MATH230 Homework 4
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Section: 5

1) a)

$$\underbrace{\int_{-1}^{0} (c+y)\,dy}_{cy+\frac{y^2}{2}\Big|_{-1}^{0}} + \underbrace{\int_{0}^{1} (c-y)\,dy}_{cy-\frac{y^2}{2}\Big|_{0}^{1}}$$

$$0-\left(-c+\frac{1}{2}\right)+\left(c-\frac{1}{2}\right) = c-\frac{1}{2}+c-\frac{1}{2}=1 \rightarrow 2c-1=1 \rightarrow c=1$$

b) $F_y(t) = \int_{0}^{t} f(y)\,dy$

For $0<t<-1$

$$\int_{-1}^{t} (1+y)\,dy = y+\frac{y^2}{2}\Big]_{-1}^{t} = t+\frac{t^2}{2}-\left(-1+\frac{1}{2}\right) = t+\frac{t^2}{2}+\frac{1}{2}$$

$$\int_{0}^{t} (1-y)\,dy = y-\frac{y^2}{2}\Big]_{0}^{t} = t-\frac{t^2}{2}$$

$$F_y(t) = \begin{cases} 1 & t \geq 1 \\ \frac{1}{2}+t-\frac{t^2}{2} & 1>t>0 \\ t+\frac{t^2}{2}+\frac{1}{2} & 0 \geq t > -1 \\ 0 & t \leq -1 \end{cases}$$

c)

$$\int_{0}^{1/2} (1-y)\,dy = y-\frac{y^2}{2}\Big]_{0}^{1/2} = \frac{1}{2}-\frac{1}{8}=\frac{3}{8}$$

2) $X \sim N(\mu, \sigma^2) \Rightarrow f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

a) $E(x) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx \Rightarrow E(x) = E(x-\mu+\mu) = E(x-\mu) + E(\mu) = E(x-\mu) + \mu$

$$E(x-\mu) = \int_{-\infty}^{\infty} (x-\mu) \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{\sigma t}{\sqrt{2\pi}} e^{\frac{-1}{2}t^2} dt = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{\frac{-1}{2}t^2} dt$$

$\hookrightarrow \frac{x-\mu}{\sigma} = t$

$\frac{dx}{\sigma} = dt$

$$= \frac{\sigma}{2\pi} \int_{-\infty}^{\infty} e^{-u} du = \frac{\sigma}{2\pi} \cdot \frac{e^{-u}}{-1} = \frac{\sigma}{2\pi} \cdot \frac{e^{\frac{-1}{2}t^2}}{-1} = \frac{-\sigma}{2\pi} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \Bigg|_{-\infty}^{\infty}$$

$= 0 - 0 = 0$

$\frac{1}{2}t^2 = u$

$\to t\,dt = du$

a) $E(x) = E(x-\mu+\mu) = 0 + \mu = \mu$

b) part is in the last page

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3) cdf method:

$$F_U(u) = P(U \leq u) = P(x^2 \leq u) = P(x \leq \sqrt{u}) = F_x(\sqrt{u}) = \int_0^{\sqrt{u}} \frac{t^3}{4} dt = \frac{t^4}{16} \Bigg|_0^{\sqrt{u}} = \frac{u^2}{16}$$

pdf of U: $\frac{d}{du} F_U(u) = \frac{d}{du}\left(\frac{u^2}{16}\right) = \frac{2u}{16} = \frac{u}{8}$

$0 < x < 4$ interval

$$\int_0^4 \frac{u}{8} du = \frac{u^2}{16} \Bigg|_{u=0}^{4} = \frac{4^2}{16} - 0 = 1 \to \text{so pdf}$$

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4) $f(x) = \lambda e^{-\lambda x} \quad x > 0$

$E(x) = \frac{1}{\lambda} \quad \lambda = 0.1$ since $E(x) = 0$

$f(x) = \frac{1}{10} e^{\frac{-x}{10}}$

$$\int_{10}^{\infty} \frac{1}{10} e^{-x/10} dx = \int_{-1}^{-\infty} -e^{u} du = \int_{-\infty}^{-1} e^{u} du = e^{u} \Big|_{u=-\infty}^{-1} = e^{-1} - 0 = \frac{1}{e}$$

$u = -x/10$
$du = -dx/10$

Binomial approximate;

$\text{Binomial}\left(\frac{1}{e}, 100\right)$

$\sigma = \sqrt{npq} = \sqrt{100\left(\frac{1}{e}\right)\left(\frac{e-1}{e}\right)}$

$\mu = np = 100/e$

$$P(X \geq 50) \sim P\left(\frac{X - \frac{100}{e}}{\sqrt{100\left(\frac{1}{e}\right)\left(\frac{e-1}{e}\right)}} \geq \frac{50 - \frac{100}{e}}{\sqrt{100\left(\frac{1}{e}\right)\left(\frac{e-1}{e}\right)}}\right) \sim 2.74$$

$P(X \geq 50) = P(Z > 2.74) = 1 - F(2.74) = F(-2.74) = 0.0031$

2b continue2)

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt = \frac{2\sigma^2}{\sqrt{\pi}} \left[\frac{-t}{2} e^{-t^2} \Big|_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-t^2} dt \right]$$

$$\lim_{t \to \infty} \frac{-t/2}{e^{t^2}} = \frac{\frac{1}{2}}{2te^{t^2}} = 0$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \sigma^2$$

5) $\beta = \frac{1}{\lambda}$, $f(x) = \frac{1}{\beta} e^{-x/\beta}$ density function

$$\text{Profit} = \begin{cases} 3 & x < 1 \\ -1 & x > 1 \end{cases}$$

$$E(x) = \int_0^1 x f(x)\, dx + \int_1^\infty x f(x)\, dx = \int_0^1 3 \cdot \frac{1}{\beta} e^{\frac{-x}{\beta}} dx + \int_1^\infty (-1) \cdot \frac{1}{\beta} e^{\frac{-x}{\beta}} dx$$

$$= 3 \int_0^1 e^{-u} du + (-1) \int_1^\infty e^{-u} du = 3 \cdot \left[\frac{e^{\frac{-x}{\beta}}}{-1} \Big|_0^1 \right] - 1 \cdot \left[\frac{e^{\frac{-x}{\beta}}}{-1} \Big|_1^\infty \right]$$

$\hookrightarrow \frac{x}{\beta} = u$, $\frac{dx}{\beta} = du$

$$= 3 \cdot \left[-e^{-1/\beta} + 1 \right] - \left[0 + e^{-1/\beta} \right] = -3e^{-1/\beta} + 3 - e^{-1/\beta} = 3 - 4e^{-1/\beta}$$

If $\boxed{\beta = 2}$ $\rightarrow$ profit is $3 - 4e^{-1/2} \cong 0.58$

maximize

If $\beta = 4 \rightarrow$ profit is $3 - 4e^{-1/4} \cong -0.12$

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6) $p(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ for relative max, derivative at location $= 0$.

$$\frac{d}{dx} p'(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} (\alpha-1) \cdot x^{\alpha-2} e^{-x/\beta} + \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \cdot \frac{-1}{\beta} e^{-x/\beta} = 0$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta} \cdot x^{\alpha-2} \Big(\underbrace{(\alpha-1) - \frac{x}{\beta}}_{=0} \Big) = 0$$

$\alpha - 1 = x/\beta$ $\quad x = (\alpha-1) \cdot \beta$

If $\alpha = 1$: $p(x) = \frac{1}{1 \cdot \beta} \cdot x^0 \cdot e^{-x/\beta} = \frac{1}{\beta} e^{-x/\beta} \Rightarrow x \sim \exp(\lambda = 1/\beta)$ x exponentially distributed when $\lambda = 1/\beta$

If $0 < \alpha < 1$

$\alpha - 1 < 0$ and $x^{\alpha-1} < 1$, when x increases it goes to 0, x minimum gets larger.

4) Uniform Distribution:

$$p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else} \end{cases} \xrightarrow{cdf} F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^1 dx & 0 < x < 1 \\ 1 & x > 1 \end{cases} \Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

$$F_y(t) = P(y \leq t) = P(\underbrace{-\ln x \leq t}_{\substack{\ln x > -t \\ x \geq e^{-t}}}) = P(x \geq e^{-t}) = 1 - P(x < e^{-t}) = 1 - F_x(e^{-t})$$

For density:

$$\frac{d}{dt} F_y(t) = \frac{d}{dt}(1 - F_x(e^{-t})) = 0 - (-1 \cdot e^{-t} \cdot F'_x(e^{-t})) = e^{-t} \cdot F'_x(e^{-t})$$

$$F'_x(t) = P_x(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases} \Rightarrow \frac{P_x(e^{-t})}{0 < e^{-t} < 1} = \begin{cases} 1 & 0 < e^{-t} < 1 \\ 0 & \text{else} \end{cases}$$

$$-t < \ln 1 = 0, \quad t > 0$$

$$P(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & y < 0 \end{cases}$$

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2) -- continue

b)
$$Var(X) = \sigma^2 = E(x - \mu)^2 = E(x^2) - [E(x)]^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx - \underbrace{[E(x)]^2}_{\mu^2}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)^2 e^{-t^2} dt - \mu^2$$

$\rightarrow x - \mu / \sqrt{2}\sigma = t, \quad dt = dx / \sqrt{2}\sigma$

$$= \frac{1}{\sqrt{\pi}} \left[2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + 2\sqrt{2}\sigma\mu \int_{-\infty}^{\infty} t e^{-t^2} dt + \mu^2 \int e^{-t^2} dt \right] - \mu^2$$

$t^2 = u, \ 2t\,dt = du$

$$= \frac{1}{\sqrt{\pi}} \left[2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + 2\sqrt{2}\sigma\mu \left[\frac{-1}{2} e^{-t^2} \Big|_{-\infty}^{\infty} \right] + \mu^2 (\pi) - \mu^2 \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + 0 + \mu^2 (\pi) - \mu^2 \right] = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + \frac{\mu^2\sqrt{\pi}}{\sqrt{\pi}} - \mu^2$$

--- continue after 4. question