Describing Convex Polyhedral Growth

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Abstract

We outline a method of describing convex polyhedra in computer graphics, using both algorithmic and mathematical models. Each polyhedron is described in our method with a unique set of planes that define the polyhedron's faces. These planes thereby define the polyhedron's shape, extent, and orientation. This definition allows for a computationally easy discrete growth process in which a polyhedron grows in directions perpendicular to some or all of its faces, preserving orientation and morphology.

Motivation

Many crystals, such as amethyst and quartz, naturally exhibit hexagonal structure. We aim to model these crystalline structures and their growth processes. Canonical models represent one of these crystals as a convex hexagonal polyhedron with defining crystallographic axes: three equal horizontal axes and one perpendicular vertical axis.

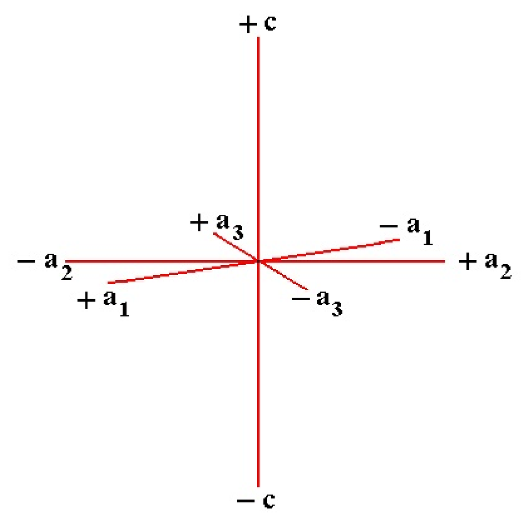

Figure 1: The crystallographic axes of a hexagonal crystal system. [1]

To maintain the standard ratio seen in quartz, we typically require $c = 1.1a$.

References

[1] "The Hexagonal Crystal System." *Hexagonal Crystal System I*, metafysica.nl/hexagonal_1.html.

[2] D. P. Woodruff. "How Does Your Crystal Grow? A Commentary on Burton, Cabrera, and Frank (1951) 'The Growth of Crystals and the Equilibrium Structure of their Surfaces'," *Philosophical Transactions of the Royal Society A* (Royal Society Publishing), 2014.

Methods

For our model, we define the extent of a convex polyhedron by a unique set of planes $\{P_n\}_{n=0}^{12}$ with equations $a_n x + b_n y + c_n z = d_n$ where $a_n, b_n, c_n, d_n \in R$. We say a plane P_i is a **neighbor** of or is **adjacent** to P_j, denoted $P_i \sim P_j$, if $i \neq j$ and P_i and P_j share an edge of the polyhedron. If we number the planes of a polyhedron in clockwise order (as viewed from the top), then

$$P_i \sim \begin{cases} P_1, P_5, P_6, P_{12} & \text{for } i = 0 \\ P_{i+1}, P_{i-1}, P_{i+6}, P_{12} & \text{for } i = 1, \ldots, 4 \\ P_0, P_4, P_{11}, P_{12} & \text{for } i = 5 \\ P_7, P_{11}, P_0 & \text{for } i = 6 \\ P_{i+1}, P_{i-1}, P_{i-6} & \text{for } i = 7, \ldots, 10 \\ P_6, P_{10}, P_5 & \text{for } i = 11 \\ P_0, P_1, P_2, P_3, P_4, P_5 & \text{for } i = 12, \end{cases}$$

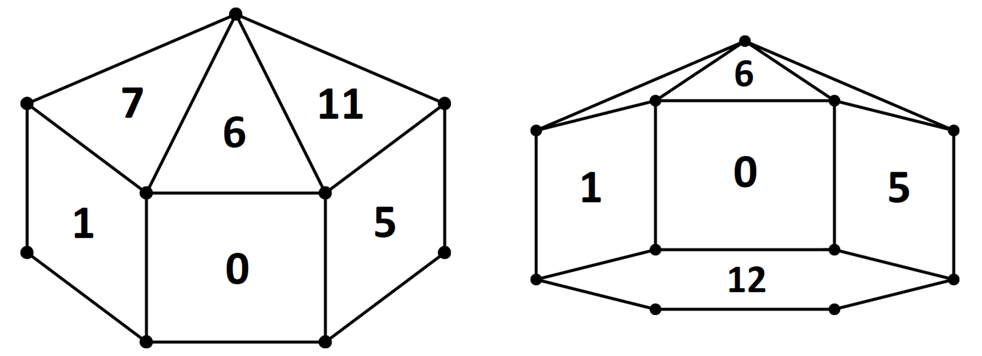

Figure 2: Top (left) and bottom view (right) of a crystal with labelled planes.

and each plane has a corresponding set of neighbors. Consider sets of three planes $\{P_i, P_j, P_k\}$ that satisfy i, j, k unique, $P_i \sim P_j$, and $P_i \sim P_k$. We find the point of intersection of these three planes by solving the linear system

$$\begin{cases} a_i x + b_i y + c_i z = d_i \\ a_j x + b_j y + c_j z = d_j \\ a_k x + b_k y + c_k z = d_k. \end{cases}$$

For each set, this point of intersection is a vertex of the polyhedron.
Doing this for all possible sets of three adjacent planes (while avoiding redundant computations) yields all the vertices of the polyhedron. For canonical hexagonal polyhedrons, this will yield 13 vertices, which are used to draw the polyhedron.

To model growth, we can now simply move some or all of its associated planes. Geometrically, this is done by adding a scalar to the d coefficient of the plane's equation. That is, for each plane P with equation $ax + by + cz = d$, we move P by replacing it with a new plane P' with equation $ax + by + cz = d + C$ for some $C \in R$.

Results: Uniform Growth

When we apply our methodology to move all of a polyhedron's planes the same amount (i.e. we add the same C to every plane's d coefficient), we model uniform growth (Figure 3).

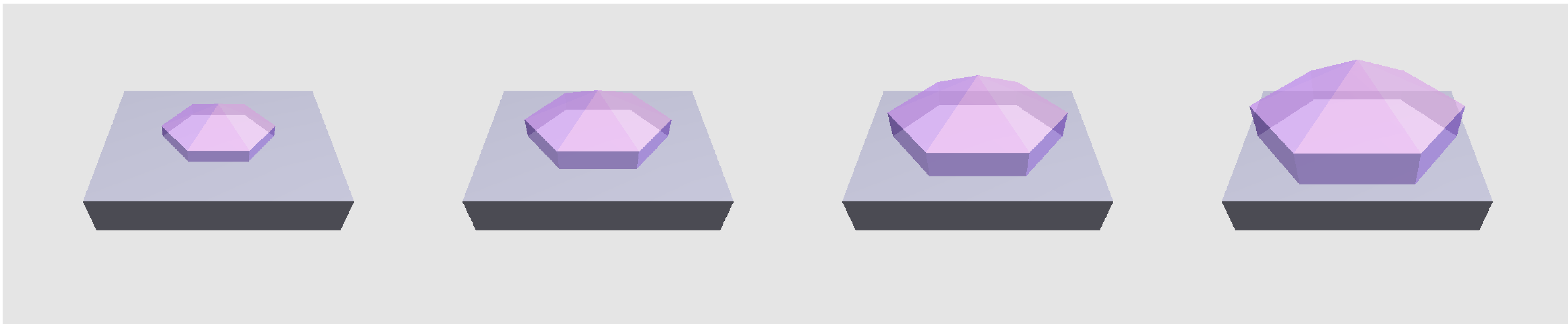

Figure 3: Sequence of uniform growth for $0 \leq C \leq 0.9$ with step size $s = 0.3$.

Results: Nonuniform Growth

Alternatively, Figure 4 shows uneven growth of a polyhedron, in which a select number of the planes are fixed while the remainder continue to move and simulate growth. We can also fix or unfix planes planes during the growth process.

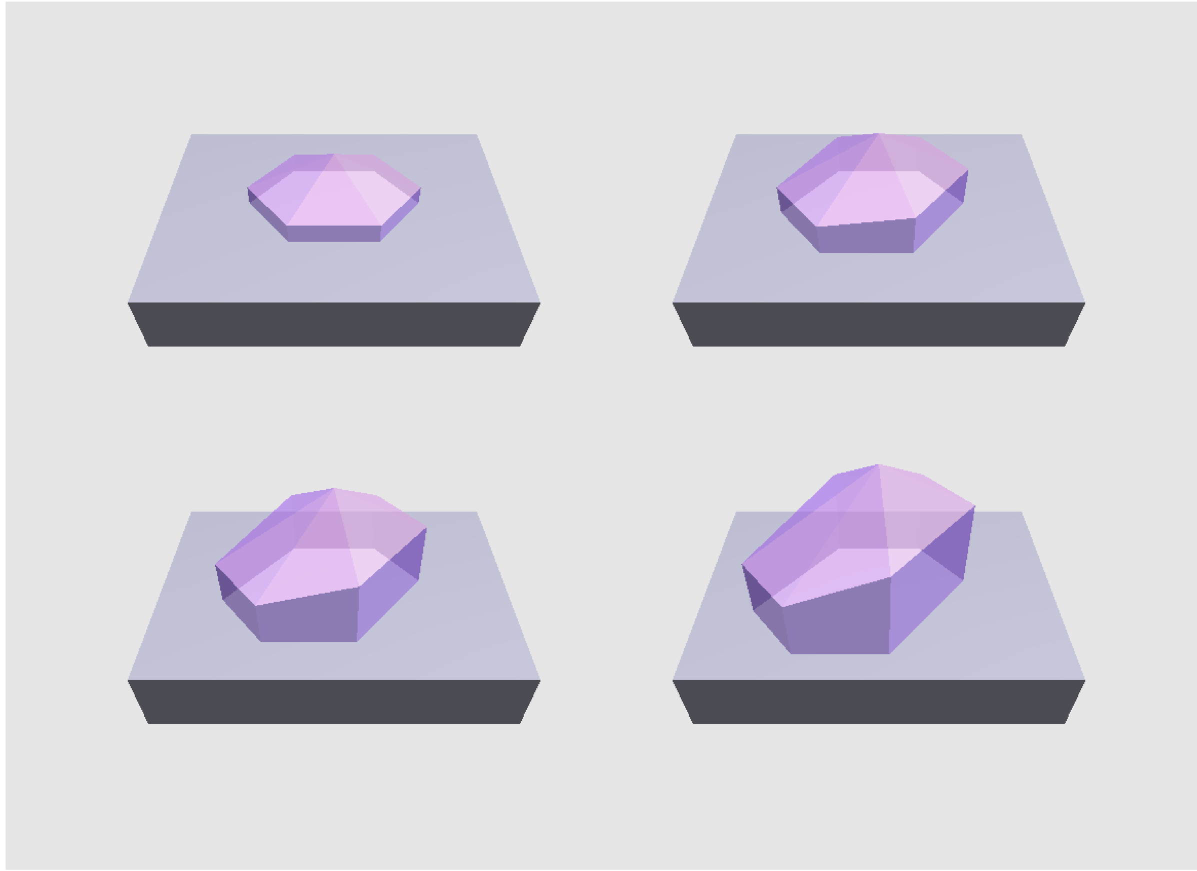

Figure 4: Sequence of nonuniform growth with step-size $s = 0.3$ from $0 \leq C \leq 0.9$ with all but two fixed planes. Non-fixed planes grow at an equal rate.

Future Work

Theoretically, molecules accrue on the exposed surface area A of a crystal, so we may be able to base growth on A. [2] Since this accrual is non-linear, let $m \in R$ and V be the crystal's volume, so that over time t,

$$\frac{dV}{dt} = mA. \tag{1}$$

We can then move the planes appropriately to reflect changes in V. We also hope to apply this method to the modeling of group aggregation and growth on a substrate. During group growth, crystals bump into one another and can no longer grow in those directions. Fixing the appropriate planes in place while allowing others to move can reflect the resulting growth pattern. Since these collisions affect A, we note that equation (1) is quite possibly analytically unsolvable, so a tempting computational model to use would be Euler's Method.