BMO 2021 - Problem 1

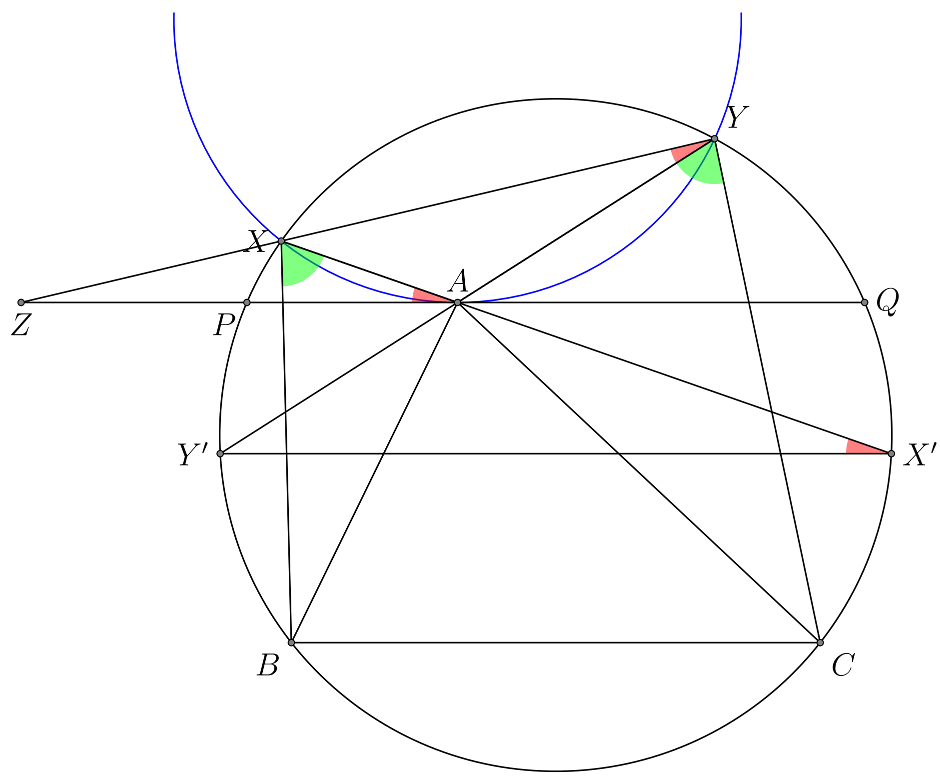
Let ABC be a triangle with $AB < AC$. Let ω be a circle passing through B, C and assume that A is inside ω. Suppose X, Y lie on ω such that $\angle BXA = \angle AYC$. Suppose also that X and C lie on opposite sides of the line AB and that Y and B lie on opposite sides of the line AC.

Show that, as X, Y vary on ω, the line XY passes through a fixed point.

Solution 1. Extend XA and YA to meet ω again at X' and Y' respectively. We then have that:

$$\angle Y'YC = \angle AYC = \angle BXA = \angle BXX'.$$

so $BCX'Y'$ is an isosceles trapezium and hence $X'Y' \parallel BC$.



Let ℓ be the line through A parallel to BC and let ℓ intersect ω at P, Q with P on the opposite side of AB to C. As $X'Y' \parallel BC \parallel PQ$ then

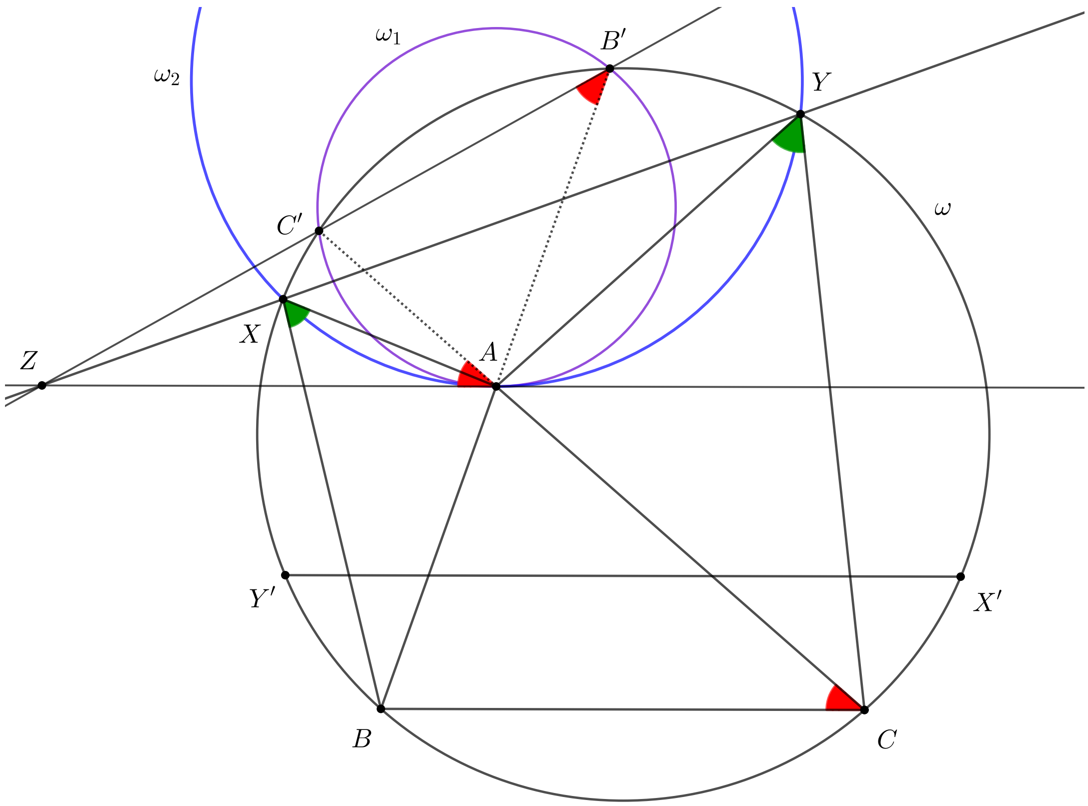
$$\angle XAP = \angle XX'Y' = \angle XYY' = \angle XYA$$

which shows that ℓ is tangent to the circumcircle of triangle AXY. Let XY intersect PQ at Z. By power of a point we have that

$$ZA^2 = ZX \cdot ZY = ZP \cdot ZQ.$$

As P, Q are independent of the positions of X, Y, this shows that Z is fixed and hence XY passes through a fixed point.

Solution 2. Let B' and C' be the points of intersection of the lines AB and AC with ω respectively and let ω_1 be the circumcircle of the triangle $AB'C'$. Let ε be the tangent to ω_1 at the point A. Because $AB < AC$ the lines $B'C'$ and ε intersects at a point Z which is fixed and independent of X and Y.



We have

$$\angle ZAC' = \angle C'B'A = \angle C'B'B = \angle C'CB\,.$$

Therefore, $\varepsilon \parallel BC$.

Let X', Y' be the points of intersection of the lines XA,YA with ω respecively. From the hypothesis we have $\angle BXX' = \angle Y'YC$. Therefore

$$\widehat{BX'} = \widehat{Y'C} \implies \widehat{BC} + \widehat{CX'} = \widehat{Y'B} + \widehat{BC} \implies \widehat{CX'} = \widehat{Y'B}$$

and so $X'Y' \parallel BC \parallel \varepsilon$. Thus

$$\angle XAZ = \angle XX'Y' = \angle XYY' = \angle XYA\,.$$

From the last equality we have that ε is also tangent to the circmucircle ω_2 of the triangle XAY.

Consider now the radical centre of the circles $\omega, \omega_1, \omega_2$. This is the point of intersection of the radical axes $B'C'$ (of ω and ω_1), ε (of ω_1 and ω_2) and XY (of ω and ω_2).

This must be point Z and therefore the variable line XY passes through the fixed point Z.

BMO 2021 - Problem 2

Find all functions $f:(0,+\infty)\to(0,+\infty)$ such that

$$f(x+f(x)+f(y))=2f(x)+y$$

holds for all $x,y\in(0,+\infty)$.

Solution 1. We will show that $f(x)=x$ for every $x\in\mathbb{R}^+$. It is easy to check that this function satisfies the equation.

We write $P(x,y)$ for the assertion that $f(x+f(x)+f(y))=2f(x)+y$.

We first show that f is injective. So assume $f(a)=f(b)$. Now $P(1,a)$ and $P(1,b)$ show that

$$2f(1)+a=f(1+f(1)+f(a))=f(1+f(1)+f(b))=2f(1)+b$$

and therefore $a=b$.

Let $A=\{x\in\mathbb{R}^+ : f(x)=x\}$. It is enough to show that $A=\mathbb{R}^+$.

$P(x,x)$ shows that $x+2f(x)\in A$ for every $x\in\mathbb{R}^+$. Now $P(x,x+2f(x))$ gives that

$$f(2x+3f(x))=x+4f(x)$$

for every $x\in\mathbb{R}^+$. Therefore $P(x,2x+3f(x))$ gives that $2x+5f(x)\in A$ for every $x\in\mathbb{R}^+$.

Suppose $x,y\in\mathbb{R}^+$ such that $x,2x+y\in A$. Then $P(x,y)$ gives that

$$f(2x+f(y))=f(x+f(x)+f(y))=2f(x)+y=2x+y=f(2x+y)$$

and by the injectivity of f we have that $2x+f(y)=2x+y$. We conlude that $y\in A$ as well.

Now since $x+2f(x)\in A$ and $2x+5f(x)=2(x+2f(x))+f(x)\in A$ we deduce that $f(x)\in A$ for every $x\in\mathbb{R}^+$. I.e. $f(f(x))=f(x)$ for every $x\in\mathbb{R}^+$.

By injectivity of f we now conclude that $f(x)=x$ for every $x\in\mathbb{R}^+$.

Solution 2. As in Solution 1, f is injective. Furthermore, letting $m = 2f(1)$ we have that the image of f contains (m, ∞). Indeed, if $t > m$, say $t = m + y$ for some $y > 0$, then $P(1, y)$ shows that $f(1 + f(1) + f(y)) = t$.

Let $a, b \in \mathbb{R}$. We will show that $f(a) - a = f(b) - b$. Define $c = 2f(a) - 2f(b)$ and $d = a + f(a) - b - f(b)$. It is enough to show that $c = d$. By interchanging the roles of a and b in necessary, we may assume that $d \geqslant 0$.

From $P(a, y)$ and $P(b, y)$, after subtraction, we get

$$f(a + f(a) + f(y)) - f(b + f(b) + f(y)) = 2f(a) - 2f(b) = c\,. \tag{1}$$

so for any $t > m$ (picking y such that $f(y) = t$ in (1)) we get

$$f(a + f(a) + t) - f(b + f(b) + t) = 2f(a) - 2f(b) = c\,. \tag{2}$$

Now for any $z > m + b + f(b)$, taking $t = z - b - f(b)$ in (2) we get

$$f(z + d) - f(z) = c\,. \tag{3}$$

Now for any $x > m + b + f(b)$ from (3) we get that

$$2f(x + d) + y = 2f(x) + y + 2c\,.$$

Also, for any x large enough, ($x > \max\{m + b + f(b), m + b + f(b) + c - d\}$ will do), by repeated application of (3), we have

$$\begin{aligned} f(x + d + f(x + d) + f(y)) &= f(x + f(x + d) + y) + c \\ &= f(x + f(x) + y + c) + c \\ &= f(x + f(x) + y + c - d) + 2c\,. \end{aligned}$$

(In the first equality we applied (3) with $z = x + f(x+d) + y > x > m+b+f(b)$, in the second with $z = x > m+b+f(b)$ and in the third with $z = x+f(x)+y-c+d > x+c-d > m+b+f(b)$.)

In particular, now $P(x + d, y)$ implies that

$$f(x + f(x) + y + c - d) = 2f(x) + y = f(x + f(x) + y)$$

for every large enough x. By injectivity of f we deduce that $x + f(x) + y + c - d = x + f(x) + y$ and therefore $c = d$ as required.

It now follows that $f(x) = x + k$ for every $x \in \mathbb{R}^+$ and some fixed constant k. Substituting in the initial equation we get $k = 0$.

BMO 2021 - Problem 3

Let a, b and c be positive integers satisfying the equation

$$(a,b)+[a,b]=2021^c\,.$$

If $|a-b|$ is a prime number, prove that the number $(a+b)^2+4$ is composite.

Here, (a,b) denotes the greatest common divisor of a and b, and $[a,b]$ denotes the least common multiple of a and b.

Solution. We write $p=|a-b|$ and assume for contradiction that $q=(a+b)^2+4$ is a prime number.

Since $(a,b)\mid[a,b]$, we have that $(a,b)\mid 2021^c$. As (a,b) also divides $p=|a-b|$, it follows that $(a,b)\in\{1,43,47\}$. We will consider all 3 cases separately:

(1) If $(a,b)=1$, then $1+ab=2021^c$, and therefore

$$q=(a+b)^2+4=(a-b)^2+4(1+ab)=p^2+4\cdot 2021^c\,. \tag{1}$$

(a) Suppose c is even. Since $q\equiv 1 \bmod 4$, it can be represented uniquely (up to order) as a sum of two (non-negative) squares. But (1) gives potentially two such representations so in order to have uniqueness we must have $p=2$. But then $4|q$ a contradiction.

(b) If c is odd then $ab=2021^c-1\equiv 1 \bmod 3$. Thus $a\equiv b \bmod 3$ implying that $p=|a-b|\equiv 0 \bmod 3$. Therefore $p=3$. Without loss of generality $b=a+3$. Then $2021^c=ab+1=a^2+3a+1$ and so

$$(2a+3)^2=4a^2+12a+9=4\cdot 2021^c+5\,.$$

So 5 is a quadratic residue modulo 47, a contradiction as

$$\left(\frac{5}{47}\right)=\left(\frac{47}{5}\right)=\left(\frac{2}{5}\right)=-1\,.$$

(2) If $(a,b)=43$, then $p=|a-b|=43$ and we may assume that $a=43k$ and $b=43(k+1)$, for some $k\in\mathbb{N}$. Then $2021^c=43+43k(k+1)$ giving that

$$(2k+1)^2=4k^2+4k+4-3=4\cdot 43^{c-1}\cdot 47-3\,.$$

So -3 is a quadratic residue modulo 47, a contradiction as

$$\left(\frac{-3}{47}\right)=\left(\frac{-1}{47}\right)\left(\frac{3}{47}\right)=\left(\frac{47}{3}\right)=\left(\frac{2}{3}\right)=-1\,.$$

(3) If $(a,b)=47$ then analogously there is a $k\in\mathbb{N}$ such that

$$(2k+1)^2=4\cdot 43^c\cdot 47^{c-1}-3\,.$$

If $c>1$ then we get a contradiction in exactly the same way as in (2). If $c=1$ then $(2k+1)^2=169$ giving $k=6$. This implies that $a+b=47\cdot 6+47\cdot 7=47\cdot 13\equiv 1 \bmod 5$. Thus $q=(a+b)^2+4\equiv 0 \bmod 5$, a contradiction.

BMO 2021 - Problem 4

Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel performs exactly one of the following moves:

(a) He clears every piece of rubbish from a single pile.
(b) He clears one piece of rubbish from each pile.

However, every evening, a demon sneaks into the warehouse and performs exactly one of the following moves:

(a) He adds one piece of rubbish to each non-empty pile.
(b) He creates a new pile with one piece of rubbish.

What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?

Solution 1. We will show that he can do so by the morning of day 199 but not earlier.If we have n piles with at least two pieces of rubbish and m piles with exactly one piece of rubbish, then we define the value of the pile to be

$$V = \begin{cases} n & m = 0\,, \\ n + \frac{1}{2} & m = 1\,, \\ n+1 & m \geqslant 2\,. \end{cases}$$

We also denote this position by (n, m). Implicitly we will also write k for the number of piles with exactly two pieces of rubbish.

Angel's strategy is the following:

(i) From position $(0, m)$ remove one piece from each pile to go position $(0, 0)$. The game ends.
(ii) From position $(n, 0)$, where $n \geqslant 1$, remove one pile to go to position $(n-1, 0)$. Either the game ends, or the demon can move to position $(n-1, 0)$ or $(n-1, 1)$. In any case V reduces by at least $1/2$.
(iii) From position $(n, 1)$, where $n \geqslant 1$, remove one pile with at least two pieces to go to position $(n-1, 1)$. The demon can move to position $(n, 0)$ or $(n-1, 2)$. In any case V reduces by (at least) $1/2$.
(iv) From position (n, m), where $n \geqslant 1$ and $m \geqslant 2$, remove one piece from each pile to go to position $(n-k, k)$. The demon can move to position $(n, 0)$ or $(n-k, k+1)$. In any case V reduces by at least $1/2$. (The value of position $(n-k, k+1)$ is $n + \frac{1}{2}$ if $k = 0$, and $n - k + 1 \leqslant n$ if $k \geqslant 1$.)

So during every day if the game does not end then V is decreased by at least $1/2$. So after 198 days if the game did not already end we will have $V \leqslant 1$ and we will be in one of positions $(0, m), (1, 0)$. The game can then end on the morning of day 199.

We will now provide a strategy for demon which guarantees that at the end of each day V has decreased by at most $1/2$ and furthermore at the end of the day $m \leqslant 1$.

(i) If Angel moves from $(n, 0)$ to $(n-1, 0)$ (by removing a pile) then create a new pile with one piece to move to $(n-1, 1)$. Then V decreases by $1/2$ and and $m = 1 \leqslant 1$
(ii) If Angel moves from $(n, 0)$ to $(n-k, k)$ (by removing one piece from each pile) then add one piece back to each pile to move to $(n, 0)$. Then V stays the same and $m = 0 \leqslant 1$.
(iii) If Angels moves from $(n, 1)$ to $(n-1, 1)$ or $(n, 0)$ (by removing a pile) then add one piece to each pile to move to $(n, 0)$. Then V decreases by $1/2$ and $m = 0 \leqslant 1$.
(iv) If Angel moves from $(n, 1)$ to $(n-k, k)$ (by removing a piece from each pile) then add one piece to each pile to move to $(n, 0)$. Then V decreases by $1/2$ and $m = 0 \leqslant 1$.

Since after every move of demon we have $m \leqslant 1$, in order for Angel to finish the game in the next morning we must have $n = 1, m = 0$ or $n = 0, m = 1$ and therefore we must have $V \leqslant 1$. But now inductively the demon can guarantee that by the end of day N, where $N \leqslant 198$ the game has not yet finished and that $V \geqslant 100 - N/2$.

Solution 2.

Define Angel's score S_A to be $S_A = 2n + m - 1$. The Angel can clear the rubbish in at most $\max\{S_A, 1\}$ days. The proof is by induction on (n, m) in lexicographic order.

Angel's strategy is the same as in Solution 1 and in each of cases (ii)-(iv) one needs to check that S_A reduces by at least 1 in each day. (Case (i) is trivial as the game ends in one day.)

Now define demon's score S_D to be $S_D = 2n - 1$ if $m = 0$ and $S_D = 2n$ if $m \geqslant 1$. The claim is the if $(n, m) \neq (0, 0)$, then the demon can ensure that Angel requires S_D days to clear the rubbish.

Again, demon's strategy is the same as in the Solution by PSC and in each of cases (i)-(iv) one needs to check that S_D reduced by at most 1 in each day.