

COMP 361: Numerical Methods:
Assignment 3

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1 Newton's method to compute the cube root of 6

Problem: Show how to use Newton's method to compute the cube root of 5. Numerically carry out the first 10 iterations of Newton's method, using $x_0 = 1$. Analytically determine the fixed points of the Newton iteration and determine whether they are attracting or repelling. If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. Draw the x_{k+1} versus x_k diagram, again taking $x_0 = 1$, and draw enough iterations in the diagram, so that the long time behavior is clearly visible. For which values of x_0 will Newton's method converge?

Solution:

1.1 10 first iterations using Newton's method

Cube root of 5 is a zero of the function $g(x)$ such that:
 $g(x) = x^3 - 5$

The first 10 iterations using Newton's method were carried out by using Python with Jupyter Notebook. Check out the source code and presentation in directory: `program.Problem1.ipynb`

```
# The main algorithm
def newton_raphson(f, diff, init_x, max_iter=1000):
    x = init_x
    estimates = []
    listX = [x]
    for i in range(max_iter):
        deltaX = -f(x)/diff(x)
        x = x + deltaX
        listX.append(x)
        estimates.append(x)
    return x, listX, estimates
```

The results after computing are as following:

1.2 Determine fixed points

Newton's method: $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$
Applying Newton's method

$$\begin{aligned} \Rightarrow f(x) &= x - \frac{x^3 - 5}{3x^2} = \frac{3x^3 - x^3 + 5}{3x^2} \\ &= \frac{2x^3 + 5}{3x^2} \end{aligned}$$

To find the fixed points, let $x = f(x)$

$$\begin{aligned}
x &= \frac{2x^3 + 5}{3x^2} \\
3x^3 &= 2x^3 + 5 \\
x^3 &= 5 \\
\Rightarrow x &= \sqrt[3]{5}
\end{aligned}$$

The fixed point of the Newton iteration is $x = \sqrt[3]{5}$

1.3 Fixed point analytically

$$f'(x) = \left(\frac{2x^3 + 5}{3x^2}\right)' = \left(\frac{2}{3}x + \frac{5}{3x^2}\right)' = \frac{2}{3} - \frac{10}{3x^3}$$

$$\Rightarrow |f'(x^*)| = |f'(\sqrt[3]{5})| = \left|\frac{2}{3} - \frac{10}{3(\sqrt[3]{5})^3}\right| = \left|\frac{2}{3} - \frac{10}{15}\right| = 0$$

$|f'(x^*)| = 0 \Rightarrow$ the fixed point is **attracting** and it is converging **quadratically**.

1.4 Fixed point iteration diagram

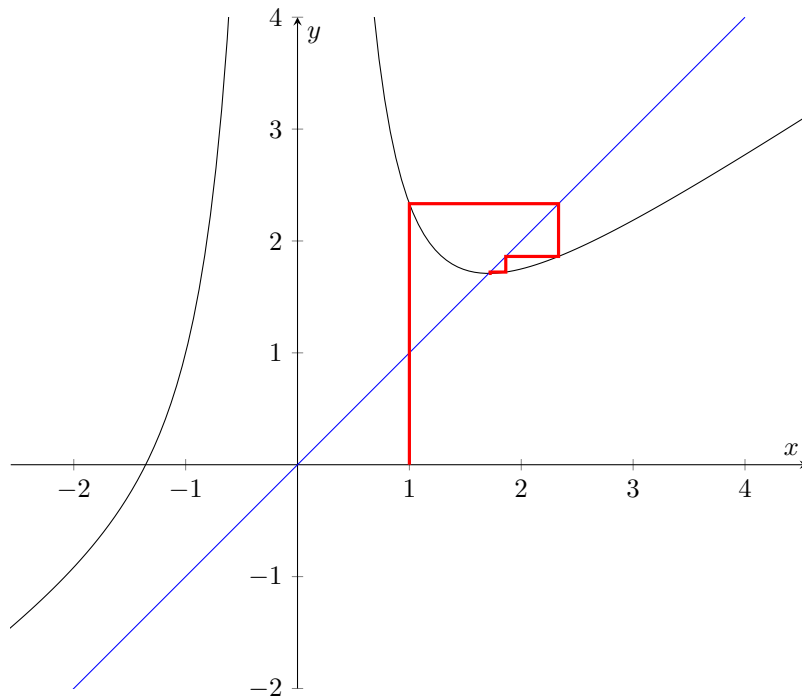


Figure 1: Fixed point iteration diagram

2 Chord method to compute the cube root of 5

Problem: Also use the Chord method to compute the cube root of 5. Numerically carry out the first 10 iterations of the Chord method, using $x^0 = 1$. Analytically determine the fixed points of the Chord iteration and determine whether they are attracting or repelling. If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. If the convergence is linear then determine analytically the rate of convergence. Draw the x^{k+1} versus x^k diagram, as in the Lecture Notes, again taking $x^0 = 1$, and draw enough iterations in the diagram, so that the long time behavior is clearly visible. (If done by hand then make sure that your diagram is sufficiently accurate, for otherwise the graphical results may be misleading.)

Do the same computations and analysis for the Chord Method when $x^0 = 0.1$. More generally, analytically determine all values of x^0 for which the Chord method will converge to the cube root of 5.

Solution:

2.1 With $x_{(0)} = 1$

2.1.1 Numerically carry out first 10 iterations

Python with Jupyter Notebook was being used to calculate the first 10 iterations, the algorithm is below. To check out the full source code, got to file in the directory of program.Problem2.ipynb

```
# Main algorithm
def chord(f, diff, init_x, max_iter=1000):
    x = init_x
    estimates = []
    listX = [x]
    for i in range(max_iter):
        deltaX = -f(x)/diff(init_x)
        x = x + deltaX
        listX.append(x)
        estimates.append(x)
    return x, listX, estimates
```

The results are as following:

```
Xi = 1 X(i+1) = 2.333333333333333
Xi = 2.333333333333333 X(i+1) = -0.23456790123456672
Xi = -0.23456790123456672 X(i+1) = 1.4364009049609154
Xi = 1.4364009049609154 X(i+1) = 2.115184017622358
Xi = 2.115184017622358 X(i+1) = 0.6274038354390186
Xi = 0.6274038354390186 X(i+1) = 2.2117476794083464
Xi = 2.2117476794083464 X(i+1) = 0.271918086443556
Xi = 0.271918086443556 X(i+1) = 1.9318829289112252
Xi = 1.9318829289112252 X(i+1) = 1.1951766954547978
Xi = 1.1951766954547978 X(i+1) = 2.2927610409500208
```

2.1.2 Determine the fixed points

Cube root of 5 is a zero of function $g(x)$ Thus, let $g(x) = x^3 - 5$

Definition 1. Chord method $x^{(k+1)} = x^k - \frac{g(x^k)}{g'(x^{(0)})}$

Apply Chord method with $g(x)$ and $x_0 = 1$:

$$f(x) = x - \frac{x^3 - 5}{3x_0^2} = x - \frac{x^3 - 5}{3} = \frac{3x - x^3 + 5}{3}$$

To find the fixed point of $f(x)$, let $x = f(x)$

$$\begin{aligned} x &= f(x) \\ \implies x &= \frac{3x - x^3 + 5}{3} \\ \implies 3x &= 3x - x^3 + 5 \\ \implies 5 - x^3 &= 0 \\ \implies x &= \sqrt[3]{5} \end{aligned}$$

The fixed point of $f(x)$ is $x = \sqrt[3]{5}$

2.1.3 Analyze fixed points of Chord iteration:

$$\begin{aligned} f'(x) &= \left(x - \frac{x^3 - 5}{3}\right)' \\ &= 1 - x^2 \\ \implies |f'(x_*)| &= |1 - x_*^2| = |1 - (\sqrt[3]{5})^2| = 1.924 > 1 \end{aligned}$$

$|f'(x_*)| > 1$; thus the fixed point is **diverge**

2.1.4 Fixed point iteration diagram

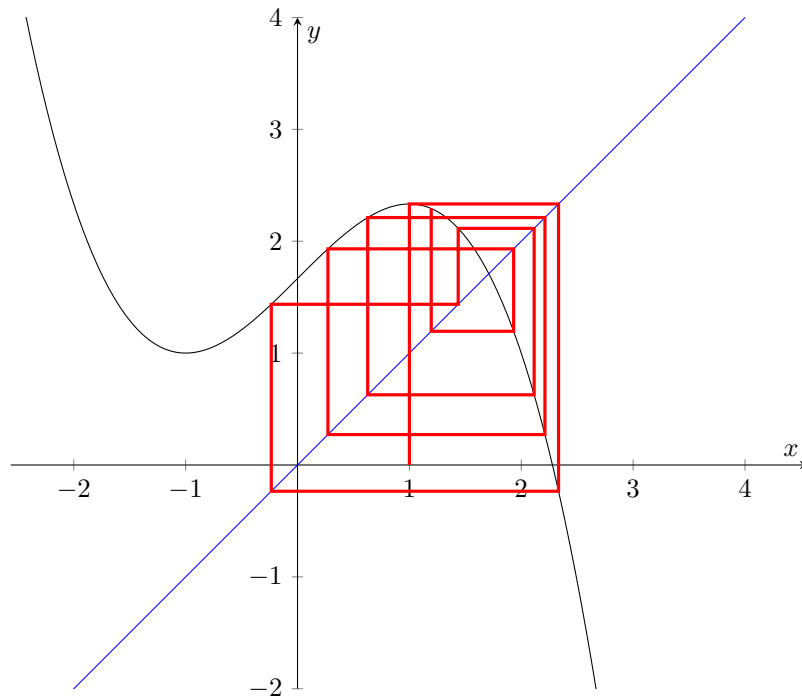


Figure 2: Fixed point iteration no.1 to no. 10 diagram

2.2 With $x^0 = 0.1$

2.2.1 Numerically carry out first 10 iterations

Python with Jupyter Notebook was being used to calculate the first 10 iterations, the algorithm is the same as above question. To check out the full source code, got to file in the directory of program.Problem2.ipynb

The results are as following:

Iteration no. 1 $X_i = 0.1$ $X_{i+1} = 166.733333333333$

Iteration no. 2 $X_i = 166.733333333333$ $X_{i+1} = -154505913.52345666$

Iteration no. 3 $X_i = -154505913.52345666$ $X_{i+1} = 1.229459037686188e+26$

Iteration no. 4 $X_i = 1.229459037686188e+26$ $X_{i+1} = -6.194709380101413e+79$

Iteration no. 5 $X_i = -6.194709380101413e+79$ $X_{i+1} = 7.923946873048754e+240$

Note: More than iteration 5, the result is too big to be computed.

2.2.2 Determine the fixed points

Let $g(x) = x^3 - 5$

Apply Chord method with $g(x)$ and $x_0 = 0.1$:

$$f(x) = x - \frac{x^3 - 5}{3x_0^2} = x - \frac{x^3 - 5}{0.03} = \frac{0.03x - x^3 + 5}{0.03}$$

To find the fixed point of $f(x)$, let $x = f(x)$

$$\begin{aligned} x &= f(x) \\ \implies x &= \frac{0.03x - x^3 + 5}{0.03} \\ \implies 0.03x &= 0.03x - x^3 + 5 \\ \implies x^3 - 5 &= 0 \\ \implies x &= \sqrt[3]{5} \end{aligned}$$

The fixed point of $f(x)$ is $x = \sqrt[3]{5}$

2.2.3 Analyze fixed points of Chord iteration:

$$\begin{aligned} f'(x) &= \left(x - \frac{x^3 - 5}{0.03}\right)' \\ ||f'(x_*)| &= |1 - 100x_*^2| = |1 - 100(\sqrt[3]{5})^2| = 281.401 > 1 \end{aligned}$$

$|f'(x_*)| > 1$; thus the fixed point is **diverge**

2.2.4 Fixed point iteration diagram

The result is too big that it is insufficient to draw such iteration graph

2.3 Analyze the condition of $x^{(0)}$ to make the Chord method converge

To find the fixed point, the equation is:

$$\begin{aligned} x &= x - \frac{x^3 - 5}{3x_0^2} \\ \implies x^3 - 5 &= 0 \implies x = \sqrt[3]{5} \end{aligned}$$

Conclude: the value of the fixed point does not depend on the value of x_0 in Chord method. It is always $\sqrt[3]{5}$

The Chord method converges when:

$$\begin{aligned} \implies |f'(x_*)| < 1 &\implies |(x - \frac{x^3 - 5}{3x_0^2})'| < 1 \\ \implies |1 - \frac{1}{3x_0^2}(3x^2)| < 1 &\implies |1 - \frac{x^2}{x_0^2}| < 1 \textbf{(1)} \\ \frac{x^2}{x_0^2} &\geq 0 \forall x_0 \neq 0 \textbf{(2)} \end{aligned}$$

$$\begin{aligned} \text{From (1) and (2)} &\implies 0 \leq \frac{x^2}{x_0^2} < 1 \\ \implies 0 \leq \frac{(\sqrt[3]{5})^2}{x_0^2} < 1 &\implies \frac{\sqrt[3]{5}^2}{x_0^2} - 1 < 0 \\ \implies \frac{\sqrt[3]{5}^2 - x_0^2}{x_0^2} < 0 &\text{However, } x_0^2 > 0 \forall x_0 \neq 0 \\ \implies \sqrt[3]{5}^2 - x_0^2 < 0 &\implies x_0^2 > \sqrt[3]{5}^2 \implies x_0 > \sqrt[3]{5} \end{aligned}$$

Conclusion: The Chord method converges only when $x_0 > \sqrt[3]{5}$