

**COMP 361/5611 - Elementary Numerical Methods**  
**Assignment 4 - Due Friday, April 3, 2020**

**Problem 1. (40%)**

Consider the unique interpolating polynomial  $p_n(x)$  of degree  $n$  or less that interpolates a function  $f(x)$  at  $n + 1$  *equally spaced interpolation points*

$$\{x_0, x_1, x_2, \dots, x_n\}$$

on an interval  $[a, b]$ , taking  $x_0 = a$  and  $x_n = b$ .

Write a program to do the following: Use the Lagrange basis functions  $\ell_i(x)$ ,  $i = 0, 1, 2, \dots, n$  on Pages 169-172 of the Lecture Notes to evaluate  $p_n(x)$  at  $M + 1$  *equally spaced sampling points*

$$\{y_0, y_1, y_2, \dots, y_M\},$$

where  $y_0 = a$  and  $y_M = b$ , and where  $M$  is much larger than  $n$ . Estimate the maximum interpolation error

$$\max_{[a,b]} |f(x) - p_n(x)|,$$

by computing the *approximate maximum interpolation error*

$$\max_{0 \leq i \leq M} |f(y_i) - p_n(y_i)|.$$

Specifically, do the above for each of the following cases :

$$f(x) = \sin(\pi x), \text{ on the interval } [-1, 1], \text{ (i.e., } a = -1 \text{ and } b = 1),$$

$$f(x) = \frac{1}{1+x^2}, \text{ on the interval } [-2, 2],$$

$$f(x) = \frac{1}{1+x^2}, \text{ on the interval } [-5, 5],$$

successively using  $n = 2, 4, 8, 16$ . In each case use  $M = 500$ .

For each of these 12 cases print the approximate maximum interpolation error.

Also, for each of the three functions  $f(x)$ , give a graph that shows  $f(x)$  and the polynomials  $p_n(x)$ ,  $n = 2, 4, 8, 16$ .

In addition, for the case  $f(x) = \sin(\pi x)$  on the interval  $[-1, 1]$ , use the Lagrange Interpolation Theorem on Page 176 and the Table on Page 183 of the Lecture Notes to derive an upper bound on the maximum interpolation error for  $n = 2, 4, 8, 16$ . Compare this upper bound to the actual (approximate) maximum interpolation error found above.

Give a concise summary and discussion of your findings.

**Problem 2. (30%)**

Give complete details on the derivation of the five-point centered approximation to the second derivative of a function  $f(x)$ , for the example on Pages 236-237 of the Lecture Notes. Also, give complete details on using Taylor expansions to determine the leading error term, as given on Page 237.

**Problem 3. (30%)**

Derive the local *Three-point Gauss Quadrature Formula* for integrating a function  $f(x)$  over the reference interval  $[-1, 1]$ . (This formula uses the roots of the orthogonal polynomial  $e_3(x)$ .)

Use the corresponding *composite formula* to integrate the function  $f(x) = \sin(\pi x)$  over the interval  $[0, 1]$ , using  $N = 2, 4, 8, 16, \dots$ , equally spaced subintervals in  $[0, 1]$ . List the observed errors (the difference between the numerical integral and the exact integral) in a Table.

How many function evaluations are needed for the error to be less than  $10^{-7}$  ?

Present your findings in a concise form, and briefly compare them to the corresponding results in Assignment 1.