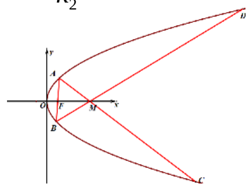
解析几何（4）抛物线解答（4）

2024-01-06

（2010四川）已知F为抛物线$y^2=4x$的焦点，M点的坐标为$(4,0)$,过点F作斜率为k_1的直线于抛物线交于A、B两点，延长AM、BM交抛物线于C、D两点，设直线CD的斜率为k_2.⑴ 求$\dfrac{k_1}{k_2}$的值；

（2）求直线AB于直线CD夹角θ的取值范围.



解：⑴ 设$A(a^2,2a),B(b^2,2b),C(c^2,2c),D(d^2,2d)$

由A,F,B三点共线得$\dfrac{2a-2b}{a^2-b^2}=\dfrac{2}{a+b}=\dfrac{2a}{a^2-1}$即$ab=-1$

由A,M,C三点共线得$\dfrac{2}{a+c}=\dfrac{2a}{a^2-4}$得$c=-\dfrac{4}{a}$,同理$d=-\dfrac{4}{b}=4a,\therefore \dfrac{k_1}{k_2}=\dfrac{\dfrac{2}{a+b}}{\dfrac{2}{c+d}}=\dfrac{-\dfrac{4}{a}+4a}{a-\dfrac{1}{a}}=4$

（2）由⑴得$\tan\theta=\left|\dfrac{4(a-\dfrac{1}{a})-(a-\dfrac{1}{a})}{1+4(a-\dfrac{1}{a})^2}\right|=\left|\dfrac{3k}{1+4k^2}\right|=\dfrac{3}{|4k+\dfrac{1}{k}|}\le\dfrac{3}{4}$(其中$k=a-\dfrac{1}{a}\in R$)

$\therefore \theta$的取值范围为$(0,\arctan\dfrac{3}{4}]$

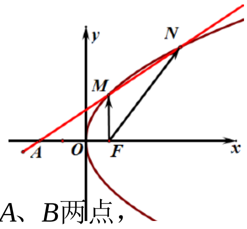
（2013B）在平面直角坐标系xOy内，点F的坐标为$(1,0)$，点A、B在抛物线$y^2=4x$上，满足$\overrightarrow{OA}\cdot\overrightarrow{OB}=-4$, $|\overrightarrow{FA}|-|\overrightarrow{FB}|=4\sqrt{3}$，则$\overrightarrow{FA}\cdot\overrightarrow{FB}=$______.

$2013Bkey:$设$A(a^2,2a),B(b^2,2b)$,则$\overrightarrow{OA}\cdot\overrightarrow{OB}=a^2b^2+4ab=-4,\therefore ab=-2$

$|\overrightarrow{FA}|-|\overrightarrow{FB}|=a^2-b^2=a^2-\dfrac{4}{a^2}=4\sqrt{3},\therefore a^2+\dfrac{4}{a^2}=\sqrt{(a^2-\dfrac{4}{a^2})^2+16}=8$

$\therefore \overrightarrow{FA}\cdot\overrightarrow{FB}=(a^2-1)(b^2-1)+4ab=a^2b^2-(a^2+b^2)+1+4ab=-(a^2+\dfrac{4}{a^2})-3=-11$

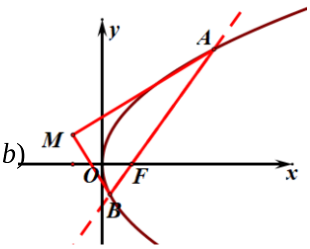
（2018Ⅰ）8.设抛物线$C:y^2=4x$的焦点为F，过点$(-2,0)$且斜率为$\dfrac{2}{3}$的直线与C交于M、N两点，则$\overrightarrow{FM}\cdot\overrightarrow{FN}=$（ ） $A.5$ $B.6$ $C.7$ $D.8$



$key:M(m^2,2m),N(n^2,2n)$,则$\dfrac{2m-2n}{m^2-n^2}=\dfrac{2}{m+n}=\dfrac{2m}{m^2+2}=\dfrac{2}{3}$得$m=1,n=2$

$\therefore \overrightarrow{FM}\cdot\overrightarrow{FN}=(m^2-1)(n^2-1)+4mn=8$,选$D$

（2018Ⅲ）16.已知点$M(-1,1)$和抛物线$C:y^2=4x$，过C的焦点且斜率为k的直线于C交于A、B两点，若$\angle AMB=90°$,则$k=$_______.



$key:A(a^2,2a),B(b^2,2b)$，由A,F,B三点共线得$\dfrac{2a-2b}{a^2-b^2}=\dfrac{2}{a+b}=\dfrac{2a}{a^2-1}$得$ab=-1$

$\therefore \angle AMB=90°,\therefore \overrightarrow{MA}\cdot\overrightarrow{MB}=(a^2+1)(b^2+1)+(2a-1)(2b-1)=2+a^2+b^2-3-2(a+b)$

$=(a+b)^2-2(a+b)+1=0,\therefore a+b=1,\therefore k=\dfrac{2}{a+b}=2$

（2022Ⅱ）10.已知O为坐标原点，过抛物线$C:y^2=2px(p>0)$的焦点F的直线与C交于A,B两点，其中A在第一象限，点$M(p,0)$,若$|AF|=|AM|$，则（ ）

$A.$直线AB的斜率为$2\sqrt{6}$ $B.|OB|=|OF|$ $C.|AB|>4|OF|$ $D.\angle OAM+\angle OBM<180°$

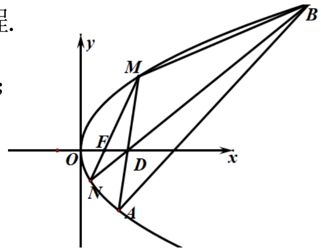
$2022Ⅱ key:$由已知得$A(\dfrac{3p}{4},\dfrac{\sqrt{6}}{2}p),\therefore k_{AB}=\dfrac{\dfrac{\sqrt{6}}{2}p}{\dfrac{3}{4}p-\dfrac{p}{2}}=2\sqrt{6}$, A对；

由A,F,B共线得$2\sqrt{6}=\dfrac{y_B}{\dfrac{y_B^2}{2p}-\dfrac{p}{2}}$即$\sqrt{6}y_B^2-py_B-\sqrt{6}p^2=0,y_B=-\dfrac{\sqrt{6}}{3}p,\therefore B$错

$|AB|=x_A+x_B+p=\dfrac{3p}{4}+\dfrac{p}{3}+p=\dfrac{25p}{12}>2p=4|OF|$，$C$对

$\overrightarrow{AO}\cdot\overrightarrow{AM}=(-\dfrac{3p}{4},-\dfrac{\sqrt{6}p}{2})\cdot(\dfrac{p}{4},-\dfrac{\sqrt{6}p}{2})=\dfrac{21p^2}{16}>0,\overrightarrow{BO}\cdot\overrightarrow{BM}=(-\dfrac{p}{3},\dfrac{\sqrt{6}p}{3})\cdot(\dfrac{2p}{3},\dfrac{\sqrt{6}p}{3})=\dfrac{4p^2}{9}>0,\therefore D$对

(2022甲)20.设抛物线$C:y^2=2px(p>0)$的焦点为F，点$D(p,0)$,过F的直线交C于M、N两点，当直线MD垂直于x轴时,$|MF|=3$.(1) 求C的方程；（2）设直线MD、ND与C的另一个交点分别为A、B，记直线MN、AB的倾斜角分别为α、β,当$\alpha-\beta$取得最大值时，求直线AB的方程.



2022甲（1）由已知得$M(p,\sqrt{2}p)$,且$\dfrac{p^2}{4}+2p^2=9$得$p=2,\therefore C$的方程为$y^2=4x$;

（2）设$M(m^2,2m),N(n^2,2n),A(a^2,2a),B(b^2,2b)$,

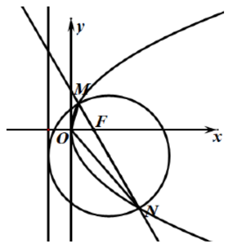
由M,F,N共线得$\dfrac{2m-2n}{m^2-n^2}=\dfrac{2}{m+n}=\dfrac{2m}{m^2-1}$得$mn=-1$

由M,D,A三点共线得$\dfrac{2}{m+a}=\dfrac{2m}{m^2-2}$得$a=-\dfrac{2}{m}$,同理$b=-\dfrac{2}{n}=2m$,

$\therefore\tan\alpha=k_{MN}=\dfrac{2}{m+n}=\dfrac{2m}{m^2-1},\tan\beta=k_{AB}=\dfrac{2}{a+b}=\dfrac{2}{2m-\dfrac{2}{m}}=\dfrac{m}{m^2-1}$记为$k$

$\therefore\tan(\alpha-\beta)=\dfrac{k}{1+2k^2}\le\dfrac{1}{2\sqrt{2}}$（当且仅当$k=\dfrac{\sqrt{2}}{2}$时，取$=$）$,\therefore AB$的方程为$x=\sqrt{2}y+4$

(2023II)10.设O为坐标原点，直线$y=-\sqrt{3}(x-1)$过抛物线$C:y^2=2px(p>0)$的焦点，且与C交于M、N两点，l为C的准线，则（　）$A.p=2$　$B.|MN|=\dfrac{8}{3}$　$C.$以MN为直径的圆与l相切　$D.\triangle OMN$为等腰三角形　AC



key:由$\dfrac{p}{2}=1$得$p=2,A$对；

由$\begin{cases}y=-\sqrt{3}(x-1)\\y^2=4x\end{cases}$消去$x$得$3x^2-10x+3=0,\therefore|MN|=2|3-\dfrac{1}{3}|=\dfrac{8}{3},B$对；

$\overrightarrow{OM}\cdot\overrightarrow{ON}=x_Mx_N+3(x_M-1)(x_N-1)=-3<0,|\overrightarrow{OM}|\ne|\overrightarrow{ON}|,D$错

$|QQ_1|=\dfrac{|MM_1|+|NN_1|}{2}=\dfrac{|MF|+|NF|}{2}=\dfrac{1}{2}|MN|$($Q$为$MN$的中点)$,C$对

(1999A）已知$A(1,2)$,过点$(5,-2)$的直线与抛物线$y^2=4x$交于另外两点B,C,那么$\triangle ABC$是（ ）

$A.$锐角三角形　$B.$钝角三角形　$C.$直角三角形　$D.$答案不确定

1999A:设$B(b^2,2b),(c^2,2c)$，由$B,C,m(5,-2)$共线得$\dfrac{2b-2c}{b^2-c^2}=\dfrac{2}{b+c}=\dfrac{2c+2}{c^2-5}$即$bc+b+c+5=0$

$\therefore k_{AB}\cdot k_{AC}=\dfrac{2b-2}{b^2-1}\cdot\dfrac{2c-2}{c^2-1}=\dfrac{4}{(b+1)(c+1)}=\dfrac{4}{bc+b+c+1}=-1,\therefore$选$C$

(2002A）已知点$A(0,2)$和抛物线$y^2=x+4$上两点B、C，使得$AB\perp BC$，则点C的纵坐标的取值范围为____.

2002A:设$B(b^2-4,b),C(c^2-4,c)(b\ne2,$且$c\ne2)$

则$k_{BA}\cdot k_{BC}=\dfrac{b-2}{b^2-4}\cdot\dfrac{b-c}{b^2-c^2}=\dfrac{1}{(b+2)(b+c)}=-1,\therefore c=-\dfrac{1}{b+2}-b\in(-\infty,0]\cup[4,+\infty)$

(2006山西）抛物线的顶点在原点，焦点在x轴的正半轴上，直线$x+y-1=0$与抛物线相交于A、B两点，且$|AB|=\dfrac{8\sqrt{6}}{11}$.在抛物线上是否存在一点$C$,使$\triangle ABC$为正三角形_____；若存在，$C$点的坐标是_______.

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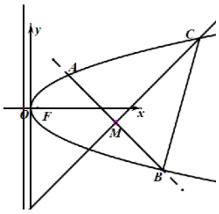
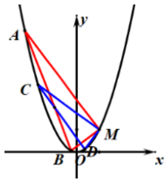


2006山西key:设抛物线方程$y^2=2px(p>0)$，联立$x+y-1=0$得：$y^2+2py-2p=0$

$\therefore |AB|=\sqrt{2}\cdot\sqrt{4p^2+8p}=\dfrac{8\sqrt{6}}{11}$得$p=\dfrac{2}{11}$

AB的中点$M(\dfrac{13}{11},-\dfrac{2}{11})$, 且$\therefore \overrightarrow{MA}=(-\dfrac{4\sqrt{3}}{11},\dfrac{4\sqrt{3}}{11})$, $\therefore \overrightarrow{MC}=(\dfrac{12}{11},\dfrac{12}{11})$, $\therefore C(\dfrac{25}{11},\dfrac{10}{11})$



（2011江西）抛物线$y=x^2$上的点$M(1,1)$，以M为直角顶点作抛物线的内接$\triangle MAB$、$\triangle MCD$，则直线AB与CD的交点坐标为______.$(-1,2)$

设$A(a,a^2),B(b,b^2)$，则$k_{MA}\cdot k_{MB}=\dfrac{a^2-1}{a-1}\cdot\dfrac{b^2-1}{b-1}=(a+1)(b+1)=-1$即$ab+a+b+2=0$

$\therefore l_{AB}:y-a^2=\dfrac{a^2-b^2}{a-b}(x-a)=(a+b)(x-a)$即$y=(a+b)x-ab=(a+b)x+a+b+2=(a+b)(x+1)+2$

$\therefore AB$经过定点$(-1,2)$,同理CD也经过$(-1,2)$

（2012A,2020新疆）直线$x-2y-1=0$与抛物线$y^2=4x$交于A,B两点，C为抛物线上的一点，$\angle ACB=90°$，则点C的坐标为_______.

解：由$\begin{cases}x-2y-1=0\\y^2=4x\end{cases}$消去$x$得$y^2-8y-4=0$, $\therefore\begin{cases}y_A+y_B=8\\y_Ay_B=-4\end{cases}$

$key1$:$\triangle ABC$的外接圆方程为:$(x-9)^2+(y-4)^2=100$,

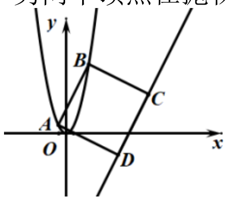
由$\begin{cases}y^2=4x\\(x-9)^2+(y-4)^2=100\end{cases}$消去$x$得:$\dfrac{y^4}{16}-\dfrac{7}{2}y^2-8y-3=0$

即$y^4-56y-128y-48=(y^2-8y-4)(y^2+8y+12)=0$, $\therefore y=-2,or,-6$

$key2$:$\because k_{CA}\cdot k_{CB}=\dfrac{y_A-y_C}{x_A-x_C}\cdot\dfrac{y_B-y_C}{x_B-x_C}=\dfrac{4}{y_A+y_C}\cdot\dfrac{4}{y_B+y_C}=\dfrac{16}{y_C^2+8y_C-4}=-1$

得$y_C=-2,or,-6$, $\therefore C(-2,1)$, 或$(-6,9)$

（2005竞赛）若正方形$ABCD$的一条边在直线$y=2x-17$上，另两个顶点在抛物线$y=x^2$上.则该正方形的面积为______.

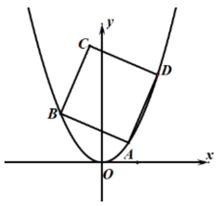


2005key:设$l_{AB}:y=2x+m$代入$y=x^2$得$x^2-2x-m=0$

$\therefore |AB|=\sqrt{5}\cdot\sqrt{4+4m}=\dfrac{m+17}{\sqrt{5}}$得$m=3$或63

$\therefore S_{ABCD}=20(1+m)=80$或1280

（1998上海）已知在抛物线$y=x^2$上有一个正方形的三个顶点A、B、C，求这种正方形面积的最小值.



$key1$:设$A(a,a^2)(a>0),B(b,b^2),(d,d^2)(d>a>0>b)$, AD的斜率为$k(k>0)$,

则$k=\dfrac{a^2-d^2}{a-d}=a+d, k_{AB}=a+b=-\dfrac{1}{k}$,

$\therefore |AD|=\sqrt{1+k^2}\cdot(k-2a)=|AB|=\sqrt{1+\dfrac{1}{k^2}}\cdot(2a+\dfrac{1}{k})$得$a=\dfrac{k^3-1}{2k(k+1)}$

$\therefore S_{ABCD}=(1+k^2)(k-\dfrac{k^3-1}{k^2+k})^2=\dfrac{(1+k^2)^3}{k^2(1+k)^2}$

$=(\dfrac{1}{k^2}+k^2+2)\cdot\dfrac{1+k^2}{(1+k)^2}\geq 4\cdot\dfrac{\dfrac{(1+k)^2}{1+1}}{(1+k)^2}=2$（当且仅当$k=1$时，取$=$）

（2023Ⅰ）在直角坐标系xOy中，点P到x轴的距离等于点P到点$(0,\dfrac{1}{2})$的距离，记动点P的轨迹为W.

(1) 求W的方程；（2）已知矩形$ABCD$有三个顶点在W上，证明：矩形$ABCD$的周长大于$3\sqrt{3}$.

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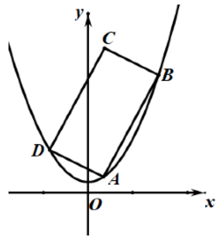


(1) 解：W的方程为$x^2=y-\dfrac{1}{4}$;

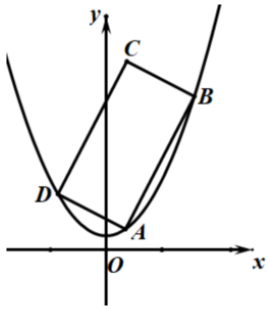
(2) 证明：不妨设A,B,D在W上，且$A(a,a^2+\dfrac{1}{4})(a>0)$

设$l_{AB}:y-a^2-\dfrac{1}{4}=k(x-a)(k>0)$代入$W$得$x_B=k-a>a$得$k>2a$,同理得$x_D=-\dfrac{1}{k}-a$

$\therefore$矩形$ABCD$的周长为$2[\sqrt{1+k^2}(k-2a)+\sqrt{1+\dfrac{1}{k^2}}(\dfrac{1}{k}+2a)]$

$=2\sqrt{1+k^2}(k-2a+\dfrac{1}{k^2}+\dfrac{2a}{k})$记为$f(k)$

当$0<k\le 1$时，$f(k)=2\sqrt{1+k^2}(k+\dfrac{1}{k^2}+(\dfrac{1}{k}-1)\cdot 2a]\ge 2\sqrt{1+k^2}\cdot(k+\dfrac{1}{k})$



$=\dfrac{2(1+k^2)^{\frac{3}{2}}}{k}$记为$p(k)$，则$p'(k)=\dfrac{2\sqrt{1+k^2}}{k^2}(2k^2-1)>0\Leftrightarrow k>\dfrac{\sqrt{2}}{2}$

$\therefore p(k)_{\min}=p(\dfrac{\sqrt{2}}{2})=3\sqrt{3}$,而$f(\dfrac{\sqrt{2}}{2})=2\sqrt{\dfrac{3}{2}}(2+\dfrac{\sqrt{2}}{2})=2\sqrt{6}+\sqrt{3}>3\sqrt{3}$

当$k>1$时，$f(k)=2\sqrt{1+k^2}(k-2a+\dfrac{1}{k^2}+\dfrac{2a-k}{k}+1)$

$=2\sqrt{1+k^2}[(k-2a)(1-\dfrac{1}{k})+\dfrac{1}{k^2}+1]>2\sqrt{1+k^2}(1+\dfrac{1}{k^2})$记为$q(k)$

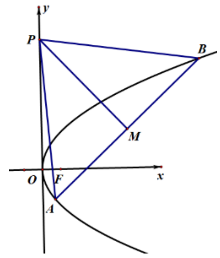
则$q'(k)=\dfrac{2\sqrt{1+k^2}(k^2-2)}{k^3}>0\Leftrightarrow k>\sqrt{2},\therefore q(k)_{\min}=q(\sqrt{2})=2\sqrt{3}\cdot\dfrac{3}{2}=3\sqrt{3},\therefore f(k)>3\sqrt{3}$

综上：矩形$ABCD$的周长大于$3\sqrt{3}$，证毕

变式1. 已知斜率为k的直线l与抛物线$y^2=4x$交于A、B两点，y轴上的点P使得$\triangle ABP$是等边三角形.

(1) 若$k>0$，证明：点P在y轴正半轴上；（2）当$|OP|$取到最大值时，求实数k的值.

（Ⅰ）证明：设$A(a^2,2a),B(b^2,2b)(b>a),P(0,p)$,则



$k=\dfrac{2a-2b}{a^2-b^2}=\dfrac{2}{a+b}>0$,且$M(\dfrac{a^2+b^2}{2},a+b)$

$\because\triangle ABP$是正三角形$,\therefore MP\perp AB$,且$|MP|=\dfrac{\sqrt{3}}{2}|AB|$

$\therefore\sqrt{1+(-\dfrac{a+b}{2})^2}\cdot\dfrac{a^2+b^2}{2}=\dfrac{\sqrt{3}}{2}\cdot\sqrt{1+(\dfrac{a+b}{2})^2}\cdot(2b-2a)$即$a^2+b^2=2\sqrt{3}(b-a)$

而$\dfrac{p-(a+b)}{0-\dfrac{a^2+b^2}{2}}=-\dfrac{1}{k}=-\dfrac{a+b}{2}$得$p=a+b+\dfrac{(a+b)(a^2+b^2)}{4}>0$得证

（Ⅱ）由（Ⅰ）得$|OP|=(a+b)(1+\dfrac{a^2+b^2}{4})$

而$a+b=\dfrac{2}{k},\therefore\dfrac{(a+b)^2+(b-a)^2}{2}=a^2+b^2=2\sqrt{3}(b-a)$得$b-a=2\sqrt{3}-2\sqrt{3-\dfrac{1}{k^2}}$,或$2\sqrt{3}+2\sqrt{3-\dfrac{1}{k^2}}$（舍去）

$\therefore|OP|=\dfrac{2}{k}(4-\sqrt{3}\cdot\sqrt{3-\dfrac{1}{k^2}})=8t-2\sqrt{3}t\sqrt{3-t^2}$（令$t=\dfrac{1}{k}\in[0,\sqrt{3}]$）记为$p(t)$,

则$p'(t)=8-2\sqrt{3}\cdot\sqrt{3-t^2}-2\sqrt{3}t\cdot\dfrac{1}{2}\cdot\dfrac{-2t}{\sqrt{3-t^2}}=8-4\sqrt{3}\cdot\sqrt{3-t^2}+\dfrac{6\sqrt{3}}{\sqrt{3-t^2}}>0,\therefore|OP|$的最大值为$\dfrac{8\sqrt{3}}{3}$.

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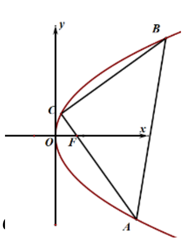
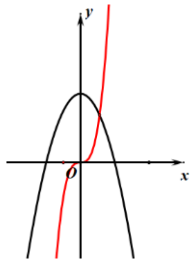
变式 2（1）抛物线$y^2=2px(p>0)$的焦点为F,是否存在内接等腰直角三角形，使该三角形的一条直角边过F？若存在，有几个？若不存在，说明理由.

解：设$C(2pc^2,2pc)(c>0),A(2pa^2,2pa)(a<0),B(2pb^2,2pb)(b>c)$





由C,F,A共线得$\dfrac{1}{a+c}=\dfrac{2pa-2pc}{2pa^2-2pc^2}=\dfrac{2pa}{2pa^2-\frac{p}{2}}$即$ac=-\dfrac{1}{4}$

由$CA\perp CB$得$\dfrac{1}{b+c}=-(a+c)$即$b=-\dfrac{1}{a+c}-c=\dfrac{c(4c^2+3)}{1-4c^2}$,

由$|CA|=|CB|$得$|CB|=\sqrt{(a+c)^2+1}\cdot 2p(b^2-c^2)=2p(c-a)(b-$

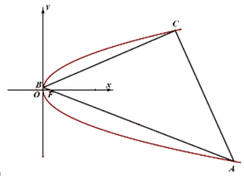
$=|CA|=2p(a^2+c^2)+p=2p(a^2+c^2+\dfrac{1}{2})=2p(c-a)^2$

$\Leftrightarrow(\dfrac{c(4c^2+3)}{1-4c^2}-c)(\dfrac{c(4c^2+3)}{1-4c^2}+c)=c+\dfrac{1}{4c}\Leftrightarrow 4\sqrt{2}c^{\frac{2}{3}}=\sqrt{1-4c^2}>0$有1个解

$\therefore$内接等腰直角三角形有2个

（2）抛物线$y^2=2px(p>0)$的焦点为F,是否存在内接等腰直角三角形，使该三角形的斜边过F？若存在，有几个？若不存在，说明理由.

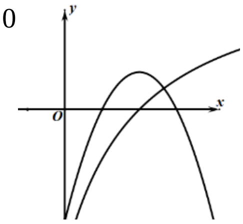
$key1$：设$A(2pa^2,2pa)<B(2pb^2,2pb),C(2pc^2,2pc)(c>b>0>a),k_{CB}=k>0$



则$\dfrac{1}{c+b}=\dfrac{2pc-2pb}{2pc^2-2pb^2}=k$得$b=\dfrac{1}{k}-c>0$,同理$a=-k-c$得$0<k<\dfrac{1}{c}$,

由$|CA|=|CB|$得$\sqrt{1+k^2}(2pc-2pa)=\sqrt{1+\dfrac{1}{k^2}}(2pc-2pb)$即$2c=\dfrac{1+k+k^2}{k}=1+k+$ k $-$ $k(1-k)$

由B,F,A共线得$\dfrac{1}{a+b}=\dfrac{2pb}{2pb^2-\frac{p}{2}}$得$ab=-\dfrac{1}{4}=(\dfrac{1}{k}-c)(-k-c)=c^2+c(k-\dfrac{1}{k})-1=0$

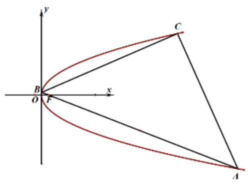


即$k-\dfrac{1}{k}=\dfrac{1}{c}-c=\dfrac{2k(1-k)}{k^3+1}-\dfrac{k^3+1}{2k(1-k)}\Leftrightarrow\dfrac{-k^3+2k^2+2k-1}{2k(1-k)}=\dfrac{2k(1-k)}{k^3+1}$

$\Leftrightarrow(k+1)^2(k^2-k+1)(-(k^2-k+1)+2k)=4k^2(1-k)^2$

$\Leftrightarrow(k+\dfrac{1}{k}-1)(-k-\dfrac{1}{k}+3)=4\cdot\dfrac{k+\frac{1}{k}-2}{k+\frac{1}{k}+2}$（令$t=k+\dfrac{1}{k}\geq 2(0<k<1)$）

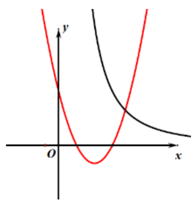
$\Leftrightarrow(t-1)(3-t)(t+2)=4(t-2)\Leftrightarrow(t-1)(3-t)=\dfrac{4(t-2)}{t+2}$,如图，只有1个解



$\therefore$内接等腰直角三角形有2个.

$key2$：设$A(2pa^2,2pa),B(2pb^2,2pb)(a<0<b)$,

由A,B,F三点共线得：$\dfrac{1}{a+b}=\dfrac{2pa-2pb}{2pa^2-2pb^2}=\dfrac{2pa}{2pa^2-\frac{p}{2}}$得$ab=-\dfrac{1}{4}$,



由AB的中点$D(p(a^2+b^2),p(a+b))$，则$\overrightarrow{DB}=(p(b^2-a^2),p(b-a))$,

$\overrightarrow{DC}=(p(b-a),p(a^2-b^2)),\therefore C(p(a^2+b^2+b-a),p(a^2-b^2+a+b))$,

$\therefore p^2(a^2-b^2+a+b)^2=2p^2(a^2+b^2+b-a)\Leftrightarrow(a+b)^2(a-b+1)^2=2(a^2+b^2+b-a)$（令$t=b-a=b+\dfrac{1}{b}\geq 2$）

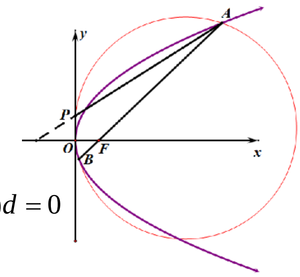
则$(t^2-4)(t+1)^2=2(t^2-2+t)\Leftrightarrow t^3+2t^2-5t-10=0\Leftrightarrow t(t+1)^2=6t+10\Leftrightarrow(t+1)^2=6+\dfrac{10}{t}$有1个解

$\therefore$等腰直角三角形有2个

变式3(1) 设AB是抛物线$y^2=2px$的焦点弦，且AB与x轴不垂直，P是y轴上异于O的一点，满足O、P、A、B四点共圆，点A、B、P的纵坐标分别为y_1、y_2、y_0.则$\dfrac{y_1+y_2}{y_0}=$________.

key: 设$A(2pa^2,2pa),B(2pb^2,2pb)$,

由A,F,B共线得: $ab=-\dfrac{1}{4}$,



设$OPAB$的外接圆方程为: $x^2+y^2+dx+ey=0$

则$4p^2a^4+4p^2a^2+2pa^2d+2pae=0$即$2pa^3+2pa+ad+e=0$

同理$2pb^3+2pb+bd+e=0,\therefore 2p(a-b)(a^2+ab+b^2)+2p(a-b)+(a-b)d=0$

$\therefore d=-2p(a^2+ab+b^2)-2p,\therefore e=2pab(a+b)$,

$\therefore \dfrac{y_1+y_2}{y_0}=\dfrac{2p(a+b)}{-e}=\dfrac{1}{-ab}=4$

(2) 已知抛物线$C:y^2=4x$的焦点为F，准线l与x轴交于点M，点P在抛物线上，直线PF与抛物线C交于另一点A.(Ⅰ) 设直线MP、MA的斜率分别为k_1、k_2,求证：k_1+k_2为常数;

(Ⅱ) (ⅰ) 设$\triangle PMA$的内切圆圆心为$G(a,b)$，半径为r，试用r表示点G的横坐标a;

(ⅱ) 当$\triangle PMA$的内切圆的面积为$\dfrac{1}{2}\pi$时，求直线PA的方程.

key:(Ⅰ) 设$A(t^2,2t),P(p^2,2p)$,由P,F,A共线得: $\dfrac{2p-2t}{p^2-t^2}=\dfrac{2}{p+t}=\dfrac{2p}{p^2-1}$得$tp=-1$

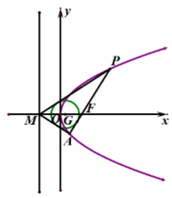
$\therefore k_1+k_2=\dfrac{2p}{p^2+1}+\dfrac{2t}{t^2+1}=2\cdot\dfrac{pt^2+p+p^2t+t}{p^2t^2+p^2+t^2+1}=2\cdot\dfrac{-t+p-p+t}{p^2+t^2+2}=0$为常数

(Ⅱ) $key1$: 设$\angle PMx=\theta$,由 (1) 得: $\tan\theta=\dfrac{2p}{p^2+1},\cos\theta=\dfrac{1+p^2}{\sqrt{(1+p^2)^2+4p^2}}$

则$|PM|=\dfrac{|PF|}{\cos\theta}|AM|=\dfrac{|AF|}{\cos\theta},\therefore |MP|+|MA|+|AP|=(1+\dfrac{1}{\cos\theta})|AP|=\dfrac{p^2+1}{p^2}\sqrt{p^4+6p^2+1}+(p+\dfrac{1}{p})^2$

$\therefore S_{\triangle PMA}=\dfrac{1}{2}[(p+\dfrac{1}{p})\sqrt{p^2+\dfrac{1}{p^2}+6}+(p+\dfrac{1}{p})^2]\cdot r=\dfrac{1}{2}\cdot 2\cdot(2p+\dfrac{2}{p})$,

$\therefore r=\dfrac{4}{u+\sqrt{u^2+4}}=\sqrt{u^2+4}-u(u=p+\dfrac{1}{p}),\therefore \dfrac{4}{r}=\sqrt{u^2+4}+u,\therefore r+\dfrac{4}{r}=2\sqrt{u^2+4}$



$\therefore (a+1)\sin\theta=r$即$a+1=\dfrac{r\sqrt{p^4+6p^2+1}}{2p}=\dfrac{r}{2}\sqrt{u^2+4}=\dfrac{r^2}{4}+1,\therefore a=\dfrac{1}{4}r^2$

$key2$: 由$l_{PA}:2px-(p^2-1)y-2p=0;l_{PM}:2px-(p^2+1)y+2p=0$

$\therefore l_{PG}$: $\dfrac{2px-(p^2+1)y+2p}{\sqrt{4p^2+(p^2+1)^2}}=\dfrac{-2px+(p^2-1)y+2p}{\sqrt{4p^2+(p^2-1)^2}}$令$y=0$得$\dfrac{1+a}{1-a}=\dfrac{\sqrt{(p^2+1)^2+4p^2}}{p^2+1}$

得$a=\dfrac{\sqrt{(1+p^2)^2+4p^2}-p^2-1}{\sqrt{(1+p^2)^2+4p^2}+p^2+1}=\dfrac{\sqrt{u^2+4}-u}{\sqrt{u^2+4}+u}=\dfrac{4}{(\sqrt{u^2+4}+u)^2}(u=p+\dfrac{1}{p})$

$\therefore r=(a+1)\sin\theta=(a+1)\cdot\dfrac{2p}{\sqrt{(1+p^2)^2+4p^2}}=\dfrac{4p}{\sqrt{(1+p^2)^2+4p^2}+p^2+1}=\dfrac{4}{\sqrt{u^2+4}+u},\therefore a=\dfrac{1}{4}r^2$

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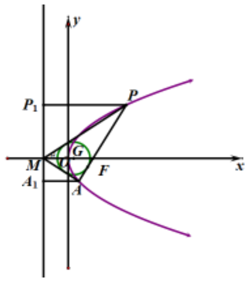
$key3:$ 设$\angle PMx=\theta$, 直线PA的方程为$x=ty+1$, 则



$S_{\triangle PMA}=\frac{1}{2}(1+\frac{1}{\cos\theta})|PA|=\frac{1}{2}(1+\frac{1}{\cos\theta})\cdot\sqrt{1+t^2}|y_A-y_P|\cdot r=\frac{1}{2}\cdot 2|y_A-y_P|$

$\therefore(1+\frac{1}{\cos\theta})\cdot\sqrt{1+t^2}\cdot r=2$

由$\frac{r}{1-a}=\sin\angle PFx=\frac{1}{\sqrt{1+t^2}}$得$\frac{1-a}{\sqrt{1+t^2}}=r,\therefore\cos\theta=\frac{1-a}{1+a}$

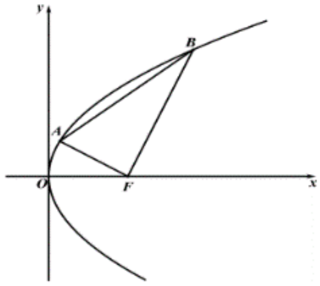
而$r=(1+a)\sin\theta$即$\sin\theta=\frac{r}{1+a},\therefore a=\frac{1}{4}r^2$

（Ⅱ）由已知得$r=\frac{\sqrt{2}}{2},\therefore a=\frac{1}{8},t=\pm\sqrt{\frac{17}{32}}$

$\therefore PA$的方程为$x=\pm\sqrt{\frac{17}{32}}y+1$

(3) 如图，已知点$F(1,0)$, A,B为抛物线$y^2=4x$上不同的两点（B在A的右上方，F在直线AB的下方），满足$\angle BAF=\angle AFO+45°$.①证明：$A,B$的中点$C$位于某定直线上；

②记$\triangle ABF$的内切圆、外接圆的半径分别为r,R，求$\frac{R}{r}$的最小值.



key:（Ⅰ）设$A(a^2,2a),B(b^2,2b)$,

延长BA交x轴于点M, 则$\angle BAF=\angle AMF+\angle AFO=45°+\angle AFO,\therefore\angle AMF=45°$

$\therefore k_{AB}=\frac{2a-2b}{a^2-b^2}=\frac{2}{a+b}=1$, 即$a+b=2(0<a<1<b),\therefore AB$的中点在直线$y=2$上

（Ⅱ）由（Ⅰ）得AB的方程为$y=x-a^2+2a$

$\therefore S_{\triangle ABF}=\frac{1}{2}\sqrt{2}(b^2-a^2)\cdot\frac{|1-a^2+2a|}{\sqrt{2}}=(2-2a)(-a^2+2a+1)=\frac{1}{2}(\sqrt{2}(b^2-a^2)+a^2+1+b^2+1)\cdot r$

$=(a-\sqrt{2}-1)^2\cdot r$得$r=\frac{2(1-a)(-a^2+2a+1)}{(\sqrt{2}+1-a)^2}=\frac{2(1-a)(a-1+\sqrt{2})}{\sqrt{2}+1-a}$

由$2R=\frac{|BF|}{\sin\angle BAF}$（而$\tan\angle BAF=\frac{1+\frac{2a}{1-a^2}}{1-\frac{2a}{1-a^2}}=\frac{-a^2+2a+1}{-a^2-2a+1},\therefore\sin\angle BAF=\frac{-a^2+2a+1}{\sqrt{2}(a^2+1)}$）

得$R=\frac{1}{2}\cdot\frac{(2-a)^2+1}{\frac{-a^2+2a+1}{\sqrt{2}(a^2+1)}}=\frac{\sqrt{2}(a^2+1)(a^2-4a+5)}{2(-a^2+2a+1)}$

$\therefore\frac{R}{r}=\frac{\frac{\sqrt{2}(a^2+1)(a^2-4a+5)}{2(-a^2+2a+1)}}{\frac{2(1-a)(a-1+\sqrt{2})}{\sqrt{2}+1-a}}=\frac{\sqrt{2}(a^2+1)(a^2-4a+5)}{4(1-a)(a-1+\sqrt{2})^2}$(令$t=1-a\in(0,1)$)

$=\frac{\sqrt{2}(t^2-2t+2)(t^2+2t+2)}{4t(\sqrt{2}-t)^2}=\frac{\sqrt{2}(t^4+4)}{4t(\sqrt{2}-t)^2}=\frac{\sqrt{2}[(t+\frac{2}{t})^2-4)}{4(t+\frac{2}{t}-2\sqrt{2})}$

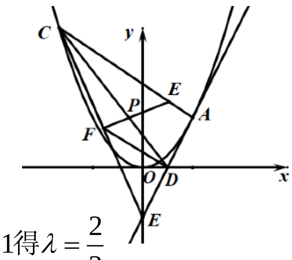
$=\frac{\sqrt{2}}{4}\cdot\frac{u^2+4\sqrt{2}u+4}{u}=\frac{\sqrt{2}}{4}(u+\frac{4}{u}+4\sqrt{2})\geq 2+\sqrt{2}$(当且仅当$u=2$时取$=$)(令$u=t+\frac{2}{t}-2\sqrt{2}\in(3-2\sqrt{2},+\infty)$)

解析几何（4）抛物线解答（4）

2024-01-06

五、切线问题

（2005A）过抛物线$y=x^2$上一点$A(1,1)$作抛物线的切线交x轴于D，交y轴于B，点C在抛物线上，E在线段AC上，$\dfrac{AE}{EC}=\lambda_1$，F在线段BC上，$\dfrac{BF}{FC}=\lambda_2$，且$\lambda_1+\lambda_2=1$，线段CD与EF交于P，当C在抛物线上移动时，求P的轨迹方程.



解:由$l_{AD}:\dfrac{1+y}{2}=x$得$D(\dfrac{1}{2},0),B(0,-1),\therefore D$为$AB$的中点，

$\therefore \overrightarrow{CD}=\dfrac{1}{2}\overrightarrow{CA}+\dfrac{1}{2}\overrightarrow{CE}=\dfrac{1+\lambda_1}{2}\overrightarrow{CE}+\dfrac{1+\lambda_2}{2}\overrightarrow{CF}$

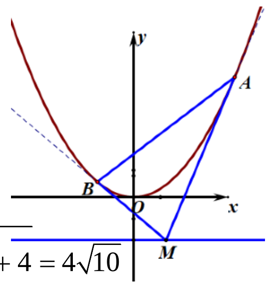
设$\overrightarrow{CP}=\lambda\overrightarrow{CD}$,则$\overrightarrow{CP}=\dfrac{\lambda(1+\lambda_1)}{2}\overrightarrow{CA}+\dfrac{\lambda(1+\lambda_2)}{2}\overrightarrow{CE},\therefore \dfrac{\lambda(1+\lambda_1)}{2}+\dfrac{\lambda(1+\lambda_2)}{2}=\dfrac{3\lambda}{2}=1$得$\lambda=\dfrac{2}{3}$

设$P(x,y)$，则$\begin{cases}x-x_C=\dfrac{2}{3}(\dfrac{1}{2}-x_C)\\ y-y_C=\dfrac{2}{3}(-y_C)\end{cases}$得$\begin{cases}x_C=3x-1\neq 1\\ y_C=3y\end{cases}$，$\therefore 3y=(3x-1)^2$即$y=\dfrac{1}{3}(3x-1)^2(x\neq\dfrac{2}{3})$即为所求的

（2008 山东）22．如图，设抛物线方程为$x^2=2py(p>0)$，M为直线$y=-2p$上任意一点，过M引抛物线的切线，切点分别为A,B．（I）求证：A,M,B三点的横坐标成等差数列；

（II）已知当M点的坐标为$(2,-2p)$时，$|AB|=4\sqrt{10}$．求此时抛物线的方程；

（III）是否存在点M，使得点C关于直线AB的对称点D在抛物线$x^2=2py(p>0)$上，其中，点C满足$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{OB}$（$O$为坐标原点）．若存在，求出所有适合题意的点$M$的坐标；若不存在，请说明理由．



(1) 证明：设$A(2pa,2pa^2),B(2pb,2pb^2)$，则$l_{MA}:2pax=p(2pa^2+y)$即$2ax=2pa^2+y$

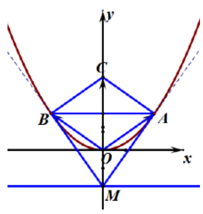
联立$l_{MB}:2bx=2pb^2+y$得$M(p(a+b),2pab),\therefore 2pab=-2p$即$ab=-1$,

$\therefore x_A+x_B=2pa+2pb=2x_M$，证毕

(2) 由 (1) 得$p(a+b)=2$即$a+b=\dfrac{2}{p}$，且$ab=-1$,

$\therefore |AB|=\sqrt{(2pa-2pb)^2+(2pa^2-2pb^2)^2}=2p|a-b|\cdot\sqrt{1+(a+b)^2}=2p\sqrt{1+\dfrac{4}{p^2}}\cdot\sqrt{\dfrac{4}{p^2}+4}=4\sqrt{10}$

得$p=1,or,2,\therefore$抛物线方程为$x^2=2y$,或$x^2=4y$



(3) 假设存在，由 (1) 得$C(2p(a+b),2p(a^2+b^2))(ab=-1)$，且$l_{AB}:y=(a+b)x+2p$

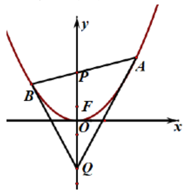
设$D(2pd,2pd^2)$，$p(a^2+b^2+d^2)=(a+b)\cdot p(a+b+d)+2p$得$d^2=d(a+b)$

当$d\neq 0$时，$d=a+b,CD\perp x$轴,$\therefore AB//x$轴,$\therefore a+b=0=d$矛盾；

当$d=0$时，$D(0,0)$,符合.$\therefore$存在，$M(0,-2p)$

（2004 湖南）（21）如图,过抛物线$x^2=4y$的对称轴上任一点$P(0,m)(m>0)$作直线与抛物线交于A、B两点，点Q是点P关于原点的对称点.（I）设点P分有向线段$\overrightarrow{AB}$所成的比为λ，证明：$\overrightarrow{QP}\perp(\overrightarrow{QA}-\lambda\overrightarrow{QB})$；

（II）设直线AB的方程是$x-2y+12=0$,过A、B两点的圆C与抛物线在点A处有共同的切线，求圆C的方程.



(1) 证明：设$A(2a,a^2),B(2b,b^2)$,由a,p,b三点共线得$\dfrac{a^2-b^2}{2a-2b}=\dfrac{a+b}{2}=\dfrac{a^2-m}{2a}$即$ab=-m$

且$\overrightarrow{AP}=\lambda\overrightarrow{PB}$,且$Q(0,-m),\lambda=-\dfrac{a}{b}$

$\therefore \overrightarrow{QP}\cdot(\overrightarrow{QA}-\lambda\overrightarrow{QB})=(0,2m)\cdot((2a,a^2+m)-\lambda(2b,b^2+m))=2m(a^2+m-\lambda(b^2+m))$

$=2m(a^2-ab+\dfrac{a}{b}(b^2-ab))=0,\therefore \overrightarrow{QP}\perp(\overrightarrow{QA}-\lambda\overrightarrow{QB})$

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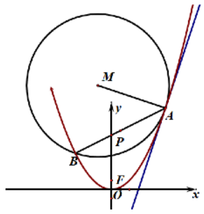


（2）解：由$\begin{cases}x^2=4y\\x-2y+12=0\end{cases}$得$A(6,9),B(-4,4)$,

则A处切线方程为$:6x=2(y+9)$即$3x-y-9=0$

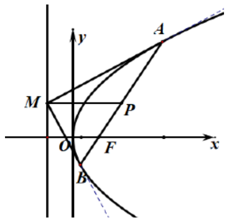
$\therefore$圆C方程为$(x-6)^2+(y-9)^2+\mu(3x-y-9)=0$

$\therefore 10^2+5^2+\mu(-12-4-9)=0$得$\mu=3,\therefore$圆$C$方程为$x^2+y^2-3x-21y+90=0$

（2012甘肃）10.M是抛物线$y^2=2px(p>0)$的准线上任意点，过M作抛物线的切线l_1,l_2，切点分别为A、B（A在x轴上方）.（1）证明：直线AB过定点；（2）设AB的中点为P，求$|MP|$的最小值.

⑴ 证明：设$A(2pa^2,2pa),B(2pb^2,2pb)$，则$l_1:2pay=p(2pa^2+x)$即$2ay=2pa^2+x$

$l_2:2by=2pb^2+x$



设$M(-\dfrac{p}{2},m)$,则$\begin{cases}2pa^2-2ma-\dfrac{p}{2}=0\\2pb^2-2mb-\dfrac{p}{2}=0\end{cases},\therefore\begin{cases}a+b=\dfrac{m}{p}\\ab=-\dfrac{1}{4}\end{cases}$

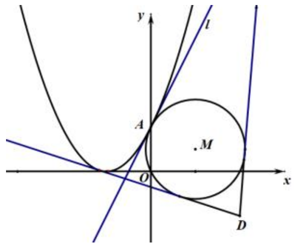
$\therefore l_{AB}:y-2pa=\dfrac{2pb-2pa}{2pb^2-2pa^2}(x-2pa^2)=\dfrac{1}{a+b}(x-2pa^2)$

即$(a+b)y-2pab=x$即$\dfrac{m}{p}y+\dfrac{p}{2}=x$经过定点$(\dfrac{p}{2},0)$,证毕

（2）解：由 ⑴ 得：$P(p(a^2+b^2),p(a+b))$即$(\dfrac{m^2}{p}+\dfrac{p}{2},m)$

$\therefore|MP|=\dfrac{m^2}{p}+p\geq p$(当$m=0$时，取$=$)，$\therefore|MP|$的最小值为$p$

（2012大纲）已知抛物线$C:y=(x+1)^2$与圆$M:(x-1)^2+(y-\dfrac{1}{2})^2=r^2(r>0)$有一个公共点，且在$A$处两曲线的切线为同一直线$l$.⑴ 求$r$的值；（2）设$m$、$n$是异于$l$切于$C$及$M$都相切的两条直线，$m$、$n$的交点为$D$，求$D$到$l$的距离.



2012Ⅰ:⑴ 设$A(a,(a+1)^2)$,则$(a-1)^2+((a+1)^2-\dfrac{1}{2})^2=r^2$

抛物线C在A处的切线方程为$\dfrac{(a+1)^2+y}{2}=ax+a+x+1$即$y=2(a+1)x+1-a^2$

$\therefore 2(a+1)=-\dfrac{a-1}{(a+1)^2-\dfrac{1}{2}}$得$a=0,\therefore r^2=1+\dfrac{1}{4}$即$r=\dfrac{\sqrt{5}}{2}$

（2）由 ⑴ 得$l:y=2x+1$,

由$\dfrac{|2(a+1)-\dfrac{1}{2}+1-a^2|}{\sqrt{1+4(a+1)^2}}=\dfrac{\sqrt{5}}{2}$得$((a+1)^2-4(a+1)+\dfrac{1}{2})^2=\dfrac{5}{4}(4(a+1)^2+1)=5(a+1)^2+\dfrac{5}{4}$

即$(a+1)^4-8(a+1)^3+12(a+1)^2-4(a+1)-1=0$即$a^2(a^2-4a-6)=0$得$a=0,or,a=2\pm\sqrt{10}$

由$\begin{cases}y=2(3+\sqrt{10})x-13-4\sqrt{10}\\y=2(3-\sqrt{10})x-13+4\sqrt{10}\end{cases}$得$D(2,-1),\therefore d=\dfrac{|4+1+1|}{\sqrt{5}}=\dfrac{6\sqrt{5}}{5}$

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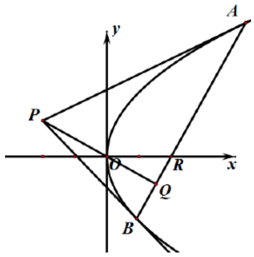
（2014A）如图，平面直角坐标系xOy中，P是不在x轴上的一个动点，满足条件：过P可作抛物线$y^2=4x$的两条切线，两切点连线l_P与PO垂直，设直线l_P与PO，x轴的交点分别为Q,R.

(1) 求证：R为定点； (2) 求$\dfrac{|PQ|}{|QR|}$的最小值.



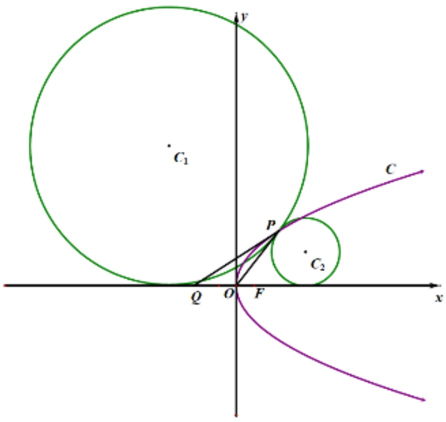
（2014A）key:由$l_{PA}:y_Ay=2(x+x_A);l_{PB}:y_By=2(x+x_B)$,

得$l_{AB}:y_Py=2(x_P+x)$令$y=0$得$x_R=-x_P$,

$\because PO\perp AB,\therefore \dfrac{y_P}{x_P}\cdot\dfrac{2}{y_P}=\dfrac{2}{x_P}=-1$即$x_P=-2,\therefore x_R=2,\therefore R$为定点

(Ⅱ)（投影）$\dfrac{|PQ|}{|QR|}=\dfrac{\dfrac{|4x_P-y_P^2|}{\sqrt{y_P^2+4}}}{\dfrac{|\overrightarrow{OR}\cdot(y_P,2)|}{\sqrt{y_P^2+4}}}=\dfrac{8+y_P^2}{2|y_P|}\ge 2\sqrt{2}$(当且仅当$y_P^2=8$时，取$=$)，$\therefore$所求最小值为$2\sqrt{2}$

（2016A）如图所示，在平面直角坐标系xOy中，F是x轴正半轴上的一个动点以F为焦点、O为顶点的抛物线C.设P是第一象限内C上的一点，Q是x轴负半轴上一点，使得PQ为C的切线，且$|PQ|=2$.圆C_1,C_2均与直线OP相切于点P，且均与x轴相切.求点F的坐标，使圆C_1与C_2的面积之和取到最小值.



key:设抛物线$C:y^2=2px(p>0)$，且$P(2pt^2,2pt)(t>0)$,

则PQ方程为:$2pty=p(x+2pt^2)$得$x_Q=-2pt^2$,

$\therefore|PQ|^2=16p^2t^4+4p^2t^2=4$即$p^2t^2(4t^2+1)=1$

设圆C_1,C_2的半径分别为r_1,r_2，设$\angle POx=2\theta$,则$\tan 2\theta=\dfrac{1}{t}$,

$\therefore\dfrac{r_2}{|OP|}=\tan\dfrac{\theta}{2},\dfrac{r_1}{|OP|}=\tan\dfrac{\pi-2\theta}{2}$

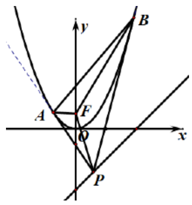
$\therefore S_1+S_2=\pi\cdot OP^2\cdot(\tan^2\theta+\dfrac{1}{\tan^2\theta})=\pi\cdot 4p^2t^2(t^2+1)\cdot(4t^2+2)$

$=8\pi\cdot\dfrac{(t^2+1)(2t^2+1)}{4t^2+1}(u=4t^2+1$则$t^2=\dfrac{u-1}{4})$

$=\pi(u+\dfrac{3}{u}+4)\ge(4+2\sqrt{3})\pi$(当且仅当$t=\dfrac{\sqrt{\sqrt{3}-1}}{2}$时取=)，此时$p=\dfrac{2}{\sqrt{3-\sqrt{3}}},F(\dfrac{1}{\sqrt{3-\sqrt{3}}},0)$

（2005 江西）22．如图，设抛物线$C:y=x^2$的焦点为 F，动点 P 在直线$l:x-y-2=0$上运动，过 P 作抛物线 C 的两条切线 PA、PB，且与抛物线 C 分别相切于 A、B 两点.

（1）求$\triangle APB$的重心 G 的轨迹方程；（2）证明：$\angle PFA=\angle PFB$.



(1) 解：设$A(a,a^2),B(b,b^2)$,则联立$l_{PA}:\dfrac{a^2+y}{2}=ax$即$2ax=a^2+y$与$l_{PB}:2bx=b^2+y$

得$P(\dfrac{a+b}{2},ab),\therefore\dfrac{a+b}{2}-ab-2=0$

设$G(x,y)$，则$\begin{cases}x=\dfrac{a+b+\dfrac{a+b}{2}}{3}\text{即}a+b=2x\\ y=\dfrac{a^2+b^2+ab}{3}\text{即}a^2+b^2+ab=(a+b)^2-ab=3y\text{即}ab=4x^2-3y\end{cases}$

$\therefore x-4x^2+3y-2=0$即$y=\dfrac{4}{3}x^2-\dfrac{1}{3}x+\dfrac{2}{3}$记为$G$的轨迹方程

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（2）证明：由（1）得 $\dfrac{\overrightarrow{FA}}{|\overrightarrow{FA}|}+\dfrac{\overrightarrow{FB}}{|\overrightarrow{FA}|}=\dfrac{(a,a^2-\frac{1}{4})}{a^2+\frac{1}{4}}+\dfrac{(b,b^2-\frac{1}{4})}{b^2+\frac{1}{4}}$

$=\dfrac{1}{(a^2+\frac{1}{4})(b^2+\frac{1}{4})}(ab^2+\frac{a}{4}+a^2b+\frac{b}{4},(a^2-\frac{1}{4})(b^2+\frac{1}{4})+(a^2+\frac{1}{4})(b^2-\frac{1}{4}))$

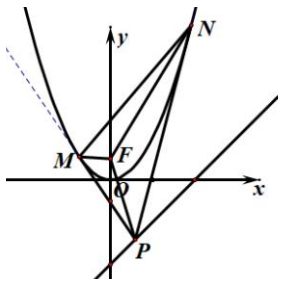
$=\dfrac{1}{(a^2+\frac{1}{4})(b^2+\frac{1}{4})}((a+b)(ab+\frac{1}{4}),2a^2b^2-\frac{1}{8})=\dfrac{ab+\frac{1}{4}}{(a^2+\frac{1}{4})(b^2+\frac{1}{4})}(a+b,2(ab-\frac{1}{4}))$

$\overrightarrow{FP}=(\dfrac{a+b}{2},ab-\dfrac{1}{4}),\therefore \dfrac{\overrightarrow{FA}}{|\overrightarrow{FA}|}+\dfrac{\overrightarrow{FB}}{|\overrightarrow{FA}|}//\overrightarrow{FP},\therefore \angle AFB$的平分线为直线$PF,\therefore \angle PFA=\angle PFB,$证毕

（2016湖北）过抛物线$x^2=2py(p>0)$外一点P向抛物线作两条切线，切点为M、N，F为抛物线的焦点，

证明：（Ⅰ）$|PF|^2=|MF|\cdot|NF|$；（Ⅱ）$\angle PFM=\angle PFN$.

key:设$P(s,t),M(2pm,2pm^2),N(2pn,2pn^2)$,则$l_{PM}:2mx=2pm^2+y;l_{PN}:2nx=2pn^2+y$



$\therefore\begin{cases}2ms=2pm^2+t\\2ns=2pn^2+t\end{cases},\therefore\begin{cases}m+n=\dfrac{s}{p}\\mn=\dfrac{t}{2p}\end{cases}$

(1)$|MF|\cdot|NF|=(2pm^2+\dfrac{p}{2})(2pn^2+\dfrac{p}{2})=4p^2(m^2n^2+\dfrac{1}{4}(m^2+n^2)+\dfrac{1}{16})$

$=s^2+t^2-pt+\dfrac{p^2}{4}=s^2+(t-\dfrac{p}{2})^2=|PF|^2$

(2)$\dfrac{\overrightarrow{FM}}{|\overrightarrow{FM}|}+\dfrac{\overrightarrow{FN}}{|\overrightarrow{FN}|}=\dfrac{(2pm,2pm^2-\frac{p}{2})}{2pm^2+\frac{p}{2}}+\dfrac{(2pn,2pn^2-\frac{p}{2})}{2pn^2+\frac{p}{2}}=\dfrac{(m,m^2-\frac{1}{4})}{m^2+\frac{1}{4}}+\dfrac{(n,n^2-\frac{1}{4})}{n^2+\frac{1}{4}}$

$=\dfrac{1}{(m^2+\frac{1}{4})(n^2+\frac{1}{4})}(\dfrac{s(2t+p)}{4p^2},\dfrac{4t^2-p^2}{8p^2})=\dfrac{2t+p}{4p^2(m^2+\frac{1}{4})(n^2+\frac{1}{4})}(s,\dfrac{2t-p}{2})//\overrightarrow{PF}=(-s,\dfrac{p}{2}-t)$

$\therefore PF$是$\angle MFN$的平分线

$key2$:综合（1）（2）得:$\triangle PFM\sim\triangle NFP\Leftrightarrow\dfrac{|PM|}{|PN|}=\dfrac{|PF|}{|NF|}=\dfrac{|FM|}{|PF|}$

由（1）得:$\dfrac{|PF|}{|NF|}=\dfrac{|FM|}{|PF|}$成立，此时$\dfrac{|PM|}{|PN|}=\dfrac{|PF|}{|NF|}=\dfrac{\sqrt{|MF|\cdot|NF|}}{|NF|}=\sqrt{\dfrac{|MF|}{|NF|}}$

$\Leftrightarrow\dfrac{\sqrt{4m^2+1}\cdot|2pm-s|}{\sqrt{4n^2+1}\cdot|2pn-s|}=\sqrt{\dfrac{2pm^2+\frac{p}{2}}{2pn^2+\frac{p}{2}}}=\sqrt{\dfrac{4m^2+1}{4n^2+1}}$

$\Leftrightarrow|2pm-s|=|2pn-s|\Leftrightarrow(2pm-s)^2-(2pn-s)^2=2p(m-2)(2p(m+n)-2s)$

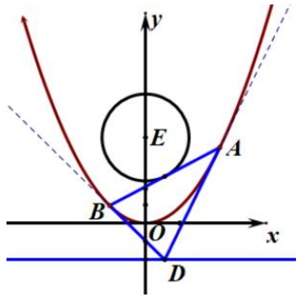
$=2p(m-n)(2p\cdot\dfrac{s}{p}-2s)=0$成立$,\therefore\triangle PFM\sim\triangle NFP,\therefore\angle PMF=\angle QMF,\angle MPF=\angle FNP,\angle PMF=\angle FPN,$

解析几何（4）抛物线解答（4）

2024-01-06

(2019Ⅰ)21.已知曲线$C:y=\dfrac{x^2}{2}$，D为直线$y=-\dfrac{1}{2}$上的动点，过D作C的两条切线，切点分别为A、B.

(1) 证明：直线AB过定点；(2) 若以$E(0,\dfrac{5}{2})$为圆心的圆与直线AB相切，且切点为线段AB的中点，求四边形$ADBE$的面积.



(1) 证明：设$A(2a,2a^2),B(2b,2b^2)$，则联立$l_{AD}:y+2a^2=2ax$与$l_{DB}:y+2b^2=2bx$

得$D(a+b,2ab)$，且$2ab=-\dfrac{1}{2}$，

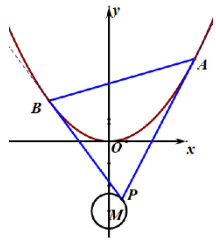
$\therefore l_{AB}:y-2a^2=\dfrac{2a^2-2b^2}{2a-2b}(x-2a)=(a+b)(x-2a)$即$y=(a+b)x+\dfrac{1}{2}$经过定点$(0,\dfrac{1}{2})$，

(2) 由(1)得：$\dfrac{\frac{5}{2}-a^2-b^2}{-(a+b)}\cdot(a+b)=a^2+b^2-\dfrac{5}{2}=-1$即$a^2+b^2=\dfrac{3}{2}$，$\therefore(a+b)^2=1$

$$S_{ADBE}=\frac{1}{2}\sqrt{1+(a+b)^2}\cdot|2a-2b|\cdot\frac{2+(a+b)^2+1}{\sqrt{1+(a+b)^2}}=\sqrt{(a+b)^2+1}\cdot((a+b)^2+1)=2\sqrt{2}$$

(2021乙)21.已知抛物线$C:x^2=2py(p>0)$的焦点为F，且F与圆$M:x^2+(y+4)^2=1$上点的距离的最小值为4.

(1) 求p；(2) 若点P在M上，PA、PB是C的两条切线，A、B是切点，求$\triangle PAB$面积的最大值.



解：(1) 由已知得：$4+\dfrac{p}{2}-1=4$得$p=2$

(2) 设$A(2a,a^2),B(2b,b^2)$，则$l_{PA}:2ax=2(a^2+y)$即$ax=a^2+y$与$l_{PB}:bx=b^2+y$

得$P(a+b,ab)$，$\therefore(a+b)^2+(ab+4)^2=1$

令$a+b=\cos\theta,ab+4=\sin\theta$，则$\sin\theta-4=ab\le(\dfrac{a+b}{2})^2=\dfrac{1}{4}\cos^2\theta$得$\sin\theta\in[-1,1]$，

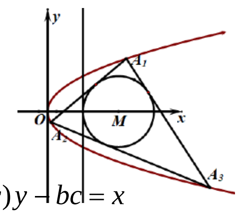
且$(a-b)^2=(a+b)^2-4ab=\cos^2\theta-4(\sin\theta-4)=17-\sin^2\theta-4\sin\theta$，

$$\therefore S_{\triangle PAB}=\frac{1}{2}\left\|\begin{vmatrix}2a & a^2 & 1\\ 2b & b^2 & 1\\ a+b & ab & 1\end{vmatrix}\right\|=\frac{1}{2}|(a-b)^3|=\frac{1}{2}(-(\sin\theta+2)^2+21)^{\frac{3}{2}}=20\sqrt{5}$$（当$\sin\theta=-1$时，取$=$）

$\therefore\triangle PAB$面积的最大值$20\sqrt{5}$

(2021甲)20.抛物线C的顶点为坐标原点O，焦点在x轴上，直线$l:x=1$交C于P、Q两点，且$OP\perp OQ$，已知点$M(2,0)$，且$\odot M$与l相切.(1) 求C及$\odot M$的方程；

(2) 设A_1、A_2、A_3是C上的三个点，直线A_1A_2、A_1A_3均与$\odot M$相切，判断直线A_2A_3与$\odot M$的位置关系，并说明理由.



解：(1) 设抛物线C方程为$y^2=2px(p>0)$，$\because OP\perp OQ$，$\therefore\sqrt{2p}=1$即$p=\dfrac{1}{2}$

且$r=2-1=1$，$\therefore C$的方程为$y^2=x$，$\odot M$的方程为$(x-2)^2+y^2=1$

(2) 设$A_1(a^2,a),A_2(b^2,b),A_3(c^2,c)$（不妨设$a>b>c$）

则$l_{A_1A_2}:y-a=\dfrac{a-b}{a^2-b^2}(x-a^2)$即$(a+b)y-ab=x$，$l_{A_1A_3}:(a+c)y-ac=x$，$l_{A_2A_3}:(b+c)y-bc=x$

由A_1A_2与$\odot M$相切得$\dfrac{|2+ab|}{\sqrt{(a+b)^2+1}}=1$即$a^2b^2-a^2-b^2+2ab+3=0$即$(a^2-1)b^2+2ab+3-a^2=0\cdots$①

同理$(a^2-1)c^2+2ac+3-a^2=0\cdots②$, $\therefore\begin{cases}b+c=\dfrac{-2a}{a^2-1}\\ bc=\dfrac{3-a^2}{a^2-1}\end{cases}$

$\therefore M$到直线A_2A_3的距离$d=\dfrac{|2+bc|}{\sqrt{(b+c)^2+1}}=\dfrac{|2+\dfrac{3-a^2}{a^2-1}|}{\sqrt{\dfrac{4a^2}{(a^2-1)^2}+1}}=\dfrac{a^2+1}{a^2+1}=1$, $\therefore A_2A_3$与$\odot M$相切

$key2$: 设$A_1(a^2,a),A_2(b^2,b),A_3(c^2,c),k_{A_1A_2}=k_1,k_{A_1A_3}=k_2$,

则$l_{A_1A_2}:y-a=k_1(x-a^2)$与圆$M$相切得$\dfrac{|k_1(2-a^2)+a|}{\sqrt{1+k_1^2}}=1$即$(a^4-4a^2+3)k_1^2+2a(2-a^2)k_1+a^2-1=0$

同理$(a^4-4a^2+3)k_2^2+2a(2-a^2)k_2+a^2-1=0$, $\therefore\begin{cases}k_1+k_2=\dfrac{2a(a^2-2)}{a^4-4a^2+3}\\ k_1k_2=\dfrac{1}{a^2-3}\end{cases}$

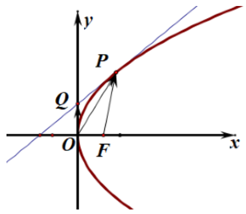
而$k_{A_1A_2}=\dfrac{a-b}{a^2-b^2}=\dfrac{1}{a+b}=k_1$得$b=\dfrac{1}{k_1}-a$, 同理$c=\dfrac{1}{k_2}-a$

$\therefore b+c=\dfrac{1}{k_1}+\dfrac{1}{k_2}-2a=\dfrac{-2a}{a^2-1}, bc=\dfrac{1}{k_1k_2}-a(\dfrac{1}{k_1}+\dfrac{1}{k_2})+a^2=\dfrac{3-a^2}{a^2-1}$

$\therefore l_{A_2A_3}:y-b=\dfrac{b-c}{b^2-c^2}(x-b^2)$即$(b+c)y-bc=x$即$(a^2-1)x-2ay-a^2+3=0$

$\therefore M$到A_2A_3的距离$d=\dfrac{|2a^2-2-a^2+3|}{\sqrt{(a^2-1)^2+4a^2}}=1$, $\therefore\odot M$与直线A_2A_3相切

（2021A）在平面直角坐标系xOy中，抛物线$\Gamma:y^2=2px(p>0)$的焦点为F，过Γ上一点P（异于O）作Γ的切线与y轴交于点Q，若$|FP|=2,|FQ|=1$, 则$\overrightarrow{OP}\cdot\overrightarrow{OQ}=$_______.



key: 设$P(2pt^2,2pt)$，则$|FP|=2pt^2+\dfrac{p}{2}=2\cdots①$

且$l_{PQ}:2pty=p(x+2pt^2)$即$2ty=x+2pt^2$

令$x=0$得$y_Q=pt$, $\therefore|FQ|=\sqrt{\dfrac{p^2}{4}+p^2t^2}=1$即$p^2(t^2+\dfrac{1}{4})=1\cdots②$

由①②得$p=1,t^2=\dfrac{3}{4}$, $\therefore\overrightarrow{OP}\cdot\overrightarrow{OQ}=2p^2t^2=\dfrac{3}{2}$

（2022Ⅰ）11.已知O为坐标原点，点$A(1,1)$在抛物线$C:x^2=2py(p>0)$上，过点$B(0,-1)$的直线交C于P,Q两点，则（　）$A.C$的准线为$y=-1$　　$B.$直线AB与C相切　$C.|OP|\cdot|OQ|>|OA|^2$　　$D.|BP|\cdot|BQ|>|BA|^2$

2022Ⅰ$key:C:x^2=y.\therefore A$错；A处切线$x=\dfrac{y+1}{2}$经过点B;

设$P(p,p^2),Q(q,q^2)(p+q)$，则$\dfrac{p^2-q^2}{p-q}=\dfrac{p^2+1}{p}$得$pq=-1$

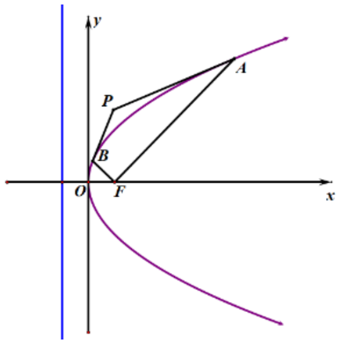
$\therefore|OP|\cdot|OQ|=\sqrt{(p^2+p^4)(q^2+q^4)}=\sqrt{p^2q^2(p^2q^2+p^2+q^2+1)}=\sqrt{p^2+q^2+2}>\sqrt{-2pq+2}=2>|OA|^2$

设$l_{PQ}:y=kx-1$代入C得$:x^2-kx+1=0,\therefore\begin{cases}x_P+x_Q=k,\\ x_Px_Q=1\end{cases}$, 且$\Delta=k^2-4>0,\therefore|BP|\cdot|BQ|=(1+k^2)\cdot1>5=|BA|^2$ $\therefore$选BCD

变式 1（1）如图，直线l经过抛物线$y^2=2px(p>0)$的焦点F，且与抛物线交于P、Q两点,由P、Q分别作抛物线的切线交于M,如果$|PF|=a,|QF|=b.$则$|MF|$的值为（　　）　D

$A.a+b\ \ B.\dfrac{a+b}{2}\ \ C.ab\ \ D.\sqrt{ab}$

（2）抛物线$C:x^2=2py(p>0)$的焦点为F，A、B是抛物线C上的两点，若$\angle AFB=90°$，求抛物线在A、B处的切线的交点P的轨迹方程.



$key1:$设$A(2pa^2,2pa),B(2pb^2,2pb)$,

$\because \angle AFB=90°,\therefore \overrightarrow{FA}\cdot\overrightarrow{FB}=(2pa^2-\dfrac{p}{2})(2pb^2-\dfrac{p}{2})+4p^2ab=0$

即$a^2b^2-\dfrac{1}{4}(a+b)^2+\dfrac{3}{2}ab+\dfrac{1}{16}=0\cdots(*)$

PA方程为$2ay=2pa^2+x$；PB方程为:$2by=2pb^2+x$, 设$P(s,t)$

则$\begin{cases}2at=2pa^2+s\\2bt=2pb^2+s\end{cases},\therefore a,b$是方程$:2px^2-2tx+s=0$的两根$,\therefore\begin{cases}a+b=\dfrac{t}{p}\\ab=\dfrac{s}{2p}\end{cases}$, 且$\Delta=4(t^2-2ps>0)$

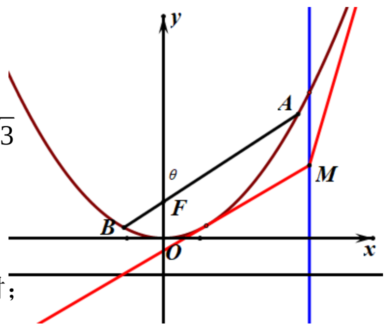
代入(*)得:$4s^2-4t^2+12s+p^2=0,\therefore$点$P$的轨迹方程为:$4x^2-4y^2+12x+p^2=0(y^2-2px>0)$

（3）（多选题）抛物线 $C:x^2=2py(p>0)$ 的准线方程为 $y=-1$，过焦点 F 的直线 l 交抛物线 C 于 A,B 两点，则（ BD　）A.C 的方程为 $x^2=2y$　　B. $|AB|+2|BF|$ 的最小值为 $4+2\sqrt{3}$

C.过点 $M(4,2)$ 且与抛物线仅有一个公共点的直线有且仅有 2 条

D.过点 A,B 分别作 C 的切线，交于点 $P(x_0,y_0)(x_0\neq0)$，则直线 PE,PA,PB 的斜率满足 $\dfrac{2}{k_{PF}}=\dfrac{1}{k_{PA}}+\dfrac{1}{k_{PB}}$

$key:$由已知得$C:x^2=4y,A$错



$B:key1:|AF|\cos\theta+2=|AF|$得$|AF|=\dfrac{2}{1-\cos\theta},\therefore|BF|=\dfrac{2}{1+\cos\theta}$

$\therefore|AB|+2|BF|=|FA|+3|FB|=\dfrac{2}{1-\cos\theta}+\dfrac{6}{1+\cos\theta}\geq\dfrac{(\sqrt{2}+\sqrt{6})^2}{2}=4+2\sqrt{3}$

$key2:A(2a,a^2),B(2b,b^2),\therefore\dfrac{b^2-a^2}{2b-2a}=\dfrac{b+a}{2}=\dfrac{a^2-1}{2a}$得$ab=-1$

$\therefore|AB|+2|BF|=a^2+1+b^2+1+2(b^2+1)=\dfrac{1}{b^2}+3b^2+4\geq4+2\sqrt{3},\therefore B$对；

点M在抛物线外，可作3条直线于抛物线只要一个公共点，C错；

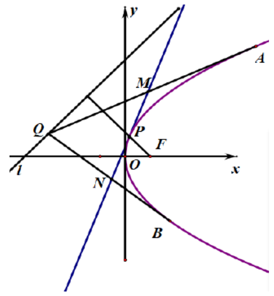
由$\begin{cases}ax=a^2+y\\bx=b^2+y\end{cases}$得$P(a+b,-1),\therefore\dfrac{1}{k_{PA}}+\dfrac{1}{k_{PB}}-\dfrac{2}{k_{PF}}=\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{2}{\dfrac{-2}{a+b}}=-(a+b)+a+b=0,\therefore D$对

（4）已知抛物线$E:y^2=2px(p>0)$和直线$l:x-y+4=0,P$是抛物线E上的点，且点P到y轴的距离与到直线l的距离之和有最小值$\dfrac{5\sqrt{2}}{2}-1$.（Ⅰ）求抛物线E的方程；

（Ⅱ）设$Q\in l$,过点Q作抛物线E的两条切线，切点分别记为A,B，抛物线E在点P处的切线与QA,QB分别交于M,N两点，求$\triangle QMN$外接圆面积的最小值.

解:（Ⅰ）距离之和为$d+|PF|-\dfrac{p}{2}\geq\dfrac{-\dfrac{p}{2}+4}{\sqrt{2}}-\dfrac{p}{2}=\dfrac{5\sqrt{2}}{2}-1$得$p=2$,

$\therefore$抛物线E的方程为$y^2=4x$



（Ⅱ）设$A(a^2,2a),B(b^2,2b),P(c^2,2c)$

则QA方程为$ay=x+a^2$，QB方程为$by=x+b^2$，$\therefore Q(ab,a+b),\therefore a+b=ab+4$

MN方程为：$cy=x+c^2$，$\therefore M(ac,a+c),N(bc,b+c)$

设$\triangle QMN$的外接圆方程为：$x^2+y^2+dx+ey+f=0$

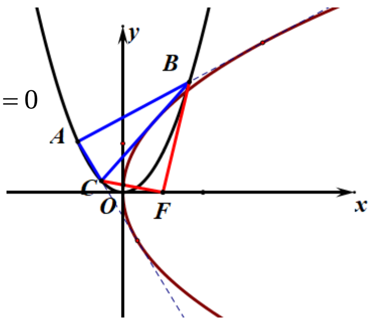
$$\therefore\begin{cases}a^2b^2+(a+b)^2+dab+e(a+b)+f=0\\a^2c^2+(a+c)^2+dac+e(a+c)+f=0\\b^2c^2+(b+c)^2+dbc+e(b+c)+f=0\end{cases}\text{得}\begin{cases}d=-ab-bc-ca-1\\e=abc-a-b-c\\f=ac+bc+ab\end{cases}$$

$$\therefore r^2=\frac{d^2+e^2-4f}{4}=\frac{1}{4}[(2(a+b)^2-10(a+b)+25)c^2+2(a+b)^2-10(a+b)+25]\ge\frac{1}{2}(a+b-\frac{5}{2})^2+\frac{25}{8}$$

（当且仅当$(a,b,c)=(\frac{1}{2},-3,0)$或$(-3,\frac{1}{2},0)$时，取$=$），$\therefore\triangle QMN$外接圆面积的最小值为$\frac{25\pi}{8}$

（2022新疆）如图，已知$\triangle ABC$内接于抛物线$E:x^2=y$，且边AB、AC所在直线分别与抛物线$M:y^2=4x$相切，F为抛物线M的焦点.求证：（1）边BC所在直线与抛物线M相切；（2）A、C、B、F四点共圆.

证明：（1）设$A(a,a^2),B(b,b^2),C(c,c^2)$，则$l_{AB}:y-a^2=\frac{a^2-b^2}{a-b}(x-a)$即$y=(a+b)x-ab$



代入M方程得：$\frac{a+b}{4}y^2-y-ab=0,\therefore\Delta_1=1+ab(a+b)=0$

同理$1+ac(a+c)=0,\therefore ab(a+b)-(ac(a+c)=a(b-c)(a+b+c)=0,\therefore a+b+c=0$

而$l_{BC}:y=(b+c)x-ab$代入M方程得：$\frac{b+c}{4}y^2-y-bc=0$

$\therefore\Delta_3=1+bc(b+c)=1-abc=\Delta_1=0,\therefore BC$与抛物线$M$相切，证毕

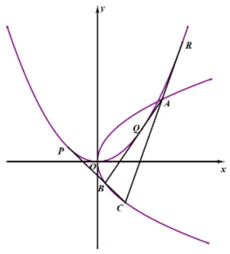
（2）如图，$\tan\angle BAC=-\frac{k_{AC}-k_{AB}}{1+k_{AC}k_{AB}}=-\frac{(a+c)-(a+b)}{1+(a+c)(a+b)}=-\frac{c-b}{1+bc}$

$$\tan\angle BFC=\frac{k_{CF}-k_{FB}}{1+k_{FC}k_{FB}}=\frac{\frac{c^2}{c-1}-\frac{b^2}{b-1}}{1+\frac{b^2c^2}{(c-1)(b-1)}}=\frac{c^2(b-1)-b^2(c-1)}{(c-1)(b-1)+b^2c^2}=\frac{(b-c)(b+c-bc)}{bc-b-c+b^2c^2+1}$$

$$=\frac{(b-c)(-a-\frac{1}{a})}{\frac{1}{a}+a+\frac{1}{a^2}+1}=\frac{b-c}{\frac{1}{a}+1}=\frac{b-c}{bc+1},\therefore\tan\angle BFC+\tan\angle BAC=0$$

$\therefore\angle BFC+\tan\angle BAC=\pi,\therefore A,C,B,F$四点共圆

变式1（1）已知抛物线$y^2=2px(p>0)$的内接$\triangle ABC$的三条边所在直线均与抛物线$x^2=2py$相切，求证：A,B,C三点的纵坐标之和为0.



（1）证明：设$A(2pa^2,2pa),B(2pb^2,2pb),C(2pc^2,2pc)(a\ne b,b\ne c,c\ne a)$

则AB方程为$y-2pa=\frac{2pa-2pb}{2pa^2-2pb^2}(x-2pa^2)$即$(a+b)y-2pab=x$

代入$x^2=2py$得：$\frac{a+b}{2p}x^2-x-2pab=0,\therefore\Delta_1=1+4ab(a+b)=0$

同理：$1+4bc(b+c)=0$，且$1+4ac(a+c)=0$，

$\therefore 1+4ab(a+b)-1-4bc(b+c)=4b(a-c)(a+b+c)=0$

$\therefore a+b+c=0,\therefore A,B,C$三点的纵坐标之和为$2p(a+b+c)=0$得证

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（2）已知$\triangle ABC$的三边AB、BC、CA是抛物线$x^2=2py(p>0)$的三条切线，求证:$\triangle ABC$的垂心在一条直线上.

（2）证明：设AB、BC、CA与抛物线$x^2=2py$的切点$D(2pd,2pd^2),E(2pe,2pe^2),F(2pf,2pf^2)$

则AB方程为$:2dx=y+2pd^2,BC$方程为$:2ex=y+2pe^2,CA$方程为$2fx=y+2pf^2$

$\therefore A(p(d+f),2pdf),B(p(d+e),2pde),C(p(e+f),2pef)$

$\therefore BC$边上的高线方程为$:x-p(d+f)+2e(y-2pdf)=0$

同理：AB边上的高线方程为$:x-p(e+f)+2d(y-2pef)=0$

$\therefore -p(d-e)+(2e-2d)y_H=0$即$y_H=-\dfrac{p}{2},\therefore$得证