The Hadamard Walk

Test for $\underline{N=5}$ to compare with the code

↪ choosing the initial spin state $|0\rangle$

$\underline{N=1}$ taking initial state as $|n\rangle$

$$\psi_1 = H|0\rangle \otimes |n\rangle \qquad \left(\text{also } H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

where H represents the Hadamard matrix $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\psi_1 = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |n\rangle = \tfrac{1}{\sqrt{2}}\left((|0\rangle \otimes |n\rangle) + (|1\rangle \otimes |n\rangle)\right)$$

Then the shift operator S is applied

$$S = \left(|0\rangle\langle 0| \otimes \sum_m |m-1\rangle\langle m|\right) + \left(|1\rangle\langle 1| \otimes \sum_m |m+1\rangle\langle m|\right)$$

$$S \cdot \psi_1 = \tfrac{1}{\sqrt{2}}\left((|0\rangle \otimes |n-1\rangle) + (|1\rangle \otimes |n+1\rangle)\right)$$

$\underline{\underline{N=2}}$

$$\psi_2 = H\left[\tfrac{1}{\sqrt{2}}\left((|0\rangle \otimes |n-1\rangle) + (|1\rangle \otimes |n+1\rangle)\right)\right]$$

$$= \tfrac{1}{\sqrt{2}}\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |n-1\rangle\right) + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes |n+1\rangle\right)\right)$$

$$= \tfrac{1}{2}\left[(|0\rangle \otimes |n-1\rangle) + (|1\rangle \otimes |n-1\rangle) + (|0\rangle \otimes |n+1\rangle) - (|1\rangle \otimes |n+1\rangle)\right]$$

$$S \cdot \psi_2 = \tfrac{1}{2}\left[(|0\rangle \otimes |n-2\rangle) + (|1\rangle \otimes |n\rangle) + (|0\rangle \otimes |n\rangle) - (|1\rangle \otimes |n+2\rangle)\right]$$

$\underline{\underline{N=3}}$

$$\psi_3 = H\left[\tfrac{1}{2}\left(|0\rangle|n-2\rangle + |1\rangle|n\rangle + |0\rangle|n\rangle - |1\rangle|n+2\rangle\right)\right]$$

$$= \tfrac{1}{2}\left[\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|n-2\rangle + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|n\rangle + \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|n\rangle - \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|n+2\rangle\right]$$

$$S \cdot \psi_3 = \tfrac{1}{2\sqrt{2}}\left[|0\rangle|n-3\rangle + |1\rangle|n-1\rangle + |0\rangle|n-1\rangle - \cancel{|1\rangle|n+1\rangle}\right.$$

$$\left. + |0\rangle|n-1\rangle + \cancel{|1\rangle|n+1\rangle} - |0\rangle|n+1\rangle + |1\rangle|n+3\rangle\right]$$

$$= \tfrac{1}{2\sqrt{2}}\left[|0\rangle|n-3\rangle + |1\rangle|n-1\rangle + 2|0\rangle|n-1\rangle - |0\rangle|n+1\rangle + |1\rangle|n+3\rangle\right]$$

$\underline{\underline{N=4}}$

$$\psi_4 = H\left[\tfrac{1}{2\sqrt{2}}\left[|0\rangle|n-3\rangle + |1\rangle|n-1\rangle + 2|0\rangle|n-1\rangle - |0\rangle|n+1\rangle + |1\rangle|n+3\rangle\right]\right]$$

$$= \tfrac{1}{2\sqrt{2}}\left[\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|n-3\rangle + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|n-1\rangle + 2\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|n\overset{-1}{\cancel{+1}}\rangle\right.$$

$$\left. - \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|n+1\rangle + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|n+3\rangle\right]$$

$$S \cdot \psi_4 = \tfrac{1}{4}\left[|0\rangle|n-4\rangle + |1\rangle|n-2\rangle + |0\rangle|n-2\rangle - |1\rangle|n\rangle + 2|0\rangle\overset{|n-2\rangle}{\cancel{|n\rangle}}\right.$$

$$+ 2|1\rangle|n\cancel{+2}\rangle - |0\rangle|n\rangle - |1\rangle|n+2\rangle + |0\rangle|n+2\rangle$$

$$\left. - |1\rangle|n+4\rangle\right]$$

$$= \tfrac{1}{4}\left[\cancel{|0\rangle|n-4\rangle + |1\rangle|n-2\rangle + |0\rangle|n-2\rangle - |1\rangle|n\rangle + |0\rangle|n\rangle}\right.$$

$$\left.\cancel{+ |1\rangle|n+2\rangle + |0\rangle|n+2\rangle - |1\rangle|n+4\rangle}\right]$$

$$= \tfrac{1}{4}\left[|0\rangle|n-4\rangle + |1\rangle|n-2\rangle + 3|0\rangle|n-2\rangle - |0\rangle|n\rangle + |1\rangle|n\rangle\right.$$

$$\left. + |0\rangle|n+2\rangle - |1\rangle|n+2\rangle - |1\rangle|n+4\rangle\right]$$

$\underline{\underline{N=5}}$

$$\psi_5 = H\Big[\tfrac{1}{4}\big[|0\rangle|n-4\rangle + 3|0\rangle|n-2\rangle + |1\rangle|n-2\rangle - |0\rangle|n\rangle + |1\rangle|n\rangle + |0\rangle|n+2\rangle - |1\rangle|n+2\rangle - |1\rangle|n+4\rangle\big]\Big]$$

$$= \tfrac{1}{4}\Big[\Big(\tfrac{|0\rangle+|1\rangle}{\sqrt2}\Big)|n-4\rangle + 3\Big(\tfrac{|0\rangle+|1\rangle}{\sqrt2}\Big)|n-2\rangle + \Big(\tfrac{|0\rangle-|1\rangle}{\sqrt2}\Big)|n-2\rangle - \Big(\tfrac{|0\rangle+|1\rangle}{\sqrt2}\Big)|n\rangle + \Big(\tfrac{|0\rangle-|1\rangle}{\sqrt2}\Big)|n\rangle + \Big(\tfrac{|0\rangle+|1\rangle}{\sqrt2}\Big)|n+2\rangle - \Big(\tfrac{|0\rangle-|1\rangle}{\sqrt2}\Big)|n+2\rangle - \Big(\tfrac{|0\rangle-|1\rangle}{\sqrt2}\Big)|n+4\rangle\Big]$$

$$S\cdot\psi_5 = \tfrac{1}{4\sqrt2}\big[|0\rangle|n-5\rangle + |1\rangle|n-3\rangle + 3|0\rangle|n-3\rangle + 3|1\rangle|n-1\rangle + |0\rangle|n-3\rangle - |1\rangle|n-1\rangle - \cancel{|0\rangle|n-1\rangle} - |1\rangle|n+1\rangle + \cancel{|0\rangle|n-1\rangle} - |1\rangle|n+1\rangle + |0\rangle|n+1\rangle + |1\rangle|n+3\rangle - \cancel{|0\rangle|n+1\rangle} + |1\rangle|n+3\rangle - |0\rangle|n+3\rangle + |1\rangle|n+5\rangle\big]$$

$$= \tfrac{1}{4\sqrt2}\big[|0\rangle|n-5\rangle + |1\rangle|n-3\rangle + 4|0\rangle|n-3\rangle + 2|1\rangle|n-1\rangle - 2|1\rangle|n+1\rangle + 2|1\rangle|n+3\rangle - |0\rangle|n+3\rangle + |1\rangle|n+5\rangle\big]$$

making probability measurements

① $I\otimes|n-5\rangle\langle n-5| \rightarrow \tfrac{1}{4\sqrt2}|0\rangle|n-5\rangle$

$\big(\tfrac{1}{4\sqrt2}\langle0|\langle n-5|\big)\big(\tfrac{1}{4\sqrt2}|0\rangle|n-5\rangle\big) = \tfrac{1}{32} = \underline{\underline{0.03125}}$

② $I\otimes|n-3\rangle\langle n-3| \rightarrow \tfrac{1}{4\sqrt2}|1\rangle|n-3\rangle + \tfrac{1}{\sqrt2}|0\rangle|n-3\rangle$

$\big(\tfrac{1}{\sqrt2}\cancel{|0\rangle|n-3\rangle}\langle0|\langle n-3| + \tfrac{1}{4\sqrt2}\langle1|\langle n-3|\big)\big(\tfrac{1}{4\sqrt2}|1\rangle|n-3\rangle + \tfrac{1}{\sqrt2}|0\rangle|n-3\rangle\big)$

$= \tfrac{1}{2} + \tfrac{1}{32} = \underline{\underline{0.53125}}$

③ $I\otimes|n-1\rangle\langle n-1| \rightarrow \tfrac{1}{2\sqrt2}|1\rangle|n-1\rangle$

$\big(\tfrac{1}{2\sqrt2}\langle1|\langle n-1|\big)\big(\tfrac{1}{2\sqrt2}|1\rangle|n-1\rangle\big) = \tfrac{1}{8} = \underline{\underline{0.125}}$

④ $I\otimes|n+1\rangle\langle n+1| \rightarrow -\tfrac{1}{2\sqrt2}|1\rangle|n+1\rangle$

$\big(-\tfrac{1}{2\sqrt2}\langle1|\langle n+1|\big)\big(-\tfrac{1}{2\sqrt2}|1\rangle|n-1\rangle\big) = \tfrac{1}{8} = \underline{\underline{0.125}}$

⑤ $I\otimes|n+3\rangle\langle n+3| \rightarrow \tfrac{1}{2\sqrt2}|1\rangle|n+3\rangle - \tfrac{1}{4\sqrt2}|0\rangle|n+3\rangle$

$\big(\tfrac{1}{2\sqrt2}\langle1|\langle n+3| - \tfrac{1}{4\sqrt2}\langle0|\langle n+3|\big)\big(\tfrac{1}{2\sqrt2}|1\rangle|n+3\rangle - \tfrac{1}{4\sqrt2}|0\rangle|n+3\rangle\big)$

$= \tfrac{1}{8} + \tfrac{1}{32} = \underline{\underline{0.15625}}$

⑥ $I\otimes|n+5\rangle\langle n+5| \rightarrow \tfrac{1}{4\sqrt2}|1\rangle|n+5\rangle$

$\hookrightarrow \underline{\underline{0.03125}}$

producing probability plot like this →
which correctly aligns with code.