9^{th} Iranian Geometry Olympiad

October 14, 2022

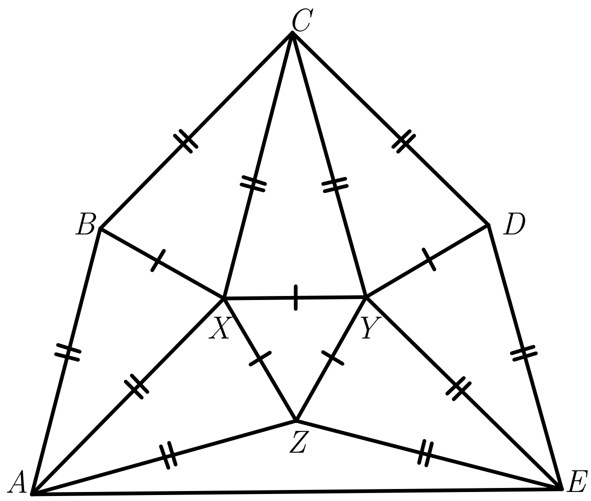
Contest problems with solutions
(Draft Version)

Contents

Elementary Level

Problems

Problem 1. Find the angles of the pentagon $ABCDE$ in the figure below.



($\rightarrow$ p.5)

Problem 2. An isosceles trapezoid $ABCD$ ($AB \parallel CD$) is given. Points E and F lie on the sides BC and AD, and the points M and N lie on the segment EF such that $DF = BE$ and $FM = NE$. Let K and L be the foot of perpendicular lines from M and N to AB and CD, respectively. Prove that $EKFL$ is a parallelogram.

($\rightarrow$ p.6)

Problem 3. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$ and $\angle BDE = \angle EAC = 30°$. Find the possible values of $\angle BEC$.

($\rightarrow$ p.7)

Problem 4. Let AD be the internal angle bisector of triangle ABC. The incircles of triangles ABC and ACD touch each other externally. Prove that $\angle ABC > 120°$. (Recall that the incircle of a triangle is a circle inside the triangle that is tangent to its three sides.)

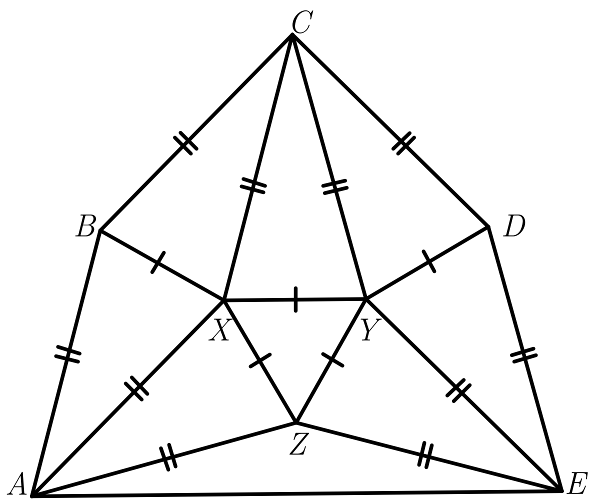
($\rightarrow$ p.8)

Problem 5. a) Do there exist four equilateral triangles in the plane such that each two have exactly one vertex in common, and every point in the plane lies on the boundary of at most two of them?
b) Do there exist four squares in the plane such that each two have exactly one vertex in common, and every point in the plane lies on the boundary of at most two of them?
(Note that in both parts, there is no assumption on the intersection of interior of polygons.)

($\rightarrow$ p.9)

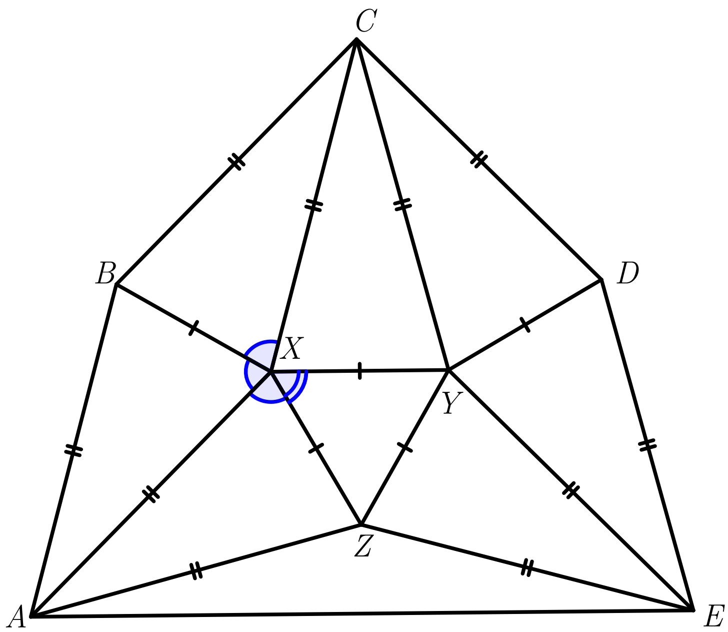
Solutions

Problem 1. Find the angles of the pentagon $ABCDE$ in the figure below.



Proposed by Morteza Saghafian - Iran

Solution.



It's easy to see that the triangles $\triangle CBX, \triangle CXY, \triangle CYD, \triangle ABX, \triangle AXZ, \triangle EDY, \triangle EYZ$ are all congruent. So $360° = \angle CXY + \angle CXB + \angle BXA + \angle AXZ + \angle YXZ = 4\angle CXY + 60°$. So $\angle CXY = 75$. So $\angle ABC = \angle CDE = 150°$ and $\angle BCD = 90°$. So $\angle BAE + \angle DEA = 540° - \angle ABC - \angle BCD - \angle CDE = 150°$.Because $\angle BAE$ and $\angle DEA$ are equal, then they are each $75°$

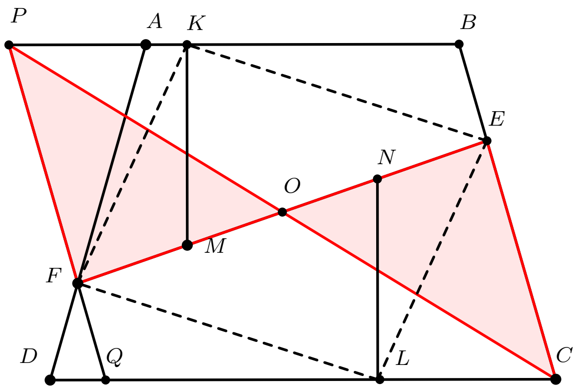
Problem 2. An isosceles trapezoid $ABCD$ ($AB \parallel CD$) is given. Points E and F lie on the sides BC and AD, and the points M and N lie on the segment EF such that $DF = BE$ and $FM = NE$. Let K and L be the foot of perpendicular lines from M and N to AB and CD, respectively. Prove that $EKFL$ is a parallelogram.

Proposed by Mahdi Etesamifard - Iran

Solution. Let the line at F parallel to BC intersect AB, CD in P and Q respectively.
Since $BP \| QC$ and $PQ \| BC$ the quadrilateral $PBCQ$ is a parallelogram.
We have $FP = AF$ because $\measuredangle FPA = \measuredangle FAP$, hence $FP = AF = EC$
Letting O be the midpoint of MN :

$$OF = OE, FP = EC, \measuredangle PFO = \measuredangle CEO$$

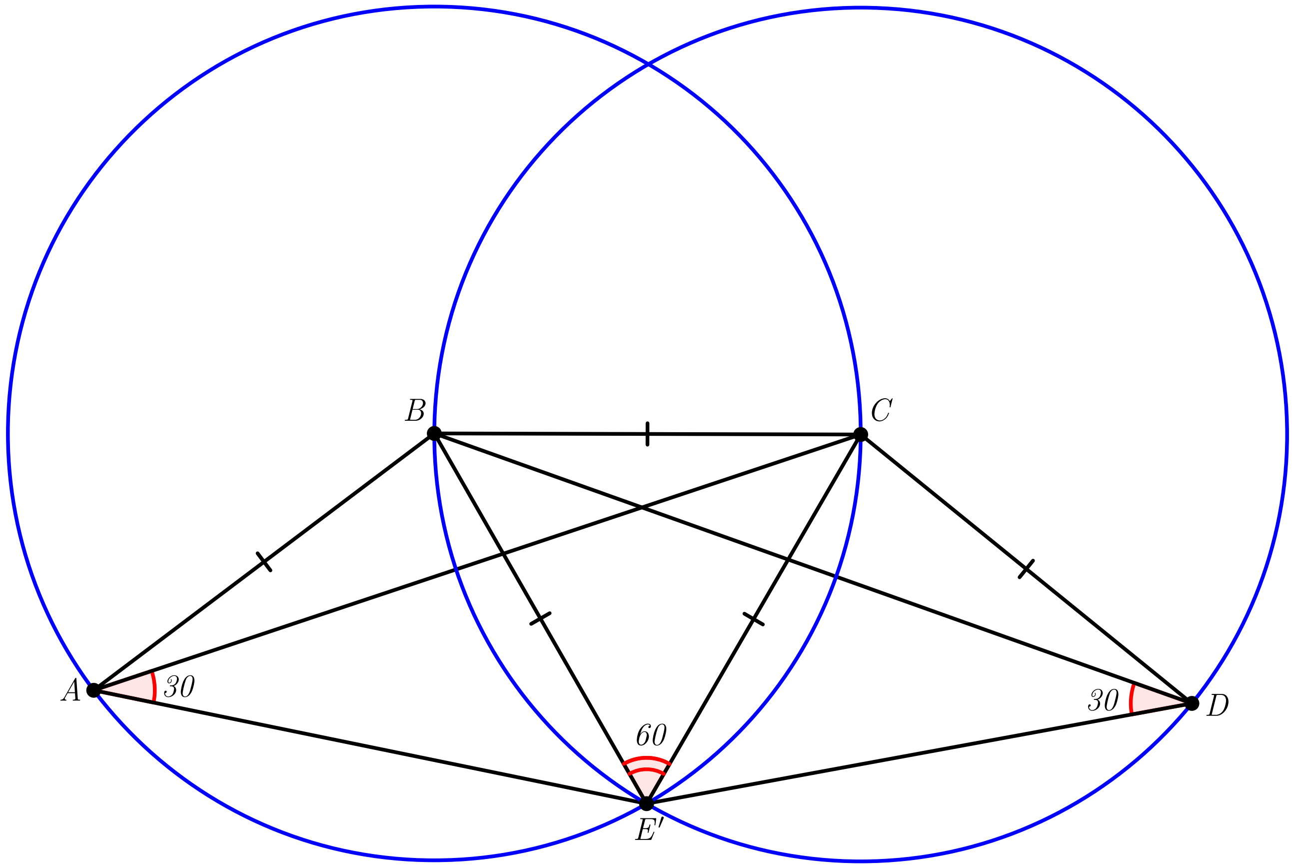
in fact $PFO \cong CEO$. Now excluding the points A and D the figure is completely symmetric with respect to O hence $FK = EL, FK \| EL$, therefore $FKEL$ is a parallelogram.



Problem 3. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$ and $\angle BDE = \angle EAC = 30°$. Find the possible values of $\angle BEC$.

Proposed by Josef Tkadlec - Czech Republic

Solution. Answer: $60°$.

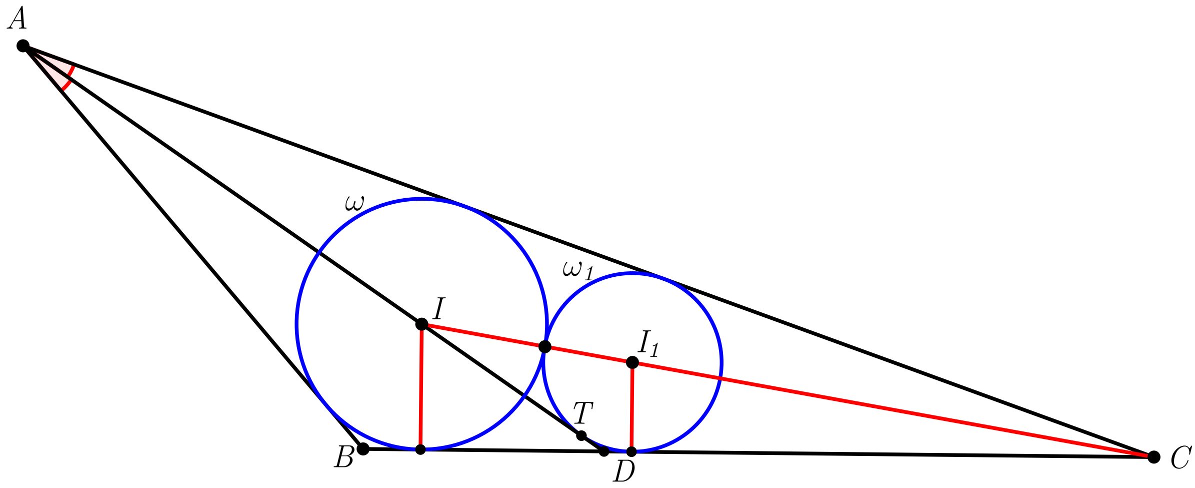


Fix A, B, C, D such that $AB = BC = CD$ and $ABCD$ is convex. Note that the point E is uniquely determined by the angle conditions.
Let E' be a point such that BCE' is equilateral (and A, E', D all lie on the same side of BC).
By inscribed angles we have $\angle BDE' = \frac{1}{2}\angle BCE' = 30°$ and likewise $\angle E'AC = 30°$. Hence $E' \equiv E$ and $\angle BEC = \angle BE'C = 60°$.

Problem 4. Let AD be the internal angle bisector of triangle ABC. The incircles of triangles ABC and ACD touch each other externally. Prove that $\angle ABC > 120°$. (Recall that the incircle of a triangle is a circle inside the triangle that is tangent to its three sides.)

Proposed by Volodymyr Brayman - Ukraine

Solution. Denote by $\omega(I, r)$ and $\omega_1(I_1, r_1)$ the incircles of triangles ABC and ACD, respectively. Let T be the point where ω_1 touches AD. Point I_1 lies on the segment CI, so the distance from I_1 to BC is less than the distance from I to BC. Hence $r_1 < r$. Consider right triangle II_1T ($\angle ITI_1 = 90°$). Since $II_1 = r + r_1 > 2r_1 = 2I_1T$, we have $\angle TI_1I > 60°$. On the other hand, $\angle TI_1I + 90° = \angle AIC = \frac{1}{2}\angle ABC + 90°$. Thus $\frac{1}{2}\angle ABC > 60°$, or $\angle ABC > 120°$.



Remark. It can be shown that triangle ABC satisfies condition of the problem if and only if $2\cos\frac{B}{2} = 1 - \sin\frac{C}{2}$.

Problem 5. a) Do there exist four equilateral triangles in the plane such that each two have exactly one vertex in common, and every point in the plane lies on the boundary of at most two of them?
b) Do there exist four squares in the plane such that each two have exactly one vertex in common, and every point in the plane lies on the boundary of at most two of them?
(Note that in both parts, there is no assumption on the intersection of interior of polygons.)

Proposed by Hesam Rajabzadeh - Iran

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Solution. a) The answer is no. We begin with a lemma.

Lemma. *Given two segments AB, CD of equal length, there is at most one point Z such that the (clockwise or counter-clockwise) rotation of angle $60°$ with center Z maps AB to CD.*

Proof. Consider Cartesian coordinate system on the plane and suppose that in this coordinate system the vector $\overrightarrow{AB}$ lies on x-axis and points to the right. Suppose that there is Z such that the rotation of $60°$ with center Z maps exactly A to C and B to D (by symmetry, other cases are similar). Therefore, this rotation maps vector $\overrightarrow{AB}$ to $\overrightarrow{CD}$. In particular, $\overrightarrow{CD}$ makes angle $60°$ with the positive direction of x-axis. We claim that such point Z is unique. Note that both triangles ZAC, ZBD are equilateral, so Z lies on the perpendicular bisectors of AC, BD. This intersection point is unique unless $AC \parallel BD$. In this case, Z is the intersection point of lines AB, CD.
Now, if there is another point Z' such that the rotation of $60°$ with center Z' maps exactly A to D and B to C, by similar arguments $\overrightarrow{DC}$ makes angle $60°$ with the positive direction of x-axis which is a contradiction with the angle between $\overrightarrow{CD}$ and the positive direction of x-axis. The proof of lemma is complete. □

We return to the main problem. Assume to the contrary that there exist four equilateral triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4$ satisfying the conditions. The conditions imply that there are six points in the plane; each is the vertex of exactly two of $\triangle_i$'s.
First suppose that $\triangle_i$'s are mutually non-congruent. Suppose that $\triangle_1 = XAC$ and $\triangle_2 = XBD$, and the labels of the vertices are so that the counter-clockwise rotation of $60°$ with center X maps A to C and B to D, respectively. So this rotation will map vector $\overrightarrow{AB}$ to $\overrightarrow{CD}$ (in particular $AB = CD$).
Next, denote the common vertex of $\triangle_3, \triangle_4$ by Y. We have several cases.

- $\triangle_3 = YAB, \triangle_4 = YCD$. In this case, we have

  $$YB = YA = AB = CD = YC = YD.$$

  So $\triangle_3 \equiv \triangle_4$, contradicting our assumption.

- The remaining two triangles are $\triangle_3 = YAD, \triangle_4 = YBC$. In this case, either $\triangle YBD \equiv \triangle YAC$ or $\triangle YCD \equiv \triangle YBA$. In the former case, we have $AC = BD$ and so $\triangle_1 \equiv \triangle_2$. This leads to contradiction with the assumption of triangles not being congruent. In the latter case, we get a rotation of angle $60°$ with center Y maps AB to CD. Note that the point X has the same property so in view of lemma we must have $X = Y$. This leads to a contradiction with the assumptions as X is a common vertex of all triangles.

Now suppose that two equilateral triangles $\triangle_1 = XAC, \triangle_2 = XBD$ are congruent. This implies that points A, B, C, D lie on the same circle with center X. So in particular, X lies on the perpendicular bisectors of any pair from $\{A, B, C, D\}$. Assume that the labels of the points are chosen so that the counter-clockwise rotation of $60°$ with center X maps A to C and B to D,

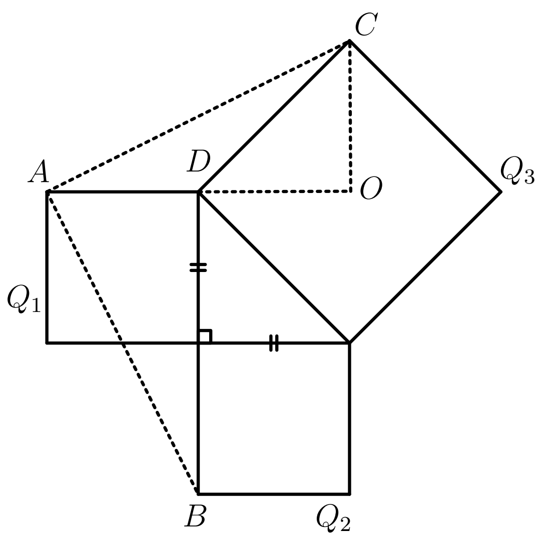
respectively. By arguments similar to what we have done before, the angle between lines AB, CD is $60°$.
Denote the common vertex of the remaining two triangles by Y. We have several cases.

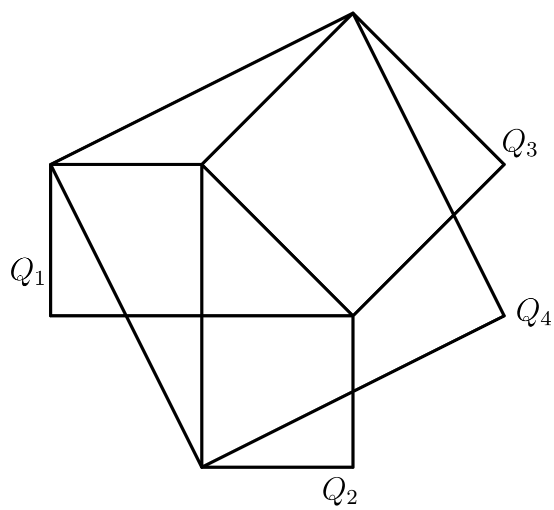
- The remaining two triangles are $\triangle_3 = YAB, \triangle_4 = YCD$. So, Y lies on the perpendicular bisector of AB, CD. Since AB is not parallel to CD, their perpendicular bisectors meet at the unique point X and so $X = Y$. This leads to contradiction as $X = Y$ became a vertex of all four triangles.

- The remaining two triangles are $\triangle_3 = YAD, \triangle_4 = YBC$. So, Y lies on the perpendicular bisector of AD, BC. Now, if $AD \nparallel BC$, similar to the previous item, we get $X = Y$, contradiction. But if $AD \parallel BC$, since $\angle BYC = \angle DYA = 60°$, Y should be the intersection point of AC, BD and A, X, D are collinear. This also leads to a contradiction, as point X is on the boundary of three triangles XBD, XAC, AYD.

So the proof is complete.

b) The answer is yes. First consider three squares Q_1, Q_2, Q_3, as figure below.





Note, if we denote the vertices of the squares by A, B, C, D (as in the left figure above) and the center of Q_3 by O, then right-angled triangles ABD and AOC are congruent. So, $AC = AB$ and

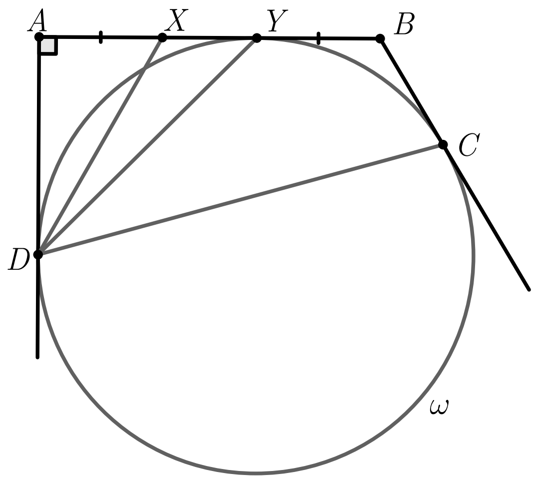
$$\angle CAB = \angle CAO + \angle DAB = 90°.$$

Therefore, A, C, B are three vertices of a square, say Q_4. Then, Q_1, Q_2, Q_3, Q_4 satisfy all required conditions (see figure right above).

Intermediate Level

Problems

Problem 1. In the figure below we have $AX = BY$. Prove that $\angle XDA = \angle CDY$.



(→ p.15)

Problem 2. Two circles ω_1 and ω_2 with equal radius intersect at two points E and X. Arbitrary points C, D lies on ω_1, ω_2. Parallel lines to XC, XD from E intersect ω_2, ω_1 at A, B, respectively. Suppose that CD intersect ω_1, ω_2 again at P, Q, respectively. Prove that $ABPQ$ is concyclic.

(→ p.16)

Problem 3. Let O be the circumcenter of triangle ABC. Arbitrary points M and N lie on the sides AC and BC, respectively. Points P and Q lie in the same half-plane as point C with respect to the line MN, and satisfy $\triangle CMN \sim \triangle PAN \sim \triangle QMB$ (in this exact order). Prove that $OP = OQ$.

(→ p.17)

Problem 4. We call two simple polygons P, Q *compatible* if there exists a positive integer k such that each of P, Q can be partitioned into k congruent polygons similar to the other one. Prove that for every two even integers $m, n \geq 4$, there are two compatible polygons with m and n sides. (A simple polygon is a polygon that does not intersect itself.)

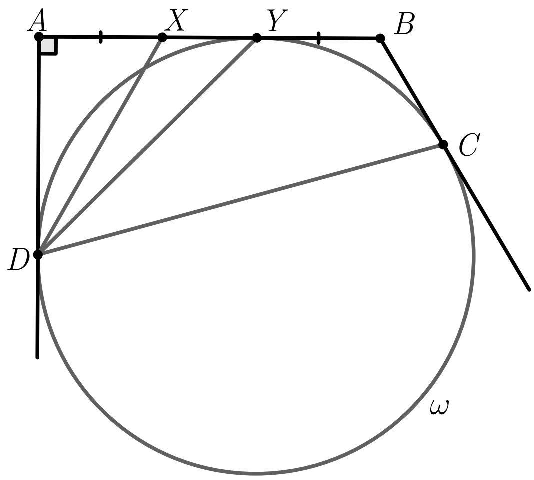
(→ p.18)

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω with center O. Let P be the intersection of two diagonals AC and BD. Let Q be a point lying on the segment OP. Let E and F be the orthogonal projections of Q on the lines AD and BC, respectively. The points M and N lie on the circumcircle of triangle QEF such that $QM \parallel AC$ and $QN \parallel BD$. Prove that the two lines ME and NF meet on the perpendicular bisector of segment CD.

(→ p.19)

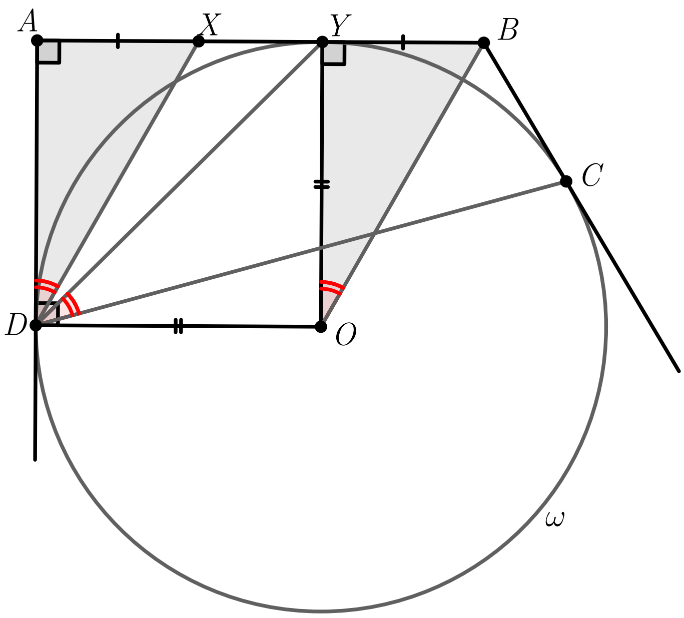
Solutions

Problem 1. In the figure below we have $AX = BY$. Prove that $\angle XDA = \angle CDY$.



Proposed by Iman Maghsoudi - Iran

Solution. Let O be the circumcenter of ω. Since $OY \perp AB, OD \perp AD$, we obtain that $AYOD$ is a square, therefore $OY = AD, \angle OYB = \angle DAX = 90°$, it implies that triangles DAX, OYB are congruent. Hence $\angle XDA = \angle BOY = \angle CDY$.



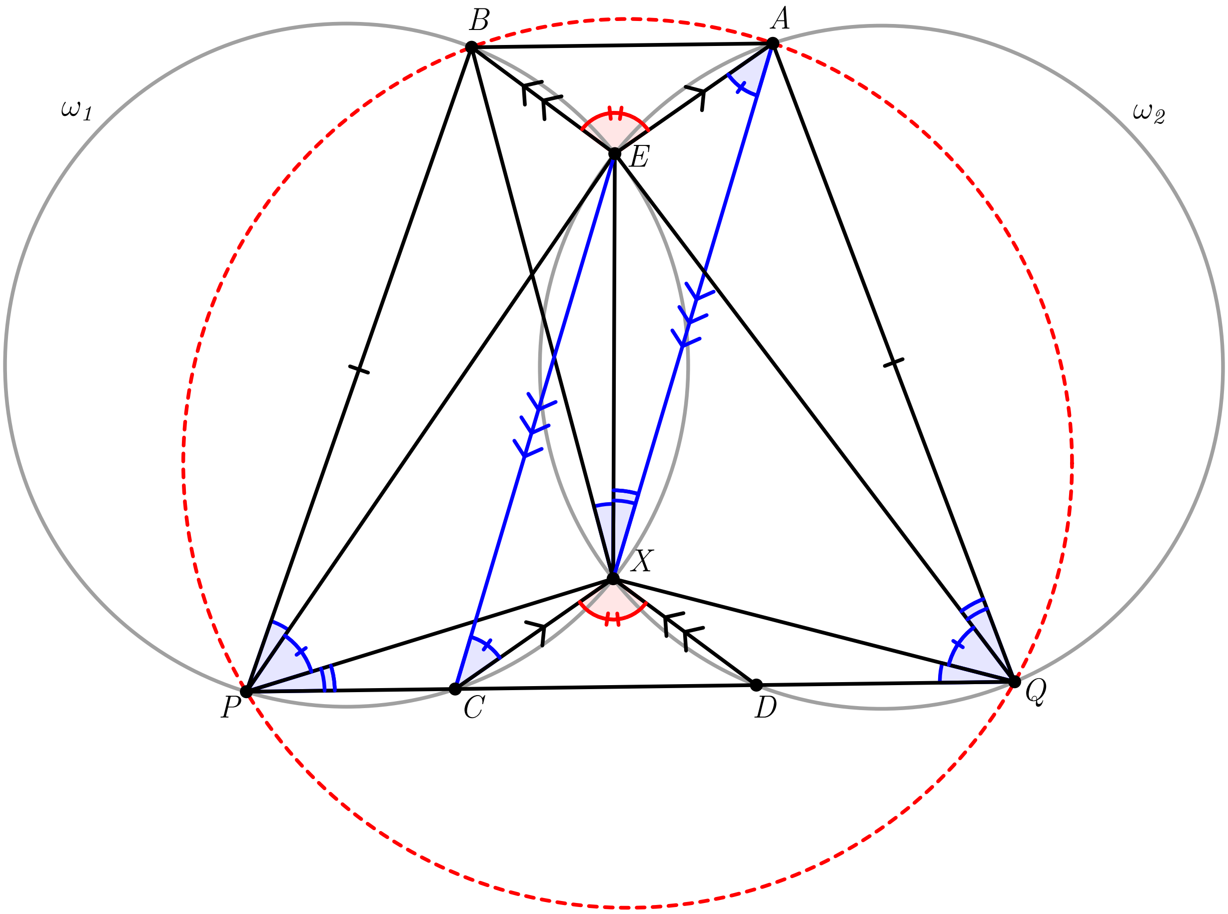
Problem 2. Two circles ω_1 and ω_2 with equal radius intersect at two points E and X. Arbitrary points C, D lies on ω_1, ω_2. Parallel lines to XC, XD from E intersect ω_2, ω_1 at A, B, respectively. Suppose that CD intersect ω_1, ω_2 again at P, Q, respectively. Prove that $ABPQ$ is concyclic.

Proposed by Ali Zamani - Iran

Solution. Since two circles are equal and $EA \parallel XC$, we have $\angle ECX = \frac{\widehat{EX}}{2} = \angle EAX = 180° - \angle AXC$, it shows that $AX \parallel EC$ and $AXCE$ is parallelogram. Similarly $BEDX$ is parallelogram. From these we get $AE = XC, BE = XD, \angle AEB = \angle CXD$, then two triangles EAB, XCD are congruent and $AB \parallel CD$.
On the other hand

$$\begin{aligned}\angle AQP = \angle AQD &= \angle AQE + \angle EQX + \angle XQD \\ &= \angle XPC + \angle EPX + \angle BPE \\ &= \angle BPC = \angle BPQ\end{aligned}$$

Hence $ABPQ$ is isosceles trapezoid and we are done.



Problem 3. Let O be the circumcenter of triangle ABC. Arbitrary points M and N lie on the sides AC and BC, respectively. Points P and Q lie in the same half-plane as point C with respect to the line MN, and satisfy $\triangle CMN \sim \triangle PAN \sim \triangle QMB$ (in this exact order). Prove that $OP = OQ$.

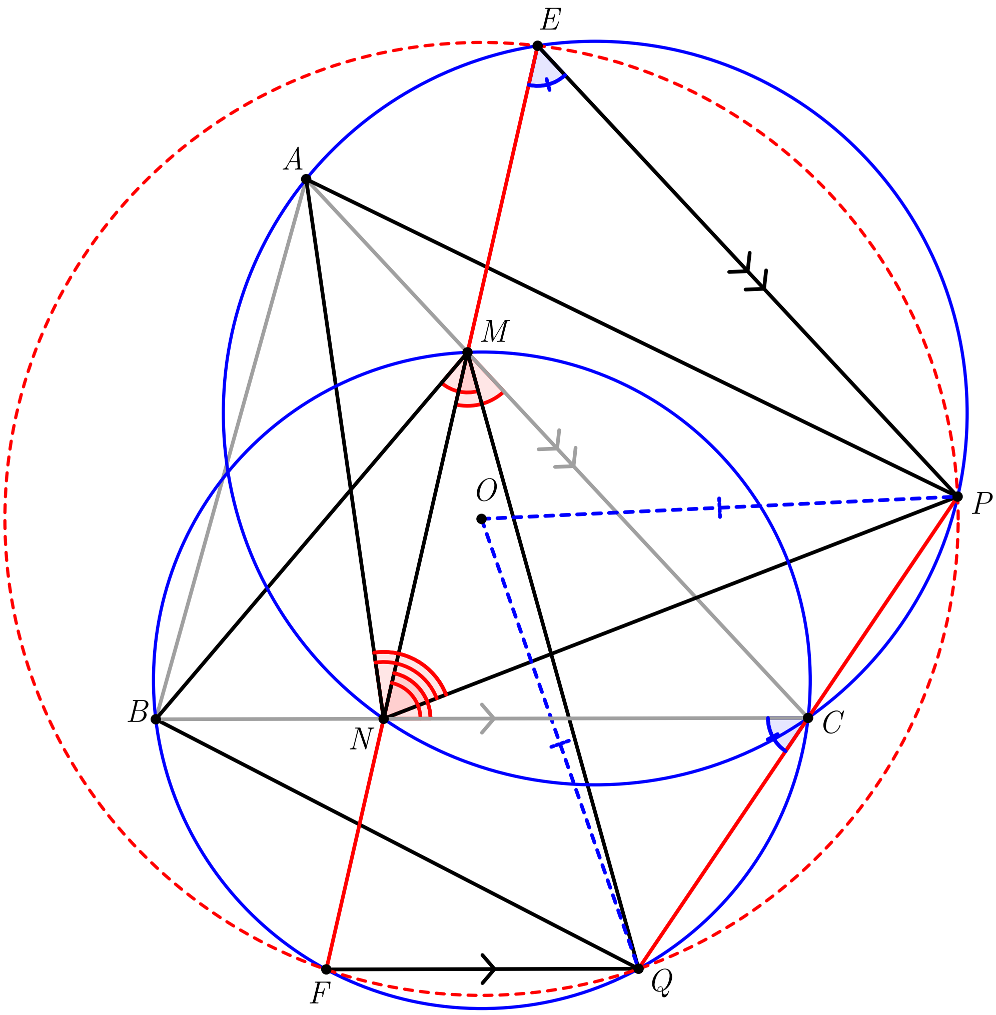
Proposed by Medeubek Kungozhin - Kazakhstan

Solution. From the statement we get quadrilaterals $ANCP$ and $BMCQ$ are concyclic, so

$$\begin{aligned}\angle ACP = \angle ANP &= 180^\circ - \angle APN - \angle PAN\\ &= 180^\circ - \angle ACB - \angle BMQ\\ &= 180^\circ - \angle ACB - \angle BCQ\end{aligned}$$

It implies that the points P, C, Q are collinear.
Let MN meet circumcircles of triangles ANC, BMC again at E, F. We have

$$\angle ANP = \angle ENC, \angle CMF = \angle BMQ$$



It shows that quadrilaterals $AEPC, BQFC$ are isosceles trapezoids, then O lies on perpendicular bisectors of EP, FQ(since O lies on perpendicular bisectors of AC, BC). Also we have

$$\angle NEP = \angle NCQ = 180 - \angle CQF = 180 - \angle PQF$$

Therefore the quadrilateral $EPQF$ is inscribed in a circle centered at O and this completes the proof.

Problem 4. We call two simple polygons P, Q *compatible* if there exists a positive integer k such that each of P, Q can be partitioned into k congruent polygons similar to the other one. Prove that for every two even integers $m, n \geq 4$, there are two compatible polygons with m and n sides. (A simple polygon is a polygon that does not intersect itself.)

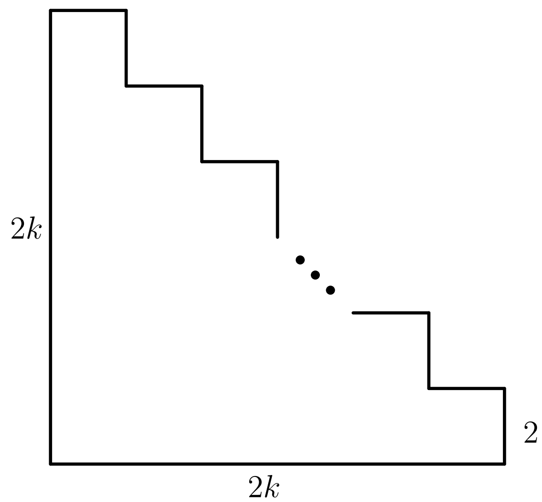
Proposed by Hesam Rajabzadeh - Iran

Solution. We begin the solution with a lemma.

Lemma. *Compatibility is an equivalence relation.*

Proof. We only need to check transitivity, i.e. whenever a polygon Q is compatible to both polygons P and R, then P is compatible with R. The other properties are trivial from the definition.
Clearly, if P can be partitioned into m congruent polygons $Q_1, \dots, Q_m$ similar to Q, and Q can be partitioned into n congruent polygons $R_1, \dots, R_n$ similar to R, then one can partition each of Q_i's into n congruent polygons $R_{i,1}, \dots, R_{i,n}$ similar to R. Note that every pair of polygons of the from $R_{i,j}$ are congruent (because they have equal area), so $\{R_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a partition of P into mn congruent polygons similar to R. Similarly, one can show that R can be partitioned into mn congruent polygons similar to P and the proof is complete. $\square$

In view of above lemma, it suffices to introduce a polygon P such that for every even number d there is simple d-gon compatible to P. We claim that one can take P to be a square.
For every $k \geq 1$, consider the following staircase $(2k+2)$-gon with two sides of length $2k$ and $2k$ sides of length 2 and denote it by P_k.



Obviously, P_k can be partitioned into $4\frac{k(k+1)}{2} = 2k(k+1)$ unit squares. On the other hand, one can partition a rectangle with side lengths $2k, 2k+2$ into two polygons congruent to P_k, and a square with side length $2k(k+1)$ can be partitioned into $k(k+1)$ of such rectangles. Putting these together, a square can be partitioned into $2k(k+1)$ polygons congruent to P_k. This shows that for every $k \geq 2$, P_k is compatible with the unit square and so by the lemma, for every pair $k, k' \in \mathbb{N}$ of even numbers, P_k and P'_k are compatible and the proof is complete.

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω with center O. Let P be the intersection of two diagonals AC and BD. Let Q be a point lying on the segment OP. Let E and F be the orthogonal projections of Q on the lines AD and BC, respectively. The points M and N lie on the circumcircle of triangle QEF such that $QM \parallel AC$ and $QN \parallel BD$. Prove that the two lines ME and NF meet on the perpendicular bisector of segment CD.

Proposed by Tran Quang Hung - Vietnam

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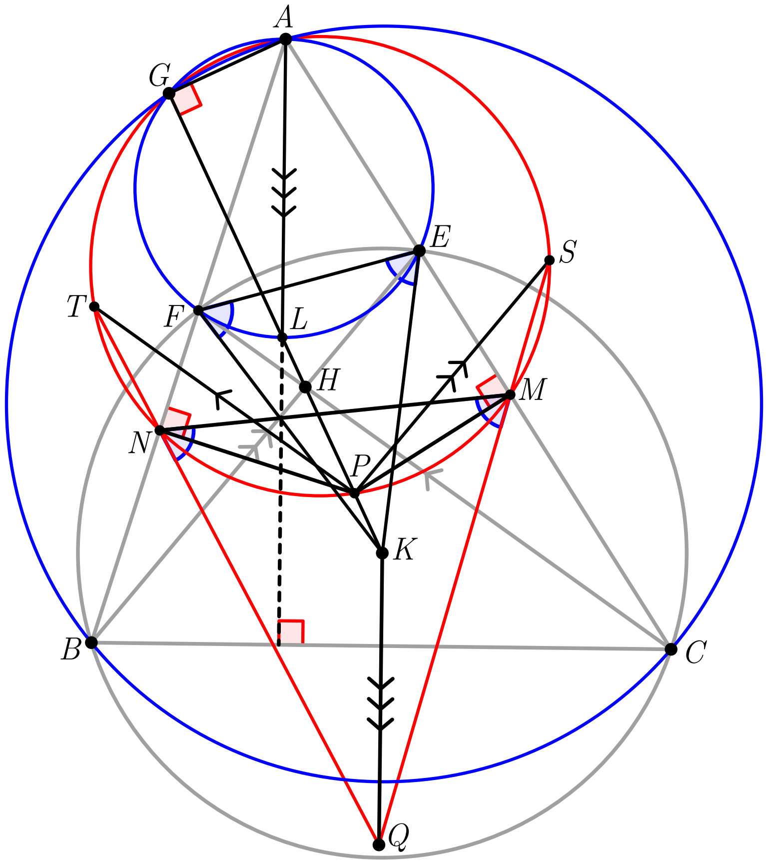
Solution. We reformulate the problem into a triangle version as follows
Problem. Let ABC be a triangle. The circle (K) passing through B, C cuts the segments CA, AB again at E, F, respectively. BE meets CF at H. Let P be a point lying on the segment KH. The circle with diameter PA meets CA, AB again at M, N, respectively. Points S, T lie on the circle with diameter PA such that $PS \parallel BE, PT \parallel CF$. Prove that MS and NT intersect on the perpendicular bisector of BC.



Two circumcircles of triangles AEF and ABC meet again at G. Easily seen $AG \perp GK$ (from the property of Miquel point) so G lies on the circle with diameter AP. Let GK meet the circumcircle of triangle AEF again at L, then AL is the diameter of circumcircle of triangle AEF, therefore $AL \perp BC$. Let MS meet NT at Q, we need to prove that $KQ \perp BC$. Indeed, we see that $PS \parallel BE$ and $PM \perp AC$, angles chasing give us

$$\begin{aligned}\angle NMQ &= \angle NMP + \angle MSP + \angle MPS \\ &= 180^\circ - \angle MPN + 90^\circ - \angle BEC \\ &= \angle BAC - \angle BEC + 90^\circ \\ &= 90^\circ - \angle EBA \\ &= \angle FEK.\end{aligned}$$

Similarly, $\angle EFK = \angle MNQ$. From these, triangles QMN and KEF are isosceles. Hence $\triangle QMN \sim \triangle KEF$. We have $\triangle GEF \sim \triangle GMN$ (since G is the center of spiral similarity which transforms $MN \mapsto EF$), so $\triangle GFK \sim \triangle GNQ$, this implies that $\triangle GFN \sim \triangle GKQ$. Hence, we have angle chasing again

$$\angle GKQ = \angle GFN = 180^\circ - \angle GFA = 180^\circ - \angle GLA = \angle ALP.$$

This leads to $KQ \parallel AL$. We are done.

Advanced Level

Problems

Problem 1. Four points A, B, C, and D lie on a circle ω such that $AB = BC = CD$. The tangent line to ω at point C intersects the tangent line to ω at point A and the line AD at points K and L. The circle ω and the circumcircle of triangle KLA intersect again at M. Prove that $MA = ML$

($\rightarrow$ p.25)

Problem 2. We are given an acute triangle ABC with $AB \neq AC$. Let D be a point on BC such that DA is tangent to the circumcircle of triangle ABC. Let E and F be the circumcenters of triangles ABD and ACD, respectively, and let M be the midpoint of EF. Prove that the line tangent to the circumcircle of AMD through D is also tangent to the circumcircle of ABC.

($\rightarrow$ p.26)

Problem 3. In triangle ABC ($\angle A \neq 90°$), let O, H be the circumcenter and the foot of the altitude from A respectively. Suppose M, N are midpoints of BC, AH respectively. Let D be the intersection of AO and BC and let H' be the reflection of H about M. Suppose that the circumcircle of $OH'D$ intersects the circumcircle of BOC at E. Prove that NO and AE are concurrent on the circumcircle of BOC.

($\rightarrow$ p.28)

Problem 4. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Its diagonals intersect at a point P. The line passing through P parallel to AB intersects AD and BC at Q and R, respectively. Exterior angle bisectors of angles DBA, DCA intersect at X. Let S be the foot of X onto BC. Prove that if quadrilaterals $ABPQ, CDQP$ are circumscribed, then $PR = PS$.

($\rightarrow$ p.31)

Problem 5. Let ABC be an acute triangle inscribed in a circle ω with center O. Points E, F lie on its sides AC, AB, respectively, such that O lies on EF and $BCEF$ is cyclic. Let R, S be the intersections of EF with the shorter arcs AB, AC of ω, respectively. Suppose K, L are the reflection of R about C and the reflection of S about B, respectively. Suppose that points P and Q lie on the lines BS and RC, respectively, such that PK and QL are perpendicular to BC. Prove that the circle with center P and radius PK is tangent to the circumcircle of RCE if and only if the circle with center Q and radius QL is tangent to the circumcircle of BFS.

($\rightarrow$ p.33)

Solutions

Problem 1. Four points A, B, C, and D lie on a circle ω such that $AB = BC = CD$. The tangent line to ω at point C intersects the tangent line to ω at point A and the line AD at points K and L. The circle ω and the circumcircle of triangle KLA intersect again at M. Prove that $MA = ML$

Proposed by Mahdi Etesamifard - Iran

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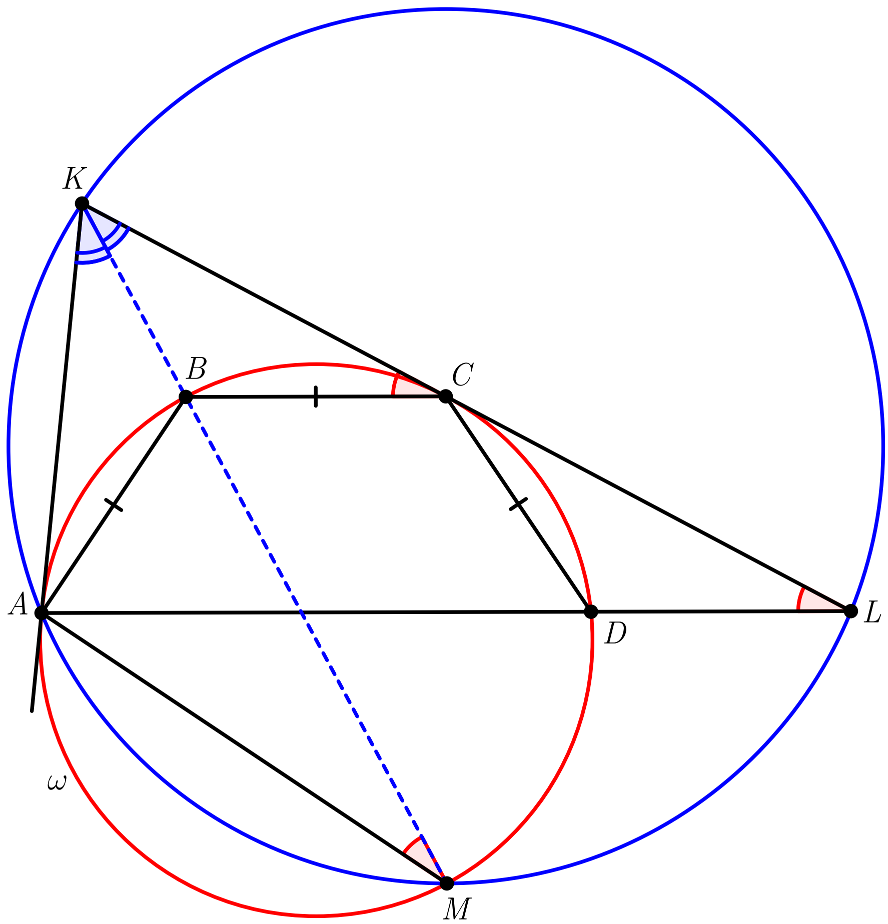
Solution. Observe that $BC \| AD$, an easy angle chase gives :

$$\measuredangle KMA = \measuredangle KLA = \measuredangle KCB = \frac{\widehat{BC}}{2} = \frac{\widehat{AB}}{2} = \measuredangle BMA$$

thus points K, B and M are collinear. But the two triangles KBC and KBA are congruent hence $\measuredangle AKB = \measuredangle CKB$ which implies that $MA = ML$ as desired.



Problem 2. We are given an acute triangle ABC with $AB \neq AC$. Let D be a point on BC such that DA is tangent to the circumcircle of triangle ABC. Let E and F be the circumcenters of triangles ABD and ACD, respectively, and let M be the midpoint of EF. Prove that the line tangent to the circumcircle of AMD through D is also tangent to the circumcircle of ABC.

Proposed by Patrik Bak - Slovakia

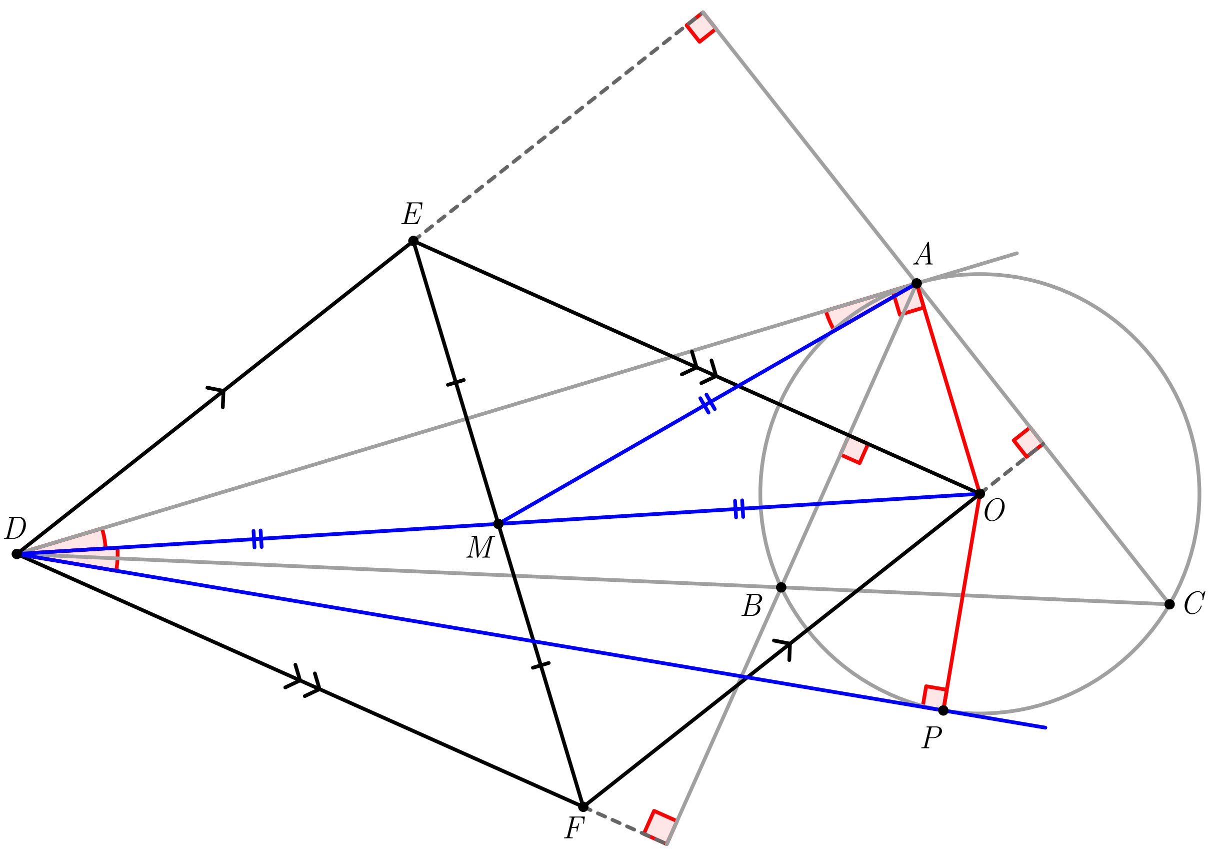
Solution. Without loss of generality assume $AB < AC$. This means that B lies between D and C. Triangle DBA is obtuse with the obtuse angle at B. Therefore, E lies in the half-plane DBC containing A. Similarly, angle DAC, equal to $180° - \angle CBA$, is obtuse, therefore F lies in the opposite half-plane. Simple angle chasing gives

$$\angle CDE = \angle BDE = 90° - \frac{1}{2}\angle DEB = 90° - \angle DAB = 90° - \angle ACB$$

and

$$\begin{aligned}\angle FDC = \angle FDB &= 90° - \frac{1}{2}\angle DFC \\ &= 90° - (180° - \angle DAC) \\ &= \angle DAC - 90° \\ &= \angle DAB + \angle BAC - 90° \\ &= \angle ACB + \angle BAC - 90° \\ &= 90° - \angle CBA\end{aligned}$$

These two equalities show that $DE \perp AC$ and $DF \perp AB$, respectively.



Denote by O the circumcenter of triangle ABC. Since FO is the perpendicular bisector of AC, we have $FO \perp AC$, therefore $DE \parallel FO$. Analogously, EO is the perpendicular bisector of AB,

and so $DF \parallel EO$. Together, we have that $DFOE$ is a parallelogram, therefore M is the midpoint of DO. Since $DA \perp AO$, we have $MD = MO = MA$.
Let P be the reflection of A in line DMO. Clearly, P lies on the circumcircle of ABC. Since $DA \perp AO$, then $OP \perp DP$, and so line DP is tangent to the circumcircle of ABC. Finally, simple angle chasing shows that $\angle PDM = \angle MDA = \angle DAM$, therefore DP is also tangent to the circumcircle of AMD, which concludes the proof.

Problem 3. In triangle ABC ($\angle A \neq 90°$), let O, H be the circumcenter and the foot of the altitude from A respectively. Suppose M, N are midpoints of BC, AH respectively. Let D be the intersection of AO and BC and let H' be the reflection of H about M. Suppose that the circumcircle of $OH'D$ intersects the circumcircle of BOC at E. Prove that NO and AE are concurrent on the circumcircle of BOC.

Proposed by Mehran Talaei - Iran

Solution 1. Let the line OH intersect the circumcircle BOC in F

Claim 1. *F, D, E are collinear*

Proof. Since $OM \perp BC$ and $MH = MH'$ we have $\measuredangle OHH' = \measuredangle OH'H$. But $OH'ED$ is cyclic and $\measuredangle OH'D = \measuredangle OED$. The claim is easily proven considering the fact that $OB = OC$ □

Now let A' be the $A-$*antipode* in the circumcircle ABC.It's easy to see that

$$OH.OF = OB^2 = OC^2 = OA^2$$

since FH is the bisector of $\measuredangle BFC$, so OA is tangent to the circumcircle AHF, thus $\measuredangle HAO = \measuredangle HFA$ or

$$\measuredangle A'AF = \measuredangle OAF = \measuredangle OHA \tag{1}$$

D lies on the radical axis of the two circumcircles ABC and BOC :

$$DF.DE = DB.DC = DA'.DA \tag{2}$$

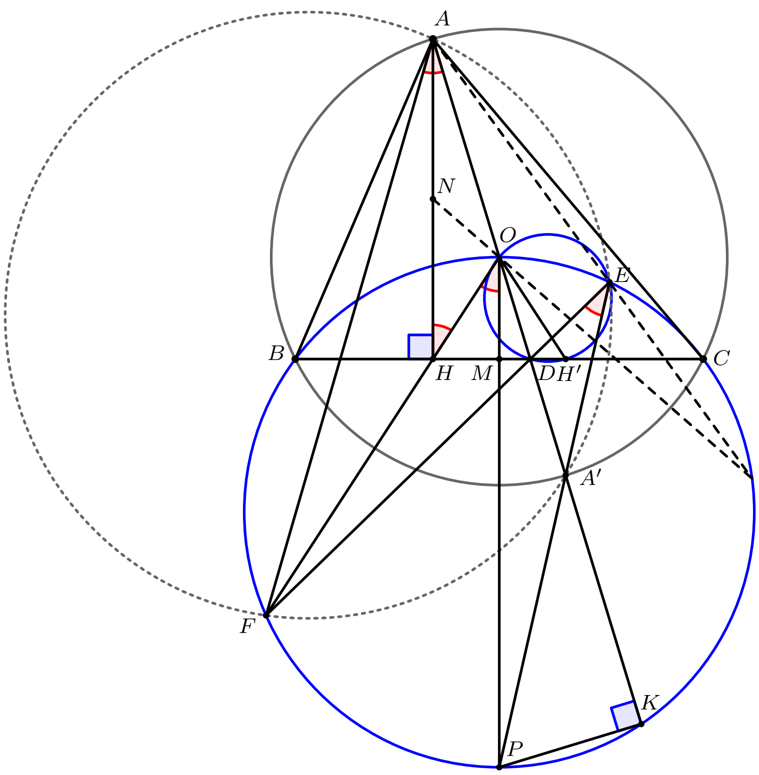
by (2) the quadrilateral $AEA'F$ is cyclic :

$$\measuredangle A'EF = \measuredangle A'AF \tag{3}$$

letting OM intersect the circumcircle BOC in P we have $OP \| AH$

$$\measuredangle POF = \measuredangle OHA \tag{4}$$

Combining (1) , (3) , (4) points E, A', P are collinear



Let AO intersect the circumcircle BOC for the second time in K

Claim 2. $(AA', DK) = -1$

Proof. We have $\measuredangle PBC = \measuredangle PCB = \measuredangle A$, PB and PC are tangent to the circumcircle ABC. Hence the line BC is the polar of P wrt the circumcircle ABC and by *La Hire's Theorem* the polar of D wrt the circumcircle ABC passes through P. Since $\measuredangle PKO = 90$, the line PK is this polar and the claim is proved! $\square$

Now projecting from O onto the line AH we have:

$$-1 = (\infty N, AH) = (PX, KF)$$

where X is the intersection point of NO and the circumcircle BOC. Taking a look at claim (2) and projecting from E

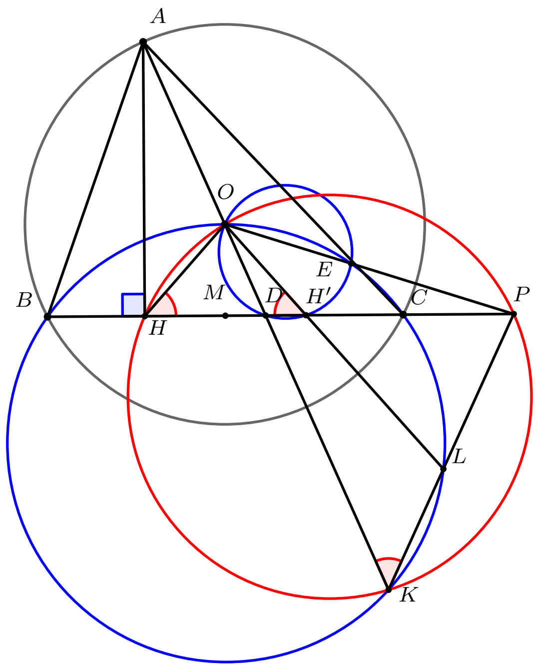
$$-1 = (AA', DK) = (YP, FK)$$

Where Y is the intersection point of AE and the circumcircle BOC Considering the last two projections $X \equiv Y$.

Solution 2. Let OD and OH' intersect the circumcircle BOC in K and L respectively.

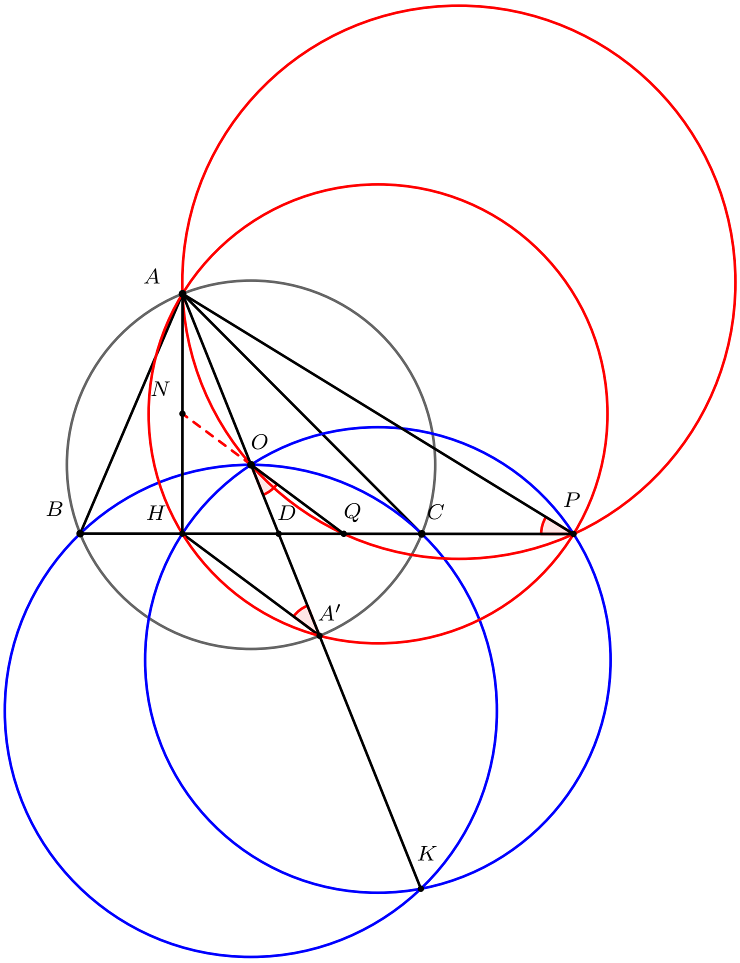
Easy to see that $\measuredangle OKL = \measuredangle OH'D = \measuredangle OHD$.

Define $P = KL \cap BC$. Then the quadrilateral $HPOK$ is cyclic.



Apply the inversion about the circumcircle ABC. The following problem is the figure after mapping :

Problem 1. *In triangle ABC the line AO intersects side BC and circumcircle BOC in D, K respectively. H is the foot of the $A-altitude$. The line BC intersects the circumcircle OHK in P and Q is the second intersection of this line with the circumcircle AOP. Prove that OQ bisects AH.*



Proof.
Let A' be the $A-$*antipode* in the circumcircle ABC.
D is the radical center of the three circumcircles OHK, ABC, BOC, hence

$$DH.DP = DO.DK = DA'.DA$$

implying that the quadrilateral $AHA'P$ is cyclic

$$\measuredangle HA'A = \measuredangle HPA = \measuredangle QOD$$

therefore the lines HA' and OQ are parallel.
But O is the midpoint of AA' thus QO bisects AH as desired ! $\square$

Problem 4. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Its diagonals intersect at a point P. The line passing through P parallel to AB intersects AD and BC at Q and R, respectively. Exterior angle bisectors of angles DBA, DCA intersect at X. Let S be the foot of X onto BC. Prove that if quadrilaterals $ABPQ, CDQP$ are circumscribed, then $PR = PS$.

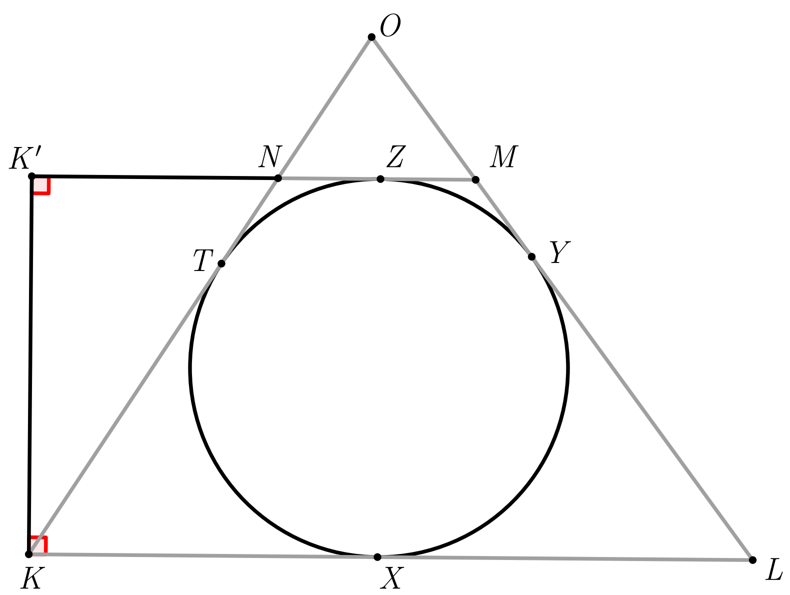
Proposed by Dominik Burek - Poland

Solution. We start with the following simple lemma.

Lemma 1. *Let $KLMN$ be a circumscribed trapezoid with $KL \parallel MN$. Suppose that rays $\overrightarrow{KN}, \overrightarrow{LM}$, intersect at O. Let K' be the projection of K onto MN. Then $KO = OM + K'M$.*

Proof. Let X, Y, Z, T be the points of tangency of the incircle with KL, LM, MN, NK, respectively. Then

$$KO = KT + TO = KX + YO = K'Z + OM + MY = K'Z + OM + MZ = K'M + OM$$

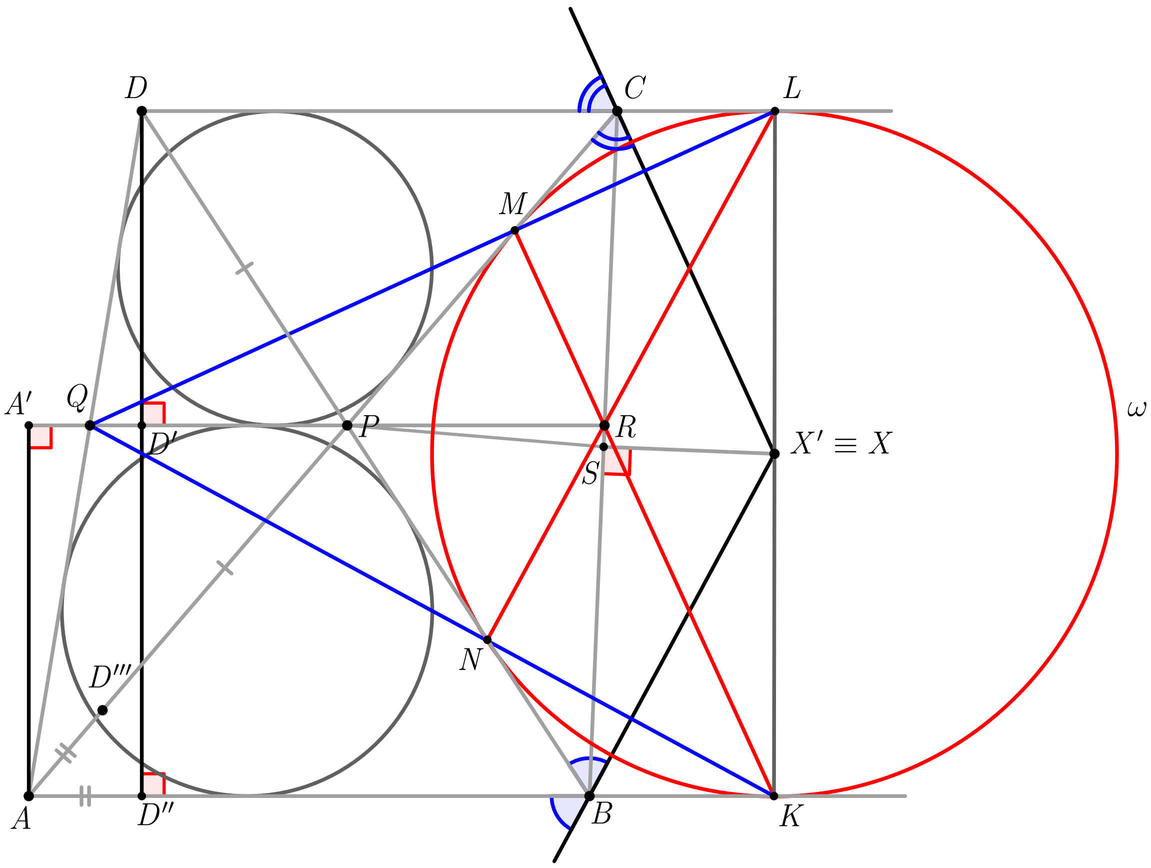


$\square$

Let A', D' be the feet of A, D onto PQ. Applying lemma for circumscribed trapezoids $ABPQ$ and $DCPQ$ we obtain $DP + A'P = AD = AP + D'P$.

Without loss of generality assume that $\angle BAD \leq 90°$. Denote the foot of D onto AB by D''. Then $AD'' = A'D' = A'P - D'P = AP - DP$.

Choose D''' on AP such that $PD''' = PD$. Then $AD''' = AP - D'''P = AP - DP = AD''$. Hence internal angle bisectors of $D''AD'''$ and DPD''' coincide with perpendicular bisectors of $D''D'''$ and DD'''. It follows that the circumcenter X' of $DD''D'''$ coincides with the A-excenter of ABP. But it lies on perpendicular bisector of DD'' as well, which is the line parallel to AB and CD equidistant from them. It follows that the A-excircle ω of ABP is tangent to CD. It is easy to see now that X' lies on the exterior angle bisectors of DBA and DCA, hence $X' = X$. Let K, L, M, N be the points of tangency of ω with AB, CD, AC, BD, respectively. Let X_∞ be the point at infinity of lines AB, CD, PR. Brianchon theorem for quadrilateral $BX_\infty CP$ shows that MK and NL intersect at R.



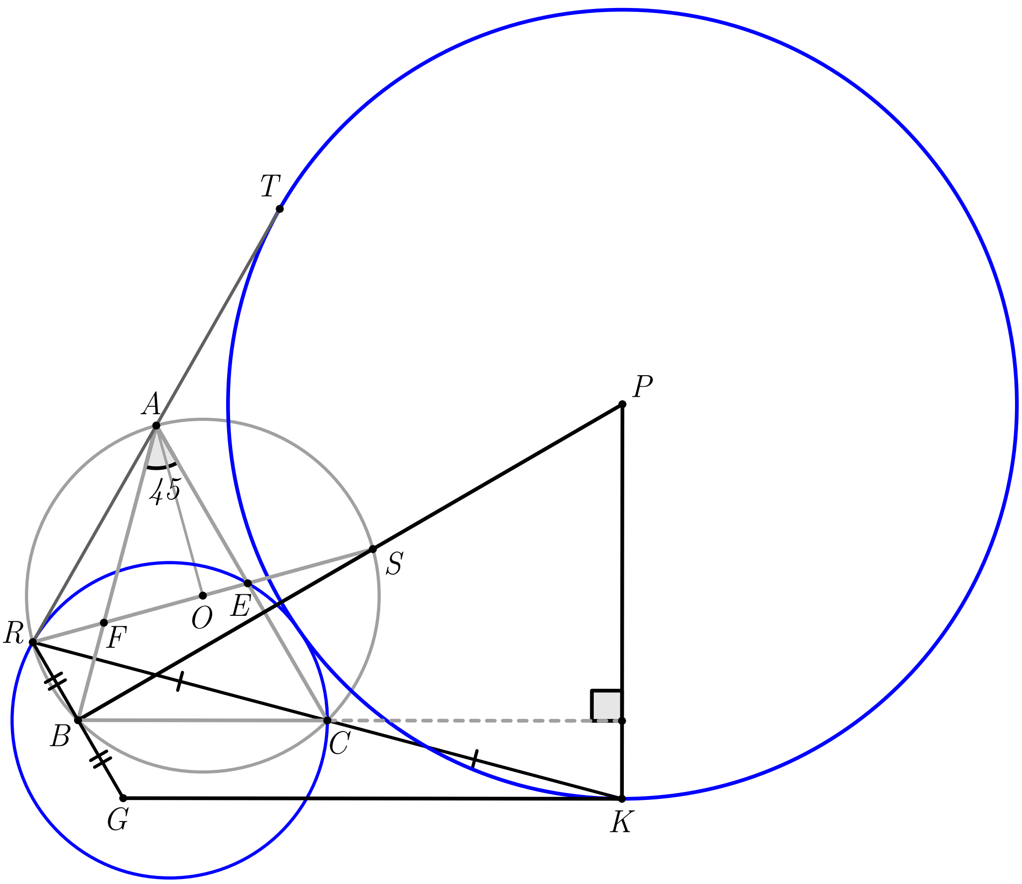
Since $PR \parallel AK$ and $AK = AM$, we have $PR = PM$. Similarly, $PN = PR$, and, obviously, $PQ = AB \cdot \frac{DP}{DB} = AB \cdot \frac{CP}{CA} = PR$. Therefore M and N lie on the circle with diameter QR, so $\angle QMR = \frac{\pi}{2}$ and $\angle RNQ = \frac{\pi}{2}$. Since KL is a diameter of ω, we have $\angle KML = \frac{\pi}{2} = \angle KNL$. It follows that Q, M, L are collinear and Q, N, K are collinear.

Since ML is polar of C with respect to ω and NK is polar of B with respect to ω, we obtain that BC is polar of Q. In particular $QX \perp BC$, so the intersection of QX with BC is the foot of X onto BC, i.e. the point S. Hence $\angle QSR = \frac{\pi}{2}$, thus S lies on the circle with diameter QR, so $PS = PR$.

Problem 5. Let ABC be an acute triangle inscribed in a circle ω with center O. Points E, F lie on its sides AC, AB, respectively, such that O lies on EF and $BCEF$ is cyclic. Let R, S be the intersections of EF with the shorter arcs AB, AC of ω, respectively. Suppose K, L are the reflection of R about C and the reflection of S about B, respectively. Suppose that points P and Q lie on the lines BS and RC, respectively, such that PK and QL are perpendicular to BC. Prove that the circle with center P and radius PK is tangent to the circumcircle of RCE if and only if the circle with center Q and radius QL is tangent to the circumcircle of BFS.

Proposed by Mehran Talaei - Iran

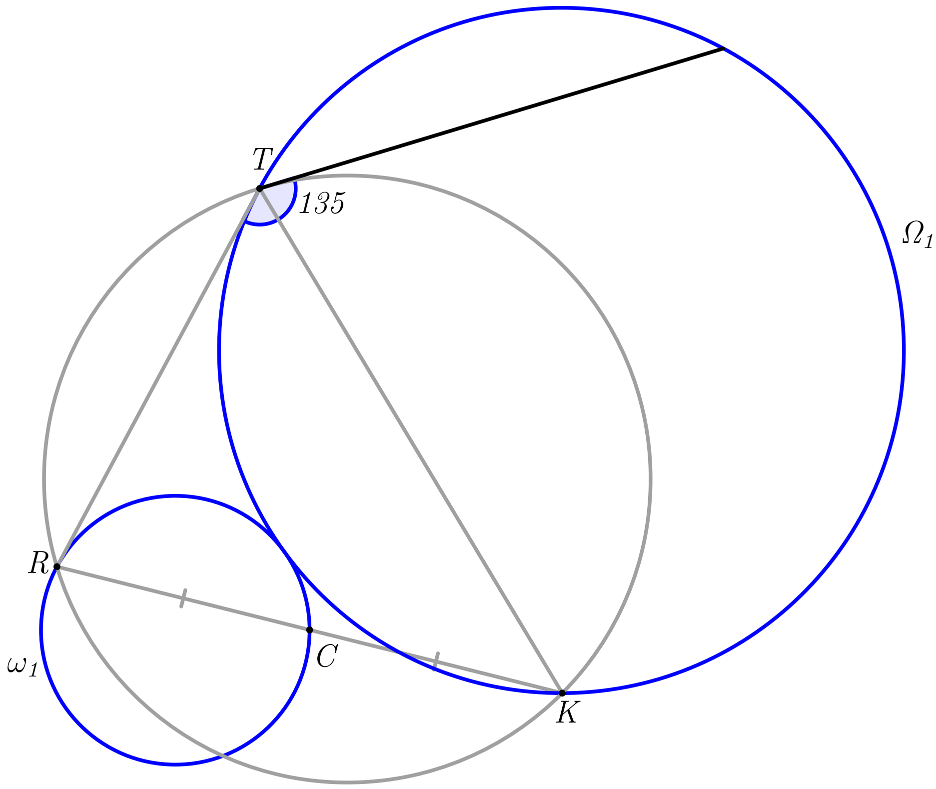
Solution. Let ω_1, ω_2 be the circumcircles of RCE and SBF and let Ω_1, Ω_2 be the circles with centers P, Q and radius PK, QL respectively. Notice that $AO \perp EF$ because $\angle OAE = 90° - \angle ABC$ and $\angle OEA = \angle ABC$. So $\overset{\frown}{AR} = \overset{\frown}{AS} = 90°$



Claim 1. *ω_1 and Ω_1 are tangent iff $\angle BAC = 45°$.*
First of all it is trivial to see this concludes the problem. Now to the proof:

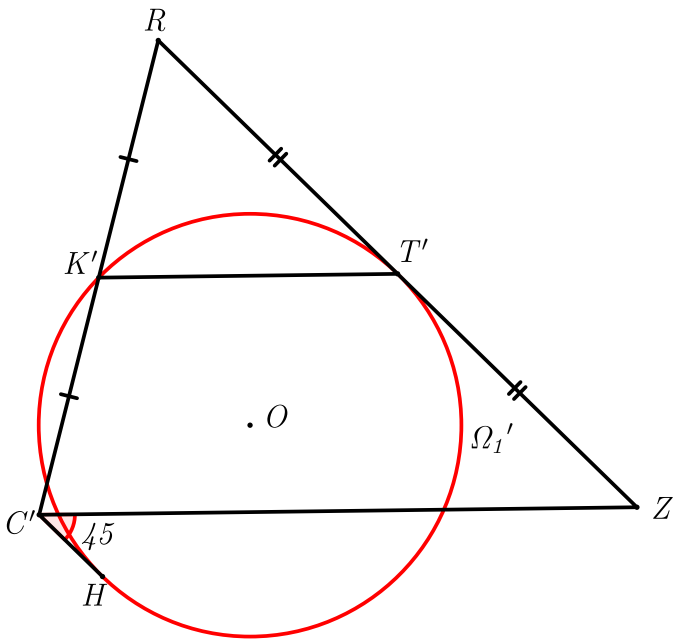
Proof. First of all notice that $\angle RBS = 90°$. Let G be the reflection of R about B. It is obvious that BS is the perpendicular bisector of RG and GK is parallel to BC so by $PK \perp BC$, we can conclude that GK is tangent to Ω_1. Now if T is the reflection of K about BS, it is clear that TR is also tangent to Ω_1. So the circumcircle of ABC maps to the circumcircle of $RTKG$ with homothety with its center at R and 2 as its scale. Note that with a little angle chasing the angle between the circumcircle of ABC and circumcircle of REC is $135°$, thus the angle between circumcircle of REC and circumcircle of RTK is also $135°$. Now we can rewrite the problem from the perspective of the triangle RTK:

Problem 1. In triangle RTK, let Ω_1 be a circle passing through K and tangent to RT at T. Also let ω_1 be a circle passing through R and C, midpoint of RK, which angle with the circumcircle of RKT is $135°$. Prove that ω_1 is tangent to Ω_1 iff $RKT = 45°$.



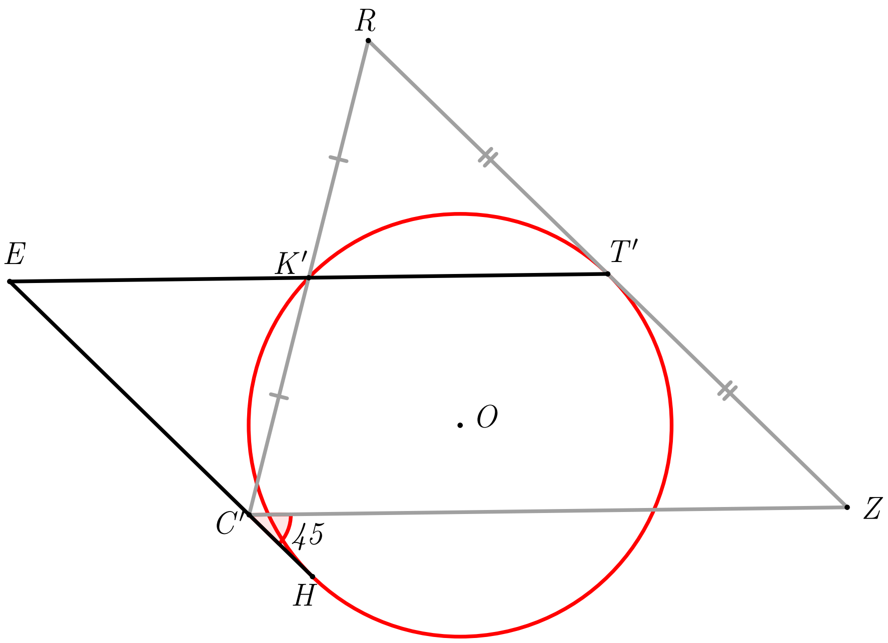
We prove the problem by performing an inversion centered at R with radius $RK.RT$ and then reflecting about the angle bisector of $K'RT'$. The following is the resulted problem: (The images of the points are denoted by primes)

Problem 2. In triangle $RK'T'$, let C' be the reflection of R about K' and Z the reflection of R about T'. Let Ω_1' be a a circle passing through K' and tangent to RT' at T' and let ω_1' be a line passing through C' which the angle between ω_1' and $C'Z$ is $45°$. Then ω_1' is tangent to Ω_1' iff $\angle RT'K' = 45°$.



Part 1. $\angle RT'K' = 45°$
Let O be the circumcenter of Ω'_1. Notice that $K'O \parallel RZ \parallel \omega'_1$. Then $K'O$ is the midline of C' in triangle $RC'Z$. Let E be the intersection of ω'_1 and $K'T'$ thus K' is the midpoint of $T'E$. So the reflection of T' about O ,H, lies on EC'. So H is the tangent point of ω'_1 and Ω'_1 because $\angle OHC = 90°$.

Part 2. ω'_1 is tangent to Ω'_1



Let α be the angle $\angle RZC'$. It's easy to see that $45° = \angle K'EH = \frac{\widehat{T'H} - \widehat{K'H}}{2}$ and $\frac{\widehat{T'H} + \widehat{K'H}}{2} = 180° - \frac{\widehat{K'T'}}{2} = 180° - \alpha$. So $\widehat{T'H} = 225° - \alpha$ and $\widehat{K'H} = 135° - \alpha$. Now we split the problem into two the cases $\alpha > 45°$ and $\alpha < 45°$.
Case 1. $\alpha > 45°$
With easy angle chasing we conclude that $\angle K'C'E > \angle K'RT'$. Therefor $K'E > K'T'$. So $(\frac{HK'}{HT'})^2 = \frac{EK'}{ET'} > \frac{1}{2}$. Thus:

$$\frac{\sin(\frac{135°-\alpha}{2})}{\sin(\frac{225°-\alpha}{2})} > \frac{1}{\sqrt{2}}$$

So if $\theta = \frac{135°-\alpha}{2}$, then $\sqrt{2}\sin\theta > \sin(\theta+45) = \frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta$, so $\sin\theta > \cos\theta$ hence $\theta > 45°$ therefor $\alpha < 45°$, which is a contradiction.
Case 2. $\alpha < 45°$
It's similar to the first case.
As shown above both cases cause a contradiction so α must be equal to $45°$, as requested. $\square$