Estonian Math Competitions 2019/2020

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Estonian Mathematical Olympiad

http://www.math.olympiaadid.ut.ee/

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools who have never been enrolled in a university or other higher educational institution, all Estonian citizens who meet the participation criteria of the forthcoming IMO may participate in these contests. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Besides the above-mentioned contests and the quiz “Kangaroo”, other regional and international competitions and matches between schools are held.

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This booklet presents selected problems of the open contests, the final round of national olympiad and the team selection contest. Selection has been made to include only problems that have not been taken from other competitions or problem sources without significant modification and seem to be interesting enough.

Selected Problems from Open Contests

O1. *(Juniors.)* Jüri writes on blackboard some consecutive integers. It is known that the total number of these integers is greater than one and the least of them is greater than 2. Mari writes on blackboard consecutive integers, too, with the same total number of them as Jüri, but the least of them equals 1. Is it possible that the product of the integers written by Jüri divided by the product of the integers written by Mari is equal to the square of some integer?

Answer: Yes.

Solution: For example, the product $8 \cdot 9$ of two consecutive integers divided by the product $1 \cdot 2$ of the first two positive integers equals 6^2.

O2. *(Juniors.)* Juku paints exactly 30 unit squares of a 10×15 table black. After that, Miku covers up exactly 4 rows and 4 columns. Can Juku ensure by the choice of the squares to be coloured that at least 10 black unit squares are left uncovered?

Answer: Yes.

Solution: Juku can paint 30 unit squares in such a way that each row contains exactly 3 black unit squares and each column contains exactly 2 of them (Fig. 1). When Miku chooses 4 rows and 4 columns, they contain at most $4 \cdot 3 + 4 \cdot 2 = 20$ black unit squares altogether. Hence at least 10 black unit squares remain uncovered.

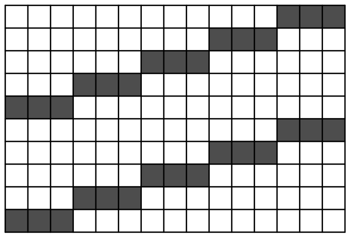


Fig. 1

O3. *(Juniors.)* Find all prime numbers p such that $\frac{p-1}{2}$ and $\frac{p+1}{4}$ are prime numbers, too.

Answer: 7 and 11.

Solution 1: Let $q = \frac{p-1}{2}$ and $r = \frac{p+1}{4}$; then $p = 4r - 1$ and $q = \frac{4r-2}{2} = 2r - 1$. Consider all remainders that can be left when r is divided by 3:

- If $r \equiv 1 \pmod 3$ then $4r - 1 \equiv 0 \pmod 3$, i.e., $4r - 1$ is divisible by 3. Thus $p = 3$. But then $r = 1$ which is not a prime.
- If $r \equiv 2 \pmod 3$ then $2r - 1 \equiv 0 \pmod 3$, i.e., $2r - 1$ is divisible by 3. Thus $q = 3$, whence $r = 2$ and $p = 7$. All three are primes indeed.
- If $r \equiv 0 \pmod 3$, i.e., r is divisible by 3, then $r = 3$. Thus $q = 5$ and $p = 11$ which are primes, too.

Consequently, p can be either 7 or 11.

Solution 2: Out of three consecutive integers $p - 1$, p, $p + 1$ one is divisible by 3. Division by 2 or 4 does not change divisibility by 3 since 2 and

4 are coprime with 3. Thus also out of integers p, $\frac{p-1}{2}$, $\frac{p+1}{4}$ one is divisible by 3. If these integers are prime, the one divisible by 3 must be 3. Consider all possible cases:

- If $p = 3$ then $\frac{p-1}{2} = 1$ but 1 is not prime.
- If $\frac{p-1}{2} = 3$ then $p = 7$ and $\frac{p+1}{4} = 2$. All three are primes indeed.
- If $\frac{p+1}{4} = 3$ then $p = 11$ and $\frac{p-1}{2} = 5$. All three are primes indeed.

Consequently, p can be either 7 or 11.

O4. *(Juniors.)* Find the value of the expression

$$\frac{1}{\left(\frac{1}{2019}\right)^2+1}+\frac{1}{\left(\frac{2}{2018}\right)^2+1}+\frac{1}{\left(\frac{3}{2017}\right)^2+1}+\ldots+\frac{1}{\left(\frac{2018}{2}\right)^2+1}+\frac{1}{\left(\frac{2019}{1}\right)^2+1}.$$

Answer: $\frac{2019}{2}$.

Solution: Group the summands into pairs: the first one together with the last one, the second one together with the second last one, etc. Adding the members of each pair gives us

$$\frac{1}{\left(\frac{i}{j}\right)^2+1}+\frac{1}{\left(\frac{j}{i}\right)^2+1}=\frac{\frac{i^2}{j^2}+1+\frac{j^2}{i^2}+1}{\left(\frac{i^2}{j^2}+1\right)\left(\frac{j^2}{i^2}+1\right)}=\frac{\frac{i^2}{j^2}+\frac{j^2}{i^2}+2}{\frac{i^2}{j^2}+\frac{j^2}{i^2}+2}=1.$$

As there are 2019 pairs in total, the sum of all numbers in the pairs is 2019. Since every summand occurs twice, the desired sum is $\frac{2019}{2}$.

O5. *(Juniors.)* A circle c with center A passes through the vertices B and E of a regular pentagon $ABCDE$. The line BC intersects the circle c the second time at point F. Prove that lines DE and EF are perpendicular.

Solution 1: The internal angles of a regular pentagon have size 108°. Thus $\angle EAB = 108^\circ$ (Fig. 2), whence $\angle EFC = \angle EFB = \frac{\angle EAB}{2} = 54^\circ$. As $\angle CDE = 108^\circ$ and $\angle FCD = \angle BCD = 108^\circ$, from the quadrilateral $CDEF$ we obtain $\angle DEF = 360^\circ - \angle FCD - \angle CDE - \angle EFC = 90^\circ$.

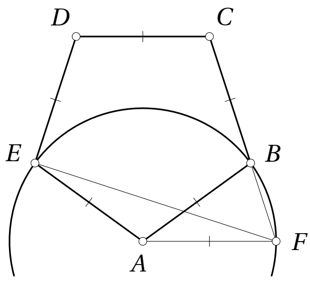


Fig. 2

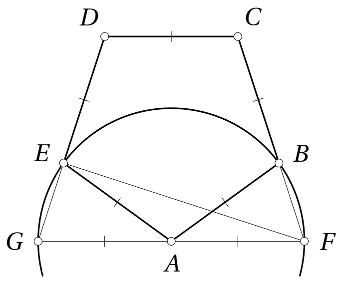


Fig. 3

Solution 2: The internal angles of a regular pentagon have size 108°. Thus $\angle ABC = 108^\circ$, whence $\angle ABF = 180^\circ - \angle ABC = 72^\circ$. As $AB = AF$, from the triangle ABF we obtain $\angle BAF = 180^\circ - 2 \cdot 72^\circ = 36^\circ$. Let G be the second intersection point of line DE with circle c (Fig. 3); by symmetry, $\angle EAG = \angle BAF = 36^\circ$. Since $\angle EAB = 108^\circ$, we have $\angle FAG = \angle BAF + \angle EAB + \angle EAG = 180^\circ$, i.e., FG is a diameter of c. Hence $\angle FEG = 90^\circ$.

O6. *(Juniors.)* Given is an $m \times m$ table with $2n$ distinct unit squares marked with a ring ($2n \leq m^2$). Juku wishes to connect these $2n$ rings into pairs using n (possibly curved) lines in a way that meets the following conditions:

(1) Each line begins from some ring and ends in some other ring;

(2) Every two unit squares visited by the same line one after another have a common side;

(3) No two lines (including their endpoints) visit a common unit square;

(4) No line visits the same unit square more than once.

Prove that the sum of the numbers of unit squares visited by the lines is either always even or always odd, no matter of how Juku draws the lines.

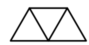
Solution 1: Color the unit squares black and white in such a way that unit squares with a common side are of different color. Each line goes from a black square to a white square and vice versa; thus whenever the endpoints of a line are in squares of equal color, the line visits an odd number of squares, and otherwise, the line visits an even number of squares. Let k squares out of the ones marked with ring be black. Among the lines drawn by Juku, let a lines have both endpoints in black squares, b lines have both endpoints in white squares, and c lines have endpoints in squares of different color. Then $2a + c = k$ and $a + b + c = n$, implying $a + b = n - k + 2a$. Hence the numbers $a + b$ and $n - k$ have equal parity.

Let the numbers of squares visited by the lines sum up to s. Then s can be expressed as the sum of $a + b$ odd numbers and c even numbers, whence s and $a + b$ have equal parity. Consequently, s and $n - k$ have equal parity. Since $n - k$ does not depend on the way Juku draws the lines, s must be always even or always odd.

Solution 2: Every line can be considered as a sequence of unit movements, each having one of four possible directions (right, left, up, down). Thus the number of unit squares visited by a line is $k + 1$ where k is the number of unit movements. Let a, b, c, d be the numbers of unit movements of the line in different directions (right, left, up and down, respectively); then the end of the line is located $a - b$ units to the right and $c - d$ units upwards from the beginning of the line. Since $a - b$ and $a + b$ have equal parity, as do $c - d$ and $c + d$, the numbers $k = (a + b) + (c + d)$ and $(a - b) + (c - d)$ have equal parity.

Let the line start in column x_A and row y_A and end in column x_B and row y_B. Then $x_B - x_A = a - b$ and $y_B - y_A = c - d$, whence the number of unit squares visited by the line has the same parity as the number $x_B - x_A + y_B - y_A + 1$. But the latter has the same parity as $x_B + x_A + y_B + y_A + 1$. Summing up these numbers for all lines, we obtain that the total number of unit squares visited by the lines has the same parity as the number $s_x + s_y + n$ where s_x and s_y are the sum of all column numbers and all row numbers, respectively, of the squares containing a ring. But this sum does not depend on the way how Juku connects the rings, which proves the desired claim.

O7. *(Juniors.)* For which positive integers n can one exactly cover an equilateral triangle of side length n by trapeziums of the shape shown in the figure, consisting of three equilateral triangles of side length 1? Trapeziums are allowed to be rotated but not to cover each other.



Answer: For all positive integers divisibly by 3.

Solution: An equilateral triangle of side length 3 can be covered by three trapeziums (Fig. 4). All equilateral triangles with side length being divisible by 3 can be partitioned into equilateral triangles of side length 3. Hence all equilateral triangles with side length being divisible by 3 can be covered by trapeziums of given shape.

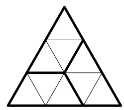


Fig. 4

On the other hand, whenever one partitions an equilateral triangle of side length n into equilateral triangles of side length 1, the number of the small triangles is n^2 since multiplying the side length by n causes the area to increase n^2 times. Consequently, the desired covering is possible only if the number n^2 is divisible by 3. The latter condition implies that n must be divisible by 3 since 3 is prime.

O8. *(Juniors.)* Find the largest remainder that can be left over when dividing the number 2019 by a three-digit natural number.

Answer: 671.

Solution: If $673 < m < 1000$ then dividing 2019 by m gives quotient 2 and remainder $2019 - 2m$. Obviously the remainder increases when m decreases. Thus in the case $m = 674$ we obtain the largest remainder 671.

Dividing 2019 by 673 gives remainder 0. Dividing 2019 by 672 or any smaller number gives remainder that does not exceed 671. Consequently, the largest remainder under the given conditions is 671.

O9. *(Juniors.)* Gandalf the Wizard added to his arsenal of magic a new trick in which he simultaneously turns each integer into some integer different from it. Call an integer *a reflecting* if, for every integer x, the numbers x and $a - x$ are turned into integers equal to each other. Is it possible that:

(a) Numbers 1001 and 1003 are both reflecting;

(b) Numbers 1000, 1003 and 1008 are all reflecting;

(c) Numbers 1002, 1004 and 1006 are all reflecting?

Answer: a) No; b) No; c) Yes.

Solution: For every integer x, denote by $G(x)$ the number into which Gandalf turns the number x.

(a) Suppose that both 1001 and 1003 are reflecting. Then, for every integer x, we have $G(x+2) = G(1003-(x+2)) = G(1001-x) = G(x)$. Hence Gandalf turns all even numbers into one and the same integer c and all odd numbers into one and the same integer c'. But $c = G(500) = G(501) = c'$, implying that all integers are turned into one and the same integer. This contradicts the assumption that Gandalf turns each integer into some other integer.

(b) Suppose that numbers 1000, 1003 and 1008 are all reflecting. Then, for every integer x,

$$\begin{aligned} G(x+3) &= G(1003-(x+3)) = G(1000-x) = G(x), \\ G(x+5) &= G(1008-(x+5)) = G(1003-x) = G(x). \end{aligned}$$

So $G(x+1) = G(x+4) = G(x+7) = G(x+10) = G(x+5) = G(x)$ for every integer x. Consequently, Gandalf again turns all integers into equal integers, contradicting the condition of the problem.

(c) Suppose Gandalf turns all even numbers into 1 and all odd numbers into 2. Then no integer is left unchanged. For every even number a, including 1002, 1004 and 1006, the numbers x and $a-x$ are either both even or both odd, whence they are turned into equal numbers.

O10. *(Juniors.)* Some numbers are printed on a spool of paper, one below another. Starting from the third number, each number is the sum of the two preceding numbers. The number 2018 occurs fourth, the number 2020 occurs eighth. Which are the first and the second numbers?

Answer: -7398 ja 4708.

Solution: Let the first and the second number be x and y, respectively. Computing the following numbers, we see that the fourth number is $x+2y$ and the eighth one is $8x+13y$. The conditions of the problem imply the system of equations

$$\begin{cases} x + 2y = 2018, \\ 8x + 13y = 2020, \end{cases}$$

solving of which gives $x = -7398$ and $y = 4708$.

O11. *(Juniors.)* Circle ω_2 is tangent to circle ω_1 at point A and passes through its center O. Point C is chosen on ω_2 in such a way that the ray AC intersects ω_1 the second time at point D, the ray OC intersects ω_1 at point E and the line DE is parallel to the line AO. Find the size of the angle DAE.

Answer: 30°.

Solution 1: Denote $\angle DAE = \alpha$; then $\angle DOE = 2\alpha$ (Fig. 5). From the isosceles triangle DOE, we obtain $\angle OED = \frac{180^\circ - 2\alpha}{2} = 90^\circ - \alpha$. Since DE and AO are parallel, $\angle AOE = 90^\circ - \alpha$.

As the common tangent to ω_1 and ω_2 at point A is perpendicular to the radius AO of ω_1, as well as to the radius of ω_2, and O lies on ω_2, the line segment AO must be a diameter of ω_2. Thus $\angle OCA = 90°$. As $OA = OD$, the line segment OC is the altitude of the isosceles triangle OAD drawn from its apex angle. This implies $\angle AOE = \angle DOE = 2\alpha$.

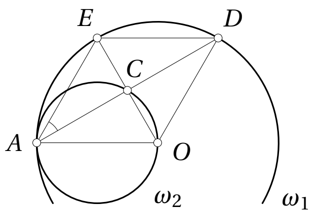


Fig. 5

Hence $90° - \alpha = 2\alpha$, implying $\alpha = 30°$.

Solution 2: As $\angle ACO = \angle DCE$ and also $\angle AOC = \angle DEC$, the triangles ACO and DCE are similar.

As the common tangent to ω_1 and ω_2 at point A is perpendicular to the radius AO of ω_1, as well as to the radius of ω_2, and O lies on ω_2, the line segment AO must be a diameter of ω_2. Thus $\angle OCA = 90°$. As $OA = OD$, the line segment OC is the altitude of the isosceles triangle OAD drawn from its apex angle. This implies $AC = DC$. Hence the triangles ACO and DCE are equal and $AO = DE$.

A quadrilateral with a pair of equal parallel sides is a parallelogram. As $AO = DO$, the quadrilateral $AODE$ is a rhombus. The diagonals of a rhombus bisect the angles at their endpoints. Thus $\angle EAD = \frac{1}{2}\angle EAO$. As $OA = OE = OD$ and $AE = OD$, the triangle AOE is equilateral. Hence the internal angles of $AODE$ are of size $60°$ and $120°$, implying $\angle EAD = 30°$.

O12. *(Juniors.)* Integers a, b, c and n are given such that $1 \leq a < b < c \leq n$. Juku and Miku play the following game on a strip of size $1 \times n$: In the beginning, squares number a, b, c contain one piece each, whereby the squares are numbered from the right to the left by consecutive integers starting from 1. On one's move, each player chooses one piece out of these three and shifts it one or more squares to the right. However, it is not allowed to move a piece to a square that contains another piece or jump over such a square; one also must not move a piece off the strip. Players move by turns, with Juku moving first. The player who cannot move loses. Which player can win regardless of the opponent's play?

Answer: Juku in the case $a + b \neq c$; Miku in the case $a + b = c$.

Solution: Firstly, note that, in any position where the number of empty squares between the leftmost and the middle piece differs from the number of empty squares in the right from the rightmost piece, one can make a move that makes these two quantities equal. Indeed, if the number of empty squares between the leftmost and the middle piece is greater than the number of empty squares right from the rightmost piece then one can move the leftmost piece, otherwise one can move the rightmost piece.

Secondly, note that every move in any position where the number of empty squares between the leftmost and the middle piece equals the number of empty squares in the right from the rightmost piece makes these two quantities different. Indeed, moving either the leftmost or the middle piece changes the number of empty squares between the leftmost and the middle piece while leaving the empty squares right from the rightmost piece unchanged; when moving the rightmost piece, it is the other way round.

Consequently, Juku can win if $c - b \neq a$ by always moving in such a way that the number of empty squares between the leftmost and the middle piece were equal to the number of empty squares right from the rightmost piece after his move. As the sum of distances of all three pieces from the right edge of the strip decreases at each move, the game must eventually end and, by the considerations above, only Miku can lose. On the other hand, if $c - b = a$ then after Juku's move the number of empty squares between the leftmost and the middle piece differs from the number of empty squares right from the rightmost piece. Analogously to the previous case, Miku can win in this position.

O13. *(Seniors.)* One writes distinct positive integers into the cells of a 3×3 table in such a way that, in each row and in each column, one number equals the sum of the other two numbers. Find the least possible total sum of the numbers written into the table.

Answer: 46.

Solution: The sum of all numbers written in the cells is at least $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$. As one number is the sum of others in every row, the sum of all numbers in every row is even. Thus the sum of all numbers in the table must be even, too. Hence the sum of all numbers in the table cannot be 45. It is possible that the sum is 46, as Fig. 6 shows.

10	6	4
8	1	7
2	5	3

Fig. 6

O14. *(Seniors.)* Find all solutions of the equation $x^3 + 3xy + y^3 = 2019$ in integers.

Answer: There are none.

Solution 1: The r.h.s. of the equation is divisible by 3 but not by 9. Assume that $3 \mid x$. Then $9 \mid x^3$ and $9 \mid 3xy$. If also $3 \mid y$ then $9 \mid y^3$, implying that the l.h.s. of the equation is divisible by 9. Thus $3 \nmid y$. But then $3 \nmid y^3$, implying that the l.h.s. of the equation is not divisible by 3. The contradiction shows that $3 \nmid x$. By symmetry, also $3 \nmid y$.

As $3 \mid 3xy$, we must have $3 \mid x^3 + y^3$. Hence also $3 \mid x^3 + 3x^2y + 3xy^2 + y^3 = (x+y)^3$, implying $3 \mid x + y$. Consequently, x and y are modulo 3 congruent to 1 and 2 in some order. Hence $x^2 \equiv y^2 \equiv 1 \pmod 3$ and $xy \equiv 2 \pmod 3$, implying $x^2 - xy + y^2 \equiv 0 \pmod 3$. We obtain $9 \mid x^3 + y^3 = (x+y)(x^2 - xy + y^2)$, whereas $3xy \equiv 6 \pmod 9$. Hence the l.h.s. of the equation is congruent to 6 modulo 9 but the r.h.s. of the equation is congruent to 3. Consequently, there are no solutions.

Solution 2: Denote $a = x + y$. Then the given equation is equivalent to $a^3 + 3xy(1 - a) = 2019$. As $3 \mid 2019$ and $3 \mid 3xy(1 - a)$, we must have $3 \mid a^3$, implying $3 \mid a$. The given equation is also equivalent to

$$(a-1)(a^2+a+1-3xy) = 2018. \tag{1}$$

Hence $a-1 \mid 2018$. As $2018 = 2 \cdot 1009$ where 1009 is prime, $a-1$ must be one of 2018, 1009, 2, 1, −1, −2, −1009, −2018. Taking into account that $3 \mid a$, we obtain four cases:

- If $a = 2019$ then $y = 2019 - x$ and substituting into (1) gives $2019^2 + 2019 + 1 - 3x(2019 - x) = 1$ which is equivalent to $x^2 - 2019x + 673 \cdot 2020 = 0$. The latter equation has no real solutions.
- If $a = 3$ then $y = 3 - x$ and substituting into (1) gives $13 - 3x(3 - x) = 1009$ which is equivalent to $x^2 - 3x - 332 = 0$. The latter equation has no integral solutions.
- If $a = 0$ then $y = -x$ and the l.h.s. of the initial equation is non-positive.
- If $a = -1008$ then $y = -1008 - x$ and substituting into (1) gives $1008^2 - 1008 + 1 + 3x(1008 + x) = -2$ which is equivalent to $x^2 + 1008x + 336 \cdot 1007 + 1 = 0$. The latter equation has no real solutions.

O15. *(Seniors.)* Let n be a positive integer. Real numbers $a_1, a_2, \ldots, a_{2n}$ satisfy the following conditions:

(1) For every $i = 1, 2, \ldots, 2n-1$, one has $0 < a_{i+1} - a_i \leq 1$;

(2) Rounding the numbers $a_1, a_2, \ldots, a_{2n}$ to the closest integer (numbers equidistant from two closest integers are rounded up) gives pairwise distinct positive integers.

Numbers $a_1, a_2, \ldots, a_{2n}$ are placed as the numerators and denominators of n fractions. Prove that the sum of the obtained fractions is greater than $\frac{n}{4}$.

Solution 1: Suppose that a numerator is greater than a denominator. Then interchanging these two numbers makes both fractions smaller. Thus we can assume w.l.o.g. that all numerators are less than all denominators.

For every $i = 1, 2, \ldots, 2n$, define $x_i = a_i - i + \frac{1}{2}$. As $a_1 < a_2 < \ldots < a_{2n}$ and rounding the numbers a_i produce pairwise distinct positive integers, we must have $a_i \geq i - \frac{1}{2}$ which implies $x_i \geq 0$. From $a_{i+1} - a_i \leq 1$, we have $x_{i+1} - x_i = a_{i+1} - a_i - 1 \leq 0$ which implies $x_1 \geq x_2 \geq \ldots \geq x_{2n}$. Hence $i < j$ always implies $\frac{a_i}{a_j} \geq \frac{a_i - x_j}{a_j - x_j} \geq \frac{a_i - x_i}{a_j - x_j} = \frac{2i-1}{2j-1}$ (the first inequality holds because of $a_i < a_j$ and $x_j \geq 0$ while the second inequality holds because of $x_i \geq x_j$). Thus it suffices to prove the desired inequality for the case where the numerators are integers $1, 3, \ldots, 2n-1$ and the denominators are integers $2n+1, 2n+3, \ldots, 4n-1$ in some order.

Denote the sum of all fractions by s. Applying AM-GM to the fractions gives $\frac{s}{n} \geq \sqrt[n]{\frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(2n+1) \cdot (2n+3) \cdot \ldots \cdot (4n-1)}}$. Thus it suffices to prove for every n the inequality $\frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(2n+1) \cdot (2n+3) \cdot \ldots \cdot (4n-1)} > \frac{1}{4^n}$. We can do it by induction on n. The claim holds for $n = 1$ since $\frac{1}{3} > \frac{1}{4}$. For the induction step, it suffices to show that $\frac{(2n+1)^2}{(4n+1)(4n+3)} > \frac{1}{4}$ which is equivalent to $(4n+2)^2 > (4n+1)(4n+3)$. The latter follows from AM-GM for $4n+1$ and $4n+3$.

Solution 2: Firstly, show as in Solution 1 that it suffices to prove the desired inequality for the case where the numerators are $1, 3, \ldots, 2n-1$ and the denominators are $2n+1, 2n+3, \ldots, 4n-1$ in some order. By applying the rearrangement inequality for $1 < 3 < \ldots < 2n-1$ and $\frac{1}{2n+1} > \frac{1}{2n+3} > \ldots > \frac{1}{4n-1}$, we conclude that the least sum of fractions is obtained if both numerators and denominators are in the increasing order. Hence it suffices to show that $\frac{1}{2n+1} + \frac{3}{2n+3} + \ldots + \frac{2n-1}{4n-1} > \frac{n}{4}$. Put the first summand together with the last one, the second one together with the second last one, etc. The fractions of every pair are of the form $\frac{n-k}{3n-k}$ and $\frac{n+k}{3n+k}$ where $-n < k < n$. The sum of these two terms is $\frac{6n^2-2k^2}{9n^2-k^2}$. As $-n < k < n$, we must have $k^2 < n^2$; we also see that both $6n^2 - 2k^2$ and $9n^2 - k^2$ are positive. Thus the inequality $9n^2 - k^2 < 12n^2 - 4k^2$, being equivalent to the valid inequality $k^2 < n^2$, is also equivalent to $\frac{6n^2-2k^2}{9n^2-k^2} > \frac{1}{2}$. Hence the sum of members of all n pairs is greater than $\frac{n}{2}$, and as each fraction occurs twice in these pairs, the sum of all fractions is greater than $\frac{n}{4}$.

Solution 3: Firstly, show as in Solution 1 that it suffices to prove the desired inequality for the case where the numerators are $1, 3, \ldots, 2n-1$ and the denominators are $2n+1, 2n+3, \ldots, 4n-1$ in some order. Replacing all denominators with their strict upper bound $4n$, all fractions become smaller and so does their sum. The sum of numerators $1 + 3 + \ldots + (2n-1)$ equals n^2. Hence the sum of all fractions is greater than $\frac{n^2}{4n}$, i.e., than $\frac{n}{4}$.

O16. *(Seniors.)* A circle c with center A passes through the vertices B and E of a regular pentagon $ABCDE$. The line BC intersects the circle c the second time at point F. Point G on the circle c is chosen in such a way that $FB = FG$ and $B \neq G$. Prove that the lines AB, EF and DG meet in a common point.

Solution 1: The internal angles of a regular pentagon have size $108°$. Thus $\angle ABC = 108°$, implying $\angle ABF = 72°$ (Fig. 7). As $AB = AF$, we have $\angle AFB = 72°$ and $\angle BAF = 36°$. Since $FG = FB$, $AG = AB$ and $AF = AF$, the triangles AFB and AFG are equal, implying that $\angle FAG = 36°$. As $\angle EAB = 108°$, we obtain $\angle GAE = 2 \cdot 36° + 108° = 180°$, implying that E, A ja G are collinear.

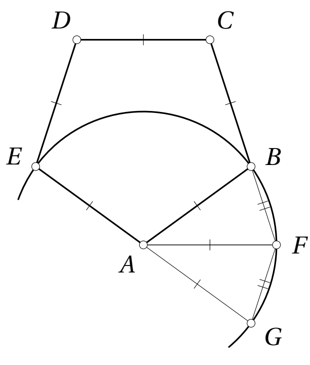


Fig. 7

Let the lines AB and EF meet at point K (Fig. 8). As $\angle AFC = 180° - \angle FCD$, we have $AF \parallel CD$. Consequently also $AF \parallel BE$, whence

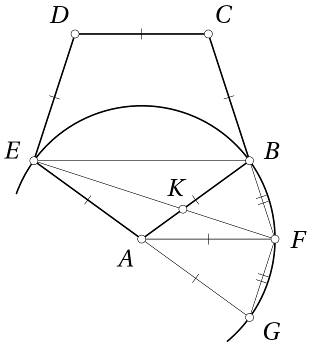


Fig. 8

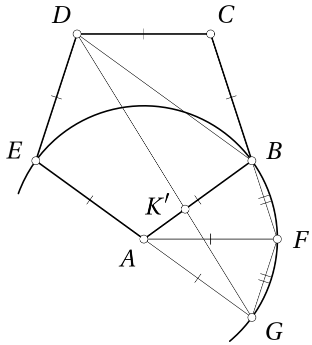


Fig. 9

similarity of triangles BKE and AKF implies $\frac{BK}{AK} = \frac{BE}{AF}$.

Let the lines AB and DG meet at point K' (Fig. 9). Lines AG and AE coincide because E, A and G are collinear. Consequently, $BD \parallel AG$, whence similarity of triangles $BK'D$ and $AK'G$ implies $\frac{BK'}{AK'} = \frac{BD}{AG}$.

Since $BD = BE$ and $AF = AG$, we have $\frac{BK}{AK} = \frac{BK'}{AK'}$, implying $K = K'$.

Solution 2: The internal angles of a regular pentagon have size $108°$. Thus $\angle ABC = 108°$ (Fig. 10), implying $\angle ABF = 72°$. As $AB = AF$, we have $\angle AFB = 72°$. Since $FG = FB$, $AG = AB$ and $AF = AF$, the triangles AFB and AFG are equal, implying also $\angle AFG = 72°$. Therefore $\angle CFG = \angle BFG = 2 \cdot 72° = 144°$, whereas $\angle FCA = \angle BCA = \frac{180° - \angle ABC}{2} = 36° = 180° - 144°$. Consequently, $GF \parallel CA$, implying also $GF \parallel DE$.

Let the lines AB and EF meet at point K (Fig. 11). As $CE \parallel BA$, we obtain $\frac{BF}{KF} = \frac{BC}{KE}$. By assumptions, $BF = GF$ and $BC = DE$; hence also $\frac{GF}{KF} = \frac{DE}{KE}$. As $\angle KFG = \angle KED$ because of $GF \parallel DE$, the triangles KFG and KED are similar. Thus $\angle FKG = \angle EKD$. Consequently, points G, K and D are collinear, meaning that the line DG passes through K.

Remark: The claim follows directly from Desargues's theorem.

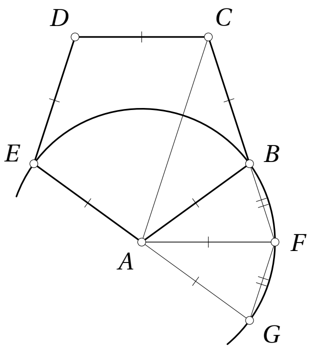


Fig. 10

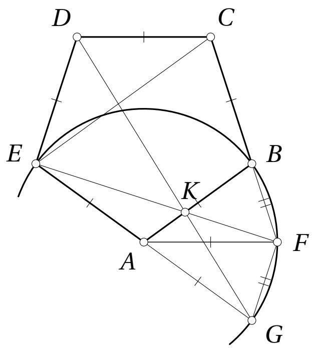


Fig. 11

O17. *(Seniors.)* Grandpa has a finite number of empty dustbins in his attic. Each dustbin is a rectangular parallelepiped with integral side lengths. A dustbin can be thrown away into another dustbin iff the side lengths of these dustbins can be set into one-to-one correspondence in such a way that the side lengths of the first dustbin are less than the corresponding side lengths of the other dustbin. For saving space, grandpa wants to throw away as many dustbins as possible. He has the following plan: while there exist yet some dustbins that can be thrown into each other, find the longest chain of such dustbins (i.e., the first throwable into the second, the second throwable into the third etc.) and throw all of them except the largest one sequentially into the next one, then do the same with remaining dustbins, etc. Is it necessarily true that as many dustbins as possible will be thrown away as the result of this process, if it is known that at each step there is a unique way of choosing the longest chain of dustbins throwable into each other among the free dustbins and no dustbin is large enough to contain two other dustbins unless they have been placed into one another?

Answer: No.

Solution: Suppose grandfather has 6 dustbins with sizes $20 \times 20 \times 20$, $19 \times 19 \times 19$, $16 \times 16 \times 16$, $21 \times 18 \times 15$, $18 \times 15 \times 12$ and $17 \times 14 \times 11$. The first dustbin can contain the second one, the second can contain the third or the fifth, the fifth can contain the sixth. The fourth also can contain the fifth. It is impossible to throw the first and the fourth into each other, the second and the fourth into each other, the third and the fourth into each other, the third and the fifth into each other, or the third and the sixth into each other. In the longest chain of dustbins that can be thrown into each other is 4 dustbins: the sixth can be thrown into the fifth, which can be thrown into the second, which can be thrown into the first. As the remaining two dustbins cannot be thrown into each other, 3 dustbins in total are not thrown away. However, by throwing the third dustbin into the second, the second into the first, the sixth into the fifth and the fifth into the fourth, only 2 dustbins are not thrown away. Hence grandfather's algorithm does not provide an optimal solution.

O18. *(Seniors.)* Given a tuple of consecutive positive integers, one forms all pairs of members of it such that the first member is less than the second member. The percentage of these pairs where the second member is divisible by the first one is called the *degree of divisibility* of the tuple. For every integer $n > 1$, denote the largest possible degree of divisibility of a tuple of n consecutive positive integers by $j(n)$.

Does there exist an integer $n > 1$ such that $j(n+1) > j(n)$?

Answer: Yes.

Solution 1: The largest percentage of pairs with the second term being divisible by the first term is achieved in the case of tuple $(1, 2, \ldots, n)$. Indeed, consider an arbitrary tuple of the form $(x+1, x+2, \ldots, x+n)$ where

$x > 0$. For any i, multiples of i in $(1, 2, \ldots, n)$ are every ith term starting from the number i, multiples of $x+i$ in $(x+1, x+2, \ldots, x+n)$ are every $(x+i)$th term starting from $x+i$. The latter multiples occuring more seldom while the first occurrence being at the same position implies the same or smaller total number.

It is easy to check that the degree of divisibility of $(1, 2, 3, 4, 5)$ is $\frac{5}{10}$ and the degree of divisibility of $(1, 2, 3, 4, 5, 6)$ is $\frac{8}{15}$. By the above, $j(6) > j(5)$.

Solution 2: Calculation shows that the degrees of divisibility of tuples $(1, 2, 3, 4, 5)$, $(2, 3, 4, 5, 6)$, $(3, 4, 5, 6, 7)$ and $(4, 5, 6, 7, 8)$ are $\frac{5}{10}$, $\frac{3}{10}$, $\frac{1}{10}$ and $\frac{1}{10}$, respectively. If the first term of a quintuple is 5 or larger then no two terms can divide each other since the largest term is less than twice larger than the least term. Hence $j(5) = \frac{5}{10} = \frac{1}{2}$. As the degree of divisibility of $(1, 2, 3, 4, 5, 6)$ is $\frac{8}{15}$, we have $j(6) \geq \frac{8}{15} > \frac{1}{2} = j(5)$.

O19. *(Seniors.)* Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} xy + x + y = z, \\ yz + y + z = x, \\ zx + z + x = y. \end{cases}$$

Answer: $(-1, -1, -1)$, $(0, 0, 0)$, $(0, -2, -2)$, $(-2, 0, -2)$, $(-2, -2, 0)$.

Solution 1: Denote $x + 1 = a$, $y + 1 = b$ and $z + 1 = c$. Adding 1 to the sides of all equations and factorizing in the left gives

$$\begin{cases} ab = c, \\ bc = a, \\ ca = b. \end{cases} \tag{2}$$

If $a = 0$ then the first and the third equation of (2) imply $b = 0$ and $c = 0$. Analogously, if $b = 0$ or $c = 0$ then a, b, c are all equal to zero. This case leads to the solution $(-1, -1, -1)$ to the initial system. If none of a, b, c is zero then pairwise multiplication of the equations of (2) and each time reducing similar factors from both sides gives $a^2 = b^2 = c^2 = 1$. Thus each of the numbers a, b, c is either 1 or -1. The case $a = b = c = 1$ satisfies (2) and leads to the solution $(0, 0, 0)$ of the initial system. The cases with exactly two of numbers a, b, c being equal to -1 also satisfy (2) and lead to the solutions $(-2, -2, 0)$, $(-2, 0, -2)$ and $(0, -2, -2)$ of the initial system. Multiplying the corresponding sides of all equations of (2) and reducing by factor abc gives $abc = 1$, whence there are no more solutions (the value -1 cannot occur an odd number of times).

Solution 2: Substituting z from the first equation into the second one gives the equation $y(xy + x + y) + y + (xy + x + y) = x$ which is equivalent to $(x + 1)y(y + 2) = 0$. Thus we have $x = -1$ or $y = 0$ or $y = -2$.

- If $x = -1$ then the first equation implies $z = -1$. Substituting into the third equation gives $y = -1$. Hence the solution $(-1, -1, -1)$.

- If $y = 0$ then the first equation implies $z = x$. Substituting into the third equation gives $x^2 + 2x = 0$. Thus $x = z = 0$ or $x = z = -2$, leading to the solutions $(0, 0, 0)$ and $(-2, 0, -2)$.
- If $y = -2$ then the first equation implies $z = -x - 2$. Substituting it into the third equation gives $x^2 + 2x = 0$. Thus either $x = 0$ and $z = -2$ or $x = -2$ and $z = 0$ which lead to the solutions $(0, -2, -2)$ and $(-2, -2, 0)$.

O20. *(Seniors.)* Find all pairs (x, y) of integers such that

$$\sqrt{x + 2019} - \sqrt{x} = \sqrt{y}.$$

Answer: $(1009^2, 1)$, $(336^2 \cdot 3, 3)$, $(335^2, 9)$, $(673, 673)$, $(0, 2019)$.

Solution 1: As x and y occur under square root, only non-negative solutions can exist. Bringing $\sqrt{x}$ to the right, squaring both sides and collecting similar terms gives $2019 = y + 2\sqrt{xy}$ which is equivalent to the initial equation. Thus $2\sqrt{xy}$ is an integer. If $x = 0$ then $y = 2019$, the case $y = 0$ leads to contradiction. Assume in the rest that both x and y are positive.

Let a^2 and b^2 be the largest perfect squares dividing x and y, respectively; then $x = a^2c$ and $y = b^2c'$ where both c and c' are square-free. As $a^2c \cdot b^2c'$ is a perfect square, also $c \cdot c'$ must be a perfect square; this is possible only if c and c' have the same prime factors, i.e., $c = c'$. Hence $2019 = b^2c + 2\sqrt{a^2b^2c^2} = b^2c + 2abc = bc(2a + b)$.

Since $2019 = 3 \cdot 673$ where both factors are prime, we have the following cases, taking into account that $b < 2a + b$:

- $b = 1$, $c = 1$, $2a + b = 2019$, implying $a = 1009$, $x = 1009^2$ and $y = 1$;
- $b = 1$, $c = 3$, $2a + b = 673$, implying $a = 336$, $x = 336^2 \cdot 3$ and $y = 3$;
- $b = 1$, $c = 673$, $2a + b = 3$, implying $a = 1$, $x = 673$ and $y = 673$;
- $b = 3$, $c = 1$, $2a + b = 673$, implying $a = 335$, $x = 335^2$ and $y = 9$.

Solution 2: As x and y occur under square root, only non-negative solutions can exist. Bringing $\sqrt{x}$ to the right, squaring both sides and collecting similar terms gives $2019 - y = 2\sqrt{xy}$ which is equivalent to the initial equation. Hence $y \leq 2019$. After squaring once more and rearranging terms, we obtain $2019^2 = y(2 \cdot 2019 - y + 4x)$. Thus $y \mid 2019^2$. Case study leads to the following solutions: (1) $y = 1$, $x = 1009^2$; (2) $y = 3$, $x = 336^2 \cdot 3$; (3) $y = 9$, $x = 335^2$; (4) $y = 673$, $x = 673$; (5) $y = 2019$, $x = 0$.

O21. *(Seniors.)* Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f\left(xf(y) + y\right) = f\left(x^2 + y^2\right) + f(y)$$

for all real numbers x and y.

Answer: $f(x) = 0$.

Solution: Substituting $x = 0$ into the equation gives $f(y) = f(y^2) + f(y)$, implying $f(y^2) = 0$. Thus $f(x) = 0$ for all non-negative real numbers x. In particular $f(x^2+y^2)=0$, which allows to simplify the initial equation as

$$f\left(xf(y)+y\right)=f(y). \tag{3}$$

Suppose that $f(c)\neq 0$ for some negative real number c. Substituting $y=c$ into (3) gives $f(z)=f(c)$ for all z because the expression $xf(c)+c$ obtains all real values. By the above, there exists z such that $f(z)=0$; hence $f(c)=0$, contradicting the choice of c. Consequently, $f(x)=0$ for every real number x. Clearly the function $f(x)=0$ satisfies the given equation.

O22. *(Seniors.)* The bisector of the internal angle at vertex B of triangle ABC and the line through point C perpendicular to the side BC intersect at point D. Let M and N be the midpoints of the line segments BC ja BD, respectively. Given that N lies on the side AC and $\frac{AM}{BC}=\frac{CD}{BD}$, find all possibilities of what the sizes of the angles of the triangle ABC can be.

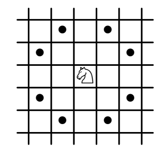
Answer: $30^\circ, 60^\circ, 90^\circ$ and $15^\circ, 30^\circ, 135^\circ$.

Solution: By assumptions, MN is the midsegment of triangle BCD such that $MN \parallel CD$ (Fig. 12). Thus $\angle BMN=\angle BCD=90^\circ$, implying $NB=NC$. Denote $\gamma=\angle ABN=\angle NBC$. The assumption $\frac{AM}{BC}=\frac{CD}{BD}$ is equivalent to the assertion that the length of AM equals the height drawn from the right angle of the right triangle BCD. Let E be the foot of this altitude; let K be the projection of M to BD and L the point of intersection of lines MK and AB (Fig. 13). As M bisects BC and $MK \parallel CE$, MK is the midsegment of triangle BEC, implying $CE=2MK$. As BK is an altitude and an angle bisector of triangle BML, it is also a median, i.e., $ML=2MK$. Thus $MA=CE=ML$, whence either $L=A$ or the triangle AML is isosceles with apex angle at M.

Consider the case $L=A$ (Fig. 14). As BD is the axis of symmetry of triange BML, we have $\angle BAC=\angle BAN=\angle BLN=\angle BMN=90^\circ$. From the right triangle ABC, we get $3\gamma=90^\circ$, implying $\gamma=30^\circ$. The sizes of internal angles of the triangle ABC are $30^\circ, 60^\circ, 90^\circ$.

It remains to study the case $L\neq A$, $ML=MA$; as the angle BLM is acute, A lies on line segment BL. Let J be the foot of the altitude drawn from vertex M of the triangle MCN (Fig. 15). Since the triangle BNC is isosceles, triangles BMN and CNM are equal, whence $JM=KM=\frac{1}{2}LM=\frac{1}{2}AM$. In the right triangle AJM, leg is twice shorter than the hypotenuse, implying $\angle JAM=30^\circ$. On the other hand, $\angle JAM=\angle LAM-\angle LAC=\angle ALM-\angle LAC=\angle KMB-(\angle ABC+\angle BCA)=(90^\circ-\gamma)-3\gamma=90^\circ-4\gamma$. Hence $90^\circ-4\gamma=30^\circ$, implying $\gamma=15^\circ$. The sizes of internal angles of the triangle ABC are $15^\circ, 30^\circ, 135^\circ$.

O23. *(Seniors.)* Let n and m be positive integers. On one turn, an *n-m-knight* can move either horizontally by n squares and vertically by m squares or vertically by n squares and horizontally by m squares. (For instance, the usual chess knight, all possible target squares of one move of which are depicted



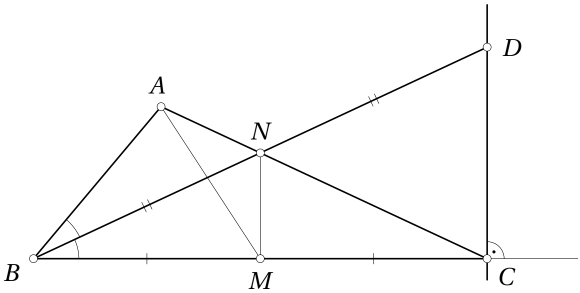


Fig. 12

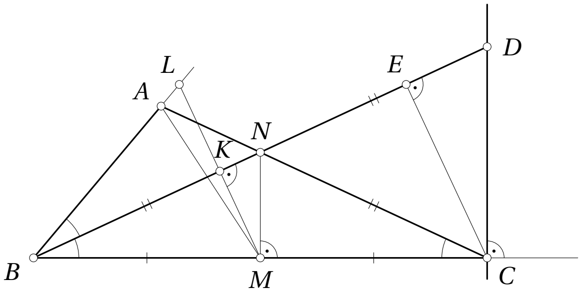


Fig. 13

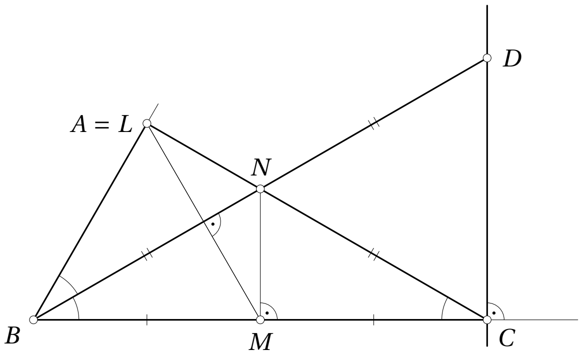


Fig. 14

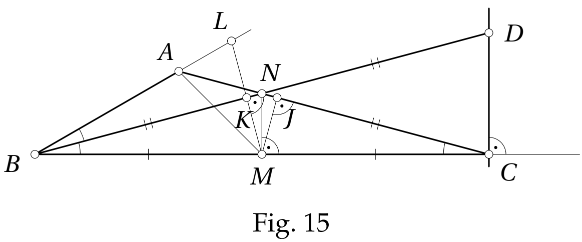


Fig. 15

by bullets in the figure, is a 1-2-knight.) Can an n-m-knight on an infinite in every direction chessboard return to the initial square in exactly 2019 turns?

Answer: No.

Solution: Consider three cases:

- *Exactly one of the numbers n, m is odd.* Color the squares like on a chessboard. Every move changes the color of the square where the knight is, whence after an odd number of moves, the knight is on a square of the opposite color. Thus the knight cannot be on the initial square after 2019 moves.
- *Both numbers n and m are odd.* Color the horizontal lines of the board alternately black and white. Again, every move changes the color of the square where the knight is. Hence, similarly to the previous case, the knight cannot be on the initial square after 2019 moves.
- *Both numbers n and m are even.* Let $n = 2^k p, m = 2^l q$ where p and q are odd. W.l.o.g., assume that $k \leq l$ and the knight starts from square $(0, 0)$. Obviously, the knight only visits squares with coordinates of the form $(2^k u, 2^k v)$ since 2^k divides the length of the step in either direction. Suppose that the knight is on the initial square after 2019 moves. Shortening all moves 2^k times while retaining their directions, we obtain a route of a p-$2^{l-k}q$-knight which in 2019 moves returns to the initial square. But p is odd which means that such route does not exist by previous cases. The contradiction shows that the knight cannot be on the initial square after 2019 moves.

Selected Problems from the Final Round of National Olympiad

F1. *(Grade 9.)* Prove that, for any natural number n, either $3^{2n} - 3^{n+1} + 3^n - 3$ or $3^{2n} - 3^{n+1} + 3^n + 1$ is divisible by 32.

Solution: Note that $3^{2n} - 3^{n+1} + 3^n - 3 = (3^n - 3)(3^n + 1)$. If n is odd then $3^n - 3 = 3(3^{n-1} - 1) = 3\left(3^{\frac{n-1}{2}} - 1\right)\left(3^{\frac{n-1}{2}} + 1\right)$. As all powers of 3 are odd, $3^{\frac{n-1}{2}} - 1$ and $3^{\frac{n-1}{2}} + 1$ are consecutive even numbers. One of these numbers must be divisible by 4, whence their product is divisible by 8. Thus $8 \mid 3^n - 3$, implying that $4 \mid 3^n + 1$. Consequently, $32 \mid 3^{2n} - 3^{n+1} + 3^n - 3$.

Assume now n being even. Note that

$$3^{2n} - 3^{n+1} + 3^n + 1 = 3^{2n} - 3 \cdot 3^n + 3^n + 1 = (3^n)^2 - 2 \cdot 3^n + 1 = (3^n - 1)^2.$$

Similarly to the previous case, $3^n - 1 = \left(3^{\frac{n}{2}} - 1\right)\left(3^{\frac{n}{2}} + 1\right)$ where factors in the r.h.s. are consecutive even numbers. Hence $8 \mid 3^n - 1$, implying that $64 \mid 3^{2n} - 3^{n+1} + 3^n + 1$. Consequently, $32 \mid 3^{2n} - 3^{n+1} + 3^n + 1$.

Remark: The problem can be solved by a study of residues modulo 32.

F2. *(Grade 9.)* There are the same number of boys and girls in a class. It is known that 60% of pupils do sports and $\frac{5}{9}$ of pupils doing sports are boys. It is also known that $\frac{1}{3}$ of pupils doing sports go to math club and $\frac{2}{15}$ of girls neither do sports nor go to math club. On the other hand, $\frac{2}{15}$ of boys both do sports and go to math club. What percentage of girls go to math club?

Answer: 60%.

Solution: There are $\frac{3}{5} \cdot \frac{1}{3} = \frac{1}{5}$ of pupils who both do sports and go to math club, whereas $\frac{1}{2} \cdot \frac{2}{15} = \frac{1}{15}$ of pupils are boys who both do sports and go to math club. Thus $\frac{1}{5} - \frac{1}{15} = \frac{2}{15}$ of pupils are girls who both do sports and go to math club. Girls who do sports constitute $\frac{3}{5}\left(1 - \frac{5}{9}\right) = \frac{4}{15}$ of all pupils. Hence girls who do sports but do not go to math club constitute $\frac{4}{15} - \frac{2}{15} = \frac{2}{15}$ of all pupils. Girls who neither do sports nor go to math club constitute $\frac{1}{2} \cdot \frac{2}{15} = \frac{1}{15}$ of all pupils. Thus girls not going to math club constitute $\frac{2}{15} + \frac{1}{15} = \frac{1}{5}$ of all pupils. Other girls who constitute $\frac{1}{2} - \frac{1}{5} = \frac{3}{10}$ of all pupils go to math club. They constitute $\frac{3}{10} : \frac{1}{2} = \frac{3}{5} = 60\%$ of all girls.

F3. *(Grade 9.)* A wheel with radius r rolls, without sliding, along the inner side of a circle of radius $2r$. In the beginning, the wheel is at the lowermost position. Find the trajectory of the point initially uppermost on the wheel.

Answer: the horizontal diameter of the circle.

Solution: Let O be the center of the circle; as the radius of the circle equals the diameter of the wheel, O is always located on the boundary of the wheel. Let A be the point of tangency between the wheel and the circle in the beginning. As the wheel rolls without sliding, the point of tangency covers equal distances along the wheel and along the circle. Also point O must cover the same distances along the wheel since it remains diametrically opposite to the point of tangency.

Consider the situation when the center of the wheel has rotated around O by angle $\phi \leq 90°$. Let the center of the wheel and the point of tangency now be Q and A', respectively, and let P be the new location of the point initially uppermost on the wheel (Fig. 16). Then $r \cdot \angle OQP = 2r \cdot \phi$, whence $\angle OQP = 2\phi$. As O, Q and A' are collinear, we also have $\angle OA'P = \phi = \angle AOA'$. Consequently, $OA \parallel PA'$. By Thales' theorem, $\angle OPA' = 90°$, implying that P lies on the horizontal diameter of the circle.

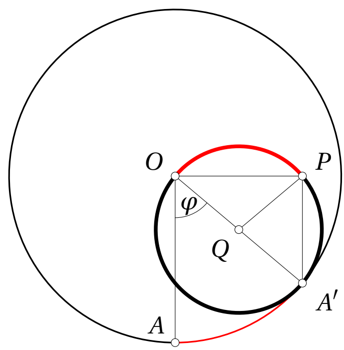


Fig. 16

After the center of the wheel has rotated around O by $180°$, the same point of the wheel is located at O as initially. Like in the previous case, we see that P lies on the horizontal diameter of the circle also if $\phi > 90°$. As any point on the boundary of the wheel eventually reaches the circle, the trajectory in question constitutes the whole diameter.

F4. *(Grade 9.)* Does there exist an integer $n \geq 3$ such that some 3 diagonals of a regular n-gon meet in one point that is neither a vertex nor the center of the n-gon? If yes then find the least such n.

Answer: Yes, 8.

Solution: If 3 diagonals of an n-gon meet in one point that is not a vertex of the n-gon then these diagonals have 6 endpoints in total, implying $n \geq 6$. If $n = 6$ then the only way to leave the endpoints of every two diagonals to different sides of the third diagonal is connecting each vertex to the opposite one (Fig. 17), but the obtained diagonals meet in the center of the n-gon. Suppose that there exist 3 diagonals satisfying the conditions for $n = 7$. Let A be the vertex that is not an endpoint of any of the diagonals. Let B be a vertex next to A and let C be the other endpoint of the diagonal whose one endpoint is B. Two endpoints of diagonals must lie on the same side of BC as A and two endpoints must lie on the other side. There is only one possibility to connect these points with two intersecting diagonals (Fig. 18). As these diagonals are symmetric w.r.t. the perpendicular bisector of AB, their common point P lies on the perpendicular bisector of AB. As C also lies on the perpendicular bisector and $C \neq P$, diagonal BC could pass through P only if B were also located on the perpendicular bisector of BC, which is not the case. Hence finding the required 3 diagonals is impossible for $n = 7$.

For $n = 8$, draw one diagonal from some vertex to the opposite vertex. Adding two shorter diagonals symmetrically w.r.t. the first diagonal, all three intersect in one point inside the polygon that is not its center. (Fig. 19).

F5. *(Grade 10.)* A room of the shape of a rectangular parallelepiped has vertical walls covered by mirrors. A laser beam of diameter 0 enters the room from one corner and moves horizontally along the bisector of that corner. After reflecting from some wall, the beam continues moving horizontally according to the laws of reflection (i.e. the bisector of the angle between the imaginary continuation of the trajectory of the beam before reflection and the real continuation trajectory is along the wall). When the beam reaches a corner, it will return along the way it arrived.

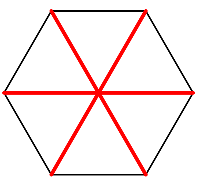


Fig. 17

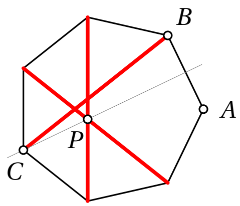


Fig. 18

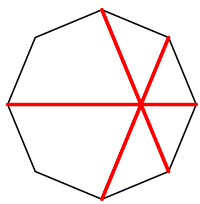


Fig. 19

(a) Prove that if the ratio of the side lengths of the floor of the room is rational then the beam eventually returns to the point of entrance.

(b) Prove that if the ratio of the side lengths of the floor of the room is irrational then the beam never returns to the point of entrance.

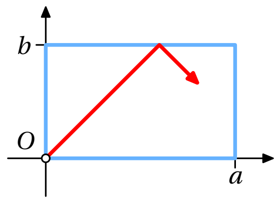


Fig. 20

Solution: Let a and b be side lengths of the floor. We use planar coordinate system whose both axes go along sides of the floor (Fig. 20). After reflection from a wall, one coordinate of the beam starts changing to the opposite direction while retaining its speed. As the beam enters the room along a corner bisector, both coordinates change with the same speed. W.l.o.g., let the speed be 1, then the coordinates change with period $2a$ and $2b$, respectively.

(a) If $\frac{a}{b}$ is rational then $\frac{a}{b} = \frac{m}{n}$ where m and n are integers. Denote $t = 2a \cdot n = 2b \cdot m$. Then at the time instance t, both coordinates have gone through a whole number of periods which means that the beam has reached the point of entrance.

(b) Suppose that the beam reaches the point of entrance at a time instance t. Then $\frac{t}{2a}$ and $\frac{t}{2b}$ are integers; denote $n = \frac{t}{2a}$ and $m = \frac{t}{2b}$, implying $a = \frac{t}{2n}$ and $b = \frac{t}{2m}$. Thus $\frac{a}{b} = \frac{m}{n}$. Consequently, in order to reach the point of entrance, $\frac{a}{b}$ must be rational.

F6. *(Grade 10.)* Prove or disprove: For every integer $n > 0$, there is a polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ satisfying the following conditions:

(1) All coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are positive real numbers;

(2) At least one of the coefficients is $\frac{1}{n}$;

(3) The value of the polynomial is integer for every integer x.

Answer: The claim is true.

Solution: The product of any n consecutive integers is divisible by n. Thus the conditions are satisfied by the polynomial obtained by removing parentheses and collecting terms in the expression $\frac{1}{n}(x+1)\ldots(x+n)$.

F7. *(Grade 10.)*

(a) Given a convex quadrilateral $ABCD$ with $AD < AB$ and $CD < CB$, is the internal angle at B always less than the internal angle at D?

(b) The same question for a non-convex quadrilateral.

Answer: a) Yes; b) No.

Solution:

(a) Consider the triangles ADB and CDB (Fig. 21). The claim $AD < AB$ implies $\angle ABD < \angle ADB$ because longer side is opposite to larger

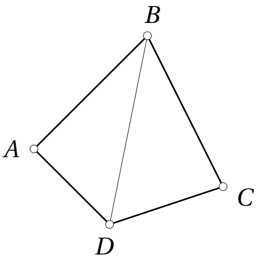


Fig. 21

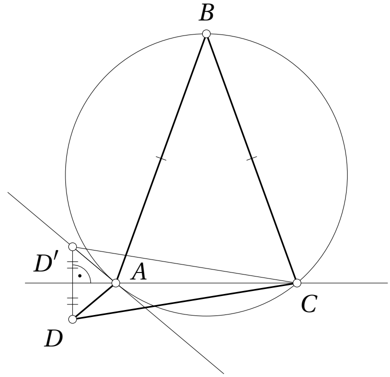


Fig. 22

angle. Similarly, $CD < CB$ implies $\angle CBD < \angle CDB$. As $ABCD$ is convex, $\angle ABD + \angle CBD = \angle ABC$ and $\angle ADB + \angle CDB = \angle ADC$. Hence adding the two inequalities gives $\angle ABC < \angle ADC$.

(b) Let points A, B, C be such that $AB = BC > AC$. Choose point D' on the line tangent to the circumcircle of the triangle ABC at A in such a way that B and D' lie on the same side of AC and the inequalities $AD' < AB$ ja $CD' < BC$ hold (the last inequality is possible since $AC < BC$); let D be the reflection of D' from AC (Fig. 22). Then both assumptions $AD < AB$ and $CD < CB$ hold. But the claim $\angle ABC < \angle ADC$ is not true: since D' lies outside the circumference of triangle ABC, we have $\angle ABC > \angle AD'C = \angle ADC$. Hence the hypothesis does not hold for non-convex quadrilaterials.

F8. *(Grade 10.)* Find all positive integers n such that one can choose $n - 3$ non-intersecting diagonals of a regular n-gon that divide the n-gon into triangles in such a way that every chosen diagonal is a side of minimal length in some triangle.

Answer: $3 \cdot 2^k$ where k is any natural number.

Solution: Let Δ be the triangle containing the center O of the n-gon (colored in Fig. 23; if the center lies on a diagonal then choose either of the triangles having this diagonal as a side). Let d be any side of Δ. If d is a diagonal of the n-gon then d separates Δ from a neighboring triangle whose other two sides are shorter than d. By assumption, d has to be a side of minimal length in Δ. But if d is a side of the n-gon then d is also a side of minimal length in Δ. Thus all sides of Δ are equally minimal, meaning that Δ is equilateral. Consequently, there is a constant number of sides of n-gon between the endpoints of every side of Δ. Hence $n = 3s_0$ for a positive integer s_0.

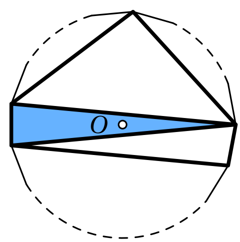


Fig. 23

Consider now an arbitrary triangle Δ' neighboring Δ. Let d' be any of its two sides not common with Δ. If d' is a diagonal of the n-gon then d' separates Δ' from a third triangle whose other sides are shorter than d'. Thus d' must be a side of minimal length in Δ'. But if d' is a side of the n-gon then d' is also a side of minimal length in Δ'. Hence the sides of Δ' not common with Δ have equal length and there must be the same number of sides of the n-gon between the endpoints of these sides of Δ'. Consequently $s_0 = 2s_1$ where s_1 is a positive integer.

If the sides of equal length of Δ' are sides of the n-gon then $s_1 = 1$ and $n = 3 \cdot 2$. Otherwise we can continue similarly to get $s_1 = 2s_2$ where either $s_2 = 1$ or $s_2 = 2s_3$, etc. Thus $n = 3 \cdot 2^k$ for a natural number k. A construction for every n of the form $3 \cdot 2^k$ follows from the argumentation.

F9. *(Grade 10.)* Four boxes with masks and means of disinfection are transported to participants of a carnival. Weighing the boxes pairwise results in six quantities, four largest of which are 125 kg, 120 kg, 110 kg and 101 kg. Find all possibilities of what can be the weights of the four boxes.

Answer: 72 kg, 53 kg, 48 kg, 38 kg or 67.5 kg, 57.5 kg, 52.5 kg, 33.5 kg.

Solution: Let the masses of the boxes be a, b, c and d kilograms, whereby $a \le b \le c \le d$. Clearly the largest result of pairwise weighing occurs in the case of two heaviest boxes, i.e., $c + d = 125$. The second largest result occurs in the case of the heaviest and the third heaviest box, i.e., $b + d = 120$, because $a + c \le b + c \le b + d$ and $a + d \le b + d$. Similarly we see that the smallest result occurs in the case of two lightest boxes and the second smallest result occurs in the case of the lightest and the third lightest boxes. Thus the total mass of the second and the third heaviest boxes is one of the middle values, i.e., $b + c$ is either 101 or 110. Consider these cases one by one, taking into account that $a + d$ must be the other among 101 and 110.

- Let $b + c = 101$. As $c = 125 - d$ and $b = 120 - d$, we obtain $(120 - d) + (125 - d) = 101$, implying $d = 72$. Thus $c = 125 - 72 = 53$, $b = 120 - 72 = 48$ and $a = 110 - 72 = 38$.
- Let $b + c = 110$. As $c = 125 - d$ and $b = 120 - d$, we obtain $(120 - d) + (125 - d) = 110$, implying $d = 67.5$. Thus $c = 125 - 67.5 = 57.5$, $b = 120 - 67.5 = 52.5$ and $a = 101 - 67.5 = 33.5$.

F10. *(Grade 11.)* Let $p_1, p_2, p_3, \ldots$ be all prime numbers in increasing order. Prove that $p_2 + p_4 + \ldots + p_{2n} > 3n^2 - 2n + 1$ for every positive integer n.

Solution: For any positive integer k, six consecutive integers $6k, 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5$ can contain at most two prime numbers. Hence $p_i \ge p_{i-2} + 6$ for all $i \ge 5$, implying $p_{2i} \ge p_4 + 6(i - 2) = 6i - 5$ for all $i \ge 2$. Thus $p_2 + p_4 + \ldots + p_{2n} > 2 + (7 + 13 + \ldots + (6n - 5)) = 3n^2 - 2n + 1$.

F11. *(Grade 11.)* The polynomial $x^3 + px + q$, where p and q are real numbers and at least one of them is non-zero, has a real root a that satisfies

$a^2 \le -\frac{4}{3}p$. Prove that this polynomial has a real root different from a.

Solution: The assumption $a^3 + pa + q = 0$ implies $q = -a(a^2 + p)$, whence $x^3 + px + q = (x - a)(x^2 + ax + a^2 + p)$. The discriminant of $x^2 + ax + a^2 + p$ is $D = a^2 - 4(a^2 + p) = -(3a^2 + 4p)$; the assumption $a^2 \le -\frac{4}{3}p$ implies $D \ge 0$. Hence there are real numbers b and c such that $x^2 + ax + a^2 + p = (x - b)(x - c)$, so the polynomial $x^3 + px + q$ has roots b and c. If $a = b = c$ then $x^3 + px + q = (x - a)^3 = x^3 - 3ax^2 + 3a^2x + a^3$. Hence $-3a = 0$, $3a^2 = p$ and $a^3 = q$, implying $p = q = 0$. This contradicts the assumption. Consequently, $x^3 + px + q$ has a real root different from a.

Remark: The problem can also be solved by standard means of calculus.

F12. *(Grade 11.)* A quadrilateral $ABCD$ has incenter I. Diagonals AC and BD intersect at point P. Given that I lies inside the triangle PAB (not on its side), prove that the area of the triangle PAB is greater than the area of any of triangles PBC, PCD and PDA.

Solution: Denote the area of any triangle Δ by S_Δ. Let D' be the reflection of D from AC and $\alpha = \angle APB$ (Fig. 24). We have

$$S_{PAB} = \frac{1}{2} \cdot PA \cdot PB \cdot \sin\alpha, \tag{4}$$

$$S_{PBC} = \frac{1}{2} \cdot PB \cdot PC \cdot \sin(180^\circ - \alpha) = \frac{1}{2} \cdot PB \cdot PC \cdot \sin\alpha, \tag{5}$$

$$S_{PCD} = \frac{1}{2} \cdot PC \cdot PD \cdot \sin\alpha, \tag{6}$$

$$S_{PDA} = \frac{1}{2} \cdot PD \cdot PA \cdot \sin(180^\circ - \alpha) = \frac{1}{2} \cdot PD \cdot PA \cdot \sin\alpha. \tag{7}$$

As the bisector of the angle DAB passes through I that lies inside the triangle PAB, we have $\angle DAC < \angle DAI = \angle BAI < \angle BAC$, implying

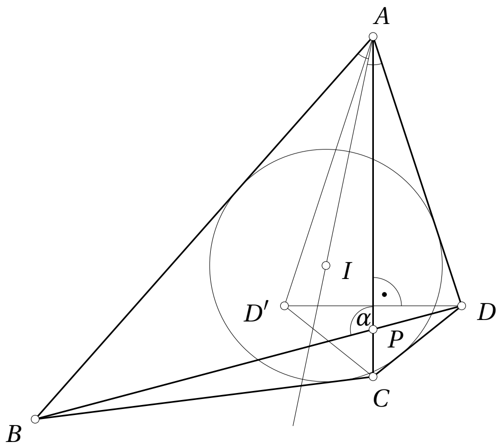


Fig. 24

$\angle D'AC < \angle BAC$. By interchanging the roles of A and C, we similarly obtain $\angle D'CA < \angle BCA$. Hence D' lies inside the triangle ABC, implying that $S_{ADC} = S_{AD'C} < S_{ABC}$. Adding (6) and (7) gives $S_{PCD} + S_{PDA} = \frac{1}{2} \cdot (PC + PA) \cdot PD \cdot \sin\alpha$, i.e., $S_{ACD} = \frac{1}{2} \cdot AC \cdot PD \cdot \sin\alpha$. Analogously from (4) and (5) we obtain $S_{ABC} = \frac{1}{2} \cdot AC \cdot PB \cdot \sin\alpha$. Hence the inequality $S_{ADC} < S_{ABC}$ implies $PD < PB$.

Interchanging the roles of A and B and the roles of C and D similarly gives $PC < PA$. Now (4) and (5) together imply $S_{PAB} > S_{PBC}$, (4) and (7) together imply $S_{PAB} > S_{PDA}$, (5) and (6) together imply $S_{PBC} > S_{PCD}$, and (6) and (7) together imply $S_{PDA} > S_{PCD}$. Hence the desired claim follows.

F13. *(Grade 11.)* Some knights are on a playground. Each knight has three properties: *speed*, *smartness* and *sightliness*. For every knight, each property takes a certain integral value x such that $1 \leq x \leq n$. A knight A can win a knight B if the speed, smartness and sightliness of A are all greater than those of B. It is known that none among the knights on the playground can win any of the other knights and every two knights differ by at least one property. Find the largest possible number of knights on the playground.

Answer: $3n^2 - 3n + 1$.

Solution: For each knight, we can define a unique non-negative integer x such that the properties of the knight are $a + x$, $b + x$ and $c + x$ and $\min(a, b, c) = 1$. Call the vector (a, b, c) the *base triple* of the knight.

If two knights had the same base triple then either their all properties would be equal or one of them could win the other one. Both cases are excluded by the assumptions. Thus each knight has a unique base triple, implying that there are at most as many knights as possible base triples.

The number of triples of integers $1, \ldots, n$ is n^3. Among them, $(n-1)^3$ triples do not contain the value 1. Thus the number of possible base triples is $n^3 - (n-1)^3 = 3n^2 - 3n + 1$. This number is achieved if, for every base triple, there is exactly one knight whose values of properties coincide with the components of this base triple.

F14. *(Grade 11.)* Let $a > 2$ be an integer. Let

$$\begin{aligned} x &= (a-1) \cdot a^{a-2} + (a-2) \cdot a^{a-3} + \ldots + 2 \cdot a^1 + 1 \cdot a^0, \\ y &= 1 \cdot a^{a-2} + 2 \cdot a^{a-3} + \ldots + (a-2) \cdot a^1 + (a-1) \cdot a^0. \end{aligned}$$

Prove that $x - 1$ is divisible by $y + 1$.

Solution: Note that $x + y = a \cdot a^{a-2} + a \cdot a^{a-3} + \ldots + a \cdot a^1 + a \cdot a^0 = a^{a-1} + a^{a-2} + \ldots + a^1$. Hence

$$\begin{aligned} x + 2y &= a^{a-1} + 2 \cdot a^{a-2} + 3 \cdot a^{a-3} + \ldots + (a-1) \cdot a^1 + (a-1) \cdot a^0 \\ &= a\left(1 \cdot a^{a-2} + 2 \cdot a^{a-3} + \ldots + (a-1) \cdot a^0\right) + (a-1) \\ &= ay + (a-1). \end{aligned}$$

As $x + 2y = ay + (a-1)$ is equivalent to $x - 1 = (a-2)(y+1)$, the claim follows.

F15. *(Grade 12.)* Does there exist a positive integer n such that the first and the second digit of 2^n are 3 and 9, respectively?

Answer: Yes.

Solution: Let m be the smallest integer such that $1.024^m \geqslant 3.9$. Then $1.024^m = 1.024^{m-1} \cdot 1.024 < 3.9 \cdot 1.024 = 3.9 + 3.9 \cdot 0.024 < 3.9 + 4 \cdot 0.025 = 4$. Thus the first two digits of $2^{10m} = 1.024^m \cdot 10^{3m}$ are indeed 3 and 9.

Remark: The number m in the solution equals 58 and the corresponding power is $2^{580} \approx 3.957286 \cdot 10^{174}$. The least integer satisfying the conditions of the problem is $2^{95} = 39614081257132168796771975168$.

F16. *(Grade 12.)* Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for every real number a, the function $g(x) = f(x+a)$ is either an even or odd function.

Answer: All constant functions $f(x) = c$.

Solution: Denote $f(0) = c$. We show that $f(x) = c$ for every real number x. Fix a real number a arbitrarily; by assumption, the function $g(x) = f\left(x + \frac{a}{2}\right)$ is either an even or odd function. If this g is even then

$$f(a) = f\left(\frac{a}{2} + \frac{a}{2}\right) = g\left(\frac{a}{2}\right) = g\left(-\frac{a}{2}\right) = f\left(-\frac{a}{2} + \frac{a}{2}\right) = f(0) = c.$$

Analogously in the case of odd g we obtain

$$f(a) = f\left(\frac{a}{2} + \frac{a}{2}\right) = g\left(\frac{a}{2}\right) = -g\left(-\frac{a}{2}\right) = -f\left(-\frac{a}{2} + \frac{a}{2}\right) = -f(0) = -c.$$

Thus if g defined as in the problem is even for all choices of a then $f(x) = c$ for every real number x. If $g(x) = f(x+a)$ is an odd function for some a then $f(a) = f(0+a) = g(0) = 0$. But by the above, $f(a) = c$ or $f(a) = -c$; hence $c = 0$. This implies that, for every real number x, $f(x) = 0 = c$.

On the other hand, if f is a constant function then $g(x) = f(x+a)$ is the same constant function for every a. Constant functions are even. Hence constant functions satisfy the conditions of the problem.

F17. *(Grade 12.)* The diagonals of a tangential quadrilateral $ABCD$ intersect at point P. The side AB is longer than any other side of $ABCD$. Prove that the angle APB is obtuse.

Solution: Let a, b, c, d be the lengths of the tangent line segments at vertices A, B, C, D, respectively, and $\alpha = \angle APB$ (Fig. 25). We have $a + b > b + c$ and $a + b > a + d$ as AB is the longest side, implying $a > c$ and $b > d$. By the law of cosines:

$$\begin{aligned}
(a+b)^2 &= PA^2 + PB^2 - 2 \cdot PA \cdot PB \cdot \cos\alpha, \\
(b+c)^2 &= PB^2 + PC^2 + 2 \cdot PB \cdot PC \cdot \cos\alpha, \\
(c+d)^2 &= PC^2 + PD^2 - 2 \cdot PC \cdot PD \cdot \cos\alpha, \\
(d+a)^2 &= PD^2 + PA^2 + 2 \cdot PD \cdot PA \cdot \cos\alpha.
\end{aligned}$$

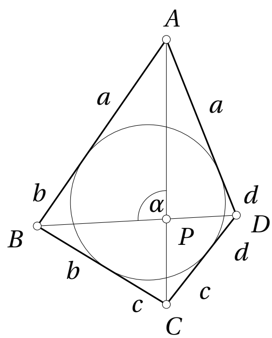


Fig. 25

Adding the first and the third equation and subtracting the second and the fourth equation gives

$2(ab+cd-bc-da) = -2\left(PA \cdot PB + PB \cdot PC + PC \cdot PD + PD \cdot PA\right) \cdot \cos\alpha$. The l.h.s. can be expressed as $2(a-c)(b-d)$ which is positive since $a > c$ and $b > d$. The parenthesized expression in the r.h.s. is also positive. Hence $\cos\alpha$ must be negative. This means that the angle APB is obtuse.

F18. *(Grade 12.)* On a horizontal line, one colors $2k$ points red and, in the right of them, $2k$ points blue. On every move, one chooses two points of different color, such that there is exactly one colored point between them, and interchanges the colors of the chosen points. How many different configurations can one obtain using these moves?

Answer: $\left(C_{k+l}^{k}\right)^2$.

Solution: Enumerate the colored points by positive integers from the left to the right. Every move can influence two points with the same parity, whereby the total number of red or blue points with this parity does not change. Thus in each configuration that can be achieved there are k red and l blue points with each parity. The number of such configurations is $\left(C_{k+l}^{k}\right)^2$ since there are C_{k+l}^{k} possibilities to choose k red points from among $k+l$ points with an odd number and C_{k+l}^{k} possibilities to choose k red points from among $k+l$ points with an even number.

On the other hand, all configurations with k red and l blue points with each parity can be achieved. Consider the points with even numbers. The number of the rightmost point with an even number that must become red is at least $2k$; after moving the rightmost in the initial configuration red point to the right until it reaches its desired position, this point and all points in the right of it are of the desired color. Next we can move the rightmost red point that is not yet at its desired position similarly to its desired position, etc. When all red points with an even number are at their desired position, other points with an even number are also of the right color. Similarly we act with points with an odd number.

F19. *(Grade 12.)* Let n be a positive integer, $n \geq 3$. In a regular n-gon, one draws a maximal set of diagonals, no two of which intersect in the interior of the n-gon. Every diagonal is labelled with the number of sides of the n-gon between the endpoints of the diagonal along the shortest path. Find the maximum value of the sum of the labels.

Answer: $\frac{n^2-9}{4}$ for odd n, $\frac{n^2-8}{4}$ for even n.

Solution 1: Consider an arbitrary set of diagonals satisfying the conditions. Label the sides of the n-gon with number 1. The diagonals partition the n-gon into triangles; let Δ be a triangle that contains the centroid of the n-gon. Let s, t, u be the labels of the sides of Δ (Fig. 26). The triangle Δ divides the interior of the n-gon outside Δ into three regions each bounded by one side of Δ and the respective sides of the n-gon (if a side of the n-gon coincides with a side of Δ then the respective region contains 0 triangles).

Consider the region bounded by the side labeled with s and the respective section of the boundary of the n-gon. Let us repeatedly cut out triangles of this region whose two sides lie on the boundary of the remaining part of the n-gon. (Such triangle must exist by the pigeonhole principle as the initial number of triangles in the region is $s-1$ while the number of sides of the n-gon in this region is s. Every cut reduces both the number of triangles and the number of sides by one, whence a similar argument holds for the subsequent steps.)

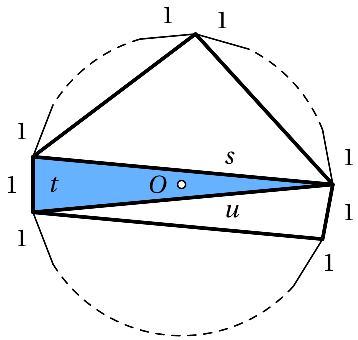


Fig. 26

As all triangles that are cut out are on the same side of the centroid of the polygon, the label of the longest side of each triangle equals the sum of labels on the two shorter sides. Cutting out a triangle removes two sides from the boundary of the region and introduces a new side labeled with the sum of the removed sides. Therefore, the sum of the labels on the boundary of the region is constant during the process. Let m_i be the greatest number on the boundary of the region after i triangles have been removed. As the boundary is initially labeled with s ones and in every step the number of sides is reduced by one then after i steps there are $s-i$ sides remaining. Therefore $s \geq m_i + (s-i-1)$ from which $m_i \leq i+1$. On the other hand, the label of the diagonal d_i along which the cut is made on the i-th step satisfies $d_i \leq m_i$. Hence the sum of all labels of the diagonals in this region (including the side of Δ labeled with s) is not greater than $2+3+\ldots+s$.

As the same is valid for the remaining two regions, the total sum of labels does not exceed $(2+3+\ldots+s)+(2+3+\ldots+t)+(2+3+\ldots+u)$. We know $s+t+u=n$; w.l.o.g., $\frac{n}{2} \geq s \geq t \geq u \geq 1$. If $u>1$ then $t<\left\lfloor\frac{n}{2}\right\rfloor$, whence a new triple (s', t', u') with $u'=u-1$ also satisfies conditions: Replace t with $t+1$ and change the order of s and $t+1$ if needed. The total sum increases since $t+1>u$. In the case of $u=1$, the only possibility is $s=t=\frac{n-1}{2}$ if n is odd and $s=\frac{n}{2}, t=\frac{n}{2}-1$ if n is even. Hence the total sum of labels does not exceed $2\cdot\left(2+3+\ldots+\frac{n-1}{2}\right)=\frac{n^2-9}{4}$ for odd n and $\left(2+3+\ldots+\left(\frac{n}{2}-1\right)\right)+\left(2+3+\ldots+\frac{n}{2}\right)=\frac{n^2-8}{4}$ for even n. These sums can be achieved by drawing all diagonals connecting a fixed vertex with all non-neighboring vertices.

Solution 2: Call the minimal number of sides of the n-gon between the endpoints of a given diagonal its *length*. The parts of the n-gon that a diagonal divides it into are called *short flank* and *long flank* (if the parts contain the same number of sides then both may be called short and long flank).

For every positive integer k, let a_k be the number of drawn diagonals whose length is at least k. Note that the sum of lengths of all drawn diagonals is $a_1 + a_2 + \ldots + a_{\lfloor \frac{n}{2} \rfloor}$. Indeed, represent all lengths of diagonals in the form $1 + 1 + \ldots + 1$; aligning the sums left, the kth column from the left contains exactly a_k ones and there are $\lfloor \frac{n}{2} \rfloor$ non-empty columns in total.

We prove that, for every $k \geq 2$, $a_k \leq n - 2k + 1$. The claim holds for $k = \frac{n}{2}$ since all diagonals of length $\frac{n}{2}$ intersect in the centroid of the n-gon. Assume in the rest that $k < \frac{n}{2}$. Let M be an arbitrary set of non-intersecting diagonals with length at least k which cannot be extended by any other such diagonal without introducing an intersection. Let d be the longest diagonal in M. Clearly M contains a diagonal of length k in the long flank of d, otherwise one could add a new diagonal of length k to the short flank of the shortest diagonal in the long flank of d. If the length of d is greater than k then M contains a diagonal of length k also in the short flank of d. Hence M contains at least 2 diagonals of length k. In the short flanks of these diagonals, there are $2(k-1)$ vertices not connected by any diagonals of M. The remaining $n - 2k + 2$ vertices can be joined by at most $n - 2k + 1$ non-intersecting diagonals, implying $a_k \leq n - 2k + 1$.

As $a_1 = n - 3 = (n-1) - 2$, the sum of the lengths of all drawn diagonals is at most $\left((n-1) + (n-3) + \ldots + \left(n - 2\lfloor \frac{n}{2} \rfloor + 1\right)\right) - 2$. This simplifies to $\frac{n^2-8}{4}$ in the case of even n and to $\frac{n^2-9}{4}$ in the case of odd n. If one draws all diagonals from a fixed vertex to the non-neighboring vertices then $a_k = n - 2k + 1$ for every $k = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$. Hence the bounds found above are achieved.

Solution 3: Define the notions *length*, *short flank* and *long flank* as in Solution 2. We show by induction that the sum of lengths of any diagonal d and all diagonals in the short flank of d does not exceed $2 + 3 + \ldots + s$, where s is the length of d. The claim holds for $s = 2$ since there are no diagonals in the short flank of a diagonal of length 2. Assume now that the claim holds for diagonals shorter than d. Let d' of length s' be the longest diagonal drawn in the short flank of d. By the induction hypothesis, the sum of lengths of the diagonal d' and all diagonals in the short flank of d' does not exceed $2 + 3 + \ldots + s'$. As there are exactly $s - 2$ diagonals drawn in the short flank of d and exactly $s' - 2$ of them are in the short flank of d', the number of diagonals that lie in the short flank of d and also in the long flank of d' must be $s - s' - 1$. By the choice of d', the sum of lengths of these diagonals is at most $s'(s - s' - 1)$ which does not exceed $(s'+1) + \ldots + (s-1)$ (the equality holds in the case $s' = s - 1$). Consequently, the sum of lengths of diagonal d and all diagonals in the short side of d does not exceed $2 + 3 + \ldots + s$. This completes the proof of the claim.

Now let d of length s be the longest drawn diagonal. Let d' of length s' be the longest diagonal in the long flank of d. By the claim proven above, the sum of lengths of the diagonal d and all diagonals in the short flank of d

does not exceed $2+3+\ldots+s$ and the sum of lengths of the diagonal d' and all diagonals in the short flank of d' does not exceed $2+3+\ldots+s'$. The total number of these diagonals is $(s-1)+(s'-1)$. As $n-3$ diagonals are drawn, the number of diagonals lying in the long flank of both d and d' is $n-s-s'-1$. By the choice of d and d', the sum of lengths of these diagonals is at most $s'(n-s-s'-1)$ that does not exceed $\left((s+1)+\ldots+\left\lfloor\frac{n}{2}\right\rfloor\right)+\left((s'+1)+\ldots+\left\lfloor\frac{n-1}{2}\right\rfloor\right)$. Thus the sum of lengths of all drawn diagonals is $\left(2+3+\ldots+\left\lfloor\frac{n}{2}\right\rfloor\right)+\left(2+3+\ldots+\left\lfloor\frac{n-1}{2}\right\rfloor\right)$ which simplifies to $\frac{n^2-8}{4}$ in the case of even n and $\frac{n^2-9}{4}$ in the case of odd n. These bounds are achieved as the sum of lengths of diagonals drawn from a fixed vertex to all non-neighboring vertices equals $\left(2+3+\ldots+\left\lfloor\frac{n}{2}\right\rfloor\right)+\left(2+3+\ldots+\left\lfloor\frac{n-1}{2}\right\rfloor\right)$.

Solution 4: Define the notions *length*, *short flank* and *long flank* as in Solution 2. Consider any set of diagonals with the maximal sum of lengths.

Suppose initially that, in the partition of the n-gon into triangles, there exists a triangle ABC whose all sides are diagonals of the n-gon; let the lengths of BC, CA and AB be a, b, c, respectively. W.l.o.g., $a \le b \le c$. Then AB and AC lie in the long flank of BC (otherwise $a = b + c$). Let BCD be the triangle of the partition in the short flank of BC (Fig. 27) and let the length of AD be d. Among the diagonals AB and AC, the one that lies in the short side of AD is shorter than AD, implying that $d > b$ or $d > c$. In either case, $d > a$. Thus replacing BC with AD in the partition leads to a larger sum of lengths. This contradicts the assumption.

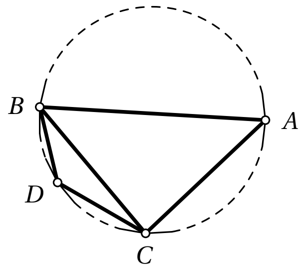


Fig. 27

Consequently, each triangle of the partition has at least one side that is a side of the n-gon. This means that, starting from any drawn diagonal d_1 of length 2, we can find a diagonal d_2 with a common endpoint with d_1, then a new diagonal d_3 with a common endpoint with d_2 etc., in such a way that the lengths of the diagonals increase with step 1 until $\left\lfloor\frac{n}{2}\right\rfloor$ and after that decrease with step 1. Hence the sum of lengths of all diagonals is at most $2\cdot\left(2+3+\ldots+\frac{n-1}{2}\right)=\frac{n^2-9}{4}$ for odd n and at most $2\cdot\left(2+3+\ldots+\left(\frac{n}{2}-1\right)\right)+\frac{n}{2}=\frac{n^2-8}{4}$ for even n. The proof immediately gives a construction that achieves the bounds.

Solution 5: Label each side of the n-gon with number 1. If the polygon with at least 5 sides is partitioned into triangles using non-intersecting diagonals then at least two triangles have two sides on the boundary of the polygon. After removing two such triangles, we can repeat the procedure in the remaining polygon. After repeating the procedure k times, the remaining polygon has exactly $n-2k$ sides since at each step a new side appears instead of two sides at two places. The sum of labels of the sides of the remaining polygon is n or less since every cut replaces two sides with a diagonal with label not greater than the sum of the labels of the two sides.

Consider arbitrary 4 sides in the polygon remaining after k steps. As the sum of labels of the other $n-2k-4$ sides is at least $n-2k-4$, the sum of labels of the chosen 4 sides is at most $2k+4$. Hence the labels of the diagonals along which the two cuts are made in the next step sum up to at most $2k+4=2(k+2)$. Using this argument for $k=0,1,\ldots,\frac{n-1}{2}-2$ in the case of odd n, we conclude that the sum of labels of the drawn diagonals, for odd n, does not exceed $2\cdot\left(2+3+\ldots+\frac{n-1}{2}\right)=\frac{n^2-9}{4}$. Similarly for even n, the sum of labels of $n-4$ drawn diagonals does not exceed $2\cdot\left(2+3+\ldots+\left(\frac{n}{2}-1\right)\right)=\frac{n^2}{4}-\frac{n}{2}-2$. As the label of the last diagonal is at most $\frac{n}{2}$, the total sum of labels is at most $\frac{n^2-8}{4}$. The sums are achieved by drawing diagonals from a fixed vertex to all non-neighboring vertices.

Selected Problems from the IMO Team Selection Contest

S1. There are 2020 inhabitants in a town. Before Christmas, they are all happy; but if an inhabitant does not receive any Christmas card from any other inhabitant, he or she will become sad. Unfortunately, there is only one post company which offers only one kind of service: before Christmas, each inhabitant may appoint two different other inhabitants, among which the company chooses one to whom to send a Christmas card on behalf of that inhabitant. It is known that the company makes the choices in such a way that as many inhabitants as possible will become sad. Find the least possible number of inhabitants who will become sad.

Answer: 674.

Solution: Partition 2019 inhabitants into 673 groups, each containing 3 inhabitants. Suppose that each inhabitant appoints two other members of the same group. As no group member is appointed thrice, the company cannot send three cards to one group member. Hence in every group, two different members get a card and at most one member will become sad. The inhabitant who does not belong to any group will become sad, too. Altogether, at most 674 inhabitants will become sad.

We show now that this is the least possible number. More precisely, we show that, in the case of n inhabitants, the company can make $\lceil\frac{n}{3}\rceil$ inhabitants sad. Call inhabitants X and Y *competitors* if somebody appoints them together to the company. By conditions of the problem, there are as many pairs of competitors as inhabitants or less (if several inhabitants appoint the same pair or some inhabitant appoints no pair). Thus it suffices to prove that, in the case of n inhabitants and at most n pairs of competitors, there must be $\lceil \frac{n}{3} \rceil$ inhabitants among which no two are competitors. The claim holds trivially for $n = 0, 1, 2$. Now let $n \geq 3$ and assume the claim being valid for $n - 3$ inhabitants. W.l.o.g., assume that there are exactly n pairs of competitors. Then there are exactly $2n$ instances of an inhabitant belonging to a pair of competitors. Consider two cases:

- If every inhabitant has exactly 2 competitors then choose an arbitrary inhabitant X and leave out X along with both competitors. Among the remaining $n - 3$ inhabitants, there are at most $n - 3$ pairs of competitors (besides two pairs containing X, removing either competitor canceled one more pair). By the induction hypothesis, one can find $\lceil \frac{n-3}{3} \rceil$ remaining inhabitants with no pair of competitors. Adding X to them results in $\lceil \frac{n}{3} \rceil$ inhabitants with no pair of competitors.
- If an inhabitant X has at most 1 competitor then there must exist an inhabitant Y with at least 3 competitors. Let Z be the competitor of X if X has a competitor and an arbitrary inhabitant different from X and Y otherwise. After leaving out X, Y and Z, we have $n - 3$ inhabitants and at most $n - 3$ pairs of competitors (removing Y cancels at least 3 pairs of competitors). By the induction hypothesis, one can find $\lceil \frac{n-3}{3} \rceil$ remaining inhabitants with no pair of competitors. Adding X to this set results in a group of $\lceil \frac{n}{3} \rceil$ inhabitants with no pairs of competitors.

This proves the desired claim.

S2. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f\left(x^3 + y^3\right) = f\left(x^3\right) + 3x^3 f(x) f(y) + 3 f(x) \left(f(y)\right)^2 + y^6 f(y).$$

Answer: $f(x) = 0$ and $f(x) = x^3$.

Solution: Substituting $x = y = 0$ gives $3\left(f(0)\right)^3 = 0$ which implies

$$f(0) = 0. \tag{8}$$

Substituting $x = 0$ into the given equation and applying (8) gives

$$f(y^3) = y^6 f(y). \tag{9}$$

Replacing the first term of the r.h.s. of the the given equation by (9) leads to

$$f(x^3 + y^3) = x^6 f(x) + 3x^3 f(x) f(y) + 3 f(x) (f(y))^2 + y^6 f(y). \tag{10}$$

Swapping the variables in (10) and subtracting the result from (10) gives

$$3x^3 f(x) f(y) + 3 f(x) (f(y))^2 = 3y^3 f(y) f(x) + 3 f(y) (f(x))^2. \tag{11}$$

By collecting similar terms in (11) and factorizing, we get

$$f(x) f(y) \left(f(x) - x^3 - f(y) + y^3\right) = 0. \tag{12}$$

Suppose that $f(x) = f(y)$ for some $x \neq y$. By injectivity, $x^3 \neq y^3$, whence $f(x) - x^3 - f(y) + y^3 \neq 0$. By (12), we get $f(x) f(y) = 0$, which implies $f(x) = f(y) = 0$. Consequently, f cannot take non-zero values repeatedly.

Suppose now that $f(b) = 0$ for some $b \neq 0$. Substituting $y = b$ into the original equation gives $f(x^3 + b^3) = f(x^3)$ for all x. If $f(a) \neq 0$ for some a then taking $x = \sqrt[3]{a}$ in the last equality contradicts the previous paragraph as $b^3 \neq 0$. Thus $f(x) = 0$ for all x. This function satisfies the given equation.

It remains to consider the case where $f(x) = 0$ implies $x = 0$. By substituting $y = 1$ into (12), we get $f(x) - x^3 - f(1) + 1 = 0$ for all $x \neq 0$. In other words, $f(x) = x^3 + c$ for all $x \neq 0$ where $c = f(1) - 1$. After substituting into (9), simplifying gives $c = y^6 c$ for all y which is possible only if $c = 0$. Hence $f(x) = x^3$ for all x. This function also satisfies the original equation.

S3. For every positive integer x, denote by $\kappa(x)$ the number of composite numbers not greater than x. Find all positive integers n such that

$$(\kappa(n))! \cdot \operatorname{lcm}(1, 2, \ldots, n) > (n-1)!.$$

Answer: $2, 3, 4, 5, 7, 9$.

Solution: The inequality holds for $n = 2, 3, 4, 5, 7, 9$ and does not hold for $n = 1, 6, 8, 10, 11, 12$. Assume in the rest that $n \geq 13$. By definition of $\kappa(n)$, there exists exactly $n - 1 - \kappa(n)$ prime numbers not greater than n; these are the primes dividing $\operatorname{lcm}(1, 2, \ldots, n)$. We have $n - 1 - \kappa(n) \geq 6$ as $n \geq 13$. Let $q_1, \ldots, q_{n-1-\kappa(n)}$ be all prime powers in the canonical representation of $\operatorname{lcm}(1, 2, \ldots, n)$. W.l.o.g., $q_1 > q_2 > \ldots > q_{n-1-\kappa(n)}$. As at least 5 numbers among $q_1, \ldots, q_6$ are odd, in the case of odd n we have $q_1 q_2 q_3 q_4 q_5 q_6 \leq n(n-1)(n-2)(n-4)(n-6)(n-8)$ and in the case of even n similarly $q_1 q_2 q_3 q_4 q_5 q_6 \leq n(n-1)(n-3)(n-5)(n-7)(n-9)$. But since $(n-3)(n-5)(n-7)(n-9) < (n-2)(n-4)(n-6)(n-8)$ and $n(n-8) < (n-3)(n-5)$, we anyway obtain

$$q_1 q_2 q_3 q_4 q_5 q_6 < (n-1)(n-2)(n-3)(n-4)(n-5)(n-6) = \frac{(n-1)!}{(n-7)!}.$$

As $q_6 < n - 6$, the inequality $q_i < n - i$ holds for every $i > 6$, whence

$$q_7 \ldots q_{n-1-\kappa(n)} \leq (n-7) \ldots (\kappa(n) + 1) = \frac{(n-7)!}{(\kappa(n))!}.$$

Consequently, $\operatorname{lcm}(1, 2, \ldots, n) = q_1 q_2 \ldots q_{n-1-\kappa(n)} < \frac{(n-1)!}{(\kappa(n))!}$, contradicting the original inequality. Hence the inequality holds for $n = 2, 3, 4, 5, 7, 9$ only.

S4. Given a triangle Δ with circumradius R and inradius r, prove that the area of the circle with radius $R + r$ is more than 5 times greater than the area of the triangle Δ.

Solution: Let the area of the triangle Δ be S. Among the triangles with fixed circumradius, the one with largest perimeter is equilateral (as can be easily inferred from Jensen's inequality). Hence $S = \frac{a+b+c}{2} \cdot r \leq \frac{3\sqrt{3}}{2} Rr$. By Euler's inequality, $R \geq 2r$. Thus $\frac{R}{r} + \frac{r}{R} \geq 2 + \frac{1}{2}$, implying $R^2 + r^2 \geq \frac{5}{2} Rr$. Consequently, $\pi (R + r)^2 \geq \pi \cdot \frac{9}{2} Rr > 5 \cdot \frac{3\sqrt{3}}{2} Rr \geq 5S$.

S5. With an expression that uses an operator $*$, one can make the following transformations:

(1) Rewrite an expression of the form $x * (y * z)$ as $((1 * x) * y) * z$;

(2) Rewrite an expression of the form $x * 1$ as x.

The transformations may be performed only on the entire expression and not on the subexpressions. For example, $(1 * 1) * (1 * 1)$ may only be rewritten using the first kind of transformation as $((1 * (1 * 1)) * 1) * 1$ (taking $x = 1 * 1$, $y = 1$ and $z = 1$), but not as $1 * (1 * 1)$ or $(1 * 1) * 1$ (in the latter two cases the second kind of transformation would have been applied just to the left or right subexpression of the form $1 * 1$).

Denote $A_n = \underbrace{1 * (1 * (1 * (\ldots * (1 * 1) \ldots)))}_{n \text{ ones}}$. For which positive integers n can the expression A_n be transformed to an expression that does not contain occurrences of the $*$ operator?

Answer: 1, 2, 3, 4.

Solution: The final result can only be 1. It can only come from the expression $1 * 1$ via transformation (2). Any intermediate expression of the form $1 * x$ can also be obtained only via transformation (2) from the expression $(1 * x) * 1$, and any intermediate result of the form $(1 * x) * y$ can appear only via transformation (2) from the expression $((1 * x) * y) * 1$. Any intermediate result of the form $((1 * x) * y) * z$ can only be obtained from the expression $x * (y * z)$ via transformation (1), because if it were obtained from the expression $(((1 * x) * y) * z) * 1$ via transformation (2) then the latter could have only been obtained from a longer expression via transformation (2), which in turn could only be obtained from a longer expression etc., none of which can be represented in the form $1 * (1 * (1 * (\ldots * (1 * 1) \ldots)))$. Therefore the final result uniquely determines all previous expressions:

$$
\begin{aligned}
\underline{1} &\overset{(2)}{\Longleftarrow} \underline{1 * 1} \overset{(2)}{\Longleftarrow} (1 * 1) * 1 \overset{(2)}{\Longleftarrow} ((1 * 1) * 1) * 1 \overset{(1)}{\Longleftarrow} \underline{1 * (1 * 1)} \\
&\overset{(2)}{\Longleftarrow} (1 * (1 * 1)) * 1 \overset{(2)}{\Longleftarrow} ((1 * (1 * 1)) * 1) * 1 \overset{(1)}{\Longleftarrow} (1 * 1) * (1 * 1) \\
&\overset{(2)}{\Longleftarrow} ((1 * 1) * (1 * 1)) * 1 \overset{(1)}{\Longleftarrow} 1 * ((1 * 1) * 1) \overset{(2)}{\Longleftarrow} (1 * ((1 * 1) * 1)) * 1 \\
&\overset{(2)}{\Longleftarrow} ((1 * ((1 * 1) * 1)) * 1) * 1 \overset{(1)}{\Longleftarrow} ((1 * 1) * 1) * (1 * 1) \\
&\overset{(1)}{\Longleftarrow} \underline{1 * (1 * (1 * 1))} \overset{(2)}{\Longleftarrow} (1 * (1 * (1 * 1))) * 1 \\
&\overset{(2)}{\Longleftarrow} ((1 * (1 * (1 * 1))) * 1) * 1 \overset{(1)}{\Longleftarrow} (1 * (1 * 1)) * (1 * 1) \\
&\overset{(2)}{\Longleftarrow} ((1 * (1 * 1)) * (1 * 1)) * 1 \overset{(1)}{\Longleftarrow} (1 * 1) * ((1 * 1) * 1) \\
&\overset{(2)}{\Longleftarrow} ((1 * 1) * ((1 * 1) * 1)) * 1 \overset{(1)}{\Longleftarrow} 1 * (((1 * 1) * 1) * 1) \\
&\overset{(2)}{\Longleftarrow} (1 * (((1 * 1) * 1) * 1)) * 1 \overset{(2)}{\Longleftarrow} ((1 * (((1 * 1) * 1) * 1)) * 1) * 1 \\
&\overset{(1)}{\Longleftarrow} (((1 * 1) * 1) * 1) * (1 * 1).
\end{aligned}
$$

However, the expression $(((1*1)*1)*1)*(1*1)$ cannot be an intermediate result based on what has been showed earlier. Therefore all expressions that can be transformed into 1 are shown in the chain above.

S6. Let n be an integer such that $n \geq 3$. On a plane, n points are chosen, no three of which are collinear. Consider all triangles with vertices at these points; denote minimal among the internal angles of these triangles by α. Find the greatest possible value of α and determine all configurations of n points for which it is achieved.

Answer: $\frac{180^\circ}{n}$; all sets of vertices of regular n-gons.

Solution: Let m be the number of points in the convex hull of the chosen set of points. As the sum of the sizes of the internal angles of an m-gon is $(m-2)\cdot 180^\circ$, minimal among the internal angles of the convex hull is of size at most $\frac{m-2}{m}\cdot 180^\circ$. Clearly $m \leq n$, whence minimal among the internal angles of the convex hull is also bounded by $\frac{n-2}{n}\cdot 180^\circ$.

Denote the chosen points by $A_0, A_1, \ldots, A_{n-1}$ in such a way that minimal internal angle of the convex hull is at A_{n-1} and a line through this point that rotates counterclockwise passes over the remaining points in order $A_0, A_1, \ldots, A_{n-2}$ (Fig. 28 depicts the situation for $n = 11$). Then

$$\angle A_0A_{n-1}A_1 + \angle A_1A_{n-1}A_2 + \ldots + \angle A_{n-3}A_{n-1}A_{n-2} = \angle A_0A_{n-1}A_{n-2},$$

whereby $A_0A_{n-1}A_{n-2}$ is minimal among the internal angles of the convex hull. As $\angle A_0A_{n-1}A_{n-2} \leq \frac{n-2}{n}\cdot 180^\circ$, there must exist an angle among $A_0A_{n-1}A_1, A_1A_{n-1}A_2, \ldots, A_{n-3}A_{n-1}A_{n-2}$ whose size is bounded by $\frac{180^\circ}{n}$. Thus $\alpha \leq \frac{180^\circ}{n}$.

To achieve $\alpha = \frac{180^\circ}{n}$, all estimations above must hold as equalities:

- Minimal internal angle of the convex hull must have size $\frac{m-2}{m}\cdot 180^\circ$, implying that all internal angles of the convex hull have equal size;
- $\frac{m-2}{m} = \frac{n-2}{n}$, implying $m = n$, i.e., the convex hull includes all n points;
- angles $A_0A_{n-1}A_1, A_1A_{n-1}A_2, \ldots, A_{n-3}A_{n-1}A_{n-2}$ must have size exactly $\frac{180^\circ}{n}$ and, by symmetry, the same holds in the case of any cyclic reenumeration of points.

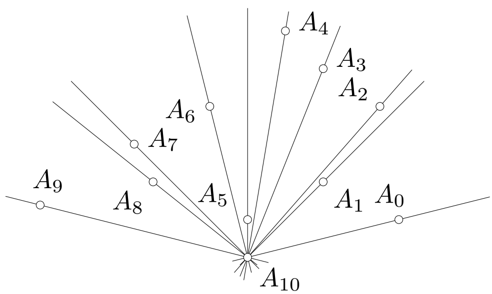


Fig. 28

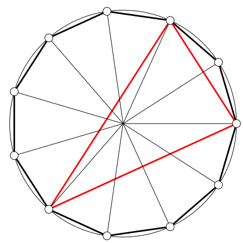


Fig. 29

From the latter, we obtain $\angle A_i A_{i+2} A_{i+1} = \frac{180^\circ}{n} = \angle A_{i+2} A_i A_{i+1}$, implying $A_i A_{i+1} = A_{i+1} A_{i+2}$ for every $i = 0, 1, \ldots, n-3$. Consequently, $A_0, A_1, \ldots, A_{n-1}$ are vertices of a regular n-gon.

On the other hand, internal angles of triangles whose vertices are arbitrary three vertices of a regular n-gon are inscribed angles subtending an arc of size at least $\frac{360^\circ}{n}$ of the circumcircle of the n-gon (Fig. 29). Thus all internal angles of these triangles are of size at least $\frac{180^\circ}{n}$.

S7. Let p, q be prime numbers and a be an integer such that $p > 2$ and $a \not\equiv 1 \pmod q$ but $a^p \equiv 1 \pmod q$. Prove that

$$(1 + a^1)(1 + a^2) \ldots (1 + a^{p-1}) \equiv 1 \pmod q.$$

Solution 1: As $a^p \equiv 1 \pmod q$ while $a \not\equiv 1 \pmod q$, the case $q = 2$ is impossible. Thus, the desired equation is equivalent to

$$(1 + a^0)(1 + a^1)(1 + a^2) \ldots (1 + a^{p-1}) \equiv 2 \pmod q. \tag{13}$$

Removing parentheses in the l.h.s. of (13) gives all monomials of the form $a^{i_1 + \ldots + i_k}$ where $\{i_1, \ldots, i_k\} \subseteq \{0, 1, \ldots, p-1\}$. As $a^p \equiv 1 \pmod q$, the sums in the exponents can be replaced with their remainders modulo p. We claim that each possible remainder is produced the same number of times provided that we leave out the empty set and the whole set $\{0, 1, \ldots, p-1\}$. Indeed, for each tuple $(i_1, \ldots, i_k)$ where $0 < k < p$, we can find l such that $kl \equiv 1 \pmod p$ and form a new tuple $(i_1 + l, \ldots, i_k + l)$ for which $(i_1 + l) + \ldots + (i_k + l) \equiv i_1 + \ldots + i_k + 1 \pmod p$. Clearly different tuples lead to different new tuples. Thus, for any remainder i, there are at least as many tuples with sum congruent to $i + 1$ as tuples with sum congruent to i. Doing this p times, we get back to the beginning and hence there are equal number of tuples giving each remainder.

Let this constant number of tuples be c. Then (13) is equivalent to

$$a^0 + a^{0+1+\ldots+(p-1)} + c(a^0 + a^1 + \ldots + a^{p-1}) \equiv 2 \pmod q.$$

As $p \mid \frac{(p-1)p}{2} = 0 + 1 + \ldots + (p-1)$, we have $a^{0+1+\ldots+(p-1)} \equiv 1 \pmod q$. Moreover, as $(a^0 + a^1 + \ldots + a^{p-1})(a-1) = a^p - 1 \equiv 0 \pmod q$ while $a - 1 \not\equiv 0 \pmod q$, we also have $a^0 + a^1 + \ldots + a^{p-1} \equiv 0 \pmod q$. Consequently, (13) is equivalent to $1 + 1 \equiv 2 \pmod q$ which holds trivially.

Solution 2: Like in Solution 1, prove that q is odd. By assumptions, the order of a modulo q divides p and does not equal 1. Hence the order of a modulo q must be p. Thus $1, a, a^2, \ldots, a^{p-1}$ are pairwise incongruent modulo q. This implies that the residue classes of $1, a, a^2, \ldots, a^{p-1}$ are all distinct roots of the polynomial $x^p - 1$ in $\mathbb{Z}_q$. Thus in $\mathbb{Z}_q$ we have $x^p - 1 = (x-1)(x-a)\left(x-a^2\right)\ldots\left(x-a^{p-1}\right)$. After substituting $x = -1$ and dividing all factors in the r.h.s. except the first one by -1, we obtain an equivalent congruence $-2 \equiv -2(1+a)\left(1+a^2\right)\ldots\left(1+a^{p-1}\right) \pmod q$. As $q \neq 2$, division by -2 is possible and gives the desired congruence.

Solution 3: Rewrite each factor $1 + a^i$ in the form $\frac{1-a^{2i}}{1-a^i}$. Since $p - 1$ is even and $a^p \equiv 1 \pmod{q}$, we obtain

$$\begin{aligned}
&\left(1-a^2\right)\left(1-a^4\right)\ldots\left(1-a^{2(p-1)}\right)\\
&\quad= \left(1-a^2\right)\left(1-a^4\right)\ldots\left(1-a^{p-1}\right)\left(1-a^{p+1}\right)\left(1-a^{p+3}\right)\ldots\left(1-a^{2p-2}\right)\\
&\quad\equiv \left(1-a^2\right)\left(1-a^4\right)\ldots\left(1-a^{p-1}\right)\left(1-a^{1}\right)\left(1-a^{3}\right)\ldots\left(1-a^{p-2}\right)\\
&\quad= \left(1-a^1\right)\left(1-a^2\right)\ldots\left(1-a^{p-1}\right) \pmod{q}.
\end{aligned}$$

As $a^p \equiv 1 \pmod{q}$, the least exponent i for which $a^i \equiv 1 \pmod{q}$ must divide p; as $a \not\equiv 1 \pmod{q}$, the least exponent must equal p. Therefore none of the factors $1 - a^1, 1 - a^2, \ldots, 1 - a^{p-1}$ is divisible by q. This means that the congruence above can be reduced by these factors, i.e.,

$$\frac{(1-a^2)(1-a^4)\ldots\left(1-a^{2(p-1)}\right)}{(1-a^1)(1-a^2)\ldots\left(1-a^{p-1}\right)} \equiv \frac{(1-a^1)(1-a^2)\ldots\left(1-a^{p-1}\right)}{(1-a^1)(1-a^2)\ldots\left(1-a^{p-1}\right)} \pmod{q}.$$

This proves the claim.

Problems Listed by Topic

Number theory: O1, O3, O8, O9, O14, O18, O20, F1, F5, F10, F15, S3, S7
Algebra: O4, O10, O15, O19, O21, F2, F6, F9, F11, F14, F16, S2
Geometry: O5, O11, O16, O22, F3, F7, F12, F17, S4
Discrete mathematics: O2, O6, O7, O12, O13, O17, O23, F4, F8, F13, F18, F19, S1, S5, S6