Eigenvalue Bounds on Regular Graphs

{BethAnna Jones, Supervisor: Dr. Cesar Aguilar}

State University of New York at Geneseo

eaj9@geneseo.edu, aguilar@geneseo.edu

Introduction

Combinatorial graphs can be used to represent and analyze an endless number of complex systems. Understanding these systems can come from studying the graphs and their corresponding eigenvalues and what they have to say about structure. We investigate the bound on these descriptive eigenvalues to better understand the minimum eigenvalue gap for regular graphs.

Background

A **graph** G consists of two sets V and E where the edge set E is a set of two-element subsets of the vertex set V. If there exists an edge $\{u, v\} \in E$, we say u and v are **adjacent**. Every graph can be represented using these adjacencies between vertices in an $n \times n$ **adjacency matrix** A where $n = |V|$.

Patterns in the values and distribution of the eigenvalues of A reflect patterns in the structure of the graph. Let the eigenvalues of a graph G be labeled such that

$$\lambda_{\min} \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1.$$

The **dominant eigenvalue** λ_1 and its corresponding eigenvector $\mathbf{v}_1$ reveal significant information such as the graph's density, diameter, connectivity, and number of components.

We define the **degree** of a vertex $v \in V$ as the number of vertices adjacent to v. When every vertex in a graph G has degree k, we say G is **k-regular**.

It is known that the dominant eigenvalue for every k-regular graph is k with the all-ones vector e as its eigenvector. This is not very informative, so as an alternative we look to the second dominant eigenvalue Λ which also can provide a lot of information.

Note that the entries of the main diagonal of an adjacency matrix are all zero and thus $\mathrm{tr}(A) = 0$. Since the trace of a matrix is also a sum of its eigenvalues, the eigenvalues of A must be both positive and negative except for the trivial empty graph in which $|E| = 0$ and has all zero eigenvalues. Since dominance is based on magnitude, either $\Lambda = \lambda_{\min}$ or $\Lambda = \lambda_2$ with respective eigenvectors $\mathbf{v}_{\min}$ or $\mathbf{v}_2$.

The magnitude of the difference $\lambda_1 - \lambda_2$ known as the **minimal eigenvalue gap**, or spectral gap, is currently an active topic of research today and can indicate a graph's density and connectivity.

References

[1] M. Meringer, Fast Generation of Regular Graphs and Construction of Cages, Journal of Graph Theory 30, 137-146, 1999.

[2] B.D. McKay, B.D. and A. Piperno, Practical Graph Isomorphism II, Journal of Symbolic Computation, 60 (2014), pp. 94-112

[3] Clemens Brand, Barry Guiduli, and Wilfried Imrich. *Characterization of Trivalent Graphs with Minimal Eigenvalue Gap.* Croatica Chemica Acta, 80(2):193-201, 2007.

Acknowledgements

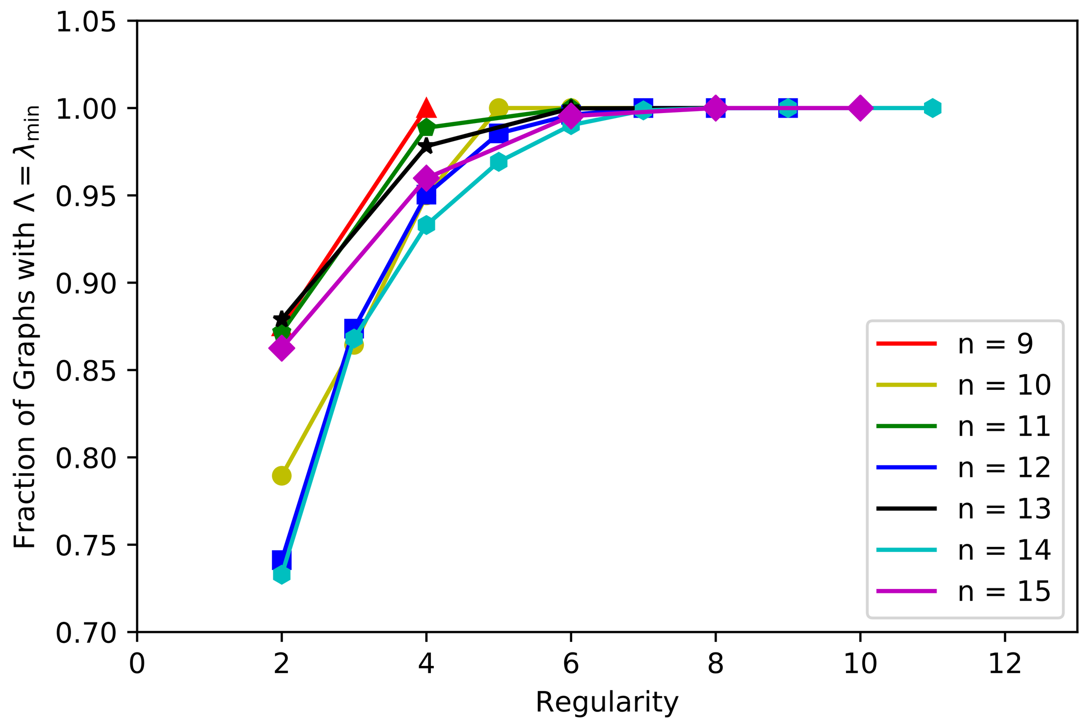
This work was supported by the National Science Foundation under Grant No. ECCS-1700578.

Methods

All connected regular graphs on $9 \leq n \leq 24$ were generated using the genreg [1] program, returned in nauty's ASCII g6format [2], and converted into their corresponding adjacency matrices using the numpy package of Python. From these, we were then able to calculate their eigenvalues and determine eigenvalue bounds.

Results 1

To better understand the minimal eigenvalue gap of a graph with moderate k, we first investigate the bounds of the extreme cases. Combining data from Figure 1 (left), we examine $k = 3$ and $k = n - 4$. It can be shown using the Handshaking Lemma that for any graph G with odd n, G must have even k. Therefore we considered only connected graphs with even n when analyzing eigenvalue bounds.



	$k = 3$		$k = n-4$	
n	$\lambda_{\min}$	λ_2	$\lambda_{\min}$	λ_2
10	$[-3, -2]$	$[1, 2.778]$	$[-4, -2]$	$[1, 2]$
12	$[-3, -2]$	$[1.532, 2.832]$	$[-4, -2.532]$	$[0, 2]$
14	$[-3, -2.090]$	$[1.414, 2.895]$	$[-4, -2.414]$	$[1, 2]$
16	$[-3, -2.039]$	$[1.732, 2.916]$	$[-4, -2.732]$	$[0, 2]$
18	$[-3, -2]$	$[1.732, 2.938]$	$[-4, -2.732]$	$[1, 2]$
20	$[-3, -2.063]$	$[1.935, 2.948]$	$[-4, -2.935]$	$[0, 2]$
22	$[-3, -2.043]$	$[2, 2.959]$	$[-4, -3]$	$[1, 2]$
24	$[-3, -2]$	$[2, 2.965]$	$[-4, -3]$	$[0, 2]$

Figure 1: The fraction of graphs on n vertices with $\Lambda = \lambda_{\min}$ (left) and computed bounds on the potential second dominant eigenvalues $\lambda_{\min}$ and λ_2 for the extreme regularities $k = 3$ and $k = n - 4$.

Results 2

From Figure 1, $\Lambda = \lambda_{\min}$ on all n for $k = n - 4$. In fact, we found that for all connected graphs if $k \geq n - 4$, then $\Lambda = \lambda_{\min}$. Note that as n increases for $k = 3$, spectral gap $\lambda_1 - \lambda_2$ decreases as λ_2 approaches k. Figure 3 shows a typical example of the distribution of eigenvalues for these extreme regularities on $n = 20$.

Brand, Guiduli and Imrich showed that there is a unique 3-regular graph that maximizes λ_2 for each n [3]. Let $x(n)$ be the number of 3-regular graphs that minimize λ_2. We found that for $n \leq 18$, there is also a unique 3-regular graph that minimizes λ_2. However, for $n \geq 20$ there is no longer a unique graph that minimizes λ_2. Due to round-off error, it is difficult to be sure if it is true for $n = 20$ as well.

n	$x(n)$
10	1
12	1
14	1
16	1
18	1
20	2
22	4
24	5

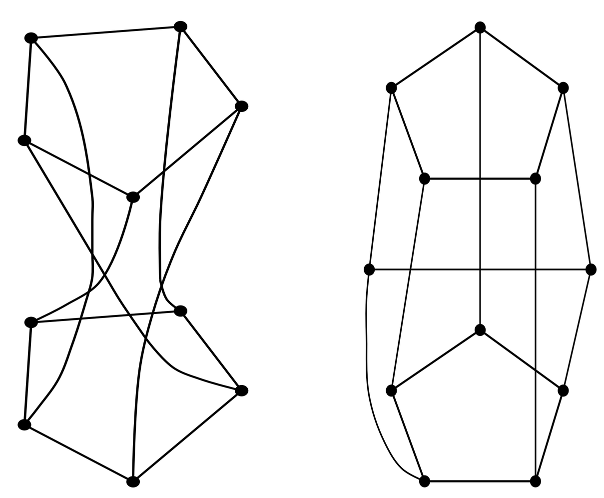


Figure 2: The number of 3-regular graphs that minimize λ_2 x(n) (left), and those unique graphs for $n = 10$ (middle) and $n = 12$ (right) that minimize λ_2.

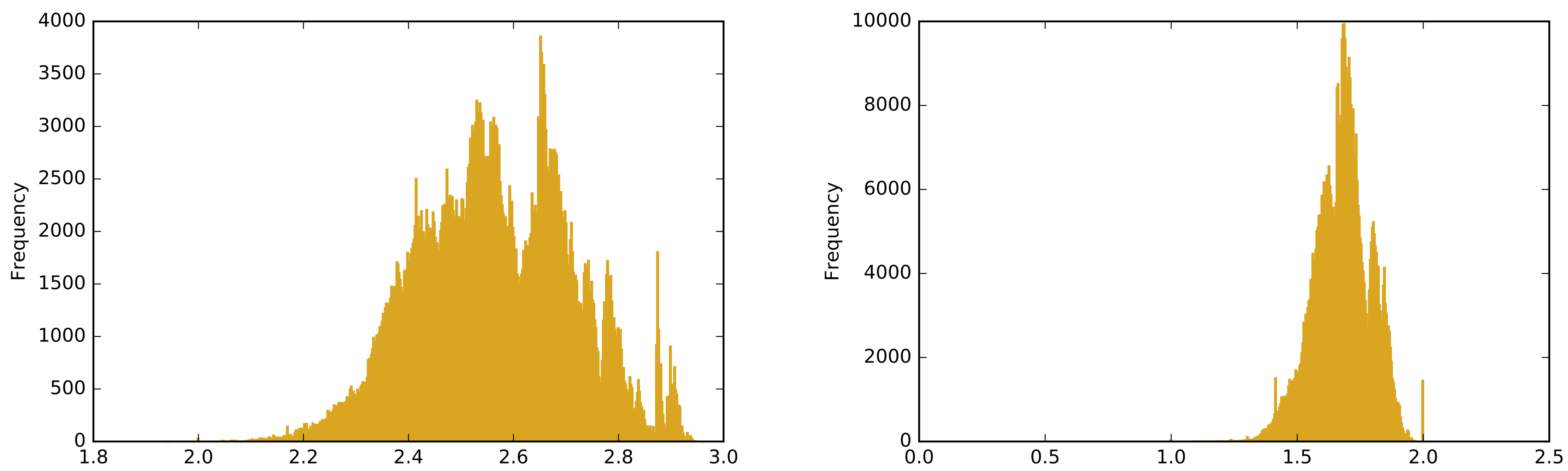


Figure 3: Distribution of λ_2 for $n = 20$ and $k = 3$ (left) and $k = n - 4$ (right). Values are normally distributed and more narrow for $k = n - 4$ across all examples of n. Note the multiple occurrences of $\lambda_2 = 2$.

Conclusion

There are distinct patterns in the bounds of $\lambda_{\min}$ and λ_2. More concrete bounds on these eigenvalues could help us define a bound on the minimal eigenvalue gap. Future research should focus on further defining the bounds on $\lambda_{\min}$ and λ_2. Questions could include:

- Is -3 a universal upper bound for $\lambda_{\min}$? How close does λ_2 approach k as n increases?
- What characteristics define the unique graphs that minimize λ_2?