{R}^3\} \\
 &= \{x(1, 0, 0, -1) + y(0, 1, 0, -1) + z(0, 0, 1, -1), (x, y, z) \in \mathbb{R}^3\} \\
 &= \text{Vect} \left(\underbrace{(1, 0, 0, -1)}_{\text{non CL de } \rightarrow}, \underbrace{(0, 1, 0, -1)}_{\text{non colinéaire}}, \underbrace{(0, 0, 1, -1)}_{\text{non colinéaire}} \right) \Rightarrow \text{échelonné} \Rightarrow \text{libre}
 \end{aligned}$$

$$2) E = \{(x, y, z) \in \mathbb{R}^3 : x-y+2z=y+z=0\}$$

$$\begin{aligned}
 &= \{(3y, y, -y, y) \in \mathbb{R}^4\} \\
 &= \text{Vect}(3, 1, -1)
 \end{aligned}$$

$$4) E = \{(an+b)_{n \in \mathbb{N}} \mid (a, b) \in \mathbb{R}^2\}$$

$$\begin{aligned}
 &= \text{Vect} \left(\underbrace{(n)_{n \in \mathbb{N}}}_{\text{"x a"}}, \underbrace{(1)_{n \in \mathbb{N}}}_{\text{"x b"}} \right) \\
 &\quad \xrightarrow{\text{non colinéaire car si colinéaire à 1 alors suite cst}}
 \end{aligned}$$

$$4) \text{ Soit } P \in \mathbb{R}_n[X]$$

$$P(2)=0 \Leftrightarrow \exists Q \in \mathbb{R}_{n-1}[X] \quad P = (X-2)Q$$

$$\Leftrightarrow \exists (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1} : P = (X-2) \sum_{k=0}^{n-1} a_k X^k$$

$$\Leftrightarrow \exists (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1} : P = \sum_{k=0}^{n-1} a_k X^k (X-2)$$

$$E = \text{Vect}(X^k(X-2))_{0 \leq k \leq n-1}$$

$$\text{Pour tout } k \in \{0, \dots, n-1\} \quad \deg X^k (X-2) = k+1$$

Donc la famille est constituée càd de polynômes de degré \neq , elle est libre

$$5) \{f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) : f' + 2f = 0\}$$

$$= \text{Vect} \left(\underbrace{\mathbb{R} \rightarrow \mathbb{R}}_{\substack{x \mapsto e^{-2x} \\ \neq 0}} \right)$$

$$6) \{f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) : f'' + f = 0\} = \text{Vect} \left(\underbrace{\cos, \sin}_{\text{non col}} \right) = 0$$

Ex 6:

1) Soit $n \in \mathbb{N}^*$ et $\alpha_1, \dots, \alpha_n$ n réels distincts

$\forall (f_{\alpha_1}, \dots, f_{\alpha_n})$ libres

Soit $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ tq $\sum_{i=1}^n \lambda_i f_{\alpha_i} = 0$

Quitte à permuter $\alpha_1 < \dots < \alpha_n$

On a $\forall x \in \mathbb{R} \sum_{i=1}^n \lambda_i e^{\alpha_i x} = 0$

donc $\forall x \in \mathbb{R}, \sum_{i=1}^n \lambda_i e^{(\alpha_i - \alpha_n)x} = 0$

$\forall x \in \mathbb{R}, \sum_{i=1}^n \lambda_i e^{(\alpha_i - \alpha_n)x} + \lambda_n = 0$

puis on fait tendre x vers $+\infty$, $\lambda_n = 0$

Supposons l'absurde que $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$

Soit $i_0 = \max \underbrace{\{k \in [1, n] : \lambda_k \neq 0\}}_{\text{partie finie non vide}}$

On a : $\forall x \in \mathbb{R} \sum_{i=1}^{i_0} \lambda_i e^{\alpha_i x} = 0$

donc $\forall x \in \mathbb{R} : \underbrace{\sum_{i=1}^{i_0-1} \lambda_i e^{(\alpha_i - \alpha_{i_0})x}}_{\xrightarrow{n \rightarrow +\infty} \lambda_{i_0}} + \lambda_{i_0} = 0$

Donc $\lambda_{i_0} = 0$ Absurde

Donc $(\lambda_1, \dots, \lambda_n) = (0, 0, 0)$

2) Soit $n \in \mathbb{N}^*$ et $\alpha_1, \dots, \alpha_n$ n réels distincts

$\mu_g(g_{\alpha_1}, \dots, g_{\alpha_n})$ libre

Soit $i \in \llbracket 1, n \rrbracket$

Si $g_{\alpha_i} \in \text{Vect}(g_{\alpha_j})_{j \in \llbracket 1, n \rrbracket \setminus \{i\}}$
pas dérivable dérivable en α_i dérivable en α_i

Soit $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ tq $\sum_{i=1}^n \lambda_i g_{\alpha_i} = 0$

Soit $i \in \llbracket 1, n \rrbracket$

Si $\lambda_i \neq 0$ alors $g_{\alpha_i} = -\sum_{k=1}^n \underbrace{\frac{\lambda_k}{\lambda_i}}_{\substack{\text{non dérivable} \\ \text{en } \alpha_i}} g_{\alpha_k}$
dérivable en α_i

\rightarrow Absurde

$\rightarrow \lambda_i = 0$