数列（3）数列性质解答（3）

2024-03-30

（2019高考）设$a,b\in R$,数列$\{a_n\}$满足$a_{n+1}=a_n^2+b,a_1=a$,则（　）A

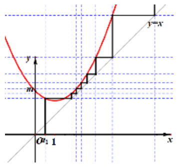
$A.$当$b=\frac{1}{2}$时，$a_{10}>10$　$B.$当$b=\frac{1}{4}$时，$a_{10}>10$　$C.$当$b=-2$时，$a_{10}>10$　$D.$当$b=-4$时，$a_{10}>10$

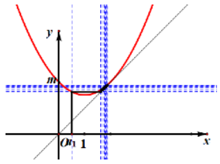
key:（归纳，递推）选A: $a_2=a^2+\frac{1}{2}\ge\frac{1}{2},a_3\ge\frac{1}{4}+\frac{1}{2}=\frac{3}{4},a_4\ge\frac{9}{16}+\frac{1}{2}>1$

$a_5>1+\frac{1}{2}=\frac{3}{2},a_6>\frac{9}{4}+\frac{1}{2}>2,a_7>4,a_8>16,a_9>256,a_{10}>10$

变式 1（1）若数列$\{a_n\}$满足$a_1=\frac{1}{2},a_{n+1}=\frac{1}{2}a_n^2-a_n+m$,若对任意的正整数都有$a_n<2$，则实数$m$的最大值为（　C）

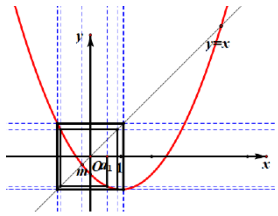
$A.\frac{1}{2}$　$B.1$　$C.2$　$D.4$





$key:a_2=-\frac{3}{8}+m<2$得$m<\frac{19}{8}$

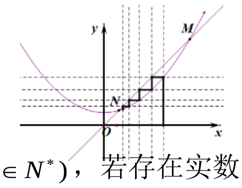
设$f(x)=\frac{1}{2}x^2-x+m=x\Leftrightarrow x^2-4x+2m=0$,得$\Delta=16-8m<0$即$m>2$



当$m>2$时，如图,$\{a_n\}$无上界；

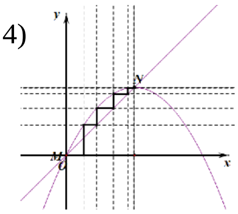
当$m=2$时，如图，$f(x)=x\Leftrightarrow x=2,\therefore a_n<2$

当$m<2$时，如图，



（2）已知数列$\{a_n\}$满足$a_1=a>0$，$a_{n+1}=-a_n^2+ta_n(n\in N^*)$，若存在实数$t$，使$\{a_n\}$单调递增，

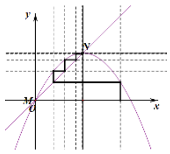
则a的取值范围是（ A　）A．$(0,1)$　B．$(1,2)$ C．$(2,3)$　D．$(3,4)$



$key:a_2=-a^2+ta>a>0$得$0<a<t-1(t>1)$,

设$f(x)=-x^2+tx$，则$f(x)=x\Leftrightarrow x=0,or,t-1$

$\therefore t-1<\frac{t}{2}$即$1<t<2,\therefore$由蛛网图得$a\in(0,1)$



（3）设$a,b\in R$,无穷数列$\{a_n\}$满足:$a_1=a,a_{n+1}=-a_n^2+ba_n-1,n\in N^*$,则下列说法中不正确的是（　D）

A.b=1 时,对任意实数a,数列$\{a_n\}$单调递减　B. $b=-1$时,存在实数a,使得数列$\{a_n\}$为常数列

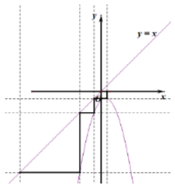
C. $b=-4$时,存在实数a,使得$\{a_n\}$不是单调数列　D.b=0 时,对任意实数a,都有$a_{2020}>-2^{2018}$

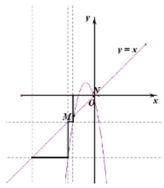
key:递推函数$f(x)=-x^2+bx-1$

$A.$由蛛网图知，正确

$B.$不动点为$x=-1$,故$a=-1$时,$a_n=-1$

$C.f(x)=-x^2-4x-1=x\Leftrightarrow x=\frac{-5\pm\sqrt{21}}{2}$,由蛛网图知$C$正确





（4）数列$\{a_n\}$满足$a_{n+1}={a_n}^2-2a_n$，若$\{a_n\}$单调递增，则首项a_1的范围是________．$(-\infty,-1)\cup(3,+\infty)$

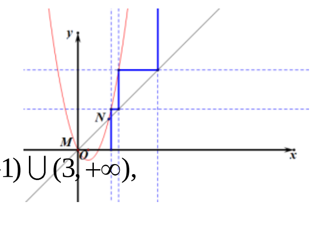
key:设$f(x)=x^2-2x$，则$f(x)=x\Leftrightarrow x=0,or,3$

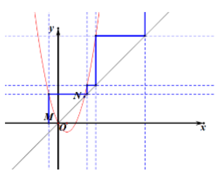
由$a_2=a_1^2-2a_1>a_1$得$a_1<0,or,a_1>3$

由$a_3=a_2^2-2a_2>a_2$得$a_2=a_1^2-2a_1<0$得

$0<a_1<2,or,a_2=a_1^2-2a_1>3$得$a_1>3,or,a_1<-1,\therefore a_1\in(-\infty,-1)\cup(3,+\infty)$,

由蛛网图得$a_1\in(-\infty,-1)\cup(3,+\infty)$.





（5）已知数列$\{a_n\}$满足：$a_1=1,a_{n+1}=\frac{1}{8}{a_n}^2+m(n\in N^*)$，若对任意的正整数$n$均有$a_n<4$，则实数$m$的最大值

是____________． 2

key: 设函数$f(x)=\frac{1}{8}x^2+m$, 则$f(x)$的不动点为$4-2\sqrt{4-2m}(0<m\le 2)$

当$m=2$时, $4>a_2=\frac{1}{8}+2>a_1\ge 1$, 若$1\le a_k<a_{k+1}<4$, 则$1\le f(1)<f(a_k)<f(a_{k+1})<f(4)=4$

若$m>2$，则$a_2>\frac{1}{8}+2, a_2-a_1>1$,

若$a_{k+1}>a_k\ge 1$, 则$f(a_{k+1})>f(a_k)>f(1)$即$a_{k+2}>a_{k+1}\ge 1$

$a_{n+1}-a_n=\frac{1}{8}a_n^2-a_n+m>\frac{1}{8}-1+2>1, \therefore a_n=(a_n-a_{n-1})+\cdots+(a_2-a_1)+1\ge n$, 不合

（6）对于数列$\{a_n\}$, 若任意正整数n，均满足$|a_n|\le M$（M为常数），则称数列$\{a_n\}$有界，已知数列$\{a_n\}$

满足递推关系$a_n=|Aa_{n-1}^2-1|$, 且$a_1=1$，若数列$\{a_n\}$有界，则A的取值范围是_____.

key: 设$f(x)=|Ax^2-1|$，则$a_{n+1}=f(a_n)$

当$A<0$时，$f(x)=-Ax^2+1=x\Leftrightarrow -Ax^2-x+1=0$

当$\Delta=1+4A<0$即$A<-\frac{1}{4}$时，如图1，不合；

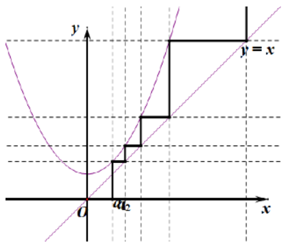
当$A=-\frac{1}{4}$时, $f(x)=\frac{1}{4}x^2+1=x\Leftrightarrow x=2$, 如图，符合

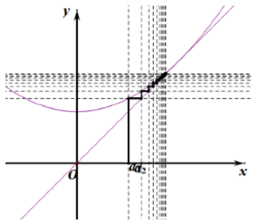
当$-\frac{1}{4}<A<0$时，如图2, $1\le\frac{1+\sqrt{1+4A}}{-2A}$得$-\frac{1}{4}<A<0$;

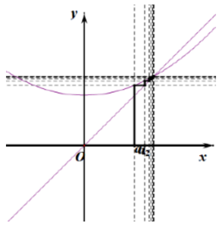
当$A=0$时，$f(x)=1$, 符合；

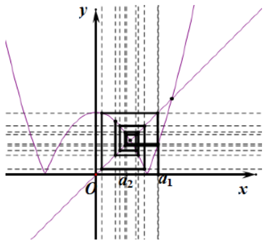
当$A>0$时，如图3, $1\le\frac{1+\sqrt{1+4A}}{2A}$得$0<A\le 2$.

综上A的取值范围为$[-\frac{1}{4},2]$









（2014浙江竞赛）设数列$\{a_n\}$定义为$a_1=a, a_{n+1}=1+\frac{1}{a_1+a_2+\cdots+a_n-1}, n\ge 1$,

求所有实数a，使得$0<a_n<1, n\ge 2$.

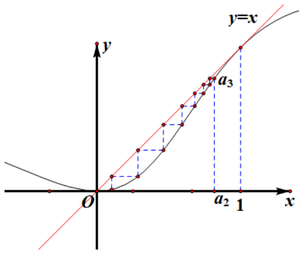
key: 由$a_1=a, a_2=1+\frac{1}{a-1}=\frac{a}{a-1}\in(0,1)$得$a<0$，由$a_{n+1}=1+\frac{1}{S_n-1}$得$S_n-1=\frac{1}{a_{n+1}-1}$,

$\therefore a_n=S_n-S_{n-1}=\frac{1}{a_{n+1}-1}-\frac{1}{a_n-1}(n\ge 2)$即$a_{n+1}=\frac{a_n^2}{a_n^2-a_n+1}(n\ge 2)$

$key1$: 若$0<a_n<1(n\ge 2)$, 则$a_{n+1}=\frac{a_n^2}{a_n^2-a_n+1}\in(0,1), \therefore a_n\in(0,1)(n\ge 2), \therefore a$的取值范围为$(-\infty,0)$

$key2$: 设$f(x)=\frac{x^2}{x^2-x+1}$, 则$f(x)=x\Leftrightarrow x=1,0$

由$f(x)=\frac{x^2}{x^2-x+1}=1+\frac{x-1}{x^2-x+1}=1+\frac{1}{t+\frac{1}{t}+1}(t=x-1)$



当$t=x-1\ge 1$即$x\ge 2$时，$f(x)$递减；$t=x-1\in(0,1)$即$1<x<2$时，$f(x)$递增；

当$t=x-1\in(-1,0)$即$0<x<1$时，$f(x)$递增；当$x<0$上，$f(x)$递减

且$f(x)<x(0<x<1)$, 如图, $\therefore 0<a_{n+1}<a_n<1(n\ge 2)$

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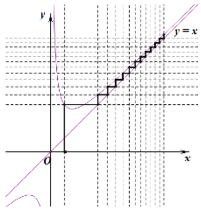
变式：已知数列$\{a_n\}$满足$a_1=1, a_{n+1}=a_n+\dfrac{c}{a_n}(n\in N^*)$，若$a_{n+1}>a_n$，则实数$c$的取值范围为____.

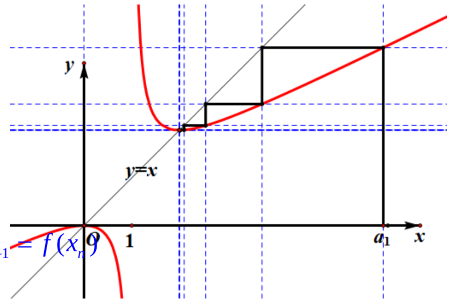


$key: a_2=1+c>1$得$c>0$;

（1984全国）设$a>2$，给定数列$\{x_n\}$满足$x_{n+1}=\dfrac{x_n^2}{2x_n-2}, x_1=a$，求证：

（Ⅰ）$2<x_{n+1}<x_n$;（Ⅱ）如果$a\le 3$，那么$x_n\le 2+\dfrac{1}{2^{n-1}}(n\in N^*)$;

（Ⅲ）如果$a>3$，那么当$n\ge\dfrac{\lg\frac{a}{3}}{\lg\frac{4}{3}}$时，必有$x_{n+1}<3$.



（1984全国）证明:（Ⅰ）（不动点方法）设$f(x)=\dfrac{x^2}{2x-2}(x\ge 2)$，则$x_{n+1}=f(x_n)$

而$f(x)=\dfrac{1}{2}(x-1+\dfrac{1}{x-1}+2)$在$x>2$上递增，且$f(x)=x\Leftrightarrow x=2$，且$f(x)\le x$，

由$x_2=f(x_1)=f(a)>f(2)=2, x_2=f(x_1)=\dfrac{a^2}{2a-2}<a\Leftrightarrow a>2$得$2<x_2<x_1$成立，

若$2<x_{k+1}<x_k$，则$f(2)<f(x_{k+1})<f(x_k)$即$2<x_{k+2}<x_{k+1}$，$\therefore 2<x_{n+1}<x_n(n\in N^*)$都成立

（Ⅱ）由（Ⅰ）得：$x_{k+1}-2=\dfrac{(x_k-2)^2}{2x_k-2}$，$\therefore\dfrac{x_{k+1}-2}{x_k-2}=\dfrac{x_k-2}{2(x_k-1)}=\dfrac{1}{2}(1-\dfrac{1}{x_k-1})<\dfrac{1}{2}, k\in N^*$

$\therefore x_n-2=\dfrac{x_n-2}{x_{n-1}-2}\cdots\dfrac{x_2-2}{x_1-2}(x_1-2)\le(\dfrac{1}{2})^{n-1}(a-2)\le\dfrac{1}{2^{n-1}}(\because a\le 3)$，而$x_1-2=a-2\le 1=\dfrac{1}{2^{1-1}}$，

$\therefore x_n-2\le\dfrac{1}{2^{n-1}}$即$x_n\le 2+\dfrac{1}{2^{n-1}}, n\in N^*$.

（Ⅲ）由（Ⅰ）得$\{x_n\}$递减，而$x_1=a>3$，若$x_N\ge 3$，则$x_n\ge 3(n=1,2,\cdots,N)$

则$\dfrac{x_{k+1}}{x_k}=\dfrac{x_k}{2x_k-2}=\dfrac{1}{2}(1+\dfrac{1}{x_k-1})<\dfrac{1}{2}(1+\dfrac{1}{3-1})=\dfrac{3}{4}(k=1,2,\cdots,N)$

$\therefore x_{N+1}=\dfrac{x_{N+1}}{x_N}\cdot\dfrac{x_N}{x_{N-1}}\cdots\dfrac{x_2}{x_1}\cdot x_1<(\dfrac{3}{4})^N\cdot a\le 3$，只需$(\dfrac{3}{4})^N\le\dfrac{3}{a}\Leftrightarrow N\lg\dfrac{3}{4}\le\lg\dfrac{3}{a}\Leftrightarrow N\ge\dfrac{\lg\frac{3}{a}}{\lg\frac{3}{4}}=\dfrac{\lg\frac{a}{3}}{\lg\frac{4}{3}}$,

$\therefore$当$n>\dfrac{\lg\frac{a}{3}}{\lg\frac{4}{3}}$时，$x_{n+1}<3$，得证

（2008Ⅰ）设函数$f(x)=x-x\ln x$，数列$\{a_n\}$满足$0<a_1<1, a_{n+1}=f(a_n)$.

（1）证明：函数$f(x)$在区间$(0,1)$是增函数；

（2）证明:$a_n<a_{n+1}<1$;（3）设$b\in(a_1,1)$，正数$k\ge\dfrac{a_1-b}{a_1\ln b}$，证明:$a_{k+1}>b$.

2008Ⅰ 证明：（1）由$f'(x)=1-\ln x-1=-\ln x>0\Leftrightarrow 0<x<1$，$\therefore f(x)$在$(0,1)$上是增函数；

（2）由$\lim\limits_{x\to 0^+}f(x)=0, f(1)=1$，$\because 0<a_1<1$，由（1）得$a_2=f(a_1)\in(0,1)$

且$a_2-a_1=-a_1\ln a_1>0$，$\therefore 0<a_1<a_2<1$成立；

若$0<a_k<a_{k+1}<1$，则$0<f(a_k)<f(a_{k+1})<f(1)$

而$f(1)=1, f(a_k)=a_{k+1}>0, f(a_{k+1})=a_{k+2}$，$\therefore 0<a_{k+1}<a_{k+2}<1$成立，$\therefore 0<a_n<a_{n+1}<1, n\in N^*$

数列（3）数列性质解答（3）

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（3）由（2）得数列$\{a_n\}$是递增数列，且$\dfrac{a_{n+1}}{a_n}=1-\ln a_n$，

设$a_1,a_2,\cdots,a_k\le b$,则$1-\ln a_n\ge 1-\ln b(n=1,2,\cdots,k)$

$\therefore a_{k+1}=\dfrac{a_{k+1}}{a_k}\cdot\dfrac{a_k}{a_{k-1}}\cdots\dfrac{a_2}{a_1}\cdot a_1\ge(1-\ln b)^k\cdot a_1\ge(1-k\ln b)a_1>b$

$\because 1-\ln b\ge 1-\ln b$,若$(1-\ln b)^k\ge 1-k\ln b$成立，则$(1-\ln b)^{k+1}\ge(1-k\ln b)(1-\ln b)$

$=1-(k+1)\ln b+k\ln^2 b\ge 1-(k+1)\ln b$也成立,$\therefore (1-\ln b)^n-n\ln b(n\in N^*)$

$\therefore$只要$k\ge\dfrac{a_1-b}{a_1\ln b}$,$\therefore$正数$k\ge\dfrac{a_1-b}{a_1\ln b}$时,$a_{k+1}>b$,证毕

（2016浙江）设数列$\{a_n\}$满足$|a_n-\dfrac{a_{n+1}}{2}|\le 1$.

（Ⅰ）证明:$|a_n|\ge 2^{n-1}(|a_1|-2)$；（Ⅱ）若$|a_n|\le(\dfrac{3}{2})^n,n\in N^*$,证明:$|a_n|\le 2,n\in N^*$.

key:（Ⅰ）由三角形不等式得：由$1\ge|a_n-\dfrac{a_{n+1}}{2}|\ge||a_n|-\dfrac{|a_{n+1}|}{2}|$得$-1\le\dfrac{|a_{n+1}|}{2}-|a_n|\le 1$

$\therefore 2(|a_n|+1)\ge|a_{n+1}|\ge 2(|a_n|-1)$,$\therefore|a_{n+1}|-2\ge 2(|a_n|-2)$

当$|a_1|-2\le 0$时，显然成立

当$|a_1|>2$时，若$|a_n|-2>0$,则$|a_{n+1}|-2\ge 2(|a_n|-2)>0$,

$\because|a_1|-2>0$成立,$\therefore|a_n|-2>0$(这就是数学归纳法)

$\therefore|a_n|-2=\dfrac{|a_n|-2}{|a_{n-1}|-2}\cdots\cdot\dfrac{|a_2|-2}{|a_1|-2}\cdot(|a_1|-2)\ge 2^{n-1}(|a_1|-2)(n\ge 2)$,

$\therefore|a_n|\ge 2^{n-1}(|a_1|-2)+2\ge 2^{n-1}(|a_1|-2),n\in N^*$

（Ⅱ）$\because|a_n|\le(\dfrac{3}{2})^n$,$\therefore|a_1|\le\dfrac{3}{2}\le 2$.假设存在$k\in N^*(k\ge 4)$,使得$|a_k|>2$,

则由（Ⅰ）得:当$m>k$时,$|a_m|-2=\dfrac{|a_m|-2}{|a_{m-1}|-2}\cdots\cdot\dfrac{|a_{k+1}|-2}{|a_k|-2}\cdot(|a_k|-2)\ge 2^{m-k}(|a_k|-2)(n\ge 2)$,

$\therefore|a_m|\ge 2^{n-k}(|a_k|-2)+2$,

令$m=n+k$,则$|a_{n+k}|>2^n(|a_k|-2)+2>2^n(|a_k|-2)$

而$2^n(|a_k|-2)<(\dfrac{3}{2})^n\Leftrightarrow|a_k|-2<(\dfrac{3}{4})^n$（$\because$当$n\to+\infty$时,$(\dfrac{3}{4})^n\to 0$)不恒成立

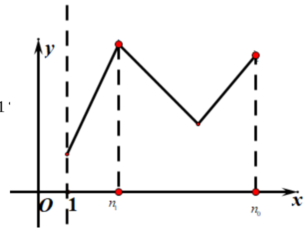
$\therefore|a_n|\le 2(n\in N^*)d$得证

（2016北京）20.设数列$A:a_1,a_2,\cdots,a_N(N\ge 2)$.如果对小于$n(2\le n\le N)$的每个正整数$k$都有$a_k<a_n$，则称$n$是数列$A$的一个“$G$时刻”.记$G(A)$是数列$A$的所有“$G$时刻”组成的集合.

（Ⅰ）对数列$A:-2,2,-1,1,3$,写出$G(A)$的所有元素；

（Ⅱ）证明：若数列A中存在a_n使得$a_n>a_1$,则$G(A)\ne\Phi$；

（Ⅲ）证明：若数列A满足$a_n-a_{n-1}\le 1(n=2,3,\cdots,N)$,则$G(A)$的元素个数不小于$a_N-a_1$.

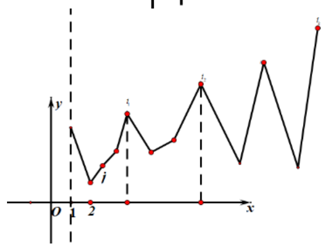


2016北京（1）解：由题意得:$G(A)=\{2,5\}$

（2）证明：设存在$n_0\ge 2$,使得$a_{n_0}>a_1$

当$2\le n<n_0$时，$a_n<a_{n_0}$,则$n_0\in G(A)$,$\therefore G(A)\ne\Phi$;

当$2\le n<n_0$时，存在最小的$n_1\in[2,n_0)$,使得$a_{n_1}>\max\{a_1,a_{n_0}\}$,



则$a_i<a_{n_1}(i=1,2,\cdots n_1)$,$\therefore n_1\in G(A)$,$\therefore$综上$G(A)\ne\Phi$

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(3) 证明：当$a_N \le a_1$时，$a_N - a_1 \le 0,$命题成立；
当$a_N > a_1$时，由（2）得$G(A) \ne \Phi$；
记数列A的所有"G时刻"为$i_1, i_2, \cdots, i_k\,(i_1 < i_2 < \cdots < i_k)$,
则$a_{i_1} > a_1 \ge a_j\,(j = 1, 2, \cdots, i_1 - 1)$, 则$a_{i_1} - a_1 \le a_{i_1} - a_{i_1 - 1} \le 1$,
由$a_{i_2} > a_{i_1} \ge a_j\,(j = 1, 2, \cdots, i_2 - 1)$, 则$a_{i_2} - a_{i_1} \le a_{i_2} - a_{i_2 - 1} \le 1$
同理$a_{i_3} - a_{i_2} \le 1, \cdots, a_{i_k} - a_{i_{k-1}} \le 1$
$\therefore k \ge (a_{i_k} - a_{i_{k-1}}) + (a_{i_{k-1}} - a_{i_{k-2}}) + \cdots + (a_{i_2} - a_{i_1}) + (a_{i_1} - a_1) = a_{i_k} - a_1$
若$N \in G(A)$，则$a_{i_k} = a_N, \therefore k \ge a_N - a_1$；
若$N \notin G(A)$, 则$a_N \le a_{i_k}$, 则$G(A)$增加了一个G时刻，矛盾，证毕

(2019北京)20.已知数列$\{a_n\}$，从中选取第i_1项、第i_2项、…、第i_m项$(i_1 < i_2 < \cdots < i_m)$, 若$a_{i_1} < a_{i_2} < \cdots < a_{i_m}$,
则称新数列$a_{i_1}, a_{i_2}, \cdots, a_{i_m}$为$\{a_n\}$的长度为$m$的递增子列.规定：数列$\{a_n\}$的任意一项都是$\{a_n\}$的长度为1的递增子列.（Ⅰ）写出数列$1, 8, 3, 7, 5, 6, 9$的一个长度为4的递增子列；（Ⅱ）已知数列$\{a_n\}$的长度为$p$的递增子列的末项的最小值为$a_{m_0}$，长度为$q$的递增子列的末项的最小值为$a_{n_0}$.若$p < q$，求证：$a_{m_0} < a_{n_0}$；
(Ⅲ) 设无穷数列$\{a_n\}$的各项均为正整数，且任意两项均不相等.若$\{a_n\}$的长度为s的递增子列末项的最小值为$2s-1$，且长度为s末项为$2s-1$的递增子列恰有2^{s-1}个$(s = 1, 2, \cdots)$, 求数列$\{a_n\}$的通项公式.

(2019北京）（Ⅰ）解：长度为4的递增子列为$:1, 3, 5, 6.$（不唯一）
（Ⅱ）证明：由已知得$\{a_n\}$的长度为p的递增子列为$: a_{i_1}, a_{i_2}, \cdots, a_{i_p}$, 且$(a_{i_p})_{\min} = a_{m_0}$
$\{a_n\}$的长度为q的递增子列为$: a_{j_1}, a_{j_2}, \cdots, a_{j_q}$, 且$(a_{j_q})_{\min} = a_{n_0}$
$\because p < q, \therefore \{a_n\}$的长度为$q$的递增子列的末项最小时，
长度为q的递增子列为$: a_{j_1}, a_{j_2}, \cdots, a_{j_p}, a_{j_p+1}, \cdots, a_{i_q}\,(a_{i_q} = a_{n_0})$
$\therefore a_{m_0} \le a_{j_p} < a_{n_0}, \therefore a_{m_0} < a_{n_0}$,
(Ⅲ) 解：若$2s$在$\{a_n\}$中，则$2s$必在$2s-1$之前；
若末项为$2s+1$的长度为$s+1$的递增子列，
若数列$\{a_n\}$中有$2n-1, 2n$, 则$2n$在$2n-1$之前,
$\therefore \{a_n\}: (2,1), (4,3), \cdots, (2n, 2n-1), \cdots \therefore a_{2k} = 2k-1, a_{2k-1} = 2k,$

$$\therefore a_k = \begin{cases} k-1, k\text{为偶数}, \\ k+1, k\text{为奇数} \end{cases} = \frac{1+(-1)^k}{2} \cdot (k-1) + \frac{1-(-1)^k}{2} \cdot (k+1) = k - (-1)^k$$

(1997A) 已知数列$\{x_n\}$满足$x_{n+1} = x_n - x_{n-1}\,(n \ge 2), x_1 = a, x_2 = b,$ 记$S_n = \sum_{i=1}^{n} x_i$, 则下列结论正确的是（　　）

$A.x_{100} = -a, S_{100} = 2b - a$　　$B.x_{100} = -b, S_{100} = 2b - a$　　$C.x_{100} = -b, S_{100} = b - a$　　$D.x_{100} = -a, S_{100} = b - a$

1997$Akey: x_1 = a, x_2 = b, x_3 = b - a, x_4 = -a, x_5 = -b, x_6 = a - b, x_7 = a, x_8 = b,$
$\therefore x_{n+6} = x_n, \therefore x_{100} = x_4 = -a, S_6 = 0, S_{100} = a_1 + a_2 + a_3 + a_4 = 2b - a,$ 选A

(2006北京)20.在数列$\{a_n\}$中，若a_1, a_2是正整数，且$a_n = |a_{n-1} - a_{n-2}|, n = 3, 4, 5, \cdots$, 则称$\{a_n\}$为“绝对等差数列”.
（Ⅰ）举出一个前5项不为零的“绝对差数列”（只要求写出前10项）；
（Ⅱ）若“绝等差数列”$\{a_n\}$中，$a_{20} = 3, a_{21} = 0,$ 数列$\{b_n\}$满足$b_n = a_n + a_{n+1} + a_{n+2}, n = 1, 2, 3 \cdots$, 分别判断当$n \to \infty$时，$a_n$与$b_n$的极限是否存在，如果存在，求出其极限值；
(Ⅲ) 证明：任何“绝对差数列”中总含有无穷多个零的项.

数列（3）数列性质解答（3）

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（Ⅰ）解:$8,7,1,6,5,1,4,3,1,2,\cdots$

（Ⅱ）解:$\because a_{20}=3,a_{21}=0,\therefore a_{22}=3,a_{23}=3,a_{24}=0,a_{25}=3,$

若$a_{20+3k}=3,a_{21+3k}=0,a_{22+3k}=3$

则$a_{20+3(k+1)}=|a_{20+3k+2}-a_{20+3k+1}|=3,a_{21+3(k+1)}=|a_{23+3k}-a_{22+3k}|=0,a_{22+3(k+1)}=|a_{24+3k}-a_{23+3k}|=3,$

$\therefore a_{20+3n}=a_{20}=3,a_{21+3n}=a_{21}=0,a_{22+3n}=a_{22}=3,n\in N^*,\therefore$当$n\ge 20$时，$b_n=a_n+a_{n+1}+a_{n+2}=6,\therefore \lim\limits_{n\to\infty}b_n=6,$

（Ⅲ）证明:$\because a_1,a_2\in N^*,\therefore a_n\in N,$

若$a_k,a_{k+1}\in N^*$,则$a_{k+2}=|a_{k+1}-a_k|\le\min\{a_k-1,a_{k+1}-1\},\therefore\exists n_0\in N^*$,使得$a_{n_0-1}=a\ne 0,a_{n_0}=0,$

若$a_{n_0+3k}=0,a_{n_0+3k+1}=a,a_{n_0+3k+2}=a$,则$a_{n_0+3(k+1)}=|a_{n_0+3k+2}-a_{n_0+3k+1}|=0,$

$a_{n_0+3(k+1)+1}=|a_{n_0+3k+3}-a_{n_0+3k+2}|=a,a_{n_0+3(k+1)+2}=|a_{n_0+3k+4}-a_{n_0+3k+3}|=a,\therefore a_{n_0+3k}=0(k\in N^*)$,证毕

（202101学考）已知数列$\{a_n\}$的前n项和为S_n，且满足$a_1=-2,a_{n+1}=1-\dfrac{1}{a_n},n\in N^*$，

则（　　）$A.a_{40}<a_{100}$　$B.a_{40}>a_{100}$　$C.S_{40}<S_{100}$　$D.S_{40}>S_{100}$

（202101学考）key:（周期性）$a_1=-2,a_2=\dfrac{3}{2},a_3=\dfrac{1}{3},a_4=-2,\therefore T=3$

$\therefore a_{40}=a_1=a_{100},S_{40}=13\times(-2+\dfrac{3}{2}+\dfrac{1}{3})-2>S_{100}=33(-2+\dfrac{3}{2}+\dfrac{1}{3}-2),\therefore$选$D$

（202107）　22. 已知整数数列$\{a_n\}$的前 n 项和为 S_n，且 $a_{n+2}=|a_{n+1}-a_n|,n\in N^*$．若对任意给定的$a_1$，存在正整数 n_0，使得 $S_{3n+n_0}-S_{n_0}<4n+1$ 对任意正整数 n 成立，则 a_3 的取值集合是________.

202107key:由$S_{3n+n_0}-S_{n_0}=a_{n_0+1}+\cdots+a_{n_0+3n}<4n+1$

若$a_3=k\ge 3$,取$a_1=k,a_2=0,a_3=k,a_4=k,a_5=0,a_6=k,\cdots$,则$S_{3n+n_0}-S_{n_0}=2kn\ge 6n>4n+1$不合；

若$a_3=0$,取$a_1=3,a_2=3$,则$a_3=0,a_4=3,a_5=3,a_6=0,\cdots$,则$S_{3n+n_0}-S_{n_0}=6n>4n+1$,不合；

若$a_3=1,\forall a_1=m$,取$a_2=m-1$,则$a_3=1,a_4=m-2,a_5=m-3,a_6=1,\cdots,1,1,0,\cdots,$

$\therefore S_{3n+n_0}-S_{n_0}=3n<4n+1$

若$a_3=2,\forall a_1=m$,取$a_2=m-2$,则$a_3=2,a_4=m-4,a_5=m-6,a_6=2,a_7=m-8,\cdots,2,2,0,\cdots$

$\therefore S_{3n+n_0}-S_{n_0}=4n<4n+1,\therefore a_3$的取值集合为$\{1,2\}$

变式 1（1）已知数列$\{x_n\}$满足$x_{n+1}=|x_n-x_{n-1}|(n\ge 2)$,如果$x_1=1,x_2=a$,当数列$\{x_n\}$的周期最小时,该数列前2022项的和是________.

key:若$a<0$,由$x_{n+1}=|x_n-x_{n-1}|\ge 0$,所以$\{x_n\}$不可能是周期数列；

若$0<a<1$，则$x_3=|1-a|\in(0,1),\Rightarrow x_n\in(0,1)(n>1)\Rightarrow x_n\ne 1$

若$a=0$,则$x_1=1,x_2=0,x_3=1,x_4=1,x_5=0,x_6=1,\therefore T=3$

若$a>1$,则$x_1=1,x_2=a,x_3=a-1<x_2,x_4=1,x_5=|2-a|<x_2$,不是周期数列；

若$a=1$,则$x_1=1,x_2=1,x_3=0,x_4=1,x_5=1,x_6=0,\therefore T=3,\therefore S_{2022}=1349$

(2011北京)20.若数列$A_n=a_1,a_2,\cdots,a_n(n\ge 2)$满足$|a_{k+1}-a_k|=1(k=1,2,\cdots,n-1)$,数列$A_n$为$E$数列，记$S(A_n)=a_1+a_2+\cdots+a_n$.（Ⅰ）写出一个满足$a_1=a_5=0$,且$S(A_5)>0$的$E$数列$A_n$;

（Ⅱ）若$a_1=12,n=2000$,证明：E数列A_n是递增数列的充要条件是$a_n=2011$;

（Ⅲ）对任意给定的整数$n(n\ge 2)$，是否存在首项为0的E数列A_n，使得$S(A_n)=0$?如果存在，写出一个满足条件的E数列A_n；如果不存在，说明理由.

数列（3）数列性质解答（3）

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（Ⅰ）解：由已知得$:a_1=0,|a_{k+1}-a_k|=1(k=1,2,\cdots,n-1)$,

$A_5=0,1,0,1,0,1,S(A_5)=2$,或者$A_5=0,1,2,1,0,S(A_5)=4.$

（Ⅱ）证明：①充分性$:\because a_n=2011,a_1=12,n=2000,\therefore|a_2-12|=1,$

若$a_2=11$,由$a_{k+1}-a_k=\pm1$得若$a_{k+1}-a_k=1,\therefore a_{2000}=11+1999=2010\neq2011,$

$\therefore a_2=13,a_{2000}=2011=13+1999,\therefore a_{k+1}-a_k=1>0,\therefore A_n$是递增数列

②必要性$:\because a_1=12,n=2000,$且A_n是递增数列$,\therefore|a_{k+1}-a_k|=a_{k+1}-a_k=1$

$\therefore a_{2020}=a_1+1999=2011.$由①②可知$:E$数列$A_n$是递增数列的充要条件是$a_n=2011$

（Ⅲ）解$:\because a_1=0,$设$b_k=a_{k+1}-a_k$,则$b_k=\pm1(k=1,2\cdots,n-1),$

$\therefore a_2=a_1+b_1=b_1,a_3=b_2+a_2=b_1+b_2,a_4=a_3+b_3=b_1+b_2+b_3,\cdots,a_n=b_1+b_2+\cdots+b_{n-1},$

$\therefore S_n=(n-1)b_1+(n-2)b_2+\cdots+b_{n-1}=\dfrac{n(n-1)}{2}-[(1-b_1)(n-1)+(1-b_2)(n-2)+\cdots+(1-b_{n-1})\cdot1]$

$(\because b_k=\pm1,\therefore1-b_k=0,or,2,\therefore(1-b_1)(n-1)+(1-b_2)(n-2)+\cdots+(1-b_{n-1})\cdot1$是偶数

$\therefore$当$n=4m,or,4m-3(m\in N^*)$时$,\dfrac{n(n-1)}{2}$是偶数$,S_n$可以为0,

当$n=4m-1,or,4m-2(m\in N^*)$时$,\dfrac{n(n-1)}{2}$时奇数$,S_n$不为0,

当$n=4m(m\in N^*)$时，$a_k=\cos\dfrac{k}{2}\pi$满足$|a_{k+1}-a_k|=1,$且$S(A_n)=0;$

当$n=4m-3$时，$a_k=\cos\dfrac{k}{2}\pi$满足$|a_{k+1}-a_k|=1,$且$S(A_n)=0;$

当$n=4m-2$时，$S(A_n)\neq0;$当$n=4m-1$时，$S(A_n)\neq0.$