PHY324: Determining the gravitational acceleration constant using the small angle motion of a Kater pendulum

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1 Abstract

In this paper, we produce a value for the gravitational acceleration g in the room MP235 using a Kater pendulum. In particular, we measure the relation between the small angle periods on each pivot of the pendulum with respect to the adjustment of a fine mass, fit both period relations to a line and discuss the validity of such a fit. We then find the intersection period to be $T = 2.00753 \pm 0.00004$ s, which with the measured distance between pivots of $L = 0.999 \pm 0.001$ m as the effective length, we compute $g = 9.79 \pm 0.01$ ms^{-2}. We then consider an analysis of various systematic errors that could cause several discrepancies in the data, such as high reduced chi squared values and possibly non-Gaussian residuals.

2 Introduction

It is a well-known fact that a simple pendulum of length l in a uniform gravitational field with acceleration g undergoes simple harmonic motion for small angles, with a period of:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

In the real world, no pendulum is simple, as the mass is not concentrated perfectly at a distance l from the pivot, but rather distributed throughout the lever's body. However, the equations of motion for small angles are the same, with the length of the pendulum being replaced with an effective length $l = l_e$ that has the same formula for its period, but much more difficult to determine [1].

The Kater pendulum serves to remedy this issue, as apendulum with two pivots on either side, variable mass adjustors on its body, and a large fixed mass on one end for asymmetry, as pictured in Figure 1. In principal, the periods on both pivots–T_u for when the large mass is on top and T_d for when its on the bottom–vary based on the mass positions on the pendulum, but for some special combination of mass adjustments, $T_u = T_d$. It is here exactly where the effective length of both pendulums is equal. Since a pendulum oscillating about a pivot an effective length from one pivot should have the same period, by symmetry this effective length should be the distance between the two pivots.

Hence, if we can find the period T that both pivots agree on, the effective length is just the distance L between the pivots, which lets us calculate the gravitational acceleration where the experiment is being performed, as:

$$g = 4\pi^2 \frac{L}{T^2} \tag{1}$$

It is impractical to adjust the masses until the periods are exactly equal, so we can instead first adjust the coarser masses so the periods are as close as possible, and then fit the relation between the position of the fine mass on the rod x with the periods linearly to $T_u \approx m_u x + b_u$ and $T_d \approx m_d x + b_d$, which is justified as a local approximation of the real relation. We can then compute the intersection x_0 just by solving for the equations as:

$$x_0 = -\frac{b_d - b_u}{m_d - m_u} \tag{2}$$

This also lets us solve for the common period $T \approx m_u x_0 + b_u = m_d x_0 + b_d$, which with a measurement of the pivot distance, can give us a good value for g in our lab.

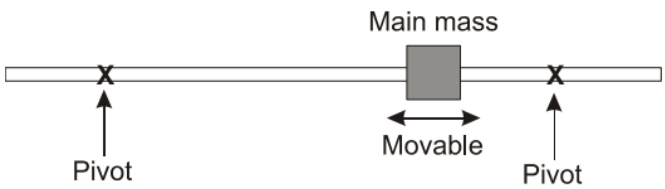


Figure 1: A sketch of a Kater pendulum's pivots and an adjustable mass on it, which we in our case will be composed of a coarse and fine mass adjustor (see next section or [1]).

3 Materials and Methods

We used a standard Kater pendulum as supplied by UofT. This pendulum has two knife edge pivots and two adjustable masses between them–one is much larger for bigger and coarser changes and the other one is meant for finer adjustments in the mass distribution. The relative positions of these masses were measured using the engraved scale on the pendulum, which is in the 1 cm range of precision for the coarse mass, while being in the 0.5 mm range of precision for the fine mass adjustments.

These edges were suspended on a secure shelf on the wall designed to support the knife edge without sliding for small angles, as shown in Figure **??**. Note that one of the key advantages of this type of pivot is that the effect of friction in damping the pendulum is minimal, as the contact point of the pivot is very small, which was evidenced by the pendulum's amplitude barely changing for small angles. The knife edge was also supported in the direction perpendicular to the plane of motion, so that off-plane motion was reduced. Even so, there still were some residual vibrations in the pendulum that made it move forward and backwards very slightly.

Figure 2: Kater pendulum on upper pivot.

To measure the distance between the pivots, a very precise steel scale–up to 0.5 mm of precision–with a vernier scale-fitted cathetometer was used to measure the distance from one knife edge to some half-way reference point, and then from that point to the other knife edge. The sum of these two distances gave us a distance of $L = 99.9$ cm with a standard uncertainty ± 0.1 cm between the two pivots, accounting for the sum and the precision of the scale.

To measure the pendulum oscillation, a black box equipped with an infrared source and sensor was placed so that the pendulum end's trajectory passes through the beam of light, interrupting the signal momentarily. The detection of this interruption was counted by a connected counter-timer, two of which is then counted as a full period. In relation to the time kept track of by an internal 1 MHz crystal oscillator, the counter timer measured the time taken for 8, 16, or 32 oscillations based on the setting, up to 6 decimal digits of precision. For all relevant measurements, we used the 32 oscillations setting so that the error in the period measurement is minimized, since each setting has the same absolute error in the total time measurement, which we discuss in the next section.

For each mass position, a pair of time measurements for each pivot was made for the pendulum's 32 oscillations with a maximum displacement of 1.5 ± 0.5 cm from the equilibrium. As convention, we refer to the measurement made from the pivot where the large mass is on the top side of the pivot as the up measurements, and vice versa for the down measurements. We started by first moving the coarse adjustment mass–with the fine adjustment mass at relative 0 cm–until the periods were as close as possible, around where the coarse mass was at the ≈ 36 cm mark on the scale. After this, we adjusted the fine mass for various positions with respect to a reference mark on it, measuring the up and down times for each mass position.

This gave us a set of 11 time measurements, which we divided by 32 to give us the periods. These were in the vicinity of the intersection point, which we found to be somewhere between 3.75 and 5.00 cm on the relative position scale due to the two periods swapping order in size. We chose not to go out of this range as then the linear fit would not be a good representation of the model due to the more complicated relationship nonlocally.

4 Constants and Uncertainties

During the experiment, we noticed that the time measurements for 32 oscillations, although they were precise to 6 decimals, had variation in the last three digits. To quantify this, we sampled 5 time measurements for a single fine mass position after the coarse adjustment was tuned, and took the standard deviation of the sample as our standard uncertainty for all the time measurements. It would be more effective to do this for each data point, but due to time constraints we only performed one sampling, giving us a standard uncertainty of ± 0.000246 s.

To get the uncertainties on the periods T_u and T_d, we then just scale this uncertainty down with the time, dividing by 32. This gives us a standard uncertainty on the measured periods of $\pm 7.70 \times 10^{-6}$ s. The uncertainties on the mass positions x are given by the standard precision error of the measurement, which in our case is $\pm 5.000 \times 10^{-4}$ m in SI units given that the most precise increment is 0.5 mm.

When fitting the linear relation to the periods versus mass positions, the standard uncertainties of the slopes and the y-axis intercepts were computed by the Python curvefitting method through its covariance matrix, using the aforementioned absolute uncertainties on the period. The uncertainty on the intersection x_0 can be computed via the standard deviation propagation of uncertainty [3], that if a quantity $u = f(u_1, ..., u_n)$, with the x_i quantities all having uncertainties, then the uncertainty on u is just:

$$\Delta u = \sqrt{\sum_{i=1}^{n} (\frac{\partial f}{\partial u_i} \Delta u_i)^2}$$

Using this for the formula given in (2) for x_0 and the uncertainties on the fitting parameters, since $\Delta(m_u - m_d) = \sqrt{\Delta m_u^2 + \Delta m_d^2}$ and likewise for the b_u and b_d, we get that Δx_0 is just, after simplification:

$$\Delta x_0 = x_0 \sqrt{\frac{\Delta(m_u - m_d)^2}{(m_u - m_d)^2} + \frac{\Delta(b_u - b_d)^2}{(b_u - b_d)^2}} \tag{3}$$

Since $T = m_u x_0 + b_u$, we can then compute:

$$\Delta T = \sqrt{x_0^2 \Delta m_u^2 + m_u^2 \Delta x_0^2 + \Delta b_u^2} \tag{4}$$

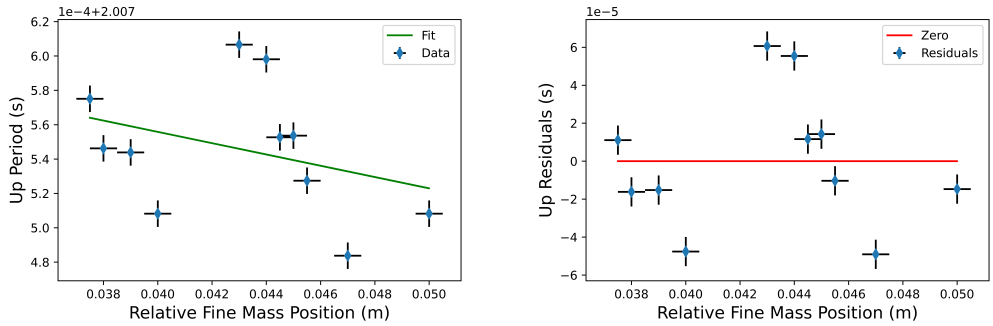
as the propagated uncertainty. Once again, we can do the same when calculating the gravitational acceleration, which gives us simplified uncertainty:

$$\Delta g = g \sqrt{\frac{\Delta L^2}{L^2} + 2 \frac{\Delta T^2}{T^2}} \tag{5}$$

Equipped with these computations, we can proceed with analysis of the results.

5 Results and Analysis

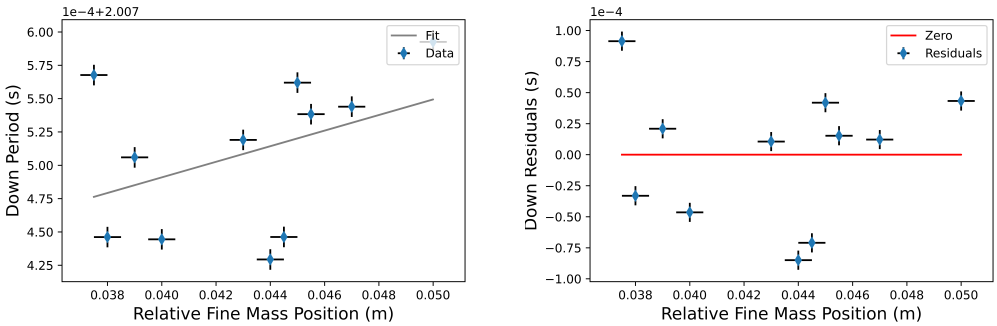
As mentioned before, we fit the period-position data to a line for both the up periods and down periods. This is outlined in Figures 3 and 4 on the next page, with fitting parameters, uncertainties, and chi squared values specified.



(a) Up period data with linear fit.

(b) Residuals of data with fit.

Figure 3: Graph of the period of the upward facing Kater pendulum related to the position of the fine mass, fitted to a line with parameters $m_u = (-3.3 \pm 0.6) \times 10^{-3}$ sm^{-1} and $b_u = 2.00769 \pm 0.00003$ m, with residuals. The reduced chi squared is $\chi^2/\text{DOF} = 23.78$.



(a) Down period data with linear fit.

(b) Residuals of data with fit.

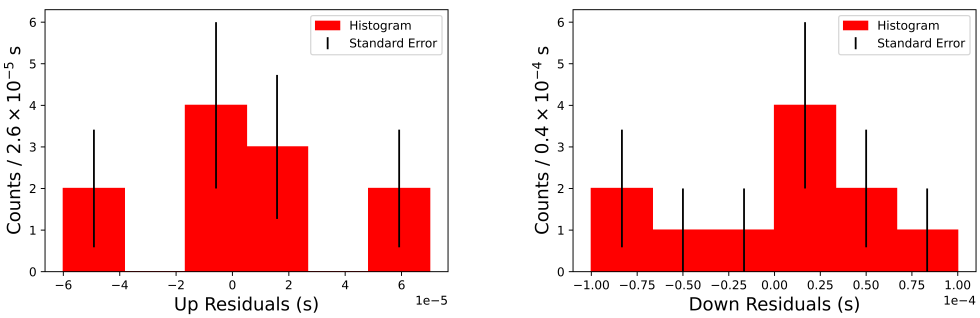
Figure 4: Graph of the period of the downward facing Kater pendulum related to the position of the fine mass, fitted to a line with parameters $m_d = (5.8 \pm 0.6) \times 10^{-3}$ sm^{-1} and $b_d = 2.00726 \pm 0.00003$ m, with residuals. The reduced chi squared is $\chi^2/\text{DOF} = 53.24$.

For reference, the axis for the graphs are shifted by 2.007 to capture the behaviour in the range of values we are looking at. Note that the scaling of the graphs are slightly different–the down periods are increasing faster than the up periods are decreasing in relation to the fine mass on average. Looking at either graph, we see that the residuals are not very close to zero within their uncertainty intervals, so we expect a very high chi squared value for both, which is indeed the case. This can be attributed either to underestimating random errors, using an insufficient model for the actual fit, or additional systematic errors that contribute to the variation.

The first of these possibilities is very unlikely, considering the precision of the instruments we are using. That is, the measurement of the relative positions is precise to a half millimeter, while the period measurement only really fluctuates because of the variation in the time measurements made by the counter-timer–which again is very precise in counting and measuring time, up to 1 : 100000 in relative error [1].

On the other hand, the data points in either graph seem to be randomly dispersed above and below the fitted lines, which suggests that it would be unlikely for any other fitting models to perform any better. To capture this behaviour for a larger dataset, we would want to show that there is no dependence among the residuals–that is, they are randomly distributed. To do this, we would plot the residuals in a histogram, which we would expect to resemble a Gaussian of some width centered at 0 if this was the case.

However, considering we only have 11 points in a very small range, this is not a very useful method of evaluating the fit's strength, as changing the bin size and range slightly drastically modifies the shape. Regardless, if we did this, it looks as in Figure 5, where (a) is that of the up period residuals and (b) is for the down period residuals. We chose the range in both cases to ensure all points were just included, and the number of bins to be 6, as the sweet spot where some shape is visible instead of just a single block or uniform small blocks. The standard uncertainty of histograms, given by the square root of the count for each count, is used for the error bars.



(a) Up residuals histogram.

(b) Down residuals histogram.

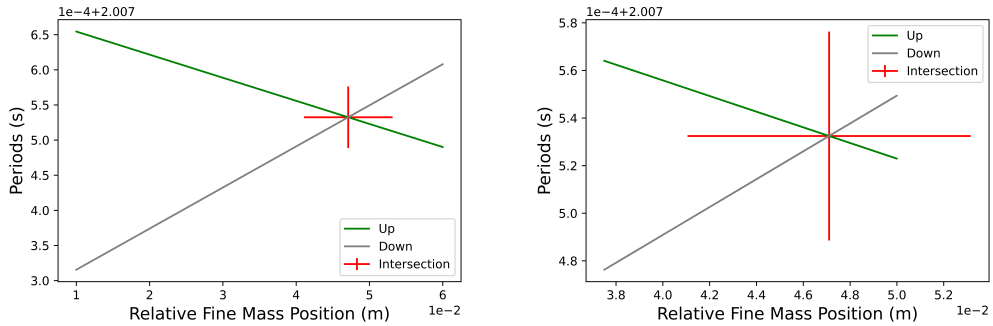
Figure 5: Histograms of the up residuals in the range $(-6.0 \times 10^{-5}, 6.5 \times 10^{-5})$ s and the down residuals in the range $(-1.0 \times 10^{-4}, 1.0 \times 10^{-4})$ s with 6 bins.

Both histograms do not qualitatively resemble Gaussians, which could mean there is room for model improvement, possibly with a polynomial fit or some power law. However, this could just as easily result in overfitting considering the extremely small pool of data we are working with, which also means that we cannot really decide whether or not the data residuals would actually fit a Gaussian if we had a much larger data set. Even so, we expect the model to not properly fit a line, as it is really only the local behaviour we want to extract.

This leaves us with the most lilkely and plausible explanation of systematic errors causing this discrepancy, among other things. We will discuss these in more detail in the next section with the results and analysis of this section.

On that note, using these values for the slopes and intercepts, we can compute the expected intersection position and its uncertainty, using equation (2) and (3), as $x_0 = 0.0471 \pm 0.006$ m, which as expected lies in the range from 4 to 5 cm. The corresponding period calculation with uncertainty using equation (4) gives us a common pivot period of $T = 2.00753 \pm 0.00004$ s, which is also very close to the point where the order of the periods was swapped when we were performing measurements. This is also very close to the expected value for the period of the pendulum, which was designed to have a period very close to 2 s for calibration purposes.

This period has an exceedingly low uncertainty value, due to the small relative uncertainties in the fitted parameters. It is important to note, however, that these are much larger within the range of variation of our data. We can visualize this by looking at the intersection error bars on a larger scale in comparison with the scale of the data points, as outlined in Figure 6.



(a) Fit lines zoomed out.

(b) Fit lines zoomed in to range of data.

Figure 6: Up and down period fit lines with their corresponding intersection, depicted with error bars at a larger scale and at the scale of the data, both with shifted y axes. This gives a heuristic explanation for why the uncertainty seems very small.

Even with the shift of the axes up, the error bars in the slightly zoomed out frame are much smaller than when we only consider the range of the data. This error width also suggests that the actual intersection really could've occurred nearly anywhere in the interval from 4 to 5 cm, as the error bar almost covers it. This also explains possibly why our data does not seem to fit to anything in particular in that interval except for a vague resemblance of an upward or downward trend corresponding to locally behaving like a line.

Equipped with our result for the intersecting period T and corresponding effective length L, we can compute an effective value for gravitational acceleration g and its uncertainty, using equation (1) and (5), as $g = 9.79 \pm 0.01$ ms^{-2}. This is very close to the accepted value of $g_n = 9.80665$ ms^{-2} [2], with a percentage error of about 0.1%. Since they are within a 5% range of each other, we can be fairly confident that the actual gravitational acceleration in the lab is very close to g_n, as we would expect considering we are on the surface of the Earth. Note that they do not agree within uncertainty intervals, as the relative uncertainty on g is slightly smaller than the percent error.

This does not really suggest anything more than possible agreement or coincidence between the values of gravitational acceleration. The small difference motivates considering systematic errors.

6 Systematic Errors

During the actual data acquisition, there were several degrees of freedom that we attempted to minimize when ensuring the data was valid. For example, the aforementioned off-plane motion of the pendulum was reduced as the knife edge was designed to not pivot in that direction on its support. However, considering that the oscillation amplitude was around 1 cm in the plane of motion, we noticed that there were vibrations in the pendulum that made it very briefly oscillate at an order of magnitude of about 0.3 cm from the tip in the off-plane direction at some moments (usually at the beginning).

This could have been due to several effects, such as very slight forces applied by us during the release in the off-plane direction, or the very slight curvature in the pendulum's supporting structure, or possibly even small turbulent effects from the air that caused it to vibrate. Regardless of what, this off-plane oscillation clearly is not negligible considering it is about a third of the normal oscillation. Hence, it would have systematically produced a shift in the period measurement to be that of a projected spherical pendulum with very high resistance in the off-plane direction given by the pendulum's support, which is a much more chaotic and unpredictable system, and so would have less certainty in a period measurement.

As described earlier as well, friction was one of the controls that was minimized by the knife edge acting as a pivot, due to the extremely small point of contact. Noting that this edge was not very sharp and had various spots where it had dulled, it is entirely possible that friction had a small effect on the pendulum's motion. This would be very unnoticeable, however, for small oscillations, as the friction component acting against the pendulum motion is extremely small in this case due to the angling. That is, since the ratio of the displacement of the pendulum to the length is less than 1/100, the fraction of friction resisting the motion is of order 1/100 of the friction acting on the edge, which already is so small. This is evidenced in the lack of damping in the pendulums motion.

An additional damping effect would be that of air resistance, but once again, this would only be significant for large velocities, which only occur for large amplitudes of oscillation. This is again clear from the fact that the pendulum did not decay visibly in the whole span of data collection.

Another possibility is that the pendulum's motion would also involve its very slight deformation due to the elasticity, which again is unlikely considering the extremely small oscillations–we would not expect a large enough torque for such a visible deformation to occur. More plausibly, the temperature fluctuations could result in very slight changes in the measurements of the length and the relative positions, which would add to the error in both measurements.

Finally, measurement of the distance between pivots could be slightly affected by calibration problems with the cathetometer–it needs to be level and the user should be level with it when aligning something with its center. This would introduce larger errors into the length and so the gravitational acceleration. Likewise, the counter-timer is calibrated using the Kater pendulum oscillations beforehand, which with the previously described possible errors in the pendulum's motion could result in amplification of these errors.

7 Conclusion

In brief, finding the intersection point of the upward and downward facing small angle oscillation periods–$T = 2.00753 \pm 0.00004$ s of the Kater pendulum with respect to a relative mass adjustor's position on its body let us successfully calculate a good value for the gravitational acceleration in the provided lab of $g = 9.79 \pm 0.01$ ms^{-2}, which is in a 0.1% range of the standard gravitational acceleration g_n. This, along with the period being very close to 2 s as expected per calibration of the counter-timer, suggests that this method is successful in determining a very powerful estimate of gravitational acceleration in a given location, in spite of the plenty systematic errors involved in the actual experiment.

This was exactly the motivation that historically resulted in this pendulum being an innovation for gravimetric analysis of the Earth made by Kater [1]. With more sophisticated equipment and precise measuring techniques, this would be a very strong tool in measuring the gravitational field on different parts of the Earth, which is very important for a variety of purposes, including navigation.

References

[1] G.F. West (2005), "The Kater Pendulum." Last accessed 31 March 2023 at `https://www.physics.utoronto.ca/~phy224_324/experiments/kater-pendulum/Kater-Pendulum.pdf`.

[2] *NIST Reference on Constants, Units, and Uncertainty* (2019), "standard acceleration of gravity. Last accessed 1 April 2023 at `https://physics.nist.gov/cgi-bin/cuu/Value?g`.

[3] Vern Lindberg (2000), "Uncertainties and Error Propagation." Last accessed 31 March 2023 at `https://www.geol.lsu.edu/jlorenzo/geophysics/uncertainties/Uncertaintiespart2.html`.