Project Evaluation Methods (PEM)

Chapter 2: Monte Carlo Simulation

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2.1. **Description of the methodology**

The **Monte Carlo simulation** consists of two main steps:

- <u>Step 1</u>
  Imitate the performance of the real system by using probability distributions to randomly generate various events that occur in the system. (therefore, a simulation model synthesizes the system by building it up component by component and event by event);

- <u>Step 2</u>
  Repeat several times the following procedure (which, is known as **simulation run**):
  - randomly generate an event, ***e***;
  - test the simulated system with the event ***e***, which allows to obtain a statistical observation on the performance of the system: ***o***.

2.1. **Description of the methodology**

Monte Carlo simulation – some considerations

- <u>Step 1</u> – **Development of a representative model of the real system under study**
  - Requires a deep knowledge of the system under study.
  - In many cases, the system is divided into a set of components which are connected by a master flow diagram, where the components themselves may be broken down into subcomponents, and so on.
  - The system is decomposed into a set of elements for which operating rules may be given. These operating rules predict the events that will be generated by the corresponding elements, perhaps in terms of probability distributions.

- <u>Step 2</u> – **Test the model**
  - This testing can be done partially by performing a gross version of the simulation on a computer and checking whether each input is received from the appropriate source and whether each output is acceptable to the next submodel.
  - The individual components of the model also should be tested alone to verify that their internal performance is reasonably consistent with reality.

2.1. **Description of the methodology**

Monte Carlo simulation – some considerations (cont'd)

- Rather than describe the overall behaviour of the system directly, the simulation model describes the operation of the system in terms of individual events of the individual components of the system. In particular, the system is divided into elements whose behaviour can be predicted, at least in terms of probability distributions, for each of the various possible states of the system and its inputs. The interrelationships among the elements also are built in the model.
- Simulated statistical experiments are performed on a computer because simulation runs typically require generating and processing a vast data set.
- Due to statistical error, it is impossible to guarantee that the solution produced by the simulation is indeed the optimal one, but it should be at least near optimal if the simulated experiment was designed properly.

2.1. **Description of the methodology**

Monte Carlo simulation – some considerations (cont'd)

- **When do we use Monte Carlo simulation?**

  Typically we use simulation when the stochastic system involved is too complex to be analyzed satisfactorily by analytical models. If it is possible to construct an analytical model, this approach usually is superior to simulation. However, many problems are so complex that they cannot be solved analytically. Thus, even though simulation is an expensive procedure (in terms of computer time), it often provides the only practical approach to a problem.

- The functional areas of a company where the Monte Carlo simulation has been applied are:
  - Production; Corporate planning; Engineering; Finance; Research and development; Marketing; Data processing; Personnel.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

A coin-flipping game (*)

Suppose a player has the opportunity to play the following game:

- Repeatedly flip an unbiased coin until the difference between the number of heads tossed and the number of tails tossed is three.

The player must pay €1 for each flip of the coin, but he receives €8 at the end of each play of the game. The player is not allowed to quit during a play of the game.
The player should participate in this game?

Note
Given the data of the game, the player will win money if the number of flips required is fewer than eight (therefore, the player will lose money if more than eight flips are required).

(*) Example adapted from Hillier, Frederick S. and Lieberman, Gerald J.; *Introduction to Operations Research*, 10th ed., McGraw-Hill, 2015.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

One possible solution

Simulate the game many times until it becomes clear whether it is worthwhile to play for money. The simulation can be performed in two versions: Version I e Version II.

Version I

Is to play the game on a trial basis (that is, without any commitment).

- For example, for half an hour, repeatedly flip a coin and record the earnings or losses that occur.

Version II

Is to simulate the game on the computer.

- Instead of flipping a coin, the computer generates a number; therefore, the computer generates a sequence of random digits, each corresponding to a flip of a coin.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Version II (Simulate the game on the computer)

- The probability distribution for the outcome of a flip is:

  P(outcome of a flip be a "head")=1/2 ;

  P(outcome of a flip be a "tail")=1/2.

- For example, if there are 10 possible values for a random digit, each having a probability of 1/10, then it is possible to affect five of these values to the outcome "head" (for example, 0, 1, 2, 3, 4) and the other five values to the outcome "tail" (in this example, would be 5, 6, 7, 8, 9).

- The computer would simulate the playing of the game by examining each new random digit generated and labeling it a head or a tail, according to its value. It would continue doing this, recording the outcome of each simulated play of the game, as long as desired.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Version II (Simulate the game on the computer)

Example of the simulation of the game using the computer

- List of random digits generated by the computer:
  8, 1, 3, 7, 2, 7, 1, 6, 5, 5, 7, 9, 0, 0, 3, 4, 3, 5, 6, 8, 5, 8, 9, 4, 8, 0, 4, 8, 6, 5, 3, 5, 9, 2, 5, 7, 9, 7, 2, 9, 3, 9, 8, 5, 8, 9, 2, 5 , 7, 6, 9, 7, 6, 0, 7, 3, 9, 8, 2, 7, 1, 0, 3, 2, 6, 2, 7, 1, 3, 7, 0, 4, 4, 1, 8, 3, 2, 1, 3, 9, 5, 9, 0, 5, 0, 3, 8, 7, 8, 9, 5, 4, 0, 8, 3, 8, 0, 1.
- Given the list of random digits, it was possible to simulate 14 plays of the game (see Table 1).
- Considering N_i as the number of flips required for the play of the game i, i=1,…,14, the average number of flips of a play of the game $(\bar{N})$ is given by:

$$\bar{N} = \frac{\sum_{i=1}^{14} N_i}{14} = \frac{11+5+5+9+\ldots+3+9+7}{14} = 7$$

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Version II (Simulate the game on the computer)

Example of the simulation of the game using the computer

Table 1

Play of the game	Random digits of the play game	Number of flips of the play game
1	8,1,3,7,2,7,1,6,5,5,7	11
2	9,0,0,3,4	5
3	3,5,6,8,5	5
4	8,9,4,8,0,4,8,6,5	9
5	3,5,9,2,5,7,9	7
6	7,2,9,3,9,8,5	7
7	8,9,2,5,7	5
8	6,9,7	3
9	6,0,7,3,9,8,2,7,1,0,3,2,6,2,7,1,3	17
10	7,0,4,4,1	5
11	8,3,2,1,3	5
12	9,5,9	3
13	0,5,0,3,8,7,8,9,5	9
14	4,0,8,3,8,0,1	7

- The average number of flips would seem to indicate that, on average, the player should win **€8-€7=€1** each time he plays the game. Therefore, if the player does not have a relatively high aversion to risk, it appears that he should decide to play this game, preferably many times.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Comment on the obtained solution

This example serves to illustrate one of the most common mistakes in the use Monte Carlo simulation, which is, in some cases, the conclusions are based on overly small samples (in this example, the conclusion was based on a sample size 14!). Thus, it is necessary to answer the following question:

What should be the number of simulated plays of the game (*M*) in order to draw valid conclusions from the experience?

Considering

N – the random variable representing the number of flips required for a play of the game;

N_i – the number of flips required for the play of the game i, , i=1,2,…,*M*; and

$\bar{N}$ – the estimate obtained by Monte Carlo simulation for the number of flips required for a play of the game;

is possible to calculate the standard deviation, as well as the average value of $\bar{N}$.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

What should be the number of simulated plays of the game (*M*) in order to draw valid conclusions from the experience? (cont'd)

I. The Average Value of the estimate, obtained by simulation, for the number of flips of a play of the game is given by:

$$E(\bar{\mathrm{N}}) = E\left(\sum_{\mathrm{i}=1}^{M} \mathrm{N_i} \middle/ M\right) \underset{(i)}{=} \frac{1}{M}\sum_{\mathrm{i}=1}^{M} E(\mathrm{N_i}) \underset{(ii)}{=} \frac{1}{M}\sum_{\mathrm{i}=1}^{M} E(\mathrm{N}) = \frac{1}{M}\left(M \times E(\mathrm{N})\right) = E(\mathrm{N})$$

Justifications:
(i) $\mathrm{N_i}$`s are independents; *(ii)* $\mathrm{N_i}$`s are identically distributed.

II. The Variance of the estimate, obtained by simulation, for the number of flips of a play of the game is given by :

$$var(\bar{\mathrm{N}}) = var\left(\sum_{\mathrm{i}=1}^{M} \mathrm{N_i} \middle/ M\right) \underset{(i)}{=} \frac{1}{M^2}\sum_{\mathrm{i}=1}^{M} var(\mathrm{N_i}) \underset{(ii)}{=} \frac{1}{M^2}\sum_{\mathrm{i}=1}^{M} var(\mathrm{N}) = \frac{1}{M^2}\left(M \times var(\mathrm{N})\right) = \frac{var(\mathrm{N})}{M}$$

Justifications:
(i) $\mathrm{N_i}$`s are independents; *(ii)* $\mathrm{N_i}$`s are identically distributed.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

What should be the number of simulated plays of the game (*M*) in order to draw valid conclusions from the experience? (cont'd)

III. The Standard Deviation of the estimate, obtained by simulation, for the number of flips of a play of the game is given by:

$$\sigma(\bar{\mathrm{N}}) = \sqrt{\frac{var(\mathrm{N})}{M}} = \frac{\sqrt{var(\mathrm{N})}}{\sqrt{M}} = \frac{\sigma(\mathrm{N})}{\sqrt{M}}$$

Given the expression of the standard deviation of the estimate, obtained by simulation, for the number of flips of a play of the game, we can say that the standard deviation of the estimate (that is, the uncertainty associated with the number of flips of a play of the game) is inversely proportional to the square root of *M* (that represents the number of simulated plays of the game). In particular, for increasing the accuracy by a factor 10 (that is, reduce the standard deviation by a factor 10), *M* must increase by a factor 100.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

What should be the number of simulated plays of the game (*M*) in order to draw valid conclusions from the experience? (cont'd)

It is also possible to provide a Confidence Interval for the estimate, obtained by simulation, for the number of flips of a play of the game:

a. Consider the variable $V = \dfrac{\bar{N} - E(\bar{N})}{\sigma(\bar{N})}$.

b. Given the Central Limit Theorem, V follows, approximately, the normal distribution (*0,1*). Considering the symmetry of this distribution, we can write:

$$P(-1{,}96 \leq V \leq 1{,}96) = 0{,}95.$$

This expression is equivalent to $P(E(\bar{N}) - 1{,}96 \times \sigma(\bar{N}) \leq \bar{N} \leq E(\bar{N}) + 1{,}96 \times \sigma(\bar{N})) = 0{,}95.$

Therefore, the interval for the estimate with 95% confidence is given by:

$$E(\bar{N}) - 1{,}96 \times \sigma(\bar{N}) \leq \bar{N} \leq E(\bar{N}) + 1{,}96 \times \sigma(\bar{N}).$$

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Comment on the obtained solution

Given the expression of the standard deviation of the estimate, obtained by simulation, for the number of flips of a play of the game (see slide 30), we can say that the standard deviation of the estimate is inversely proportional to the square root of M. In other words, a large increase in M is required to yield a relatively small increase in the precision of the estimate.
In the simulated game, we consider M=14 plays of the game and the accuracy value associated with the estimate of the number of flips of a play of the game is given by :

$$\sigma(\bar{N}) = \frac{\sigma(N)}{\sqrt{M}} = \frac{3{,}679}{\sqrt{14}} = 0{,}983.$$

Auxiliary calculations: $E(\bar{N}) = E(N) = 7$

$\sigma(N) = \sqrt{\frac{\sum_{i=1}^{14}(N_i-\bar{N})^2}{13}}$=3,679

The interval for the estimate with 95% confidence is as follows:

$$E(\bar{N}) - 1{,}96 \times \sigma(\bar{N}) \leq \bar{N} \leq E(\bar{N}) + 1{,}96 \times \sigma(\bar{N}) \Leftrightarrow$$

$$\Leftrightarrow 7 - 1{,}96 \times 0{,}983 \leq \bar{N} \leq 7 + 1{,}96 \times 0{,}983 \Leftrightarrow \mathbf{5{,}073 \leq \bar{N} \leq 8{,}927}$$

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Comment on the obtained solution (cont'd)

The analysis of the interval for the estimate with 95% confidence, leads to the following conclusion:

- the estimate of the number of flips of a play of the game can be greater than 8. That is, there is a possibility of the player lose money with their participation in the game.

In fact, repeating the simulation with M=100 states that the average number of flips of a play of the game is **9** (which confirms that the simulation experience with M=14 was very limited). In this case, the average number of flips seems to indicate that, on average, the player shall receive **€8–€9=–€1**, that is, the player shall lose 1€. Consequently, the player must not participate in the game.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Conclusion

To ensure reasonable accuracy to the estimate of the number of flips of a play of the game is necessary assign a large number to *M* (that is, a large number of simulation runs), which makes simulation procedure very costly in terms computing. However, it is possible to apply Variance – reduction techniques (see section 2.3.), which modify the original problem, reduce the standard deviation of the estimate ($\sigma(\bar{N})$) and allow to achieve a good precision with a smaller number of simulation runs.

Remark

The results presented in slides 29 and 30 are valid for any system that is studied using the Monte Carlo simulation.

2.1. **Description of the methodology**

Monte Carlo Simulation – Example 1

Version II (Simulate the game on the computer)

Formal definition of the simulation model on the game

- The **stochastic system being simulated** is the successive flipping of the coin for a play of the game.
- The information about the system that defines its current status (i.e., the state of the system) is **D(*t*)=number of "heads" minus number of "tails", after *t* flips**.
- The **events** that change the state of the system are the flipping of a "head" or the flipping of a "tail".
- The **event generation mechanism** is the generation of a random digit:
  if the generated random digit is 0, 1, 2, 3 or 4 $\Rightarrow$ flipping of a "head";
  if the generated random digit is 5, 6, 7, 8 or 9 $\Rightarrow$ flipping of a "tail".
- The **state transition mechanism** is defined by

$$\mathbf{D}(t)=\begin{cases}\mathbf{D}(t-1)+1, & \text{if the flipping } t \text{ is a "head"}\\ \mathbf{D}(t-1)-1, & \text{if the flipping } t \text{ is a "tail"}\end{cases}$$

- The simulated game ends at the first value of ***t*** where **D(*t*)=±3**, where the resulting sampling observation for the simulated experiment is **8–*t*** (i.e., the amount won, positive or negative, for that play of the game).

2.2. **Implementation of a simulation model**

As we saw in the previous section, the implementation of a simulation model requires random numbers to obtain random observations from probability distributions, which are essential for the generation of events relevant to the system under study.

To generate random numbers you can use:

I. a physical device such as a spinning disk or an electronic randomizer.
Several tables of random numbers have been generated with this process. In particular, we highlight a table containing 1 million of random digits, published by the Rand Corporation (The Rand Corporation, A Million Random Digits with 100,000 Normal Deviates. Copyright, The Free Press, Glencoe, IL, 1955); or

II. a computer, which has a random number generator.

- **Random Number Generator** is an algorithm that produces sequences of numbers that follow a specified probability distribution and possess the appearance of randomness.

In this course we use the generator of Excel.

2.2. **Implementation of a simulation model**

Random Number Generator – some considerations

- The reference to sequences of numbers means that the algorithm produces many random numbers in a serial manner.
- Although an individual user may need only a few of the random numbers, generally the algorithm must be capable of producing many random numbers.
- Since the numbers generated following a certain probability distribution, it is possible to associate a probability information to the occurrence of each of the numbers produced by the algorithm. This observation leads to the following definition:

  Random Number
  Represents a random observation from some form of a Uniform Distribution, so that all possible numbers are equally likely.

  NOTE
  When we are interested in some other probability distribution, we shall refer to these numbers as **random observations** from that distribution.

2.2.1. **Random Number Generation**

Random Numbers can be divided into two main categories:

Random Integer Numbers and **Uniform Random Number**.

- **Random Integer Numbers**
  Is a random observation from a Discretized Uniform Distribution over some range $\underline{n}, \underline{n}+1, \ldots, \overline{n}$. The probabilities for this distributions are

$$P(\underline{n}) = P(\underline{n}+1) = \ldots = P(\overline{n}) = \frac{1}{\overline{n}-\underline{n}+1}.$$

Note
Usually, $\underline{n}=0$ or 1.

- **Uniform Random Numbers**
  Is a random observation from a (continuous) Uniform Distribution over some interval $[a, b]$. The probability density function of this Uniform Distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Note
When a and b are not specified, they are assumed to be a=0 and b=1.

2.2.1. **Random Number Generation**

- **COMMENT**

Strictly speaking, the numbers generated by the computer should not be called random numbers because they are **predictable** and **reproducible** (which sometimes is advantageous), given the random number generator being used. Therefore, they are sometimes given the name **pseudo– random numbers**. However, the important point is that they satisfactorily play the role of random numbers in the simulation if the method used to generate them is valid.

2.2.1. **Random Number Generation**

Various relatively sophisticated statistical procedures have been proposed for testing whether a generated sequence of numbers has an acceptable appearance of randomness (that is, each successive number in the sequence have an equal probability of taking on any one of the possible values and that it be statistically independent of the other numbers in the sequence.

There are several random number generators available, of which the most popular are the Congruential Methods, namely

- Mixed Congruential Method;
- Multiplicative Congruential Method;
- Additive Congruential Method.

The detailed study of the Congruential Methods is outside of the scope of this course. To study these methods, for example, you may consult Hillier, Frederick S. and Lieberman, Gerald J.; *Introduction to Operations Research*, 9th ed., McGraw-Hill, 2009.

In this course, we use the random number generator of Excel.

2.2.2. **Random Observations Generation from a Probability Distribution**

Random observations generation from a probability distribution depends on the distribution under study. In this Course, will be applied Excel functions to generate random observations from a specific probability distribution.

- **Random Observations from a simple discrete distribution**

  The generation of random observations from a simple discrete distribution consists of allocate the possible values of a random number to the various numbers in the probability distribution in direct proportion to the respective probabilities of those numbers. This type of generation was applied in Example 1 of Monte Carlo simulation (A coin-flipping game - see slides 22, 23 and 24). Another example of application is the following:

  Example

  Consider the throw of two dice. It is known that the probability of throwing a **2** is 1/36 (as is the probability of throwing a **12**), the probability of throwing a **3** is 2/36, and so on. Therefore, 1/36 of the possible values of a random integer number should be associated with throwing a **2**, 2/36 of the values with throwing a **3**, and so forth.

2.2.2. Random Observations Generation from a Probability Distribution

- **Random Observations from a Known Probability Distribution**

The generation of random observations from complicated probability distributions, whether discrete or continuous, is done using excel generator functions. The following probability distributions will be studied:

- Exponential distribution;
- Gamma distribution;
- Normal distribution;
- Chi-Square distribution;
- Triangular distribution;
- Log-Normal distribution;
- Binomial distribution;
- Poisson distribution;
- (…)

2.2.3. **Model Validation**

As mentioned earlier, a simulation model is composed of a large number of elements, rules and logical links. Thus, many approaches can occur with smallest errors (despite the individual components of the model have been tested), which can lead to gross distortions in the output of the model.

This observation, leads to suggest that after writing and perfecting the simulation program, we should test the validity of the model in order to predict, with some accuracy, the behaviour of the simulated system.

How to test the validity of the model?

- If some form of the real system has already been in operation, then these performance data should be compared with the corresponding data output of the model.

  How is made the comparison between the two data sets?

  - Sometimes, it is possible to apply statistical tests to verify whether differences in means, variances and probability distributions (generated by the two data sets) are statistically significant.
  - If the data are not susceptible to statistical analysis then we should contact people who know the system's behaviour, which can analyze the two data sets.

2.2.3. **Model Validation**

How to test the validity of the model? (cont'd)

- If there are no historical data to the real system, then we can develop a *practical test* to collect some real data (which are compared with the output of the simulation model).

  Notes

  i. To perform a *practical test* may be necessary:

     - build a small prototype of a version of the system and put it into operation; or
     - modify, temporarily, an existing system in order to correspond the characteristics of the system under study.

     In either case, the development of a *practical test* is, normally, a costly and time-consuming process.

  ii. If there are no historical data of the system under study (real data or created from a *practical test*) then the only way to validate the model is to ask the assistance of experts to test the credibility of the output of the model for various situations.

2.3. **Variance-Reducing Techniques**

As noted above, to obtain a reasonable accuracy for the performance of the model we need a larger number of simulation runs. Thus, generally, a considerable computer time is required to implement a Monte Carlo simulation procedure.
In order to mitigate this disadvantage, special techniques have been developed to increase the accuracy (that is, reduce the variance) of sample the estimators. These variance-reducing techniques modify the original problem and will help achieve a reasonable accuracy with a smaller number of simulation runs.

In the literature there are various techniques to reduce the variance of which are: *Antithetic Variable Technique*, *Control Variate Technique*, *Importance Sampling*, *Stratified Sampling*, *Moment Matching*, *Using Quasi-Random Sequences* and *Representative Sampling through a Tree*. Because all of these techniques tend to be rather sophisticated, it is not possible to explore them deeply here.

In this course, only one technique is presented: *Antithetic Variable Technique*. The other techniques can be found in Hull (2000) (*) and in Trigeorgis (2000) (**).

(*) Hull, John C. 2000. Options, Futures and other Derivatives. *Prentice-Hall International, Inc., fourth edition.*
(**) Trigeorgis, L. 2000. Real Options: Managerial Flexibility and Strategy in Resource Allocation. *Mit Press, London.*

2.3.1. *Antithetic Variable Technique*

This technique may be presented as follows:

- In each simulation run i, i=1, …,*M*,
  - generate a pair of random observations (from the probability distribution under study) with high negative correlation: $(O_{1,i}, O_{2,i})$;
  - determine the arithmetic mean of the pair of observations: $\hat{O}_i = \dfrac{O_{1,i} + O_{2,i}}{2}$;

  Note

  This procedure produces good results because when the value of a point is higher than the true value (under study) the other tends to be less and vice versa).

- The final estimate of the value in study, called $\bar{O}$, is the arithmetic mean of the averages $\hat{O}_1, \hat{O}_2, ..., \hat{O}_M$, that is, $\bar{O} = \sum_{i=1}^{M} \hat{O}_i \Big/ M$.

Particular case – Method of Complementary Random Numbers

In each simulation run i, i=1, …,*M*, we generate a uniform random number r_i and consider its complementary random number $1-r_i$. Using the pair of uniform random numbers we generate a pair of random observations from the probability distribution under study: $(O_{1,i}, O_{2,i})$. The remaining steps are identical to the procedure described in *Antithetic Variable Technique*.

2.3.3. **Example**

Consider a random variable, X, that follows the Exponential distribution with parameter α=1. For the variable X, we know that the average value is equal to $E(X)=1/\alpha=1$ and the variance is equal to $Var(X)=1/\alpha^2=1$.

Supposing that the average value of the random variable X is not known. In this example, we intend to apply the Monte Carlo simulation and variance-reducing techniques for estimating the average value of X. We divided the exercise into two parts:

- **Part I**, which corresponds to the application of the Monte Carlo approach without variance-reducing techniques;
- **Part II**, which corresponds to the application of the Monte Carlo approach with the *Method of Complementary Random Numbers*.

2.3.3. **Example**

Obtaining an estimate for the mean value of X – Part I

- The procedure of Monte Carlo simulation consists of the following steps:

  Step 1 – Generate M uniform random numbers: $r_1, r_2, \ldots, r_M$;

  Step 2 – Generate M random observations from the Exponential distribution with α=1:

  $$x_i = -\ln(1 - r_i),\ i=1, \ldots, M;$$

  Step 3 – Determine the estimate for the mean value of the random variable X:

  $$\overline{x} = \sum_{i=1}^{M} x_i \Big/ M.$$

- Summary table
  [MC Simulation (M=10)(*)]

Table 2

i	Uniform random number (r_i) [seed=443]	Random observation from the Exponential distribution with α =1 [x_i = - ln(1 - r_i)]
1	0,045	0,046
2	0,546	0,790
3	0,159	0,173
4	0,393	0,500
5	0,716	1,259
6	0,447	0,592
7	0,333	0,406
8	0,411	0,529
9	0,544	0,785
10	0,817	1,697
Estimate for the mean value of the Exponential distribution with α =1		**0,678**

(*) Only ten runs (M=10) were considered to illustrate the Monte Carlo simulation and the variance-reducing techniques. However, in a practical exercise M would have to assume a very high value.

2.3.3. **Example**

Comment on the estimate obtained for the mean value of X – Part I

- The estimated mean value of X is 0.678, which is well below the true average value (which is equal to 1).
- The standard deviation of the sample average is $\sigma(\overline{\mathrm{x}}) = \frac{\sigma(\mathrm{X})}{\sqrt{M}} = \frac{1}{\sqrt{10}} = 0.316$ (see slide 29), that represents the uncertainty associated with the estimate of the average value of X.
- Because the standard deviation is inversely proportional to $\sqrt{M}$, this sample size would need to be quadrupled to reduce this standard deviation by one-half (that is, the value of M would have to be changed from 10 to 4×10=40).
- These observations suggest the need for other techniques to obtain more precise estimates and more efficiently.

2.3.3. **Example**

Obtaining an estimate for the mean value of X – Part II

- The procedure of Monte Carlo simulation with the *method of complementary random numbers* consists of the following steps:

  Step 1 – Generate M=10 uniform random numbers: $r_1, r_2, \ldots, r_{10}$;

  Step 2 – Consider the complementary numbers to the numbers generated in **Step 1**: $(1-r_1), (1-r_2), \ldots, (1-r_{10})$;

  Step 3 – For each pair of complementary uniform random numbers, $[r_i, (1-r_i)]$, i=1, …, 10, generate a pair of random observations from the Exponential distribution with α=1: $[O_{1,i}, O_{2,i}]=[-\ln(1-r_i), -\ln(r_i)]$, i=1, …,10

  Step 4 – For each pair of random observations $[O_{1,i}, O_{2,i}]$, determine the arithmetic mean $\hat{O}_i = (O_{1,i}+O_{2,i})/2$, i=1, …, 10

  Step 5 – Determine the estimate for the mean value of the random variable X:

$$\bar{x} = \sum_{i=1}^{10} \hat{O}_i \Big/ 10.$$

2.3.3. **Example**

Obtaining an estimate for the mean value of X – Part II (cont'd)

- Summary Table

  [MC simulation with the *method of complementary random numbers* (M=10) [(*)]]

Table 3

i	Uniform random number (r_i) [seed=443]	Random observation from the Exponential distribution with α =1 [$O_{1,i}$= - ln(1 - r_i)]	Complementary uniform random number (1-r_i)	Complementary random observation from the Exponential distribution with α =1 [$O_{2,i}$ = - ln(r_i)]	$\hat{O}_i = (O_{1,i}+O_{2,i})/2$
1	0,045	0,046	0,955	3,094	1,570
2	0,546	0,790	0,454	0,604	0,697
3	0,159	0,173	0,841	1,839	1,006
4	0,393	0,500	0,607	0,933	0,716
5	0,716	1,259	0,284	0,334	0,796
6	0,447	0,592	0,553	0,806	0,699
7	0,333	0,406	0,667	1,098	0,752
8	0,411	0,529	0,589	0,890	0,709
9	0,544	0,785	0,456	0,609	0,697
10	0,817	1,697	0,183	0,202	0,950
Estimate for the mean value of the Exponential distribution with α =1					0,859

[(*)] Only ten runs (M=10) were considered to illustrate the Monte Carlo simulation and the variance-reducing techniques. However, in a practical exercise M would have to assume a very high value.

2.3.3. **Example**

Comment on the estimates of the average value of X - Parts I and II

- As mentioned in previous slides, only ten runs were considered to ilustrate the Monte Carlo simulation and the variance-reducing techniques. In the table below are the estimates for the average value of the random variable X considering *M*=250, 500, 1000:

Table 4

Estimates for the average value of the random variable X		
Number of simulations runs (*M*)	Monte Carlo simulation without variance-reducing techniques	Monte Carlo simulation with the *Method of Complementary Random Numbers*
250	**0,928**	**0,962**
500	**0,935**	**0,975**
1000	**0,964**	**0,996**

2.3.3. **Example**

<u>**Comment on the estimates of the average value of X - Parts I and II**</u> (cont'd)

- In this exercise, the application of the variance-reducing technique *Method of complementary random numbers* leds to more precise estimates of the average value of the random variable X than that obtained with the procedure of Monte Carlo simulation (without additional techniques).

- Typically, in comparison with the Monte Carlo simulation (without additional techniques), the variance-reducing technique ensure
  - → more precise estimates with equal computation time; or
  - → estimates of equal accuracy with less computation time.

- Even though additional analysis may be required to incorporate one or more of these variance-reducing techniques into the simulation study, the rewards should not be forgone readily. At least, the *method of complementary random numbers* can be applied simply by repeating the original simulation run, substituting the complements of the original uniform random numbers to generate the corresponding random observations.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project(*)

- Description of the investment project

A company from the computer area is planning to develop a new model of personal computer. For this reason, the management department of the company would like to know the likely financial performance and the extent of financial risk of this investment.

Initial outlays of €50000 are required in arranging contracts for the supply of components, tooling up and hiring computer technicians who would assemble the new PCs. Are also known overheads and marketing costs, which are estimated at €500 per PC sold.

(*) Example adapted from Dayananda, D., Irons, R., Harrison, S., Herbohn, J.; Rowland, P. *Capital Budgeting – Financial Appraisal of Investment Projects*, University Press, Cambridge, 2002.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Description of the investment project (cont'd)

The management department of the computer firm has identified four uncertain variables which will be important in determining the performance of this project:

i. **PC sales in each year**;

ii. **Market price, per PC**;

iii. **Component cost, per PC**; and

iv. **Labour cost, per PC**.

The management department adopts a planning horizon of five years for the project (because, management considers that after five years the PC will have run its useful life) and a discount rate of 7%.

The management department decided that the Triangular distribution is an acceptable approximation for the four variables. The parameters of the Triangular distribution for each of the four uncertain variables are indicated in table 5.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Description of the investment project (cont'd)

Table 5

Variables	Pessimistic Value (i)	Modal Value (ii)	Optimistic Value (iii)
Sales (PCs a year)	50	100	130
Market Price (euros per PC)	2200	2500	3000
Component Cost (euros per PC)	1200	1000	900
Labour Cost (euros per PC)	350	300	200

(i) The pessimistic value is that value which would lead to the poorest investment outcome (e.g., the lowest PC price or the highest component cost);

(ii) The modal value represents the highest point in the probability distribution (for discrete variables, the modal value represents the most likely value);

(iii) The optimistic value is the most favourable value in terms of investment outcome.

Note
In practice, probability distributions are usually specified for only the most important cash flow variables (in this example, we consider four variables). But it is hoped that these variables capture most of the investment uncertainty.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Description of the investment project (cont'd)

Given the variables selected to define project performance, the expression of net cash flow for each year t, t=0, 1,…,5, is given by

$$\mathbf{NCF_t = SR_t - CO_t - OC_t}\text{, where:}$$

- $\mathbf{SR_t = P_t \times NPC_t}$ represents the gross sales receipts from PC sales in year t;
- $\mathbf{OC_t = NPC_t \times (CPC_t + LC_t + OMC_t)}$ represents the operational cost in year t;
- $\mathbf{CO_t}$ represents the capital outlay in year t;
- $\mathbf{NCF_t}$ represents the net cash flow in year t;
- $\mathbf{P_t}$ represents the sale price of a PC in year t;
- $\mathbf{NPC_t}$ represents the number of PCs sold in the year t;
- $\mathbf{CPC_t}$ represents the component cost, per PC, in year t;
- $\mathbf{LC_t}$ represents the labour cost, per PC, in year t;
- $\mathbf{OMC_t}$ represents the overhead and marketing costs, per PC, in year t.

2.4. **Case studies of assessment using specific software**

<u>Example 1</u> – Evaluation of a computer project (cont'd)

- <u>Description of the investment project</u> (cont'd)

  After calculating the net cash flow for each year t, t=0,1,…,5, we can compute the Net Present Value (NPV):

$$\sum_{t=0}^{5} NCF_t \big/ (1+r)^t,$$

  where r is the discount rate.

  The firm wishes to obtain information on the NPV of the project. If the study points to a positive NPV, then the project will be rated as acceptable.

- <u>Objective</u>
  Present estimates for the NPV of the project using Monte Carlo simulation.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Estimates for the NPV of the project – Application of Monte Carlo Simulation

  To obtain estimates of the NPV of the project, using Monte Carlo simulation, it is necessary to repeat *M* times the following procedure(*):

Step 1 – Generate random observations from the variables that are crucial to the net cash flow of the project

The variables NPC_t, P_t, CPC_t and LC_t, t=1,2,3,4,5, follow the Triangular distribution with parameters (*a*, *b*, *c*) (see Table 6 – slide 70).

Step 2 – For each year t, calculate the net cash flow of the project:

$$NCF_t = SR_t - CO_t - OC_t\,,\ t=0,1,2,3,4,5.$$

Step 3 – Calculate the NPV of project: $NPV = \sum_{t=0}^{5} NCF_t \big/ (1+r)^t$

(*) To illustrate the Monte Carlo simulation, we consider *M* = 100. However, in a practical exercise *M* would have to assume a very high value.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- **Step 1** - Generate random observations from the variables that are crucial to the net cash flow of the project
  - Generate random observations from the **variable** $\mathbf{NPC_t}$, which follows the Triangular distribution with *a*=50, *b*=100 and *c*=130;
  - Generate random observations from the **variable** $\mathbf{P_t}$, which follows the Triangular distribution with *a*=2200, *b*=2500 and *c*=3000;
  - Generate random observations from the **variable** $\mathbf{CPC_t}$, which follows the Triangular distribution with *a*=900, *b*=1000 and *c*=1200;
  - Generate random observations from the **variable** $\mathbf{LC_t}$, which follows the Triangular distribution with *a*=200, *b*=300 and *c*=350.

  - **Note**
    The excel generator function **NTrandTriangular** was used to generate random observations from Triangular distribution.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- **Step 2** - For each year t, calculate the net cash flow of the project
  - Exemplification of a simulation run

    Suppose that the variables NPC_t, P_t, CPC_t and LC_t, in each year t, t=1, …,5, assume the values shown in Table 6.

Table 6

Year [t]	**Sales** (PCs per year) [variable NPC_t]	**Market Price** (Euros per PC) [variable P_t]	**Component Cost** (Euros per PC) [variable CPC_t]	**Labour Cost** (Euros per PC) [variable LC_t]
1	82	2636,32	988,25	299,25
2	112	2692,4	988,75	277,3
3	103	2732,49	946,3	298,67
4	63	2545,71	975,8	291,4
5	91	2564,59	983,58	297,75

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- **Step 2** - For each year t, calculate the net cash flow of the project (cont'd)
  - Exemplification of a simulation run (cont'd)

Given the values generated for each of the four variables, it is possible to determine, using the model defined on slide 58, the project's net cash flow for each year t, t = 0,1, ..., 5, (considering CO_0 = 50000, $OMC_0=OC_0=SR_0$= 0 and CO_t = 0, t = 1, ..., 5):

Table 7

Year [t]	**Capital Outlay** (Euros per year) [CO_t]	**Overhead and Marketing Costs** (Euros per year) [OMC_t]	**Operational Costs** (Euros per year) [$OC_t=NPC_t\times(CPC_t+LC_t+OMC_t)$]	**Gross Sales Receipts** (Euros per year) [$SR_t=P_t\times NPC_t$]	**Net Cash Flow** (Euros per year) [$NCF_t=SR_t-CO_t-OC_t$]
0	50000,00	0,00	0,00	0	-50000
1	0,00	500,00	146575,00	216178,24	69603,24
2	0,00	500,00	197797,60	301548,8	103751,20
3	0,00	500,00	179731,91	281446,47	101714,56
4	0,00	500,00	111333,60	160379,73	49046,13
5	0,00	500,00	162101,03	233377,69	71276,66

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- **Step 3** – Calculate the NPV of project
  - Exemplification of a simulation run (cont'd)

  Given the net cash flows shown in Table 7, the Net Present Value of the project is given by:

$$NPV = \sum_{t=0}^{5} NCF_t \Big/ (1+r)^t =$$

$$= -50000 + \frac{69603{,}24}{1{,}07} + \frac{103751{,}20}{1{,}07^2} + \frac{101714{,}56}{1{,}07^3} + \frac{49046{,}13}{1{,}07^4} + \frac{71216{,}66}{1{,}07^5} =$$

$$= 276935{,}78.$$

The goal is to repeat the simulation run M=100 times to obtain a sample of 100 estimates for the NPV of the project.

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Results obtained by Monte Carlo simulation

The application of the Monte Carlo simulation procedure, described in slide 62, leads to the estimates for the value of the NPV presented in Table 8.

Table 8

Run K	NPV	Run K	NPV	Run K	NPV	Run K	NPV	Run K	NPV
1	263565,33	21	326901,69	41	212774,25	61	247230,88	81	242963,10
2	209761,06	22	282481,73	42	224107,27	62	256671,92	82	231388,10
3	229605,45	23	311136,13	43	219689,45	63	237012,53	83	322453,72
4	229535,39	24	238928,81	44	224804,31	64	187718,66	84	218853,42
5	231772,23	25	184961,93	45	265400,48	65	262430,74	85	285529,07
6	219533,77	26	266520,23	46	260122,60	66	232150,57	86	180213,24
7	203226,19	27	267253,75	47	287099,78	67	233645,45	87	293269,41
8	285277,56	28	201506,29	48	164017,43	68	192001,66	88	205960,23
9	255644,18	29	248154,72	49	207085,51	69	315324,65	89	270332,96
10	175976,93	30	286118,29	50	253576,79	70	210414,03	90	233422,62
11	264216,75	31	244429,28	51	242697,07	71	186015,66	91	228944,51
12	239329,04	32	212476,00	52	245385,63	72	266318,12	92	197032,95
13	297470,35	33	271610,67	53	153851,14	73	260062,16	93	283887,23
14	206846,06	34	222572,98	54	272704,94	74	187481,30	94	165367,19
15	256708,96	35	287767,74	55	221197,77	75	229907,90	95	212079,92
16	261079,26	36	206449,27	56	205602,00	76	287005,30	96	206739,50
17	226718,53	37	236639,79	57	252445,47	77	301003,63	97	216377,60
18	300395,95	38	238982,32	58	192238,52	78	202853,91	98	250454,33
19	147908,20	39	275705,99	59	189208,83	79	194987,75	99	260660,55
20	299653,95	40	209979,42	60	254346,50	80	251474,99	100	177460,41

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Results obtained by Monte Carlo simulation(cont'd)

  The sample of estimates for the NPV of the project may be subject to a descriptive statistical analysis (see Table 9), which allows us to state that:

  - The NPV values are between €147908,20 and €326901,69;
  - The estimate for the NPV of the project is €237982,60 (which corresponds to the sample mean of 100 values of the NPV of the project);
  - (…); and
  - The project may be classified as acceptable, since the range of NPV values comprises only positive values.

Table 9

Sample of estimates for the NPV - Descriptive Statistics	
Average	237982,5978
Standard error	3914,829043
Median	236826,1622
Mode	#N/D
Standard deviation	39148,29043
Sample variance	1532588644
Kurtosis	-0,468121394
Asymmetry	0,047004181
Interval	178993,493
Minimum	147908,196
Maximum	326901,689
Sum	23798259,78
Score	100

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Results obtained by Monte Carlo simulation(cont'd)

  The absolute frequencies can be summarized by a histogram(see chart 1). However, it is more useful to express the information about the NPV in cumulative relative frequencies (see chart 2), since, for any value of NPV (see the x-axis), the y-axis indicates an estimate of the probability of NPV be this value or less. Thus, the cumulative relative frequency curve provides us with a good deal of information about the likely financial performance. As examples, we have the following readings:

  - it would appear that a payoff of less than €183706,89 is highly unlikely, with an estimated probability of only about 0,07;
  - a payoff of more than €309002,34 is extremely unlikely, since the estimated probability is 1−0,96=0,04;
  - it is possible to indicate the probability (estimate) of the payoff be between two values, for example, between €201606,24 and €237404,94:

$$P(NPV \leq 237404{,}94) - P(NPV \leq 201606{,}24) = 0{,}51 - 0{,}17 = 0{,}34.$$

2.4. **Case studies of assessment using specific software**

Example 1 – Evaluation of a computer project (cont'd)

- Results obtained by Monte Carlo simulation(cont'd)

Chart 1

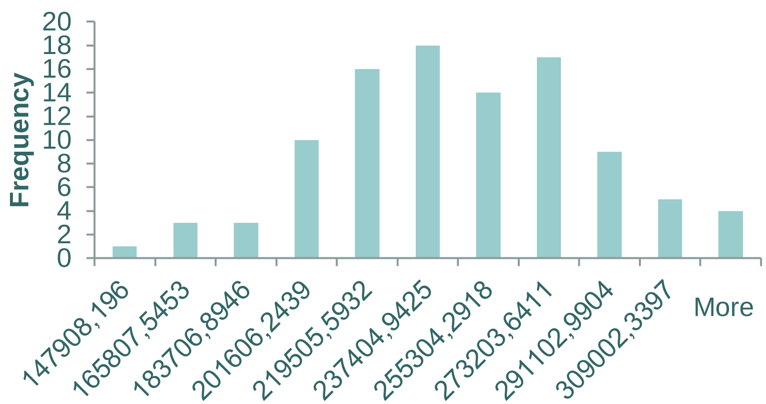
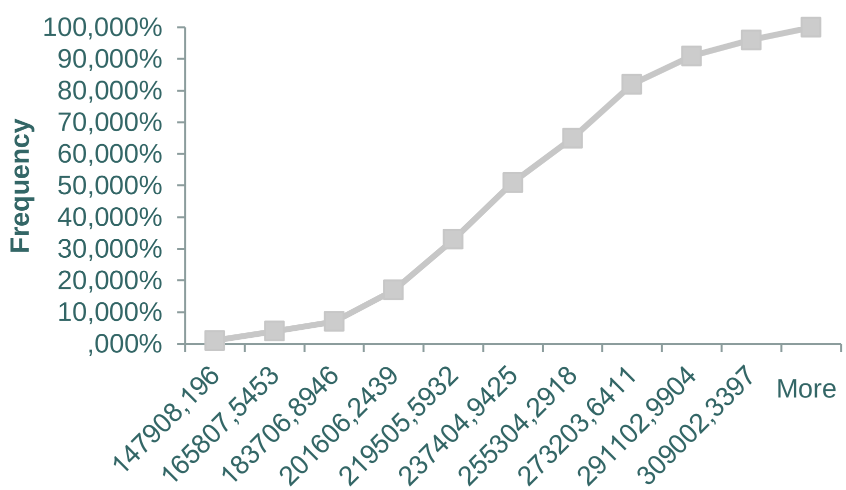


Chart 2



Note

The cumulative relative frequency curve may be viewed as an estimate of the cumulative (probability) density function (CDF) of the NPV variable. The term 'estimate' is used because it is derived from sampling, not from a precise knowledge of the overall behaviour of NPV (the statistical concept of a 'population'). However, in practice we may refer to the curve, if somewhat loosely in statistical terms, as the CDF.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline[(*)]

- Description of the investment project

An airline company is planning to introduce a new country run, to provide two return services a week (104 services per year) to a rural city with a population of 30000 persons.
The airline will use a forty-passenger aircraft, which it can purchase for 4.2 million euros.

An airstrip owned by the local government in the rural area is available without charge, but the company will have to carry out restoration of the disused airstrip and terminal at a cost of €200000.

[(*)] Example adapted from Dayananda, D., Irons, R., Harrison, S., Herbohn, J.; Rowland, P. *Capital Budgeting – Financial Appraisal of Investment Projects*, University Press, Cambridge, 2002.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Objective

The company intends to develop a financial model to

i. simulate the net cash flows from this project over seven-year period; and
ii. provide an estimate of the net present value of the investment.

- Relevant information about the project

  - The demand of passengers in year **t** (one-way flights) is estimated according to the expression $\mathbf{5000+300\times(t-1)+w}$, $\mathbf{t}=1,\ldots,7$, where **w** denotes a random component that follows the Normal distribution with mean value of zero and standard deviation of 400 passengers.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Relevant information about the project (cont'd)
  - In addition to passenger services, the company has a contract to transport mail to the rural city. The estimated revenue from the mail service in year **t** is given by the expression **€200000+v**, **t**=1,…, 7, where **v** denotes a random component that follows the Normal distribution with mean value of zero and standard deviation of €10000.
  - The cost of fuel per flight in each year t, t = 1, ..., 7, is a random variable that follows the Log-Normal distribution with mean value of €1500 and standard deviation of €100.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Relevant information about the project (cont'd)
  - The personnel outlay in year **t** is estimated according to the expression **€250000+q**, **t**=1,…,7, where **q** denotes a random component that follows the Normal distribution with mean value of zero and standard deviation of €25000.
  - The annual maintenance cost for the aircraft is estimated at 5% of the initial price (that is, is equal to 0,05×€4200000=€210000).
  - The cost of booking and other services in year **t**=1,…,7 is €10 per passenger flight.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Relevant information about the project (cont'd)
  - The price of one-way airline ticket is a random variable that follows the Log-Normal distribution with mean value of €280 and standard deviation of €20.
  - The appropriate discount rate is 8%.
  - Company tax is 30% of the annual operating surplus.
  - The aircraft and ground facilities can be depreciated at a rate of 10% per annum for taxation purposes (that is, is equal to 0,1×€4400000=€440000).
  - After seven years, the aircraft has a salvage value of 1 million of euros.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Expression of the Net Cash Flow in each year

  Taking into account the data associated with the project the expression of the net cash flow in each year **t** = 0, 1, ..., 7 is given by

$$\mathbf{NCF_t = P_t \times NP_t + Mail_t - CO_t - OC_t - TaxValue_t + SV_t}\text{, where:}$$

- $\mathbf{P_t}$, denotes the price, per passenger, of an one-way airline ticket in year **t**;
- $\mathbf{NP_t = 5000 + 300 \times (t-1) + w}$, represents the estimated number of passengers in year **t**;
- $\mathbf{Mail_t = 200000 + v}$, represents the estimated gross revenue from courier services in year **t**;
- $\mathbf{Fuel_t}$, denotes the cost of fuel per flight in year **t**;

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Expression of the Net Cash Flow in each year (cont'd)
  - $\mathbf{OC_t=250000+q+210000+208^{(*)}\times Fuel_t+10\times NP_t}$, represents the operational costs in year **t**;
  - $\mathbf{CO_t}$, represents the capital outlay in year **t**: $\mathbf{CO_0}$=€4400000; $\mathbf{CO_1}$=…=$\mathbf{CO_7}$=€0;
  - $\mathbf{SV_t}$, represents the value of the aircraft after **t** years of the project (fully taxable): $\mathbf{SV_1}$=…=$\mathbf{SV_6}$=€0; $\mathbf{SV_7}$=€1000000;
  - $\mathbf{TaxValue_t=0,3\times(P_t\times NP_t+Mail_t+SV_t-OC_t-440000)}$ represents the company tax in year **t**.

(*) In each year, are estimated 104 round trip services, that is, 2×104 =208 one-way services.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Expression of the NPV of the investment

After determining the net cash flow for each year **t**= 0,1,...,7, it is possible to calculate the Net Present Value (NPV) of the project:

$$\sum_{t=0}^{7} NCF_t \big/ 1{,}08^t .$$

The company wishes to obtain information on the NPV of the project using Monte Carlo simulation. If the study points to a positive estimate for the NPV, then the project will be rated as acceptable.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Estimates for the NPV of the project – Application of Monte Carlo Simulation

To obtain estimates of the NPV of the project, using Monte Carlo simulation, it is necessary to repeat *M* times the following procedure(*):

Step 1 – Generate random observations from the variables that are crucial to the net cash flow of the project

The variables $\mathbf{P_t}$, $\mathbf{Fuel_t}$, **w**, **v** and **q**, **t**=1, …,7.

Step 2 – For each year **t**, calculate the net cash flow of the project:

$$\mathbf{NCF_t = P_t \times NP_t + Mail_t - CO_t - OC_t - TaxValue_t + SV_t}\,,\ t=0, \ldots, 7.$$

Step 3 – Calculate the NPV of project:

$$\sum_{t=0}^{7} NCF_t \Big/ 1{,}08^t.$$

(*) To illustrate the Monte Carlo simulation, we consider *M* = 1000. However, in a practical exercise *M* would have to assume a high value.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Results obtained by Monte Carlo simulation

The sample of estimates for the NPV of the project may be subject to a descriptive statistical analysis (see Table 10), which allows us to state that:

- The NPV values are between -€388496,92 and €1028870,89;
- The estimate for the NPV of the project is €321875,51 (which corresponds to the sample mean of 1000 values of the NPV);
- (…); and
- Since the range of NPV values comprises negative values, to classify the project, as an acceptable or not, we must determine the percentage of negative values for NPV.

Table 10

Sample of Estimates for the NPV - Descriptive Statistics	
Average	321875,5112
Standard Error	7166,969826
Median	330417,0397
Mode	#N/D
Standard Deviation	226639,4857
Sample Variance	51365456481
Kurtosis	0,014265331
Asymmetry	-0,112495274
Interval	1417367,82
Minimum	-388496,9245
Maximum	1028870,895
Sum	321875511,2
Score	1000

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Results obtained by Monte Carlo simulation (cont'd)

  The histogram and the cumulative relative frequency curve are presented in Chart 3. According to the cumulative curve, we have the following readings:

  - it would appear that a positive value for the NPV is highly likely, since the estimated probability is approximately equal to 0,9 (because the estimated probability of a negative value for the NPV is between 0,072 and 0,1 – see table 11);
  - (…)
  - the probability (estimate) of the NPV be between €160161,59 and €617377,01 is equal to:

    $P(NPV \leq 617377{,}01) - P(NPV \leq 160161{,}59) = 0{,}913 - 0{,}229 = 0{,}684$ (see table 11).

Conclusion

Since the estimated probability of a negative NPV is less than 0,1, with some risk, the project can be classified as acceptable.

2.4. **Case studies of assessment using specific software**

Example 2 – Evaluation of flight project of an airline (cont'd)

- Results obtained by Monte Carlo simulation (cont'd)

Chart 3

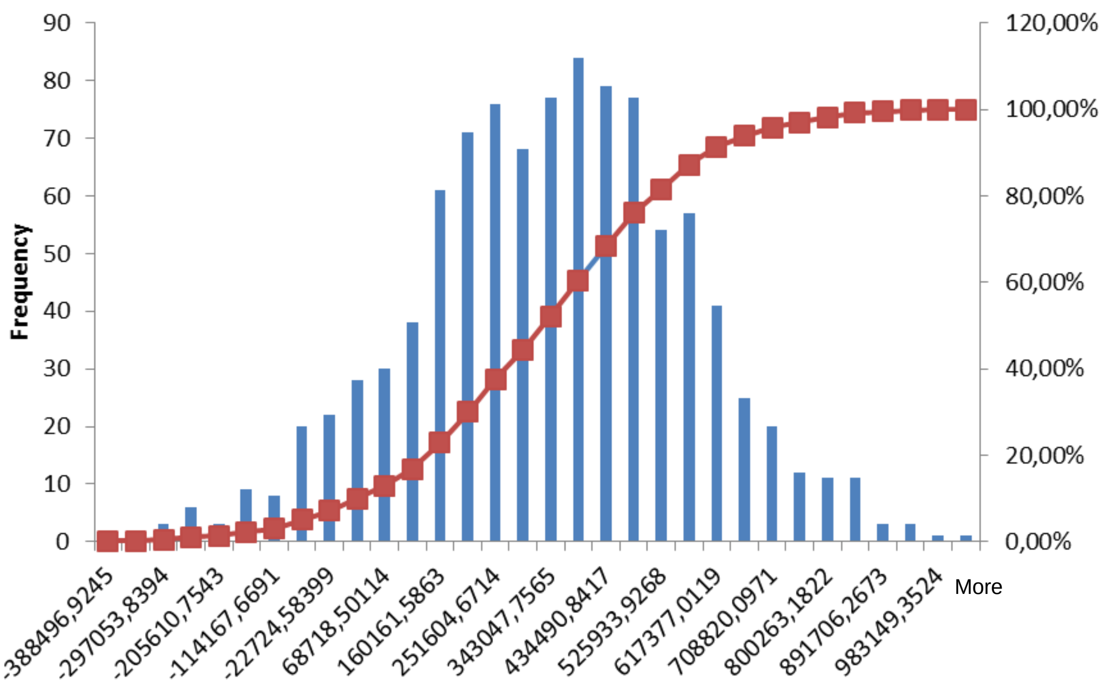


Table 11

Block	*Frequency*	*% cumulative*
-388496,9245	1	0,10%
-342775,3819	0	0,10%
-297053,8394	3	0,40%
-251332,2968	6	1,00%
-205610,7543	3	1,30%
-159889,2117	9	2,20%
-114167,6691	8	3,00%
-68446,12656	20	5,00%
-22724,58399	22	7,20%
22996,95857	28	10,00%
68718,50114	30	13,00%
114440,0437	38	16,80%
160161,5863	61	22,90%
205883,1288	71	30,00%
251604,6714	76	37,60%
297326,214	68	44,40%
343047,7565	77	52,10%
388769,2991	84	60,50%
434490,8417	79	68,40%
480212,3842	77	76,10%
525933,9268	54	81,50%
571655,4694	57	87,20%
617377,0119	41	91,30%
663098,5545	25	93,80%
708820,0971	20	95,80%
754541,6396	12	97,00%
800263,1822	11	98,10%
845984,7248	11	99,20%
891706,2673	3	99,50%
937427,8099	3	99,80%
983149,3524	1	99,90%
More	1	100,00%

2.5. **Monte Carlo Simulation – conclusion**

To finish the chapter on Monte Carlo simulation are presented, in Table 12, its main advantages.

Table 12

Monte Carlo Simulation – main advantages
Simple and flexible technique that allows the modeling of complex systems in a realistic way;
The model developed can be used to study the behavior of the simulated system for multiple hypotheses or management policies;
Powerful tool to generate the probability distributions;
Provides a standard error for the estimate generated by the simulation (see slide 29);
More efficient than other techniques when the value under study depends on several variables. Since the computational time increases linearly with the number of the underlying variables, in the case of the simulation, but exponentially to most other techniques.

2.5. **Monte Carlo Simulation – conclusion**

With regard the most significant disadvantages associated with the Monte Carlo simulation, they are listed in Table 13.

Table 13

Monte Carlo Simulation – main disadvantages
In most cases, requires large computational effort in the development and testing of the model;
Requires a large number of simulation runs so as to obtain a reasonable accuracy. However, it is possible to apply variance- reducing techniques (see slides 44, 45, …, 52) so as to reduce the standard deviation of the estimated values and to achieve a good precision with a smaller number of runs (M);
Basically, Monte Carlo simulation is only applied in the evaluation of investment projects that do not involve intermediate decisions;
Inefficient when the value of the investment project depends on different current values of a variable.