初等函数（II）三角解答（1）

一．三角变换

①三角函数定义、象限上的符号、特殊角三角函数值、三角函数线

②同角三角函数关系: $\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha},\sin^2\alpha+\cos^2\alpha=1$

③诱导公式:以下$k\in Z$

周期性: $\sin(2k\pi+\alpha)=\sin\alpha,\cos(2k\pi+\alpha)=\cos\alpha,\tan(k\pi+\alpha)=\tan\alpha,$

奇偶性: $\sin(-\alpha)=-\sin\alpha$，$\cos(-\alpha)=\cos\alpha$，$\tan(-\alpha)=-\tan\alpha$

$\sin(\pi-\alpha)=\sin\alpha,\cos(\pi-\alpha)=-\cos\alpha,\tan(\pi-\alpha)=-\tan\alpha,\sin(\pi+\alpha)=\sin\alpha,\cos(\pi+\alpha)=-\cos\alpha,$

$\sin(\dfrac{\pi}{2}-\alpha)=\cos\alpha,\cos(\dfrac{\pi}{2}-\alpha)=\sin\alpha,\tan(\dfrac{\pi}{2}-\alpha)=\dfrac{1}{\tan\alpha};\sin(\dfrac{\pi}{2}+\alpha)=\cos\alpha,\cos(\dfrac{\pi}{2}+\alpha)=-\sin\alpha,$

$\tan(\dfrac{\pi}{2}+\alpha)=-\dfrac{1}{\tan\alpha};\sin(\dfrac{3\pi}{2}-\alpha)=-\cos\alpha,\cos(\dfrac{3\pi}{2}-\alpha)=-\sin\alpha,\tan(\dfrac{3\pi}{2}-\alpha)=\dfrac{1}{\tan\alpha};$

$\sin(\dfrac{3\pi}{2}+\alpha)=-\cos\alpha,\cos(\dfrac{3\pi}{2}+\alpha)=\sin\alpha,\tan(\dfrac{3\pi}{2}+\alpha)=-\dfrac{1}{\tan\alpha}$

（2）特殊角三角函数值: $k\in Z,m\in Z$

$\sin\dfrac{k\pi}{6}=\_\_\_\_\_,\cos\dfrac{k\pi}{6}=\_\_\_\_\_\_,\tan\dfrac{k\pi}{6}=\_\_\_\_\_(k=6m\pm1);\sin\dfrac{k\pi}{4}=\_\_\_\_\_,\cos\dfrac{k\pi}{4}=\_\_\_\_\_\_,$

$\tan\dfrac{k\pi}{4}=\_\_\_\_\_(k=4m\pm1);\sin\dfrac{k\pi}{3}=\_\_\_\_\_,\cos\dfrac{k\pi}{3}=\_\_\_\_\_\_,\tan\dfrac{k\pi}{3}=\_\_\_\_\_(k=3m\pm1)$

（2）和差倍角公式

$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta;\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

变式①： $a\sin\alpha+b\cos\alpha=\_\_\_\_\_\_\_\_\_\_$

*② $\sin\alpha\pm\sin\beta=\_\_\_\_\_\_\_\_\_\_\_\_\_,\cos\alpha\pm\cos\beta=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$

*③ $\sin\alpha\cos\beta=\_\_\_\_\_\_\_\_\_\_\_\_\_\_,\cos\alpha\cos\beta=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_,\sin\alpha\sin\beta=\_\_\_\_\_\_\_\_\_\_\_\_$

$\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}\Rightarrow\tan\alpha\pm\tan\beta=\tan(\alpha\pm\beta)(1\mp\tan\alpha\tan\beta)$

$\sin2\alpha=2\sin\alpha\cos\alpha\Rightarrow\sin\alpha\cos\alpha=\dfrac{1}{2}\sin2\alpha$

$\cos2\alpha=\cos^2\alpha-\sin^2\alpha=2\cos^2\alpha-1=1-2\sin^2\alpha$，变式: $\tan2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha},\sin2\alpha=\dfrac{2\tan\alpha}{1+\tan^2\alpha},\cos2\alpha=\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$

升幂公式: $1+\cos\alpha=2\cos^2\dfrac{\alpha}{2},1-\cos\alpha=2\sin^2\dfrac{\alpha}{2}$

降幂公式: $\cos^2\alpha=\dfrac{1+\cos2\alpha}{2},\sin^2\alpha=\dfrac{1-\cos2\alpha}{2}$，变形: $\sin\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1-\cos\alpha}{2}},\cos\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1+\cos\alpha}{2}},\tan\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{\sin\alpha}=\dfrac{\sin\alpha}{1+\cos\alpha}$

2．三角变换的基本出发点：①角关系：互余、互补，和、差、倍；②名称关系：切化弦，

③结构特征：分式（分子分母尽量化积），根式，高次降幂

例 1（1）①在$\triangle ABC$中，$\sin A=\dfrac{3}{5},\sin B=\dfrac{12}{13}$，则$\sin C=____________.\ \dfrac{63}{65},or,\dfrac{33}{65}$

②已知 $\cos\theta-\sin\theta=\dfrac{7\sqrt{2}}{25},\theta\in(\pi,2\pi)$，求 $\sin(\dfrac{\theta}{2}+\dfrac{\pi}{8})$ 的值.

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解：$\therefore \cos(\theta+\frac{\pi}{4})=1-2\sin^2(\frac{\theta}{2}+\frac{\pi}{8})=\frac{7}{25},\therefore \sin(\frac{\theta}{2}+\frac{\pi}{8})=\pm\frac{3}{5}$

$\because \pi<\theta<2\pi,\therefore \frac{\theta}{2}+\frac{\pi}{8}\in(\frac{5\pi}{8},\frac{9\pi}{8})\subseteq(\frac{\pi}{2},\frac{7\pi}{6}),\therefore \sin(\frac{\theta}{2}+\frac{\pi}{8})\in(-\frac{1}{2},1)$ $\quad\therefore \sin(\frac{\theta}{2}+\frac{\pi}{8})=\frac{3}{5}$

③已知$\frac{5\cos\alpha-\sin\alpha}{\sin\alpha+2\cos\alpha}=\frac{16}{5}$,则$\frac{\sin\alpha\cos\alpha-1}{2-\sin^2\alpha}=$__________; $-\frac{13}{19}$

变式：已知$\sin\beta=\frac{3}{5}(\frac{\pi}{2}<\beta<\pi)$，且$\sin(\alpha+\beta)=\cos\alpha$,则$\sin^2\alpha+\sin\alpha\cos\alpha-2\cos^2\alpha=$_____. $-\frac{9}{5}$

$key:\sin(\alpha+\beta)=\sin\alpha\cdot(-\frac{4}{5})+\cos\alpha\cdot\frac{3}{5}=\cos\alpha$得$\tan\alpha=-\frac{1}{2}$

$\therefore \sin^2\alpha+\sin\alpha\cos\alpha-2\cos^2\alpha=\frac{\sin^2\alpha+\sin\alpha\cos\alpha-2\cos^2\alpha}{\cos^2\alpha+\sin^2\alpha}=\frac{\tan^2\alpha+\tan\alpha-2}{\tan^2\alpha+1}=-\frac{9}{5}$

（2）①已知$\sin\alpha+\cos\alpha=\frac{7}{5}$,则$\tan\alpha=$__________. $\frac{3}{4},or,\frac{4}{3}$

②已知$\theta\in(\frac{\pi}{2},\pi),\frac{1}{\sin\theta}+\frac{1}{\cos\theta}=2\sqrt{2}$,则$\sin(2\theta-\frac{\pi}{3})=$________.

$key:\theta\in(\frac{\pi}{2},\pi),\frac{1}{\sin\theta}+\frac{1}{\cos\theta}=2\sqrt{2}\Rightarrow \sin\theta+\cos\theta=2\sqrt{2}\sin\theta\cos\theta=\sqrt{2}\cdot[(\sin\theta+\cos\theta)^2-1]$

$\Rightarrow \sin\theta+\cos\theta=\sqrt{2}\sin(\theta+\frac{\pi}{4})=-\frac{\sqrt{2}}{2},or,\sqrt{2}(舍去),\therefore \theta=\frac{11\pi}{12},\therefore \sin(2\theta-\frac{\pi}{3})=-1$

③已知$\sin\alpha+2\cos\alpha=\frac{11}{5}(0<\alpha<\pi)$，则$\tan\alpha=$____________. $\frac{7}{24},or,\frac{3}{4}$

④函数$f(x)=(1-\cos x)(1+\sin x)(x\in[0,\frac{\pi}{2}])$的值域为________. $[0,2]$

变式1（1）已知实数x,y满足$x^2+y^2=1$,则$z=\frac{x+y}{1-xy}$的最大值为_______.

$key:$令$x=\cos\theta,y=\sin\theta$，则$x=\frac{\cos\theta+\sin\theta}{1-\cos\theta\sin\theta}=\frac{\cos\theta+\sin\theta}{1-\frac{(\cos\theta+\sin\theta)^2-1}{2}}$

$$=\frac{2t}{3-t^2}=\begin{cases}0,t=0\\ \frac{2}{\frac{3}{t}-t},t\in[-\sqrt{2},0)\cup(0,\sqrt{2}]\end{cases}\le 2\sqrt{2}(其中t=\cos\theta+\sin\theta=\sqrt{2}\sin(\theta+\frac{\pi}{4})\in[-\sqrt{2},\sqrt{2}])$$

（2）函数$y=2x-\sqrt{1-x^2}$的值域为_______;

$key:$令$x=\cos\theta(\theta\in[0,\pi])$，则$y=2\cos\theta-\sin\theta=\sqrt{5}\cos(\theta+\arctan\frac{1}{2})\in[-\sqrt{5},2]$

函数$y=\sqrt{3+x}+2\sqrt{1-x}$的值域为_______

$key1:$由$-3\le x\le 1$得$-2\le x+1\le 2$令$x+1=2\cos\theta(\theta\in[0,\pi])$

则$y=\sqrt{2+2\cos\theta}+2\sqrt{2-2\cos\theta}=2\cos\frac{\theta}{2}+4\sin\frac{\theta}{2}=2\sqrt{5}\sin(\frac{\theta}{2}+arc\tan\frac{1}{2})\in[2,2\sqrt{5}]$

$key2$:由$-3\le x\le 1$得$x+3\in[0,4]$令$x+3=4\cos^2\theta(\theta\in[0,\frac{\pi}{2}])$

则$y=2\cos\theta+4\sin\theta\in[2,2\sqrt{5}]$

（3）设 x,y 为实数，若 $4x^2+y^2+xy=1$. 则 $(x+y)\in$ ______; $[-\frac{4}{\sqrt{15}},\frac{4}{\sqrt{15}}]$

$(x^2+2y^2)_{\max}=$ ______, $(x^2+2y^2)_{\min}=$ ______；（基本型：缺 xy 项） $\frac{4}{9+\sqrt{51}},\frac{4}{9-\sqrt{51}}$

$key:1=4x^2+y^2+xy=4x^2+y^2+2\cdot\lambda x\cdot\frac{1}{2\lambda}y\le 4x^2+y^2+\lambda^2x^2+\frac{1}{4\lambda^2}y^2=(4+\lambda^2)x^2+(1+\frac{1}{4\lambda^2})y^2$

（其中$2(4+\lambda^2)=1+\frac{1}{4\lambda^2}$即$\lambda^2=\frac{-7+\sqrt{51}}{4}$）$=\frac{9+\sqrt{51}}{4}(x^2+2y^2),\therefore x^2+2y^2\ge\frac{4}{9+\sqrt{51}}$

$1=4x^2+y^2+xy=4x^2+y^2-2\cdot(-\lambda x)\cdot\frac{1}{2\lambda}y\ge 4x^2+y^2-(\lambda^2x^2+\frac{1}{4\lambda^2}y^2)=(4-\lambda^2)x^2+(1-\frac{1}{4\lambda^2})y^2$

（其中$2(4-\lambda^2)=1-\frac{1}{4\lambda^2}$即$\lambda^2=\frac{7-\sqrt{51}}{4}$）$=\frac{9-\sqrt{51}}{4}(x^2+2y^2),\therefore x^2+2y^2\le\frac{4}{9-\sqrt{51}}$

$key2$:由$1=4x^2+y^2+xy=\frac{(2x+y)^2+(2x-y)^2}{2}+\frac{(2x+y)^2-(2x-y)^2}{8}=\frac{5}{8}(2x+y)^2+\frac{3}{8}(2x-y)^2=1$

令$\begin{cases}\sqrt{\frac{5}{8}}(2x+y)=\cos\theta\\ \sqrt{\frac{3}{8}}(2x-y)=\sin\theta\end{cases}$得$\begin{cases}x=\frac{\sqrt{2}}{2}(\frac{1}{\sqrt{5}}\cos\theta+\frac{1}{\sqrt{3}}\sin\theta)\\ y=\sqrt{2}(\frac{1}{\sqrt{5}}\cos\theta-\frac{1}{\sqrt{3}}\sin\theta)\end{cases}$

$\therefore x^2+2y^2=\frac{9}{10}\cos^2\theta+\frac{3}{2}\sin^2\theta-\frac{7}{\sqrt{15}}\sin\theta\cos\theta=\frac{6}{5}-\frac{3}{10}\cos 2\theta-\frac{7}{2\sqrt{15}}\sin 2\theta\in[\frac{6}{5}-\frac{2\sqrt{51}}{15},\frac{6}{5}+\frac{2\sqrt{51}}{15}]$

$key3$:令$x^2+2y^2=S$，则$x=\sqrt{S}\cos\theta,y=\frac{\sqrt{S}}{\sqrt{2}}\sin\theta,$

$\therefore 1=4S\cos^2\theta+\frac{S}{2}\sin^2\theta+\frac{S}{\sqrt{2}}\sin\theta\cos\theta,\therefore S=\frac{1}{2(1+\cos 2\theta)+\frac{1}{4}(1-\cos 2\theta)+\frac{1}{2\sqrt{2}}\sin 2\theta}$

$=\frac{1}{\frac{9}{4}+\frac{7}{4}\cos 2\theta+\frac{1}{2\sqrt{2}}\sin 2\theta}\in[\frac{4}{9+\sqrt{51}},\frac{4}{9-\sqrt{51}}]$

$(x^2-y^2)_{\max}=$ ______, $(x^2-y^2)_{\min}=$ ______. $\frac{2}{3+2\sqrt{6}},\frac{2}{3-2\sqrt{6}}$ （基本型：缺 xy 项）

$key:1=4x^2+y^2-2\cdot(-\lambda x)\cdot(\frac{1}{2\lambda}y)\ge 4x^2+y^2-(\lambda^2x^2+\frac{1}{4\lambda^2}y^2)$

$=(4-\lambda^2)x^2+(1-\frac{1}{4\lambda^2})y^2$（其中$4-\lambda^2=-(1-\frac{1}{4\lambda^2})$即$\lambda^2=\frac{5\pm 2\sqrt{6}}{2}$）

$=\frac{3\pm 2\sqrt{6}}{2}(x^2-y^2),\therefore\frac{2}{3-2\sqrt{6}}\le x^2-y^2\le\frac{2}{3+2\sqrt{6}},$

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$(x^2-xy)_{\max}=$______,$(x^2-xy)_{\min}=$______；$\dfrac{3+2\sqrt{6}}{15},\dfrac{3-2\sqrt{6}}{15}$（基本方法：消掉 xy 项）

$key1:1=4x^2+y^2+xy=\lambda x^2+(4-\lambda)x^2+y^2+xy\ge \lambda x^2+(2\sqrt{4-\lambda}+1)xy$（其中$\lambda+2\sqrt{4-\lambda}+1=0$即$\lambda=-3-2\sqrt{6}$）

$=(-3-2\sqrt{6})(x^2-xy),\therefore x^2-xy\ge\dfrac{1}{-3-2\sqrt{6}}$

$1=4x^2+y^2+xy=\lambda x^2+(4-\lambda)x^2+y^2+xy\ge \lambda x^2+(-2\sqrt{4-\lambda}+1)xy$（其中$\lambda-2\sqrt{4-\lambda}+1=0$即$\lambda=-3+2\sqrt{6}$）

$=(-3+2\sqrt{6})(x^2-xy),\therefore x^2-xy\le\dfrac{1}{-3+2\sqrt{6}}$

$key2:$令$t=x^2-xy,$则$t+1=5x^2+y^2$

由$1=4x^2+y^2+xy=4x^2+y^2+2\cdot\lambda x\cdot\dfrac{1}{2\lambda}y\le(4+\lambda^2)x^2+(1+\dfrac{1}{4\lambda^2})y^2$

$=\dfrac{9+\sqrt{6}}{10}(5x^2+y^2)$（其中$4+\lambda^2=5(1+\dfrac{1}{4\lambda^2})$即$\lambda^2=\dfrac{1+\sqrt{6}}{2}$）$\therefore t+1=5x^2+y^2\ge\dfrac{10}{9+\sqrt{6}}$

由$1=4x^2+y^2+xy=4x^2+y^2-2\cdot(-\lambda x)\cdot\dfrac{1}{2\lambda}y\ge(4-\lambda^2)x^2+(1-\dfrac{1}{4\lambda^2})y^2$

$=\dfrac{9-\sqrt{6}}{10}(5x^2+y^2)$（其中$4-\lambda^2=5(1-\dfrac{1}{4\lambda^2})$即$\lambda^2=\dfrac{-1+\sqrt{6}}{2}$），$\therefore t+1=5x^2+y^2\le\dfrac{10}{9-\sqrt{6}}$

$(2x+y+xy)\in$__________.

$\because 1=4x^2+xy+y^2=(2x+y)^2-3xy,\therefore xy=\dfrac{1}{3}[1-(2x+y)^2]$

$\therefore 2x+y+xy=(2x+y)+\dfrac{1}{3}[1-(2x+y)^2]=-\dfrac{1}{3}(2x+y-3)^2+\dfrac{10}{3}$

$\in[-\dfrac{1+2\sqrt{10}}{5},\dfrac{-1+2\sqrt{10}}{5}]$$(\because 2x+y\in[-\sqrt{\dfrac{8}{5}},\sqrt{\dfrac{8}{5}}])$

（4）①设a、b是非零实数,$x\in R$,若$\dfrac{\sin^4 x}{a^2}+\dfrac{\cos^4 x}{b^2}=\dfrac{1}{a^2+b^2}$,则$\dfrac{\sin^{2022}x}{a^{2020}}+\dfrac{\cos^{2022}x}{b^{2020}}=$____.

①由已知得$(\dfrac{\sqrt{a^2+b^2}\sin^2 x}{a})^2+(\dfrac{\sqrt{a^2+b^2}\cos x}{b})^2=1$令$\dfrac{\sqrt{a^2+b^2}\sin^2 x}{a}=\cos\theta,\dfrac{\sqrt{a^2+b^2}\cos^2 x}{b}=\sin\theta$

$\therefore\sin^2 x=\dfrac{a\cos\theta}{\sqrt{a^2+b^2}},\cos^2 x=\dfrac{b\sin\theta}{\sqrt{a^2+b^2}},\therefore 1=\dfrac{a}{\sqrt{a^2+b^2}}\cos\theta+\dfrac{b}{\sqrt{a^2+b^2}}\sin\theta=\sin(\theta+\varphi)(\cos\varphi=\dfrac{b}{\sqrt{a^2+b^2}},\sin\varphi=\dfrac{a}{\sqrt{a^2+b^2}})$

$\therefore\theta+\varphi=2k\pi+\dfrac{\pi}{2}(k\in Z),\therefore\cos\theta=\sin\varphi=\dfrac{a}{\sqrt{a^2+b^2}},\sin\theta=\cos\varphi=\dfrac{b}{\sqrt{a^2+b^2}}$

$\therefore\sin^2 x=\dfrac{a^2}{a^2+b^2},\cos^2 x=\dfrac{b^2}{a^2+b^2},\therefore\dfrac{\sin^{2022}x}{a^{2020}}+\dfrac{\cos^{2022}x}{b^{2020}}=\dfrac{(\dfrac{a^2}{a^2+b^2})^{1011}}{a^{2020}}+\dfrac{(\dfrac{b^2}{a^2+b^2})^{1011}}{b^{2020}}=\dfrac{1}{(a^2+b^2)^{1010}}$

②设θ_1,θ_2为锐角，且$\dfrac{\sin^{2020}\theta_1}{\cos^{2018}\theta_2}+\dfrac{\cos^{2020}\theta_1}{\sin^{2018}\theta_2}=1$,则$\theta_1+\theta_2=$_____.

②由$\dfrac{\sin^{2020}\theta_1}{\cos^{2018}\theta_2}+\underbrace{\cos^2\theta_2+\cdots+\cos^2\theta_2}_{1009}\ge1010\sin^2\theta_1;\dfrac{\cos^{2020}\theta_1}{\sin^{2018}\theta_2}+\underbrace{\sin^2\theta_2+\cdots+\sin^2\theta_2}_{1009}\ge1010\cos^2\theta_1$

$\therefore 1010=\dfrac{\sin^{2020}\theta_1}{\cos^{2018}\theta_2}+\dfrac{\cos^{2020}\theta_1}{\sin^{2018}\theta_2}+1009\ge1010,\therefore\sin\theta_1=\cos\theta_2,\sin\theta_2=\cos\theta_1,\therefore\theta_1+\theta_2=\dfrac{\pi}{2}$

（5）①已知 $\alpha,\beta\in(\frac{3\pi}{4},\pi)$， $\sin(\alpha+\beta)=-\frac{3}{5}$， $\sin(\beta-\frac{\pi}{4})=\frac{12}{13}$，则 $\cos(\alpha+\frac{\pi}{4})=$_____. $-\frac{56}{65}$

②设 $\sin(\frac{\pi}{4}+\theta)=\frac{1}{3}$， 则 $\sin 2\theta=$___________.

$key1:\sin(\frac{\pi}{4}+\theta)=\frac{\sqrt{2}}{2}(\sin\theta+\cos\theta)=\frac{1}{3}\Rightarrow\sin\theta+\cos\theta=\frac{\sqrt{2}}{3}\Rightarrow(\sin\theta+\cos\theta)^2=1+\sin 2\theta=\frac{2}{9},\therefore\sin 2\theta=-\frac{7}{9}$

$key2:\sin 2\theta=-\cos(2\theta+\frac{\pi}{2})=-(1-2\sin^2(\theta+\frac{\pi}{4}))=-\frac{7}{9}$

③ 设 α 为锐角，若 $\cos(\alpha+\frac{\pi}{6})=\frac{4}{5}$， 则 $\sin(2\alpha+\frac{\pi}{12})=$______. $\frac{17\sqrt{2}}{50}$

④设 $\alpha,\beta\in(0,\pi),\sin(\alpha+\beta)=\frac{5}{13},\tan\frac{\alpha}{2}=\frac{1}{2}$， 则 $\cos\beta$ 的值是________.

$key:$ 由 $\tan\alpha=\frac{2\cdot\frac{1}{2}}{1-\frac{1}{4}}=\frac{4}{3}>0,\therefore\cos\alpha=\frac{3}{5},\sin\alpha=\frac{4}{5}$，且 $\alpha\in(\frac{\pi}{4},\frac{\pi}{3}),\therefore\alpha+\beta\in(\frac{\pi}{4},\frac{4\pi}{3})$,

$\because\sin(\alpha+\beta)=\frac{3}{5}\in(0,\frac{\sqrt{2}}{4}),\therefore\alpha+\beta\in(\frac{3\pi}{4},\pi),\therefore\cos\beta=\cos((\alpha+\beta)-\alpha)=-\frac{12}{13}\cdot\frac{3}{5}+\frac{5}{13}\cdot\frac{4}{5}=-\frac{16}{65}$

⑤已知 $\cos(\alpha-\frac{\beta}{2})=-\frac{1}{9},\sin(\frac{\alpha}{2}-\beta)=\frac{2}{3}$，且 $\frac{\pi}{2}<\alpha<\pi,0<\beta<\frac{\pi}{2}$，则 $\cos(\alpha+\beta)=$________.

$key:\because\alpha\in(0,\frac{\pi}{2}),\beta\in(-\pi,-\frac{\pi}{2}),\therefore\alpha-\frac{\beta}{2}\in(\frac{\pi}{4},\pi),\frac{\alpha}{2}-\beta\in(-\frac{\pi}{4},\frac{\pi}{2})$

$\therefore\sin(\alpha-\frac{\beta}{2})=\frac{4\sqrt{5}}{9},\cos(\frac{\alpha}{2}-\beta)=\frac{\sqrt{5}}{3}$,

$\therefore\sin\frac{\alpha+\beta}{2}=\sin[(\alpha-\frac{\beta}{2})-(\frac{\alpha}{2}-\beta)]=\frac{4\sqrt{5}}{9}\cdot\frac{\sqrt{5}}{3}-(-\frac{1}{9})\cdot\frac{2}{3}=\frac{22}{27}$

$\therefore\cos(\alpha+\beta)=1-2\sin^2\frac{\alpha+\beta}{2}=-\frac{239}{729}$

⑥已知 $\tan(2\alpha+\frac{\pi}{6})=\frac{4}{3},\alpha\in(-\frac{\pi}{2},0)$， 则 $\sin(\alpha+\frac{\pi}{12})=$____, $\tan\alpha=$_________

$\because\tan(2\alpha+\frac{\pi}{6})=\frac{4}{3}>1,\alpha\in(-\frac{\pi}{2},0),\therefore 2\alpha+\frac{\pi}{6}\in(-\pi,-\frac{\pi}{2})$，且 $\cos(2\alpha+\frac{\pi}{6})=-\frac{3}{5}$

$\therefore\sin(\alpha+\frac{\pi}{12})=-\sqrt{\frac{1-(-\frac{3}{5})}{2}}=-\frac{2\sqrt{5}}{5},\tan(\alpha+\frac{\pi}{12})=-2,\therefore\tan\alpha=\tan(\alpha+\frac{\pi}{12}-\frac{\pi}{12})=-\frac{6+5\sqrt{3}}{3}$

（6）①设 $\alpha\in(0,\frac{\pi}{2}),\beta\in(0,\frac{\pi}{2})$，且 $\tan\alpha=\frac{1+\sin\beta}{\cos\beta}$， 则（ C ）

A. $3\alpha-\beta=\frac{\pi}{2}$　B. $3\alpha+\beta=\frac{\pi}{2}$　C. $2\alpha-\beta=\frac{\pi}{2}$　D. $2\alpha+\beta=\frac{\pi}{2}$

②已知 $\tan\alpha\tan\beta=\frac{7}{3},\tan\frac{\alpha+\beta}{2}=\frac{\sqrt{2}}{2}$，则 $\cos(\alpha+\beta)=$_____, $\cos(\alpha-\beta)=$_____

$$\cos(\alpha+\beta)=\frac{1-\tan^2\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}}=\frac{1}{3},\therefore\tan\alpha\tan\beta=\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}=\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)}=\frac{\cos(\alpha-\beta)-\frac{1}{3}}{\cos(\alpha-\beta)+\frac{1}{3}}=\frac{7}{3}$$

$$\therefore\frac{2\cos(\alpha-\beta)}{\frac{2}{3}}=\frac{10}{-4},\therefore\cos(\alpha-\beta)=-\frac{5}{6}$$

③已知α、β为锐角，且$\frac{1+\sin\alpha-\cos\alpha}{\sin\alpha}\cdot\frac{1+\sin\beta-\cos\beta}{\sin\beta}=2$，则$\tan\alpha\tan\beta=$______.

$$\frac{1+\sin\alpha-\cos\alpha}{\sin\alpha}\cdot\frac{1+\sin\beta-\cos\beta}{\sin\beta}=\frac{2\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\cdot\frac{2\sin^2\frac{\beta}{2}+2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}$$

$$=\frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\cdot\frac{\sin\frac{\beta}{2}+\cos\frac{\beta}{2}}{\cos\frac{\beta}{2}}=(1+\tan\frac{\alpha}{2})(1+\tan\frac{\beta}{2})=2,\therefore\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}=1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2},$$

$$\therefore\tan\frac{\alpha+\beta}{2}=1,\therefore\tan\alpha\tan\beta=1$$

$key2:\sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}+\sin\frac{\alpha}{2}\cos\frac{\beta}{2}=2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$得$\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$,

$$\therefore\alpha+\beta=\frac{\pi}{2}$$

（7）①若$\cos x\cos y+\sin x\sin y=\frac{1}{2},\sin 2x+\sin 2y=\frac{2}{3}$,则$\sin(x+y)=$_________ $\frac{2}{3}$.

②在$\triangle ABC$中,$\begin{cases}3\sin A+4\cos B=6\\3\cos A+4\sin B=1\end{cases}$,则$\cos(A+B)=$________,$\sin A=$________.

key:由$(3\sin A+4\cos B)^2+(3\cos A+4\sin B)^2=25+24\sin(A+B)=37$得$\sin(A+B)=\frac{1}{2}$

$(\because A+B\in(0,\pi))$得$A+B=\frac{\pi}{6},or,\frac{5\pi}{6}$,

若$A+B=\frac{\pi}{6}$，则$A,B\in(0,\frac{\pi}{6}),\therefore 3\cos A>\frac{3\sqrt{3}}{2},4\sin B>0,\therefore 3\cos A+4\sin B>1$

$$\therefore A+B=\frac{5\pi}{6},\therefore\cos(A+B)=-\frac{\sqrt{3}}{2}$$

由$16=(4\cos B)^2+(4\sin B)^2=(6-3\sin A)^2+(1-3\cos A)^2=46-36\sin A-6\cos A$

得$6\sin A+\cos A=5,\therefore 1-\sin^2 A=\cos^2 A=(5-6\sin A)^2=25-60\sin A+36\sin^2 A$

即$37\sin^2 A-60\sin A+24=0,\therefore\sin A=\frac{30\pm2\sqrt{3}}{37}$

③已知$\frac{\sin\alpha\cdot 2\sin\alpha\cos 2\beta}{\sin^2\alpha\cos^2 2\beta-\cos^2\alpha\sin^2 2\beta}=2,\sin\beta\neq0,\sin\alpha-k\cos\beta=0$,则$k=$　（　）

A．$\sqrt{2}$　　B．$-\sqrt{2}$　C．$\sqrt{2},or,-\sqrt{2}$　　D．以上都不对

$key:\dfrac{\sin\alpha\cdot 2\sin\alpha\cos 2\beta}{\sin^2\alpha\cos^2 2\beta-\cos^2\alpha\sin^2 2\beta}-2$

$=\dfrac{2\sin^2\alpha\cos 2\beta}{\sin^2\alpha\cos^2 2\beta-(1-\sin^2\alpha)\sin^2 2\beta}-2=\dfrac{2\sin^2\alpha\cos 2\beta}{\sin^2\alpha-\sin^2 2\beta}-2=0$

即$4\sin^2\beta\cos^2\beta=\sin^2 2\beta=\sin^2\alpha(1-\cos 2\beta)=2\sin^2\alpha\sin^2\beta,\therefore k=\pm\sqrt{2}$

④若$\sin\alpha\cos\beta=\dfrac{1}{3}$,则$\cos\alpha\sin\beta$的取值范围为______

设$t=\cos\alpha\sin\beta$,则$t+\dfrac{1}{3}=\sin(\alpha+\beta)\in[-1,1],\therefore t\in[-\dfrac{4}{3},\dfrac{2}{3}]$

$t-\dfrac{1}{3}=\sin(\beta-\alpha)\in[-1,1],\therefore t\in[-\dfrac{2}{3},\dfrac{4}{3}],\therefore t\in[-\dfrac{2}{3},\dfrac{2}{3}]$

⑤（2018 河南）已知$\cos(\alpha+\beta)=\cos\alpha+\cos\beta$,则$\cos\alpha$的取值范围为______.

$key:(\cos\alpha-1)\cos\beta-\sin\alpha\sin\beta=\cos\alpha,\therefore\dfrac{|\cos\alpha|}{\sqrt{(\cos\alpha-1)^2+\sin^2\alpha}}\le 1$得$\cos\alpha\in[-1,-1+\sqrt{2}]$

⑥设$\dfrac{\pi}{12}\le z\le y\le x$,且$x+y+z=\dfrac{\pi}{2}$，则$(\cos x\sin y\cos z)_{\max}=$____,$(\cos x\sin y\cos z)_{\min}=$____.

$\because x\ge y\ge z\ge\dfrac{\pi}{12}$,且$3z\le\dfrac{\pi}{6}+x\le x+y+z=\dfrac{\pi}{2}\le 3x$,且$\dfrac{\pi}{2}\ge x+2\cdot\dfrac{\pi}{12},\therefore x\in[\dfrac{\pi}{6},\dfrac{\pi}{3}],z\in[\dfrac{\pi}{12},\dfrac{\pi}{6}]$,

$\therefore\cos x\sin y\cos z=\cos x\cdot\dfrac{1}{2}[\sin(y+z)+\sin(y-z)]\ge\dfrac{1}{2}\cos^2 x\ge\dfrac{1}{8}$

$\cos x\sin y\cos z=\cos z\cdot\dfrac{1}{2}[\sin(y+x)+\sin(y-x)]\le\dfrac{1}{2}\cos^2 z\le\dfrac{1}{2}\cdot(\dfrac{\sqrt{6}+\sqrt{2}}{4})^2=\dfrac{2+\sqrt{3}}{8}$

⑦已知$\alpha,\beta,\gamma\in[0,\dfrac{\pi}{2}]$,则$\cos\alpha+2\cos\beta+\cos\gamma-\cos(\alpha+\gamma)-2\cos(\beta+\gamma)$的最大值为____.

$key:$原式$=\cos(\dfrac{\alpha+\alpha+\gamma}{2}+\dfrac{\alpha-(\alpha+\gamma)}{2})-\cos(\dfrac{\alpha+\alpha+\gamma}{2}-\dfrac{\alpha-(\alpha+\gamma)}{2})$

$+2(\cos\beta-\cos(\beta+\gamma))+\cos\gamma=2\sin\dfrac{2\alpha+\gamma}{2}\sin\dfrac{\gamma}{2}+4\sin\dfrac{2\beta+\gamma}{2}\sin\dfrac{\gamma}{2}+\cos(\dfrac{\gamma}{2}+\dfrac{\gamma}{2})$

$(\because\alpha,\beta,\gamma\in[0,\dfrac{\pi}{2}],\therefore\dfrac{2\alpha+\gamma}{2},\dfrac{2\beta+\gamma}{2}\in[0,\dfrac{3\pi}{4}])$

$\le 2\sin\dfrac{\gamma}{2}+4\sin\dfrac{\gamma}{2}+\cos^2\dfrac{\gamma}{2}-\sin^2\dfrac{\gamma}{2}=-2\sin^2\dfrac{\gamma}{2}+6\sin\dfrac{\gamma}{2}+1=-2(\sin\dfrac{\gamma}{2}-\dfrac{3}{2})^2+\dfrac{11}{2}\le 3\sqrt{2}(\because\sin\dfrac{\gamma}{2}\in[0,\dfrac{\sqrt{2}}{2}])$

⑧已知$\dfrac{\sin a}{\sin b}=\dfrac{\sin c}{\sin d}=\dfrac{\sin(a-c)}{\sin(b-d)},a,b,c,d\in(0,\pi)$,证明：$a=b,c=d$.

若$\dfrac{a}{b}=\dfrac{c}{d}$,则（合比定理）$\dfrac{a+b}{b}=\dfrac{c+d}{d}$;分比定理:$\dfrac{a}{a-b}=\dfrac{c}{c-d}$;合分比定理:$\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$

等比定理:$\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{a+c}{b+d}=\dfrac{a-c}{b-d}$

$key:\because a,b,c,d\in(0,\pi),\therefore\dfrac{\sin a}{\sin b}=\dfrac{\sin c}{\sin d}=\dfrac{\sin(a-c)}{\sin(b-d)}>0$,且$\dfrac{a-c}{2},\dfrac{b-d}{2}\in(-\dfrac{\pi}{2},\dfrac{\pi}{2}),a-c\ne 0,b-d\ne 0$,

$\therefore\cos\dfrac{a-c}{2},\cos\dfrac{b-d}{2}\ne 0,\sin\dfrac{a-c}{2}\ne 0,\sin\dfrac{b-d}{2}\ne 0$,

由 $\dfrac{2\sin\frac{a+c}{2}\cos\frac{a-c}{2}}{2\sin\frac{b+d}{2}\cos\frac{b-d}{2}}=\dfrac{\sin a+\sin c}{\sin b+\sin d}=\dfrac{\sin a}{\sin b}=\dfrac{\sin c}{\sin d}=\dfrac{\sin(a-c)}{\sin(b-d)}=\dfrac{2\sin\frac{a-c}{2}\cos\frac{a-c}{2}}{2\sin\frac{b-d}{2}\cos\frac{b-d}{2}}$

$\therefore\dfrac{\sin\frac{a+c}{2}}{\sin\frac{b+d}{2}}=\dfrac{\sin\frac{a-c}{2}}{\sin\frac{b-d}{2}}=\dfrac{\sin\frac{a+c}{2}+\sin\frac{a-c}{2}}{\sin\frac{b+d}{2}+\sin\frac{b-d}{2}}=\dfrac{2\sin\frac{a}{2}\cos\frac{c}{2}}{2\sin\frac{b}{2}\cos\frac{d}{2}}=\dfrac{\sin\frac{a+c}{2}-\sin\frac{a-c}{2}}{\sin\frac{b+d}{2}-\sin\frac{b-d}{2}}=\dfrac{2\cos\frac{a}{2}\sin\frac{c}{2}}{2\cos\frac{b}{2}\sin\frac{d}{2}},\therefore\tan\frac{a}{2}\cdot\tan\frac{d}{2}=\tan\frac{b}{2}\cdot\tan\frac{c}{2}$

由 $\dfrac{2\cos\frac{a+c}{2}\sin\frac{a-c}{2}}{2\cos\frac{b+d}{2}\sin\frac{b-d}{2}}=\dfrac{\sin a-\sin c}{\sin b-\sin d}=\dfrac{\sin a}{\sin b}=\dfrac{\sin c}{\sin d}=\dfrac{\sin(a-c)}{\sin(b-d)}=\dfrac{2\sin\frac{a-c}{2}\cos\frac{a-c}{2}}{2\sin\frac{b-d}{2}\cos\frac{b-d}{2}}$

$\therefore\dfrac{\cos\frac{a+c}{2}}{\cos\frac{b+d}{2}}=\dfrac{\cos\frac{a-c}{2}}{\cos\frac{b-d}{2}}=\dfrac{\cos\frac{a+c}{2}+\cos\frac{a-c}{2}}{\cos\frac{b+d}{2}+\cos\frac{b-d}{2}}=\dfrac{2\cos\frac{a}{2}\cos\frac{c}{2}}{2\cos\frac{b}{2}\cos\frac{d}{2}}=\dfrac{\cos\frac{a+c}{2}-\cos\frac{a-c}{2}}{\cos\frac{b+d}{2}-\cos\frac{b-d}{2}}=\dfrac{-2\sin\frac{a}{2}\sin\frac{c}{2}}{-2\sin\frac{b}{2}\sin\frac{d}{2}},\therefore\tan\frac{a}{2}\tan\frac{c}{2}=\tan\frac{b}{2}\tan\frac{d}{2}$

$\therefore\tan^2\frac{a}{2}=\tan^2\frac{b}{2},\therefore a=b,\therefore\tan\frac{c}{2}=\tan\frac{d}{2},\therefore c=d$得证

（8）已知$\tan(\alpha-\beta)=\dfrac{1}{2},\cos\beta=-\dfrac{7\sqrt{2}}{10},\alpha,\beta\in(0,\pi)$,求$2\alpha-\beta$. $-\dfrac{3\pi}{4}$

解:$\because\cos\beta=-\dfrac{7\sqrt{2}}{10}<0,\beta\in(0,\pi),\therefore\tan\beta=-\dfrac{1}{7}$,且$\beta\in(\dfrac{\pi}{2},\pi)$

$\therefore\tan\alpha=\tan(\alpha-\beta+\beta)=\dfrac{\frac{1}{2}-\frac{1}{7}}{1-\frac{1}{2}\cdot(-\frac{1}{7})}=\dfrac{1}{3}\in(0,\dfrac{\sqrt{3}}{3})$,且$\alpha\in(0,\pi),\therefore\alpha\in(0,\dfrac{\pi}{6})$

$\therefore 2\alpha-\beta\in(-\pi,-\dfrac{\pi}{6})$

而$\tan(2\alpha-\beta)=\tan(\alpha+\alpha-\beta)=\dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}=1,\therefore 2\alpha-\beta=-\dfrac{3\pi}{4}$

②若$3\sin^2\alpha+2\sin^2\beta=1,3\sin 2\alpha-2\sin 2\beta=0,\alpha,\beta\in(-\dfrac{\pi}{2},0)$,求$\alpha+2\beta$. $\dfrac{\pi}{2}$

2（1） 若$\sin 76^\circ=m$,则$\cos 7^\circ=$__________.

$key:\cos 7^\circ=\sqrt{\dfrac{1+\cos 14^\circ}{2}}=\sqrt{\dfrac{1+m}{2}}$

（2） $\dfrac{(1+\sqrt{3}\tan 65^\circ)\sin 25^\circ}{\sqrt{1+\sin 100^\circ}}=$__________; $\sqrt{2}$

（3） $(\dfrac{1}{\sin^2 10^\circ}-\dfrac{3}{\sin^2 80^\circ})\cdot\dfrac{1}{\sin 70^\circ}=$__________.

原式$=\dfrac{(\cos 10^\circ-\sqrt{3}\sin 10^\circ)(\cos 10^\circ+\sqrt{3}\sin 10^\circ)}{\sin^2 10^\circ\cos^2 10^\circ\cos 20^\circ}=\dfrac{4\sin 20^\circ\sin 40^\circ}{\frac{1}{4}\sin^2 20^\circ\cos 20^\circ}=32$

$key2:原式=\dfrac{\cos^2 10^\circ-3\sin^2 10^\circ}{\sin^2 10^\circ\cos^2 10^\circ\cos 20^\circ}=\dfrac{\dfrac{1+\cos 20^\circ}{2}-\dfrac{3(1-\cos 20^\circ)}{2}}{\dfrac{1}{4}\sin^2 20^\circ\cos 20^\circ}=\dfrac{-1+2\cos 20^\circ}{\dfrac{1}{8}\sin 20^\circ\sin 40^\circ}$

$=16\cdot\dfrac{-\cos 60^\circ+\cos 20^\circ}{\sin 20^\circ\sin 40^\circ}=16\cdot\dfrac{\cos(40^\circ-20^\circ)-\cos(40^\circ+20^\circ)}{\sin 20^\circ\sin 40^\circ}=32$

（4） $\dfrac{1}{\cos 50^\circ}+\tan 10^\circ=$______

$\dfrac{1}{\sin 40^\circ}+\tan 10^\circ=\dfrac{1}{\sin 40^\circ}+\dfrac{1}{\tan 80^\circ}=\dfrac{2\cos 40^\circ+\cos 80^\circ}{\sin 80^\circ}$

$=\dfrac{2\cos(60^\circ-20^\circ)+\cos(60^\circ+20^\circ)}{\sin 80^\circ}=\dfrac{\dfrac{3}{2}\cos 20^\circ+\dfrac{\sqrt{3}}{2}\sin 20^\circ}{\sin 80^\circ}=\dfrac{\sqrt{3}\sin 80^\circ}{\sin 80^\circ}=\sqrt{3}$

（或$=\dfrac{2\cos(30^\circ+10^\circ)+\sin 10^\circ}{\cos 10^\circ}=\dfrac{\sqrt{3}\cos 10^\circ-\sin 10^\circ+\sin 10^\circ}{\cos 10^\circ}=\sqrt{3}$）

（5） $\sin 18^\circ=$________.

由$\sin 2\times 18^\circ=\sin(90^\circ-3\times 18^\circ)$得$2\sin 18^\circ\cos 18^\circ=\cos 3\times 18^\circ$

$=\cos 18^\circ\cdot(1-2\sin^2 18^\circ)-\sin 18^\circ\cdot 2\sin 18^\circ\cos 18^\circ$

即$4\sin^2 18^\circ+2\sin 18^\circ-1=0,\therefore\sin 18^\circ=\dfrac{\sqrt{5}-1}{4}$

（6）①$\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=$____________ $\dfrac{1}{16}$

$key1:原式=\dfrac{1}{2}\cdot\dfrac{\sin 20^\circ}{\sin 20^\circ}\cos 20^\circ\cos 40^\circ\cos 80^\circ=\dfrac{1}{16}\cdot\dfrac{\sin 160^\circ}{\sin 20^\circ}=\dfrac{1}{16}$

$key2$:（三倍角，积化和差）$=\dfrac{1}{2}\cdot\dfrac{1}{4}\cos(3\times 20^\circ)=\dfrac{1}{16}$

②$\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ=$_____________ $\dfrac{3}{16}$

$key1$:（三倍角，积化和差）$=\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{4}\sin(3\times 20^\circ)=\dfrac{3}{16}$

（7） $\sin^2 33^\circ+\cos^2 63^\circ+\cos 57^\circ\sin 27^\circ=$_____ $\dfrac{3}{4}$

$key1:原式=\dfrac{1-\cos 66^\circ}{2}+\dfrac{1+\cos 126^\circ}{2}+\dfrac{1}{2}(\sin(27^\circ+57^\circ)+\sin(27^\circ-57^\circ))=\cdots=\dfrac{3}{4}$

$key2:设A=\sin^2 33^\circ+\cos^2 63^\circ+\cos 57^\circ\sin 27^\circ, B=\cos^2 33^\circ+\sin^2 63^\circ+\sin 57^\circ\cos 27^\circ$

$$\therefore\begin{cases}A+B=2+\sin 84^\circ\\ A-B=-\cos 66^\circ+\cos 126^\circ+\sin(-30^\circ)=-\dfrac{1}{2}-\sin 84^\circ\end{cases},\therefore A=\dfrac{3}{4}$$

$key3:原式=\sin^2 33^\circ+\sin^2 27^\circ+\sin 33^\circ\sin 27^\circ=(\sin 120^\circ)^2=\dfrac{3}{4}$（正弦、余弦定理）