



Estonian Math Competitions

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Estonian Mathematical Olympiad

<http://www.math.olympiaadid.ut.ee/>

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools who have never been enrolled in a university or other higher educational institution, all Estonian citizens who meet the participation criteria of the forthcoming IMO may participate in these contests. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Besides the above-mentioned contests and the quiz “Kangaroo”, other regional and international competitions and matches between schools are held.

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This booklet presents selected problems of the open contests, the final round of national olympiad and the team selection contest. Selection has been made to include only problems that have not been taken from other competitions or problem sources without significant modification and seem to be interesting enough.

Selected Problems from Open Contests

O1. (*Juniors.*) A new number is formed by writing all integers from 1 through 2018 next to each other, but leaving out all of the digits 8. Is the resulting number divisible by 3?

Answer: No.

Solution: A number and the sum of its digits are equal modulo 3. Consecutive positive integers are congruent to 1, 2, 0, 1, 2, 0, ... modulo 3, thus the corresponding sums of digits are also congruent to 1, 2, 0, 1, 2, 0, ... modulo 3. We see that each three consecutive sums of digits give a number divisible by 3 when summing them up. As 2016 is divisible by 3, the sum of sums of digits of numbers 1, 2, ..., 2016 is also divisible by 3. The sum of the digits of the last two numbers 2017 and 2018 is also divisible by 3, so the number we get when concatenating numbers 1, 2, ..., 2018 is divisible by 3. In the numbers 1, 2, ..., 1000 the digit 8 can be found in the hundreds, tens and ones places 100 times each. The same is true for the numbers 1001, 1002, ..., 2000. Thus the number of digits 8 occurring in numbers 1, 2, ..., 2000 is divisible by 3. In the numbers 2001, 2002, ..., 2018 the digit 8 occurs twice. Thus the sum of all digits 8 is not divisible by 3, which means that removing them makes the sum of digits of the big number not divisible by 3 and thus the concatenated number itself is also not divisible by 3.

O2. (*Juniors.*) Three monks are meditating in a cave. Each monk lies on two consecutive days of the week and tells the truth every other day of the week. No two monks lie on the same day. On Monday, one monk says: "Yesterday I was lying." The next day, the second monk replies: "An interesting coincidence, I also lied yesterday." On what day of the week will no monk lie?

Answer: Monday.

Solution: If the first monk were lying on Mondays, then his lying days would be Monday and Tuesday, because on Sunday the falseness of the sentence "Yesterday I was lying" would imply speaking truth. Because two monks cannot lie on the same day, the second monk would have to be truthful on Mondays and Tuesdays, which cannot let him claim on Tuesday "Yesterday I was lying". Thus the first monk must be speaking truth on Mondays and based on his sentence "Yesterday I was lying" he would have to be lying on Sundays.

If the second monk were lying on Mondays, then on Tuesdays his sentence "Yesterday I was lying" would be true. Thus the lying days of the second monk would be Sunday and Monday. But this is not possible, because the first monk lies on Sundays. Thus the second monk is being truthful on Mondays. As the second monk's Tuesday sentence "Yesterday I was lying" turns out to be false, the second monk must be lying on Tuesdays.

If the third monk were lying on Mondays, then he would also have to lie on Sundays or Tuesdays. This is not possible, because on these days the first and second monk are already lying. Thus on Mondays no monk lies.

O3. (*Juniors.*) Positive integers n , m and k are such that n divides $\text{lcm}(m, k)$ and m divides $\text{lcm}(n, k)$. Prove that $n \cdot \gcd(m, k) = m \cdot \gcd(n, k)$.

Solution 1: Let p be an arbitrary prime and let α , β and γ be its exponents in the canonical representation of numbers n , m and k , respectively. It suffices to show that the exponent of p in the canonical representation of numbers $n \cdot \gcd(m, k)$ and $m \cdot \gcd(n, k)$ is the same. Consider three cases.

Let $\alpha < \beta$. Because the number $\text{lcm}(n, k)$ is divisible by m , the greater of the numbers α and γ must be at least as big as β ; this is only possible if $\beta \leq \gamma$. From $\alpha < \beta \leq \gamma$ we get that the exponent of p in the canonical representation of $n \cdot \gcd(m, k)$ is $\alpha + \beta$ and in the canonical representation of $m \cdot \gcd(n, k)$ is $\beta + \alpha$ which is the same.

Let $\alpha > \beta$. By swapping the places of n and m in the previous analysis we analogously get that the exponent of p in the canonical representation of $m \cdot \gcd(n, k)$ and $n \cdot \gcd(m, k)$ is the same.

Let $\alpha = \beta$. Then the exponents of p in the canonical representation of n and m are equal, because of which the exponents of $\gcd(m, k)$ and $\gcd(n, k)$ are the same as well. Thus the exponents of p in the canonical representations of $n \cdot \gcd(m, k)$ and $m \cdot \gcd(n, k)$ are the same as well.

Solution 2: By assumption, the number $\text{lcm}(n, k)$ is divisible by m . But $\text{lcm}(n, k)$ is divisible by k as well. Thus $\text{lcm}(n, k)$ is divisible by $\text{lcm}(m, k)$. By swapping the places of n and m , we also get that $\text{lcm}(m, k)$ is divisible by $\text{lcm}(n, k)$. Thus $\text{lcm}(m, k) = \text{lcm}(n, k)$. We now obtain $\frac{m}{\gcd(m, k)} = \frac{\text{lcm}(m, k)}{k} = \frac{\text{lcm}(n, k)}{k} = \frac{n}{\gcd(n, k)}$, from where the problem statement follows.

O4. (*Juniors.*) Jüri and Mari's mother is expecting twins. If boys are born, then Jüri will have exactly k times more brothers than sisters. If girls are born, then Mari will have exactly l times less sisters than brothers. However, a boy and girl are born. How many times more brothers than sisters does the newborn boy have and how many times less sisters than brothers does the newborn girl have?

Answer: l and k .

Solution 1: The newborn boy has the same number of sisters as Mari would have if two girls had been born, because instead of the second girl he has sister Mari. The brothers of this boy are the brothers of Mari right before his birth. Mari would have had the same brothers also if two girls had been born. By assumption, Mari would have had l times less sisters than brothers, so the newborn boy now has l times more brothers than sisters. Analogously the newborn girl has as many brothers as Jüri would have had if two boys had been born, because instead of the missing second boy she

has brother Jüri. The sisters of the newborn girl are the sisters of Jüri before her birth. By assumption, Jüri would have had k times more brothers than sisters, thus the newborn girl has k times more sisters than brothers.

Solution 2: Let there be x boys and y girls in the family before the birth of the twins. If twins were both boys, then the family would have $x + 2$ boys and y girls, implying that Jüri would have $x + 1$ brothers and y sisters. If twins were both girls, then the family would have x boys and $y + 2$ girls, so Mari would have x brothers and $y + 1$ sisters. Problem statement gives us the system of equations $x + 1 = ky$, $y + 1 = \frac{x}{k}$. By solving it we get that $x = \frac{(k+1)l}{k-1}$ and $y = \frac{l+1}{k-1}$. If 1 boy and 1 girl are added, then each boy has $\frac{(k+1)l}{k-1}$ brothers and $\frac{l+1}{k-1} + 1$ or, equivalently, $\frac{k+1}{k-1}$ sisters, that is, l times more brothers than sisters, and each girl has $\frac{l+1}{k-1}$ sisters and $\frac{(k+1)l}{k-1} + 1$ or, equivalently, $\frac{k(l+1)}{k-1}$ brothers, that is, k times less sisters than brothers.

O5. (*Juniors.*) Point M lies on the diagonal BD of parallelogram $ABCD$ such that $MD = 3BM$. Lines AM and BC intersect in point N . What is the ratio of the area of triangle MND to the area of parallelogram $ABCD$?

Answer: $\frac{1}{8}$.

Solution: Denote the area of Δ as S_{Δ} . Since the altitudes dropped to side AD in triangles AND and ABD are equal due to AD and BC being parallel (Fig. 1), we have $S_{AND} = S_{ABD}$. Thus $\frac{S_{AND}}{S_{ABCD}} = \frac{S_{ABD}}{S_{ABCD}} = \frac{1}{2}$. Since AMD and NMB are similar due to parallel sides, we have $\frac{|MN|}{|MA|} = \frac{|MB|}{|MD|} = \frac{1}{3}$.

Thus $\frac{|MN|}{|AN|} = \frac{|MN|}{|MA| + |MN|} = \frac{1}{4}$. As triangles

MND and AND share a common altitude dropped from D , also $\frac{S_{MND}}{S_{AND}} = \frac{|MN|}{|AN|} = \frac{1}{4}$. Hence, $\frac{S_{MND}}{S_{ABCD}} = \frac{S_{MND}}{S_{AND}} \cdot \frac{S_{AND}}{S_{ABCD}} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$.

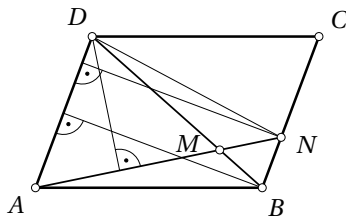


Fig. 1

O6. (*Juniors.*) The numbers from 1 through 9 are written in a 3×3 grid of cells. Every cell has one number and no number is written more than once. Find the largest number of pairs of adjacent (sharing a side) cells such that the number in the first cell of the pair is divisible by the number in the other cell in the pair.

Answer: 9.

Solution 1: In Fig. 2 there are 9 such pairs of cells. Let us show that this is the maximum possible number. For this we first count all possible pairs formed out of integers 2, ..., 9, where the first number is divisible by the second one. If the smaller number in such pair is 2, then the greater one is either 4, 6 or 8, and if the smaller number is 3, then the greater is 6 or 9, and if the smaller number is 4, then the greater number is 8. Altogether

| | | |
|---|---|---|
| 2 | 6 | 3 |
| 8 | 1 | 9 |
| 4 | 5 | 7 |

Fig. 2

| | | |
|--|---|---|
| | | 2 |
| | 1 | |
| | | |

Fig. 3

| | | |
|--|---|---|
| | | |
| | 1 | 2 |
| | | |

Fig. 4

| | | |
|--|--|---|
| | | |
| | | 5 |
| | | |

Fig. 5

| | | |
|--|--|---|
| | | 7 |
| | | 1 |
| | | 5 |

Fig. 6

we found $3 + 2 + 1 = 6$ pairs (the smaller number in the pair cannot be 5 or greater). Let us assume that numbers have been arranged in such a way that we get 10 required pairs of cells. Because we can create up to 6 such pairs with numbers 2 to 9, at least 4 of them must be formed using the number 1. This is only possible if 1 is in the middle cell. Then we get exactly 4 pairs using number 1, which means that to get 10 pairs we must also use all the 6 pairs listed above using numbers 2 to 9. But now there are no spots for number 2: if 2 were located in the corner (Fig. 3), then we could not get 3 pairs using number 2, but if it were located in the middle cell of an edge row or column (Fig. 4), then the neighbouring cell would have number 1 and thus we could still not create 3 pairs with number 2 such that the second number would be greater than 2.

Solution 2: In the example in Fig. 2 there are 9 such pairs. Assume that in some arrangement there are 10 required pairs of cells. Because we can pick two neighbouring cells in 12 ways, in at most 2 places we may have neighbouring cells in which neither number is divisible by the other one.

Numbers 5 and 7 are only divisible by 1 and themselves, no other single digit number given is divisible by them. Thus 5 and 7 cannot be located in the middle square, because the middle square has 4 neighbours and 3 of those give indivisible pairs. If 5 would be located on the edge but not in the corner (Fig. 5), then for a similar reason one of the neighbouring squares would definitely need to include 1. Because two other neighbouring squares form indivisible pairs, the number 7 cannot add any other indivisible pair. This is only possible if the only neighbours of 7 are 1 and 5. But this means that 5, 1 and 7 are all pairwise neighbours, which is not possible.

Therefore 5 must be in a corner. Analogously 7 must be in a corner. There are now 4 options to choose two neighbouring squares such that one of them contains 5 or 7. In at least 2 of these there must be a divisible pair, thus the neighbouring square must include 1. For this to be possible 5 and 7 cannot be in the opposite corners and 1 must be in the square between them; without loss of generality let 5, 1 and 7 be in the rightmost column in this order (Fig. 6). Number 9 cannot add any new indivisible pairs. Because 9 only forms divisible pairs with numbers 1 and 3, the neighbours of 9 can only be 1, 3, 5 and 7. Because of the locations of 1, 5 and 7 only one of them can be a neighbour of 9. Thus 9 could only have 2 neighbours and 9 would have to be located in a corner, but then none of 1, 5 and 7 could be its neighbour. The contradiction shows that 10 pairs is not achievable.

O7. (*Juniors.*) a) Are there two distinct positive integers whose sum of squares is the cube of an integer? b) The same question, but fourth power instead of a cube.

Answer: a) Yes; b) Yes.

Solution 1: a) For example, 10 and 5 are suitable, because $10^2 + 5^2 = 5^3$.
b) For example, 20 and 15 are suitable, because $20^2 + 15^2 = 5^4$.

Solution 2: We know that $3^2 + 4^2 = 5^2$. By multiplying both sides of the equation by 5^{10} we get that $3^2 \cdot 5^{10} + 4^2 \cdot 5^{10} = 5^2 \cdot 5^{10}$ which reduces to $(3 \cdot 5^5)^2 + (4 \cdot 5^5)^2 = 5^{12}$. As $5^{12} = (5^3)^4 = (5^4)^3$, the example is suitable for both parts of the problem.

O8. (*Juniors.*) A pentagon can be divided into equilateral triangles. Find all the possibilities that the sizes of the angles of this pentagon can be.

Answer: $(120^\circ, 120^\circ, 120^\circ, 120^\circ, 60^\circ)$, $(240^\circ, 120^\circ, 60^\circ, 60^\circ, 60^\circ)$, and $(300^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$.

Solution: If the polygon can be divided into equilateral triangles, then the sizes of all of its internal angles are of the form $x \cdot 60^\circ$, where x is an integer. Here x can only be 1, 2, 4 or 5 because the sizes of internal angles are in the range between 0° and 360° (non-inclusive) and differ from 180° . The sum of the angles of a pentagon, however, is $(5 - 2) \cdot 180^\circ = 9 \cdot 60^\circ$. The equation $x_1 + x_2 + x_3 + x_4 + x_5 = 9$ has, without taking the order into account, three solutions $(2, 2, 2, 2, 1)$, $(4, 2, 1, 1, 1)$ and $(5, 1, 1, 1, 1)$, in which the values of all variables are in the set $\{1, 2, 4, 5\}$. Thus the solutions of the problem could be $(120^\circ, 120^\circ, 120^\circ, 120^\circ, 60^\circ)$, $(240^\circ, 120^\circ, 60^\circ, 60^\circ, 60^\circ)$ and $(300^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$. The corresponding examples are shown on Figures 7, 8 and 9.

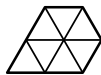


Fig. 7

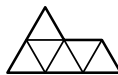


Fig. 8

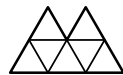


Fig. 9

O9. (*Juniors.*) The naughty Juku deleted two digits from the ten-digit number *2018*2019 (asterisks indicate deleted digits) written on the board. Find all possibilities of the original number if it is known that the original number is divisible by 99.

Answer: 7201862019.

Solution 1: Let the first and second digit to be deleted be x and y respectively. As $99 = 9 \cdot 11$ and 9 and 11 are relatively prime, the number is divisible by 99 if and only if it is divisible by 9 and 11. So the sum of its digits $x + 2 + 0 + 1 + 8 + y + 2 + 0 + 1 + 9$ and therefore also $x + y + 5$ has to be divisible by 9. From this we get $x + y = 4$ or $x + y = 13$. For the original number to be divisible by 11 the sum of its digits with alternating signs has to be divisible by 11, that is, $x - 2 + 0 - 1 + 8 - y + 2 - 0 + 1 - 9$ or, equivalently, $x - y - 1$ has to be divisible by 11. Here the only option is $x - y = 1$. This shows that x and y have different parity, thus $x + y = 4$ is not possible. From the system $x + y = 13$, $x - y = 1$ we get $x = 7$ and $y = 6$. Therefore the original number on the board was 7201862019.

Solution 2: Let the first and second number that were erased be x and y respectively. As $99 = 100 - 1$, the number is divisible by 99 if and only if the sum of the numbers formed of consecutive pairs of digits in it (that is, the sum of base 100 digits) is divisible by 99. Thus the original number is divisible by 99 if and only if $(10x + 2) + 1 + (80 + y) + 20 + 19$ is divisible by 99. This holds if and only if $10x + y$ is congruent to 76 modulo 99. Because $10 \leq 10x + y \leq 99$, the only possibility for this is if $10x + y = 76$, which in turn gives $x = 7$ and $y = 6$ as the only options. Thus the original number was 7201862019.

O10. (*Juniors.*) The numbers on a computer screen appear one after the other. Whenever the numbers a and b appear consecutively, the number $ab - 1$ appears next. The numbers 1 and 2 appear first. Find the 2018th number that will appear on the screen.

Answer: 0.

Solution: Because the first numbers are 1 and 2, the first nine numbers appearing on the screen are 1, 2, 1, 1, 0, -1 , -1 , 0, -1 . As the 8th and the 9th number are the same as the 5th and the 6th number and every number is only dependent on the previous two, the next numbers start copying those having been on the screen three steps earlier. As $2018 = 5 + 3 \cdot 671$, the 2018th number is the same as the 5th number, that is, 0.

O11. (*Juniors.*) Grandma brings Mari and Jüri a pack of 22 candies – four different colours and yellow occurs more than any other colour – and tells them to divide them nicely. Since she doesn't specify what "nicely" means, Mari offers Jüri the following game: they will take turns until the pack is empty, during each move Jüri takes two candies (or the only remaining candy), while Mari takes one candy of each colour still left in the bag. Since Jüri is still little and only understands that two are more than one, the game seems to him to be very useful and he agrees. Of course his big sister allows Jüri to go first. When all the candies are out of the pack, Mari and Jüri compare the candies they took and discover that they got an equal number of candies. Find the maximum and minimum possible number of yellow candies in the beginning.

Answer: 16 and 8.

Solution: As there are 22 candies, each kid gets 11 of them. According to Jüri's move rule, in order to get 11 candies he must take 2 candies during the first 5 moves and 1 candy during the last move. Because this candy has to be the last one in the pack, Mari can make exactly 5 moves.

Mari's move rule dictates that she can only get a maximum of 1 yellow candy every turn, that is, 5 yellow ones in total. Because Jüri can take up to 11 yellow candies, there cannot be more than 16 yellow candies in the pack originally. This number is indeed possible, and the colours of the rest of the candies are not relevant. To see that, suppose that Jüri only takes yellow

candies. As the remaining 6 candies do not have all the same colour, Mari can consume them in 5 moves and both end with 11 candies. We now show that there must be at least 8 yellow candies originally, by studying three cases separately.

If Mari takes 3 candies or less on her first move, then there must be 2 or less candies of some colour for Jüri to be able to exhaust that colour with his opening move. Thus there must be at least 20 candies of three colours. But if there are a maximum of 7 yellow candies, then there can only be a maximum of 19 candies of three colours initially.

If Mari takes 4 candies with her first move, but 2 or less with her second one, then before her second move two colours must be exhausted. During her first move Mari gets 2 candies of these colours and during his first two moves Jüri gets at most 2 candies of these colours with each move. Thus there can be a maximum of 6 candies of these colours, implying that there must be at least 16 candies of other colours. Thus there must be at least 8 yellow candies.

If Mari obtains 4 candies with her first move and 3 or 4 during her second move, then in order to get 11 candies in total she must get only 4 or 3 candies during the rest of the game. In both cases during her last two moves she gets 1 candy per move. Thus before her last two moves three colours must be exhausted. With the first three moves Mari gets $11 - 2$, i.e., 9 candies, out of which 3 are yellow, thus she will have 6 candies of other colours. Jüri can use his first 4 moves to take at most $4 \cdot 2 = 8$ candies of these colours. Thus there can be a maximum of 14 non-yellow candies, implying that there must be at least 8 yellow candies.

Having 8 yellow candies is possible, for example if the numbers of candies for each colour are 8, 7, 4, 3, respectively, and Jüri always takes the colour of which there are least candies available, then Mari during her moves gets 4, 3, 2, 1, 1 candies respectively, for a total of 11 candies.

O12. (*Seniors.*) Jüri and Mari play the following game. Jüri starts by drawing a random triangle on a piece of paper. Mari then draws a line on the same paper that goes through the midpoint of one of the midsegments of the triangle. Then Jüri adds another line that also goes through the midpoint of the same midsegment. These two lines divide the triangle into four pieces. Jüri gets the piece with maximum area (or one of those with maximum area) and the piece with minimum area (or one of those with minimum area), while Mari gets the other two pieces. The player whose total area is bigger wins. Does either of the players have a winning strategy, and if so, who has it?

Answer: No.

Solution: The midpoint of the midsegment of the triangle is located on the median of the triangle. Indeed, let D , E and F be the midpoints of sides BC , CA and AB of triangle ABC respectively and K be the point of intersec-

tion of median AD and midsegment EF (Fig. 10). Triangles AEK and ACD are similar, since the corresponding sides are parallel, hence $\frac{|EK|}{|EF|} = \frac{|AE|}{|AC|} = \frac{1}{2}$, that is, the median bisects the midsegment.

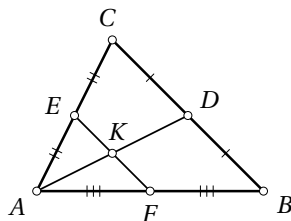


Fig. 10

Mari can avoid loss by drawing the extension of a median of the triangle during her move. This move respects the rules. The line divides the triangle into two regions with the same area.

Let Jüri's second move divide the first of these regions into two pieces with areas S_1 and S_2 and the second one into two pieces with areas S_3 and S_4 . W.l.o.g., let S_1 be the greatest area among these four. Since $S_1 + S_2 = S_3 + S_4$, the area S_2 must be the least among these areas. Thus Jüri gets precisely half of the original triangle and Mari gets the second half.

Jüri can also avoid loss. If Mari draws the extension of the median during her move, then based on the previous part neither player wins. If Mari draws any other line through the midpoint of some midsegment, then Jüri can draw the median himself and thus both again get an equal total area.

Thus neither of the two has a winning strategy.

Remark: The answer stays the same if one would modify the rules in such a way that on his first move Jüri also chooses the point inside the triangle through which the two lines must go (instead of the midpoint of the midsegment). Through any point inside the triangle, a line that divides the triangle into two parts of equal area can be drawn. Indeed, by starting with a random line through this point and rotating it 180° around that point, the area on one side of the line at one point turns from greater to smaller and vice versa. Since it is a continuous process, there will be a situation where areas on either side of the line are equal. By using this line instead of the median line, both players can avoid loss analogously to the solution above.

O13. (*Seniors.*) A calculator found in the attic has keys 1 through 9 and an operation key \otimes where $x \otimes y$ stands for $x + \frac{xy}{x-y}$. By pressing the operation button the first time, the screen keeps the previous number, at every next press the number $x \otimes y$ appears, where y is the number that was on the screen right before pressing the button and x is the number that was there before entering y . For example, if the user presses keys 2, 2, \otimes , 3, 3, \otimes , 4, 4, \otimes , then the screen displays numbers 2, 22, 22, 3, 33, -44 (the result of $22 \otimes 33$), 4, 44, -22 (the result of $(-44) \otimes 44$). If the user presses the operation key right in the beginning, presses operation key twice in a row or tries to do a calculation the result of which is not an integer, then the calculator breaks. Can the user get the number 2018 on the screen on this calculator?

Answer: No.

Solution: Let's assume it is possible to get the number 2018 on the screen. Because the calculator is missing the 0-key, it is not possible to enter 2018 directly and the number must instead be formed as a result of a $x \otimes y$ operation, where x and y are some integers and y can be entered on the calculator. As $x \otimes y = x + \frac{xy}{z-y} = \frac{x^2}{x-y}$, it must hold that $2018(x-y) = x^2$. Thus the number x^2 is divisible by 2018. Because $2018 = 2 \cdot 1009$, the number x^2 must be divisible by both 2 and 1009; as these are prime numbers, also x must be divisible by them. All in all x must be divisible by 2018, i.e., $x = 2018z$ for some integer z . By substituting this into $2018(x-y) = x^2$ and simplifying we get $2018z - y = 2018z^2$, implying $y = 2018z(1-z)$. In order to be able to enter y on the calculator it must be positive, hence $z(1-z) > 0$. But if $z \leq 0$ then $1-z \geq 1$ and $z(1-z) \leq 0$, and if $z \geq 1$ then $1-z \leq 0$ and $z(1-z) \leq 0$ again.

O14. (*Seniors.*) Let n and k be positive integers such that $k \leq n$. Prove the inequality $\frac{1}{k} + \frac{1}{k+1} + \dots + \frac{1}{n} \geq \frac{2(n-k+1)}{n+k}$.

Solution 1: Applying AM-HM to $k, k+1, \dots, n$ gives $\frac{n-k+1}{\frac{1}{k} + \frac{1}{k+1} + \dots + \frac{1}{n}} \leq \frac{k+(k+1)+\dots+n}{n-k+1}$. As $k, k+1, \dots, n$ form an arithmetic sequence, the r.h.s. equals the AM of the first and the last terms, i.e., $\frac{k+n}{2}$. Taking reciprocals and multiplying by $n-k+1$ now leads to the desired result.

Remark: The inequality used in the solution can also be obtained by applying Cauchy-Schwarz to $\sqrt{k}, \dots, \sqrt{n}$ and their reciprocals.

Solution 2: For arbitrary positive a and b , we have $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}$ since $(a+b)^2 \geq 4ab$. If k and n have different parity then applying this inequality to all pairs $(\frac{1}{k+i}, \frac{1}{n-i})$ gives $\frac{1}{k} + \frac{1}{k+1} + \dots + \frac{1}{n-1} + \frac{1}{n} \geq \frac{n-k+1}{2} \cdot \frac{4}{k+n} = \frac{2(n-k+1)}{k+n}$. If k and n have the same parity then, analogously, $\frac{1}{k} + \frac{1}{k+1} + \dots + \frac{1}{n-1} + \frac{1}{n} \geq \frac{n-k}{2} \cdot \frac{4}{k+n} + \frac{1}{\frac{k+n}{2}} = \frac{2(n-k)}{k+n} + \frac{2}{k+n} = \frac{2(n-k+1)}{k+n}$.

O15. (*Seniors.*) Polygon $A_0A_1 \dots A_{n-1}$ satisfies the following: $A_0A_1 \leq A_1A_2 \leq \dots \leq A_{n-1}A_0$ and $\angle A_0A_1A_2 = \angle A_1A_2A_3 = \dots = \angle A_{n-2}A_{n-1}A_0$ (all angles are internal angles). Prove that this polygon is regular.

Solution: Let l be a line that passes through A_0 and bisects the angle $A_{n-1}A_0A_1$, and define B_i for each $i = 0, 1, \dots, n-1$ as the foot of the perpendicular dropped from the point A_i onto the line l (Figures 11 and 12 depict the situation for $n = 6$ and $n = 7$, respectively). In addition, let us also denote $\alpha = \angle A_0A_1A_2 = \angle A_1A_2A_3 = \dots = \angle A_{n-2}A_{n-1}A_0$ and $\beta = \angle A_{n-1}A_0A_1$ and define $A_n = A_0$. Whenever $0 \leq i < \frac{n}{2}$, the lines A_iA_{i+1} and $A_{n-i}A_{n-(i+1)}$ are at the same angle with respect to l , because their direction is obtained by turning l by angle $(180^\circ - \frac{\beta}{2}) + i \cdot (180^\circ - \alpha)$. In particular, $A_{\frac{n-1}{2}}A_{\frac{n+1}{2}} \perp l$ for an odd n . Thus $\frac{B_iB_{i+1}}{A_iA_{i+1}} = \frac{B_{n-i}B_{n-(i+1)}}{A_{n-i}A_{n-(i+1)}}$.

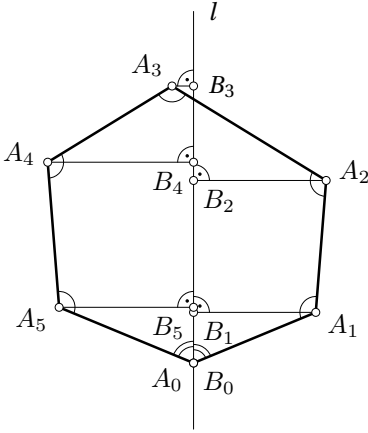


Fig. 11

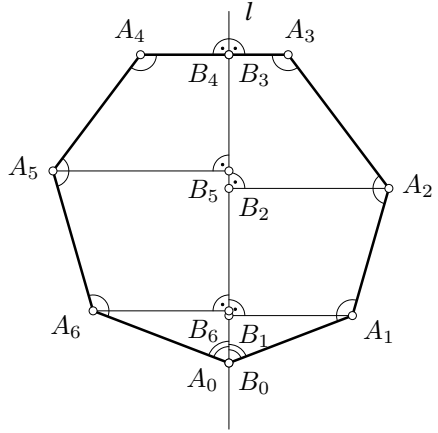


Fig. 12

Because $A_i A_{i+1} \leq A_{n-i} A_{n-(i+1)}$, the inequality $0 \leq i < \frac{n}{2}$ implies also $|B_i B_{i+1}| \leq |B_{n-i} B_{n-(i+1)}|$.

Let us assume that in the polygon $A_0 A_1 \dots A_{n-1}$ not all sides have equal lengths. Then $A_0 A_1 < A_0 A_{n-1}$, implying also $B_0 B_1 < B_0 B_{n-1}$ by the above. Adding all inequalities $B_i B_{i+1} \leq B_{n-i} B_{n-(i+1)}$, where $0 \leq i < \frac{n}{2}$, and taking the strict inequality for $i = 0$ into account gives, for even n , $B_0 B_1 + B_1 B_2 + \dots + B_{\frac{n}{2}-1} B_{\frac{n}{2}} < B_n B_{n-1} + B_{n-1} B_{n-2} + \dots + B_{\frac{n}{2}+1} B_{\frac{n}{2}}$. This, however, is impossible, because both sides of the inequality must be equal to $B_0 B_{\frac{n}{2}}$. For an odd n we analogously get $B_0 B_1 + B_1 B_2 + \dots + B_{\frac{n-1}{2}} B_{\frac{n-1}{2}+1} < B_n B_{n-1} + B_{n-1} B_{n-2} + \dots + B_{\frac{n-1}{2}+1} B_{\frac{n-1}{2}}$. This is also impossible, because it would mean that $B_0 B_{\frac{n-1}{2}} < B_0 B_{\frac{n-1}{2}+1}$, but $A_{\frac{n-1}{2}} A_{\frac{n-1}{2}+1} \perp l$ implies $B_{\frac{n-1}{2}} = B_{\frac{n-1}{2}+1}$. This contradiction shows us that $A_0 A_1 = A_1 A_2 = \dots = A_{n-1} A_0$.

To prove that the polygon is regular, we consider a polygon $A_i A_{i+1} A_{i+2} A_{i+3}$ where $0 \leq i \leq n-3$. Let X be the point of intersection of lines $A_i A_{i+1}$ and $A_{i+2} A_{i+3}$ (Fig. 13). As $\angle X A_{i+1} A_{i+2} = 180^\circ - \alpha = \angle X A_{i+2} A_{i+1}$, the triangle $X A_{i+1} A_{i+2}$ is isosceles with $X A_{i+1} = X A_{i+2}$. As $A_i A_{i+1} = A_{i+2} A_{i+3}$, the triangle $X A_i A_{i+3}$ is isosceles as well, implying that $\angle X A_i A_{i+3} = \angle X A_{i+3} A_i$. Hence

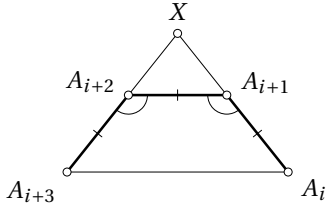


Fig. 13

$\angle A_i A_{i+1} A_{i+2} + \angle A_{i+2} A_{i+3} A_i = \angle A_{i+1} A_{i+2} A_{i+3} + \angle A_{i+3} A_i A_{i+1}$, implying that points A_i, A_{i+1}, A_{i+2} and A_{i+3} form a cyclic polygon. By starting from points A_0, A_1, A_2 and taking in more points one at a time we get that all points A_0, A_1, \dots, A_{n-1} are all located on the same circumcircle. From the equality of sides we get that the consecutive points form equal arcs. Thus $A_0 A_1 \dots A_{n-1}$ is regular.

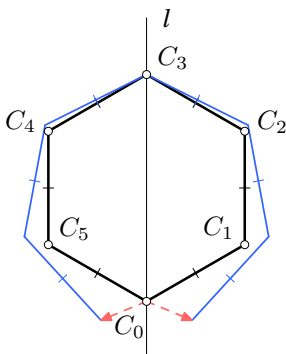


Fig. 14

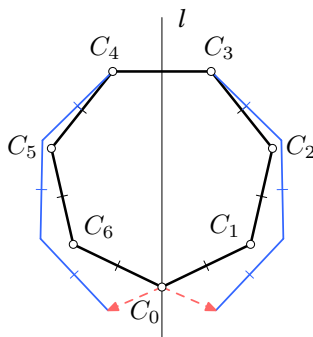


Fig. 15

Remark: The equality of angle $\angle A_{n-1}A_0A_1$ to others can also be proved as follows. Assuming the contrary, consider a regular n -gon $C_0C_1 \dots C_{n-1}$, the sides of which are equal to the sides of polygon $A_0A_1 \dots A_{n-1}$ and which has an axis of symmetry l at vertex C_0 . By increasing or decreasing angles by equal amounts at vertices C_1, \dots, C_{n-1} and preserving the axis of symmetry l we should get a shape equal to the polygon $A_0A_1 \dots A_{n-1}$. This is not possible, because vertex C_0 moves to different directions from the axis of symmetry l (Figures 14 and 15 depict the cases of $n = 6$ and $n = 7$ respectively). Thus the polygon $A_0A_1 \dots A_{n-1}$ must be regular.

O16. (*Seniors.*) Find all integers $n > 1$ for which the $n \times n$ grid of unit squares with the four corner squares removed can be tiled with the hook shaped tromino shown in the figure.



Answer: all integers greater than 1 that are not divisible by 3.

Solution 1: By removing all corner squares from $n \times n$ grid we are left with $n^2 - 4$ unit squares. If n is divisible by 3, then n^2 is also divisible by 3, meaning $n^2 - 4$ is not. Thus the area cannot be divided into trominoes.

In the rest of the cases n is congruent to 1, 2, 4 or 5 modulo 6. Case $n = 2$ is trivial, because the remaining area can be divided into 0 trominoes. For cases $n = 4$, $n = 5$ and $n = 7$ suitable tilings have been shown on Figures 16, 17 and 18, respectively. We now show that for greater n the existence of tiling can be concluded from the existence of tiling for $n - 6$.

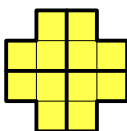


Fig. 16

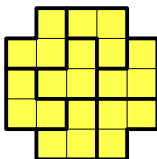


Fig. 17

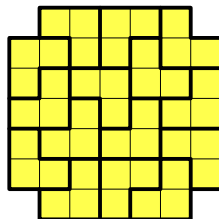


Fig. 18

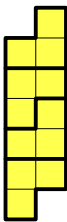


Fig. 19

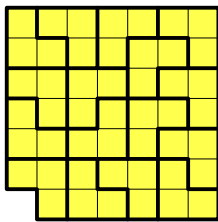


Fig. 20



Fig. 21

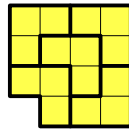


Fig. 22

If n is odd, then we tile a strip of width 6 along two sides of the grid so that the given area for $n - 6$ remains, which in turn can be tiled by the induction hypothesis. Firstly, we tile two similar shapes consisting of 12 unit squares at the ends of the strip as shown in Fig. 19. From the corner part of the strip we can remove a 7×7 square without one corner square; tiling of this has been shown in Fig. 20. The remaining rectangles have dimensions $(n - 9) \times 6$ and $6 \times (n - 9)$; as $n - 9$ is even, these can be divided into 2×3 and 3×2 rectangles respectively, which can in turn be divided into trominoes as shown in Fig. 21.

If n is even, then we tile a strip of width 3 surrounding the whole area so that we are again left with the original shape for $n - 6$, which can be divided by induction hypothesis. From the four corners of the strip we can tile 4×4 squares without one corner square as shown in Fig. 22. The remaining four rectangles have dimensions $(n - 8) \times 3$ and $3 \times (n - 8)$. As $n - 8$ is even, these areas can be divided into 2×3 and 3×2 rectangles, which can be divided into trominoes.

Remark: The way of dividing the strip of width 6 used for odd n can also be used for even n greater than 10, because a $6 \times k$ rectangle can be divided into trominoes for an odd k as well if $k \geq 5$. If $n \leq 10$, then the approach does not work, because $k = n - 9 \leq 1$, and thus the cases $n = 8$ and $n = 10$ must be handled individually.

Solution 2: For an even n greater than 6 not divisible by 3 the given area can also be divided into trominoes in the following way: If $n = 6k + 4$ for some positive integer k , then we cut out a square of size $6k \times 6k$ from the middle of the given area. This square can be cut into 2×3 rectangles, which in turn can be divided into trominoes as shown on Fig. 21. From the corners of the remaining area we cut out four trominoes so that we are left with $6k \times 2$ and $2 \times 6k$ rectangles. These can also be divided into rectangles of size 3×2 and 2×3 . If $n = 6k + 2$ for some positive integer k , then we cut out a square of size $6(k - 1) \times 6(k - 1)$ from the middle of the given area. The square can be split into 2×3 rectangles. From the remaining area we cut out four pieces that can be obtained by removing one corner square of a 4×4 square. These can be divided into trominoes as shown on Fig. 22. The remaining $6(k - 1) \times 4$ and $4 \times 6(k - 1)$ rectangles can also be divided into 3×2 and 2×3 rectangles.

O17. (*Seniors.*) Call a number *beautiful* if it differs from a square of an integer by less than 10%. Prove that the equation $x^3 + y^3 = z^2$ has infinitely many solutions where x, y and z are beautiful positive integers.

Solution 1: Every positive integer either is a perfect square or falls into the interval between the squares of two consecutive integers. We notice that for example if $n > 100$, then $\frac{(n+1)^2}{n^2} = 1 + \frac{2n+1}{n^2} < 1 + \frac{3n}{n^2} = 1 + \frac{3}{n} < 1.1$ and $\frac{(n-1)^2}{n^2} = 1 - \frac{2n-1}{n^2} > 1 - \frac{2n}{n^2} = 1 - \frac{2}{n} > 0.9$. Thus distinct sufficiently large consecutive perfect squares differ by less than 10%. Hence the integers between them also differ by less than 10%. Thus starting from some integer all integers are beautiful. We note that for each positive integer a , $\{x = a^2, y = 2a^2, z = 3a^3\}$ is a solution for the given equation. Based on the previous part there are infinitely many beautiful integers a and for all of them integers $a^2, 2a^2$ and $3a^3$ are also beautiful.

Solution 2: The number 50 differs from 7^2 by $\frac{1}{49}$ which is less than 10%. Thus 50 is beautiful. The number 500 differs from 22^2 by $\frac{16}{484}$ which is less than 10% again. Thus 500 is beautiful, too. As $50^3 + 50^3 = 250000 = 500^2$, the equation of the problem has at least one solution where the values of all variables are beautiful integers.

If n is beautiful then, for each positive integer k , the number k^2n is also beautiful. Indeed, if n differs from m^2 by less than 10% then k^2n differs from $(km)^2$ by the same percentage. Now if (x, y, z) satisfies the equation $x^3 + y^3 = z^2$ then (a^4x, a^4y, a^6z) satisfies this equation, because $(a^4x)^3 + (a^4y)^3 = a^{12}(x^3 + y^3) = a^{12}z^2 = (a^6z)^2$. Since both coefficients a^4 and a^6 are squares of integers for every integer a , we can obtain infinitely many solutions of the given equation in beautiful integers.

O18. (*Seniors.*) The physical education teacher has lined up the pupils in such a way that for every three consecutive pupils, the height of the first pupil is between the heights of the second and third pupils. After the teacher orders every second pupil to take a step forward, two lines form. Prove that both lines are sorted by height.

Solution: Let p_k be the height of the k th pupil in the original line. Without loss of generality let $p_1 < p_2$. Let us show by induction on k that for odd k , $p_{k+2} < p_k < p_{k+1}$, and for even k , $p_{k+2} > p_k > p_{k+1}$. The statement holds for $k = 1$: for p_1 to be between p_2 and p_3 , we must have $p_3 < p_1$ as $p_1 < p_2$ by assumption. Assume that the statement holds for $k - 1$. Then if k is even, then $k - 1$ is odd and by induction hypothesis $p_{k+1} < p_{k-1} < p_k$, from which $p_k > p_{k+1}$. As p_k must be between p_{k+1} and p_{k+2} , the only option is $p_{k+2} > p_k$, i.e., the statement holds for k . But if k is odd, then $k - 1$ is even and by the induction hypothesis $p_{k+1} > p_{k-1} > p_k$, from which $p_k < p_{k+1}$. Because p_k must be between p_{k+1} and p_{k+2} , the only option is $p_{k+2} < p_k$, i.e., the statement again holds for k . The statement of the problem can now be concluded from the following: Since $p_k > p_{k+2}$ for

every odd k and $p_k < p_{k+2}$ for every even k , the pupils with odd numbers have been sorted decreasingly by height and pupils with even numbers have been sorted increasingly by height.

O19. (*Seniors.*) Positive integers a and b are such that $\frac{5a^4+a^2}{b^4+3b^2+4}$ is an integer. Prove that a is composite.

Solution: If b is even, then b^2 and b^4 are both divisible by 4, thus also $b^4 + 3b^2 + 4$ is divisible by 4. If b is odd, then b^2 is congruent to 1 modulo 4, thus b^4 is congruent to 1 modulo 4 and $3b^2$ congruent to 3 modulo 4. Hence $b^4 + 3b^2 + 4$ is divisible by 4. Altogether we get that the denominator of the fraction is divisible by 4 for each integer b . For the fraction to be an integer, the numerator must also be divisible by 4.

If a is odd, then $a^2 \equiv 1 \pmod{4}$, implying $5a^4 \equiv 1 \pmod{4}$. Thus $5a^4 + a^2 \equiv 2 \pmod{4}$ and $5a^4 + a^2$ is not divisible by 4. Thus a must be even. The only even number that is not a composite number is 2. If $a = 2$, then $5a^4 + a^2 = 84$. In the cases $b = 1$ and $b = 2$ we have $b^4 + 3b^2 + 4 = 8$ and $b^4 + 3b^2 + 4 = 32$, respectively, but 84 is not divisible by either of the two. But if $b \geq 3$, then $b^4 + 3b^2 + 4 \geq 112 > 84$, implying that 84 is not divisible by $b^4 + 3b^2 + 4$. Thus a must be a composite number.

Remark: Numbers (a, b) satisfying the problem statement do exist. For example if $a = 8$, $b = 2$ then $\frac{5a^4+a^2}{b^4+3b^2+4} = \frac{20544}{32} = 641$.

O20. (*Seniors.*) A calculator found in the attic has as an operation key $*$, such that $x * y = (x + 1) * (-xy) = \frac{1}{y * x}$ for any integers x and y (one cannot use the operation key $*$ with fractions), and $x * y$ is always a real number. Find all possible operations that $*$ might denote.

Answer: $x * y = 1$ and $x * y = -1$.

Solution: Let us define $0 * 0 = c$. From the equation $x * y = \frac{1}{y * x}$ we get $c = \frac{1}{c}$, implying that $c = 1$ or $c = -1$. By substituting $y = 0$ in $x * y = (x + 1) * (-xy)$, we obtain the equality $x * 0 = (x + 1) * 0$, which holds for each integer x . By applying it a sufficient number of times in one direction or the other, we see that $x * 0 = 0 * 0 = c$ for any integer x . The equality $x * y = \frac{1}{y * x}$ now gives $0 * y = \frac{1}{c} = c$ for each integer y .

If now $x < 0$ and y is arbitrary, then by applying $x * y = (x + 1) * (-xy)$ a sufficient number of times we can find integers y_1, y_2, \dots, y_{-x} such that $x * y = (x + 1) * y_1 = (x + 2) * y_2 = \dots = 0 * y_{-x} = c$. From $x * y = \frac{1}{y * x}$ we can then also see for arbitrary x and $y < 0$ that $x * y = \frac{1}{c} = c$. Finally if $x > 0$ and $y > 0$, then $x * y = (x + 1) * (-xy) = c$, because $-xy < 0$. All in all we have obtained $x * y = c$ for any integers x and y . Both $x * y = 1$ and $x * y = -1$ indeed satisfy the problem statement.

O21. (*Seniors.*) Let k be a positive integer, $k > 1$. Find the smallest positive integer n such that some cells of an $n \times n$ grid of unit squares can be

coloured black in such a way that each row and each column contains exactly k black cells and no two black cells have a common side or vertex.

Answer: $4k$.

Solution: In every 2×2 square a maximum of one cell can be black. Let us consider any two consecutive rows in the grid. They must contain exactly $2k$ black cells in total. By dividing the cells of the two rows into 2×2 squares starting from the left, we see that the number of 2×2 squares must be at least $2k - 1$ (the last black cell can be alone in the rightmost column). Thus $n \geq 2 \cdot (2k - 1) + 1 = 4k - 1$.

Let us assume that $n = 4k - 1$. Out of the black cells in any two consecutive rows, at most $2k - 1$ can be in the leftmost $4k - 2$ columns. Thus one black square must be located in the last column. But then the second to last column cannot have any black squares in it. As this holds for any two consecutive rows of an $n \times n$ grid, the second to last column of the entire grid cannot contain any black cells. This, however, contradicts the assumption that each column contains k black cells. Thus $n \geq 4k$.

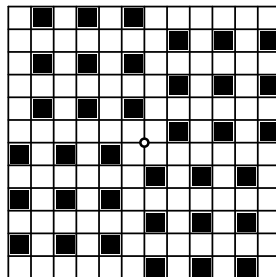


Fig. 23

One suitable construction for $n = 4k$ is the following: take four grids of size $2k \times 2k$, in which exactly those cells that are in odd rows and even columns have been coloured black, and place them around a midpoint by rotating them 90° with respect to each other (Fig. 23 displays this for $k = 3$).

Selected Problems from the Final Round of National Olympiad

F1. (Grade 9.) Find all pairs (x, y) of integers such that $x - y = xy$.

Answer: $(0, 0)$ and $(-2, 2)$.

Solution 1: The given equation is equivalent to $x(1 - y) = y$. Thus x divides y . The given equation is also equivalent to $y(x + 1) = x$. Thus also y divides x . Consequently $x = y$ or $x = -y$. If $x = y$ then the original equation implies $xy = 0$, whence $x = y = 0$. If $x = -y \neq 0$ then $y(x + 1) = x$ reduces to $x + 1 = -1$, whence $x = -2$ and, consequently, $y = 2$.

Solution 2: The given equation is equivalent to $y(x + 1) = x$. If $x + 1 = 0$, this equation would imply also $x = 0$ which is impossible. Thus $x + 1 \neq 0$, whence the equation can be written as $y = \frac{x}{x+1}$. From this, we see that $x \leq 0$, otherwise y could not be an integer. If $x = 0$ then $y = 0$, whereas $x = -1$ is impossible and $x = -2$ implies $y = 2$. If $x \leq -3$ then $\frac{x}{x+1}$ is again a fraction as $|x|$ is not divisible by a number that is only by 1 less than it. Consequently, the only pairs of integers satisfying the given equation are $(0, 0)$ and $(-2, 2)$.

Solution 3: The given equation is equivalent to $(x + 1)(y - 1) = -1$. Hence the factors $x + 1$ and $y - 1$ are 1 and -1 in some order. If $x + 1 = 1$ and $y - 1 = -1$ then $x = y = 0$, whereas $x + 1 = -1$ and $y - 1 = 1$ together imply $x = -2$ and $y = 2$.

F2. (*Grade 9.*) An athlete starts canoeing along river from point A towards point B. After reaching point B, he instantly turns around and rows back to point A along the same way. At the same moment when the athlete begins his trip, his coach starts from point B towards point A. After reaching point A, he instantly turns around and rows back to point B along the same way. The athlete and the coach meet twice during their trip, whereby the time passed from the start until the first meeting equals the time remaining between the two meetings. The water flows at a constant speed and either canoe moves at a constant speed w.r.t. the water during the whole trip. The speed of the coach w.r.t. the banks when moving downstream is exactly by one third larger than his speed when moving upstream, whereas the coach moves faster than the athlete w.r.t. the water. How many times faster than the athlete does the coach move w.r.t. the water? Find all possibilities.

Answer: $\frac{11}{5}$ and 5.

Solution: Denote the distance between A and B by s . Defining the direction from A to B as positive, let the flow speed w.r.t. the banks be v_0 . Let the absolute values of the speeds of the athlete and the coach w.r.t. the water be v_1 and v_2 , respectively. Let the time interval from the beginning of the trip until the first meeting be t . Within the time t , the canoes together cover the distance s with total speed $(v_1 + v_0) + (v_2 - v_0) = v_1 + v_2$. So $t = \frac{s}{v_1 + v_2}$.

By the first meeting, the slower w.r.t. the banks canoe has covered at most half of the distance between A and B. As the time passed from the start until the first meeting equals the time between the two meetings, the slower canoe covers at most one distance between A and B by the second meeting. Hence the second meeting takes place when the faster canoe on its journey back to its starting point catches the slower canoe.

By the assumptions, the coach's canoe is faster. Consequently the athlete has covered the distance $(v_1 + v_0) \cdot 2t$ by the second meeting. By the same time, the coach has covered the distance s with speed $v_2 - v_0$ and also the distance $(v_1 + v_0) \cdot 2t$ with speed $v_2 + v_0$. For this, the coach spends time $\frac{s}{v_2 - v_0} + \frac{(v_1 + v_0) \cdot 2t}{v_2 + v_0}$. Thus $2t = \frac{s}{v_2 - v_0} + \frac{(v_1 + v_0) \cdot 2t}{v_2 + v_0}$, whence $2t \cdot (v_2 - v_1) = s \cdot \frac{v_2 + v_0}{v_2 - v_0}$. As $t = \frac{s}{v_2 + v_1}$, we obtain $2 \cdot \frac{v_2 - v_1}{v_2 + v_1} = \frac{v_2 + v_0}{v_2 - v_0}$.

As the downward speed of the coach is by one third larger than his upward speed, either $\frac{v_2 + v_0}{v_2 - v_0} = \frac{4}{3}$ or $\frac{v_2 + v_0}{v_2 - v_0} = \frac{3}{4}$ depending on the sign of v_0 . In the former case, the obtained equation implies $\frac{v_2 - v_1}{v_2 + v_1} = \frac{2}{3}$ which reduces to $v_2 = 5v_1$, i.e., the coach is 5 times faster than the athlete w.r.t. the water. In the other case, the obtained equation implies $\frac{v_2 - v_1}{v_2 + v_1} = \frac{3}{8}$ which reduces to $5v_2 = 11v_1$, i.e., the coach is $\frac{11}{5}$ times faster than the athlete.

In order to check the validity of the solutions, it suffices to see that the speed of the athlete w.r.t. the water is larger than the speed of the flow. Indeed, in the first case $v_2 = 7v_0$ which implies $v_1 = \frac{7}{5}v_0 > v_0$, whereas in the second case $v_2 = -7v_0$ implying $v_1 = -\frac{35}{11}v_0 > |v_0|$.

F3. (Grade 9.) In quadrilateral $ABCD$ we have $AB = BC$. Point E on the line AB is such that $DB = BE$ and the line segments AD and DE are perpendicular. Prove that the perpendicular bisectors of the line segments AD , DC and CE meet in one point.

Solution 1: Let M be the midpoint of the line segment DE (Fig. 24). Then BM and DE are perpendicular because the median and the altitude drawn from the vertex angle B of isosceles triangle BDE coincide. By assumptions, AD and DE are perpendicular, too. Hence MB and AD are parallel. Consequently, B is the midpoint of line segment AE , i.e., $BA = BE$. As $BA = BC$ and $BD = BE$, the points A, D, C and E lie on a circle with centre B . The perpendicular bisectors of the chords AD , CD and CE of this circle all meet at the point B .

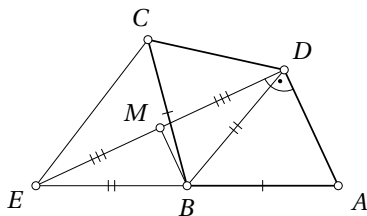


Fig. 24

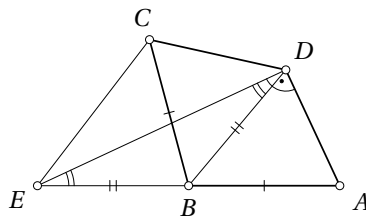
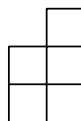


Fig. 25

Solution 2: As the triangle BDE is isosceles with vertex angle at B , we have $\angle BED = \angle BDE < 90^\circ$. As the triangle ADE is right with right angle at D , we also have $\angle AED < 90^\circ$. Consequently, points A and B lie on the same side of the point E , and because of $\angle BDE < 90^\circ = \angle ADE$, the point B must lie between points A and E (Fig. 25). Now we obtain $\angle BAD = \angle EAD = 180^\circ - \angle ADE - \angle AED = 180^\circ - 90^\circ - \angle BED = 90^\circ - \angle BDE = \angle ADE - \angle BDE = \angle BDA$. Thus the triangle BAD is isosceles with vertex angle at B , whence $BA = BD$. As $BA = BC$ and $BD = BE$, the points A, C, D and E lie on a circle with centre B . The perpendicular bisectors of the chords AD , CD and CE of this circle all meet at the point B .

F4. (Grade 9.) A rectangular grid of size $a \times h$ is bended to the lateral surface of a cylinder by bundling its sides of length h together. From the surface obtained, one starts along the edges of unit squares cutting off pieces of the shape shown in the figure (the figure is allowed to be rotated and reflected). Find the largest number of pieces that can be cut off in this way if: a) $a = 10$ and $h = 5$; b) $a = 11$ and $h = 5$.



Answer: a) 10; b) 11.

Solution: a) Colour the unit squares of the grid black and white as shown in Fig. 26. Any piece cut out from the cylinder entails precisely one black square. As there are 10 black squares in total, it is impossible to cut out more than 10 pieces. Figure 27 shows that it is possible to cut out 10 pieces.

b) Colour the unit squares of the grid black and white as shown in Fig. 28. Every piece cut out from the cylinder contains precisely two black squares (either two black squares from one row or one black square from either row). As there are $11 \cdot 2$ black squares in total, it is impossible to cut out more than 11 pieces. Figure 29 shows that it is possible to cut out 11 pieces.

Remark: Part a) can be solved also by using the colouring observed in the solution of part b) for $a = 10$.

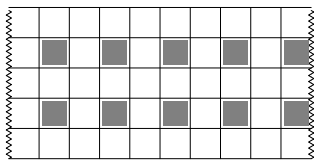


Fig. 26

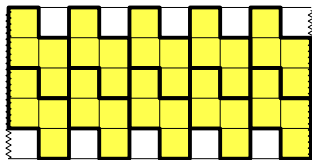


Fig. 27

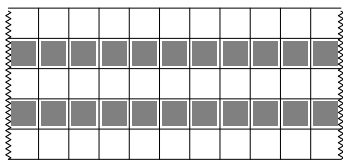


Fig. 28

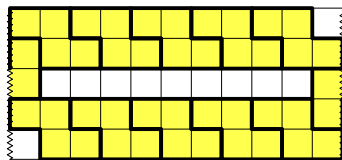


Fig. 29

F5. (Grade 10.) Find all pairs (x, y) of integers such that $x - y = \frac{x}{y}$.

Solution 1: The equation implies $y \neq 0$. As substituting $x = 0$ would imply $y = 0$, we must have also $x \neq 0$. In the case of non-zero values of variables, the given equation is equivalent to $x(y - 1) = y^2$. Thus x divides y^2 . As $y - 1$ and y^2 are relatively prime, also y^2 must divide x . Consequently $x = y^2$ or $x = -y^2$. If $x = -y^2$ then reducing the equation $x(y - 1) = y^2$ implies $y - 1 = -1$, i.e., $y = 0$ which is impossible. If $x = y^2$ then we similarly get $y - 1 = 1$. Hence $y = 2$ which implies $x = 4$.

Solution 2: Multiplying both sides by y and bringing all terms to the same side of the equation gives $y^2 - xy + x = 0$. The discriminant of the quadratic equation is $x^2 - 4x$ or, equivalently, $(x - 2)^2 - 4$. In order to make y an integer, the discriminant must be a perfect square. The only two squares of integers that differ from each other by 4 are 0 and 4. Thus $(x - 2)^2 = 4$, implying $x = 4$ or $x = 0$. In the former case $y = 2$. The second option implies $y = 0$ which is impossible. It is easy to check that $(4, 2)$ satisfies the original equation.

Solution 3: In the case of non-zero values of variables, the given equation is equivalent to $xy - x - y^2 = 0$. Rewriting this as $(x - (y + 1))(y - 1) = 1$, we see that the numbers $x - (y + 1)$ and $y - 1$ are either both 1 or both -1 . If $x - (y + 1) = 1$ and $y - 1 = 1$ then $y = 2$ and $x = 4$. In the other case $y = 0$ which is impossible by the original equation.

F6. (Grade 10.) Let a , b and c be some positive real numbers. Two banks play the following game. Initially, one bank has a euros and the other bank has b euros. On its turn, each bank steals from the other bank half of the latter's money. The players move by turns and the one who initially has less money starts. Any bank which is left with c euros or less bankrupts, which also ends the game. Prove that if the first player does not bankrupt at the very beginning then it never bankrupts.

Solution: Consider a sequence of two moves; let the beginner have a euros and the other have b euros, whereby $a > c$ and $b > c$. After the first move, the beginner has $\frac{2a+b}{2}$ euros while the other bank has $\frac{b}{2}$ euros. If $\frac{b}{2} \leq c$ then the latter bank bankrupts. Otherwise the game proceeds and, after the second move, the first bank has $\frac{2a+b}{4}$ euros while the other one has $\frac{2a+3b}{4}$ euros which is more. Note that $\frac{2a+b}{4} = \frac{a}{2} + \frac{b}{4} > \frac{c}{2} + \frac{c}{2} = c$, whence neither bank bankrupts at this move and the argument can be repeated.

Remark: One can prove by mathematical induction that, after $2k$ moves, the first player has $(\frac{1}{3} + \frac{2}{3 \cdot 4^k})a + (\frac{1}{3} - \frac{1}{3 \cdot 4^k})b$ euros, whereas the second player has $(\frac{2}{3} - \frac{2}{3 \cdot 4^k})a + (\frac{2}{3} + \frac{1}{3 \cdot 4^k})b$ euros. After $2k + 1$ moves, the first player has $(\frac{2}{3} + \frac{1}{3 \cdot 4^k})a + (\frac{2}{3} - \frac{1}{6 \cdot 4^k})b$ euros, whereas the second player has $(\frac{1}{3} - \frac{1}{3 \cdot 4^k})a + (\frac{1}{3} + \frac{1}{6 \cdot 4^k})b$ euros. Hence if the game continues infinitely then the numbers of euros the players have after an even number of moves converge to $(\frac{a+b}{3}, \frac{2(a+b)}{3})$ while the numbers of euros they have after an odd number of moves converge to $(\frac{2(a+b)}{3}, \frac{a+b}{3})$.

F7. (Grade 10.) The Youth Academy formed a committee consisting of representatives of different subjects. Each member of the committee belongs to exactly one out of 7 sections of discussion, whereby every section contains a different positive number of committee members. In votings, all members of each section are either all for or all against. In order to take a decision, there must be more committee members voting for it than those voting against it.

- Find the least possible number of committee members.
- In how many different ways can the committee take a decision in the case of this minimal number of members? (Ways of voting are different if the members of at least one section vote differently.)

Answer: a) 28; b) 60.

Solution: a) The least number of members is achieved when the sections of discussion are as small as possible, i.e., they consist of 1, 2, 3, 4, 5, 6, 7 committee members. Then the total number of committee members is 28.

b) Each section can vote either for or against independently of the other sections. As there are 7 sections, the total number of different votings is 2^7 which is 128. The voting gets stuck because of draw if exactly 14 committee members vote for; for that, at least 3 sections must vote for since 2 largest sections together have less members. The ways of making a draw with 3 sections voting for are $7 + 6 + 1$, $7 + 5 + 2$, $7 + 4 + 3$ and $6 + 5 + 3$, so there are 4 such ways of voting in total. The other ways of making a draw are obtained by switching the vote of each member, which gives another 4 ways of voting. Thus there are 120 non-draw ways of voting. By switching every vote, each of these ways of voting that results in taking the decision is mapped to another way of voting among these 120 in the case of which the decision is not taken, and vice versa. As this mapping is bijective, exactly half of all such ways of voting result in taking the decision. Hence it is possible to take a decision in 60 different ways.

F8. (Grade 10.) All side lengths of a right triangle are integers. Prove that the inradius of the triangle is an integer.

Solution: Let ABC be the given triangle with right angle at C . Let the projections of the incentre of ABC to legs BC and AC and hypotenuse AB be D , E and F , respectively (Fig. 30). We have $AE = AF$ and $BD = BF$, as well as $CD = CE = r$ where r is the inradius of ABC . Hence

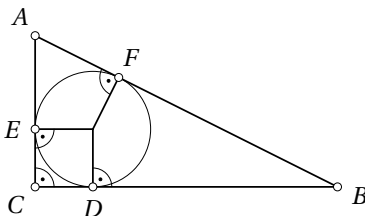


Fig. 30

$AB = AF + BF = AE + BD = AC - CE + BC - CD$. Denoting $|BC| = a$, $|AC| = b$ and $|AB| = c$, we obtain $c = a + b - 2r$, implying $r = \frac{a+b-c}{2}$.

By the Pythagorean theorem, $a^2 + b^2 = c^2$. Note that a and a^2 have equal parities, as well as b and b^2 have equal parities and c and c^2 have equal parities. Hence $a + b - c$ has the same parity as $a^2 + b^2 - c^2$, i.e., 0, meaning that $a + b - c$ is even. Consequently, r is an integer.

F9. (Grade 11.) Mari wrote a prime number that is larger than 10^{17} but smaller than $10^{17} + 10$. Find the number written by Mari.

Answer: $10^{17} + 3$.

Solution 1: Natural numbers ending with either an even digit or 5 are composite. We show that $10^{17} + 1$, $10^{17} + 7$ and $10^{17} + 9$ are composite, too. This implies that the only number that Mari could have written is $10^{17} + 3$. As $98 = 7 \cdot 14$, we have $100 \equiv 2 \pmod{7}$. Thus $10^{17} = 10 \cdot 100^8 \equiv 3 \cdot 2^8 = 3 \cdot 8 \cdot 8 \cdot 4 \equiv 3 \cdot 1 \cdot 1 \cdot 4 = 3 \cdot 4 \equiv 5 \pmod{7}$. Consequently, $10^{17} + 9 \equiv 5 + 9 = 14 \equiv 0 \pmod{7}$, implying $7 \mid 10^{17} + 9$. As $99 = 11 \cdot 9$,

we have $100 \equiv 1 \pmod{11}$. Thus $10^{17} = 10 \cdot 100^8 \equiv 10 \cdot 1^8 = 10 \pmod{11}$. Consequently, $10^{17} + 1 \equiv 10 + 1 = 11 \equiv 0 \pmod{11}$, i.e., $11 \mid 10^{17} + 1$. As $102 = 17 \cdot 6$, we have $100 \equiv -2 \pmod{17}$. Thus $10^{17} = 10 \cdot 100^8 \equiv 10 \cdot (-2)^8 = 10 \cdot 16 \cdot 16 \equiv 10 \cdot (-1) \cdot (-1) \equiv 10 \pmod{17}$. Consequently, $10^{17} + 7 \equiv 10 + 7 = 17 \equiv 0 \pmod{17}$, i.e., $17 \mid 10^{17} + 7$.

Solution 2: Natural numbers ending with either an even digit or 5 are composite. We show that $10^{17} + 1$, $10^{17} + 7$ and $10^{17} + 9$ are composite, too. This implies that the only number that Mari could have written is $10^{17} + 3$. As $7 \cdot 11 \cdot 13 = 10^3 + 1 \mid 10^{15} + 1$, the number $10^{15} + 1$ is divisible by both 7 and 11. Hence $10^{17} + 100$ is divisible by both 7 and 11. This implies that $7 \mid 10^{17} + 100 - 7 \cdot 13 = 10^{17} + 9$ and $11 \mid 10^{17} + 100 - 11 \cdot 9 = 10^{17} + 1$. By Fermat's theorem, $17 \mid 10^{17} - 10$; this implies $17 \mid 10^{17} - 10 + 17 = 10^{17} + 7$.

Remark: Indeed $10^{17} + 3$ is prime as one can see with help of computer.

F10. (Grade 11.) Prove that for any positive integers m and n , $\frac{(m+n)!}{m!n!} > mn$.

Solution: W.l.o.g., assume that $m \geq n$. Note that

$$\frac{(m+n)!}{m!n!} = \frac{1 \cdot 2 \cdot \dots \cdot m \cdot (m+1) \cdot \dots \cdot (m+n)}{(1 \cdot 2 \cdot \dots \cdot m) \cdot (1 \cdot 2 \cdot \dots \cdot n)} = \frac{m+1}{1} \cdot \frac{m+2}{2} \cdot \dots \cdot \frac{m+n}{n}.$$

Consider three cases:

1) If $n = 1$ then the expression obtained contains only one factor $\frac{m+1}{1}$. Its value $m+1$ is larger than the r.h.s. $m \cdot 1$ of the required inequality.

2) If $n = 2$ then the expression obtained contains two factors $\frac{m+1}{1}$ and $\frac{m+2}{2}$ whereas the r.h.s. of the required inequality is $2m$. As $m+2 > m+1$, it suffices to prove that $\frac{m+1}{1} \cdot \frac{m+2}{2} \geq 2m$, i.e., $(m+1)^2 \geq 4m$. The latter inequality is equivalent to $(m-1)^2 \geq 0$ which holds.

3) If $n \geq 3$ then the expression obtained contains distinct factors $\frac{m+1}{1}$, $\frac{m+2}{2}$ and $\frac{m+n}{n}$, whereby $\frac{m+1}{1} > m$, $\frac{m+2}{2} > \frac{m}{2} \geq \frac{n}{2}$ and $\frac{m+n}{n} \geq \frac{n+n}{n} = 2$. Hence the product of these three factors is larger than $m \cdot \frac{n}{2} \cdot 2$ which is mn ; the other factors are all larger than 1 and do not matter.

Remark: A more standard solution by induction on $\min(m, n)$ is possible.

F11. (Grade 11.) Ats and Pets play the following game: An integer n where $n \geq 3$ is written on the blackboard initially. On any move, Ats can either increase the number on the blackboard by 3 or decrease the number on the blackboard by 1. On any move, Pets can either increase or decrease the number on the blackboard by 2. The players take turns and the one who writes on the blackboard any number k such that $|k - n| \geq n$ wins. If a number that has been on the blackboard during the *current* game reappears as the result of a player's move, the game ends in a draw. Ats and Pets don't like to end in a draw, wherefore in the case of a draw they immediately start a new game with the number initially on the blackboard being by 1 less than at the beginning of the previous game. Pets decides before the first game who moves first in the first game, whereas in every subsequent game, the

player who must have moved in the game just ended moves first. So the boys continue until either Ats or Pets finally wins. Can any of the players finally win whatever the opponent's moves are, and if yes then who?

Answer: Yes, Ats.

Solution: Since there are finitely many numbers for which the current game continues, a repetition that ends this game must occur sooner or later if neither player wins this game. So the game must end with some result.

If $n = 3$ then Ats can win on his first move, irrespective of which player moves first. If $n \geq 4$ then let Ats decrease by 1 whenever the current number k on the blackboard satisfies $n - 1 \leq k \leq 2n - 4$ and increase by 3 otherwise. For instance, if $n = 4$ and Ats starts the game then he writes 3; if Pets now writes 5 then Ats writes 8 and wins, but if Pets writes 1 then Ats writes 4 and draws the game. If Pets starts in the case $n = 4$ by writing 6 then Ats writes 9 and wins, hence Pets must write 2 on his initial move.

We show that, when playing according to the described strategy, Ats does not lose the game for any $n \geq 4$. To this end, it suffices to show that the following claims hold: 1) Pets cannot win on his first move; 2) If Ats does not win or draw the game on his move then also Pets cannot win on his subsequent move. The first claim is obvious, so we continue with the second one. Suppose that Ats does not win or draw the game on his move. Then the number k on the blackboard before his move must satisfy $k \leq 2n - 4$ (otherwise he would win). If $k \leq n - 2$ then Ats writes $k + 3$; as $4 \leq k + 3 \leq n + 1$, Pets cannot win right after that. If $n \leq k \leq 2n - 4$ then Ats writes $k - 1$; as $n - 1 \leq k - 1 \leq 2n - 5$, Pets again cannot win immediately. Finally if $k = n - 1$ then Ats writes $n - 2$. In the case $n \geq 5$, Pets still cannot win immediately. However in the case $n = 4$, the number $n - 2$ equals 2 which must have occurred on the blackboard as observed in the previous paragraph. Consequently in the case of any $n \geq 4$, Ats can either win or transfer the game down to smaller and smaller numbers. In the case $n = 3$ at latest Ats wins.

F12. (Grade 11.) Let n be a natural number, $n \geq 3$. Arbitrary n points, no three of which are collinear, are marked in a plane. Does there definitely exist an n -gon with all vertices at marked points?

Remark: An n -gon is a closed broken line that does not intersect itself.

Answer: Yes.

Solution 1: Let A and B be marked points with the smallest and the largest x -coordinate, respectively; if there are two such points in either case then choose any of them. Let C_1, \dots, C_{m-1} be all marked points below the line AB in the order of increasing x -coordinate, and let C_{m+1}, \dots, C_{n-1} be all marked points above the line AB in the order of decreasing x -coordinate (the order of points with equal x -coordinates is not important in either case; Fig. 31 depicts an instance for $n = 11$ and $m = 6$). We show that $AC_1 \dots C_{m-1}BC_{m+1} \dots C_{n-1}A$ is a closed broken line that does

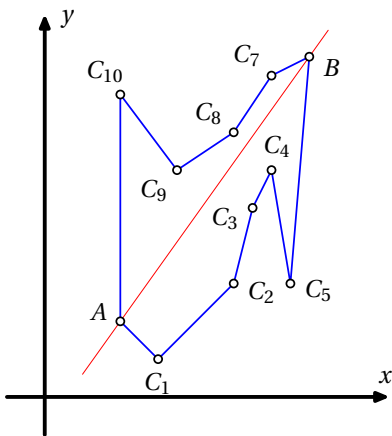


Fig. 31

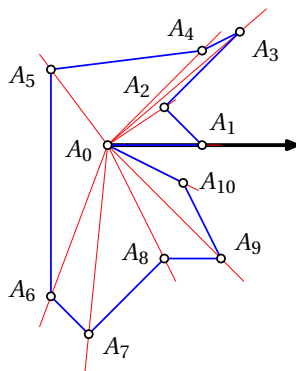


Fig. 32

not intersect itself. Denote $C_0 = C_n = A$ and $C_m = B$. Consider any two non-neighbouring segments C_iC_{i+1} and C_jC_{j+1} where $0 \leq i < j < n$. These segments do not intersect if $0 \leq i < j < m$ because the x -coordinates of C_i and C_{i+1} are less than the x -coordinates of C_j and C_{j+1} . Also C_iC_{i+1} and C_jC_{j+1} do not intersect if $m \leq i < j < n$. Finally, these segments do not intersect if $0 \leq i < m \leq j < n$ because of being separated by the line AB .

Solution 2: Choose an arbitrary marked point A_0 and a ray from A_0 to some direction. Moving the ray around the point A_0 , the ray meets all other marked points one by one; denote these points by A_1, \dots, A_{n-1} in this order. For every $i = 1, 2, \dots, n-1$, let α_i be the angle by which the ray turns when moving from the position of passing through A_i to the position of passing through A_{i+1} (or through A_1 in the case $i = n-1$). Obviously $\alpha_1 + \alpha_2 + \dots + \alpha_{n-1} = 360^\circ$, which implies that at most one of the angles α_i can be larger than 180° . Assume w.l.o.g. that $\alpha_1, \alpha_2, \dots, \alpha_{n-2}$ are less than 180° . We prove that $A_0A_1 \dots A_{n-1}A_0$ is a closed broken line that does not intersect itself. Introduce polar coordinates with ray A_0A_1 as polar axis (Fig. 32 depicts the points of the previous figure along with the positions of the rotating ray). By assumption, the polar angles of all interior points of any segment A_iA_{i+1} (where $1 \leq i \leq n-2$) are between the polar angles of its endpoints. Hence non-neighbouring segments A_iA_{i+1} and A_jA_{j+1} do not intersect if $1 \leq i < j \leq n-2$ because the polar angles of A_i and A_{i+1} are less than the polar angles of A_j and A_{j+1} . Analogously, the segments A_0A_1 and $A_{n-1}A_0$ do not intersect with any segment A_iA_{i+1} where $1 \leq i \leq n-2$.

F13. (Grade 12.) Positive integers a, b and c satisfy $10a^2 - 3ab + 7c^2 = 0$. Find the minimum value of $\gcd(a, b) \cdot \gcd(b, c) \cdot \gcd(c, a)$.

Answer: 3.

Solution: By the assumption $10a^2 + 7c^2$ is divisible by 3. If neither of a

and c is divisible by 3, then a^2 and c^2 would give remainder 1 modulo 3. The coefficients 10 and 7 give remainder 1 modulo 3, therefore $10a^2 + 7c^2$ gives remainder 2 modulo 3 and hence is not divisible by 3. Therefore, a or c is divisible by 3. If a is divisible by 3, then $10a^2$ is divisible by 3, therefore the term $7c^2$ has to be divisible by 3 as well. 7 is not divisible by the prime number 3, therefore c^2 has to be divisible by 3, implying that c has to be divisible by 3. Similar reasoning gives us that if c is divisible by 3, a has to be divisible by 3 as well. Therefore, in both cases we have got that both a and c are divisible by 3, from which $\gcd(c, a) \geq 3$. Therefore, $\gcd(a, b) \cdot \gcd(b, c) \cdot \gcd(c, a) \geq 3$. On the other hand, $a = c = 3$ and $b = 17$ satisfies the original equality and $\gcd(a, b) \cdot \gcd(b, c) \cdot \gcd(c, a) = 1 \cdot 1 \cdot 3 = 3$. Therefore, the smallest possible value for the expression $\gcd(a, b) \cdot \gcd(b, c) \cdot \gcd(c, a)$ is 3.

F14. (Grade 12.) Which number is bigger, $\sin 1 - \cos 1$ or $\frac{1}{4}$? (The arguments of trigonometric functions are given in radians.)

Answer: $\sin 1 - \cos 1$.

Solution 1: Since $\frac{\pi}{4} < 1 < \frac{\pi}{2}$, and in the first quadrant, the sine is increasing and the cosine decreasing, we have $\sin 1 - \cos 1 > \sin \frac{\pi}{4} - \cos \frac{\pi}{4} = 0$. Hence, the numbers $\sin 1 - \cos 1$ and $\frac{1}{4}$ are ordered in the same way as their squares $(\sin 1 - \cos 1)^2$ and $\frac{1}{16}$. Since $(\sin 1 - \cos 1)^2 = \sin^2 1 + \cos^2 1 - 2 \sin 1 \cos 1 = 1 - \sin 2$, it suffices to compare the numbers $1 - \sin 2$ and $\frac{1}{16}$. We show that the first number is bigger. For this, we prove the equivalent inequality $\sin 2 < \frac{15}{16}$.

Since $\pi < 3.2 = \frac{16}{5}$, we have $2 > \frac{5}{8}\pi$. Since the sine is decreasing in the second quadrant, this implies $\sin 2 < \sin \frac{5}{8}\pi$. From the half-angle sine formula we obtain $\sin \frac{5}{8}\pi = \sqrt{\frac{1 - \cos \frac{5}{4}\pi}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 + \sqrt{2}}}{2}$. It remains to compare the numbers $\frac{\sqrt{2 + \sqrt{2}}}{2}$ and $\frac{15}{16}$. From $\sqrt{2} < \frac{3}{2} < \frac{97}{64}$ we get $2 + \sqrt{2} < 2 + \frac{97}{64} = \frac{225}{64}$, yielding $\sqrt{2 + \sqrt{2}} < \frac{15}{8}$. Hence $\frac{\sqrt{2 + \sqrt{2}}}{2} < \frac{15}{16}$.

Solution 2: Using $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, observe that $(\sin 1 - \cos 1) \cdot \frac{1}{\sqrt{2}} = \sin 1 \cos \frac{\pi}{4} - \cos 1 \sin \frac{\pi}{4} = \sin(1 - \frac{\pi}{4})$. Hence, the numbers $\sin 1 - \cos 1$ and $\frac{1}{4}$ are ordered in the same way as the numbers $\sin(1 - \frac{\pi}{4})$ and $\frac{1}{4\sqrt{2}}$. We show that the first number is bigger. The sine is convex in the first quadrant, thus $\sin \frac{\pi}{16} > \frac{1}{4} \sin \frac{\pi}{4} = \frac{1}{4\sqrt{2}}$, hence it suffices to prove the inequality $\sin(1 - \frac{\pi}{4}) > \sin \frac{\pi}{16}$. Since both angles are in the first quadrant, it is enough to show $1 - \frac{\pi}{4} > \frac{\pi}{16}$. The last inequality is equivalent to $\pi < \frac{16}{5} = 3.2$, which is true.

Remark 1: There exist also solutions using power series of the sine and cosine functions.

Remark 2: The exact value of $\sin 1 - \cos 1$ is 0.30116867893975...

F15. (Grade 12.) There is a finite number of towns in a country. Some towns are connected with bidirectional flight routes. It has been decided to replace each bidirectional route with a unidirectional route between the same towns in some direction. Is it possible to do it in such a way that, for each town, the numbers of the routes originating from this town and those terminating in this town differ by at most one?

Answer: Yes.

Solution: Let us prove by induction on the number of flight routes that it is always possible. If there are 0 flight routes, there will be 0 unidirectional routes, which means that the number of routes originating and the number of routes terminating in each town are equal. Now let us assume a positive number of routes and also that for all configurations with less flight routes it is possible to replace the bidirectional routes with unidirectional routes as required. Let us inspect two cases:

1) *There exists a town X , which is the endpoint of exactly one bidirectional route.* Let the other end of that route be Y . Removing the route XY , we can replace all other bidirectional routes with unidirectional as required by the induction hypothesis. If the number of routes terminating at Y is less than the number of routes originating from there, let us add a route from X to Y ; or if the number of routes terminating at Y is greater than the number of routes originating from there, let us add a route from Y to X ; or if the number of routes terminating and originating at Y are equal, let us add a route between X and Y in an arbitrary direction. After adding this unidirectional route, the difference between the numbers of routes originating and terminating at Y is at most 1 and at X the difference is exactly 1.

2) *There does not exist a town which is an endpoint of exactly one bidirectional route.* Let us pick town X_0 which is an endpoint for some bidirectional route, denote the other endpoint of that route with X_1 . As X_1 is an endpoint for more than one bidirectional route, we can choose another route for which X_1 is an endpoint, let the other endpoint of that route be X_2 . As the number of routes is finite we will at some point reach town X_i from town X_{n-1} ; without loss of generality $i = 0$. After removing the n routes which form the cycle through n towns, we can replace the bidirectional routes with unidirectional ones as required by the induction hypothesis. By adding the n unidirectional routes $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_{n-1} \rightarrow X_0$, each town in this cycle gains one originating route and one terminating route which preserves the difference of their numbers.

F16. (Grade 12.) Find all positive integers n for which there exists a convex n -gon whose sizes of internal angles in degrees are consecutive positive integers.

Answer: 3, 5, 9, 15, 16.

Solution: Let the largest angle of the n -gon be x degrees. By assumption, $(n - 2) \cdot 180 = nx - \frac{1}{2}n(n - 1)$, i.e., we have the equation $n \cdot (180 - x) +$

$\frac{1}{2}n(n-1) = 360$. The n -gon is convex, therefore $x < 180$. Hence from this equation we obtain $\frac{1}{2}n(n-1) < 360$. Solving this inequality gives $n < \frac{1+\sqrt{2881}}{2}$; taking into account the context the problem, we have $3 \leq n \leq 27$. Henceforth, let us inspect two cases:

1) If n is odd, the second term $\frac{1}{2}n(n-1)$ of the l.h.s. of equation is divisible by n as $\frac{n-1}{2}$ is an integer. Therefore the entire l.h.s. is divisible by n , implying that 360 has to be divisible by n as well. As $360 = 2^3 \cdot 3^2 \cdot 5$, the odd divisors of 360 are 1, 3, 5, 9, 15 and 45. From these, inequality $3 \leq n \leq 27$ is satisfied by 3, 5, 9 and 15.

2) If n is even, the second term $\frac{1}{2}n(n-1)$ of the l.h.s. of equation is not divisible by n , however it is divisible by $\frac{n}{2}$. The first term is divisible by n , therefore 360 has to be divisible by $\frac{n}{2}$ but not by n . In other words, 360 is not divisible by n but 720 is. This is satisfied by all divisors of 720 which are divisible by the highest power of 2 which divides 720. As $720 = 2 \cdot 360 = 2^4 \cdot 3^2 \cdot 5$, the possible divisors are $16 \cdot 1$, $16 \cdot 3$, $16 \cdot 5$, $16 \cdot 9$, $16 \cdot 15$ and $16 \cdot 45$. The inequalities $3 \leq n \leq 27$ are satisfied only for $n = 16 \cdot 1 = 16$.

All found values satisfy the original condition. Indeed, from the equation above we get $x = 180 - \frac{360}{n} + \frac{n-1}{2}$ from which we find the smallest angle $x - (n-1)$ in degrees being $180 - \frac{360}{n} - \frac{n-1}{2}$. This is clearly positive.

Remark: The internal angles in the n -gons which satisfy the problem statement are as follows: $59^\circ \dots 61^\circ$ for $n = 3$, $106^\circ \dots 110^\circ$ for $n = 5$, $136^\circ \dots 144^\circ$ for $n = 9$, $149^\circ \dots 163^\circ$ for $n = 15$, $150^\circ \dots 165^\circ$ for $n = 16$.

Selected Problems from the IMO Team Selection Contest

S1. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which for all $x, y \in \mathbb{R}$ satisfy

$$f(x^2)f(y^2) + |x|f(-xy^2) = 3|y|f(x^2y).$$

Answer: $f(x) = 0$ and $f(x) = 2|x|$.

Solution: Substituting $-x$ for x in the original equation, we get

$$f(x^2)f(y^2) + |x|f(xy^2) = 3|y|f(x^2y). \quad (1)$$

Combining the original equation and Eq. (1) gives $|x|f(-xy^2) = |x|f(xy^2)$, which has to hold for any x and y . Substituting here 1 for x , we obtain $f(-y^2) = f(y^2)$. Therefore, for any non-negative real number x , we have $f(-x) = f(x)$; by swapping the sides of the equation, we get the same condition for negative real numbers x , i.e. f is even. Substituting in Eq. (1) 0 for both x and y , we get $(f(0))^2 = 0$, from which $f(0) = 0$. After swapping x and y in Eq. (1), we get

$$f(x^2)f(y^2) + |y|f(x^2y) = 3|x|f(xy^2). \quad (2)$$

Subtracting Eq. (2) from Eq. (1), we get

$$|x|f(xy^2) - |y|f(x^2y) = 3|y|f(x^2y) - 3|x|f(xy^2).$$

After simplifying and dividing by 4, we get

$$|x|f(xy^2) = |y|f(x^2y), \quad (3)$$

which has to hold for all x and y . Substituting 1 for y in Eq. (3) and flipping the sides of the equation gives us

$$f(x^2) = |x|f(x), \quad (4)$$

which has to hold for all x . On the other hand, substitution of Eq. (3) to Eq. (1) or to (2) gives us

$$f(x^2)f(y^2) = 2|x|f(xy^2). \quad (5)$$

Substituting $f(x^2)$ from Eq. (4) to the l.h.s. of Eq. (5) gives us the equation $|x|f(x)f(y^2) = 2|x|f(xy^2)$. For all $x > 0$, let us substitute \sqrt{x} in place of y . After simplifying we get

$$f(x^2) = \frac{1}{2} (f(x))^2. \quad (6)$$

As f is an even function and $f(0) = 0$, Eq. (6) holds for all $x \leq 0$ as well. Subtracting Eq. (4) from Eq. (6) gives the equation $\frac{1}{2} (f(x))^2 - |x|f(x) = 0$. Solving it w.r.t. $f(x)$ gives $f(x) = 0$ or $f(x) = 2|x|$. For each x , one of the two cases has to hold.

Let us assume that for some non-zero real number c , we have $f(c) = 0$. Substituting in Eq. (4) c for x , we get $f(c^2) = 0$. Substituting c for x in Eq. (5) and simplifying, we get $f(cy^2) = 0$ for all y . Therefore $f(x) = 0$ for all non-negative x and as f is an even function, also for negative x . Hence the only functions that satisfy the original condition are $f(x) = 0$ and $f(x) = 2|x|$. Substituting these into the original equation confirms their validity.

S2. Let us call a real number r *interesting*, if $r = a + b\sqrt{2}$ for some integers a and b . Let $A(x)$ and $B(x)$ be polynomial functions with interesting coefficients for which the constant term of $B(x)$ is 1, and $Q(x)$ be a polynomial function with real coefficients such that $A(x) = B(x) \cdot Q(x)$. Prove that the coefficients of $Q(x)$ are interesting.

Solution: Let

$$B(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0,$$

$$Q(x) = q_m x^m + q_{m-1} x^{m-1} + \dots + q_1 x + q_0,$$

$$A(x) = a_{n+m} x^{n+m} + a_{n+m-1} x^{n+m-1} + \dots + a_1 x + a_0.$$

By assumption, the coefficients a_i and b_j are all interesting and $b_0 = 1$. Let us prove by induction on k that all coefficients q_k are interesting. As $A(x) = B(x) \cdot Q(x)$, we get $a_0 = b_0 q_0 = q_0$, implying that q_0 is interesting. For the induction step, let us now assume that q_0, \dots, q_{k-1} are interesting. As $A(x) = B(x) \cdot Q(x)$, we get $a_k = b_k q_0 + b_{k-1} q_1 + \dots + b_1 q_{k-1} + q_k$, i.e.,

$$q_k = a_k - b_k q_0 - b_{k-1} q_1 - \dots - b_1 q_{k-1}.$$

The equations $(x + y\sqrt{2})(u + v\sqrt{2}) = (xu + 2yv) + (xv + yu)\sqrt{2}$ and $(x + y\sqrt{2}) - (u + v\sqrt{2}) = (x - u) + (y - v)\sqrt{2}$ show that the product and difference of two interesting numbers are both interesting as well. Therefore, the equation for q_k implies that q_k is interesting which completes the proof.

S3. Boeotia is comprised of 3 islands which are home to 2019 towns in total. Each flight route connects three towns, each on a different island, providing connections between any two of them in both directions. Any two towns in the country are connected by at most one flight route. Find the maximal number of flight routes in the country.

Answer: 673^2 .

Solution 1: Let each island have 673 towns. Let us denote the towns of the first, second and third island with A_i , B_j and C_k ($i, j, k = 1, 2, \dots, 673$). For each triplet (i, j, k) for which $i + j + k$ is divisible by 673, let us have a route which connects the towns A_i , B_j and C_k . For each pair of towns (A_i, B_j) on the first and second island there exists exactly one town C_k on the third island for which $i + j + k$ is divisible by 673 since the numbers $i + j + 1, i + j + 2, \dots, i + j + 673$ are 673 consecutive integers. Similarly, there exists exactly one town A_i on the first island for each pair of towns (B_j, C_k) on the second and third island for which $i + j + k$ is divisible by 673, and there exists exactly one town B_j on the second island for each pair of the towns (C_k, A_i) on the third and first island for which $i + j + k$ is divisible by 673. This construction gives a route network with the total of 673^2 routes.

Let us now show that this is the maximal number of routes. Let a , b and c be the numbers of towns on each island respectively and let us assume without loss of generality that $a \leq b \leq c$. As earlier, let us denote the towns with A_i , B_j and C_k but now $i = 1, 2, \dots, a$, $j = 1, 2, \dots, b$ and $k = 1, 2, \dots, c$. For each pair of towns (A_i, B_j) on the first and second island, there exists at most one route which connects these towns and every route in the country connects one of such pairs. Therefore the maximal number of routes is ab . As $a \leq b \leq c$ and $a + b + c = 2019$, we get $c \geq 673$ and $a + b \leq 1346$. From the AM-GM inequality, $\sqrt{ab} \leq \frac{a+b}{2}$. In summary, $ab \leq (\frac{a+b}{2})^2 \leq (\frac{1346}{2})^2 = 673^2$. Therefore the maximal number of lines in the country is 673^2 .

Solution 2: Let a , b and c be the numbers of towns on each island respectively. As in Solution 1, let us denote the towns on each island with A_i , B_j and C_k in which $1 \leq i \leq a$, $1 \leq j \leq b$ and $1 \leq k \leq c$.

Let us first show that there exists a valid construction with 673^2 routes. Let us choose $b = c = n$ with $n = 673$, and let us show by induction that for all $a = 0, 1, \dots, n$ it is possible to construct an routes s.t. the routes originating from different towns A_i are connecting different pairs of towns (B_j, C_k) and the routes originating from the same town A_i do not have common destinations on the other two islands. This is equivalent to the condition

that there are a bijections between the towns on the second and the third island. For $a = 0$ the statement holds trivially. Let us now assume that the statement is true for some $a < n$, and let us prove that the statement is true for $a + 1$ as well. Let us have a bijections between the towns of the second and the third island. Every town on the second island can be connected to another $n - a$ islands on the third island which are not yet connected to the town on the second island.

We need to show that for all the towns on the second island we can assign a partnering town among the “available” towns on the third island in such way that each town on the second island is partnered with a different town on the third island. Hall’s marriage theorem claims that such pairing exists iff for any set of towns on the second island the number of towns “available” for some town in the given set is at least the number of towns in the given set. To show the validity of the condition, let us choose any subset X of x towns on the second island. Let Y be the set of towns on the third island which are “available” for some town in X . Let the set Y contain y towns. Let us find an upper bound for the unconnected pairs of towns (B_j, C_k) for which $B_j \in X$ and $C_k \in Y$. For every town in set X , all the $n - a$ towns on the third island, which are not connected to the town, are in Y ; therefore the number of the pairs under consideration is $x(n - a)$. As for every town in Y , there are $n - a$ towns on the second island which it is not connected to. They may or may not be in X , the number of the counted pairs is $y(n - a)$ at most. Thus $x(n - a) \leq y(n - a)$ and as $n - a > 0$, we have $x \leq y$, which completes the proof for existence of the construction.

Let us now show that 673^2 is the maximal number of routes. The islands on the first and second island are connected with ab routes at most and it includes all the routes on the island. Therefore the total number of routes is not more than ab , and similarly also not more than bc or ca . Therefore, the number of routes is at most $\min(ab, bc, ca)$ which does not exceed the number $\frac{1}{3}(ab + bc + ca)$. However, $ab + bc + ca = \frac{1}{3}(ab + bc + ca) + \frac{2}{3}(ab + bc + ca) \leq \frac{1}{3}(a^2 + b^2 + c^2) + \frac{2}{3}(ab + bc + ca) = \frac{1}{3}(a + b + c)^2$, which sets the maximal number of routes to $\frac{1}{9}(a + b + c)^2 = 673^2$. (The inequality $ab + bc + ca \leq a^2 + b^2 + c^2$ can be proven, for example, using AM-GM thrice: on a^2 and b^2 , on b^2 and c^2 , and on c^2 and a^2 .)

Remark 1: The application of Hall’s marriage theorem in Solution 2 in fact proves the following general theorem: *In a balanced bipartite graph which has all vertices of equal positive degree, the vertices can be split into pairs s.t. each pair is joined with an edge.* In our case, the vertices are the towns on the second and third island, edges indicate the absence of routes between the towns on different islands.

S4. It is allowed to perform the following transformations in the plane with any integers α : (1) Transform every point (x, y) to the corresponding point $(x + \alpha y, y)$; (2) Transform every point (x, y) to the corresponding

point $(x, y + \alpha x)$. Does there exist a non-square rhombus whose all vertices have integer coordinates and which can be transformed to: a) Vertices of a square; b) Vertices of a rectangle with unequal side lengths?

Answer: a) No; b) Yes.

Solution: a) We use the term *transformation*, meaning one of the transformations described in the problem, using some integer α . We also use the term *translation*, meaning translation by a vector with integer coordinates, and the term *expansion*, meaning multiplying the coordinates of all vertices of the plane by the same integer. Note that a translation followed by a transformation is equal to a transformation followed by a translation. Indeed in the former case, every point (x, y) is mapped either to the point $(x + c + \alpha(y + d), y + d)$ or to the point $(x + c, y + d + \alpha(x + c))$, which are $(x + \alpha y + (c + \alpha d), y + d)$ and $(x + c, y + \alpha x + (\alpha c + d))$, respectively. Moreover, an expansion followed by a transformation is equal to a transformation followed by an expansion since $(xg + \alpha yg, yg) = ((x + \alpha y)g, yg)$ and $(xg, yg + \alpha xg) = (xg, (y + \alpha x)g)$.

We prove that every rhombus that can be transformed into a square is in fact already a square. To this end, consider a rhombus that can be transformed into a square. As translations preserve the properties of being a square and being a non-square rhombus, the observations made in the previous paragraph enable to assume w.l.o.g. that one of the vertices of the rhombus lies at the origin of the coordinate plane. Denote the vertices of the rhombus counter-clockwise by $(0, 0)$, (a, b) , $(a + c, b + d)$, (c, d) . Let the corresponding vertices of the square obtained be $(0, 0)$, (e, f) , $(e - f, e + f)$, $(-f, e)$. (It is easy to show that the endpoints of parallel line segments transform to the endpoints of parallel line segments. Hence the vertices of the square are listed in the order they occur on the boundary.) As expansions preserve the properties of being a square and being a non-square rhombus, the observations above enable to assume w.l.o.g. that $\gcd(a, b, c, d) = 1$.

Since every transformation preserves the greatest common divisor of the coordinates of a point, we find $\gcd(e, f) = \gcd(\gcd(e, f), \gcd(-f, e)) = \gcd(\gcd(a, b), \gcd(c, d)) = \gcd(a, b, c, d) = 1$. So $\gcd(a, b) = \gcd(e, f) = 1$, $\gcd(c, d) = \gcd(-f, e) = 1$ and $\gcd(c + a, d + b) = \gcd(e - f, e + f) = \gcd(e - f, 2f) \leq \gcd(2e - 2f, 2f) = \gcd(2e, 2f) = 2$. Since the transformation of a translation is some translation of the transformation, we see, using the translation by $(-a, -b)$, that the rhombus with vertices $(0, 0)$, (c, d) , $(c - a, d - b)$, $(-a, -b)$ is also transformable to a square. By applying the arguments above to this rhombus, we find $\gcd(c - a, d - b) \leq 2$. Since the side lengths of a rhombus are equal, we have $a^2 + b^2 = c^2 + d^2$. Therefore, $(a + c)(a - c) = (d + b)(d - b)$. If $\gcd(c + a, d + b) = 1$ then one of $c + a$ and $d + b$ is odd, whence one of $c - a$ and $d - b$ is odd and $\gcd(c - a, d - b) = 1$. Therefore, $|a + c| = |d - b|$, $|a - c| = |d + b|$.

If $\gcd(c + a, d + b) = 2$, then $\gcd(c - a, d - b) = 2$. Thus $\gcd(\frac{c+a}{2}, \frac{d+b}{2}) = 1$, $\gcd(\frac{c-a}{2}, \frac{d-b}{2}) = 1$, which together with $(a + c)(a - c) = (d + b)(d - b)$

imply the equations $|a + c| = |d - b|$, $|a - c| = |d + b|$ also in this case.

Squaring and subtracting the last equations, we find $ac + bd = 0$, meaning that the sides of the rhombus are perpendicular, i.e., it is a square.

b) For example, the rhombus with vertices $(0, 0)$, $(5, -12)$, $(5, 1)$, $(0, 13)$ is transformable to a rectangle with vertices $(0, 0)$, $(1, -2)$, $(27, 11)$, $(26, 13)$ by transformation (2) with $\alpha = 2$ followed by transformation (1) with $\alpha = 2$.

Remark: Using the reasoning of the solution of part a), it is easy to see that if the rhombus with vertex coordinates $(0, 0)$, (a, b) , $(a + c, b + d)$, (c, d) can be transformed into a rectangle then the ratio of the sides of the rectangle will be $\gcd(a, b) : \gcd(c, d)$. Hence any rhombus with $\gcd(a, b) = \gcd(c, d)$ can not be transformed to a non-square rectangle.

Rhombi with vertices having integral coordinates can be found from the diophantine equation $a^2 + b^2 = c^2 + d^2$. One simple special case of it is $d = 0$ which corresponds to Pythagorean triples. The triple $(3, 4, 5)$ does not give a rhombus that could be transformed into a rectangle. The next smallest (by the shortest side) triple is $(5, 12, 13)$, which gives the example provided in the solution of part b).

There are examples that do not come from a Pythagorean triple: the rhombus with vertices $(0, 0)$, $(13, 13)$, $(20, 30)$, $(7, 17)$ is transformable to a rectangle with vertices $(0, 0)$, $(39, -13)$, $(40, -10)$, $(1, 3)$ by applying transformation (2) with $\alpha = -2$ followed by transformation (1) with $\alpha = -2$.

Problems Listed by Topic

Number theory: O1, O3, O7, O9, O13, O19, F1, F5, F9, F13, F16

Algebra: O4, O10, O14, O17, O18, O20, F2, F6, F10, F14, S1, S2

Geometry: O5, O8, O12, O15, F3, F8, S4

Discrete mathematics: O2, O6, O11, O16, O21, F4, F7, F11, F12, F15, S3