A Review of General Relativity with Torsion as a Gauge Theory

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1 Introduction and Motivation

A common simplifying assumption in general relativistic theories takes the underlying connection ∇_a describing the transport of vectors to other vectors to be torsionless, which requires that for any pair of vector fields V^a and W^a, $V^a\nabla_a W^b - W_a\nabla_a V^b = [V,W]^b$, so the switching of the order of the connection derivatives is identical to the commutation of the natural derivatives on the fields. This is a natural assumption to make, since it ensures that the curvature tensor satisfies the Bianchi identities, which ensures the symmetry rules for many of the other physical tensors in the theory (i.e., the energy-momentum tensor).

On the other hand, this discounts some additional predictive power in the theory, such as a natural way to introduce spin-like behaviour in the gravitational field, if we treat the metric as part of a field theory. There are also important relationships between the choice of connection and the choice of gauge found in local gauge theories on flat space time. In this report, I will present a generalized notion of gauge theory that fits in the framework of general relativity with possibly non-torsionless connections.

2 General connections on principal G-bundles

Note that the idea of comparing a field at different points is not exclusive to semi-Riemannian manifolds and their tangent bundles. In particular, this concept can be defined for a class of manifolds called principal bundles. The analogy comes from what a connection does on a regular manifold (M^n, g)–it is a differential operator ∇_a on the sections of the tangent bundle and the associated sections of the tensor bundles (satisfies product rule).

The action of this connection takes tensor-valued k forms to $k+1$ forms. This then introduces a notion of horizontalness or parallelism of vector fields, which are exactly the vector fields $X^a \in \Gamma(TM)$ for which its contracted connection $X^a\nabla_a X^b = 0$, as in is trivial. Since we can always locally construct an orthonormal frame $(e_i)^a$ that is self-parallel, so $(e_i)^a\nabla_a(e_j)^b = 0$ for each i, j, b, we see that the connection constructs a notion of parallelism for orthonormal frames in general. Hence, we can define a horizontal subspace of the orthonormal frame bundle, given exactly by the frames that have the aforementioned property. Since the frame bundle is identical to the vector bundle $\pi : F \to M$ with the structure space $V = SO(n, \eta)$, which is the orthogonal group of transformations of $\mathbb{R}^n$, we can isolate the horizontal subspace as a component V^H and a vertical subspace V^V locally corresponding to the frames that the projection pushes forward to a trivial point in TM. These decompose the associated vector space completely, so $V = V^V \oplus V^H$. In other words, each frame can be split into a horizontal and vertical part uniquely, letting us define a projection map $\omega(f) = f^H$, which is the horizontal part, which is a V-valued 1-form on the principal bundle [1].

Alternatively, if we start by defining such an ω, we can reverse engineer the connection, since it gives a natural notion of horizontal and vertical subspaces, so long as it satisfies some special properties. We can generalize this notion to a principal G-bundle with some Lie group G (recall a Lie group is a topological group with a smooth multiplication and inversion). We begin by defining what this is exactly.

Definition 2.1. A smooth fibre bundle is a set (F, M, G, π) of smooth manifolds F and M with a continuous surjection (projection) $\pi : F \to M$ that satisfies some properties, where G is the fibre of the bundle and $\pi^{-1}(U)$ is called the fibre of $U \subset M$. The essential properties are that every point in M has an open neighbourhood that is diffeomorphic to $U \times G$ (local trivialization condition), in the sense that the transition maps are smooth. These transition maps correspond to choice of gauge.

A smooth principal G-bundle is a fibre bundle with the fibre G being a Lie group, and an additional condition that each $g \in G$ has an associated action on F given by diffeomorphisms $g : F \to F$ that preserve the projection, so $\pi(gf) = f$, and preserves the group property with the identity map as the identity. The equivalence relation $\sim$ under this projection implies that $M = F/\sim$.

Note in the case of general relativity, we pick $G = SO(n, g)$, which has an associated connection defined by some ω, which is a 1-form with values in the Lie algebra of $SO(n, g)$, which is $V = SO(n, \eta)$ (Minkowski metric local trivialization). We can do the same for any G, and so we decide to call ω the connection form. Observe that the group action corresponds to a local transformation of the fibres, for which we want infinitesimal generators. These come from the Lie algebra, which is the tangent space at the identity element of G, and their exponential maps.

That is, G has a natural exponential map that takes its tangent spaces to itself in a locally diffeomorphic way that preserves the group structure if we view the vector space as a Lie group [5]. Since the exponential map for open balls near the origin of the Lie algebra correspond to group elements close to the identity, for a Lie algebra element $q \in \mathfrak{g}$ and some $f \in F$, $\tilde{f} = \exp(\epsilon q) f$ represents a small transformation. If we want to know how this transformation changes f as $\epsilon \to 0$, we evaluate its derivative at 0, which gives us a vector field:

$$q^*_f = \frac{d}{d\epsilon}(\exp(\epsilon q) f)_{\epsilon=0} \tag{1}$$

We want ω to preserve this local type transformation, as in $\omega(q^*) = q$, so there is a natural association with the gauge transformations. Denoting g^A as the adjoint action of $g \in G$ on the Lie algebra [5], given by the derivative at the identity of $A_g(h) = ghg^{-1}$ we also want $g^A g^* \omega = \omega$, to ensure that the pull back on ω behaves nicely with the projection. These two conditions together are called G-invariance of ω, where this can be generalized to any output vector space (useful for later). This gives us the following definition [1].

Definition 2.2. A connection form ω for a principal G-bundle (F, M, π) is a 1-form on F with values in its Lie algebra $\mathfrak{g}$, the space of such forms denoted by $\Lambda^1(F, \mathfrak{g})$, that satisfies the condition $\omega(q) = q$ and $g^A g^* \omega = \omega$, or that ω is G-invariant.

There is freedom in how we can pick ω, as these conditions are not very strict. With ω and π, we can uniquely identify the decomposition $\Gamma(F) = N(F, \omega) \oplus N(F, \pi_*)$, where $N(F, \omega)$ is the space of ω-horizontal vector fields, and $N(F, \pi_*)$ is the space of π-vertical fields, corresponding to the respective null spaces. Again, the horizontal component of a field is denoted by $(\cdot)^H$ and the vertical with V instead.

Definition 2.3. For a W-valued k-form $\beta \in \Lambda^k(F, W)$, where W is an arbitrary vector space, we can define $\beta^H(X_1, \cdots, X_k) = \beta(X_1^H, \cdots, X_k^H)$ for $X_1, \cdots, X_k \in \Lambda(TF)$ and likewise for verticality with V. It is called horizontal if $\beta^V = 0$, which is true iff it vanishes if any one of the X_i are vertical. We can then define, in local coordinates, the associated derivative connection, where d is the standard exterior derivative associated with the smooth structure:

$$\nabla_{[b}\beta_{a_1 \cdots a_k]} = (d\beta)^H_{ba_1 \cdots a_n}$$

We also denote this $k+1$ form as $\nabla_\omega \beta$. Observe that if $\nabla_\omega \beta = 0$, then $d\beta$ must be horizontal, which provides a notion of transport on k-forms and so on k-tensors. This also lets us identify a class of k-forms closely related to ω, given by G-invariant and ω-horizontal forms. These are called basic forms, which will be useful later.

Note moreover that this idea of transport applies not only to forms with associated vector space equal to the Lie algebra, but for general vector space codomain. In fact, this lets us use the pull back of the k-forms on TF to TM via π^* to get the notion of horizontality on W-valued k-forms on the base space, and hence any tensor bundle by setting W appropriately.

3 Curvature and torsion

Immediately from the definition of the connection via ω, we can define the generalized Riemann curvature 2-form as:

$$\Omega_{ab} = \nabla_{[a}\omega_{b]} \tag{2}$$

Or $\Omega = \nabla_\omega \omega$ [1]. To get a more explicit formula, we need to be able to perform a wedge product of $\mathfrak{g}$ valued k-forms, which requires a bilinear self-map of the Lie algebra. Luckily, Lie algebras are equipped with a commutator associated with the local sections, which gives us a wedge product defined in the same way, just replacing multiplication with the Lie commutator. We get the following result for basic k-forms on $\mathfrak{g}$ from before [1].

Theorem 3.1. *Let β be a basic k-form, then $\nabla_\omega \beta = d\beta + \omega \wedge \beta$. Essentially, the differential action of ∇_ω on horizontal and group invariant forms is that of a regular exterior derivative plus a correction with ω. One can see this simple formula coming from being very simple, since β is G-invariant, implying that it uniquely determines a k-form on M since it we can always ignore the redundancy from the group action, and horizontalness makes the connection's derivative nicer.*

Using this, we can write $\Omega = d\omega + \omega \wedge \omega$, where the second term does not vanish as it would in regular forms. Note that this curvature reduces to the regular curvature, if we take the Lie group to be any subgroup of $GL(n)$ (invertible matrices), where the output of Ω_{ab} has two indices, which are given by $(\Omega_{ab})^d_c = R^d_{abc}$ as usual. The action on vector fields X and Y is also identical, as in $\Omega(X,Y) = R(X,Y)$ acts like a matrix that satisfies the identifying equation:

$$R(X,Y) = \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]} \tag{3}$$

acting on vector fields locally in that way, where $\nabla_X = X^a \nabla_a$. In fact, most theories reduce to this case of Lie groups, as the finite-dimensionality results in the Lie algebra being some subspace of the general linear group. This case is nice because it lets us define the conjugate form to the connection, associated to the projection. Denote the bundle for this case as $G(M)$.

Definition 3.2. The canonical form $\psi \in \Lambda^1(G(M), \mathbb{R}^n)$ is a 1-form given by $\psi(X_g) = g^{-1}(\pi^*(X_g))$, where the action of g is given from $\mathbb{R}^n \mapsto T_{X_g}M$. Note that we can restrict this to any subgroup with the same definition, so this form makes sense for all Lie groups of this type. Note that ψ is basic; it clearly satisfies the G-invariance, and its horizontalness comes from π^* taking X_g to 0 when it is vertical. Hence, its derivative has a nice form, and it turns out that this is a good definition for the torsion:

$$\Theta_{ab} = \nabla_{[a}\psi_{b]} \tag{4}$$

or $\Theta = \nabla_\omega \psi$. In the notation of vector fields, viewing this as an operator on vectors, we can write [1]:

$$\Theta(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{5}$$

which is the form it is normally presented in. Observe that the canonical form ψ corresponds to the vertical, but does not depend on the choice of connection ω. Moreover, in the version where structure group is $SO(n,g)$ as before, we can always pick a unique ω for which $\Theta = 0$, or that ψ is parallel, corresponding to the Levi-Civita connection, which is the special case general relativity reduces to.

4 Gauge Theories and Poincare Gravity

Observe that when we take the structure group $G = U(1)$, since the group is abelian, all commutators vanish, meaning that the wedge $\omega \wedge \omega = 0$, so $\Omega = d\omega$. Writing $\omega = -iA_\mu dx^\mu$ locally since the Lie algebra is just $i\mathbb{R}$, we see that $\Omega = -i\partial_\nu A_\mu dx^\nu \wedge dx^\mu = -i(A_{\mu,\nu} - A_{\nu,\mu})dx^\mu \wedge dx^\nu$, so $\Omega_{\mu\nu} = -iF_{\mu\nu}$, where F is the electromagnetic tensor [1]. In regular gauge theories, this quantity $F_{\mu\nu}$ is invariant under gauge transformations of A_μ via $U(1)$. The corresponding connection derivative for this choice is given by, on vector fields X^a:

$$\nabla_a X^b = (\partial_a + iA_a)X^b$$

which corresponds to the gauge covariant derivative commonly found when discussing simple $U(1)$ gauge theories like electromagnetism [2]. That is, if ω is changed in any way while remaining a connection, although the gauge potentials A_a change, and so the derivative, the visible curvature or "field strength" so to speak is not changed, which corresponds to the physical result. Note that the torsion is not defined in this case, so no degrees of freedom are taken by it. The analogy is slightly less clear in the case of $G = SO(4, \eta)$, where we now have a varying curvature and torsion dependent on the connection [1]. However, we still have a corresponding gauge potential, provided by the Christoffel symbols of the connection, such that:

$$\nabla_a X^b = \partial_a X^b + \Gamma^b_{ac} X^c$$

These replace the potential, and have the pair of torsion and curvature define the effective fields. Although these are not invariant to the changes in the symbols, for a slightly larger structure group containing translations–called the Poincare group–the field equations give the same result. In particular, consider the group $G = \mathbb{R}^4 \oplus SO(4, \eta)$, which is the group given by the semidirect sum of translations ($\mathbb{R}^4$) with the regular Lorentz hyperrotations. Alternatively, these could be viewed as the space of orthonormal affine maps of a specific orientation, which forms an affine frame bundle $\mathring{A}(M)$. It can be verified that the connection form ω on this choice of G can be pulled back through the affine map onto another form that must be a direct sum of the translation component and the rotational component. That is, given $\tilde{\omega} \in \Lambda^1(\mathring{A}(M), \cdot)$ (where the dot denotes the choice of subgroup), we can pull back using the natural affine map γ from the general affine maps to the general linear maps, and take [2]:

$$\gamma^* \tilde{\omega} = \omega \oplus \phi \tag{6}$$

The affine curvature $\tilde{\Omega}$ is related to Ω then by:

$$\tilde{\Omega} = \Omega + \nabla_\omega \phi \tag{7}$$

by taking the derivative of both sides and passing through the pull back, given it is just a translation. The second term turns into torsion if $\phi = \psi$, which means that we have a decomposition of the affine connection $\tilde{\omega}$ into the horizontal and vertical parts. In this special case, we get something called an affine connection, which is naturally identified with the affine connections under $SO(n, g)$ only. Moreover, the affine curvature is the curvature plus the torsion, which is effectively the gauge invariant quantity over the space of affine connections. Note there is no affine torsion, so all degrees of freedom in the gauge are captured by the affine curvature.

This special effect is exclusive to Lorentz gravity out of other physical theories, due to the existence of the canonical form, which arises from the representative action of the group on an underlying vector bundle. It is likely that if there were more unique linearly independent basic differential forms ψ_i associated to isomorphic actions on vector bundles in the same way, the terms $\Theta_i = \nabla_\omega \psi_i$ would give multiple torsion terms. Suggestively, as in this case of the theory, we would expect a way to extend the structure group in a way so that all these torsions get integrated into a single affine curvature.

5 Formalism for M-Projection

We did not clarify on how to explicitly determine the curvature and torsion terms on the underlying space M, as we only have a form available on the fibre bundle. We can perform a local pull back using a section from an open set U, we can expand out the components of each term as $n \times n$ matrices, writing Ω_a^b and ω_a^b as the matrix components, and as a vector Θ^a and ϕ^a, in the orthogonal basis of $SO(n, \eta)$ and $\mathbb{R}^4$ respectively [1]. This will let us write:

$$\Omega_a^b = \frac{1}{2} R^b_{a\mu\nu} \phi^\mu \wedge \phi^\nu \tag{8}$$

$$\Theta^b = \frac{1}{2} T^b_{\mu\nu} \phi^\mu \wedge \phi^\nu \tag{9}$$

Note that the ϕ^μ form a local coframe that is dual to the frame that the local section provides.

This produces a tetrad formulation of the curvature and torsion, in which we can choose conveniently orthonormal frames and coframes to measure everything. We also get generalized Bianchi identities $\nabla T^a = \Omega_b^a \wedge \phi^b$ and $\nabla \Omega_b^a = 0$, where ∇ here is just the local derivative. Note the former identity reduces to the regular antisymmetrized vanishing term for the torsionless case, and the second term is just the standard derivative Bianchi identity. Observe that the ϕ^a and ω_a^b act like gauge potentials as before, with the corresponding field strengths given by the connection gradients Ω_a^b and Θ^a, satisfying the aforementioned relations.

Various remarks before continuing, the curvature tensor does not necessarily depend fully on just the metric terms, due to the presence of the gauge Christoffels. This also is true for the torsion, which is now nonvanishing. This makes calculations generally much harder in this framework than any other.

6 Einstein-Cartan Theory and Field Equations

Recall in general relativity, the physical trajectories follow the geodesics given by the maximizers of the age between two causal points (the geometric), or alternatively as the parallel transported point given some initial velocity (the parallel). These notions only agree because of the assumption that gravity is torsion-free, which ensures that the maximizers satisfy exactly the geodesic equation. Hence, Einstein-Cartan theory cannot decide the dynamics based off of one or the other symmetrically, as they produce different results. It is more common practice to instead define an action principle that needs to be extremized, which produces the potential and metric, as well as conserved quantities, decidedly some kind of momentum that measures the trajectory instead.

The original trajectories in torsionless GR can be rederived from the Einstein-Hilbert action $1/2\kappa \int R\sqrt{-g} d^4x$ [4]. In Einstein-Cartan theory, we have a similar type action, but with the new curvature form, which makes it $\int \Omega^{ab} \wedge \eta_{ab}$, where η_{ab} refers to the Hodge dual term as per the notation given [1]. In this case, we get the canonical stress energy tensor Q_{ab} and the new spin current tensor S^c_{ab} as physical objects that the curvature and torsion gauge potentials couple to:

$$G_{ab} = \kappa Q_{ab} \tag{10}$$

$$\tau^c_{ab} = \kappa S^c_{ab} \tag{11}$$

where $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ as the standard Einstein curvature tensor, where $R_{ab} = R^c_{acb}$ and $R = R^a_a$, and $\tau^c_{ab} = T^c_{ab} + 2\delta^c_{[a}T^d_{b]d}$ is the new Einstein torsion tensor [1]. Note that the energy momentum tensor is no longer symmetric in this formalism, given the curvature tensor cannot satisfy Bianchi's original identity, and in fact it can be written in terms of its general relativistic version plus some spin current corrections. The spin current tensor is antisymmetric as well, as indicative of the Einstein torsion tensor. Observe that this coupling gives us a unique effect from spinning particles on the torsion of the system, which is exclusively pointwise due to the lack of derivatives. That is, the torsion seems to be determined completely by only behaviour at points, not in neighbourhoods, suggesting that it does not have global effects. Together, the solution metric g_{ab} and affine coefficients Γ^c_{ab} determine the compatible dynamics. Note also that when the spin current vanishes, the only possible torsion is 0, so we return to regular GR.

Note that the stress energy and the spin current are both locally conserved currents due to the nature of the Lagrangian. We can verify a more symmetric form of the field equations using the differential form version [1]:

$$\partial \mathcal{L}/\partial \omega^a = \frac{1}{2\kappa}\eta_{abc} \wedge \Omega^{bc} = -Q_a = Q_{ab}\eta^b \tag{12}$$

$$\partial \mathcal{L}/\partial \omega^{ab} = \frac{1}{2\kappa}\eta_{abc} \wedge T^c = S_{ab} = S^c_{ab}\eta_c \tag{13}$$

where the η terms again correspond to the Hodge dual terms of the respective k-coframe basis in terms of ϕ, as specified in the schema given in the computational section of [1]. This shows clearly how the stress energy and spin currents are conserved quantities via Noether's theorem.

We identify two distinct degenerate cases of the field equations and gauge potentials. The former case we already discussed, where the torsion tensor vanishes. The second scenario, referred to as teleparallel theory, derivatives of the basis elements ϕ^a with respect to each other to vanish locally via the gauge potential given by the connection, which implies that the connection form vanishes, which means the curvature vanishes. This case is dominated by the torsion term, and uniquely determines the motion through momentum conservation equations. Note that both degenerate cases, and all other cases, all predict equivalent behaviours for spinless particles. We can view GR as the geometrized form of gravity, where all of it is taken into the geometry of space-time, curving it and changing the derivative, while teleparallel gravity is the flattened version of the relativistic problem, where gravity is now a force applied through the torsion on each point in space time [4].

As with regular physical theories, we can now add more terms to the action that produce other coupling effects. The most common addition is quadratic gauge invariant type terms corresponding to the quadratic electromagnetic terms in F usually added in regular gauge theory [1].

7 Effect on Physical Predictions

According to [3], the regular Einstein-Cartan equations predict identical dynamics to the original equations in GR whenever the theory only has up to linear order in the spin current terms. This would only be possible locally in areas where the spin current is on the same scale of magnitude as the energy momentum densities and currents. An estimate of the scale at which this would be true suggests a size much smaller than a nucleon's masking radius, which although still larger than the Planck scale, indicates that the theory is basically identical to GR for literally all applications that could be thought of. On the other hand, the same estimates as suggested by [3] suggest the energy density scale is of a high enough order to match the amount predicted to be required for the unification of the force interactions at this length scale.

Many pathological or singular physical scenarios (i.e., big bang, black holes, and so on) also have more well-behaved solutions in Einstein-Cartan, so long as spin remains a dominant term at that scale. For example, in the FLRW case of a universe dominated by a dust or fluid with a uniform spin density with little to no shear effect has been predicted to have a sensible solution for all time t, avoiding singularities, so long as the shear is much smaller than the spin effect [3]. Alternatively, after plugging the adjusted solutions into the Friedmann equations, various terms corresponding to some form of torsion density come up, which seem to emulate effects of something like dark matter or dark energy, with predictions in the normal evolution case allowing for late time universe expansion [1].

8 Conclusion

To sum up what we reviewed, general relativity can be formulated with torsion to introduce intrinsic particle spin into it in a natural way, which also seems to be compatible with non-perturbed GR due to how weakly the spin seems to interact with the torsion. The redundancies in the theory can be expressed through the gauge potentials given by the connection form, which although do not produce a gauge invariant curvature or torsion separately, do indeed add up to form a gauge invariant affine curvature when viewed as a theory with Poincare group invariance. This is analogous to the way the Yang-Mills electromagnetic type theory pops out of $U(1)$ gauge invariance. Projecting the curvature and torsion of the theory back onto the base space lets us formalize the Einstein-Cartan field equations, which come out as conservation rules for the respective gauge potentials from the action principle. From this we can recover the original formulation of Einstein's equations in the torsionless degeneracy, and the flat degeneracy corresponds to the case where Einstein's equations are hidden in the torsion, which encodes the forces through teleparallel gravity dynamics. Last of all, the various predictions that the theory makes are not really testable or observable in any realistic way, due to the scales being too extreme and unreachable. It also seems less necessary for anything but figuring out how the early universe might have behaved or behaviour near and inside black holes. Altogether, this topic was very interesting to learn about and review, and gave a better understanding of what a local gauge theory resembles in a less trivial setting.

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