Theory of Algorithms III

Linear Programming

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1. satisfy a set of linear equations and/or linear inequalities involving these variables, and
2. maximize or minimize a given linear objective function.

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- triangular chocolates, called Pyramide,
- and the more decadent and deluxe Pyramide Nuit.

Q: How much of each should it produce to maximize profits?

- Every box of Pyramide has a a profit of \$1.
- Every box of Nuit has a profit of \$6.
- The daily demand is limited to at most 200 boxes of Pyramide and 300 boxes of Nuit.
- The current workforce can produce a total of at most 400 boxes of chocolate per day.

LP Formulation

`Objective function`	$\max x_1 + 6x_2$
`Constraints`	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 \leq 400$
	$x_1, x_2 \geq 0$

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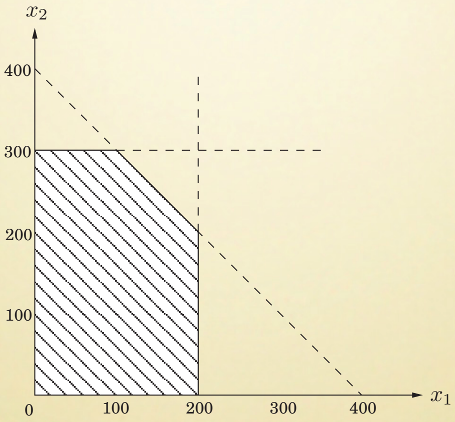
It is a convex polygon.

The Convex Polygon



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Since the goal is to maximize c, we must move the line as far up as possible, while still touching the feasible region.

The optimum solution will be the very last feasible point that the profit line sees and must therefore be a vertex of the polygon.

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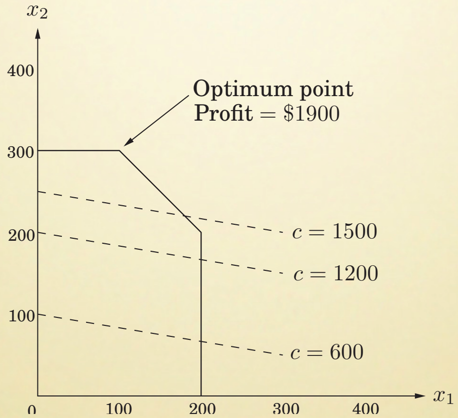

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Solving Linear Programs

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Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts.

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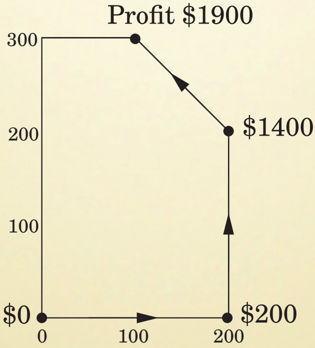
By simple geometry. Since all the vertex's neighbors lie below the line, the rest of the feasible polygon must also lie below this line.

The Example



A Notable Result

Smoothed analysis proposed by Daniel Spielman and Shanghua Teng is a way of measuring the complexity of an algorithm. It gives a more realistic analysis of the practical performance of the algorithm. It was used to explain that the simplex algorithm runs in exponential-time in the worst-case and yet in practice it is a very efficient algorithm, which was one of the main motivations for developing smoothed analysis. The authors received the 2008 Gödel Prize and the 2009 Fulkerson Prize.

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Nuit and Luxe require the same packaging machinery. Luxe uses it three times as much, which imposes another constraint
$x_2 + 3x_3 \leq 600$.

LP

$$\max x_1 + 6x_2 + 13x_3$$
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 + x_3 \leq 400$$
$$x_2 + 3x_3 \leq 600$$
$$x_1, x_2, x_3 \geq 0$$

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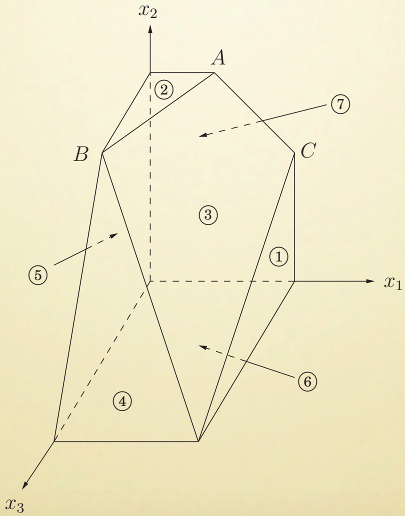
A profit of c corresponds to the plane $x_1 + 6x_2 + 13x_3 = c$.

As c increases, this profit-plane moves parallel to itself, further into the positive orthant until it no longer touches the feasible region.

The Example



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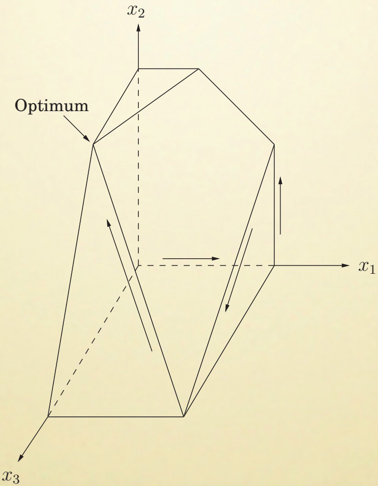
A possible trajectory

$$\frac{(0,0,0)}{\$0} \to \frac{(200,0,0)}{\$200} \to \frac{(200,200,0)}{\$1400} \to \frac{(200,0,200)}{\$2800} \to \frac{(0,300,100)}{\$3100}$$

The Example



ILP and Rounding

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With no initial surplus of carpets.

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1. Overtime. Overtime pay is 80% more than regular pay. Workers can put in at most 30% overtime.
2. Hiring and firing, costing \$320 and \$400, respectively, per worker.
3. Storing surplus production, costing \$8 per carpet per month. Currently without stored carpets on hand, and without any carpets stored at the end of year.

LP Formulations

$$\begin{aligned}
w_i &= \text{number of workers during } i\text{-th month; } w_0 = 30. \\
x_i &= \text{number of carpets made during } i\text{-th month.} \\
o_i &= \text{number of carpets made by overtime in month } i. \\
h_i, f_i &= \text{number of workers hired and fired, respectively,} \\
&\quad\ \text{at beginning of month } i. \\
s_i &= \text{number of carpets stored at end of month } i\text{; } s_0 = 0.
\end{aligned}$$

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$$x_i = 20w_i + o_i$$

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The number of workers can potentially change at the start of each month:

$$w_i = w_{i-1} + h_i - f_i$$

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The number of carpets stored at the end of each month is what we started with, plus the number we made, minus the demand for the month:

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And overtime is limited:

$$o_i \leq 6w_i$$

LP Formulation

The objective function is to minimize the total cost:

$$\min 2000 \sum_i w_i + 320 \sum_i h_i + 400 \sum_i f_i + 8 \sum_i s_i + 180 \sum_i o_i$$

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In the example, most of the variables take on fairly large values, and thus rounding is unlikely to affect things too much.

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In NP problems, finding the optimum integer solution of an LP is an important but very hard problem, called integer linear programming.

Quiz

In the shortest path problem, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the weight of a shortest path from s to t.

Shortest Path in LP

$$\max d_t$$

$$\begin{aligned} d_v \leq d_u + w(u,v) &\quad (u,v) \in E \\ d_s = 0 & \\ d_i \geq 0 &\quad i \in V \end{aligned}$$

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Q: Another formalization?

Shortest Path in LP

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Let $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$; that is, $\mathcal{S}$ is the set of all s-t cuts in the graph. Then we can model the shortest s-t path problem with the following integer program,

$$\min \sum_{e \in E} w_e x_e$$

$$\begin{aligned} \sum_{e \in \delta(S)} x_e \geq 1 \quad & S \in \mathcal{S} \\ x_e \in \{0, 1\} \quad & e \in E \end{aligned}$$

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where $\delta(S)$ is the set of all edges that have one endpoint in S and the other endpoint not in S.

- Can we relax the restriction $x_e \in \{0, 1\}$ to $0 \leq x_e \leq 1$?
- How about $x_e \geq 0$?

Duality

Product Planning Revisit

Recall:

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We take the first inequality and add it to six times the second inequality:

$$x_1 + 6x_2 \leq 2000$$

Product Planning Revisit

Multiplying the three inequalities by 0, 5, and 1, respectively, and adding them up yields

$$x_1 + 6x_2 \leq 1900$$

Multipliers

Let's investigate the issue by describing what we expect of these three multipliers, call them y_1, y_2, y_3.

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After the multiplication and addition steps, we get the bound:

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After the multiplication and addition steps, we get the bound:

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

We want the left-hand side to look like the objective function $x_1 + 6x_2$ so that the right-hand side is an upper bound on the optimum solution.

Multipliers

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3$$

if

$$\begin{gathered} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{gathered}$$

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What we want is a bound as tight as possible, so we minimize

$$200y_1 + 300y_2 + 400y_3$$

subject to the preceding inequalities. This is a new linear program!

The Dual Program

$$\begin{aligned} \min 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

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Any feasible value of this dual LP is an upper bound on the original primal LP.

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$$\begin{aligned} \min 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

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They both have value 1900 and certify each other's optimality.

Matrix-Vector Form and Its Dual

Primal LP	Dual LP
$\max c^T \mathbf{x}$	$\min \mathbf{y}^T b$
$A\mathbf{x} \leq b$	$\mathbf{y}^T A \geq c^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

Primal LP:

$$\begin{aligned} \max \ c_1x_1 + \cdots + c_nx_n & \\ a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i & \quad \text{for } i \in I \\ a_{i1}x_1 + \cdots + a_{in}x_n = b_i & \quad \text{for } i \in E \\ x_j \geq 0 & \quad \text{for } j \in N \end{aligned}$$

Dual LP:

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$$\begin{aligned} \max x_1 + 6x_2 \\ x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{aligned}$$

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Matrix-Vector Form and Its Dual

Theorem (Duality)

If a linear program has a bounded optimum, then so does its dual, and the two optimum values coincide.

Shortest Path in LP

In the shortest path problem, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to R$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the weight of a shortest path from s to t.

Shortest Path in LP

$$\min \sum_{e \in E} w_e x_e$$

$$\begin{aligned} \sum_{e \in \delta(S)} x_e &\geq 1 \quad && S \in \mathcal{S} \\ x_e &\geq 0 && e \in E \end{aligned}$$

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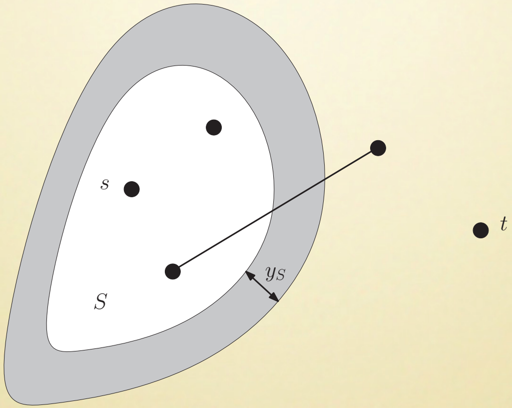
$$\max \sum_{S \in \mathcal{S}} y_S$$

$$\sum_{S \in \mathcal{S}, e \in \delta(S)} y_S \leq w_e \quad e \in E$$

$$y_S \geq 0 \quad S \in \mathcal{S}$$

The Moat



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The complementary slackness refers to a relationship between the slackness in a primal constraint and the associated dual variable.

LP and Its Dual

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$$x_1 = 100, x_2 = 300$$

$$y_1 = 0, y_2 = 5, y_3 = 1$$

Complementary Slackness

Theorem

Assume LP problem (P) has a solution x^* and its dual problem (D) has a solution y^*.

1. *If $x_j^* > 0$, then the j-th constraint in (D) is binding.*
2. *If the j-th constraint in (D) is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the i-th constraint in (P) is binding.*
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Proof.

Assignment ! $\square$

Flows in Networks

Shipping Oil

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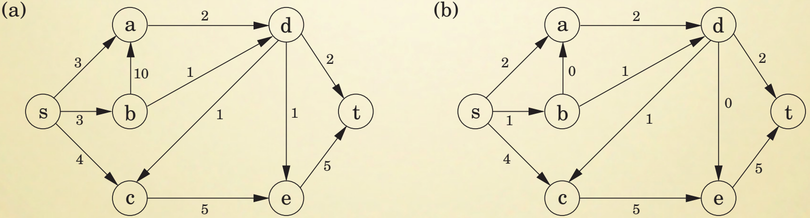
Each pipeline has a maximum capacity it can handle, and there are no opportunities for storing oil en route.

A Flow Example



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We would like to send as much oil as possible from s to t without exceeding the capacities of any of the edges.

Maximizing Flow

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1. It doesn't violate edge capacities: $0 \leq f_e \leq c_e$ for all $e \in E$.
2. For all nodes u except s and t, the amount of flow entering u equals the amount leaving

$$\sum_{(w,v)\in E} f_{wu} = \sum_{(u,z)\in E} f_{uz}$$

In other words, flow is conserved.

Maximizing Flow

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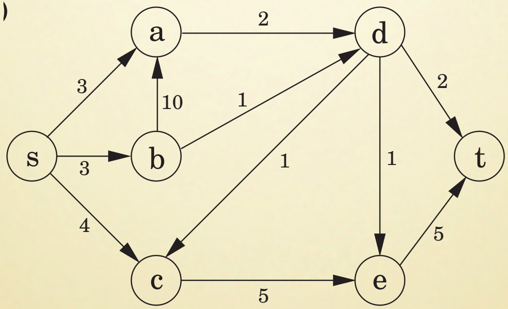
But this is a linear program! The maximum-flow problem reduces to linear programming.

The Example



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- 11 for nonnegativity (such as $f_{sa} \geq 0$)
- 11 for capacity (such as $f_{sa} \leq 3$)
- 5 for flow conservation (one for each node of the graph other than s and t, such as $f_{sc} + f_{dc} = f_{ce}$).

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The advantage of making this modification is that we can now require flow conservation at s and t as well.

Another Representation

$$\max f_{ts}$$

$$f_{ij} \leq c_{ij} \qquad (i,j) \in E$$

$$\sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 \quad i \in V$$

$$f_{ij} \geq 0 \qquad (i,j) \in E$$

A Closer Look at the Algorithm

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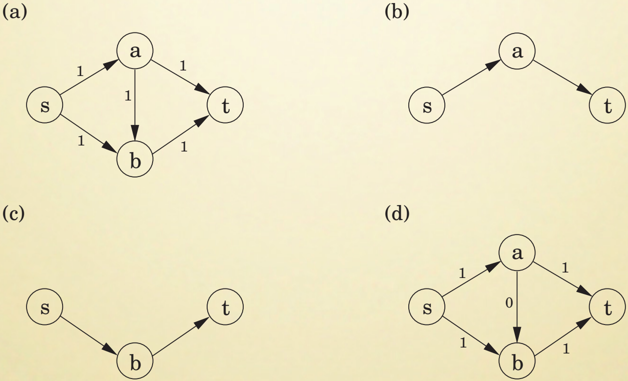
The behavior of simplex has an elementary interpretation:

- Start with zero flow.
- Repeat: choose an appropriate path from s to t, and increase flow along the edges of this path as much as possible.

A Flow Example



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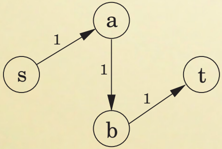
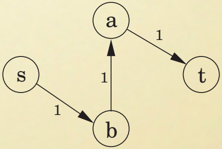
Simplex gets around this problem by also allowing paths to cancel existing flow.





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in the second case, up to f_{vu} additional units (canceling all or part of the existing flow on (v, u)).

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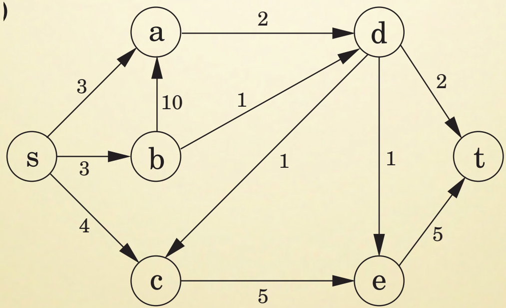

A Flow Example

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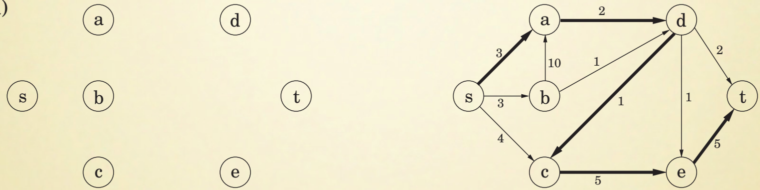

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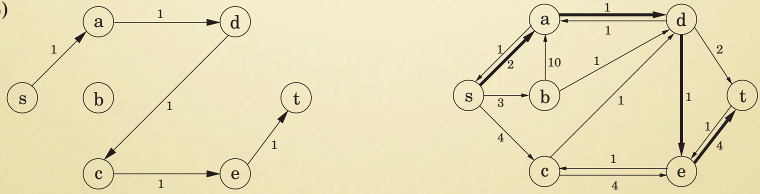

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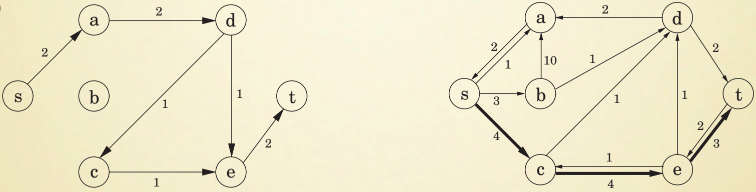

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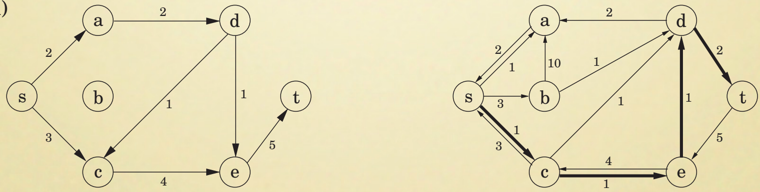
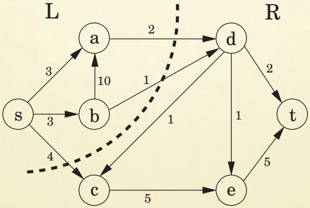


Cuts

A truly remarkable fact:

Not only does simplex correctly compute a maximum flow, but it also generates a short proof of the optimality of this flow!



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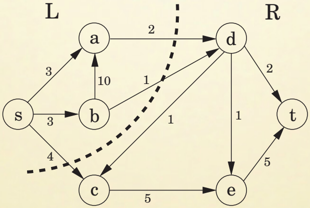

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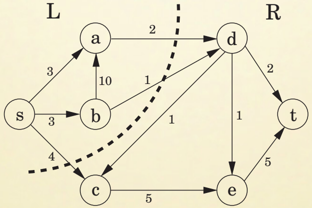

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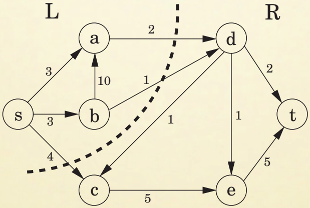

An (s,t)-cut partitions the vertices into two disjoint groups L and R such that $s \in L$ and $t \in R$. Its capacity is the total capacity of the edges from L to R, and as argued previously, is an upper bound on any flow: Pick any flow f and any (s,t)-cut (L,R). Then $\texttt{size}(f) \leq \texttt{capacity}(L,R)$.

A Certificate of Optimality

Theorem (Max-flow min-cut)
The size of the maximum flow in a network equals the capacity of the smallest (s, t)-cut.

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Therefore the net flow across (L, R) is exactly the capacity of the cut.

Efficiency

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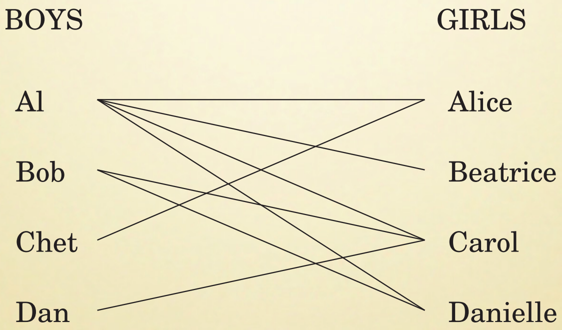
This latter bound gives an overall running time of $O(|V| \cdot |E|^2)$ for maximum flow.

Bipartite Matching



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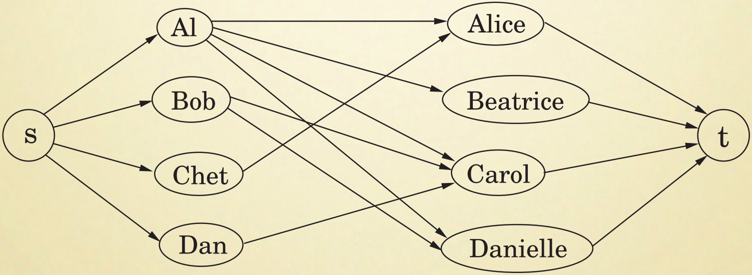

Homework

[DPV07] 7.6, 7.7, 7.8, 7.13, 7.21 and 7.23

** Min-Max Relations in LP **

LP for Max Flow

$$\max f_{ts}$$

$$f_{ij} \leq c_{ij} \qquad (i,j) \in E$$

$$\sum_{(w,i)\in E} f_{wi} - \sum_{(i,z)\in E} f_{iz} \leq 0 \quad i \in V$$

$$f_{ij} \geq 0 \qquad (i,j) \in E$$

Duality

Primal LP	Dual LP
$\max c^T \mathbf{x}$	$\min \mathbf{y}^T b$
$A\mathbf{x} \leq b$	$\mathbf{y}^T A^T \geq c^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

Primal LP:

$$\begin{aligned} \max\ & c_1 x_1 + \cdots + c_n x_n \\ a_{i1}x_1 + \cdots + a_{in}x_n &\leq b_i \quad \text{for } i \in I \\ a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \quad \text{for } i \in E \\ x_j &\geq 0 \quad \text{for } j \in N \end{aligned}$$

Dual LP:

$$\begin{aligned} \min\ & b_1 y_1 + \cdots + b_m y_m \\ a_{1j}y_1 + \cdots + a_{mj}y_m &\geq c_j \quad \text{for } j \in N \\ a_{1j}y_1 + \cdots + a_{mj}y_m &= c_j \quad \text{for } j \notin N \\ y_i &\geq 0 \quad \text{for } i \in I \end{aligned}$$

LP-Duality

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- To obtain the dual program we introduce variables d_{ij} and p_i corresponding to the two types of inequalities in the primal.
  - d_{ij}: distance labels on edges;
  - p_i: potentials on nodes.

Integer Program

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- This solution naturally defines an $s - t$ cut $(X, \overline{X})$, where X is the set of potential 1 nodes, and $\overline{X}$ the set of potential 0 nodes.

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- The distance label for each of the remaining edges can be set to either 0 or 1 without violating the first constraints; however in order to minimize the objective function value it must be set to 0.
- The objective function value is precisely the capacity of the cut $(X, \overline{X})$, and hence $(X, \overline{X})$ must be a minimum $s-t$ cut.

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- Dropping these constraints gives the dual program in the form given above. We will say that this program is the LP relaxation of the integer program.

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- Now, it can be proven that each vertex solution is integral, with each coordinate being 0 or 1.
- This follows from the fact that the constraint matrix of this program is totally unimodular, Thus, the dual program always has an integral optimal solution.