COMP 361: Numerical Methods: Assignment 3

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1 Newton's method to compute the cube root of 6

Problem: Show how to use Newtons method to compute the cube root of 5. Numerically carry out the first 10 iterations of Newtons method, using $x_0 = 1$. Analytically determine the fixed points of the Newton iteration and determine whether they are attracting or repelling. If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. Draw the x_{k+1} versus x_k diagram, again taking $x_0 = 1$, and draw enough iterations in the diagram, so that the long time behavior is clearly visible. For which values of x_0 will Newtons method converge?
Solution:

1.1 10 first iterations using Newton's method

Cube root of 5 is a zero of the function g (x) such that:
$g(x) = x^3 - 5$

The first 10 iterations using Newton's method were carried out by using Python with Jupyter Notebook. Check out the source code and presentation in directory: program.Problem1.ipynb

```
# The main algorithm
def newton_raphson(f, diff, init_x, max_iter=1000):
    x = init_x
    estimates = []
    listX = [x]
    for i in range(max_iter):
        deltaX = -f(x)/diff(x)
        x = x + deltaX
        listX.append(x)
        estimates.append(x)
    return x, listX, estimates
```

The results after computing are as following:

1.2 Determine fixed points

Newton's method: $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$
Applying Newton's method

$$\implies f(x) = x - \frac{x^3 - 5}{3x^2} = \frac{3x^3 - x^3 + 5}{3x^2} = \frac{2x^3 + 5}{3x^2}$$

To find the fixed points, let x = f (x)

$$
\begin{aligned}
x &= \frac{2x^3+5}{3x^2} \\
3x^3 &= 2x^3+5 \\
x^3 &= 5 \\
\implies x &= \sqrt[3]{5}
\end{aligned}
$$

The fixed point of the Newton iteration is $x = \sqrt[3]{5}$

1.3 Fixed point analytically

$$f'(x) = (\frac{2x^3+5}{3x^2})' = (\frac{2}{3}x + \frac{5}{3x^2})' = \frac{2}{3} - \frac{10}{3x^3}$$

$$\implies |f'(x^*)| = |f'(\sqrt[3]{5})| = |\frac{2}{3} - \frac{10}{3(\sqrt[3]{5})^3}| = |\frac{2}{3} - \frac{10}{15}| = 0$$

$|f'(x^*)| = 0 \implies$ the fixed point is **attracting** and it is converging **quadratically**.

1.4 Fixed point iteration diagram

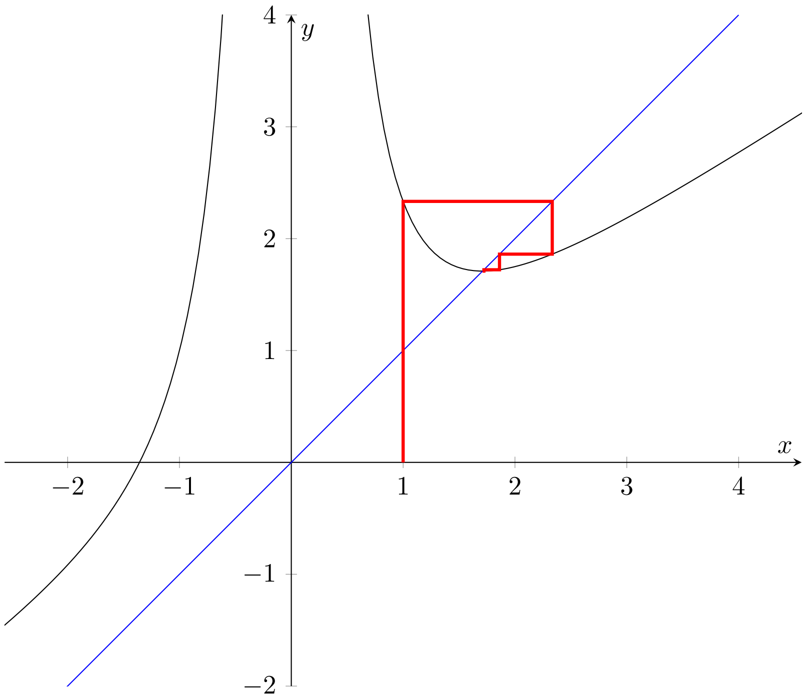

Figure 1: Fixed point iteration diagram

2 Chord method to compute the cube root of 5

Problem: Also use the Chord method to compute the cube root of 5. Numerically carry out the first 10 iterations of the Chord method, using $x^0 = 1$. Analytically determine the fixed points of the Chord iteration and determine whether they are attracting or repelling. If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. If the convergence is linear then determine analytically the rate of convergence. Draw the x^{k+1} versus x^k diagram, as in the Lecture Notes, again taking $x^0 = 1$, and draw enough iterations in the diagram, so that the long time behavior is clearly visible. (If done by hand then make sure that your diagram is sufficiently accurate, for otherwise the graphical results may be misleading.)
Do the same computations and analysis for the Chord Method when $x^0 = 0.1$. More generally, analytically determine all values of x^0 for which the Chord method will converge to the cube root of 5.
Solution:

2.1 With $x_{(0)} = 1$

2.1.1 Numerically carry out first 10 iterations

Python with Jupyter Notebook was being used to calculate the first 10 iterations, the algorithm is below. To check out the full source code, got to file in the directory of program.Problem2.ipynb

```
# Main algorithm
def chord(f, diff, init_x, max_iter=1000):
    x = init_x
    estimates = []
    listX = [x]
    for i in range(max_iter):
        deltaX = -f(x)/diff(init_x)
        x = x + deltaX
        listX.append(x)
        estimates.append(x)
    return x, listX, estimates
```

The results are as following:

Xi = 1 X(i+1) = 2.3333333333333333

Xi = 2.3333333333333333 X(i+1) = -0.23456790123456672

Xi = -0.23456790123456672 X(i+1) = 1.4364009049609154

Xi = 1.4364009049609154 X(i+1) = 2.115184017622358

Xi = 2.115184017622358 X(i+1) = 0.6274038354390186

Xi = 0.6274038354390186 X(i+1) = 2.2117476794083464

Xi = 2.2117476794083464 X(i+1) = 0.271918086443556

Xi = 0.271918086443556 X(i+1) = 1.9318829289112252

Xi = 1.9318829289112252 X(i+1) = 1.1951766954547978

Xi = 1.1951766954547978 X(i+1) = 2.2927610409500208

2.1.2 Determine the fixed points

Cube root of 5 is a zero of function g (x) Thus, let $g(x) = x^3 - 5$

Definition 1. ***Chord method*** $x^{(k+1)} = x^k - \frac{g(x^k)}{g'(x^{(0)})}$

Apply Chord method with g (x) and $x_0 = 1$:

$$f(x) = x - \frac{x^3 - 5}{3x_0^2} = x - \frac{x^3 - 5}{3} = \frac{3x - x^3 + 5}{3}$$

To find the fixed point of f (x), let x = f (x)

$$\begin{aligned} x &= f(x) \\ \implies x &= \frac{3x - x^3 + 5}{3} \\ \implies 3x &= 3x - x^3 + 5 \\ \implies 5 - x^3 &= 0 \\ \implies x &= \sqrt[3]{5} \end{aligned}$$

The fixed point of f (x) is $x = \sqrt[3]{5}$

2.1.3 Analyze fixed points of Chord iteration:

$$\begin{aligned} f'(x) &= (x - \frac{x^3 - 5}{3})' \\ &= 1 - x^2 \\ \implies |f'(x_*)| = |1 - x_*^2| = |1 - (\sqrt[3]{5})^2| &= 1.924 > 1 \end{aligned}$$

$|f'(x_*)| > 1$; thus the fixed point is **diverge**

2.1.4 Fixed point iteration diagram

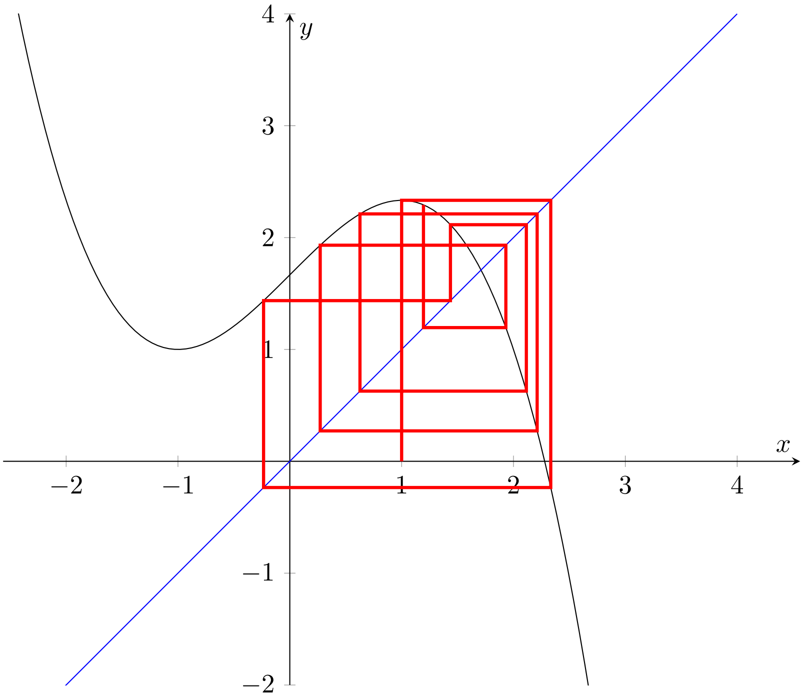

Figure 2: Fixed point iteration no.1 to no. 10 diagram

2.2 With $x^0 = 0.1$

2.2.1 Numerically carry out first 10 iterations

Python with Jupyter Notebook was being used to calculate the first 10 iterations, the algorithm is the same as above question. To check out the full source code, got to file in the directory of program.Problem2.ipynb

The results are as following:

Iteration no. 1 Xi = 0.1 X(i+1) = 166.7333333333333

Iteration no. 2 Xi = 166.7333333333333 X(i+1) = -154505913.52345666

Iteration no. 3 Xi = -154505913.52345666 X(i+1) = 1.229459037686188e+26

Iteration no. 4 Xi = 1.229459037686188e+26 X(i+1) = -6.194709380101413e+79

Iteration no. 5 Xi = -6.194709380101413e+79 X(i+1) = 7.923946873048754e+240

Note: More than iteration 5, the result is too big to be computed.

2.2.2 Determine the fixed points

Let $g(x) = x^3 - 5$

Apply Chord method with g (x) and $x_0 = 0.1$:

$$f(x) = x - \frac{x^3 - 5}{3x_0^2} = x - \frac{x^3 - 5}{0.03} = \frac{0.03x - x^3 + 5}{0.03}$$

To find the fixed point of f (x), let x = f (x)

$$\begin{aligned} x &= f(x) \\ \implies x &= \frac{0.03x - x^3 + 5}{0.03} \\ \implies 0.03x &= 0.03x - x^3 + 5 \\ \implies x^3 - 5 &= 0 \\ \implies x &= \sqrt[3]{5} \end{aligned}$$

The fixed point of f (x) is $x = \sqrt[3]{5}$

2.2.3 Analyze fixed points of Chord iteration:

$$\begin{aligned} f'(x) &= (x - \frac{x^3 - 5}{0.03})' \\ ||f'(x_*)| = |1 - 100x_*^2| &= |1 - 100(\sqrt[3]{5})^2| = 281.401 > 1 \end{aligned}$$

$|f'(x_*)| > 1$; thus the fixed point is **diverge**

2.2.4 Fixed point iteration diagram

The result is too big that it is insufficient to draw such iteration graph

2.3 Analyze the condition of $x^{(0)}$ to make the Chord method converge

To find the fixed point, the equation is:

$$x = x - \frac{x^3 - 5}{3x_0^2}$$

$$\implies x^3 - 5 = 0 \implies x = \sqrt[3]{5}$$

Conclude: the value of the fixed point does not depend on the value of x_0 in Chord method. It is always $\sqrt[3]{5}$

The Chord method converges when:

$$\implies |f'(x_(*))| < 1 \implies |(x - \frac{x^3-5}{3x_0^2})'| < 1$$

$$\implies |1 - \frac{1}{3x_0^2}(3x^2)| < 1 \implies |1 - \frac{x^2}{x_0^2}| < 1\mathbf{(1)}$$

$$\frac{x^2}{x_0^2} \geq 0 \forall x_0 \neq 0\mathbf{(2)}$$

$$\text{From (1) and (2)} \implies 0 \leq \frac{x^2}{x_0^2} < 1$$

$$\implies 0 \leq \frac{(\sqrt[3]{5})^2}{x_0^2} < 1 \implies \frac{\sqrt[3]{5}^2}{x_0^2} - 1 < 0$$

$$\implies \frac{\sqrt[3]{5}^2 - x_0^2}{x_0^2} < 0 \text{ However, } x_0^2 > 0 \forall x_0 \neq 0$$

$$\implies \sqrt[3]{5}^2 - x_0^2 < 0 \implies x_0^2 > \sqrt[3]{5}^2 \implies x_0 > \sqrt[3]{5}$$

Conclusion: The Chord method converges only when $x_0 > \sqrt[3]{5}$

3 Iteration of discrete logistic equation

Problem:
Consider the *discrete logistic equation*
$x^{k+1} = cx^k(1 - x^k)$, k = 0, 1, 2, 4,...
For each of the following values of c, determine analytically the fixed points and whether they are attracting or repelling: c = 0.70, c = 1.00, c = 1.80, c = 2.00, c = 3.30, c = 3.50, c = 3.97. (You need only consider physically meaningful fixed points, namely those that lie in the interval [0, 1].) If a fixed point is attracting then determine analytically if the convergence is linear or quadratic. If the convergence is linear then analytically determine the rate of convergence. For each case include a statement that describes the behavior of the iterations
Solution:

3.1 General analysis

$$\begin{aligned} f'(x) &= (cx_* - cx_*^2)' \\ &= c - 2cx_* \\ &= c(1 - 2x_*) \end{aligned}$$

To determine the fixed point, let $f(x) = x^* = cx^*(1 - x^*)$

$$\begin{aligned} &\implies cx_*^2 + x_* - cx_* = 0 \\ &\implies x_* = 0 \text{ and } x_* = 1 - \frac{1}{c} \end{aligned}$$

- With the fixed point x = 0, $|f'(x)| = |c(1 - 2 * 0)| = c$. Thus, the fixed point is attracting when $0 \le c < 1$, and repelling when $1 < c$
- With the fixed point $x = 1 - \frac{1}{c}$, $|f'(1 - \frac{1}{c})| = c[1 - 2(1 - \frac{1}{c})] = 2 - c$. Thus, the function is attracting when $1 < c < 3$, repelling otherwise.

The iterations are computed to demonstrate the behavior using Jupyter Notebook with Python. The full source code can be found in the directory of *program/Problem3.ipynb* The main algorithm is as following:

```
def f(c, x):
    return c*x*(1 - x)

def logistic_equation(f, c, init_x, max_iter=1000):
    x = init_x
    estimates = []
    listX = [x]
    for i in range(max_iter):
        x = f(c, x)
        listX.append(x)
        estimates.append(x)
    return x, listX, estimates
```

3.1.1 c = 0.70

c = 0.70, meaning fixed points of the function are 0 and $1 - \frac{1}{c} = \frac{-3}{7}$

- The fixed point x = 0 is attracting. $|f'(0)| = 0.7 \implies$ the rate of convergence is 0.7
- The fixed point x = 0.7 is repelling (0.7 < 1)

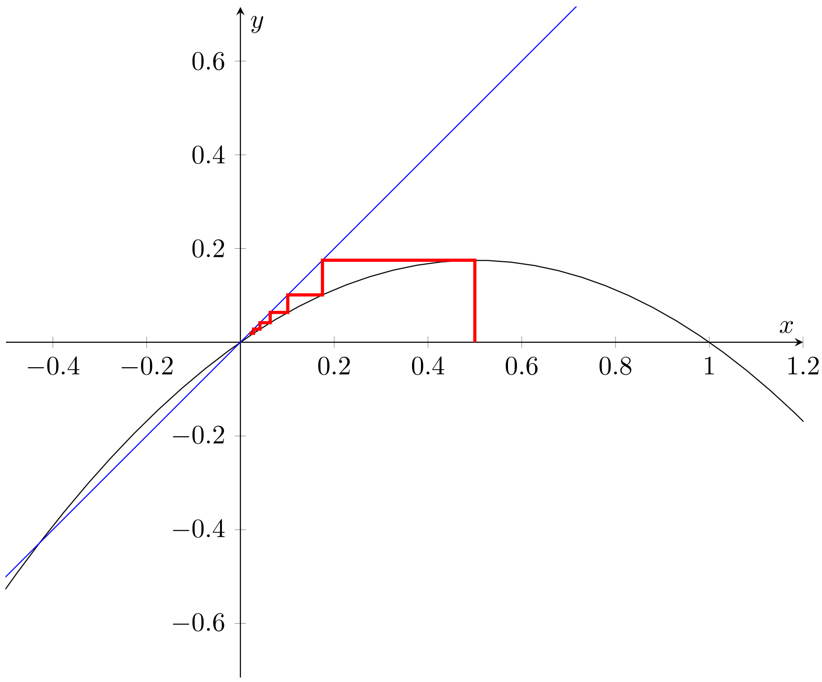

Figure 3: c = 0.7

3.1.2 c = 1.00

- The fixed point x = 0 is attracting. $|f'(0)| = 1 \implies$ the rate of convergence is 1

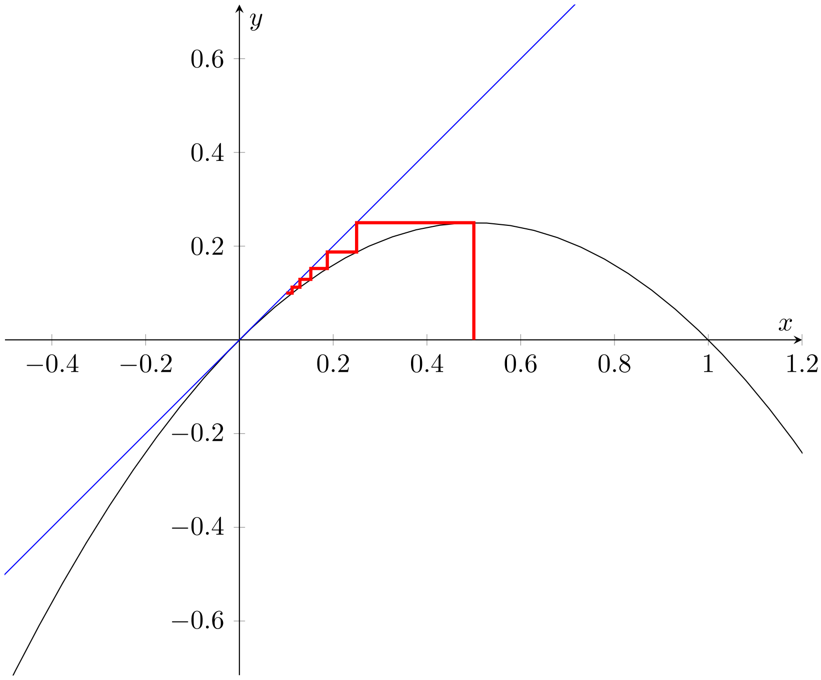

Figure 4: c = 1.00

3.1.3 c = 1.8

- The fixed point x = 0 is repelling. ($|f'(0)| = 1.8$)
- The fixed point x = 0.44 is repelling ($0.44 < 1$)

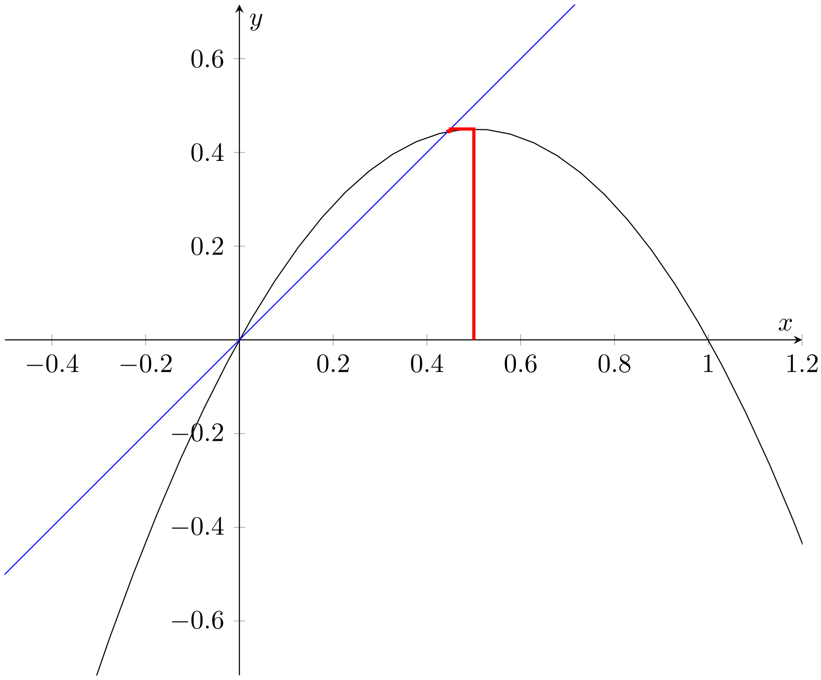

Figure 5: c = 1.80

3.1.4 c = 2.00

- The fixed point x = 0 is repelling. ($|f'(0)| = 2$)
- The fixed point x = 0.5 is repelling ($0.5 < 1$)

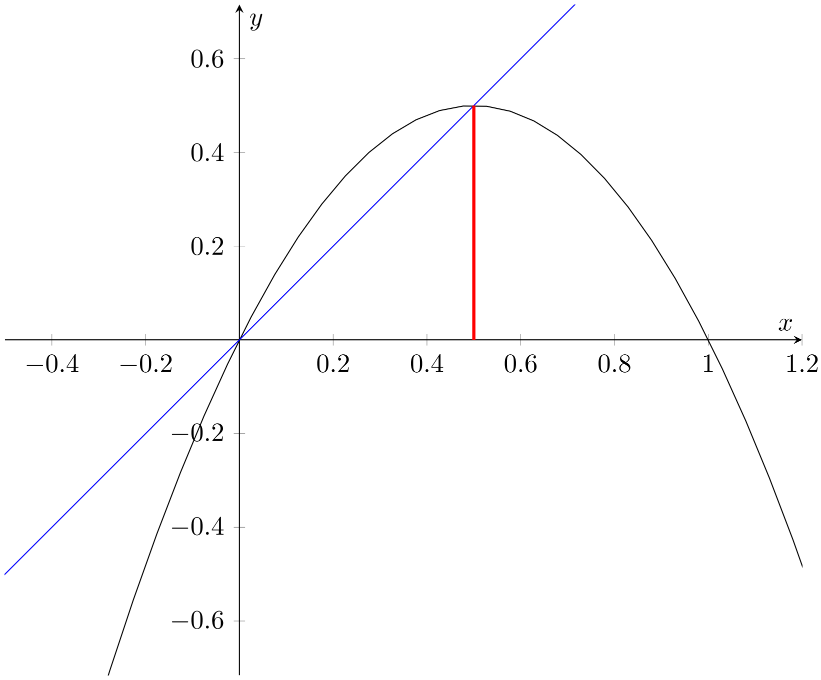

Figure 6: c = 2

3.1.5 c = 3.30

- The fixed point x = 0 is repelling. ($|f'(0)| = 3.3$)
- The fixed point x = 0.69 is repelling ($0.69 < 1$)

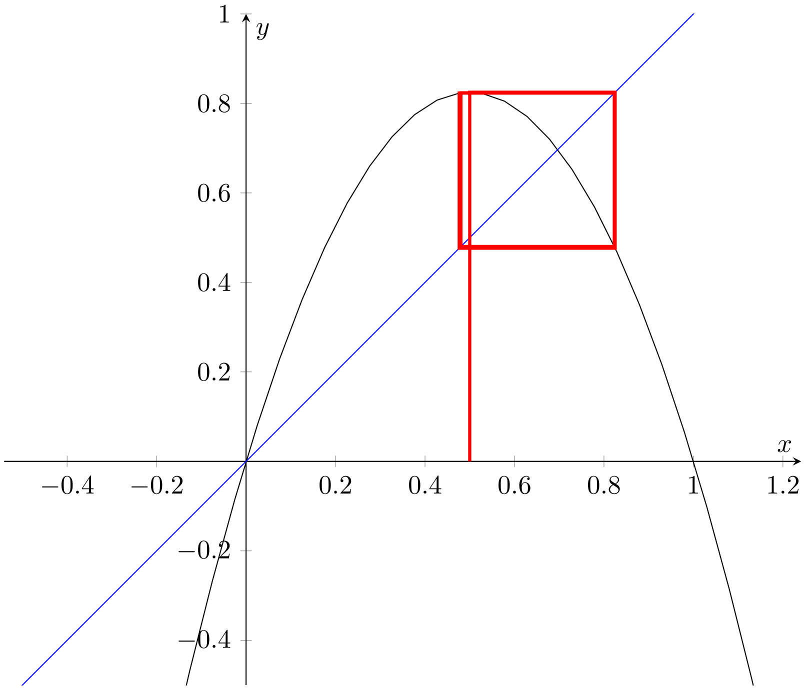

Figure 7: c = 3.3

3.1.6 c = 3.50

- The fixed point x = 0 is repelling. ($|f'(0)| = 3.5$)
- The fixed point x = 0.71 is repelling ($0.71 < 1$)

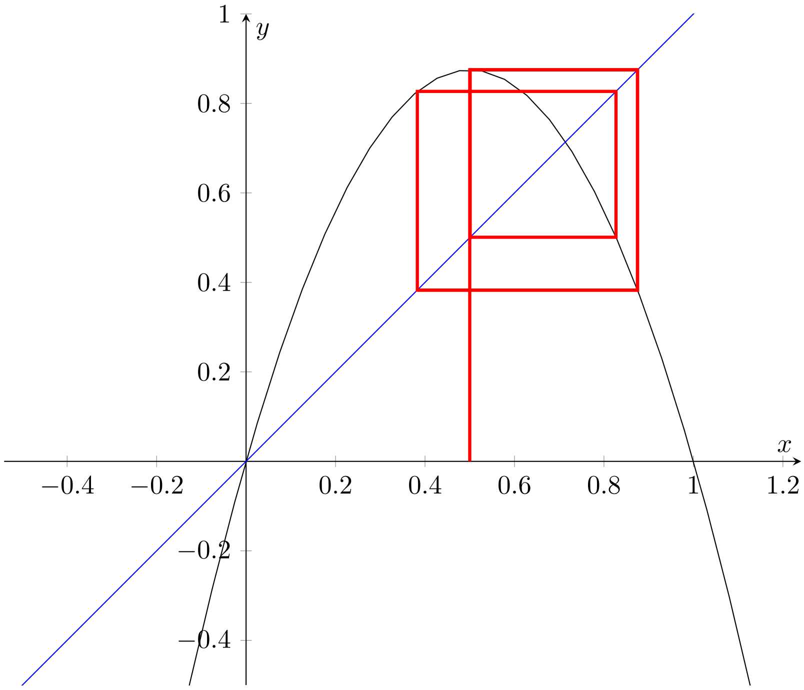

Figure 8: c = 3.5

3.1.7 c = 3.97

- The fixed point x = 0 is repelling. ($|f'(0)| = 3.97$)
- The fixed point x = 0.74 is repelling (0.74 < 1)

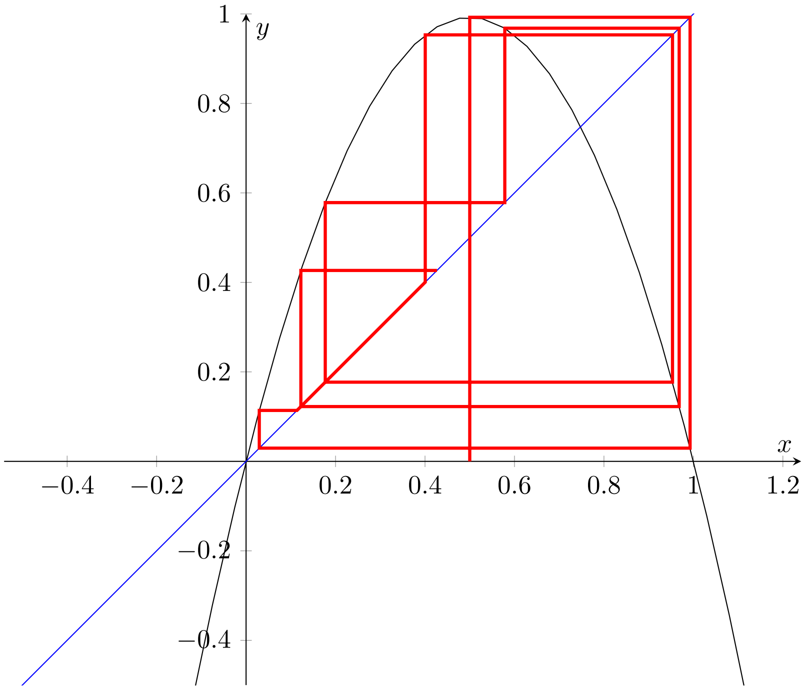

Figure 9: c = 3.97