数列（3）通项与求和解答（1）

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一、已知k阶递推关系$f(a_n,a_{n+1},\cdots,a_{n+k})=0$，求项、和.

1°.一阶递推关系:1.一阶常系数:$a_n=pa_{n-1}+q$(特例：等差、等比数列)

2.累（叠）加：$a_n=a_{n-1}+f(n)$($a_n=a_1+\sum_{i=2}^{n}(a_i-a_{i-1})=a_1+\sum_{i=2}^{n}f(i)(n\ge 2)$)

3.累乘：$a_n=f(n)a_{n-1}$($a_n=a_1\cdot\prod_{i=2}^{n}\frac{a_i}{a_{i-1}}=a_1\cdot\prod_{i=2}^{n}f(i)(n\ge 2)$)

4.归纳法

(1994A）已知数列$\{a_n\}$满足$3a_{n+1}+a_n=4(n\in N^*)$,且$a_1=9$，其前n项之和为S_n，则满足不等式$|S_n-n-6|<\frac{1}{125}$的最小整数n是（　）$A.5$　$B.6$　$C.7$　$D.8$

1994$Akey$:由已知得$a_{n+1}-1=-\frac{1}{3}(a_n-1),a_1-1=8$

$\therefore a_n=8(-\frac{1}{3})^{n-1}+1,\therefore S_n=n+\dfrac{8(1-(-\frac{1}{3})^n)}{1+\frac{1}{3}}=n+6-6(-\frac{1}{3})^n$

$\therefore|S_n-n-6|=\frac{6}{3^n}<\frac{1}{125}\Leftrightarrow 250<3^{n-1},\therefore n_{\min}=7$,选$C$

(1993A）设正数列$a_0,a_1,a_2,\cdots,a_n,\cdots$满足$\sqrt{a_na_{n-2}}-\sqrt{a_{n-1}a_{n-2}}=2a_{n-1}(n\ge 2)$且$a_0=a_1=1$,则$a_n=$________.

1993$Akey$:$\dfrac{\sqrt{a_na_{n-2}}-\sqrt{a_{n-1}a_{n-2}}}{\sqrt{a_{n-1}a_{n-2}}}=\dfrac{2a_{n-1}}{\sqrt{a_{n-1}a_{n-2}}}\Leftrightarrow\sqrt{\frac{a_n}{a_{n-1}}}-1=2\sqrt{\frac{a_{n-1}}{a_{n-2}}}$(记$b_{n-1}=\sqrt{\frac{a_{n-1}}{a_{n-2}}},n\ge 2$)

$\therefore b_n+1=2(b_{n-1}+1),b_1+1=2,\therefore b_n+1=2^n,\therefore\sqrt{\frac{a_n}{a_{n-1}}}=2^n-1,\therefore\frac{a_n}{a_{n-1}}=(2^n-1)^2,n\in N^*$

$$\therefore a_n=\begin{cases}1,n=0,\\ \frac{a_n}{a_{n-1}}\cdot\frac{a_{n-1}}{a_{n-2}}\cdots\frac{a_1}{a_0}\cdot a_0=[(2^n-1)(2^{n-1}-1)\cdots(2^1-1)]^2,n\ge 1.\end{cases}$$

(2007Ⅰ）设数列$\{a_n\}$的首项$a_1\in(0,1),a_n=\frac{3-a_{n-1}}{2},n=2,3,4,\cdots$.(1）求$\{a_n\}$的通项公式；

(2）设$b_n=a_n\sqrt{3-2a_n}$,求证：$b_n<b_{n+1}$,其中n为正整数.

2007Ⅰ（1）解:由已知得$a_n-1=-\frac{1}{2}(a_{n-1}-1),\therefore\{a_n-1\}$是首项为$a_1-1\in(-1,0)$,公比为$-\frac{1}{2}$的等比数列,

$\therefore a_n=1+(a_1-1)\cdot(-\frac{1}{2})^{n-1},n\in N^*$

(2）证明:由（1）得$a_n\in(0,1),\therefore b_n>0,b_{n+1}>0,$

$\therefore b_{n+1}^2-b_n^2=a_{n+1}^2(3-2a_{n+1})-a_n^2(3-2a_n)=(\frac{3-a_n}{2})^2a_n-a_n^2(3-2a_n)=\frac{9}{4}a_n(a_n-1)^2>0,\therefore b_{n+1}>b_n$,证毕

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(2006福建)22.已知数列$\{a_n\}$满足$a_1=1, a_{n+1}=2a_n+1(n\in N^*)$.(1) 求数列$\{a_n\}$的通项公式；

(2) 若数列$\{b_n\}$满足$4^{b_1-1}4^{b_2-1}\cdots 4^{b_n-1}=(a_n+1)^{b_n}(n\in N^*)$, 证明：数列$\{b_n\}$是等差数列；

(3) 证明:$\dfrac{n}{2}-\dfrac{1}{3}<\dfrac{a_1}{a_2}+\dfrac{a_2}{a_3}+\cdots+\dfrac{a_n}{a_{n+1}}<\dfrac{n}{2}(n\in N^*)$.

2006福建 (1) 由已知得$a_{n+1}+1=2(a_n+1)$

$\therefore\{a_n+1\}$是首项为2，公比为2的等比数列,$\therefore a_n=2^n-1$

(2) 证明：由 (1) 及已知得得:$4^{b_1+b_2+\cdots+b_n-n}=2^{nb_n}$,$\therefore b_1+b_2+\cdots+b_n=\dfrac{1}{2}nb_n+n$

$\therefore b_1+b_2+\cdots+b_n+b_{n+1}=\dfrac{1}{2}(n+1)b_{n+1}+n+1$

$\therefore b_{n+1}=\dfrac{1}{2}(n+1)b_{n+1}-\dfrac{1}{2}nb_n+1$即$(n-1)b_{n+1}=nb_n-1$,$\therefore nb_{n+2}=(n+1)b_{n+1}-1$

$\therefore nb_{n+2}-(n-1)b_{n+1}=(n+1)b_{n+1}-nb_n\Leftrightarrow b_{n+2}-b_{n+1}=b_{n+1}-b_n=\cdots=b_2-b_1$,$\therefore\{b_n\}$是等差数列

(3) 证明：由 (1) 得$a_n=2^n-1$,

$$\therefore\frac{a_n}{a_{n+1}}=\frac{2^n-1}{2^{n+1}-1}=\frac{\frac{1}{2}(2^{n+1}-1)-\frac{1}{2}}{2^{n+1}-1}=\frac{1}{2}-\frac{1}{2(2^{n+1}-1)}$$

$\therefore$要证:$\dfrac{n}{2}-\dfrac{1}{3}<\dfrac{a_1}{a_2}+\dfrac{a_2}{a_3}+\cdots+\dfrac{a_n}{a_{n+1}}<\dfrac{n}{2}$,只要证明:$0<\dfrac{1}{2^2-1}+\dfrac{1}{2^3-1}+\cdots+\dfrac{1}{2^{n+1}-1}<\dfrac{2}{3}\cdots(*)$

而$\dfrac{1}{2^2-1}+\dfrac{1}{2^3-1}+\cdots+\dfrac{1}{2^{n+1}-1}<\dfrac{2}{3}\Leftrightarrow\dfrac{1}{2^3-1}+\cdots+\dfrac{1}{2^{n+1}-1}<\dfrac{1}{3}$

由$2^n-1\geq\lambda\cdot 2^n(n\geq 3)\Leftrightarrow\lambda\leq 1-\dfrac{1}{2^n}\geq\dfrac{7}{8}$,$\therefore\lambda\leq\dfrac{7}{8}$,$\therefore 2^n-1\geq\dfrac{7}{8}\cdot 2^n=7\cdot 2^{n-3}$,$\therefore\dfrac{1}{2^n-1}\leq\dfrac{1}{7\cdot 2^{n-3}}(n\geq 3)$

$$\therefore\frac{1}{2^3-1}+\cdots+\frac{1}{2^{n+1}-1}\leq\frac{1}{7}(\frac{1}{2^0}+\frac{1}{2^1}+\cdots+\frac{1}{2^{n-2}})=\frac{1}{7}\cdot\frac{1-\frac{1}{2^{n-1}}}{1-\frac{1}{2}}<\frac{2}{7}<\frac{1}{3},\therefore(*)\text{成立，证毕}$$

(2010湖北) 已知数列$\{a_n\}$满足:$a_1=\dfrac{1}{2}$,$\dfrac{3(1+a_{n+1})}{1-a_n}=\dfrac{2(1+a_n)}{1-a_{n+1}}$,$a_na_{n+1}<0(n\geq 1)$,数列$\{b_n\}$满足:$b_n=a_{n+1}^2-a_n^2(n\geq 1)$.

(1) 求数列$\{a_n\},\{b_n\}$的通项公式； (2) 证明：数列$\{b_n\}$中的任意三项不可能成等差数列.

(2010湖北) (1) 解：由已知得$3(1-a_{n+1}^2)=2(1-a_n^2)$,$\therefore\{1-a_n^2\}$是首项为$\dfrac{3}{4}$,公比为$\dfrac{2}{3}$的等比数列，

$\therefore 1-a_n^2=\dfrac{3}{4}(\dfrac{2}{3})^{n-1}$,$\because a_na_{n+1}<0, a_1=\dfrac{1}{2}>0$,$\therefore a_n=(-1)^{n-1}\sqrt{1-\dfrac{3}{4}(\dfrac{2}{3})^{n-1}}$,

且$b_n=(1-\dfrac{3}{4}(\dfrac{2}{3})^n)-(1-\dfrac{3}{4}(\dfrac{2}{3})^{n-1})=\dfrac{1}{4}\cdot(\dfrac{2}{3})^{n-1}$

(2) 证明：假设$b_l,b_m,b_n(l<m<n)$成等差数列，

则$2b_m-(b_l+b_n)=\dfrac{1}{4}[2(\dfrac{2}{3})^{m-1}-(\dfrac{2}{3})^{l-1}-(\dfrac{2}{3})^{n-1}]=\dfrac{1}{4}\cdot(\dfrac{2}{3})^{l-1}(2(\dfrac{2}{3})^{m-l}-(\dfrac{2}{3})^{n-l}-1)=0$

$\Leftrightarrow 2(\dfrac{2}{3})^{m-l}-1-(\dfrac{2}{3})^{n-l}=0\Leftrightarrow 2^{m-l+1}\cdot 3^{n-m}-2^{n-l}=3^{n-l}$

而3^{n-l}是3的倍数,$2^{m-l+1}\cdot 3^{n-m}-2^{n-l}$不是3的倍数，矛盾,$\therefore$假设错误

$\therefore\{b_n\}$中的任意三项不可能成等差数列

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(2011大纲)20.设数列$\{a_n\}$满足$a_1=0$且$\dfrac{1}{1-a_{n+1}}-\dfrac{1}{1-a_n}=1$.⑴ 求$\{a_n\}$的通项公式;

（2）$b_n=\dfrac{1-\sqrt{a_{n+1}}}{\sqrt{n}}$，记$S_n=\sum\limits_{k=1}^{n}b_k$，证明：$S_n<1$.

2011大纲卷 ⑴ 解：由已知的数列$\{\dfrac{1}{1-a_n}\}$是首项为1，公差为1的等差数列，$\therefore \dfrac{1}{1-a_n}=n,\therefore a_n=1-\dfrac{1}{n},n\in N^*$

（2）证明：由 ⑴ 得$b_n=\dfrac{1-\sqrt{1-\dfrac{1}{n+1}}}{\sqrt{n}}=\dfrac{1}{\sqrt{n}}-\dfrac{1}{\sqrt{n+1}},\therefore S_n=\sum\limits_{k=1}^{n}b_k=\dfrac{1}{1}-\dfrac{1}{\sqrt{n+1}}<1$，证毕

变式 1（1）已知数列$\{a_n\}$满足$a_0=0$, $|a_{i+1}|=|a_i+1|\ (i\in \boldsymbol{N})$，则$|\sum\limits_{k=1}^{20}a_k|$的值不可能是（ B ）

A.2　　　　B.4　　　　C.10　　　　D.14

$key: a_{i+1}^2=a_i^2+2a_i+1,\therefore \sum\limits_{i=1}^{20}a_i=\dfrac{1}{2}\sum\limits_{i=1}^{20}(a_{i+1}^2-a_i^2-1)=\dfrac{1}{2}(a_{21}^2-a_1^2-20)=\dfrac{1}{2}a_{21}^2-\dfrac{21}{2},\therefore |\sum\limits_{i=1}^{20}a_i|=\dfrac{1}{2}|a_{21}^2-21|$

（2）已知数列$\{x_n\}$满足$x_0=0$且$|x_k+1|=|x_{k-1}+2|,k\in N^*$，则$|x_1+x_2+\cdots+x_{2021}|$的最小值是（ ）B

$A.17$　　$B.19$　　$C.69$　　$D.87$

key：由$|x_k+1|=|x_{k-1}+2|\Leftrightarrow (x_k+1)^2=(x_{k-1}+2)^2$,

$\therefore x_k^2-x_{k-1}^2=-2x_k+4x_{k-1}+3$为奇数，$\because x_0=0,\therefore x_{2n-1}$为奇数，$x_{2n}$为偶数

$\therefore x_{2022}^2=(x_{2022}^2-x_{2021}^2)+\cdots+(x_2^2-x_1^2)+(x_1^2-x_0^2)$

$=(-2x_{2022}+4x_{2021}+3)+\cdots+(-2x_2+4x_1+3)+(-2x_1+4x_0+3)=-2x_{2022}+2(x_{2021}+\cdots+x_1)+6066$

$\therefore |x_1+x_2+\cdots+x_{2021}|=|\dfrac{1}{2}(x_{2022}^2+2x_{2022})-3033|=|\dfrac{1}{2}(x_{2022}+1)^2-\dfrac{6067}{2}|$被4除余3

(2020III)17.设数列$\{a_n\}$满足$a_1=3,a_{n+1}=3a_n-4n$.⑴ 计算a_2,a_3，猜想$\{a\}$的通项公式并加以证明；
（2）求数列$\{2^n a_n\}$的前n项和S_n.

2020III解： ⑴ 由已知得$a_{n+1}-(p(n+1)+q)=3(a_n-(pn+q))$（其中$p=2,q=1$）
即$a_{n+1}-(2n+3)=3(a_n-(2n+1))$，而$a_1-(2\times 1+1)=0,\therefore a_n=2n+1$，且$a_2=5,a_3=7$
（2）由（1）得$2^n a_n=(2n+1)\cdot 2^n=(pn+q)\cdot 2^{n+1}-(p(n-1)+q)\cdot 2^n\ (p=2,q=-1)$
$=(2n-1)\cdot 2^{n+1}-(2n-3)\cdot 2^n,\therefore S_n=(2n-1)\cdot 2^{n+1}+2$