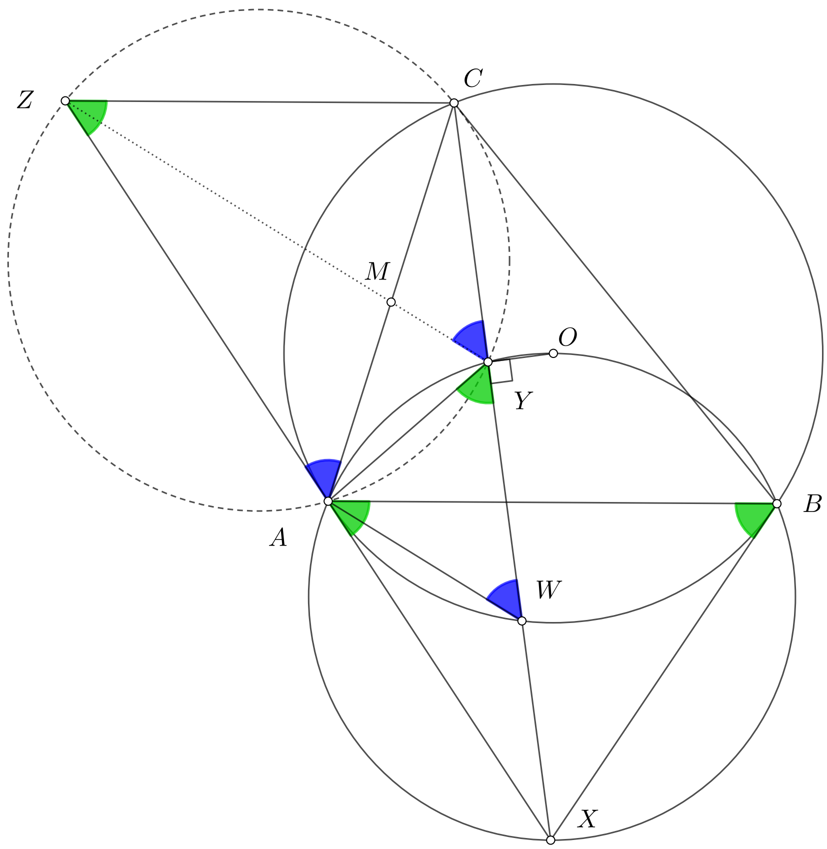
Problem 1

Problem. Let ABC be an acute triangle such that $CA \neq CB$ with circumcircle ω and circumcentre O. Let t_A and t_B be the tangents to ω at A and B respectively, which meet at X. Let Y be the foot of the perpendicular from O onto the line segment CX. The line through C parallel to line AB meets t_A at Z. Prove that the line YZ passes through the midpoint of the line segment AC.

Solution 1. Firstly observe that $OAXB$ is cyclic, with diameter OX, and Y also lies on this circle since $OY \perp XC$. Hence:

$$\angle AZC = \angle XAB = \angle ABX = \angle AYX$$

and so $CYAZ$ is cyclic.



Let M be the intersection of YZ and AC and let CY intersect ω again at W. Using the new cyclic relation we get $\angle CYZ = \angle CAZ$ and then using that ZA is tangent to ω we get $\angle CAZ = \angle CWA$, so $\angle CYM = \angle CWA$. Therefore the triangles CWA and CYM are similar. But CW is a chord of ω, and Y is the foot of the perpendicular from O, hence Y is the midpoint of CW. It follows from the similarity relation that M is the midpoint of AC, as required.

Solution 2. Let M be the midpoint of AC. We have $\angle CAZ = \angle CBA$ and $\angle ZCA = \angle BAC$ so the triangles CAZ and ABC are similar. The line CYX is the C-symmedian of triangle ABC, and ZM is the corresponding median in triangle CAZ, hence by isogonality $\angle AZM = \angle ACY$. So

$$\angle ZMA = 180^\circ - \angle AZM - \angle MAZ = 180^\circ - \angle ACY - \angle CBA \tag{1}$$

Now observe $\angle OMC = \angle OYC = 90^\circ$, so $CMYO$ is cyclic. Thus:

$$\angle CYM = \angle COM = \frac{1}{2}\angle COA = \angle CBA.$$

This shows that

$$\angle YMC = 180^\circ - \angle MCY - \angle CYM = 180^\circ - \angle ACY - \angle CBA$$

Combining this with (1) we get that $\angle YMC = \angle ZMA$ and as A, C, M are collinear, it follows that Z, M, Y are collinear as required.

Solution 3. As in Solution 2 we have that CX is the A-symmedian of triangle ABC and that triangle ABC is similar to triangle CAZ.

Let f be the spiral similarity which maps AC onto AB and let g be the reflection on the perpendicular bisector of AB. Note that f is a rotation about A by an angle of $\angle CAB$ (clockwise in our figure) followed by a homothety centered at A by a factor of AB/AC. By the similarity of triangles ABC and CAZ we have that $g(f(Z)) = C$, so actually $f(Z)$ is the other point of intersection, say C', of CZ with ω.

As in Solution 1 we have that $CYAZ$ is cyclic. Therefore, letting W be the other point of intersection of CY with ω, we have $\angle WAB = \angle WCB = \angle CAY$. We also have $\angle ACY = \angle ABW$. It follows that $f(Y) = W$.

Let $W' = g(W)$. Then $W' \in \omega$ and since CW is the A-symmedian, then CW' passes through the midpoint N of AB. Now CW' and $C'W$ intersect on the perpendicular bisector of AB and therefore they intersect on N. It follows that $N = AB \cap C'W = Af(C) \cap f(Z)f(Y)$ is the image of $M = AC \cap ZY$ under f. Since N is the midpoint of AB, then M is the midpoint of AC.

Solution 4. Let $E = AB \cap CX$ and $F = AW \cap CZ$. We have $(C, W; X, E) = -1$. Projecting from the line CX onto the line CZ from A we get that $(C, F; Z, \infty) = -1$. Thus Z is the midpoint of CF. Since also Y is the midpoint of CW, we get that ZY bisects CA.

Problem 2

Problem. Let a, b and n be positive integers with $a > b$ such that all of the following hold:

(i) a^{2021} divides n,

(ii) b^{2021} divides n,

(iii) 2022 divides $a - b$.

Prove that there is a subset T of the set of positive divisors of the number n such that the sum of the elements of T is divisible by 2022 but not divisible by 2022^2.

Solution. If $1011 \mid a$, then $1011^{2021} | n$ and we can take $T = \{1011, 1011^2\}$. So we can assume that $3 \nmid a$ or $337 \nmid a$.

We continue with the following claim:

Claim. If k is a positive integer, then $a^k b^{2021-k} \mid n$.

Proof of the Claim. We have that $n^{2021} = n^k \cdot n^{2021-k}$ is divisible by $a^{2021k} \cdot b^{2021(2021-k)}$ and taking the 2021-root we get the desired result. $\square$

Back to the problem, we will prove that the set $T = \{a^k b^{2021-k} : \ k \geqslant 0\}$ consisting of 2022 divisors of n, has the desired property. The sum of its elements is equal to

$$S = \sum_{k=0}^{2021} a^k b^{2021-k} \equiv \sum_{k=0}^{2021} a^{2021} \equiv 0 \bmod 2022\,.$$

On the other hand, the last sum is equal to $\frac{a^{2022}-b^{2022}}{a-b}$.

If $3 \nmid a$, we will prove that S is not divisible by 9. Indeed if $3 \nmid a$ then we also have $3 \nmid b$. So if $3^t \mid\mid a - b$ then, since $3^1 \mid\mid 2022$, by the Lifting the Exponent Lemma, we have that $3^{t+1} \mid\mid a^{2022} - b^{2022}$. This implies that S is not divisible by 9, therefore, 2022^2 doesn't divide S.

If $3 \mid a$, then we have $337 \nmid a$ and a similar argument shows that $337^2 \nmid S$.

Problem 3

Problem. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(y\left(f\left(x\right)\right)^3 + x\right) = x^3 f(y) + f(x)$$

for all $x, y > 0$.

Solution 1. Setting $y = \frac{t}{f(x)^3}$ we get

$$f(x+t) = x^3 f\left(\frac{t}{f(x)^3}\right) + f(x) \tag{1}$$

for every $x, t > 0$.

From (1) it is immediate that f is increasing.

Claim. $f(1) = 1$

Proof of Claim. Let $c = f(1)$. If $c < 1$, taking $x = 1$ and $y = \frac{1}{1-c^3}$ we have $y - yc^3 = 1$, so $yf(1)^3 + 1 = y$ and $f(yf(1)^3 + 1) = f(y) = 1^3 f(y)$. Thus $f(1) = 0$, a contradiction. Assume now for contradiction that $c > 1$. We claim that

$$f(1 + c^3 + \cdots + c^{3n}) = (n+1)c$$

for every $n \in \mathbb{N}$. We proceed by induction, the case $n = 0$ being trivial. The inductive step follows easily by taking $x = 1, t = c^3 + c^6 + \cdots + c^{3(k+1)}$ in (1).

Now taking $x = 1 + c^3 + \cdots + c^{3n-3}, t = c^{3n}$ in (1) we get

$$(n+1)c = f(1 + c^3 + \cdots + c^{3n}) = (1 + c^3 + \cdots + c^{3n-3})^3 f\left(\frac{c^{3n}}{(cn)^3}\right) + nc$$

giving

$$f\left(\frac{c^{3n-3}}{n^3}\right) = \frac{c}{(1 + c^3 + \cdots + c^{3n})^3} < c = f(1) \implies \frac{c^{3n-3}}{n^3} < 1\,.$$

But this leads to a contradiction if n is large enough. $\square$

Now for $x = 1$ we get $f(y+1) = f(y) + 1$ and since $f(1) = 1$ inductively we get $f(n) = n$ for every $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$, setting $x = n, y = q = m/n$ we get

$$mn^2 + n = f\left(qn^3 + n\right) = f(yf(x)^3 + x) = x^3 f(y) + f(x) = n^3 f\left(q\right) + n \implies f(q) = q\,.$$

Since f is strictly increasing with $f(q) = q$ for every $q \in \mathbb{Q}^{>0}$ we deduce that $f(x) = x$ for every $x > 0$. It is easily checked that this satisfies the functional equation.

Solution 2. We can also derive a contradiction in the case $c > 1$ as follows:

Since f is strictly increasing then

$$f(y) + f(1) = f(yf(1)^3 + 1) > f(yf(1)^3) \implies f(c^3 y) < f(y) + c$$

for every $y > 0$. So by induction we get $f(c^{3n}) < (n+1)c$ for every $n \in \mathbb{N}$. Setting $x = c^{3n}$ and $t = c^{3n+3} - c^{3n}$ in (1) we get

$$(n+2)c > f(c^{3n+3}) > f(c^{3n+3}) - f(c^{3n}) = c^{9n} f\left(\frac{c^{3n+3} - c^{3n}}{f(c^{3n})^3}\right) > c^{9n} f\left(\frac{c^{3n+3} - c^{3n}}{c^3(n+1)^3}\right)$$

But

$$\frac{c^{3n+3} - c^{3n}}{c^3(n+1)^3} = \frac{c^{3n}}{(n+1)^3} \cdot \frac{c^3 - 1}{c^3} > 1$$

for c large enough. So $(n+2)c > c^{9n+1}$ which leads to a contradiction if n is large enough.

Problem 4

Problem. Consider an $n \times n$ grid consisting of n^2 unit cells, where $n \geqslant 3$ is a given odd positive integer. First, Dionysus colours each cell either red or blue. It is known that a frog can hop from one cell to another if and only if these cells have the same colour and share at least one vertex. Then, Xanthias views the colouring and next places k frogs on the cells so that each of the n^2 cells can be reached by a frog in a finite number (possibly zero) of hops. Find the least value of k for which this is always possible regardless of the colouring chosen by Dionysus.

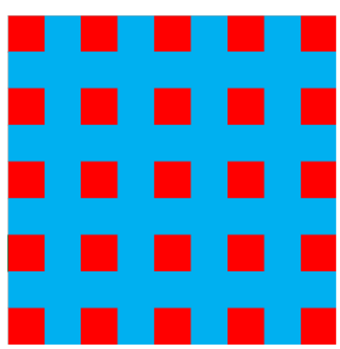
Note. Dionysus and Xanthias are characters from the play of Aristophanes 'frogs'. Dionysus is the known god of wine and Xanthias is his witty slave.

Solution 1. Let G be the graph whose vertices are all $(n+1)^2$ vertices of the grid and where two vertices are adjacent if and only if they are adjacent in the grid and moreover the two cells in either side of the corresponding edge have different colours.

The connnected components of G, excluding the isolated vertices, are precisely the boundaries between pairs of monochromatic regions each of which can be covered by a single frog. Each time we add one of these components in the grid, it creates exactly one new monochromatic region. So the number of frogs required is one more than the number of such components of G.

It is easy to check that every corner vertex of the grid has degree 0, every boundary vertex of the grid has degree 0 or 1 and every 'internal' vertex of the grid has degree $0, 2$ or 4. It is also easy to see that every component of G which is not an isolated vertex must contain at least four vertices unless it is the boundary of a single corner of the grid, in which case it contains only three vertices.

Writing N for the number of components which are not isolated vertices, we see that in total they contain at least $4N-4$ vertices. (As at most four of them contain 3 vertices and all others contain 4 vertices.) Since we also have at least 4 components which are isolated vertices, then $4N = (4N-4) + 4 \leqslant (n+1)^2$. Thus $N \leqslant \frac{(n+1)^2}{4}$ and therefore the minimal number of frogs required is $\frac{(n+1)^2}{4} + 1$.



This bound for $n = 2m+1$ is achieved by putting coordinates (x, y) with $x, y \in \{0, 1, \ldots, 2m\}$ in the cells and colouring red all cells both of whose coordinates are even, and blue all other cells. An example for $n = 9$ is shown above.

Solution 2. Consider an $n \times m$ grid with $n, m \geqslant 3$ being odd. We say that a column is of 'Type A' if, when partitioned into its monochromatic pieces, the first and last piece have the same colour with each one containing at least two cells. Otherwise we say that that it is of 'Type B'.

It is enough to show that the number F of frogs required satisfies the inequality

$$F \leqslant \frac{(m+1)(n+1)}{4} + 1 - C \qquad (1)$$

where C is the number of boundary columns of Type A.

We will proceed by induction but we first need a preliminary result.

Claim. Consider two neighbouring columns of height n which when taken alone need k and ℓ frogs respectively. Let $k + t$ be the number of frogs required when both columns are taken together. (It is allowed for t to be negative.) Then the maximum value of t is given by the following table according to the types of the two columns:

Column 1	Column 2	t
A	A	$\min\left\{\frac{\ell+1}{2}, \frac{n-k}{2}\right\}$
A	B	$\min\left\{\frac{\ell+1}{2}, \frac{n-k+2}{2}\right\}$
B	A	$\min\left\{\frac{\ell-1}{2}, \frac{n-k}{2}\right\}$
B	B	$\min\left\{\frac{\ell}{2}, \frac{n-k+1}{2}\right\}$

Proof of Claim. Note that for every two consecutive monochromatic regions of the second column, one can be covered by a frog from the first column. This is because there is a cell in the first column which neighbours both of them and a from can jump from it to the region of the corresponding colour. So the new frogs needed is at most $\frac{\ell+1}{2}$. Furthermore, if we have equality, then ℓ must be odd so its top and bottom cell have the same colour. Furthermore the neighbouring cells in the first column must be of opposite colour, so the first column is of Type A. If the first column is of Type B and the second column is of Type A, then even $\frac{\ell}{2}$ cannot be achieved. If it could, then ℓ ought to be even but this contradicts the fact that the second column is of type A.

We draw the $k-1$ horizontal lines separating the first column into monochromatics regions and suppose that those they have heights $h_1, h_2, \ldots, h_k$. Note that the cells touching these lines in the second column do not need any frog as a frog from the first column can jump to them. So the remaining cells are partitioned in columns of heights $h_1 - 1, h_2 - 2, \ldots, h_{k-1} - 2, h_k - 1$ all of whose cells to the left are the same colour. Now in each one of them we will need at most $\frac{h_1}{2}, \frac{h_2-1}{2}, \ldots, \frac{h_{k-1}-1}{2}, \frac{h_k}{2}$ frogs. Their sum is $\frac{n-k+2}{2}$ so we need at most that many frogs. Equality holds only if h_1, h_k are even and the other h_i's are odd. In that case, since their sum is equal to n which is odd we must have that k is odd. So the first column must be of Type A. Furthermore, if the second column is of Type A, then the first and last monochromatic regions need at most $\frac{h_1-1}{2}$ and $\frac{h_k-1}{2}$ new frogs respectively. So the total number of new frogs needed is at most $\frac{n-k}{2}$. $\square$

Suppose now that $m = 3$ and the middle column needs k frogs. So depending on the type of the three columns we need at most the following number of frogs:

Column 1	Column 2	Column 3	Number of Frogs
A	A	A	$k + \frac{n-k}{2} + \frac{n-k}{2} = n$
A	A	B	$k + \frac{n-k}{2} + \frac{n-k+2}{2} = n+1$
A	B	A	$k + \frac{n-k}{2} + \frac{n-k}{2} = n$
A	B	B	$k + \frac{n-k}{2} + \frac{n-k+1}{2} = n - \frac{1}{2}$
B	A	A	$k + \frac{n-k+2}{2} + \frac{n-k}{2} = n+1$
B	A	B	$k + \frac{n-k+2}{2} + \frac{n-k+2}{2} = n+2$
B	B	A	$k + \frac{n-k+1}{2} + \frac{n-k}{2} = n + \frac{1}{2}$
B	B	B	$k + \frac{n-k+1}{2} + \frac{n-k+1}{2} = n+1$

This proves (1) for the case $m = 3$ as the claim is $F \leqslant n + 2 - C$ and it can be checked that this is satisfied in all cases.

Suppose now by induction that the result is true for m and we are trying to prove it for $m + 2$. We attach two columns at the end of the table. We need to show that we need additionally at most $\frac{n+1}{2} + C_{\text{old}} - C_{\text{new}}$ number of frogs.

Suppose they need k and ℓ frogs respectively. So depending on the type of these two columns with the previous one we need at most the following additional number of frogs:

Column 1	Column 2	Column 3	Number of Frogs
A	A	A	$\frac{k+1}{2} + \frac{n-k}{2} = \frac{n+1}{2}$
A	A	B	$\frac{k+1}{2} + \frac{n-k+2}{2} = \frac{n+3}{2}$
A	B	A	$\frac{k+1}{2} + \frac{n-k}{2} = \frac{n+1}{2}$
A	B	B	$\frac{k+1}{2} + \frac{n-k+1}{2} = \frac{n+2}{2}$
B	A	A	$\frac{k-1}{2} + \frac{n-k}{2} = \frac{n-1}{2}$
B	A	B	$\frac{k-1}{2} + \frac{n-k+2}{2} = \frac{n+1}{2}$
B	B	A	$\frac{k}{2} + \frac{n-k}{2} = \frac{n}{2}$
B	B	B	$\frac{k}{2} + \frac{n-k+1}{2} = \frac{n+1}{2}$

It can now be checked (using also that n is odd) that this completes the inductive step.