Project 1

Beth Tian

March 8, 2025

1 Problem 1

A Calculate the Arithmetic Returns. Remove the mean, such that each series has 0 mean. Present the last 5 rows and the total standard deviation.

Table 1: Last 5 rows of Arithmetic Returns (Zero Mean)

	SPY	AAPL	EQIX
499	-0.011492	-0.014678	-0.006966
500	-0.012377	-0.014699	-0.008064
501	-0.004603	-0.008493	0.006512
502	-0.003422	-0.027671	0.000497
503	0.011538	-0.003445	0.015745

Table 2: Standard Deviation of Arithmetic Returns (Zero Mean)

Asset	Standard Deviation
SPY	0.008077
AAPL	0.013483
EQIX	0.015361

B Calculate the Log Returns. Remove the mean, such that each series has 0 mean. Present the last 5 rows and the total standard deviation.

Table 3: Last 5 rows of Log Returns (Zero Mean)

	SPY	AAPL	EQIX
499	-0.011515	-0.014675	-0.006867
500	-0.012410	-0.014696	-0.007972
501	-0.004577	-0.008427	0.006602
502	-0.003392	-0.027930	0.000613
503	0.011494	-0.003356	0.015725

Table 4: Standard Deviation of Log Returns (Zero Mean)

Asset	Standard Deviation
SPY	0.008078
AAPL	0.013446
EQIX	0.015270

2 Problem 2

A Calculate the current value of the portfolio given today is 1/3/2025

Total value of a portfolio

The total value of a portfolio is calculated as the sum of the values of individual positions:

$$V_p = \sum_{i=1}^{n} V_i \tag{1}$$

$$= \sum_{i=1}^{n} P_i \times Q_i \tag{2}$$

Where:

- V_p is the total portfolio value
- V_i is the value of asset i
- P_i is the price of asset i
- Q_i is the quantity (number of shares) of asset i held in the portfolio
- n is the total number of assets in the portfolio

Portfolio Composition

The weight or percentage contribution of each asset to the total portfolio value is given by:

$$w_i = \frac{P_i \times Q_i}{V_p} \times 100\% \tag{3}$$

Where:

- w_i is the weight of asset i in the portfolio (expressed as a percentage)
- $P_i \times Q_i$ is the value of the position in asset i
- V_p is the total portfolio value

B Calculate the VaR and ES of each stock and the entire portfolio at the 5% alpha level assuming arithmetic returns and 0 mean return

VaR

$$\text{VaR}_\alpha(X) = -F_X^{-1}(\alpha) \tag{4}$$

Where:

- F_X is the cumulative distribution function of X
- F_X^{-1} is the generalized inverse function (quantile function) of F_X
- α is 0.05 here.

ES

$$\begin{aligned}\text{ES}_\alpha(X) &= -\mathbb{E}[X|X \leq F_X^{-1}(\alpha)] & (5)\\ &= -\frac{1}{\alpha}\int_0^\alpha F_X^{-1}(x)dx & (6)\end{aligned}$$

Where:

- $\mathbb{E}[X|X \leq F_X^{-1}(\alpha)]$ is the conditional expectation of X given that X is less than or equal to the α-quantile

B.a Normally distributed with exponentially weighted covariance with lambda=0.97.

The Exponentially Weighted Moving Average (EWMA) covariance matrix is a time-weighted covariance estimation method that assigns higher weights to more recent observations.
Here, I choose the first 20 observations to calculate the covariance for the three stocks. Compared to 504 data points, 20 points is relative small amount. Choosing 20 points as the initial data will not influence the final covariance too much.
Then, I update the covariance matrix according to the following formula.

$$\mathbf{\Sigma}_t = \lambda\mathbf{\Sigma}_{t-1} + (1-\lambda)\mathbf{r}_t\mathbf{r}_t^T \tag{7}$$

Where:

- $\mathbf{\Sigma}_t$ is the covariance matrix at time t

- λ is the exponential decay factor ($\lambda = 0.97$)
- $\mathbf{r}_t$ is the return vector at time t

For the VaR and ES, the formula is:

$$\text{VaR}(\alpha) = -\text{PV} \cdot F^{-1}(\alpha) \cdot \sqrt{\nabla \mathbf{R}^T \mathbf{\Sigma} \nabla \mathbf{R}} \tag{8}$$

$$\text{ES}(\alpha) = -\text{PV} \cdot \frac{\phi(F^{-1}(\alpha))}{\alpha} \cdot \sqrt{\nabla \mathbf{R}^T \mathbf{\Sigma} \nabla \mathbf{R}} \tag{9}$$

Parameters:

- $\mathbf{\Sigma}$ (DataFrame): Covariance matrix
- $\text{PV} \cdot \nabla \mathbf{R}$ (dict): Position values for each stock
- α (float): Significance level

$$\nabla \mathbf{R} = \begin{pmatrix} \frac{\partial V}{\partial r_1} \\ \frac{\partial V}{\partial r_2} \\ \vdots \\ \frac{\partial V}{\partial r_n} \end{pmatrix} \tag{10}$$

$$\frac{\partial V}{\partial r_i} = w_i \cdot A_i = PV_i \tag{11}$$

- w_i is the weight of asset i in the portfolio
- A_i is the value of asset i
- PV_i is the position value of asset i

The result is:
Portfolio VaR (5%): $3,856.32
Portfolio ES (5%): $-4,835.98

Risk metrics for individual stocks:

Table 5: Risk Measures by Asset

Asset	**VaR ($)**	**ES ($)**
SPY	827.85	1,038.16
AAPL	946.08	1,186.42
EQIX	2,933.51	3,678.74

B.b T distribution using a Gaussian Copula.

T-Distribution with Gaussian Copula for Portfolio Risk Assessment

This method combines marginal t-distributions with Gaussian Copula functions to estimate portfolio risk metrics, including these main steps:

1. Fit t-distributions as marginal distributions for each asset return series
2. Model the dependency structure between assets using Gaussian Copula functions
3. Generate return scenarios through Monte Carlo simulation
4. Calculate Value-at-Risk (VaR) and Expected Shortfall (ES) from the simulated scenarios

1. Mathematical Foundation

1.1 Fitting Marginal Distributions

For all assets, including the market index (SPY), t-distributions are used to capture the heavy tails observed in financial returns:

$$R_i \sim t(\nu_i, \mu_i, \sigma_i) \tag{12}$$

where:

- ν_i is the degrees of freedom, controlling the tail thickness
- μ_i is the location parameter (typically close to zero for mean-adjusted returns)
- σ_i is the scale parameter

The parameters are estimated using Maximum Likelihood Estimation (MLE):

$$(\hat{\nu}_i, \hat{\mu}_i, \hat{\sigma}_i) = \arg\max_{\nu_i,\mu_i,\sigma_i} \sum_{t=1}^{T} \log f_t(r_{i,t}; \nu_i, \mu_i, \sigma_i) \tag{13}$$

where f_t is the probability density function of the t-distribution, and $r_{i,t}$ are the observed returns.

1.2 Modeling Dependency Structure

Convert original returns to uniform distributions using the probability integral transform:

$$U_i = F_i(R_i) \tag{14}$$

where F_i is the cumulative distribution function of the fitted t-distribution for asset i.

Transform the uniform variables to standard normal variables:

$$Z_i = \Phi^{-1}(U_i) \tag{15}$$

where Φ^{-1} is the inverse function of the standard normal distribution.

Calculate the Spearman's rank correlation matrix $\mathbf{R}$ of these transformed variables to capture the dependency structure:

$$\rho_S(X_i, X_j) = \rho(F_i(X_i), F_j(X_j)) \tag{16}$$

Spearman's correlation is preferred over Pearson correlation as it better captures non-linear dependencies between variables.

1.3 Gaussian Copula

The Gaussian Copula function is defined as:

$$C_{\mathbf{R}}(u_1, \ldots, u_n) = \Phi_{\mathbf{R}}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)) \tag{17}$$

where:

- $\Phi_{\mathbf{R}}$ is the multivariate normal distribution function with correlation matrix $\mathbf{R}$
- Φ^{-1} is the inverse function of the standard normal distribution

2. Simulation Process

2.1 Generating Correlated Random Variables

To generate correlated random variables, Cholesky decomposition of the correlation matrix $\mathbf{R}$ is used:

$$\mathbf{R} = \mathbf{L}\mathbf{L}^T \tag{18}$$

where $\mathbf{L}$ is a lower triangular matrix.

Using this decomposition, correlated random variables are generated:

$$\mathbf{Z} = \mathbf{r}\mathbf{L}^T \tag{19}$$

where $\mathbf{r}$ is a vector of independent standard normal random variables.

2.2 Simulating Returns

Convert correlated standard normal random variables to uniform distributions:

$$\mathbf{U} = \Phi(\mathbf{Z}) \tag{20}$$

Then transform to returns using the inverse function of the marginal t-distributions:

$$R_i = F_i^{-1}(U_i) \tag{21}$$

where F_i^{-1} is the inverse cumulative distribution function of the fitted t-distribution for asset i.

3. Risk Metrics Calculation

3.1 Simulating Asset Value Changes

For each asset i, calculate the current and simulated values:

$$V_{i,0} = PV_i \tag{22}$$
$$V_{i,j} = PV_i \cdot (1 + R_{i,j}) \tag{23}$$

where:

- PV_i is the current position value of asset i
- $R_{i,j}$ is the return of asset i in the j-th simulation

Calculate Profit and Loss (P&L):

$$PnL_{i,j} = V_{i,j} - V_{i,0} \tag{24}$$

3.2 Value-at-Risk (VaR) Calculation

At confidence level $1 - \alpha$, the portfolio's Value-at-Risk is:

$$VaR_\alpha = -F_{PnL}^{-1}(\alpha) \tag{25}$$

where F_{PnL}^{-1} is the empirical quantile function of the total P&L.

3.3 Expected Shortfall (ES) Calculation

The Expected Shortfall at the same confidence level is calculated as:

$$ES_\alpha = -\mathbb{E}[PnL | PnL \leq -VaR_\alpha] \tag{26}$$

which is implemented as the average of losses exceeding VaR:

$$ES_\alpha = -\frac{1}{k\alpha} \sum_{j: PnL_j \leq -VaR_\alpha} PnL_j \tag{27}$$

where k is the number of simulations.

The result is:
SPY T-distribution parameters: df=5.0000, loc=0.001139, scale=0.006611
AAPL T-distribution parameters: df=5.0000, loc=0.001428, scale=0.010530
EQIX T-distribution parameters: df=5.0000, loc=0.000786, scale=0.011735
Correlation matrix (Spearman):
Portfolio VaR (5%): \$4,074.76
Portfolio ES (5%): \$5,844.19
Risk metrics for individual stocks:

Table 6: Risk Metrics for Individual Stocks

Asset	VaR ($)	VaR (%)	ES ($)	ES (%)
SPY	722.88	1.22%	1,080.40	1.83%
AAPL	971.75	2.00%	1,384.71	2.85%
EQIX	3,216.40	2.23%	4,739.73	3.29%
Portfolio	**4,074.76**	**1.62%**	**5,844.19**	**2.32%**

B.c Historic simulation using the full history.

The historical simulation method follows these steps:

1. Calculate current portfolio value:

$$\text{PV}_0 = \sum_{i=1}^{n} h_i \cdot p_{i,0} \tag{28}$$

   where h_i is the holding quantity of asset i, and $p_{i,0}$ is the current price of asset i.

2. Sample N historical return scenarios (with replacement): The method uses the entire historical return dataset as potential scenarios, where N is the number of historical observations.

3. For each historical return scenario j, calculate simulated prices:

$$p_{i,j} = p_{i,0} \cdot (1 + r_{i,j}) \tag{29}$$

   where $r_{i,j}$ is the historical return of asset i in scenario j.

4. Calculate portfolio value under each historical scenario:

$$\text{PV}_j = \sum_{i=1}^{n} h_i \cdot p_{i,j} \tag{30}$$

5. Sort the simulated portfolio values in ascending order:

$$\text{PV}_{(1)} \leq \text{PV}_{(2)} \leq \cdots \leq \text{PV}_{(N)} \tag{31}$$

6. Determine the α-quantile of the sorted portfolio values:

$$\text{PV}_\alpha = \text{PV}_{(\lfloor \alpha \cdot N \rfloor)} \tag{32}$$

   where $\lfloor \alpha \cdot N \rfloor$ represents the integer floor of $\alpha \cdot N$.

7. Calculate VaR as the difference between current portfolio value and the α-quantile:

$$\text{VaR}_\alpha = \text{PV}_0 - \text{PV}_\alpha \tag{33}$$

8. Identify the subset of portfolio values that fall below the α-quantile (the tail):

$$\text{Tail}_\alpha = \{\text{PV}_j : \text{PV}_j \leq \text{PV}_\alpha\} \tag{34}$$

9. Calculate ES as the average loss in these tail scenarios:

$$\text{ES}_\alpha = \text{PV}_0 - \frac{1}{|\text{Tail}_\alpha|} \sum_{\text{PV}_j \in \text{Tail}_\alpha} \text{PV}_j \tag{35}$$

where $|\text{Tail}_\alpha|$ is the number of scenarios in the tail (approximately $\alpha \cdot N$).

Note that ES can also be expressed as a conditional expectation:

$$\text{ES}_\alpha = \text{PV}_0 - \mathbb{E}[\text{PV}_j | \text{PV}_j \leq \text{PV}_\alpha] \tag{36}$$

When using KDE for smoothing, ES can be calculated as:

$$\text{ES}_\alpha^{\text{KDE}} = \text{PV}_0 - \frac{1}{\alpha} \int_{-\infty}^{\text{PV}_\alpha^{\text{KDE}}} x \cdot \hat{f}(x)\, dx \tag{37}$$

where $\hat{f}(x)$ is the estimated density function from KDE and $\text{PV}_\alpha^{\text{KDE}}$ is the KDE-smoothed α-quantile.

The result is:
Number of historical scenarios used: 503
Current portfolio value: $251,862.50
Portfolio VaR (5%): $4,577.27
Portfolio VaR with KDE smoothing (5%): $4,577.27
Portfolio ES (5%): $6,059.39

Risk metrics for individual stocks:

Table 7: Risk Metrics for Individual Stocks

Asset	VaR ($)	VaR (%)	ES ($)	ES (%)
SPY	872.43	1.47%	1,080.10	1.82%
AAPL	1,069.34	2.20%	1,437.79	2.95%
EQIX	3,650.99	2.54%	4,714.89	3.27%
Portfolio	**4,577.27**	**1.82%**	**6,059.39**	**2.41%**

C Discuss the differences between the methods.

1. Comparison of Risk Estimates

Looking at the portfolio risk metrics across all three methods:

- **Normal Distribution with EWMA Covariance:** Reports the lowest VaR ($3,856.32) and ES ($4,835.98), representing 1.53% and 1.92% of the portfolio value respectively.

Table 8: Portfolio Risk Metrics Comparison

	Normal with EWMA	T-dist with Copula	Historical Simulation
VaR ($)	3,856.32	4,074.76	4,577.27
ES ($)	4,835.98	5,844.19	6,059.39
VaR (%)	1.53	1.62	1.82
ES (%)	1.92	2.32	2.41

Table 9: SPY Risk Metrics Comparison

	Normal with EWMA	T-dist with Copula	Historical Simulation
VaR ($)	827.85	722.88	872.43
ES ($)	1,038.16	1,080.40	1,080.10
VaR (%)	1.40	1.22	1.47
ES (%)	1.75	1.83	1.82

Table 10: AAPL Risk Metrics Comparison

	Normal with EWMA	T-dist with Copula	Historical Simulation
VaR ($)	946.08	971.75	1,069.34
ES ($)	1,186.42	1,384.71	1,437.79
VaR (%)	1.94	2.00	2.20
ES (%)	2.44	2.84	2.95

Table 11: EQIX Risk Metrics Comparison

	Normal with EWMA	T-dist with Copula	Historical Simulation
VaR ($)	2,933.51	3,216.40	3,650.99
ES ($)	3,678.74	4,739.73	4,714.89
VaR (%)	2.04	2.23	2.54
ES (%)	2.55	3.29	3.27

- **T-Distribution with Gaussian Copula:** Shows intermediate risk estimates with VaR ($4,074.76) and ES ($5,844.19), or 1.62% and 2.32% of portfolio value.
- **Historical Simulation:** Produces the highest risk estimates with VaR ($4,577.27) and ES ($6,059.39), equivalent to 1.82% and 2.41% of portfolio value.

This pattern, where the normal distribution method produces the lowest risk estimates and historical simulation produces the highest, is consistent across both the portfolio level and individual stock level.

2. Theoretical Explanations for the Differences

a. Normal Distribution with EWMA Covariance

The normal distribution typically underestimates tail risk because financial returns often exhibit fat tails (larger probability of extreme events) than what a normal distribution would predict. This method assumes:

- Returns follow a normal distribution (which has thinner tails than actual market data)
- Recent observations are more relevant than older ones (through EWMA weighting)
- Linear correlations between assets

The EWMA component helps make this method more responsive to recent market conditions but still suffers from the normal distribution's limitations in capturing extreme events.

b. T-Distribution with Gaussian Copula

This method addresses some of the shortcomings of the normal distribution approach by:

- Using t-distributions for individual assets, which better capture the fat tails observed in financial returns
- Using a copula function to model the dependency structure between assets
- Allowing for more flexible modeling of the joint distribution

The higher risk estimates compared to the normal distribution approach reflect the t-distribution's better ability to model extreme events. The degrees of freedom parameter (around 5 for all assets) indicates relatively fat tails in the return distributions.

c. Historical Simulation

This non-parametric approach uses actual historical returns without making distribution assumptions. It produces the highest risk estimates because:

- It directly captures actual extreme events that occurred in the historical data
- It preserves the empirical characteristics of returns without smoothing
- It implicitly incorporates the true joint distribution and dependency structure between assets

The higher estimates suggest that the actual historical data contains more extreme events than would be predicted by either the normal or t-distribution models.

3 Problem 3

A Calculate the implied volatility

The Generalized Black-Scholes-Merton model for a European call option is given by:

$$C(S, K, T, r, b, \sigma) = S \cdot e^{(b-r)T} \cdot N(d_1) - Ke^{-rT} \cdot N(d_2) \tag{38}$$

where:

$$d_1 = \frac{\ln(\frac{S}{K}) + (b + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{39}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{40}$$

And:

- C = Call option price
- S = Current stock price
- K = Strike price
- T = Time to maturity (in years)
- r = Risk-free interest rate
- b = Cost-of-carry rate (equal to r when no dividends)
- σ = Volatility of the underlying asset
- $N(\cdot)$ = Cumulative distribution function of the standard normal distribution

Given market-observed option prices, implied volatility is the value of σ that, when inserted into the Generalized Black-Scholes-Merton formula, yields the market price. Mathematically, we need to solve:

$$C_{market} = C(S, K, T, r, b, \sigma_{implied}) \tag{41}$$

For $\sigma_{implied}$, where C_{market} is the observed market price of the option.

In this example, since no dividends are paid, we set $b = r$, which means $e^{(b-r)T} = 1$. This is why the Generalized Black-Scholes-Merton model yields the same numerical results as the standard Black-Scholes model in this specific case.

In Python, I use the brent method in brentq method to calculate the result σ. The implied Volatility: 0.3351

B Calculate the Delta, Vega, and Theta. Using this information, by approximately how much would the price of the option change is the implied volatility increased by 1%. Prove it.

Generalized Black-Scholes-Merton Model Preliminaries For a European option, the Generalized Black-Scholes-Merton model:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Where:

- S = Stock price
- K = Strike price
- T = Time to maturity (in years)
- r = Risk-free interest rate
- b = Cost-of-carry rate (equal to r when no dividends)
- σ = Volatility of the underlying asset

Delta Calculation Delta represents the rate of change of option value with respect to changes in the underlying asset's price. For European options, Delta is given by:

$$\Delta_{\text{call}} = e^{(b-r)T} \cdot N(d_1)$$

$$\Delta_{\text{put}} = e^{(b-r)T} \cdot (N(d_1) - 1)$$

Where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. For our example with $S = 31$, $K = 30$, $T = 0.25$, $r = 0.10$, $b = 0.10$, and $\sigma = 0.4066$:

$$d_1 = \frac{\ln\left(\frac{31}{30}\right) + \left(0.10 + \frac{0.4066^2}{2}\right)0.25}{0.4066\sqrt{0.25}} \approx 0.4287$$

$$\Delta_{\text{call}} = e^{(0.10-0.10)0.25} \cdot N(0.4287) \approx 0.6659$$

$$\Delta_{\text{put}} = e^{(0.10-0.10)0.25} \cdot (N(0.4287) - 1) \approx -0.3341$$

Vega Calculation Vega measures the sensitivity of the option price to changes in volatility.

$$\text{Vega} = S \cdot e^{(b-r)T} \cdot \sqrt{T} \cdot n(d_1)$$

Where $n(\cdot)$ is the probability density function of the standard normal distribution. For a 1% change in volatility (0.01 in absolute terms), Vega is typically scaled:

$$\text{Vega}_{1\%} = \frac{S \cdot e^{(b-r)T} \cdot \sqrt{T} \cdot n(d_1)}{100}$$

For our example:

$$n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{0.4287^2}{2}} \approx 0.3645$$

$$\text{Vega} = 31 \cdot e^{(0.10-0.10)0.25} \cdot \sqrt{0.25} \cdot 0.3645 \approx 5.6407$$

$$\text{Vega}_{1\%} = \frac{5.6407}{100} \approx 0.0564$$

Theta Calculation Theta measures the rate of change of the option price with respect to the passage of time. For European options:

$$\Theta_{\text{call}} = -\frac{S \cdot e^{(b-r)T} \cdot \sigma \cdot n(d_1)}{2\sqrt{T}} - (b-r) \cdot S \cdot e^{(b-r)T} \cdot N(d_1) - r \cdot K \cdot e^{-rT} \cdot N(d_2)$$

$$\Theta_{\text{put}} = -\frac{S \cdot e^{(b-r)T} \cdot \sigma \cdot n(d_1)}{2\sqrt{T}} + (b-r) \cdot S \cdot e^{(b-r)T} \cdot N(-d_1) + r \cdot K \cdot e^{-rT} \cdot N(-d_2)$$

For the daily Theta, we divide by the number of days in a year (365):

$$\Theta_{\text{call}}^{\text{daily}} = \frac{\Theta_{\text{call}}}{365}$$

$$\Theta_{\text{put}}^{\text{daily}} = \frac{\Theta_{\text{put}}}{365}$$

For our example:

$$d_2 = 0.4287 - 0.4066\sqrt{0.25} \approx 0.2253$$

$$N(d_2) \approx 0.5891$$

$$\Theta_{\text{call}} = -\frac{31 \cdot e^{(0.10-0.10)0.25} \cdot 0.4066 \cdot 0.3645}{2\sqrt{0.25}} - (0.10-0.10) \cdot 31 \cdot e^{(0.10-0.10)0.25} \cdot N(0.4287) - 0.10 \cdot 30 \cdot e^{-0.10 \cdot 0.25} \cdot 0.58$$

$$\Theta_{\text{call}} \approx -5.5446$$

$$\Theta_{\text{call}}^{\text{daily}} = \frac{-5.5446}{365} \approx -0.0152$$

$$\Theta_{\text{put}} \approx -2.6186$$

$$\Theta_{\text{put}}^{\text{daily}} = \frac{-2.6186}{365} \approx -0.0072$$

Verification of Vega

Using Vega, I can estimate that a 1% increase in volatility would increase the option price by approximately \$0.0564. To prove this, I calculated:

Original option price : \$3.0000

Option price after increasing volatility by 1% : \$3.0565

Actual price change: \$0.0565

The estimated price change using Vega (\$0.0564) is extremely close to the actual calculated price change (\$0.0565), with only a small difference of \$0.0001 due to the linear approximation nature of Greeks. This confirms that Vega provides an accurate estimation of how option prices respond to changes in volatility.

C Calculate the price of the put using Generalized Black Scholes Merton. Does Put-Call Parity Hold?

$$d_1 = \frac{\ln(\frac{31}{30}) + (0.10 + \frac{0.4066^2}{2}) \cdot 0.25}{0.4066 \cdot \sqrt{0.25}} \approx 0.4287$$

$$d_2 = 0.4287 - 0.4066 \cdot \sqrt{0.25} \approx 0.2253$$

$$P = 30 \cdot e^{-0.10 \cdot 0.25} \cdot N(-0.2253) - 31 \cdot e^{(0.10-0.10)0.25} \cdot N(-0.4287)$$

$$P = 29.26 \cdot 0.4109 - 31 \cdot 0.3341$$

$$P = 12.02 - 10.36 \approx 1.2593$$

Put-Call Parity Verification
The put-call parity relationship in the Generalized Black-Scholes-Merton model states:

$$P = C + Ke^{-rT} - Se^{(b-r)T}$$

Using our parameters with call price $C = 3.00$ and $b = r$:

$$P = 3.00 + 30 \cdot e^{-0.10 \cdot 0.25} - 31 \cdot e^{(0.10-0.10)0.25}$$

$$P = 3.00 + 30 \cdot 0.9753 - 31 \cdot 1$$

$$P = 3.00 + 29.26 - 31 \approx 1.2593$$

Based on the calculations:
Put price using Generalized Black-Scholes-Merton: \$1.2593
Put price using put-call parity: \$1.2593
Difference: \$0.0000 (less than 0.0001)
Therefore, put-call parity holds perfectly in this example. Note that since $b = r$ in this case (no dividends), the generalized model gives the same results as the standard model.

D Calculate VaR and ES for a 20 trading day holding period, at alpha=5using:

D.a Delta Normal Approximation

The Delta-Normal method approximates portfolio risk using linear sensitivities (deltas) to the underlying asset's price movements. For our portfolio:

$$\text{Portfolio Delta} = \Delta_{\text{call}} + \Delta_{\text{put}} + 1$$

With the previously calculated delta values:

- $\Delta_{\text{call}} = 0.6659$
- $\Delta_{\text{put}} = -0.3341$
- Stock delta = 1

The portfolio volatility is calculated as:

$$\text{Portfolio Volatility} = |\text{Portfolio Delta}| \times S \times \sigma_{\text{stock}} \times \sqrt{\frac{\text{holding period}}{\text{trading days}}}$$

The formulas for VaR and ES are:

$$\text{VaR}_\alpha = \text{Portfolio Value} \times (-z_\alpha) \times \frac{\text{Portfolio Volatility}}{S}$$

$$\text{ES}_\alpha = \text{Portfolio Value} \times \frac{-\phi(z_\alpha)}{\alpha} \times \frac{\text{Portfolio Volatility}}{S}$$

Where z_α is the z-score at the α confidence level and $\phi(z_\alpha)$ is the standard normal PDF at z_α.

Results:

- VaR (95%, 20 days): \$5.41
- ES (95%, 20 days): \$6.78

D.b Monte Carlo Simulation

For the Monte Carlo approach, I simulated 10,000 potential stock price paths over the 20-day period using the Generalized Black-Scholes-Merton model:

$$S_T = S_0 \exp\left((b - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\right)$$

Where b is the cost-of-carry rate (equal to r in this case with no dividends) and Z is a standard normal random variable.

For each simulated stock price, I:

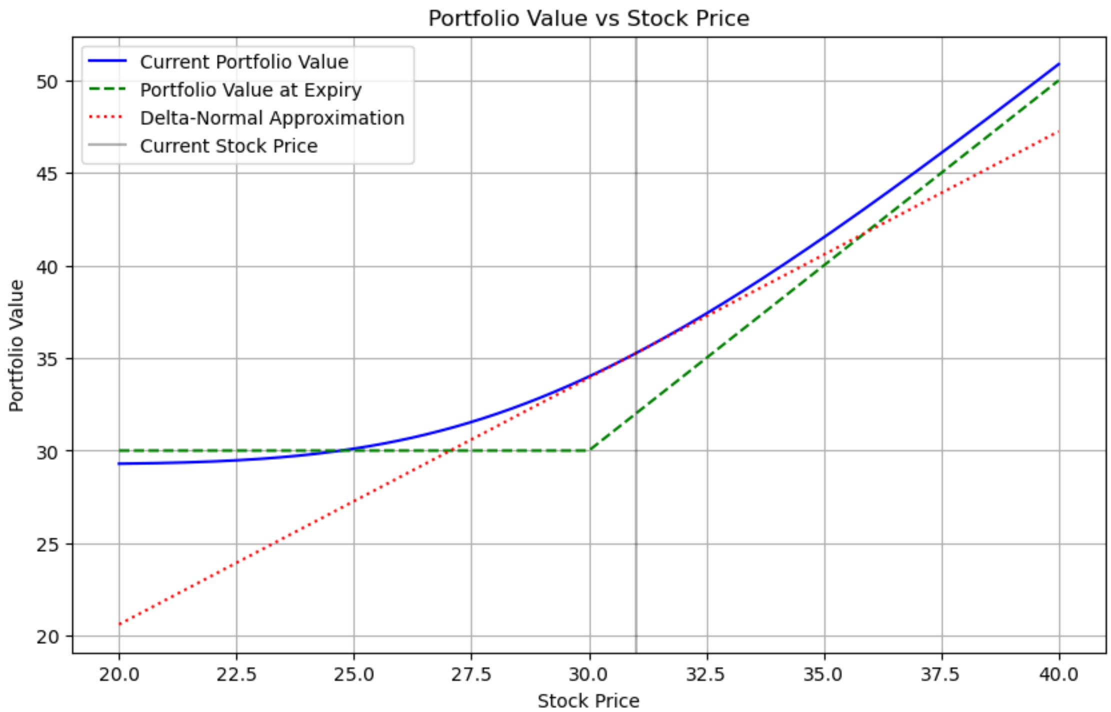
1. Calculated the remaining time to option expiration
2. Computed the new option values using the Generalized BSM model
3. Determined the new portfolio value (call + put + stock)
4. Calculated the change in portfolio value

VaR was then taken as the negative of the 5th percentile of the sorted portfolio value changes, and ES as the mean of losses beyond the VaR threshold.

Results:

- VaR (95%, 20 days): \$3.87
- ES (95%, 20 days): \$4.37

E Discuss the differences between the 2 methods.



Difference:

- The delta-normal approximation assumes that there is linear relationship between the portfolio value and stock price, while Monte Carlo simulation captures the complete nonlinear payoff structure of options and considers time decay.
- The portfolio value at expiry shows a kinked structure at the strike price (K = 30), reflecting the option payoff characteristics. The current portfolio value smooths this kink due to time value, but still maintains curvature that Delta-Normal cannot capture.
- Delta-Normal assumes portfolio returns follow a normal distribution, while Monte Carlo simulation better captures the typical skewness and fat tails of option portfolio distributions.
- The Delta-Normal approximation is relatively accurate near the current stock price (S = 31), but it increasingly diverges as the stock price moves further away. This demonstrates that Delta-Normal is only reliable for small price movements.