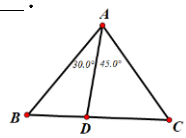
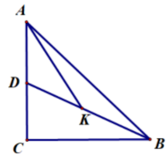
初等函数（Ⅱ）三角变换与解三角形解答（5）

2023-04-16

2（1）①如图，已知$\triangle ABC$中，$\angle BAD=30°,\angle CAD=45°,AB=3,AC=2$,则$\dfrac{BD}{DC}=$________.



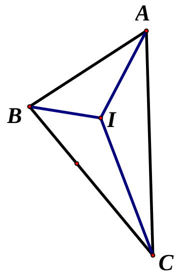
①$key:\dfrac{BD}{\frac{1}{2}}=\dfrac{AD}{\sin B},\dfrac{DC}{\frac{\sqrt{2}}{2}}=\dfrac{AD}{\sin C}\Rightarrow\dfrac{BD}{DC}=\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sin C}{\sin B}=\dfrac{\sqrt{2}}{2}\cdot\dfrac{AB}{AC}=\dfrac{3\sqrt{2}}{4}$



②如图，已知$\triangle ABC$中，$\angle C=90°,AC=6,BC=8,D$为边AC上一点，K为BD上一点，且$\angle ABC=\angle KAD=\angle AKD$,则$DC=$______.

②$key:$设$DC=x$,则$\dfrac{8}{x}=\tan(2\arctan\dfrac{3}{4})=\dfrac{\frac{3}{2}}{1-\frac{9}{16}}=\dfrac{24}{7}$得$DC=\dfrac{7}{3}$

③已知点I在$\triangle ABC$内部，AI平分$\angle BAC,\angle IBC=\angle ACI=\dfrac{1}{2}\angle BAC$,对满足上述条件的所有$\triangle ABC$，则$\triangle ABC$的三边$a,b,c$的关系为_______.

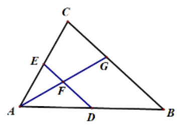


③$key:\dfrac{IB}{\sin\frac{A}{2}}=\dfrac{IA}{\sin(B-\frac{A}{2})},\dfrac{IC}{\sin\frac{A}{2}}=\dfrac{IB}{\sin(C-\frac{A}{2})}$,且$IA=IC$

$\therefore\dfrac{\sin\frac{A}{2}}{\sin(B-\frac{A}{2})}=\dfrac{\sin(C-\frac{A}{2})}{\sin\frac{A}{2}},\therefore\dfrac{1-\cos A}{2}=\dfrac{1}{2}[\cos(B-\dfrac{A}{2}-C+\dfrac{A}{2})-\cos(B-\dfrac{A}{2}+C-\dfrac{A}{2})$

$\therefore 1-\cos A=1+\cos(B+C)=\cos(B-C)+\cos 2A,\therefore 2\sin B\sin C=2\sin^2 A,\therefore bc=a^2$

（2019贵州）在$\triangle ABC$中，$AB=30,AC=20,S_{\triangle ABC}=210,D,E$分别为边$AB,AC$的中点，$\angle BAC$的平分线分别与$DE,BC$交于点$F,G$.则四边形$BGFD$的面积为_______



$key:\dfrac{3}{4}\cdot\dfrac{3}{5}\cdot 210=\dfrac{189}{2}$

（2）①若$2B=A+C$,且b边上的高$h_b=c-a$,则$A=$____,$B=$____,$C=$____.

①$key:B=\dfrac{\pi}{3}$,且$h_b=c-a=c\sin A$得$\sin C-\sin A=\sin A\sin C$

$\Leftrightarrow 2\cos\dfrac{C+A}{2}\sin\dfrac{C-A}{2}=\dfrac{1}{2}[\cos(A-C)-\cos(A+C)]\Leftrightarrow\sin\dfrac{C-A}{2}=2(1-2\sin^2\dfrac{C-A}{2}-\dfrac{1}{2})$

得$\sin\dfrac{C-A}{2}=\dfrac{1}{2}(\because\dfrac{C-A}{2}\in(-\dfrac{\pi}{2},\dfrac{\pi}{2})),\therefore C-A=\dfrac{\pi}{3},\therefore C=\dfrac{\pi}{2},A=\dfrac{\pi}{6},B=\dfrac{\pi}{3}$

②若$h_a=a$,则$\dfrac{c}{b}+\dfrac{b}{c}$的取值范围为________;$\dfrac{c}{b}+\dfrac{b}{c}+\dfrac{a^2}{bc}$的最大值为____.

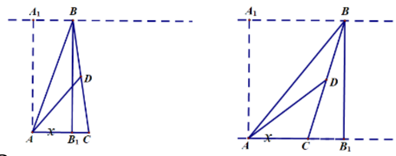
$key:$由$a=h_a=c\sin B=b\sin C$即$\sin A=\sin B\sin C,\therefore a^2=bc\sin B\sin C=bc\sin A,$

$\therefore\dfrac{c}{b}+\dfrac{b}{c}=\dfrac{b^2+c^2}{bc}=\dfrac{a^2+2bc\cos A}{bc}=\sin A+2\cos A\in[2,\sqrt{5}]$(由几何意义得$A\in(0,\arctan\dfrac{4}{3}]$)

$\therefore\dfrac{c}{b}+\dfrac{b}{c}+\dfrac{a^2}{bc}=2\sin A+2\cos A=2\sqrt{2}\sin(A+\dfrac{\pi}{4})\le 2\sqrt{2}$

（3）①若角A为锐角,且$b=1,S_{\triangle ABC}=\sqrt{3}$,则边$BC$上的中线$AD$的长的取值范围为________.

①key:如图，由$S_{\triangle ABC}=\frac{1}{2}\cdot 1\cdot h_b=\sqrt{3}$得$BB_1=h_b=2\sqrt{3}$,



设$AB_1=x>0$,则$BC=\sqrt{(1-x)^2+12},AB=\sqrt{x^2+12}$,

$$\overrightarrow{AD}^2=\frac{2\overrightarrow{AB}^2+2\overrightarrow{AC}^2-(\overrightarrow{AC}-\overrightarrow{AB})^2}{4}=\frac{1}{4}(2x^2+24+2-x^2+2x-13)$$

$$=\frac{1}{4}(x^2+2x+13)>\frac{13}{4},\therefore|\overrightarrow{AD}|\in(\frac{\sqrt{13}}{2},+\infty)$$

②等腰三角形的腰上的中线长为$\sqrt{5}$，则$\triangle ABC$的面积的最大值为________.

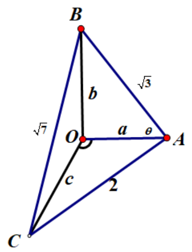
key:由$a^2+b^2=2\cdot 5+2\cdot\frac{b^2}{4}$得$a^2+\frac{1}{2}b^2=10$

$$\therefore S=\frac{1}{2}a\cdot\sqrt{b^2-\frac{a^2}{4}}=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{2}a\cdot\sqrt{b^2-\frac{a^2}{4}}\le\frac{1}{3}\cdot\frac{b^2+2a^2}{2}=\frac{10}{3}$$

（4）求值:$\sin^2\alpha+\sin^2(\frac{2\pi}{3}-\alpha)-\sin\alpha\sin(\frac{2\pi}{3}-\alpha)$　key:令$2R=1$,$\therefore$由余弦定理得:$(\sin\frac{\pi}{3})^2=\frac{3}{4}$;

变式:$\sin^2\alpha+\sin^2(\frac{\pi}{3}-\alpha)+\sin\alpha\sin(\frac{\pi}{3}-\alpha)=$____. $=(\sin\frac{2\pi}{3})^2=\frac{3}{4}$

（2014浙江竞赛）设正实数a,b,c满足$\begin{cases}a^2+b^2=3,\\a^2+c^2+ac=4,\\b^2+c^2+\sqrt{3}bc=7,\end{cases}$　则$a=$____,$b=$_____,$c=$_____



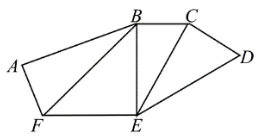
key:$\frac{2}{\sin 120^\circ}=\frac{c}{\sin\theta}$即$c=\frac{4}{\sqrt{3}}\sin\theta,a=\sqrt{3}\sin\theta,b=\sqrt{3}\cos\theta$

$$\therefore a^2+c^2+ac=3\sin^2\theta+\frac{16}{3}\sin^2\theta+4\sin^2\theta=\frac{37}{3}\sin^2\theta=4,$$

$$\therefore\sin\theta=\frac{2\sqrt{3}}{\sqrt{37}},\cos\theta=\frac{5}{\sqrt{37}},\therefore a=\frac{6\sqrt{37}}{37},b=\frac{5\sqrt{111}}{37},c=\frac{8\sqrt{37}}{37}$$

（2019北京）如图，$\angle BAF=\angle FEB=\angle EBC=\angle ECD=90^\circ,\angle ABF=30^\circ,\angle BFE=45^\circ,\angle BCE=60^\circ$,

$AB=2CD$,则$\tan\angle CDE$等于（　）$A.\frac{4\sqrt{2}}{3}$　$B.\frac{3\sqrt{2}}{8}$　$C.\frac{8\sqrt{6}}{3}$　$D.\frac{5\sqrt{2}}{6}$



（2019北京）key:$2=\frac{AB}{CD}=\frac{AB}{BF}\cdot\frac{BF}{BE}\cdot\frac{BE}{EC}\cdot\frac{EC}{CD}$

$=\cos 30^\circ\cdot\frac{1}{\sin 45^\circ}\cdot\sin 60^\circ\cdot\tan\angle CDE,\therefore$选$A$

1（1）在平面四边形$ABCD$中，$AB=1,AC=\sqrt{5},BD\perp BC,BD=2BC$,则$AD$的最小值为________.

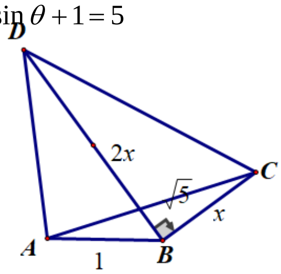
$key1$:设$BC=x$，则$AD=\sqrt{1+4x^2-2\cdot 1\cdot 2x\cdot\cos(\angle ABC-\frac{\pi}{2})}$

$$=\sqrt{1+4x^2-4x\sqrt{1-(\frac{1+x^2-5}{2x})^2}}=\sqrt{1+4x^2-2\sqrt{20-(x^2-6)^2}}$$

（由$\sqrt{5}-1<x<\sqrt{5}+1$, 令$t=x^2-6\in(-2\sqrt{5},2\sqrt{5})$)

$=\sqrt{4t+25-2\sqrt{20-t^2}}=\sqrt{(4,-2)\cdot(t,\sqrt{20-t^2})+25}\ge\sqrt{5}$

$key2$: 设$\angle DBA=\theta, BC=x$，则$BD=2x$, 且$AC^2=1+x^2-2\cdot1\cdot x\cos(\frac{\pi}{2}+\theta)=x^2+2x\sin\theta+1=5$



即$(x\sin\theta+1)^2+(x\cos\theta)^2=5$, 令$\begin{cases}x\sin\theta+1=\sqrt{5}\sin\alpha\\x\cos\theta=\sqrt{5}\cos\alpha\end{cases}$

$\therefore AD^2=1+4x^2-4x\cos\theta=1+4[(\sqrt{5}\sin\alpha-1)^2+(\sqrt{5}\cos\alpha)^2]-4\sqrt{5}\cos\alpha$

$=25-4\sqrt{5}(\cos\alpha+2\sin\alpha)\ge5$

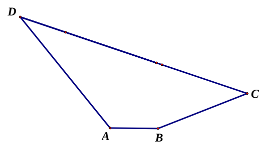
（2）①已知凸四边形$ABCD$中，$AB=2, BC=4, CD=5, DA=3$, 则四边形$ABCD$的面积S的最大值为____.

①key: 由$4+9-12\cos A=BD^2=16+25-40\cos D$得$10\cos D-3\cos A=7$

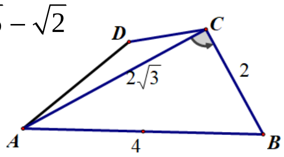
$\therefore S=S_{ABCD}=3\sin A+10\sin D$

$\therefore 49+S^2=109-60\cos(A+D)\in[49,169], \therefore S^2\in[0,120], \therefore S_{\max}=2\sqrt{30}$

② 如图, 在平面四边形$ABCD$中, $\angle A=45°, \angle B=60°, \angle D=150°, AB=2BC=4$, 则四边形$ABCD$的面积为__.



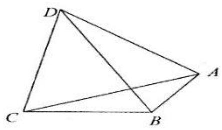
②如图, $\dfrac{2\sqrt{3}}{\frac{\sqrt{3}}{2}}=\dfrac{AD}{\sin15°}=\dfrac{CD}{\sin165°}$得$AD=CD=\sqrt{6}-\sqrt{2}$



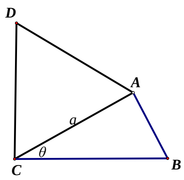
$\therefore S_{ABCD}=\frac{1}{2}(\sqrt{6}-\sqrt{2})^2\cdot\frac{1}{2}+2\sqrt{3}=2+\sqrt{3}$

③ 在平面四边形$ABCD$中，$AB=1, BC=2$,$\triangle ACD$为正三角形，则$\triangle BCD$的面积的最大值为（　）

$A.2\sqrt{3}+2$　$B.\dfrac{\sqrt{3}+1}{2}$　$C.\dfrac{\sqrt{3}}{2}+2$　$D.\sqrt{3}+1$



③key: 设$AC=a, \angle ACB=\theta\in(0,\pi)$，则$a^2+4-4a\cos\theta=1$



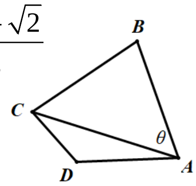
即$(a\cos\theta-2)^2+(a\sin\theta)^2=1$，令$\begin{cases}a\cos\theta-2=\cos\alpha\\a\sin\theta=\sin\alpha\end{cases}$

则$S_{\triangle BCD}=\frac{1}{2}\cdot a\cdot2\sin(60°+\theta)=\frac{1}{2}(\sqrt{3}a\cos\theta+a\sin\theta)$

$=\frac{1}{2}(2\sqrt{3}+\sin\alpha+\sqrt{3}\cos\alpha)\le\sqrt{3}+1$, 选$D$

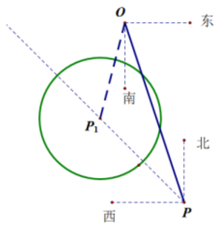
④ 在平面四边形$ABCD$中, $A=B=C=75°, BC=2$, 则AB的取值范围为_____. $(\sqrt{6}-\sqrt{2},\sqrt{6}+\sqrt{2})$

key: 如图, $\dfrac{AB}{\sin(75°+\theta)}=\dfrac{2}{\sin\theta}(\theta=\angle BAC\in(30°,75°))$得$AB=\dfrac{2\sin(75°+\theta)}{\sin\theta}=\dfrac{\sqrt{6}+\sqrt{2}}{2\tan\theta}+\dfrac{\sqrt{6}-\sqrt{2}}{2}$



$\in(\sqrt{6}-\sqrt{2},\sqrt{6}+\sqrt{2})(\because\tan\theta\in(\frac{\sqrt{3}}{3},2+\sqrt{3}))$

（2003全国）在某海滨城市附近海面有一台风,据监测,当前台风位于城市O的东偏南$\theta(\cos\theta=\dfrac{\sqrt{2}}{10})$方向300$km$的海面$P$处,并以20$km/h$的速度向西偏北45°方向移动,台风侵袭的范围为圆形区域,当前半径为60km,并以10km/h的速度不断增大,问几小时后该城市开始受到台风的侵袭？持续约多少时间？

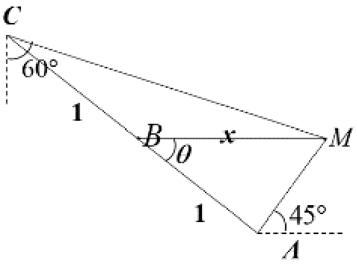


$key: OP_1^2=300^2+(20t)^2-2\cdot300\cdot20t\cos(\theta-\dfrac{\pi}{4})\le(60+10t)^2$

即$t^2-36t+288\le0$得$12\le t\le24$

变式：已知A、B、C是一条直路上的三点，AB与BC各等于1km，从三点分别遥望塔M，在A处看见塔在北偏东45°方向，在B处看塔在正东方向，在点C处看见塔南偏东60°方向，求塔到直路ABC的最短距离.

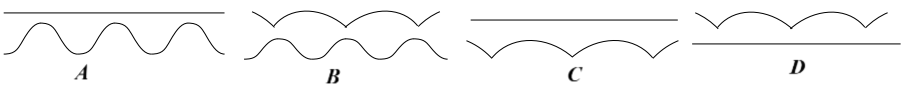
$key:\begin{cases}\dfrac{1}{\sin45°}=\dfrac{x}{\sin(\theta+45°)}\\ \dfrac{1}{\sin30°}=\dfrac{x}{\sin(\theta-30°)}\end{cases}$ 得 $\dfrac{\sin(\theta-30°)}{\sin(\theta+45°)}=\dfrac{1}{\sqrt{2}}$ 得 $\tan\theta=\sqrt{3}+1$

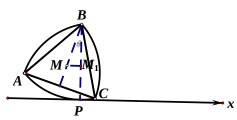


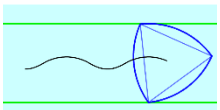
$\therefore d=x\sin\theta=\sqrt{2}\sin\theta\sin(\theta+45°)=\sin^2\theta+\sin\theta\cos\theta$

$=\dfrac{\sin^2\theta+\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}=\dfrac{\tan^2\theta+\tan\theta}{\tan^2\theta+1}=\dfrac{7+5\sqrt{3}}{13}$

(11年江西文科）如图，一个“凸轮”放置于直角坐标系x轴上方，其“底端”落在原点O处，一顶点及中心M在y轴正半轴上，它的外围由以正三角形的顶点为圆心，以正三角形的边长为半径的三段等弧组成. 今使“凸轮”沿x轴正向滚动前进，在滚动过程中“凸轮”每时每刻都有一个“最高点”. 其中心也在不断移动位置，则在“凸轮”滚动一周的过程中，将其“最高点”和“中心点”所形成的图形按上、下放置，应大致为（ ）

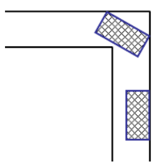






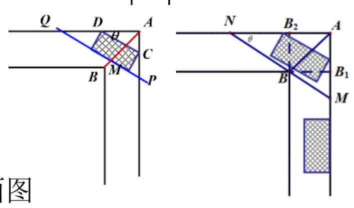
$key: d_{M\to l}=1-\dfrac{\sqrt{3}}{3}\cos\theta$在$\theta=0$附近递增

1（1）如图,某厂的一段直角通道宽3m,现有一辆载重平板车的长为4m、宽为2m,问该平板车能否通过拐角.



$key:$由$PQ=\dfrac{2}{\tan\theta}+4+2\tan\theta\ge8$

而$MN=\dfrac{3}{\sin\theta}+\dfrac{3}{\cos\theta}=\dfrac{3(\sin\theta+\cos\theta)}{\sin\theta\cos\theta}=\dfrac{3t}{\dfrac{t^2-1}{2}}=\dfrac{6}{t-\dfrac{1}{t}}\ge6\sqrt{2}>8$

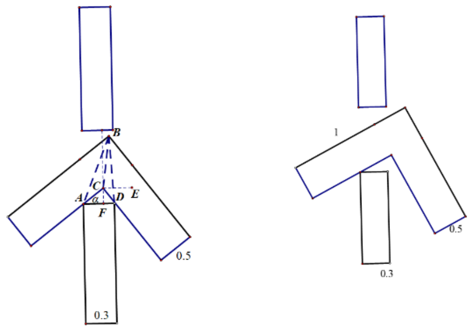


$(t=\sin\theta+\cos\theta=\sqrt{2}\sin(\theta+\dfrac{\pi}{4})\in(1,\sqrt{2}])$

$\therefore$能通过

（2）一仓库房门宽度1m、厚度为0.3m,现有一根拐角钢材,其最大水平截面图的宽度为0.5m.问能否将这根拐角钢材移进仓库？

key : 如图，设$\angle CAD=\alpha\in(0,\frac{\pi}{2})$, 则



$$d_{B\to AD}=\frac{\sqrt{2}}{2}\sin(\alpha+\frac{\pi}{4})+0.3\cos\alpha\sin\alpha$$

$$=\frac{\sqrt{2}}{2}\sin(\alpha+\frac{\pi}{4})-\frac{3}{20}\cos(2\alpha+\frac{\pi}{2})(t=\sin(\alpha+\frac{\pi}{4})\in(\frac{\sqrt{2}}{2},1])$$

$$=-\frac{3}{10}t^2+\frac{\sqrt{2}}{2}t+\frac{3}{20}=-\frac{3}{10}(t-\frac{5\sqrt{2}}{6})^2+\frac{17}{30}\le\frac{17}{30}<1$$

$\therefore$能通过