平面向量解答（7）

2023-04-02

四、角

向量夹角公式：若非零向量$\vec{a},\vec{b}$，则$\cos<\vec{a},\vec{b}>=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$，$<\vec{a},\vec{b}>$的范围为$:[0,\pi]$

$<\vec{a},\vec{b}>$为锐角$\Leftrightarrow \vec{a}\cdot\vec{b}>0$, 且$\vec{a}\nparallel\vec{b}$

$<\vec{a},\vec{b}>$为直角$\Leftrightarrow \vec{a}\cdot\vec{b}=0$

$<\vec{a},\vec{b}>$为钝角$\Leftrightarrow \vec{a}\cdot\vec{b}<0$, 且$\vec{a}\nparallel\vec{b}$

向量垂直条件:$\vec{a}\perp\vec{b}\Leftrightarrow\vec{a}\cdot\vec{b}=0$

（05理）已知向量$\vec{a}\neq\vec{e},|\vec{e}|=1$, 对任意$t\in R$, 恒有$|\vec{a}-t\vec{e}|\geq|\vec{a}-\vec{e}|$, 则（　）C

$A.\vec{a}\perp\vec{e}$　$B.\vec{a}\perp(\vec{a}-\vec{e})$　$C.\vec{e}\perp(\vec{a}-\vec{e})$　$D.(\vec{a}+\vec{e})\perp(\vec{a}-\vec{e})$

变式1(1) 已知非零向量$\vec{a},\vec{b}$满足$(\vec{a}-2\vec{b})\perp\vec{a},(\vec{b}-2\vec{a})\perp\vec{b}$, 则$<\vec{a},\vec{b}>=$____. $\dfrac{\pi}{3}$

(2) 设两个向量$\vec{e_1},\vec{e_2}$，满足$|\vec{e_1}|=2,|\vec{e_2}|=1,\vec{e_1}$与$\vec{e_2}$的夹角为$\dfrac{\pi}{3}$，记$<2t\vec{e_1}+7\vec{e_2},\vec{e_1}+t\vec{e_2}>=\theta$.

若θ为锐角，则实数t的范围为_______; $(-\infty,-7)\cup(-\dfrac{1}{2},\dfrac{\sqrt{14}}{2})\cup(\dfrac{\sqrt{14}}{2},+\infty)$

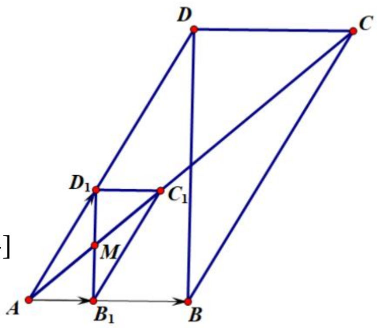
若θ为钝角，则实数t的取值范围为____. $(-7,-\dfrac{\sqrt{14}}{2})\cup(-\dfrac{\sqrt{14}}{2},-\dfrac{1}{2})$

(3) ①矩形$ABCD$中，$A(1,1),C(3,5)$, 且$|AB|=2|BC|$, 则点B的坐标为_______. $(\dfrac{17}{5},\dfrac{9}{5}),or,(\dfrac{3}{5},\dfrac{21}{5})$

②已知等腰梯形$ABCD$中，$AB//CD$，且$A(-1,1)$，$B(4,2),D(1,3)$, 则点C的坐标为_____. $(\dfrac{18}{13},\dfrac{40}{13})$

(4) 在平行四边形$ABCD$中，$\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}+\dfrac{2\overrightarrow{AD}}{|\overrightarrow{AD}|}=\dfrac{\lambda\overrightarrow{AC}}{|\overrightarrow{AC}|},\lambda\in[\sqrt{2},2]$, 则$\cos\angle ABD$的取值范围是（ ）D

$A.[\dfrac{\sqrt{6}}{4},\dfrac{\sqrt{2}}{4}]$　$B.[\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{3}}{2}]$　$C.[\dfrac{\sqrt{2}}{4},\dfrac{\sqrt{3}}{2}]$　$D.[\dfrac{\sqrt{6}}{4},\dfrac{5\sqrt{2}}{8}]$

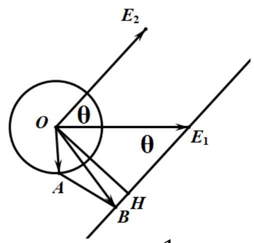


key: 设$MB_1=x$，则$1+2^2=\dfrac{1}{2}\lambda^2+2x^2$即$x^2=\dfrac{5}{2}-\dfrac{1}{4}\lambda^2$

$$\therefore\cos\angle ABD=\frac{\frac{7}{2}-\frac{1}{2}\lambda^2}{2\sqrt{\frac{5}{2}-\frac{1}{4}\lambda^2}}=t-\frac{3}{4t}(t=\sqrt{\frac{5}{2}-\frac{1}{4}\lambda^2}\in[\sqrt{\frac{3}{2}},\sqrt{2}])\in[\frac{\sqrt{6}}{4},\frac{5\sqrt{2}}{8}]$$

（201604学考17）已知平面向量$\vec{a},\vec{b}$满足$|\vec{a}|=\dfrac{\sqrt{3}}{4},\vec{b}=\vec{e_1}+\lambda\vec{e_2}(\lambda\in R)$, 其中$\vec{e_1},\vec{e_2}$为不共线的单位向量，若对符合上述条件的任意向量$\vec{a},\vec{b}$恒有$|\vec{a}-\vec{b}|\geq\dfrac{\sqrt{3}}{4}$，则$\vec{e_1},\vec{e_2}$夹角的最小值为（　）B

$A.\dfrac{\pi}{6}$　$B.\dfrac{\pi}{3}$　$C.\dfrac{2\pi}{3}$　$D.\dfrac{5\pi}{6}$

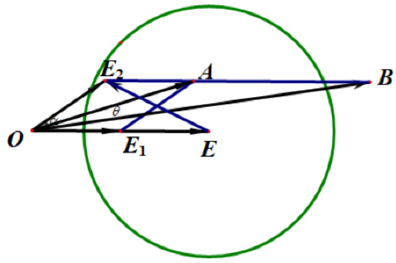


$1604key$:如图，得$\sin\theta=\dfrac{|\overline{OH}|}{1}\geq\dfrac{\sqrt{3}}{2},\therefore\theta\geq\dfrac{\pi}{3}$

（202201学考）17. 已知单位向量$\vec{e_1},\vec{e_2}$不共线，且向量$\vec{a}$满足$|\vec{a}|=\dfrac{1}{4}$.若$|\vec{a}-\lambda\vec{e_1}+(\lambda-1)\vec{e_2}|\geq\dfrac{1}{4}$对任意实数λ都成立，则向量$\vec{e_1},\vec{e_2}$夹角的最大值是（　B　）A. $\dfrac{\pi}{2}$　　B. $\dfrac{2\pi}{3}$　　C. $\dfrac{3\pi}{4}$　　D. $\dfrac{5\pi}{6}$

key:(等和线)$|\vec{a}-(\lambda\vec{e_1}+(1-\lambda)\vec{e_2})|\geq\dfrac{1}{4},\therefore<\vec{e_1},\vec{e_2}>\leq\dfrac{2\pi}{3}$

（2020浙江17题）设$\overrightarrow{e_1},\overrightarrow{e_2}$为单位向量，满足$|2\overrightarrow{e_1}-\overrightarrow{e_2}|\le\sqrt{2},\vec{a}=\overrightarrow{e_1}+\overrightarrow{e_2},\vec{b}=3\overrightarrow{e_1}+\overrightarrow{e_2}$，设$\vec{a},\vec{b}$的夹角为$\theta$，则$\cos^2\theta$的最小值为______________.



key：如图，设$<\overrightarrow{e_1},\overrightarrow{e_2}>=\alpha$，则$|\vec{a}|=\sqrt{2+2\cos\alpha}$，

$|\vec{b}|=\sqrt{10+6\cos\alpha}$，且$5-4\cos\alpha\le 2$即$\cos\alpha\in[\frac{3}{4},1]$

$$\therefore\cos\theta=\frac{2+2\cos\alpha+10+6\cos\alpha-4}{2\sqrt{2+2\cos\alpha}\cdot\sqrt{10+6\cos\alpha}}=2\sqrt{\frac{1+\cos\alpha}{5+3\cos\alpha}}$$

$$\therefore\cos^2\theta=4\cdot\frac{1+\cos\alpha}{5+3\cos\alpha}=\frac{4}{3}(1-\frac{2}{3\cos\alpha+5})\ge\frac{28}{29}$$

（2021 甲）14. 已知向量$\vec{a}=(3,1),\vec{b}=(1,0),\vec{c}=\vec{a}+k\vec{b}$．若$\vec{a}\perp\vec{c}$，则$k=$_________.　$-\frac{10}{3}$

（2021I）14. 已知向量$\vec{a}=(1,3),\vec{b}=(3,4)$，若$(\vec{a}-\lambda\vec{b})\perp\vec{b}$，则$\lambda=$__________.　$\frac{3}{5}$

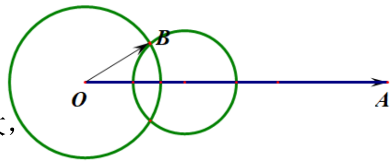
（2022 甲）13. 设向量$\vec{a},\vec{b}$的夹角的余弦值为$\frac{1}{3}$，且$|\vec{a}|=1,|\vec{b}|=3$，则$(2\vec{a}+\vec{b})\cdot\vec{b}=$__________.　11

（2022II）4. 已知向量$\vec{a}=(3,4),\vec{b}=(1,0),\vec{c}=\vec{a}+t\vec{b}$，若$<\vec{a},\vec{c}>=<\vec{b},\vec{c}>$，则$t=$（　C　）

A. -6　　B. -5　　C. 5　　D. 6

变式 2 （1）①已知$\vec{a},\vec{b}$是平面向量，满足$|\vec{a}|=4,|\vec{b}|\le 1$且$|3\vec{b}-\vec{a}|\le 2$，则$\cos<\vec{a},\vec{b}>$的最小值是（　）D

A. $\frac{11}{16}$　　B. $\frac{7}{8}$　　C. $\frac{\sqrt{15}}{8}$　　D. $\frac{3\sqrt{15}}{16}$



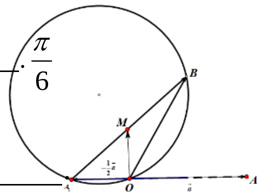
key：$|3\vec{b}-\vec{a}|\le 2\Leftrightarrow|\vec{b}-\frac{\vec{a}}{3}|\le\frac{2}{3}$，如图：$\therefore$当$|\vec{b}|=1$，且$|3\vec{b}-\vec{a}|=2$时，$<\vec{a},\vec{b}>$最大，

② 设不共线向量$\vec{\alpha},\vec{\beta},|\vec{\alpha}|=2,|\vec{\beta}|=1$，则向量$\vec{\alpha}$与$\vec{\alpha}-\vec{\beta}$的夹角的取值范围为_________.　$(0,\frac{\pi}{6}]$

key：设$|\vec{\alpha}-\vec{\beta}|=a$，则$2-1<a<1+2$即$1<a<3$，而$1=\vec{\beta}^2=(\vec{\alpha}-(\vec{\alpha}-\vec{\beta}))^2=4+a^2-4a\cos<\vec{\alpha},\vec{\alpha}-\vec{\beta}>$

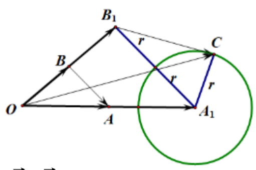
即$\cos<\vec{\alpha},\vec{\alpha}-\vec{\beta}>=\frac{3+a^2}{4a}=\frac{1}{4}(a+\frac{3}{a})\in(\frac{\sqrt{3}}{2},1)$，$\therefore<\vec{\alpha},\vec{\alpha}-\vec{\beta}>\in(0,\frac{\pi}{6})$

③已知两个不共线的非零向量$\vec{a},\vec{b}$满足$|\vec{a}|=2,|\vec{a}-\vec{b}|=1$，则向量$\vec{a},\vec{b}$夹角的最大值是_____.　$\frac{\pi}{6}$



④已知平面向量$\vec{a},\vec{b},\vec{c}$满足$|\vec{a}-\vec{b}|=|2\vec{a}-\vec{c}|\ne 0$，则$\vec{a}-\vec{b}$与$\vec{c}-2\vec{b}$所成夹角的最大值是_________

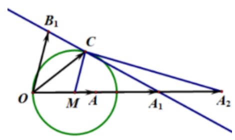
key：$\vec{a}-\vec{b}=\overrightarrow{BA}=\frac{1}{2}\overrightarrow{B_1A_1}$，且$|\overrightarrow{A_1C}|=\frac{1}{2}|\overrightarrow{B_1A_1}|$，



$\vec{c}-2\vec{b}=\overrightarrow{B_1C}$，$\therefore<\vec{a}-\vec{b},\vec{c}-2\vec{b}>=\angle A_1B_1C\le\frac{\pi}{6}$

⑤已知平面向量$\vec{a},\vec{b}$满足$|\vec{a}|=3|\vec{b}|=3$，若$\vec{c}=(2-2\lambda)\vec{a}+3\lambda\vec{b}(\lambda\in R)$，且$\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|}=\frac{\vec{c}\cdot\vec{b}}{|\vec{b}|}$，

则$\cos<\vec{a},3\vec{a}-\vec{c}>$的最小值为____.



平面向量解答（7）

2023-04-02

key: 设$\overrightarrow{OC}=\vec{c}=(1-\lambda)(2\vec{a})+\lambda(3\vec{b})=(1-\lambda)\overrightarrow{OA_1}+\lambda\overrightarrow{OB_1},|\overrightarrow{OA_1}|=6,|\overrightarrow{OB}|=3,$

由$\dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|}=\dfrac{\overrightarrow{OC}\cdot\overrightarrow{OA_1}}{|\overrightarrow{OA_1}|}=\dfrac{\vec{c}\cdot\vec{b}}{|\vec{b}|}=\dfrac{\overrightarrow{OC}\cdot\overrightarrow{OB_1}}{|\overrightarrow{OB_1}|}$得$OC$是$\angle A_1OB_1$的平分线，

$key1$: 作$CM//OB_1$交OA_1于M，则$|\overrightarrow{CM}|=\dfrac{2}{3}|\overrightarrow{OB_1}|=2,\therefore C$在以$M$为圆心，半径为2的圆上，

$\therefore\cos<\vec{a},3\vec{a}-\vec{c}>=\cos\angle AA_2M\ge\dfrac{3\sqrt{5}}{7}$

$key2$: $\vec{c}=\dfrac{1}{3}(2\vec{a})+\dfrac{2}{3}(3\vec{b})=\dfrac{2}{3}\vec{a}+2\vec{b},\therefore 3\vec{a}-\vec{c}=\dfrac{7}{3}\vec{a}-2\vec{b}$

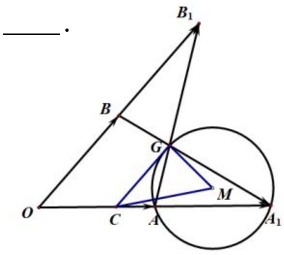
$\therefore\cos<\vec{a},3\vec{a}-\vec{c}>=\dfrac{21-6\cos\theta}{3\sqrt{53-28\cos\theta}}=\dfrac{7-2\cos\theta}{\sqrt{53-28\cos\theta}}=\sqrt{\dfrac{t^2}{14t-45}}=\sqrt{\dfrac{2}{\dfrac{14}{t}-\dfrac{45}{t^2}}}\ge\dfrac{3\sqrt{5}}{7}(t=7-2\cos\theta\in[5,9])$

（2）①已知平面单位向量$\vec{a}$，$\vec{b}$满足$|\vec{a}-\vec{b}|\le 1$．设向量$2\vec{a}+\vec{b}$与向量$\vec{a}-2\vec{b}$的夹角为θ，则$\cos\theta$的最大值为______.

key: 设$<\vec{a},\vec{b}>=\alpha$, 则$\cos\alpha\ge\dfrac{1}{2},\therefore\cos\theta=\dfrac{(2\vec{a}+\vec{b})\cdot(\vec{a}-2\vec{b})}{|2\vec{a}+\vec{b}|\cdot|\vec{a}-2\vec{b}|}=\dfrac{-3\cos\alpha}{\sqrt{25-16\cos^2\alpha}}$

$=-3\sqrt{\dfrac{\cos^2\alpha}{25-16\cos^2\alpha}}=-\dfrac{3}{4}\sqrt{-1+\dfrac{25}{25-16\cos^2\alpha}}\le 0$

②已知平面向量$\vec{a},\vec{b}$满足$|\vec{a}|=1,2\vec{a}-\vec{b}$与$2\vec{b}-\vec{a}$的夹角为$120°$，则$|\vec{b}|$的最大值是____.

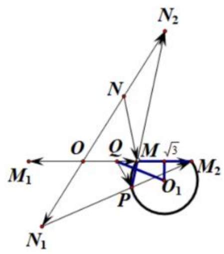


key: 如图，$CG//OB,G$为$\triangle OA_1B_1$的重心，

$\therefore\angle A_1GB_1=120°,\therefore\angle AGA_1=60°,\therefore G$的轨迹为圆弧，且R=$\dfrac{\sqrt{3}}{3}$如图，

$\therefore|\vec{b}|=\dfrac{3}{2}|\overrightarrow{CG}|\le\dfrac{3}{2}(R+|\overrightarrow{MC}|)=\dfrac{3}{2}\left(\dfrac{1}{2\cdot\dfrac{\sqrt{3}}{2}}+\sqrt{(\dfrac{\sqrt{3}}{2}-\dfrac{1}{\sqrt{3}})^2+(\dfrac{1}{3}+\dfrac{1}{2})^2}\right)=\dfrac{\sqrt{3}+\sqrt{7}}{2},$

③已知平面向量$\vec{m},\vec{n}$满足$|\vec{m}|=\sqrt{3},2\vec{m}+\vec{n}$与$2\vec{n}-\vec{m}$的夹角为$60°$，则$|\vec{m}-\vec{n}|$的取值范围为______.



key: $\vec{m}=\overrightarrow{OM},\vec{n}=\overrightarrow{ON},\overrightarrow{OM_2}=2\vec{m},\overrightarrow{ON_2}=2\vec{n},\overrightarrow{ON_1}=-\vec{n}$

$\therefore 2\vec{m}+\vec{n}=\overrightarrow{N_1M_2},2\vec{n}-\vec{m}=\overrightarrow{MN_2},\therefore\angle N_2PM_2=60°,\therefore P$的轨迹为圆弧

设$\overrightarrow{OP}=\lambda\overrightarrow{OM}+\mu\overrightarrow{ON}=\lambda\overrightarrow{OM}+\dfrac{1}{2}\mu\overrightarrow{ON_2},\therefore\lambda+\dfrac{1}{2}\mu=1;\overrightarrow{OP}=\lambda\overrightarrow{OM}+\mu\overrightarrow{ON}=\dfrac{1}{2}\lambda\overrightarrow{OM}-\mu\overrightarrow{ON_1},$

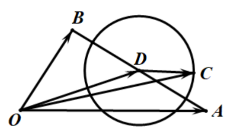
$\therefore\dfrac{1}{2}\lambda-\mu=1$得$\lambda=\dfrac{6}{5},\mu=-\dfrac{2}{5},$

取$\overrightarrow{OQ}=\dfrac{4}{5}\vec{m}$, 则$|\overrightarrow{QP}|=\dfrac{2}{5}|\vec{m}-\vec{n}|$即$|\vec{m}-\vec{n}|=\dfrac{5}{2}|\overrightarrow{QP}|\in[\dfrac{\sqrt{43}}{2}-\dfrac{5}{2},\dfrac{\sqrt{43}}{2}+\dfrac{5}{2}]$

（3）①已知非零向量$\vec{a},\vec{b},\vec{c}$满足$|\vec{a}-\vec{b}|=4$, 且$(\vec{a}-\vec{c})\cdot(\vec{b}-\vec{c})=-1$，若$\vec{a}$与$\vec{b}$的夹角为$\theta$，且$\theta\in[\dfrac{\pi}{3},\dfrac{\pi}{2}]$，则$|\vec{c}|$的取值范围是______.

key: $(\vec{a}-\vec{c})\cdot(\vec{b}-\vec{c})=\overrightarrow{CA}\cdot\overrightarrow{CB}=\overrightarrow{CD}^2-4=-1$得$|\overrightarrow{CD}|=\sqrt{3},$

$\therefore|\vec{c}|=|\overrightarrow{OC}|\le|\overrightarrow{OD}|+|\overrightarrow{DC}|=\dfrac{1}{2}\sqrt{2\overrightarrow{OA}^2+2\overrightarrow{OB}^2-(\overrightarrow{OA}-\overrightarrow{OB})^2}+\sqrt{3}$(记$|\vec{a}|=a,|\vec{b}|=b$)



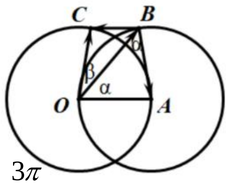
$=\dfrac{1}{2}\sqrt{2a^2+2b^2-16}+\sqrt{3}\le\dfrac{1}{2}\sqrt{64-16}+\sqrt{3}=3\sqrt{3}$

$|\vec{c}|=|\overrightarrow{OC}|\ge||\overrightarrow{OD}|-|\overrightarrow{DC}||=|\overrightarrow{OD}|-\sqrt{3}\ge 2-\sqrt{3}$

（由$|\vec{a}-\vec{b}|^2=a^2+b^2-2ab\cos\theta=16$得$\cos\theta=\frac{a^2+b^2-16}{2ab}\in[0,\frac{1}{2}]$即$\begin{cases}a^2+b^2\ge 16\\ a^2+b^2-ab\le 16\end{cases}$

$\therefore 16\ge a^2+b^2-ab\ge a^2+b^2-\frac{a^2+b^2}{2}=\frac{a^2+b^2}{2}$，且$|\overrightarrow{OD}|=\frac{1}{2}\sqrt{2a^2+2b^2-16}\ge 2$），$\therefore|\vec{c}|\in[2-\sqrt{3},3\sqrt{3}]$

②已知平面向量$\vec{a},\vec{b},\vec{c}$满足$|\vec{a}|=|\vec{a}-\vec{b}|=|\vec{c}|=1,\vec{b}^2+\vec{a}\cdot\vec{c}+\frac{\sqrt{2}}{2}|\vec{b}-\vec{c}|=\vec{b}\cdot(\vec{a}+\vec{c})$，且



$\frac{\vec{a}\cdot\vec{b}+|\vec{b}|}{\vec{b}\cdot\vec{c}}=|\vec{a}+\frac{\vec{b}}{|\vec{b}|}|$，则$(\vec{b}-\vec{c})^2=$ ________.

key：由$\vec{b}^2+\vec{a}\cdot\vec{c}+\frac{\sqrt{2}}{2}|\vec{b}-\vec{c}|=\vec{b}\cdot(\vec{a}+\vec{c})$得$(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{c})=-\frac{\sqrt{2}}{2}|\vec{b}-\vec{c}|$得$<\vec{a}-\vec{b},\vec{c}-\vec{b}>=\frac{3\pi}{4}$

由$\frac{\vec{a}\cdot\vec{b}+|\vec{b}|}{\vec{b}\cdot\vec{c}}=|\vec{a}+\frac{\vec{b}}{|\vec{b}|}|$得$\frac{1+\cos\alpha}{\cos\beta}=\sqrt{1+1+2\cos\alpha}$($<\vec{a},\vec{b}>=\alpha,<\vec{b},\vec{c}>=\beta\in[0,\pi]$)

$\therefore\cos\beta=\cos\frac{\alpha}{2},\therefore\beta=\frac{\alpha}{2},\therefore|\vec{b}|=2\cos\alpha,\therefore\frac{1}{\sin(\frac{3\pi}{4}-\alpha)}=\frac{2\cos\alpha}{\sin(\frac{3\pi}{4}-\alpha+\frac{\alpha}{2})}$得$\alpha=\frac{\pi}{3},\beta=\frac{\pi}{6}$

$\therefore|\vec{b}|=1,\therefore|\vec{b}-\vec{c}|=\sqrt{2-\sqrt{3}},\therefore|\vec{b}-\vec{c}|^2=2-\sqrt{3},$