NAME: ....

DEPT. ....

Date: December 4, 2015
Time: 17:45-19:35
Instructor: Dilek Güvenç

MATH 230 MIDTERM EXAM II

IMPORTANT

1. Check that there are 4 questions in your booklet.
2. Do **NOT** use your mobile phone as a calculator. Turn it off during the exam.
3. Show all your work. Correct results without sufficient explanation and correct notation might not get full credit.
4. Write your name on each page.

1	2	3	4	TOTAL
20	30	25	25	100

GOOD LUCK!

....

1. a) A fair die is rolled until 6 appears for the 6^{th} time. Find the probability that this will occur before the 9^{th} trial. **(10 Points)**

b) Now, a fair die is rolled 36 times. Approximate the probability that 6 appears more than 10 times. **(10 Points)**

a) X: # of rolls needed to get a 6, for the 6-th time.

$X \sim \text{Neg. Bin}(r=6,\ p=\frac{1}{6})$

$P(X<9) = P(X\le 8) = P(X=6)+P(X=7)+P(X=8)$

$= \binom{6-1}{6-1}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^0 + \binom{7-1}{6-1}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right) + \binom{8-1}{6-1}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^2$

$= \left(\frac{1}{6}\right)^6\left(1+6\cdot\frac{5}{6}+21\left(\frac{5}{6}\right)^2\right) \sim 0.0016$

b) X: # of 6's in 36 rolls. $X \sim \text{Bin}(n=36,\ p=\frac{1}{6})$

$np = 36\cdot\frac{1}{6} = 6 \qquad \sqrt{npq} = \sqrt{36\frac{1}{6}\cdot\frac{5}{6}} = \sqrt{5} = 2.24$

$P(X>10) = P(X\ge 11) \sim P\left(Z > \frac{11-0.5-6}{2.24}\right)$

$\sim P(Z>2.01) = 1-F(2.01)$

$\sim F(-2.01)$

~ 0.0222

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2. Jobs arrive to a file server according to a Poisson process with a rate $\lambda = \frac{1}{2}$ per minute.

a) Obtain the density function of T, which is the time of the arrival of the third job to the file server. (Derive the density by writing all necessary steps.) **(10 Points)**

b) Use Poisson distribution to find the probability that the third job arrival will be within 2 minutes. **(10 Points)**

c) Let X denote the number of arrivals in time interval **(0,t)**, in the Poisson process above, ($\lambda = \frac{1}{2}$ per minute). Show that **t** is an even integer, if two consecutive values of X have equal probabilities. **(10 Points)**

$\lambda = \frac{1}{2}$ a) T: time of the arrival of the 3rd job

$F_T(t) = P(T \leq t) = 1 - P(T > t)$ X: # of arrival in $(0,t)$

$= 1 - P(X \leq 2)$ $X \sim \text{Poisson}(\lambda t)$

$= 1 - P(X=0) - P(X=1) - P(X=2) = 1 - e^{-\lambda t} - (\lambda t)e^{-\lambda t} - \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$

$\Rightarrow F'_T(t) = f_T(t) = \cancel{\lambda e^{-\lambda t}} - \cancel{\lambda e^{-\lambda t}} + \cancel{\lambda^2 t e^{-\lambda t}} - \cancel{\lambda^2 t e^{-\lambda t}} + \frac{\lambda^3 t^2 e^{-\lambda t}}{2!}$

$f_T(t) = \frac{\lambda^3 t^2 e^{-\lambda t}}{\Gamma(3)}$ $t > 0 \Rightarrow T \sim \text{gamma}(\alpha = 3, \beta = \frac{1}{\lambda})$

where $\lambda = 1/2 \Rightarrow \beta = 2$

b) X: # of arrivals in interval $(0,2)$. $X \sim \text{Poisson}(\lambda t = \frac{1}{2} \cdot 2 = 1)$

$f(x) = \frac{e^{-1}}{x!}$ $x = 0, 1, 2, \ldots$ $P(T < 2) = 1 - P(T \geq 2) = 1 - P(X < 3)$

$= 1 - P(X \leq 2) = 1 - P(X=0) - P(X=1) - P(X=2)$

$\Rightarrow P(T < 2) = 1 - e^{-1} - e^{-1} - \frac{e^{-1}}{2} = 1 - (2{,}5)e^{-1}$

c) $P(X=n) = P(X=n+1)$, $f(x) = \frac{e^{-\frac{t}{2}} \left(\frac{t}{2}\right)^x}{x!}$

$\Rightarrow \cancel{n!}\frac{e^{-\frac{t}{2}} \left(\frac{t}{2}\right)^n}{n!} = \cancel{n!}\frac{e^{-\frac{t}{2}} \left(\frac{t}{2}\right)^{n+1}}{(n+1)!} \Rightarrow 1 = \frac{\frac{t}{2}}{n+1} \Rightarrow n+1 = \frac{t}{2}$

$\Rightarrow t = 2(n+1)$ (n is an integer) $\Rightarrow$ t is an even integer.

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3. Let random variable X be a Weibull with parameters α and β.

a) Show that $E(X) = \alpha^{-\frac{1}{\beta}}\Gamma(1+\frac{1}{\beta})$, $Var(X) = \alpha^{-\frac{2}{\beta}}\left\{\Gamma(1+\frac{2}{\beta}) - \left[\Gamma(1+\frac{1}{\beta})\right]^2\right\}$.

(12 Points)

b) If X has Weibull distribution with failure rate $\rho(t) = \frac{1}{\sqrt{t}}$, find the expected value and the variance of X. **(13 Points)**

a) $E(X) = \int_0^\infty x\alpha\beta x^{\beta-1} e^{-\alpha x^\beta} dx$ let $\alpha x^\beta = u$ then $\alpha\beta x^{\beta-1}dx = du$ and $x = \left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}}$

$= \int_0^\infty \frac{u^{\frac{1}{\beta}}}{\alpha^{\frac{1}{\beta}}} e^{-u} du$

$= \alpha^{-\frac{1}{\beta}} \int_0^\infty u^{\frac{1}{\beta}} e^{-u} du = \alpha^{-\frac{1}{\beta}}\Gamma(1+\frac{1}{\beta})$

$E(x^2) = \int_0^\infty x^2 \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} dx = \int_0^\infty \frac{u^{\frac{2}{\beta}}}{\alpha^{\frac{2}{\beta}}} e^{-u} du$

$= \alpha^{-\frac{2}{\beta}}\Gamma(1+\frac{2}{\beta})$

$\Rightarrow Var(X) = \alpha^{-\frac{2}{\beta}}\Gamma(1+\frac{2}{\beta}) - \alpha^{-\frac{2}{\beta}}\left[\Gamma(1+\frac{1}{\beta})\right]^2$

$= \alpha^{-\frac{2}{\beta}}\left\{\Gamma(1+\frac{2}{\beta}) - \left[\Gamma(1+\frac{1}{\beta})\right]^2\right\}$

b) If $X \sim Weibull(\alpha, \beta)$ $R(t) = P(X > t) = \int_t^\infty \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} dx = e^{-\alpha t^\beta}$

$\Rightarrow \rho(t) = \frac{\alpha\beta t^{\beta-1} \cancel{e^{-\alpha t^\beta}}}{\cancel{e^{-\alpha t^\beta}}} = \alpha\beta t^{\beta-1}$

$\alpha\beta t^{\beta-1} = \frac{1}{\sqrt{t}} = t^{-\frac{1}{2}} \Rightarrow \beta - 1 = -\frac{1}{2} \Rightarrow \beta = \frac{1}{2}$

$\alpha\beta = 1 \quad \alpha \cdot \frac{1}{2} = 1 \Rightarrow \alpha = 2$

$\Rightarrow E(X) = 2^{-2}\Gamma(1+2) = \frac{\Gamma(3)}{4} = \frac{2}{4} = \frac{1}{2}$

$Var(X) = 2^{-4}\left\{\Gamma(1+4) - [\Gamma(3)]^2\right\} = \frac{1}{16}[\Gamma(5) - 4]$

$= \frac{1}{16}(4! - 4) = \frac{20}{16} = \frac{5}{4}$

$= 1.25$

4. Joint density of (X,Y) is given by

$$f(x,y)=\begin{cases} x+y & 0<x<1 \quad 0<y<1 \\ 0 & elsewhere \end{cases}$$

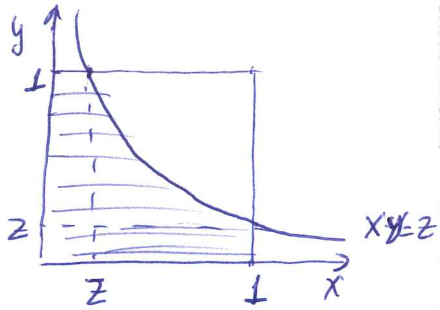
a) Find the marginal densities of X and Y. (**5 Points**)

b) Find the density of Z = XY. **(12 Points)**

c) Find Cov (X,Y). **(8 Points)**

a) $f_X(x) = \int_0^1 (x+y)\,dy = (xy + \frac{y^2}{2})\Big|_0^1 = x + \frac{1}{2} \quad 0<x<1$

$f_Y(y) = \int_0^1 (x+y)\,dx = (\frac{x^2}{2} + yx)\Big|_0^1 = y+\frac{1}{2} \quad 0<y<1$



b) $F_Z(z) = P(Z\le z) = P(XY \le z)$

$= \iint\limits_{xy\le z} (x+y)\,dy\,dx$

$= 1 - \iint\limits_{xy>z}(x+y)\,dy\,dx$

$= 1 - \int_z^1\int_{\frac{z}{x}}^1 (x+y)\,dy\,dx = 1 - \int_z^1 (xy+\frac{y^2}{2})\Big|_{\frac{z}{x}}^1 dx$

$= 1 - \int_z^1 (x+\frac{1}{2} - z - \frac{z^2}{2x^2})\,dx = 1 - (\frac{x^2}{2}+\frac{x}{2} - zx + \frac{z^2}{2x})\Big|_z^1$

$= 1 - [(\frac{1}{2}+\frac{1}{2} - z + \frac{z^2}{2}) - (\frac{z^2}{2}+\frac{z}{2} - z^2 + \frac{z}{2})]$

$= \cancel{1} - \cancel{1} + z - \cancel{\frac{z^2}{2}} + \cancel{\frac{z^2}{2}} + \frac{z}{2} - z^2 + \frac{z}{2} = 2z - z^2 \quad 0<z<1$

$\Rightarrow f_Z(z) = 2-2z \quad 0<z<1$

c) $E(XY) = E(Z) = \int_0^1 z(2-2z)\,dz = (z^2 - \frac{2z^3}{3})\Big|_0^1 = 1-\frac{2}{3}=\frac{1}{3}$

$E(X) = \int_0^1 x(x+\frac{1}{2})\,dx = (\frac{x^3}{3}+\frac{x^2}{4})\Big|_0^1 = \frac{1}{3}+\frac{1}{4}=\frac{7}{12}$

$E(Y)=\frac{7}{12} \quad \Rightarrow \quad Cov(X,Y) = \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = \frac{48-49}{144} = -\frac{1}{144}$

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