Defining a Wedge Product on Discrete Surfaces

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0.1 Motivation

Recall the double cellular decomposition of a surface $\Lambda = \Gamma \sqcup \Gamma^*$, defined as a pair of dual cellular decompositions of an oriented surface with boundary. By defining k-chains as the $\mathbb{Z}$-module of k-cells, we were able to construct k-forms as homomorphisms of the chains. The dualizing operation also produced a corresponding Hodge star. To continue with the analogy, we wish to define a wedge product on these forms that maintains good properties.

Note that there is only really one meaningful type of wedge product on a surface, since any other product is either:

1. trivial, as the sum of the forms' degrees is larger than 2 or
2. a product of a function with a k-form, which is really just multiplication of the k-form at each point by the value of the function.

This leaves the wedge between 1-forms as the only non-obvious one to define. For a smooth surface S, if we have $\alpha, \beta \in \Omega^1(S)$, we can define, at a point $p \in S$, for a pair of tangent vectors $v, w \in T_pS$, the 2-form $\alpha \wedge \beta$ via the alternating product:

$$(\alpha \wedge \beta)_p(v, w) = \alpha_p(v)\beta_p(w) - \beta_p(v)\alpha_p(w)$$

Note that the usual analogy for smooth k-forms is that they are homomorphisms on the k-chains of the surface. We could see this for the 1-forms, as they provide a linear weight to tangent vectors that can be integrated over a curve. In the case of 2-forms, each pair of vectors form a parallelopiped that we weigh instead, which we integrated over a 2 dimensional surface segment.

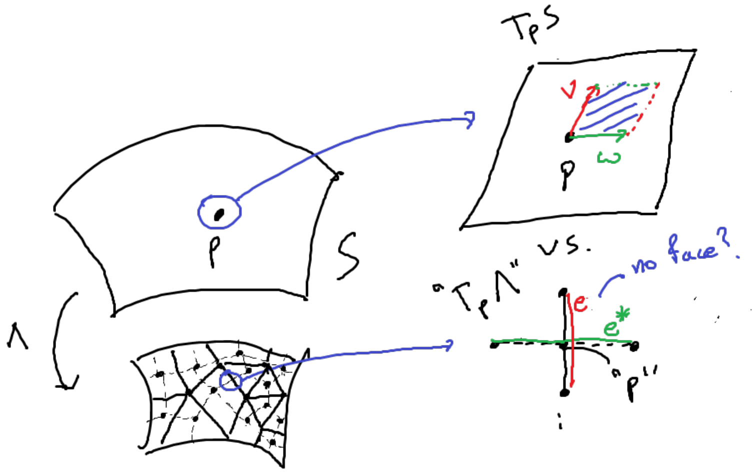

Figure 1: Face on tangent space of smooth surface S generated by (v, w) compared to the "tangent space" of Λ on edge pair intersection

Following this analogy, recall that our notion of "tangent" space on the discrete surface was really just described by intersecting dual edges e, e^*, as our respective "x" and "y" coordinates. We could take two 1-forms on our discrete surface, $\alpha, \beta \in C^1(\Lambda)$ and define a "wedge" product on pairs of dual edges:

$$(\alpha \wedge \beta)(e, e^*) = \alpha(e)\beta(e^*) - \alpha(e^*)\beta(e)$$

In the smooth case, we could always associate (v, w) to a sort of infinitesimal "face" since the surface is smooth, so there are no real restrictions. However, on the discrete surface, for this to be a 2-form, we need to assign (e, e^*) to a unique face in a natural way,. This is not usually possible in general, so we need more faces that we can use, particularly from another cellular decomposition, which we will call the diamond $\Diamond$.

1 The diamond decomposition $\Diamond$

The construction of $\Diamond$ is fairly straightforward using our motivation. From the double Λ, take every pair of dual edges (x, x') and (y, y') and define a new face on $\Diamond$, which we denote $F_e = F_{e^*}$, as the quadrilateral with x, y, x', y' as vertices, and $(x, y), (y, x'), (x', y'), (y', x)$ as edges, as outlined in Figure 2. From now on, we will refer to the edges enclosing a face on the diamond as $e_i(F)$, oriented so that $(e_1 + \cdots + e_4)(F) = \partial F$ and their head and tail as $h_{e_i} = h_i(F)$ and $t_{e_i} = t_i(F)$ for the respective e_i. To make this convenient, we can label $e_{i+4}(F) = e_i(F)$ as identical edges. It is clear that the vertices of the double and the diamond agree, while the edges

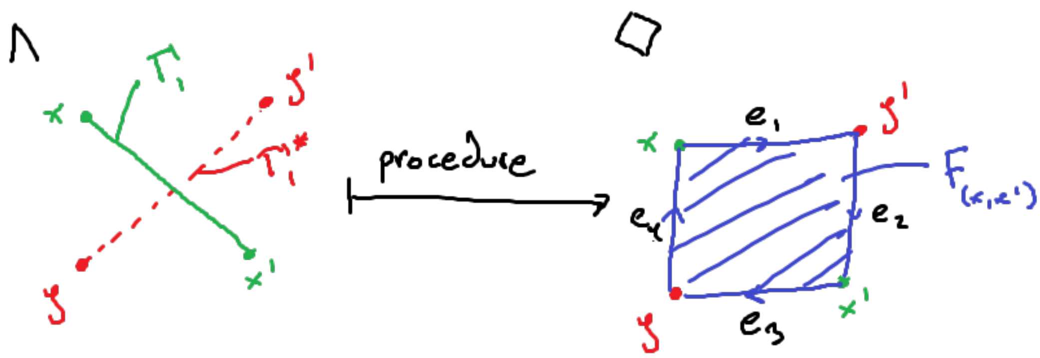

Figure 2: Constructing $\Diamond$ from Λ starting from a pair of dual (x, x') and (y, y') using our rule.

on the diamond now move between the dual graphs. Moreover, $\Diamond$ is connected, unlike Λ. The faces are also restricted to be quadrilaterals, since edge pair provides exactly 4 vertices to form a face from. Interpreting these as rhombi makes sense as well, considering the diagonals are perpendicular and intersect at a midpoint. The geometric analogy to the smooth case is also more clear if viewed this way.

We could alternatively start by defining a quadrilateral cellular decomposition $\Diamond$ that represents a lattice on our surface, and by doing the backwards procedure of connecting antipodal vertices on each face to get back to Λ. Note that this is equivalent to being able to separate the graph into bipartite pieces.

The issue is that this is not always possible, particularly when there exists an odd cycle in $\Diamond$, since there is no way to colour the vertices along the cycle with 2 colours in a way so that there are no adjacent vertices of the same colour (consider the example in 3). We can always resolve this by refining the lattice simply by splitting each face into four pieces and cutting each edge into two, which doubles the length of each cycle.

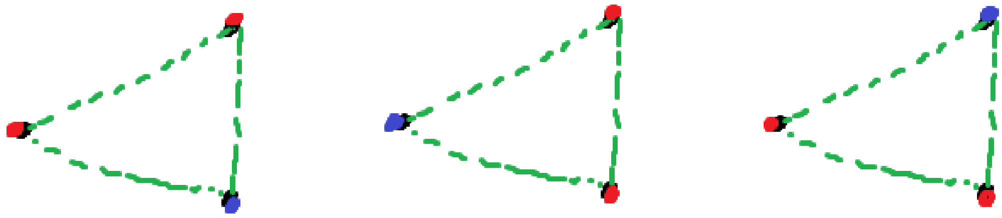

Figure 3: Simplest odd cycle has no red/blue colouring of vertices so no two colours are adjacent.

Definition 1.1. A discrete surface with boundary is a quadrilateral cellular decomposition $\Diamond$ of an oriented surface with boundary paired with a bipartite double Λ.

This alternative definition is more convenient to work with, since $\Diamond$ is like a lattice on our surface, which makes discussion of forms and chains much more natural. Note that k-chains and k-forms are defined in the same way as for Λ, as the $\mathbb{Z}$-module of the cells and respective homomorphisms. We also have a boundary operation $\partial_\Diamond$ and the corresponding coboundary $d_\Diamond$, defined in the same way as on Λ by satisfying $d_\Diamond \omega(s) = \omega(\partial_\Diamond s)$ for some k-form ω and $k+1$ chain s.

1.1 The wedge product $\wedge_\Diamond$ and averaging map A

However, we now don't have the Hodge star since we don't have a way of dualizing faces. Instead, the source of our motivation, the wedge product has an available definition, since we have our parallelopiped faces.

Definition 1.2. The wedge product $\wedge_\Diamond : C^k(\Diamond) \times C^l(\Diamond) \to C^{k+l}(\Diamond)$ is given by the following rules, for $f, g \in C^0(\Diamond)$, $\alpha, \beta \in C^1(\Diamond)$ and $\omega \in C^2(\Diamond)$:

$$(f \wedge_\Diamond g)(x) = f(x)g(x)$$

$$(f \wedge_\Diamond \alpha)(e) = \frac{f(h_e) + f(t_e)}{2}\alpha(e)$$

$$(f \wedge_\Diamond \omega)(F) = \frac{f(t_1(F)) + f(t_2(F)) + f(t_3(F)) + f(t_4(F))}{4}\omega(F)$$

$$(\alpha \wedge_\Diamond \beta)(F) = \frac{1}{4}\sum_{k=1}^{4}(\alpha(e_k(F))\beta(e_{k+1}(F)) - \beta(e_k(F))\alpha(e_{k+1}(F)))$$

for $x \in \Diamond_0$, $e \in \Diamond_1$, and $F \in \Diamond_2$.

The first three definitions are sensible, as we are basically "multiplying" a function with a k-form on that face, and as there is no natural vertex to pick the function's value on, taking the average over the function on the edge or face vertices is necessary.

The wedge of 1-forms should be similar to an alternating product on pairs of adjacent edges describing the face, and as there are four possible such edge pairings, we should provide a contribution from each one, thus the average. We can verify that the wedge product is anticommutative. Sadly, due to the averaging procedure, it is no longer non-degenerate in general.

Proposition 1.3. *$d_\Diamond$ is a graded derivation with respect to $\wedge_\Diamond$.*

Exercise 1. Prove this.

Proof. Note that we only need to prove this for products between functions and functions with 1-forms, as all other wedges are 2-forms and so have vanishing coboundary. For $f, g \in C^0(\Diamond)$ and $(x, y) \in \Diamond_1$

$$d_\Diamond(f \wedge_\Diamond g)(e) = (fg)(\partial e) = f(y)g(y) - f(x)g(x)$$

On the other hand, $(f \wedge_\Diamond d_\Diamond g)(e) = \frac{f(x)+f(y)}{2}(g(y) - g(x))$. By symmetry, the terms $f(x)g(y)/2$ and $f(y)g(x)$ cancel when we add this to $g \wedge d_\Diamond f$, while $-f(x)g(x)/2$ and $f(y)g(y)/2$ add, so we get

$$(f \wedge_\Diamond d_\Diamond g + g \wedge_\Diamond d_\Diamond f)(e) = f(y)g(y) - f(x)g(x)$$

For $\alpha \in C^1(\Diamond)$, we can compute on a face $F = (x, y, x', y') \in \Diamond_2$

$$\begin{aligned} d_\Diamond(f \wedge_\Diamond \alpha)(F) &= (f \wedge_\Diamond \alpha)((x, y) + (y, x') + (x', y') + (y', x)) \\ &= \frac{f(x) + f(y)}{2}\alpha(x, y) + \cdots + \frac{f(y') + f(x)}{2}\alpha(y', x) \end{aligned}$$

On the other hand, we can compute

$$(f \wedge_\Diamond d_\Diamond \alpha)(F) = \frac{f(x) + f(y) + f(x') + f(y')}{4}\alpha((x, y) + (y, x') + (x, y') + (y', x))$$

We can subtract this from our previous term, which gives us

$$\frac{f(x) + f(y) - f(x') - f(y')}{4}\alpha(x, y) + \cdots + \frac{f(y') + f(x) - f(x') - f(y)}{4}\alpha(y', x)$$

Separating each term into pieces of the form $-(f(x') - f(y))\alpha(x, y) = -\alpha(x, y)df(y, x')$ and $(f(x) - f(y'))\alpha(x, y) = df(y', x)\alpha(x, y)$, we get 4 positive terms of the latter form and 4 negative terms of the former. Rearranging so that the latter and former terms on alternating edges are grouped up, our expression is exactly

$$\frac{1}{4}(df(x, y)\alpha(y, x') - \alpha(x, y)df(y, x') + \cdots + df(y', x)\alpha(x, y) - \alpha(y', x)df(x, y)) = ((d_\Diamond f) \wedge_\Diamond \alpha)(F)$$

giving our desired formula. □

Since forms on $\Diamond$ are so useful, we want a way to connect them to forms on Λ. A nice way to attempt this is by define an averaging map.

Definition 1.4. The averaging map $A : C^k(\Diamond) \to C^k(\Lambda)$ is a linear map given as follows, for $f \in C^0(\Diamond)$, $\alpha \in C^1(\Diamond)$, and $\omega \in C^2(\Diamond)$:

$$A(f)(x) = f(x)$$

$$A(\alpha)(e) = \frac{1}{2}\alpha((x, y) + (y, x') + (x, y') + (y', x'))$$

$$A(\omega)(x^*) = \frac{1}{2}\omega(\sum_i F_i)$$

for $x \in \Lambda_0$ and $(x, x') \in \Lambda_1$, with $(y, y') = (x, x')^*$, and $F_i \in \Diamond_2$ are faces with x as a vertex. Clearly, functions on $\Diamond$ are just functions on Λ, hence the natural identification. 1-forms can be identified by taking the average over the two paths on a $\Diamond$ face from opposing vertices that define an edge of Λ. Likewise, 2-forms can be produced by adding the values on faces of $\Diamond$ that intersect the chosen face on Λ. This is described by Figure 4.

Moreover, observe that A preserves the coboundary structure of the forms, as in $Ad_\Diamond = d_\Lambda A$, where d_Λ refers to the coboundary on Λ. Because of this, however, A is not injective.

Example 1.5. The function $\epsilon(x) = 1$ for $x \in \Gamma$ and -1 on the dual has $d_\Diamond\epsilon(x, y) = 2$ for edges moving from Γ^* to Γ vertices and -2 otherwise, while $d_\Lambda\epsilon(x, x') = 0$ since ϵ is constant when restricted to either dual graph. Thus, we get $A(d_\Diamond\epsilon) = 0$, so we have a non-trivial kernel for A.

Recalling the "wedge" product from our motivation, now that we have the diamond, we can associate (e, e^*) to F_e, so we can define a heterogenous wedge product for 1-forms on the double.

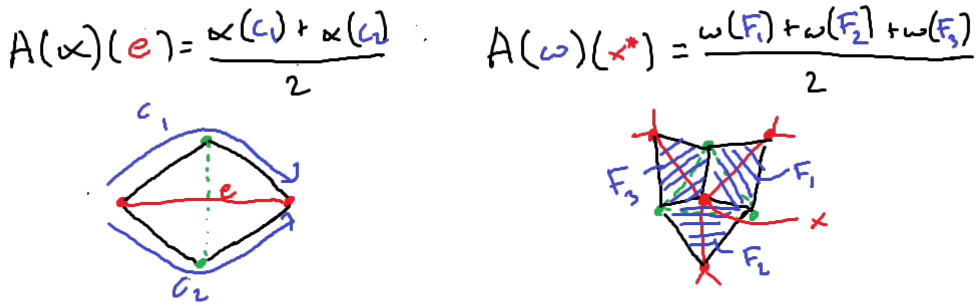

Figure 4: Visualization of the form on Λ taken by A from a 1-form α and a 2-form ω on $\diamondsuit$.

Definition 1.6. The heterogenous wedge product $\wedge_\Lambda : C^1(\Lambda)\times C^1(\Lambda) \to C^2(\diamondsuit)$ for $\alpha, \beta \in C^1(\Lambda)$ is given by:

$$(\alpha \wedge_\Lambda \beta)(F_e) = \alpha(e)\beta(e^*) - \beta(e)\alpha(e^*)$$

We can define a homogeneous wedge product between functions and k-forms on Λ in the usual way, for $f, g \in C^0(\Lambda)$, $\alpha \in C^1(\Lambda)$, and $\omega \in C^2(\Lambda)$:

1. $(f \wedge_\Lambda g)(e) = f(x)g(x)$
2. $(f \wedge_\Lambda \alpha)(x, x') = \frac{f(x)+f(x')}{2}\alpha(x, x')$
3. $(f \wedge_\Lambda \omega)(F) = f(F^*)\omega(F)$

The averaging map also produces a nice relation between the two wedges on 1-forms.

Proposition 1.7. *For any $\alpha, \beta \in C^1(\diamondsuit)$, we have $A(\alpha) \wedge_\Lambda A(\beta) = \alpha \wedge_\diamondsuit \beta$.*

Exercise 2. Prove this too.

Proof. We see that for a face $F = (x, y, x', y') \in \diamondsuit_1$ with (x, x') dual to (y, y')

$$\begin{aligned}4(A(\alpha) \wedge_\Lambda A(\beta))(F) &= \alpha((x, y) + (y, x') + (x, y') + (y', x'))\beta((y, x) + (x, y') + (y, x') + (x', y')) \\ &\quad - \beta((x, y) + (y, x') + (x, y') + (y', x'))\alpha((y, x) + (x, y') + (y, x') + (x', y'))\end{aligned}$$

After expanding, the terms with common edges cancel, of the form $\alpha(x, y)\beta(y, x)$ as they have a positive and negative term, of which there are exactly 8. The 16 terms with completely disjoint edges of the form $\alpha(x, y)\beta(y', x')$ occur twice with the argument flipped in both the positive and negative term, so they also cancel. We are left with only the 8 terms of the format $\alpha(x, y)\beta(y, x') - \beta(x, y)\alpha(y, x')$ after rearranging, which is exactly $4(\alpha \wedge_\diamondsuit \beta)(F)$. ∎

We could also attempt a homogeneous wedge on 1-forms as $A(\alpha \wedge_\Lambda \beta)$, but this does not work always since A has a nonzero kernel. Moreover, this wedge does not make d_Λ a graded derivation, so we sacrifice important properties in the $\wedge$ on the double.

1.2 Useful integral formulas

From now on, we will use the notation $\alpha(S) = \int_S \alpha$ to refer to the action of a k-form on chain S. This will make the idea of integration more clear in the context of this section. Recall that Mercat defines an inner product for 1-forms $\alpha, \beta \in C^1(\Lambda)$, given by

$$(\alpha, \beta) = \sum_{e\in\Lambda_1} \rho(e) \int_e \bar{\alpha} \int_e \beta$$

I have chosen to swap the complex conjugation choice to make it more consistent with more common convention and for later. Note that this only really makes sense for forms for which the norm $|\beta|^2 = (\beta, \beta)$ converges.

Definition 1.8. Let $L^k_n(\Lambda)$ be the set of k-forms α satisfying

$$\sum_{s \in \Lambda_k} \rho_k(s) \left| \int_s \alpha \right|^n < \infty$$

where ρ_k are weights we assign to the k-chains. Denote $L^k_\infty(\Lambda)$ as the set of k-forms that are bounded, so $\sup_{s \in \Lambda_k} |\int_s \alpha| < \infty$. We can define $L^k_n(\Diamond)$ in the same way just by replacing Λ. Note that this space is normed by the obvious choice of

$$\|\alpha\|_n = \left(\sum_{s \in \Lambda_k} \rho_k(s) \left| \int_s \alpha \right|^n \right)^{1/n}$$

Note that this makes sense and coincides with our desired 1-forms so long as $\rho(e)$ are bounded above, which we expect since there should be no accumulation points on either dual graph. In particular, for $\alpha, \beta \in L^1_2(\Lambda)$, we then get the formula:

$$(\alpha, \beta) = \int_{\Diamond_2} \bar{\alpha} \wedge *\beta$$

Recall in the smooth case, the Hodge star is meant to complete a k-form so that when wedged with another after conjugation will produce the volume form multiplied by the inner product of the pair. Since this formula works for any pair from L^k_2, we see that our choice of product is sensible.

For $\omega \in L^2_1(\Diamond)$ and $f \in L^0_\infty(\Diamond)$, we also get the following relations:

$$\int_{\Diamond_2} f \wedge_\Diamond \omega = \int_{\Gamma_2} f \wedge_\Lambda A(\omega) = \int_{\Gamma^*_2} f \wedge_\Lambda A(\omega) = \frac{1}{2} \int_{\Lambda_2} f \wedge_\Lambda A(\omega)$$

We also define another class of functions that are useful when discussing distributions.

Definition 1.9. The space of bump functions $B(\Lambda)$ are the set of all functions $f \in C^0(\Lambda)$ with compact support, defined equivalently on $\Diamond$.

In particular, recall that a harmonic function f satisfies $\Delta f(x) = 0$, and since

$$\Delta f(x) = -\int_{\Lambda_2} f \wedge_\Lambda *\Delta\chi_x$$

for every characteristic function χ_x for $x \in \Lambda_0$, we can just write the condition as being the RHS vanishing. Since the χ_x form a basis for all functions, we can weight and add them up.

Lemma 1.10 (Discrete Weyl's)**.** *$f \in C^0(\Lambda)$ is harmonic iff for every $g \in B(\Lambda)$*

$$\int_{\Lambda_2} f \wedge_\Lambda *\Delta g = 0$$

holds.

The notation so far suggests introducing functional distributions.

Definition 1.11. A distribution $F : B(\Lambda) \to \mathbb{C}$ is a linear functional on the space of compactly supported functions. The space of distributions is denoted by $B^*(\Lambda)$. Again, Λ can be replaced with $\Diamond$.

Since we are living on a discrete surface, the duality between functions and distributions is more clear than when on the smooth surface. We use the notation $F(\psi) = (F, \psi)$ to make this more clear, especially given what follows.

Theorem 1.12 (Reisz Representation)**.** *For any $F \in B^*(\Lambda)$, there exists $f \in C^0(\Lambda)$ for which any $\psi \in B(\Lambda)$ has $(F, \psi) = (f, \psi)$. Moreover, any such f uniquely defines a functional F.*

Proof. Recall that χ_x form a basis for $B(\Lambda)$, so F can be expressed completely by its action on χ_x. Define $f(x) = \overline{F(\chi_x)}$, so

$$\begin{aligned}(f, \psi) &= \int_{\Lambda_2} f \wedge *\psi \\ &= \sum_{x \in \Lambda_0} \int_{x^*} f \wedge *\psi \\ &= \sum_{x \in \Lambda_0} f(x) * \psi(x^*) \\ &= \sum_{x \in \Lambda_0} F(\chi_x)\psi(x) \\ &= F(\sum_{x \in \Lambda_0} \psi(x)\chi_x) = F(\psi)\end{aligned}$$

The other way is obvious by just defining $(F, \psi) = (f, \psi)$ ■

As before, we see that the usual notion of taking derivatives of distributions is preserved. For example, we can define ΔF by $(\Delta F, \psi) = -(F, \Delta\psi)$, which implies Weyl's lemma by our representation theorem.

Lastly, we can produce a sort of Green's theorem on $\Diamond$, if the averaging map A was a surjection. Particularly, we can then define a quotient map $B : C^1(\Lambda) \to C^1(\Diamond)/\mathrm{Ker}A$, and the diamond laplacian $\Delta_\Diamond = d_\Diamond B * d_\Lambda$ defined in the natural way over the equivalence class since A preserves the derivative structure.

Theorem 1.13 (Green's)**.** *For any $f, g \in C^0(\Diamond_0)$ and a compact domain $D \subset \Diamond_2$, we get the formula equivalence for each element of the equivalence classes*

$$\int_D f \wedge_\Diamond \Delta_\Diamond g - g \wedge_\Diamond \Delta_\Diamond f = \int_{\partial_\Diamond D} f \wedge_\Diamond B * dg - g \wedge_\Diamond B * df$$

Proof. This comes from the fact that $d_\Diamond$ is a derivation on $\wedge_\Diamond$, so

$$\begin{aligned}f \wedge_\Diamond \Delta_\Diamond g &= f \wedge_\Diamond d_\Diamond(B * d_\Lambda g) \\ &= d_\Diamond(f \wedge_\Diamond B * d_\Lambda g) - d_\Diamond f \wedge_\Diamond (B * d_\Lambda g)\end{aligned}$$

Using the boundary-coboundary rule, we see that the equation is true up to an additional term

$$\int_D d_\Diamond g \wedge_\Diamond (B * d_\Lambda f) - d_\Diamond f \wedge_\Diamond (B * d_\Lambda g)$$

Since $d_\Diamond g \wedge_\Diamond (B * d_\Lambda f) = d_\Lambda g \wedge_\Lambda *d_\Lambda f$ by applying A on both sides, and as $d_\Lambda g \wedge_\Lambda *d_\Lambda f = d_\Lambda f \wedge_\Lambda *d_\Lambda g$ by unraveling the definitions of the heterogenous wedge, this obviously vanishes. ■

1.3 Additional comments

There are various other topics that could be useful to understand on discrete surfaces, motivated by what we were able to define so far. Particularly, we could discuss analysis of classes of more interesting discrete functions, and look for denseness theorems for these classes, as one does in normal analysis.

We could also discuss a subset of distributions called tempered distributions if our space is not compact and has some form of polynomials on them. For example, if we use the pseudo-powers $z^{(n)}$ defined in Lovasz on the $\Diamond = \mathbb{Z}^2$ lattice in $\mathbb{C}$, we could define a notion of rapidly decreasing functions $f \in S(\Lambda)$ as satisfying the rule $\|z^{(n)} \wedge_\Lambda \Delta^j f\|_\infty < \infty$, as in any product of "derivatives" of f by powers should be bounded. Note that $B(\Lambda) \subset S(\Lambda)$.

We also introduced briefly the notion of weights on all cells, not just edges. This sets the stage for generalizing our theory on discrete surfaces to any higher dimensional discrete manifold.