

The Hadamard Walk

Test for $N=5$ to compare with the code

→ choosing the initial spin state $|10\rangle$

$N=1$ taking initial state as $|1u\rangle$

$$\Psi_1 = H|10\rangle \otimes |1u\rangle \quad (\text{also } H|11\rangle = \frac{|10\rangle - |11\rangle}{\sqrt{2}})$$

where H represents the Hadamard matrix $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\Psi_1 = \left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) \otimes |1u\rangle = \frac{1}{\sqrt{2}} (|10\rangle \otimes |1u\rangle + |11\rangle \otimes |1u\rangle)$$

Then the shift operator S is applied

$$S = (|10\rangle \otimes \sum_m |1u-m\rangle \langle 1u-m|) + (|11\rangle \otimes \sum_m |1u+m\rangle \langle 1u+m|)$$

$$S \cdot \Psi_1 = \frac{1}{\sqrt{2}} (|10\rangle \otimes |1u-1\rangle + |11\rangle \otimes |1u+1\rangle)$$

$N=2$

$$\begin{aligned} \Psi_2 &= H \left[\frac{1}{\sqrt{2}} (|10\rangle \otimes |1u-1\rangle + |11\rangle \otimes |1u+1\rangle) \right] \\ &= \frac{1}{\sqrt{2}} \left(\left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) \otimes |1u-1\rangle + \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right) \otimes |1u+1\rangle \right) \\ &= \frac{1}{2} \left[(|10\rangle \otimes |1u-1\rangle) + (|11\rangle \otimes |1u-1\rangle) + (|10\rangle \otimes |1u+1\rangle) - (|11\rangle \otimes |1u+1\rangle) \right] \end{aligned}$$

$$S \cdot \Psi_2 = \frac{1}{2} \left[(|10\rangle \otimes |1u-2\rangle) + (|11\rangle \otimes |1u\rangle) + (|10\rangle \otimes |1u\rangle) - (|11\rangle \otimes |1u+2\rangle) \right]$$

$N=3$

$$\begin{aligned} \Psi_3 &= H \left[\frac{1}{2} \left(|10\rangle |1u-2\rangle + |11\rangle |1u\rangle + |10\rangle |1u\rangle - |11\rangle |1u+2\rangle \right) \right] \\ &= \frac{1}{2} \left[\left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) |1u-2\rangle + \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right) |1u\rangle + \left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) |1u\rangle - \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right) |1u+2\rangle \right] \end{aligned}$$

$$\begin{aligned} S \cdot \Psi_3 &= \frac{1}{2\sqrt{2}} \left[|10\rangle |1u-3\rangle + |11\rangle |1u-1\rangle + |10\rangle |1u-1\rangle - |11\rangle |1u+1\rangle \right. \\ &\quad \left. + |10\rangle |1u-1\rangle + |11\rangle |1u+1\rangle - |10\rangle |1u+1\rangle + |11\rangle |1u+3\rangle \right] \\ &= \frac{1}{2\sqrt{2}} \left[|10\rangle |1u-3\rangle + |11\rangle |1u-1\rangle + 2|10\rangle |1u-1\rangle - |10\rangle |1u+1\rangle + |11\rangle |1u+3\rangle \right] \end{aligned}$$

$N=4$

$$\begin{aligned} \Psi_4 &= H \left[\frac{1}{2\sqrt{2}} \left[|10\rangle |1u-3\rangle + |11\rangle |1u-1\rangle + 2|10\rangle |1u-1\rangle - |10\rangle |1u+1\rangle + |11\rangle |1u+3\rangle \right] \right] \\ &= \frac{1}{2\sqrt{2}} \left[\left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) |1u-3\rangle + \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right) |1u-1\rangle + 2 \left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) |1u-1\rangle \right. \\ &\quad \left. - \left(\frac{|10\rangle + |11\rangle}{\sqrt{2}} \right) |1u+1\rangle + \left(\frac{|10\rangle - |11\rangle}{\sqrt{2}} \right) |1u+3\rangle \right] \end{aligned}$$

$$\begin{aligned} S \cdot \Psi_4 &= \frac{1}{4} \left[|10\rangle |1u-4\rangle + |11\rangle |1u-2\rangle + |10\rangle |1u-2\rangle - |11\rangle |1u\rangle + 2|10\rangle |1u-2\rangle \right. \\ &\quad \left. + 2|11\rangle |1u+2\rangle - |10\rangle |1u\rangle - |11\rangle |1u+2\rangle + |10\rangle |1u+2\rangle \right. \\ &\quad \left. - |11\rangle |1u+4\rangle \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4} \left[|10\rangle |1u-4\rangle + |11\rangle |1u-2\rangle + |10\rangle |1u-2\rangle - |11\rangle |1u\rangle + |11\rangle |1u\rangle \right. \\ &\quad \left. + |11\rangle |1u+2\rangle + |10\rangle |1u+2\rangle - |11\rangle |1u+4\rangle \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4} \left[|10\rangle |1u-4\rangle + |11\rangle |1u-2\rangle + 3|10\rangle |1u-2\rangle - |10\rangle |1u\rangle + |11\rangle |1u\rangle \right. \\ &\quad \left. + |10\rangle |1u+2\rangle - |11\rangle |1u+2\rangle - |11\rangle |1u+4\rangle \right] \end{aligned}$$

N=5

$$4S = 11 \left[\frac{1}{4} [10|1u-4\rangle + 3|10\rangle|u-2\rangle + |11\rangle|u-2\rangle + -10\rangle|u\rangle + |11\rangle|u\rangle + |10\rangle|u+2\rangle - |11\rangle|u+2\rangle - |11\rangle|u+4\rangle] \right]$$

$$= \frac{1}{4} \left[\left(\frac{10+11}{\sqrt{2}} \right) |u-4\rangle + 3 \left(\frac{|10+11\rangle}{\sqrt{2}} \right) |u-2\rangle + \left(\frac{|10-11\rangle}{\sqrt{2}} \right) |u-2\rangle \right.$$

$$- \left(\frac{|10+11\rangle}{\sqrt{2}} \right) |u\rangle + \left(\frac{|10-11\rangle}{\sqrt{2}} \right) |u\rangle + \left(\frac{|10+11\rangle}{\sqrt{2}} \right) |u+2\rangle$$

$$\left. - \left(\frac{|10-11\rangle}{\sqrt{2}} \right) |u+2\rangle - \left(\frac{|10-11\rangle}{\sqrt{2}} \right) |u+4\rangle \right]$$

$$S \cdot 4S = \frac{1}{4\sqrt{2}} \left[10\rangle|u-5\rangle + |11\rangle|u-3\rangle + 3|10\rangle|u-3\rangle + 3|11\rangle|u-1\rangle + |10\rangle|u-3\rangle - |11\rangle|u-1\rangle - 10\rangle|u-1\rangle - |11\rangle|u+1\rangle + |10\rangle|u+1\rangle + |11\rangle|u+3\rangle - 10\rangle|u+1\rangle + |11\rangle|u+3\rangle - 10\rangle|u+3\rangle + |11\rangle|u+5\rangle \right]$$

$$= \frac{1}{4\sqrt{2}} \left[10\rangle|u-5\rangle + |11\rangle|u-3\rangle + 4|10\rangle|u-3\rangle + 2|11\rangle|u-1\rangle - 2|11\rangle|u+1\rangle + 2|11\rangle|u+3\rangle - 10\rangle|u+3\rangle + |11\rangle|u+5\rangle \right]$$

making probability measurements

① $I \otimes |u-5\rangle\langle u-5| \rightarrow \frac{1}{4\sqrt{2}}|10\rangle|u-5\rangle$

$$\left(\frac{1}{4\sqrt{2}} \langle 01 \langle u-5 | \frac{1}{4\sqrt{2}} |10\rangle|u-5\rangle \right) = \frac{1}{32} = \underline{\underline{0.03125}}$$

② $I \otimes |u-3\rangle\langle u-3| \rightarrow \frac{1}{4\sqrt{2}}|11\rangle|u-3\rangle + \frac{1}{\sqrt{2}}|10\rangle|u-3\rangle$

$$\left(\frac{1}{\sqrt{2}} |10\rangle|u-3\rangle \langle 01 \langle u-3 | + \frac{1}{4\sqrt{2}} \langle 11 \langle u-3 | \left(\frac{1}{4\sqrt{2}} |11\rangle|u-3\rangle + \frac{1}{\sqrt{2}} |10\rangle|u-3\rangle \right) \right)$$

$$= \frac{1}{2} + \frac{1}{32} = \underline{\underline{0.53125}}$$

③ $I \otimes |u-1\rangle\langle u-1| \rightarrow \frac{1}{2\sqrt{2}}|11\rangle|u-1\rangle$

$$\left(\frac{1}{2\sqrt{2}} \langle 11 \langle u-1 | \frac{1}{2\sqrt{2}} |11\rangle|u-1\rangle \right) = \frac{1}{8} = \underline{\underline{0.125}}$$

④ $I \otimes |u+1\rangle\langle u+1| \rightarrow -\frac{1}{2\sqrt{2}}|11\rangle|u+1\rangle$

$$\left(-\frac{1}{2\sqrt{2}} \langle 11 \langle u+1 | \left(-\frac{1}{2\sqrt{2}} |11\rangle|u+1\rangle \right) \right) = \frac{1}{8} = \underline{\underline{0.125}}$$

⑤ $I \otimes |u+3\rangle\langle u+3| \rightarrow \frac{1}{2\sqrt{2}}|11\rangle|u+3\rangle - \frac{1}{4\sqrt{2}}|10\rangle|u+3\rangle$

$$\left(\frac{1}{2\sqrt{2}} \langle 11 \langle u+3 | -\frac{1}{4\sqrt{2}} \langle 01 \langle u+3 | \left(\frac{1}{2\sqrt{2}} |11\rangle|u+3\rangle - \frac{1}{4\sqrt{2}} |10\rangle|u+3\rangle \right) \right)$$

$$= \frac{1}{8} + \frac{1}{32} = \underline{\underline{0.15625}}$$

⑥ $I \otimes |u+5\rangle\langle u+5| \rightarrow \frac{1}{4\sqrt{2}}|11\rangle|u+5\rangle$

$$\rightarrow \underline{\underline{0.03125}}$$

producing probability

plot like this →

which correctly aligns

with code.

