PHY324: Measuring the Thermal Diffusivity of Rubber Using the Phase Differences of a Temperature Signal

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Due: 7 February 2023

1 Introduction

When studying the behaviour of temperature in a medium, the way it changes over time is of particular interest. That is, the theoretical model of temperature $T(x,t)$ predicts that temperature obeys a diffusion equation:

$$\partial_t T - m\Delta T = 0$$

where m is the thermal diffusivity of the medium we are considering T in. This coefficient is very closely related to the conductivity and specific heat of the given medium, and so quantifies an important property of the medium: how fast thermal energy diffuses through the medium. Hence, finding it is of tremendous benefit, as it lets us predict other results in material applications.

In the special case where we know the system is spherically symmetric, $\Delta T = \partial_r^2 T + \partial_r T/r$, letting us write:

$$\partial_t T = m(\partial_r^2 T + \partial_r T/r)$$

If we perform a separation of variables $T(r,t) = R(r)\exp(i\omega t)$, where ω is the angular frequency of the temperature as a signal, we get a differential equation for R of the form:

$$\partial_r^2 R + \frac{1}{r}\partial_r R + \lambda^2 R = 0$$

with $\lambda^2 = -i\omega/m$, well-known as the Bessel's equation of order zero. Famously, this has a special converging power series solution called the Bessel function of zero order J_0, for which $R(r) = AJ_0(\lambda r)$ is our solution.

Given that $J_0(\lambda r) = J_0((-i\omega/m)^{1/2}r) = \mathrm{ber}_0((\omega/m)^{1/2}r) + i\mathrm{bei}_0((\omega/m)^{1/2}r)$ as a complex function, where ber_0 and bei_0 are defined to be Kelvin functions (can be numerically computed), we see that $J_0(\lambda r)$ contributes a complex phase to the overall temperature solution $T(r,t)$. This phase ϕ is exactly given by the equation:

$$\tan(\phi) = \mathrm{bei}_0((\omega/m)^{1/2}r)/\mathrm{ber}_0((\omega/m)^{1/2}r)$$

in which we let r vary. Note particularly, due to the symmetry of the problem, we need only consider phase differences and differences in r, since the phase change from one radius to another still has this as a contribution due to shifting the origin in the radial direction to the initial point. Hence, we have:

$$\tan(\delta\phi) = \mathrm{bei}_0((\omega/m)^{1/2}\delta r)/\mathrm{ber}_0((\omega/m)^{1/2}\delta r)$$

If we fix the value of δr and let ω vary as the frequency of an external temperature source on the outer boundary $r_0 + \delta r$, we then can extract the value of m in the relationship between $\delta\phi$ and ω. The goal of this experiment is to achieve exactly this, as will be outlined in the next section.

2 Methods

To do this experiment, we need a way to achieve a spherical symmetric system, in which we know the temperature at some outer radius of a rubber material w.r.t. the origin, while we measure the temperature at the inner radius.

To satisfy this requirement, we used a piece of rubber tubing that encased a thermometer of radius $r_0 = 0.35 \pm 0.05$ cm, which had an approximately cylindrical shape and constant radius $r_1 = 0.8 \pm 0.05$ cm. Note we really only need the difference $\delta r = r_1 - r_0 = 0.45$ cm, which has an inherited uncertainty of about ± 0.1 cm. To avoid any external influences to the internal temperature, we applied a plug at the bottom with some sealant on holes.

Note that this should have marginal effect on the actual measurement of temperature changes due to an external source, but the sealant is a different material. Note only that, but some of this change in temperature is a result of diffusion across the bottom of the tube, which may contribute slightly more to the change.

To provide a consistent periodic source, we submerge the tubing periodically between two beakers of boiling and freezing water respectively. If done fast enough in the time scale given, this resembles a square wave source of temperature difference on the surface, while the measured temperature on the inside should resemble that of a low pass filtered square wave with some phase delay given by the temperature differential traveling through the rubber medium.

In our particular case, we performed this over 5 different periods, given by 120, 150, 180, 210, and 240 second periods to provide some substantial data. We would start by submerging in the hot beakers and then swapping beakers every half period to emulate a square wave. For the measurements in each trial, one of us would look at the thermometer temperature at 5 second intervals using a timer and record the result on a pre-written table, while the other would regulate the beaker changing. We made sure that each period trial had at least 3 to 4 periods worth of data measured, so had more accurate data to work with considering that the tube started at room temperature.

Note that the source temperature of the hot and cold beaker for each trial was slightly different, as we had to refill and reheat the water regularly for each period. Mainly, due to external factors, the slight variability in the structure of the ice used, and inhomogeneity in the heating, we got different values for temperature near 100°C and 0°C. Moreover, we expect that these sources' temperatures would change over each trial as a result of exchanging heat with each other and the surroundings.

To minimize this exchange, we started the experiment as soon as the water started boiling and we retrieved the ice water. The temperatures describing the source square wave are assumed to not change from the start, so we will just use the measured values of initial temperatures for the sources. This should not change the actual analysis of phase differences, as the amplitude does not influence the phase.

However, our data acquisition has some introduced uncertainties in it due to our inexact timing in the mechanical measurement of the temperature every 5 seconds (we were ± 1 second off at maximum from the actual time of measurement due to reaction and apprehension). Not just that, but there an approximate $2-3$ second delay in the swapping between beakers as it takes some time for one of us to adjust the holder, take it out and then put it in over the other beaker. Although this is also very small in hindsight of the range of the periods being a factor of 100 above this, it is still non-negligible and necessary to note.

To measure the time delays, we took the difference between the turning point of the ideal square wave at each time and the next nearest local extremum of the delayed wave. After doing this for each extremum, we take an average over these to give us the most accurate phase difference for each trial. The uncertainty on this result has contributions from the uncertainty on the time measurements and the uncertainty on the position of each turning point, amounting to around ± 4 s of error in time calculations.

Finally, to produce phase differences, we simply multiply the time delay by the angular frequency of that particular period trial, $\omega = 2\pi/T$, since the time delay corresponds to a shift in the time axis equal to that amount, hence expanding out by product with the angular frequency into the phase domain.

Given the period measurements have an uncertainty of ± 3 s as with the turning points, we can compute the relative uncertainty of ω to be equal to that of T, letting us determine that of ω and the time delays to be computable using the quadrature method.

This gives us a good estimate of the phase differences and their uncertainties. Using this, we can consider intersection points of these constant phase lines with the graph of the phase curve for $J_0(e^{-i\pi/4}x)$, which corresponds exactly to the phase of $\mathrm{ber}_0(x) + i\mathrm{bei}_0(x)$. Since the point of intersection $x = (\omega/m)^{1/2}\delta r$, we can then compute m as $m = \omega(\delta r/x)^2$ for each ω and corresponding x. Finally, taking an average should give us our result, or alternatively we can just fit the relation between x and ω with m as a parameter to extract its value.

With this methodology in hand, we can proceed to the analysis of our results.

3 Results

We first consider an example graph of the delayed signal on the inner surface of the tube, compared with the source square wave signal on the outside. Particularly, we have the temperature time graph for the $T = 120$ s period signal, given by: Note that the square wave is modelled by the Python signal package.

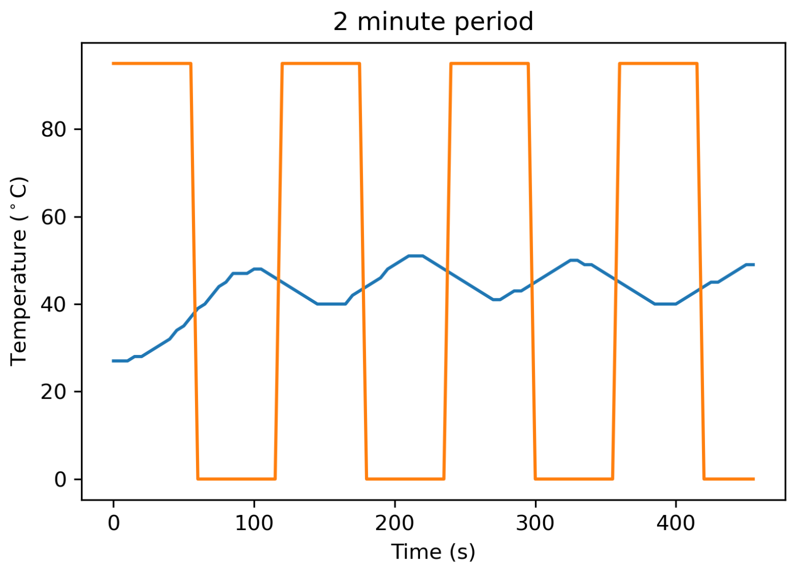

Figure 1: Original temperature signal on outer rubber tube compared to the delayed filtered signal on inner rubber tubing, for period $T = 120$ s.

Observe that there is indeed a delayed response between the turning points of the square wave and those of the sine wave. The rest of the collected data will be available in the tables in the Appendix.

When we measure this delay for each crest and trough, we get a series of time delays with uncertainty equal to the total uncertainty on the time measurements at each turning point of the square wave. If we take an average, multiply by the angular frequency $\omega = 2\pi/T$ of that system, we get our phase differences for each T. The uncertainties are given by standard quadrature computations ($\Delta\omega = 2\pi\frac{\Delta T}{T^2}$ and $\Delta\phi = \phi\sqrt{(\Delta\omega/\omega)^2 + (\Delta T/\delta t)^2}$). (next page)

These are given in the following table:

Period (T / s)	Avg. Time Delay (δt / s)	Frequency (ω / rad/s)	$\Delta\omega$ / 10^{-3} rad/s	Phase (ϕ / rad)	$\Delta\phi$ / rad
120	34.2	0.052	1.75	1.79	0.22
150	32.9	0.042	1.12	1.38	0.17
180	37.1	0.035	0.78	1.30	0.14
210	30.4	0.030	0.57	0.91	0.12
240	29.6	0.026	0.43	0.77	0.11

This gives us a series of phases with their respective angular frequencies. As of now, we cannot relate these two directly with much success, as we have to find the connection between the phase graph of $\text{bei}_0/\text{ber}_0$. To understand the behaviour of these Kelvin functions, we consider their graph:

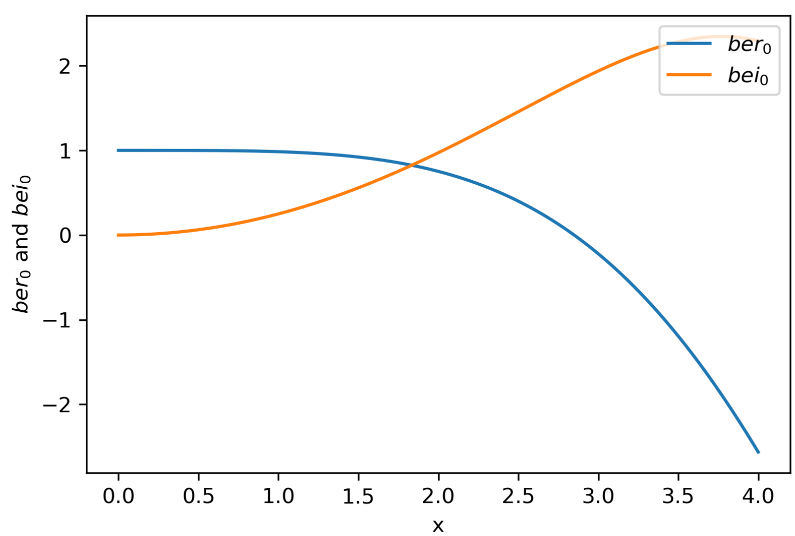

Figure 2: Graph of Kelvin functions as a function of non-dimensional positive real x.

We consider finding the x value corresponding to the phase $\phi = \phi(x)$ as the phase function of J_0. To do this, we search for the intersectionn points of the constant phase lines for each phase difference with the standard phase curve of $\text{ber}_0 + i\text{bei}_0$. (next page)

To understand this process, we look at the intersection graph:

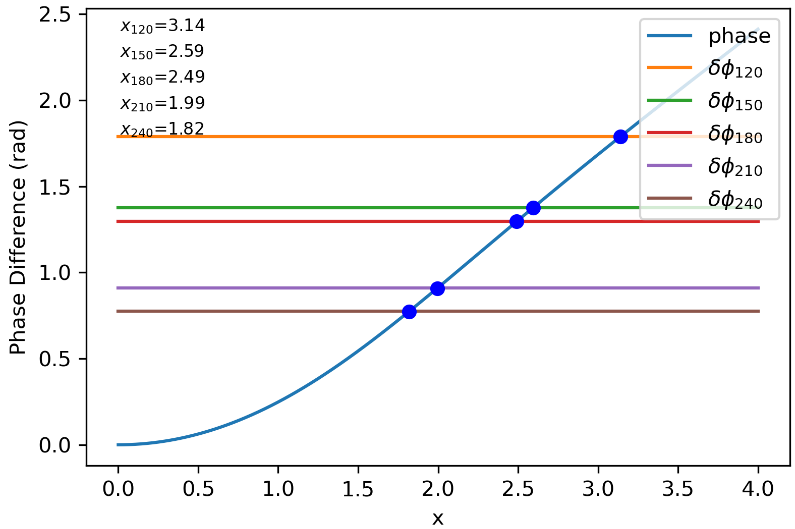

Figure 3: Graph of phase curve intersecting with phase lines and resulting intersection points.

Note that the values presented for the intersection points for each period's phase difference has been solved numerically using a Python algorithm that searches for turning points in sign for the difference between two curves, and so determines the intersection points up to the precision of the increment in the graph. In this case, we have a 4 second interval separated into 1000 increments, which has an error much smaller than the actual uncertainty, so it is negligible.

Observe moreover that the phase curve can be approximated by a shifted identity map $\phi = x + c$ for large enough x, suggesting that the uncertainty of x can be approximated to be the same in the computation. This is justified by observing that the intersections all occur on the part where it begins to resemble this line.

Now, we have a set of x values correlated to phase differences. Recall that in order for this relation to work, $x = (\omega/m)^{1/2} r$ as the argument of the right hand side for our previously derived phase equation. Alternatively, $m = \omega(r/x)^2$, letting us compute m directly for each choice of ω and intersection x. (next page)

Taking the quadrature uncertainty as before, setting the uncertainty of x to be the same of ϕ, gives us the following m values:

Diffusivity (m / 10^{-7} m^2/s)	Δm / 10^{-7} m^2/s
1.08	0.36
1.26	0.42
1.14	0.37
1.52	0.50
1.60	0.52

Note that all of these values are consistent with each and well within range of one another, indicating that there are no significant systematic errors. Moreover, this justifies taking an average of the values and their uncertainties as our final value for m.

That is, we get that $m_a = 1.321 \times 10^{-7}$ m^2/s with uncertainty $\Delta m_a = 0.432 \times 10^{-7}$ m^2/s as our value of m computed in this manner.

We consider an alternate computational method for m to better gauge how good of a value it really is, by comparing the accuracy of the model itself. To do this, instead of just direcctly computing m, we considering performing a fit of the relation $x = (\omega/m)^{1/2}r$ on to our given data of x and ω. This procedure gives us the following graph of the fit in comparison to the measurement:

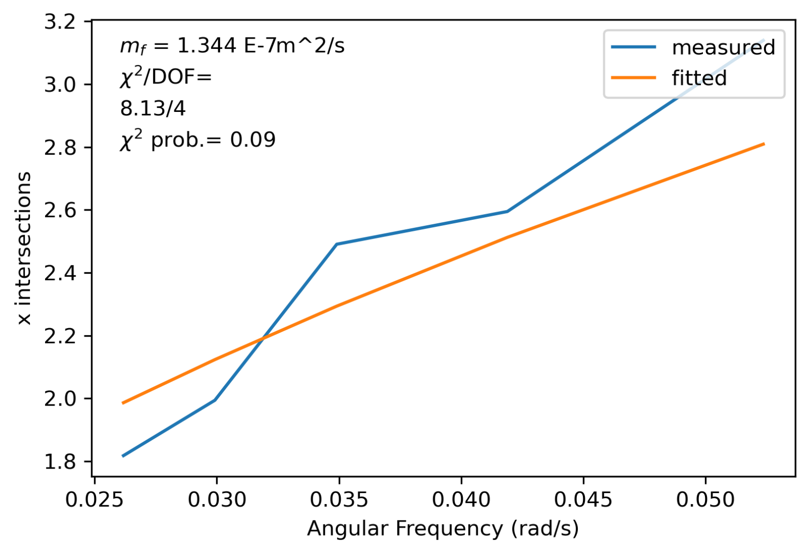

Figure 4: Fitted x vs. ω in comparison with the actual result.

with a fitted value of $m_f = 1.344 \times 10^{-7}$ m^2/s and an uncertainty $\Delta m_f = 1.32 \times 10^{-16}$ m^2/s, which is an absurdly low uncertainty considering the data, so we will just maintain the one computed for m_a as that of m_f. Observe that the χ^2 value is not tremendously high, suggesting this fit has some merit. In particular, it is only 4 above the degrees of freedom, which although is relatively large considering that there are only 4 DOF, still gives it some usability.

Moreover, the chi squared probability is not nonzero, at around 0.09, so although this does mean that there most likely is not sufficient data to properly fit, especially with the uncertainties in the experiment, it still is a somewhat valid fit. Not only that, but coinciding with m_a makes this more concrete.

On another note, an online engineering source claims a value for the diffusivity of general use rubber to be in the range of approximately 0.89-1.3 $\times 10^{-7}$ m^2/s, which is very close to both m_f and m_a, directly coinciding with m_a in its uncertainty interval [1]. Foregoing scrutiny of the source's validity, it does increase confidence in the actual value of m we produced, particularly that of m_a, specifically at room temperature.

Recall however that our rubber is not necessarily ideal–it is not perfectly cylindrical, has some sealant contained in it and possibly some impurities. This could lead to the small disparity that we observe, maybe also possibly contributing to the poor fit we produced in the second computational method.

4 Conclusion

To brief, by gathering data on the delay between a temperature square wave source on the outer surface of rubber tubing and the resulting response wave on the inner tubing–measured via thermometer–we were able to construct a relation between the phase differences and the respective angular frequencies ω.

By using the well-known radial Bessel solution to the problem, we made this indirect relationship easier to fit over by transforming into the x domain of the Kelvin functions' from the phase domain, through the computation of intersection points x between the measured phase lines and the phase curve of J_0. This gives us an easier relation to model between x and m, as $x = (\omega/m)^{1/2}r$.

Finally, we used two methods to compute the diffusivity m of the rubber tubing, first by directly computing by the equivalent equation $m = \omega(r/x)^2$, and then averaging over our values, and then just fitting to the relationship with regression.

The first method by and large produced a more accurate result, reflected in its reasonable uncertainty and its coincidence with the (allegedly) actual value that has much higher confidence. On the other hand, although the second method gave us a result with supposedly much less uncertainty, the fit was less than ideal, with a chi squared double the degrees of freedom, and a relatively low chi squared probability. This suggests that more data would be required to avoid underfitting this data and to truly verify the model.

All in all, this application of phase differences in computing an important constant describing a material shows the utility of these methods when analyzing signals. In general, things like phase differences that pop up as a consequence of the system's structure–in this case being the fact that diffusion over space is delayed–is important to recognize when building an experiment or attempting to model a phenomenon.

References

[1] Enginner's Edge (2023), *Thermal Diffusivity Table*. Last accessed 7 February 2023, `https://www.engineersedge.com/heat_transfer/thermal_diffusivity_table_13953.htm`.

5 Appendix

The temperature-time graphs for the other periods that we did not include, along with their respective square wave signals, are provided here. (next page)

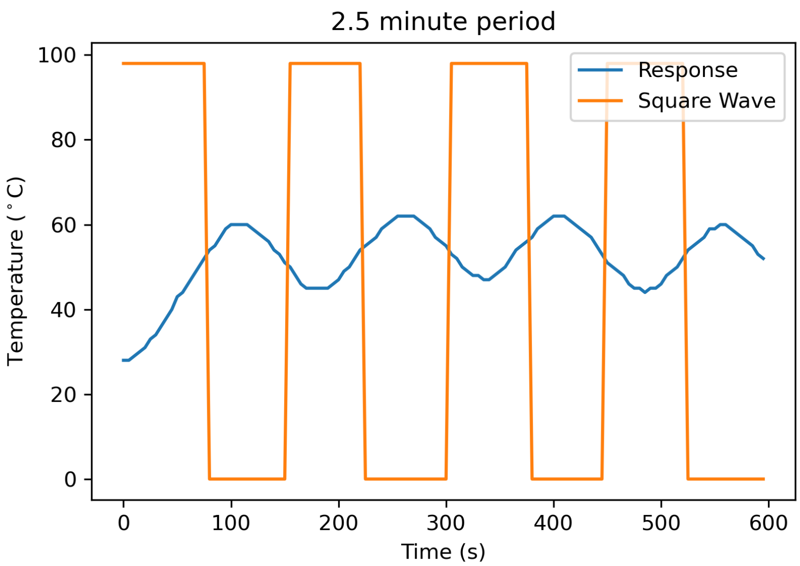

Figure 5: Inner response to square wave of period $T = 150$ s.

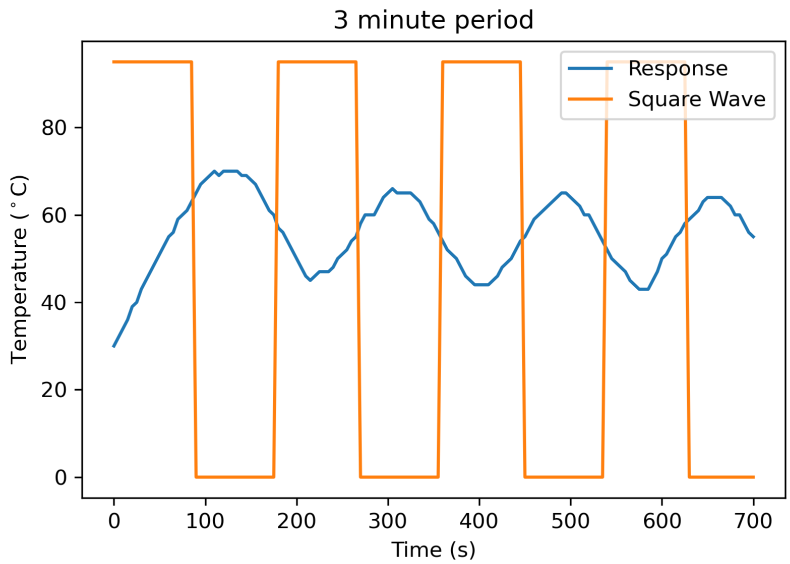

Figure 6: Inner response to square wave of period $T = 180$ s.

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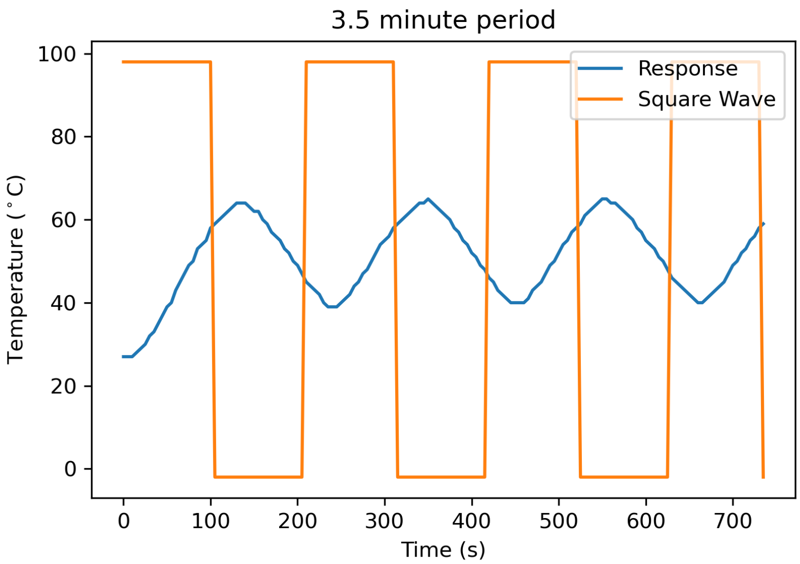

Figure 7: Inner response to square wave of period $T = 210$ s.

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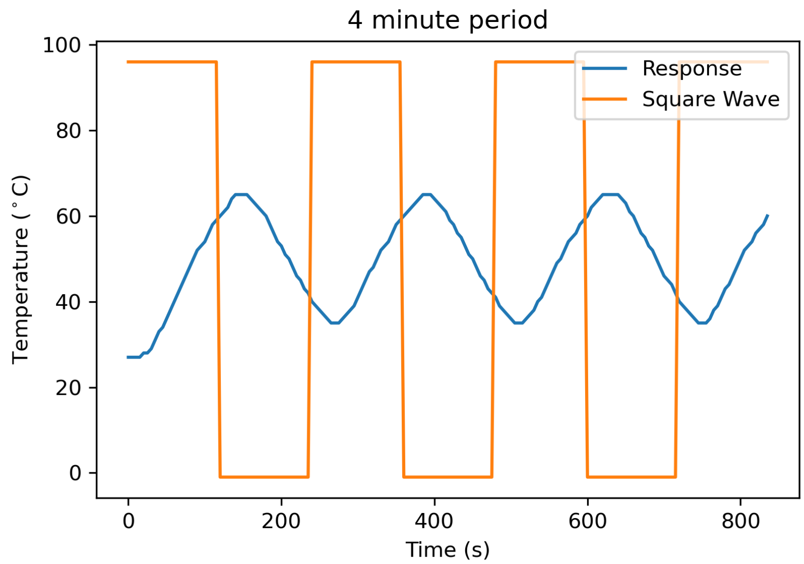

Figure 8: Inner response to square wave of period $T = 240$ s.

This concludes all the data acquisition we performed, and thus all the results of our experiment.