COMP 361: Elementary Numerical Methods
Assignment no.3

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Contents

NOTE:
The folder <program> attached in the same directory contains every program for the assignment. Consult the file README.md inside the folder for build guide and list of dependencies.
The programs were written in Python with Jupyter Notebook.

1 Question1

1.1 Problem:

Consider the unique interpolating polynomial $p_n(x)$ of degree n or less that interpolates a function f(x) at n + 1 equally spaced interpolation points

$$x_0, x_1, x_2, \ldots, x_n$$

on an interval [a, b], taking x_0 = a and $x_n = b$.
Write a program to do the following: Use the Lagrange basis functions $l_i(x), i = 0, 1, 2, \ldots n$ on Pages 169-172 of the Lecture Notes to evaluate pn(x) at M + 1 equally spaced sampling points

$$y_0, y_1, y_2, \ldots, y_M,$$

where y0 = a and yM = b, and where M is much larger than n. Estimate the maximum interpolation error

$$max_{[a,b]}|f(x)p_n(x)|$$

by computing the approximate maximum interpolation error

$$max_{0\leq i\leq M}|f(yi)pn(yi)|$$

Specifically, do the above for each of the following cases :

- $f(x) = \sin(\pi x)$, on the interval$[1, 1], i.e., a = 1$ and $b = 1)$
- $\frac{1}{1+x^2}$, on the interval [2, 2]
- $\frac{1}{1+x^2}$, on the interval [5, 5]

successively using n = 2, 4, 8, 16. In each case use M = 500.
For each of these 12 cases print the approximate maximum interpolation error.
Also, for each of the three functions f(x), give a graph that shows f(x) and the polynomials $p_n(x)$, n = 2, 4, 8, 16.
In addition, for the case $f(x) = \sin(\pi x)$ on the interval [1, 1], use the Lagrange Interpolation Theorem on Page 176 and the Table on Page 183 of the Lecture Notes to derive an upper bound on the maximum interpolation error for n = 2, 4, 8, 16. Compare this upper bound to the actual (approximate) maximum interpolation error found above. Give a concise summary and discussion of your findings.

1.2 Lagrange Interpolation

The Lagrange Interpolation was carried out by using Python with Jupyter Notebook. Check out the full source code and presentation in directory: *program/Problem_1.ipynb* Below is the main algorithm:

```
def lagrange(f, listXi, listYi, n):
    def approximate(x):
        listLi = list()
        for i in range(n):
            Li = 1
            for j in range(n):
                if(j==i):
                    continue
                Li = Li * ((x - listXi[j]) / (listXi[i] - listXi[j]))
            listLi.append(Li)

        approx = 0
        for i in range(n):
            approx = approx + listLi[i] * listYi[i]
        return approx

    return approximate
```

To get the list of iteration and the list of points to be examined. The following 2 methods were used:

```
def getListOfEvaluation(a, b, M):
    listInterpolation = list()
    fraction = (b - a)/M
    temp = a
    for i in range(M+1):
        listInterpolation.append(temp)
        temp += fraction
    return listInterpolation

def getListX(a, b, n):
    listX = list()
    fraction = (b - a)/n
    temp = a
    for i in range(n+1):
        listX.append(temp)
        temp += fraction
    return listX
```

After that, *evaluatePerror()* will approximate the maximum error of the interpolation as following:

```
def maxInterpolationError(f, realList, listP, numOfPoint):
    maxErr = 0
    for i in range(numOfPoint):
        err = abs(realList[i] - listP[i])
        if err > maxErr:
            maxErr = err
```

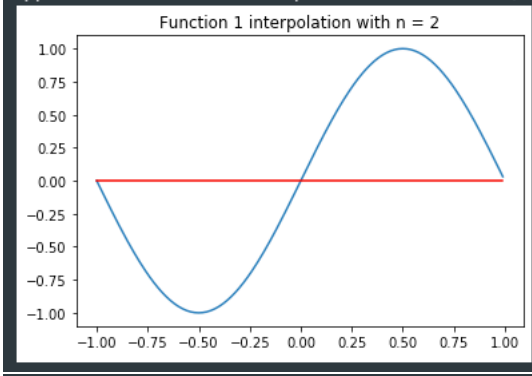
```python
    return maxErr

def evaluatePerror(f, a, b, n, M):
    listX = getListX(a, b, n)
    iterList = getListOfEvaluation(a, b, M)
    Px = lagrange(f, listX, list(map(f, listX)), n)
    listP = list(map(Px, iterList))
    realList = list(map(f, iterList))
    numOfPoint = M + 1
    return maxInterpolationErrorNew(f, realList, listP, numOfPoint)
```

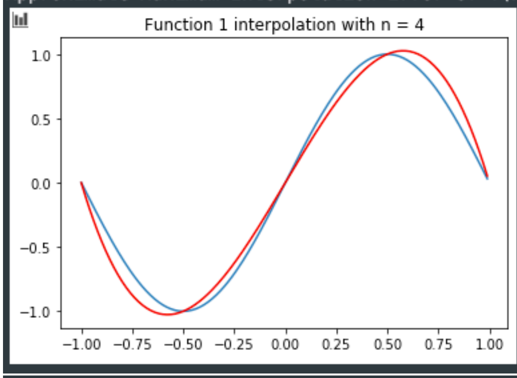
1.3 Function 1 $\sin(\pi x)$ on interval [-1, 1]

The output is as followed:

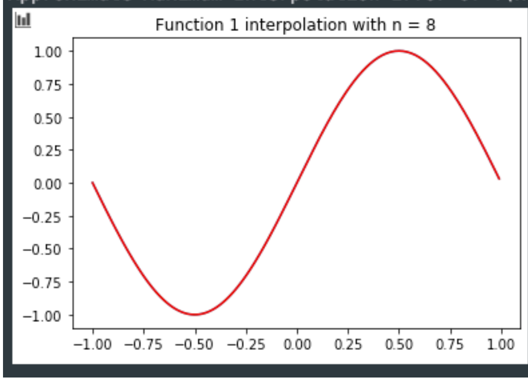
```
FUNCTION F(X) = SIN(PI*X) on interval [-1, 1]
Approximate Maximum Interpolation Error of f(x) with n= 2  is  0.9999999999999999
```



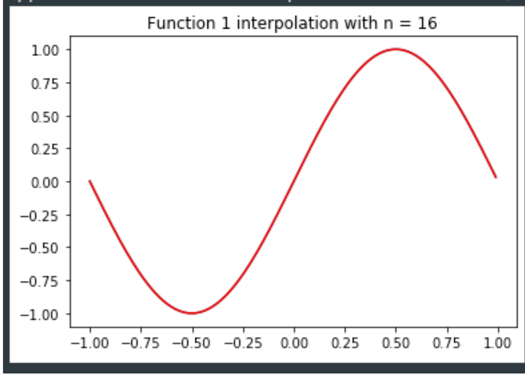

```
------------------------------------------------------------
Approximate Maximum Interpolation Error of f(x) with n= 4  is  0.1807581755118126
```




```
Approximate Maximum Interpolation Error of f(x) with n= 8  is  0.0012055380604188426
```



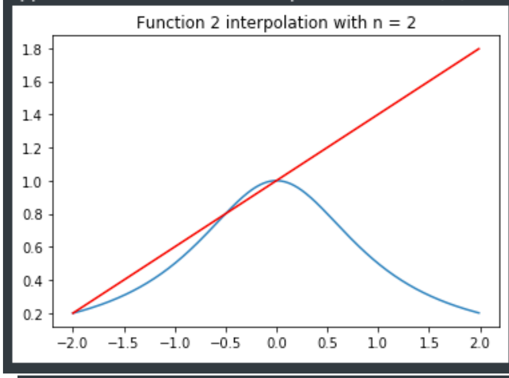
```
Approximate Maximum Interpolation Error of f(x) with n= 16  is  6.652962347697411e-10
```



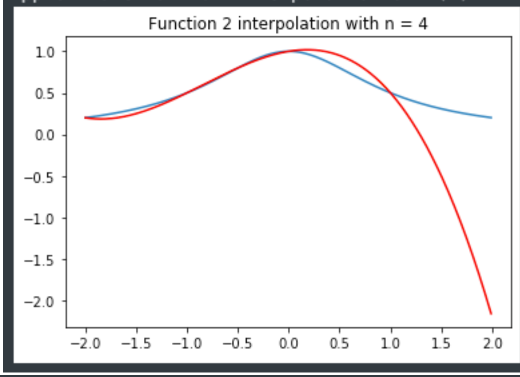
1.4 Function 2: $\frac{1}{1+x^2}$ on interval [-2, 2]

The output is as followed:

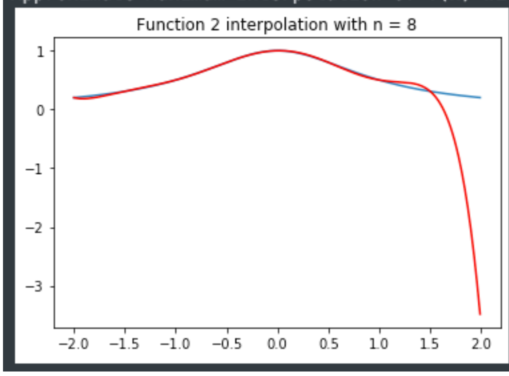
```
FUNCTION F(X) = 1/(1+ x^2) on interval [-2, 2]
Approximate Maximum Interpolation of f(x) with n= 2  is  1.6000000000000019
```

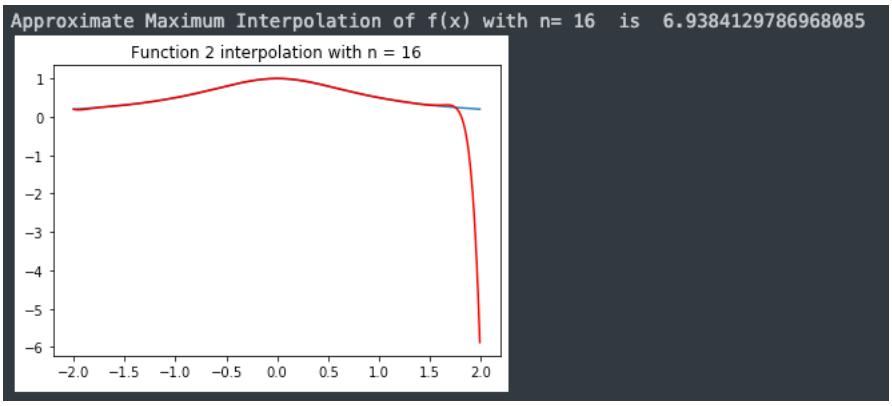



```
Approximate Maximum Interpolation of f(x) with n= 4  is  2.400000000000013
```



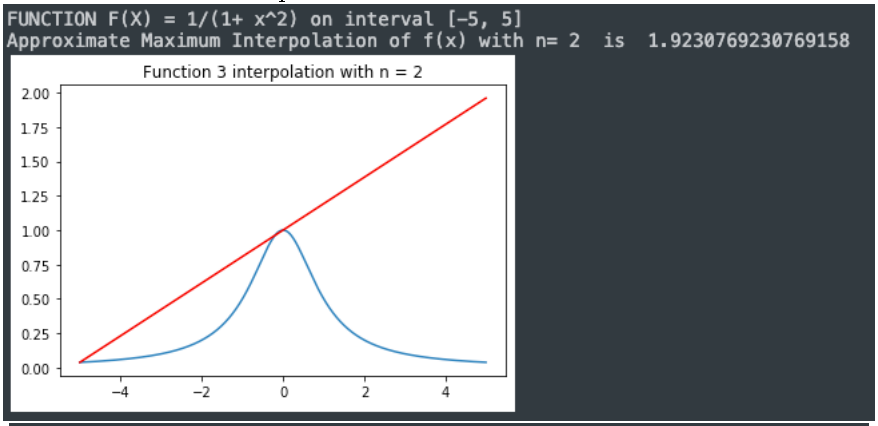

```
Approximate Maximum Interpolation of f(x) with n= 8  is  3.876923076923118
```

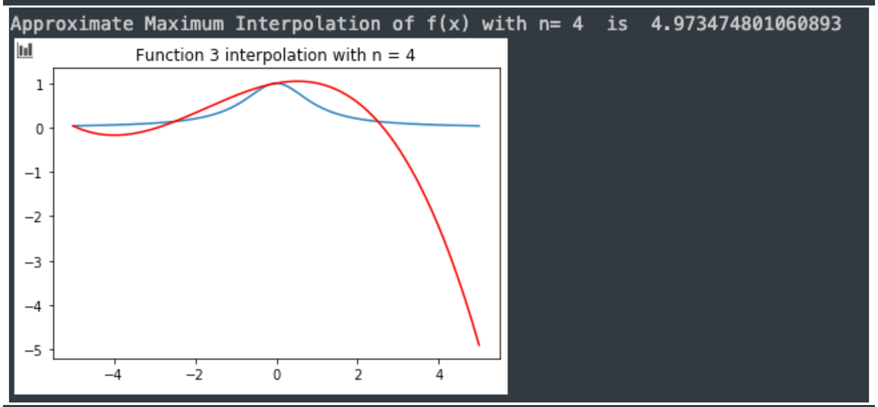


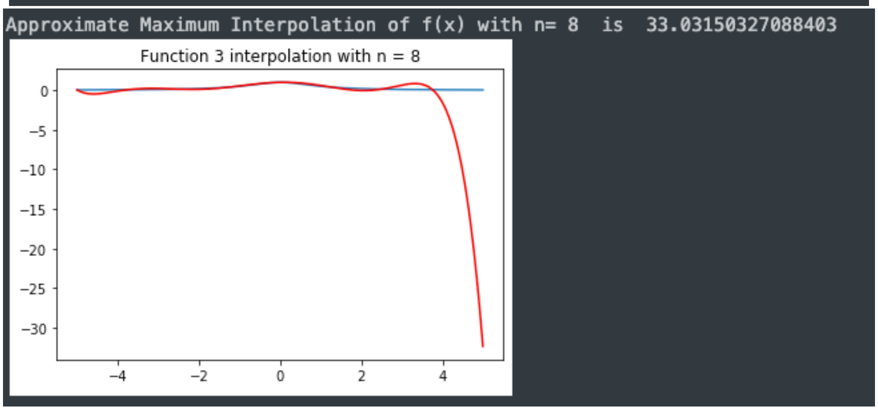

Approximate Maximum Interpolation of f(x) with n= 16 is 6.9384129786968085
Function 2 interpolation with n = 16
1
0
−1
−2
−3
−4
−5
−6
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

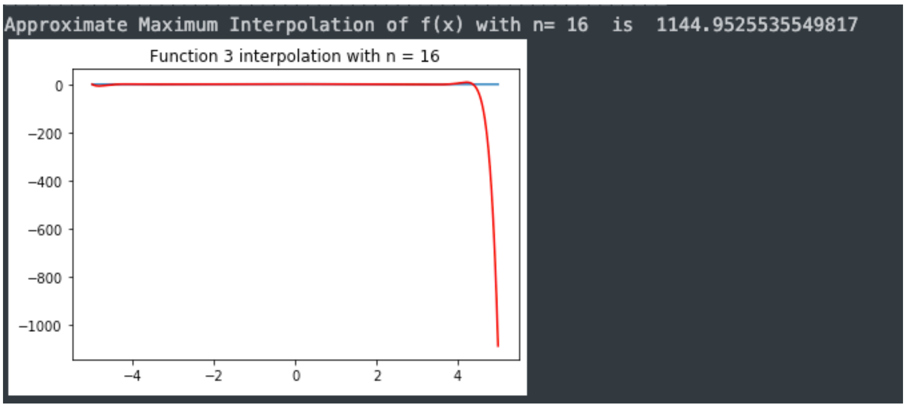
1.5 Function 3: $\frac{1}{1+x^2}$ on interval [-5, 5]

The output is as followed:








Approximate Maximum Interpolation of f(x) with n= 16 is 1144.9525535549817
Function 3 interpolation with n = 16
0
−200
−400
−600
−800
−1000
−4
−2
0
2
4

1.6 Summary:

As observed, since the approximate maximum error could be increased when n increases, it can not be concluded that the more points (n) were examined to create the interpolating function, the smaller the error would be. However, the Lagrange interpolating function does a good job on interpolating the given function in term of shape.
Another conclusion would be interpolating a function by a high degree polynomial is not a good idea.

2 Question 2

2.1 Problem

Give complete details on the derivation of the five-point centered approximation to the second derivative of a function f(x), for the example on Pages 236-237 of the Lecture Notes. Also, give complete details on using Taylor expansions to determine the leading error term, as given on Page 237.

2.2 Complete details on the derivation of the five-point centered approximation

With n = 4, m = 2, x = x_2, and reference interval:

- $x_0 = -2h$
- $x_1 = -h$
- $x_2 = 0$
- $x_3 = h$
- $x_4 = 2h$

2.2.1 $l_0(x_2)''$

$$\begin{aligned}
l_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
&= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{24h^4} \\
l_0'(x) &= \frac{1}{24h^4}((x-x1)(x-x2)(x-x3)+(x-x1)(x-x2)(x-x4) \\
&\quad +(x-x1)(x-x3)(x-x4)+(x-x2)(x-x3)(x-x4)) \\
\implies l_0''(x) &= \frac{1}{24h^4}2(6x^2+x_2x_3+x_2x_4+x_3x_4+ \\
&\quad x_1(x_2+x_3+x_4)-3x(x_1+x_2+x_3+x_4)) \\
\implies l_0''(x_2) &= \frac{1}{12h^4}(6x_2^2+x_2x_3+x_2x_4+x_3x_4+ \\
&\quad x_1(x_2+x_3+x_4)-3x_2(x_1+x_2+x_3+x_4)) \\
&\quad \text{Substitute reference interval} \\
&= \frac{1}{12h^4}(0+0+0+2h^2-h(3h)-0) \\
&= \frac{-h^2}{12h^4} \\
&= \frac{-1}{12h^2}
\end{aligned}$$

2.2.2 $l_1(x_2)''$

$$
\begin{aligned}
l_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \\
l_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{-6h^4} \\
l_1'(x) &= \frac{1}{-6h^4}((x-x_0)(x-x_2)(x-x_3)+(x-x_0)(x-x_2)(x-x_4) \\
&\quad +(x-x_0)(x-x_3)(x-x_4)+(x-x_2)(x-x_3)(x-x_4)) \\
\implies l_1''(x) &= \frac{1}{-6h^4}(2(6x^2+x_2x_3+x_2x_4+x_3x_4 \\
&\quad +x_0(x_2+x_3+x_4)-3x(x_0+x_2+x_3+x_4))) \\
\implies l_1''(x_2) &= \frac{1}{-6h^4}(2(6x_2^2+x_2x_3+x_2x_4+x_3x_4 \\
&\quad +x_0(x_2+x_3+x_4)-3x_2(x_0+x_2+x_3+x_4))) \\
&\text{Substitute reference interval} \\
&= \frac{1}{-6h^4}(2(0+0+0+2h^2-2h(3h)-0)) \\
&= \frac{2(-4h^2)}{-6h^4} \\
&= \frac{4}{3h^2} \\
&= \frac{16}{12h^2}
\end{aligned}
$$

2.2.3 $l_2(x_2)''$

$$
\begin{aligned}
l_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_2)(x_2-x_3)(x_2-x_4)} \\
l_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{4h^4} \\
\implies l_2'(x) &= \frac{1}{4h^4}((x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_1)(x-x_4) \\
&\quad +(x-x_0)(x-x_3)(x-x_4)+(x-x_1)(x-x_3)(x-x_4)) \\
\implies l_2''(x) &= \frac{1}{4h^4}(2(6x^2+x_1x_3+x_1x_4+x_3x_4 \\
&\quad +x_0(x_1+x_3+x_4)-3x(x_0+x_1+x_3+x_4))) \\
\implies l_2''(x_2) &= \frac{1}{4h^4}(2(6x_2^2+x_1x_3+x_1x_4+x_3x_4 \\
&\quad +x_0(x_1+x_3+x_4)-3x_2(x_0+x_1+x_3+x_4))) \\
&\quad \text{Substitute reference interval} \\
&= \frac{1}{4h^4}(2(0-h^2-2h^2+2h^2-2h(-h+h+2h)-0)) \\
&= \frac{1}{4h^4}(2(-h^2-4h^2)) \\
&= \frac{1}{2h^4}(-5h^2) \\
&= \frac{-5}{2h^2} \\
&= \frac{-30}{12h^2}
\end{aligned}
$$

2.2.4 $l_3(x_2)''$

$$
\begin{aligned}
l_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_2)(x_3-x_3)(x_3-x_4)} \\
l_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{-6h^4} \\
\implies l_3'(x) &= \frac{1}{-6h^4}(x-x_0)(x-x_1)(x-x_2)+(x-x_0)(x-x_1)(x-x_4) \\
&+(x-x_0)(x-x_2)(x-x_4)+(x-x_1)(x-x_2)(x-x_4) \\
\implies l_3''(x) &= \frac{1}{-6h^4}(2(6x^2+x_1x_2+x_1x_4+x_2x_4 \\
&+x_0(x_1+x_2+x_4)-3x(x_0+x_1+x_2+x_4))) \\
\implies l_3''(x_2) &= \frac{1}{-6h^4}(2(6x_2^2+x_1x_2+x_1x_4+x_2x_4 \\
&+x_0(x_1+x_2+x_4)-3x_2(x_0+x_1+x_2+x_4))) \\
&\text{Substitute reference interval} \\
&= \frac{1}{-6h^4}(2(0+0-h(2h)+0-2h(-h+2h)-0)) \\
&= \frac{1}{-6h^4}(2(-2h^2-2h^2)) \\
&= \frac{2(-4h^2)}{-6h^4} \\
&= \frac{-8h^2}{-6h^4} \\
&= \frac{16}{12h^2}
\end{aligned}
$$

2.2.5 $l_4(x_2)''$

$$
\begin{aligned}
l_4(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\
l_4(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{24h^4} \\
\implies l_4'(x) &= \frac{1}{24h^4}((x-x_0(x-x_1)(x-x_2)+(x-x_0)(x-x_1)(x-x_3) \\
&+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3))) \\
\implies l_4''(x) &= \frac{1}{24h^4}(2(6x^2+x_1x_2+x_1x_3+x_2x_3 \\
&+x_0(x_1+x_2+x_3)-3x(x_0+x_1+x_2+x_3))) \\
\implies l_4''(x_2) &= \frac{1}{24h^4}(2(6x_2^2+x_1x_2+x_1x_3+x_2x_3 \\
&+x_0(x_1+x_2+x_3)-3x_2(x_0+x_1+x_2+x_3))) \\
&\text{Substitute reference interval} \\
&= \frac{1}{24h^4}(2(0+0+(-h)h+0-h(-h+h)-0)) \\
&= \frac{2(-h^2)}{24h^4} \\
&= \frac{-1}{12h^2}
\end{aligned}
$$

2.3 Complete details on using Taylor expansions to determine the leading error term

$$f''(x_2) \approx \frac{-f_0 + 16f_1 - 30f_2 + 16f_3 - f_4}{12h^2}$$

-

$$\begin{aligned}-f_0 = -[f(x) - (2h)f^{(1)}(x) + \frac{(2h)^2}{2}f^{(2)}(x) - \frac{(2h)^3}{6}f^{(3)}(x) \\ + \frac{(2h)^4}{24}f^{(4)}(x) - \frac{(2h)^5}{120}f^{(5)}(x) \\ + \frac{(2h)^6}{720}f^{(6)}(x)]\end{aligned}$$

3 Question 3

3.1 Problem

Derive the local Three-point Gauss Quadrature Formula for integrating a function f(x) over the reference interval [1, 1]. (This formula uses the roots of the orthogonal polynomial $e_3(x)$.)
Use the corresponding composite formula to integrate the function $f(x) = \sin(\pi x)$ over the interval [0, 1], using N = 2, 4, 8, 16, ů ů ů , equally spaced subintervals in [0, 1]. List the observed errors (the difference between the numerical integral and the exact integral) in a Table.
How many function evaluations are needed for the error to be less than $10^(7)$?

3.2 Local Three-point Gauss Quadrature Formula