# Grids and Foliations on Discrete Surfaces

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### 0.1 Motivation

Recall that on integer lattices ($\Gamma = \mathbb{Z}^2$) we were able to define the notion of holomorphic functions using the vanishing loop integral property on the diamond, and found this to be identical to the original definition using finite differences on the double. We wish to do this for a general discrete surface, or at least find conditions that a surface must satisfy for this to be possible.

Particularly, in order to define an integral of a function, we need something similar to $dz$ on $\Diamond$, which would additionally require having some sort of direction for the real and imaginary axis. Essentially, we would need a set of grid lines on our surface in order to make the definitions identical. Equivalently, we could associate a pair of vector fields for the directions, which we will show is what we want.

## 1 Vector Fields and Foliations on $\Lambda$

We start by defining a vector field on $\Lambda$ in the most natural way, as a way of assigning each vertex an edge attached to it. Recall the notation for an edge $e = (t_e, h_e)$ as the head and tail vertex of the edge.

**Definition 1.1.** A discrete smooth vector field on $\Lambda$ or $\Diamond$ is a map $V : \Lambda_0 \to \Lambda_1 \cup \{0\}$ so that $t_{V(x)} = x$, where 0 denotes the vector field is not pointing to any other vertex. We use the notation $h_{V(x)} = x$ if $V(x) = 0$ to represent this idea more clearly. An example of a vector field is provided in Figure 1. We see that as in the continuous case, a vector field has an induced flow $\psi_n(x) = \psi(n, x)$ which satisfies the recursion:

$$x_{n+1} = h_{V(x_n)}$$

$$x_0 = x$$

defined for $n \in \mathbb{N}$. Paths given by the vertices along the flow for fixed $x$ are called integral curves of $V$.

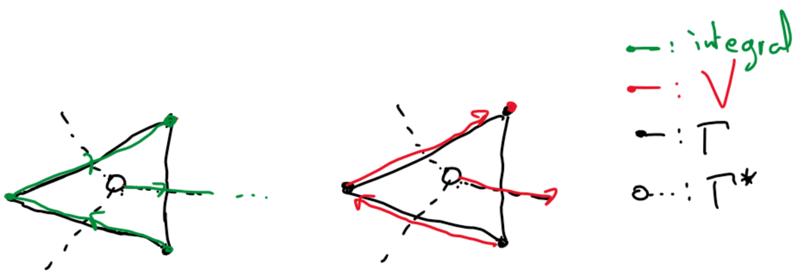


Figure 1: Vector field (in red) given on a section of our double, with corresponding integral curves.

## 1.1 Completeness with singularities and foliations

Just as in the continuous case, these may not be distinct, in the sense that two integral curves can join together. An integral curve may also not have any previous points along it, as depicted in Figure 2.


Figure 2: Scenario depicted where integral curves (green) converge into one point or do not reach a point at all.

This would prevent a way to uniquely reverse the path along an integral curve of the field, that is, a way to extend our flow to all of $\mathbb{Z}$. To allow this, we introduce the notion of a complete field.

**Definition 1.2.** A vector field $V$ is called complete with singularities at the boundary if $h_{V(x)}$ is an injection and its image contains the interior of the surface, and is said to have no singularities if this map is a bijection. Since integral curves are continued applications of $h_{V(\cdot)}$, this implies that we can always invert it to get to previous points for any interior points, since they are in the image. Note that on a compact surface, injectivity of this map is enough, since there can only be finitely many such points, and so it must be bijective as well. The reason for which we do not require boundary points to necessarily have a point "behind" it is that we do not need to extend curves past the boundary, so this weaker notion of completeness with singularities is preferred.

We see that our integral curves also partition the surface's vertices, and so are a sort of "foliation" of the surface into "curves".

**Definition 1.3.** A foliation of our discrete surface is a partition of the surface into disjoint curves called leaves $L_\alpha$, where each leaf is an integral curve of a complete (with singularities) field, or equivalently just some path (see Figure3 for reference).

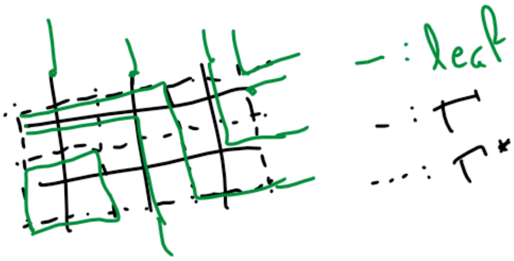


Figure 3: Example foliation with leaves labelled in green.

## 1.2 Nonsingular noncrossing fields and grid foliations

To ensure that the integral curves are actually one dimensional and not just points on the surface, we want the following property to be satisfied.

**Definition 1.4.** A vector field $V$ is called nonsingular if it is nonzero everywhere except possibly at the boundary, where it vanishes only in the case where it is in the image of $h_{V(x)}$ applied to the interior (an interior point flows to boundary point where it stops). This condition ensures that a field's flow does not get trapped at the boundary and just stops the moment it hits the boundary. Note that the integral curves of nonsingular fields will never be single points. We can combine this with another condition to get nicer integral curves.

**Definition 1.5.** A field $V$ is noncrossing if there are no two $x, y \in \Lambda_0$ for which $V(x) = \pm V(y)^*$ (no perpendicular intersections) or $V(x) = -V(y)$ (no double edges). These conditions are depicted in Figure 4.

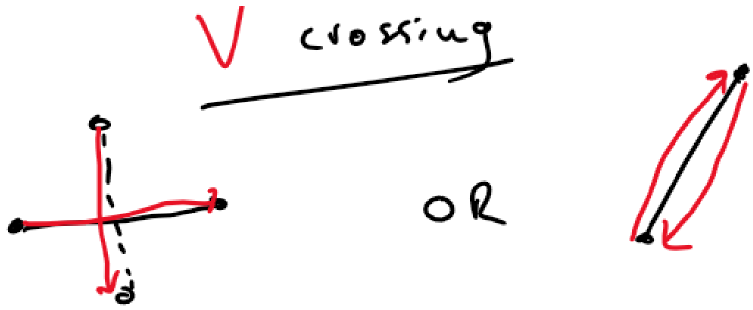


Figure 4: The scenario where a vector field is crossing itself corresponds to either dual edges being covered by the field or the same edge.

This condition, when paired with nonsingularity, gives us integral curves that do not cross each other or themselves on $\Lambda$ and are not identically just points. This also imposes a condition on the integral curves, which prevents them from being boundary curves.

**Proposition 1.6.** *If a nonsingular field $V$ has an integral curve that is a boundary of some finite region on the surface, then it cannot be noncrossing.*

**Exercise 1.** Prove this.

*Proof.* Suppose we have a nonsingular field with an integral curve $I$ that is a boundary of some finite region $\Omega_0 \subset \Gamma_2$ or $\Gamma_2^*$. Pick any point $x_0$ that is the dual to some face interior to $\Omega_0$ and observe that if $V$ were to be noncrossing, the integral curve starting at $x_0$ would not be allowed to move outside of $\Omega_0$. That is, $\psi_n(x_0)$ would be trappped in $\Omega^*$, as otherwise it would cross over $I$. This implies that $\psi_n(x)$ can only take on the value of the finitely many dual vertices of the faces in $\Omega$, which is only possible if $\psi_n(x_0)$ form a cycle, which in turn is a boundary of some region $\Omega_1$ given that $\Omega_0$ is necessarily simply connected if its boundary is a single connected component, with the number of faces dual to vertices on the same graph as the cycle strictly less than the number of faces in $\Omega_0$, as in $\Omega_1^* \subsetneq \Omega_0$. Repeating this process by picking a new $x_1 \in \Omega_1$ and so on gives us a sequence of strictly decreasing regions satisfying $\Omega_{k+1}^* \subsetneq \Omega_k$ with $\psi_n(x_{k+1})$ as a boundary.

For some large enough $k$, the region reduces to the form of a group of $N$ faces that are adjacent to each other, so that their dual vertices form a minimally connected path (removing an edge will disconnect it always), as in faces that are directly side-by-side with at least one and at most two faces adjacent to each face. This is because any other configuration would allow a noncrossing cycle, which we can instantly reduce to a smaller case. The only cycles possible while remaining inside require doubling an edge in reverse, which contradicts the noncrossing property, for $N > 1$. For $N = 1$ (a single face), a noncrossing vector field would be trapped at the vertex dual to the given face, which would imply it is singular, another contradiction. $\square$

**Definition 1.7.** A grid foliation is a foliation produced from a complete (with singularities) nonsingular noncrossing field.

This class of foliations is particularly useful, as we now have one-dimensional integral curves that do not cross each other, partition the surface, and terminate at the boundary, giving us our desired grid lines, at least in one direction. The usefulness of this comes from the way the leaves of such a foliation are spaced.

**Lemma 1.8.** *For any leaf $L \in F$ of a grid foliation, there exists at least one and at most two leaves $L_1$ and $L_2$ so any point $x \in L$ is adjacent to at least one point in $L_1$ or $L_2$, $L_1$ and $L_2$ have no pairwise adjacent points, and there are no points adjacent to $L$ not on $L_1$ or $L_2$ (nothing in between).*

*Moreover, there exists at most two leaves $\tilde{L_1}$ and $\tilde{L_2}$ that are wedged in between $L_1$ and $L$, and $L$ and $L_2$, respectively, on the dual graph. That is, any edge from $L_1$ to $L$ is dual to an edge on $\tilde{L_1}$ and likewise any edge on $\tilde{L_1}$ is dual to one between $L_1$ and $L$. The same goes for $L_2$ and $\tilde{L_2}$.*

Before we prove this, we need the following property for grid foliations, which basically states that grid foliations cannot have junctions which have more than two entrances and exits (see Figure 5).

**Proposition 1.9.** *Given a grid foliation $F$, there is no face on the surface with vertices touched by three or more leaves.*

*Proof.* Suppose this is not true, which implies that we can partition the edges into $n \geq 3$ sets on the face, based on which pair of $n$ leaves they are between along the face boundary. Since the dual of the face is covered by the leaves, we require a leaf passing through one of these sets of edges and out another, since they cannot cross the other three leaves. This prevents any other paths through the face, which means all other paths on this junction either have to bounce off, which requires at least three additional edges for the face on the edge touching any of the original three leaves (one for entering and leaving and one adjacent to the leaf). We can apply the same argument for the vertex dual to this face, and repeat endlessly to get the requirement that some face has a number of edges as large as we want. This is not possible for any discrete surface on a smooth surface, as its diamond lattice would then have arbitrarily small squares. $\square$

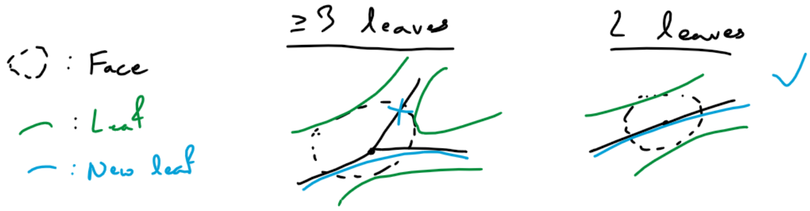


Figure 5: If more than two leaves touch a face, it will produce a sort of split into different paths a leaf can "walk" down, which cuts off access to the other paths after some leaves have already gone through the junction.

A nice corollary is that a face has boundary vertices only touched by at most two leaves of a grid foliation, and any other face that shares a common edge not on either of the leaves also touches those leaves. In fact, only two edges along a face can be left exposed, as if more than two edges are not covered, either there is a point not touched by either leaf, which again would mean that three leaves touch, or one leaf contacts the face at two disjoint edges, which would imply a "pocket" of faces that has boundary covered by one leaf everywhere but at a single edge, and would prevent any of the internal dual vertices being covered by a leaf since it would have to terminate on the interior. We can use this to prove our lemma.

*Proof of lemma.* Given a leaf $L \in F$ of a grid foliation, take $A$ to be the points adjacent to $L$, and $L_i$ the leaves covering $A$. Check first that no two $L_i$ are pairwise adjacent, as in they cannot have any adjacent vertices, as otherwise we could pick $L_i$ and $L_j$ with a region enclosed by the cycle chosen by moving along $L$, crossing over to $L_i$, to the point adjacent to $L_j$ and crossing again, and then to the point adjacent to $L$ and closing loop.

The vertices inside would have to be covered by another leaf, which cannot cross $L_i$, $L_j$ or $L$, meaning that the leaves that do this can only enter and leave through the edges in between the three. If there are any vertices from the same graphs the $L$ and $L_i$ reside on, then a leaf covering them would be trapped in this region, which would be a contradiction since it would necessarily form a boundary, so this is not possible. In the other case, the cycle encloses a face, which has a dual vertex. It is clear by our proposition that this is not allowed, so adjacency is not possible.

Our corollary implies that all the faces between each $L_i$ and $L$ can only be adjacent to each other off of the edge connected to $L$, which form together a connected component necessarily since the leaves are connected. As the surface is two dimensional and smooth, there can only at most be $n = 2$ such $L_i$, since having more than two disjoint components that lie on a smooth surface that intersect along a line would imply the surface is not smooth there. Clearly, all the vertices on $L$ would have to be on a crossing edge of one of the faces in either connected component, which implies that it has to be adjacent to a vertex on one of the leaves we pick. We take these leaves to be our desired $L_1$ and $L_2$.

Moreover, the faces in either component have dual vertices that form a distinct path, and as a leaf must cover all of these vertices, this is only possible if this path is a leaf, since it cannot cross over the leaves that the faces are bound between. We pick the corresponding $\tilde{L_1}$ and $\tilde{L_2}$. □

The essence of this result is that a grid foliation form nicely arranged evenly spaced lines that split up the space, as we would like from grid lines.

## 1.3 Existence of dual foliations

On the integer lattice, we have grid lines that go in two directions, not just one. That is, we want some sort of dual foliation.

**Definition 1.10.** Given a foliation $F$, a dual foliation $F^*$ is a foliation such that any edge $e$ on a leaf in $F$ has $e^*$ in some leaf in $F^*$, and likewise any edge on $F^*$ has a dual in $F$. As a note, observe that the dual foliation of a foliation may not even exist, since the condition is so strong and restrictive. This is because we essentially take all the edges in $F$ and dualize them to get the ones in $F^*$, which might not end up with a foliation, much less a cover of the space. However, when it does exist, it is a necessary condition that each face has boundary touched by a leaf at most twice, as otherwise when taking the dual, the dual vertex of the face will not flow along a unique integral curve.

In particular, we know grid foliations satisfy this condition, so we would expect them to be dualizable. Unfortunately, this is not true in general. Not only that, but the dual may not be a grid foliation either. When it is, however, it behaves like an integer lattice.

**Theorem 1.11** (Quad. Grid)**.** *The dual foliation of a grid foliation $F$ is a grid foliation iff the surface $\Lambda$ is a dual quadrilateral cellular decomposition, as in both $\Gamma$ and $\Gamma^*$ are composed of only quadrilateral faces (see Figure 6).*

*Proof.* The backwards direction is clear since both dual graphs being quadrilateral implies that each interior vertex is connected to exactly four other vertices. If we pick any vertex, and then choose a direction given by pairs of opposing edges (the edges connected to the vertex that do not lie on the same face), following this will produce two leaves, one we can pick for $F$ and the other for $F^*$. Repeating this for a new vertex by choosing the ones that do not intersect as part of the respective foliations, gives us two distinct grid foliations, with our desired property.

The forward direction requires some work. Start with a face and pick an edge $e$ on its boundary, from some leaf $L \in F$ with the dual edge $e^*$ on leaf $L^* \in F^*$ that passes into the face to meet its dual vertex. Moving along $L^*$ in the direction of the face (into the face) will give us another edge $(e^*)' \in L^*$, which by duality will have an edge $e'$ on some leaf $L'$ in $F$. Since both $e^*$ and $(e^*)'$ pass through our face, $e'$ and $e$ both lie on the boundary of the face, so $L$ and $L'$ are adjacent. Moreover, since there can only be exactly two edges that are not covered by $L$ and $L'$, and as there is only one configuration of edges that would permit this, given by edges connecting the heads and tails of $e$ and $e'$ respectively, this implies that our face is enclosed by exactly 4 edges. We can do this for any face, which gives our result. □

Note that we can find a weaker condition that guarantees a dual grid foliation.

**Proposition 1.12.** *The dual foliation of a grid foliation $F$ is also a grid foliation iff none of the faces on the discrete surface are triangles (see Figure 6)*

*Proof.* That is, any interior point has a dual face, which either is a triangle or necessarily has at least two boundary edges covered by leaves. If every face satisfies the latter condition only (no triangles), we get our result, since the existence of a dual would imply that the duals of the two edges on the dual face would form a leaf that passes through the interior point we started with, which implies that the leaves of the foliation do not vanish anywhere. The noncrossing property transfers from the original grid foliation, since if there was a cross, we could dual it back and would get a cross in the starting foliation, which is a contradiction. Together, we get that the dual has leaves that are integrals of a nonsingular noncrossing complete vector field. □

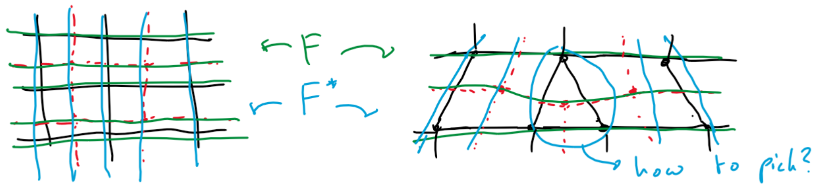


Figure 6: Left depicts a scenario where dual foliation (in blue) is possible given foliation (in green), while right shows an example with a triangle preventing this.

The existence of a pair of grid foliations is an extremely strong requirement, and restricts our attention to only quadrilateral decompositions. We will call any surface that permits this a grid foliable surface. There aren't many compact grid foliable surfaces.

**Example 1.13.** A compact surface without boundary with a quadrilateral cellular decomposition will count 4 edges for each vertex, overcounted twice for each edge since two vertices count it, so the number of edges would be exactly twice the vertices on each dual graph. Since the number of dual faces $\#(F^*) = \#V$ and vice versa, while $\#(E^*) = \#E$, Euler's characteristic is exactly

$$\chi = \#F - \#E + \#V = \#(F^*) - \#(E^*) + \#(V^*)$$

adding it to itself and observing that $\#(F^*)+\#V+\#(V^*)+\#F = 2\#V+2\#(V^*) = \#E+\#(E^*)$, so $2\chi = 0$, or $\chi = 0$ for our compact surface. The only two surfaces that satisfy this are the torus and the Klein bottle. Since the torus is the only one that is orientable, it is of particular interest to us. ■

Note that for non-compact (unbounded) surfaces, we cannot calculate $\chi$ in this way, so the same restriction does not always apply. For example, the regular plane admits a quadrilateral decomposition and a dual grid foliation pair, but its characteristic is 2.

### 1.4 Canonical 1-form

Suppose we have an orientable grid foliable surface. We can then choose a "positive" and "negative" direction along each leaf of a pair of grid foliations $(F, F^*)$ that is consistent with the orientation, as in the tangent vectors corresponding to the directed edges using our choice at each intersection point have positive orientation in the order of $F$ then $F^*$. We denote this choice of direction by a $+$ or $-$ sign given by $\mathrm{Or}_F(e)$. Note that to have positive orientation, using the complex structure to rotate the first vector by $pi/2$ should result in the second vector in the orientation, which corresponds to the dual of a positive orientation edge in $\Gamma_1$ being a positive orientation edge on $\Gamma_1^*$.

Using this, we can define a nice analogue to the standard $dz$ form on the integer lattice. We call this the canonical 1-form.

**Definition 1.14.** The canonical 1-form $d\xi_F$ of a grid foliation $F$ is a 1-form on $\Lambda$ given by

$$d\xi_F(e) = \begin{cases} \mathrm{Or}_F(e)l(e), e \in \Gamma_1 \\ i\mathrm{Or}_F(e)l(e), e \in \Gamma_1^* \end{cases}$$

For now, we will omit $F$ and assume we already have a grid foliation on a grid foliable surface. Using this in a similar way to $dz$, we can write a good condition for holomorphicity on a general grid foliable surface.

**Theorem 1.15.** *A 1-form $\omega$ is holomorphic iff it is closed and $\omega \wedge d\xi = 0$, where the wedge product is done on the double.*

*Proof.* The equation $i\omega = - * \omega$ is identical to, for $e \in \Gamma_1$ following the orientation:

$$i\omega(e) = \rho(e^*)\omega(e^*)$$

multiplying out $l(e^*)$ gives us $il(e^*)\omega(e) - l(e)\omega(e^*) = 0$, which is just the equation $(\omega \wedge d\xi)(F_e) = 0$, where $F_e$ is the face on the diamond produced by $e$. The same calculation works for positive orientation $e \in \Gamma_1^*$, except now $*\omega(e) = \rho(e^*)\omega(e^*)$ without the additional negative sign cancelling. By definition, rotating $e$ to $e^*$ is now negative orientation, since $-e^*$ rotates to $e$ under dualization. Using this, the equality $il(e^*)\omega(e) + l(e)\omega(e^*) = 0$ is just $-d\xi(e^*)\omega(e) + d\xi(e)\omega(e^*) = 0$ after multiplying $i$, so we get the same result. We can just check the negative orientation edges by using linearity and taking the negative of positive edges. □

Note that $d\xi$ may not be closed or exact, depending on the weights we assign the surface's edges. However, when it is closed, it is holomorphic, since $d\xi \wedge d\xi = 0$ via antisymmetry. A quasi-criticality condition on the surface has to be imposed for this to be possible.

**Lemma 1.16** (Quasicritical canonical surface)**.** *Suppose we have a grid foliable discrete surface with a grid foliation $F$. If for any face on $\Lambda$ we have the opposing edges along its boundary have the same length value assigned, where opposing means that they do not have common vertices, $d\xi$ is closed.*

*Proof.* Observe that on the boundary of a face $\partial A = e_1 + e_2 + e_3 + e_4$, where the length of $e_1$ is the same as $e_3$ and likewise for the other two, picked so that $e_1$ and $e_2$ follows the direction of orientation, and $e_3$ and $e_4$ oppose it. We get that:

$$d\xi(\partial A) = l(e_1) - l(e_3) + i(l(e_2) - l(e_4)) = 0$$

so $d\xi$ vanishes on all boundaries, which means that it is closed. □

Note that this limits our grid foliable surfaces to those where faces are essentially parallelograms. Not only that, but this condition necessitates that lined up opposing edges all have to have equal length, which basically means that the rhombuses have to align with each other in a way that resembles a grid distorted along the $x$ and $y$ axes given by the foliations and sheared to form the parallelograms.

## 1.5 Equivalent definitions of discrete analytic functions

Note that in some cases, we can define a 1-form $dz$ on the diamond that satisfies $A(dz) = d\xi$ ($A$ is the averaging map). In this case, we see that for any function $f$, $df \wedge d\xi = df \wedge dz$, since $A(df) = df$ and the averaging map preserves the wedge. Hence, the holomorphicity of $f$ is equivalent to the requirement that $df \wedge dz = 0$, since $df$ would then be a holomorphic 1-form. The scenario where our surface is quasi-critical would additionally imply $dz$ is closed, since the averaging map preserves the coboundary, and the kernel of the averaging map on exact 2-forms is null. Since $d(f \wedge dz) = df \wedge dz + df \wedge d^2 z = df \wedge dz = 0$, our condition is equivalent to $f \wedge dz$ being a closed 1-form.

In Lovasz, one would write this as $\oint_C f \wedge dz = 0$ on any cycle $C$, reflecting the equivalence between the notions. The key difference is that Lovasz requires at the very least introducing a closed $d\xi$, which is only possible when we have a quasi-critical grid foliable discrete surface. A sufficient condition to guarantee that $dz$ exists is that the distortion along the foliations is the same, so the discrete surface is made up of identical rhombi only, which is exactly the condition of criticality in Mercat.

## 1.6 Closing remarks and additional topics

The aforementioned results display the key advantages that quadrilateral decomposed discrete surfaces have. A particular sub-class may be especially useful and interesting, those of critical quadrilateral surfaces, since it induces a nice canonical 1-form on the double and diamond that behaves identically to how $dz$ behaves on the integer lattice.

This gives us a good way to integrate functions as if they are on the regular plane, and also a method of identifying a "derivative" function for a given function $f$ like one would in the continuous case. That is, the space of 1-forms really is a 2-dimensional module under the ring of Lovasz-critical functions with wedge product as scalar multiplication, so we can pick an orthogonal linearly independent 1-form $d\bar{z}$ under the wedge-induced inner product, for which we can then write:

$$df = g \wedge dz + h \wedge d\bar{z}$$

for critical $g$ and $h$. If $f$ is holomorphic, we would require that $df \wedge dz$ vanishes, which means the last term $h \wedge d\bar{z} \wedge dz$ vanishes since the first term vanishes given $dz \wedge dz = 0$. By criticality of $h$, it must vanish, so we are left with $df = g \wedge dz$ for some critical $g$, which we can label $g = f'$ uniquely since the kernel of the wedge on critical maps is null.

This definition turns out to be identical to the one presented in Mercat, which does not use critical functions but instead defines the derivative unique up to the alternating graph map $\varepsilon$ (since we can identify critical functions with the equivalence classes modulo $\varepsilon$). The $dz$ form is constructed using the coboundary of the critical map from the universal covering space of the surface with some singularities punctured out to C, and as the corresponding 1-form on the covering space descends to a 1-form onto the surface. We can produce a correspondence between critical maps restricted to the discrete surface and the canonical choice of $dz$, but that is beyond the scope of this discussion.