（3）①设正实数 x,y 满足 $xy=\dfrac{x-4y}{x+y}$，则 y 的最大值是________． $\sqrt{5}-2$

$key:\because xy=\dfrac{x-4y}{x+y},\therefore x+y=\dfrac{x-4y}{xy}=\dfrac{1}{y}-\dfrac{4}{x}$即$\dfrac{1}{y}-y=x+\dfrac{4}{x}\ge 4$得$y\le -2+\sqrt{5}.$

$key2:$以x为主元的二次方程

②已知正数 x,y 满足 $x+\dfrac{y}{x}=1$，则 $\dfrac{1}{x}+\dfrac{x}{y}$ 的最小值为________.

$$key1:\frac{1}{x}+\frac{x}{y}=\frac{1}{x}+\frac{x}{y}+x+\frac{y}{x}-1\ge 3;$$

$$key2:1\cdot(\frac{1}{x}+\frac{x}{y})=(x+\frac{y}{x})(\frac{1}{x}+\frac{x}{y})=2+\frac{y}{x^2}+\frac{x^2}{y}\ge 4$$

$$key3:\frac{y}{x}=1-x>0,\therefore 0<x<1,\therefore \frac{1}{x}+\frac{x}{y}=\frac{1}{x}+\frac{1}{1-x}=\frac{x+1-x}{x}+\frac{x+1-x}{1-x}=2+\frac{1-x}{x}+\frac{x}{1-x}\ge 4$$

③ 已知正实数a,b,c满足$a+b=1,c+d=1,$则$\dfrac{1}{abc}+\dfrac{1}{d}$的最小值是（ ） $A.10$ $B.9$ $C.4\sqrt{2}$ $D.3\sqrt{3}$ B

④ 已知$x,y>0,\dfrac{2x+6}{\sqrt{y}}+\dfrac{3x+5}{x}\sqrt{y}=16,$则$x\in$______, $\dfrac{x}{y}\in$_______.$[1,5],[\dfrac{1}{2},\dfrac{5}{2}]$

$$key:16=\frac{2x+6}{\sqrt{y}}+\frac{3x+5}{x}\sqrt{y}\ge 2\sqrt{\frac{(2x+6)(3x+5)}{x}}\text{得}1\le x\le 5$$

$$\text{设}t=\frac{x}{y},\ \text{则}y=\frac{x}{t},\therefore 16=\frac{2x+6}{\sqrt{x}}\cdot\sqrt{t}+\frac{3x+5}{x}\cdot\sqrt{\frac{x}{t}}=2\sqrt{t}\sqrt{x}+\frac{6\sqrt{t}}{\sqrt{x}}+\frac{3}{\sqrt{t}}\sqrt{x}+\frac{5}{\sqrt{t}\cdot\sqrt{x}}$$

$$=(2\sqrt{t}+\frac{3}{\sqrt{t}})\sqrt{x}+(6\sqrt{t}+\frac{5}{\sqrt{t}})\frac{1}{\sqrt{x}}\ge 2\sqrt{(2\sqrt{t}+\frac{3}{\sqrt{t}})(6\sqrt{t}+\frac{5}{\sqrt{t}})}\text{即}4t^2-12t+5\le 0\text{得}\frac{1}{2}\le t\le\frac{5}{2}$$

（4）①若实数 $a,b>0$，且$(a+\dfrac{1}{a})+2(b+\dfrac{1}{b})=10$，则$a+2b$ 最大值是_______9；

$\dfrac{a}{b}+\dfrac{b}{a}$ 的最大值为_______.10

$$key:10(a+2b)=(a+2b)^2+(\frac{1}{a}+\frac{2}{b})(a+2b)=(a+2b)^2+5+\frac{2b}{a}+\frac{2a}{b}\ge(a+2b)^2+9,\therefore 1\le a+2b\le 9$$

$$key2:\text{令}t=a+2b,\ \text{则}10=t+\frac{1}{t-2b}+\frac{2}{b}=t+\frac{1}{t}\cdot\frac{t-2b+2b}{t-2b}+\frac{2}{t}\cdot\frac{t-2b+2b}{b}$$

$$=t+\frac{1}{t}(5+\frac{2b}{t-2b}+\frac{2(t-2b)}{b})\ge t+\frac{9}{t}$$

②已知正实数 a,b满足 $\dfrac{1}{(2a+b)b}+\dfrac{2}{(2b+a)a}=1.$则$ab$的最大值为________.$2-\dfrac{2\sqrt{2}}{3}$

$$key:\text{（}a\text{与}b\text{关系拟线性）令}t=ab,\text{则}\frac{1}{(2a+\frac{t}{a})\frac{t}{a}}+\frac{2}{(\frac{2t}{a}+a)a}=1\text{即}(2t-1)a^4+(5t^2-6t)a^2+2t^3-2t^2=0$$

$$\therefore\Delta=t^2(5t-6)^2-16t^2(t^2-t)\ge 0\text{得}t\le 2-\frac{2\sqrt{2}}{3},or,t\ge 2+\frac{2\sqrt{2}}{3}$$

而$1=\frac{1}{2ab+b^2}+\frac{2}{2ab+b^2}<\frac{1}{2ab}+\frac{2}{2ab}=\frac{3}{2ab},\therefore ab<\frac{3}{2},\therefore ab\le 2-\frac{2\sqrt{2}}{3}$

$key2:ab=\frac{a}{2a+b}+\frac{2b}{a+2b}=\frac{1}{2+t}+\frac{2t}{1+2t}$(分母不动分子用分母表示化对勾)

③已知正数x,y满足$x+4y=x^2y^3$，则$\frac{8}{x}+\frac{1}{y}$的最小值是____.

$key1$:（拟线性）设$t=\frac{8}{x}+\frac{1}{y}$,则$x=\frac{8y}{ty-1},\therefore\frac{8y}{ty-1}+4y=\frac{4y(ty+1)}{ty-1}=\frac{64y^2}{(ty-1)^2}\cdot y^3$得$t^2=16y^2+\frac{1}{y^2}\ge 8,\therefore t\ge 2\sqrt{2}$

$key2$:（待定系数法）由已知得$\frac{1}{y}+\frac{4}{x}=xy^2,\therefore\frac{a+1}{y}+\frac{4+b}{x}=xy^2+\frac{a}{y}+\frac{b}{x}=xy^2+\frac{a}{2y}+\frac{a}{2y}+\frac{b}{x}$

$\ge 4\sqrt[4]{\frac{a^2b}{4}}=4(\sqrt{2}+1)$(其中$\begin{cases}4+b=8(a+1)\\ xy^2=\frac{a}{2y}=\frac{b}{x}\\ x+4y=by\end{cases}$ 即$\begin{cases}a=\sqrt{2}+1\\ b=12+8\sqrt{2}\end{cases}$)$,\therefore\frac{8}{x}+\frac{1}{y}\ge 2\sqrt{2}$

④已知正数a,b满足$ab^2(a+b)=4$,则$2a+b$的最小值为（ ）$A.12$ $B.8$ $C.2\sqrt{2}$ $D.\sqrt{3}$ C

$key1$:令$t=2a+b$，则有$4=a(t-2a)^2(t-a)=(4+\lambda)a\cdot(t-2a)\cdot(t-2a)\cdot\lambda(t-a)\cdot\frac{1}{\lambda(4+\lambda)}$

$\le\frac{1}{\lambda(4+\lambda)}\cdot(\frac{(\lambda+2)t}{4})^4$（其中$(4+\lambda)a=t-2a=\lambda(t-a)$即$\lambda^2+4\lambda-4=0),\therefore t\ge 2\sqrt{2}$

$key2:key:2a+b=\frac{1-\lambda}{2}b+\frac{1-\lambda}{2}b+\lambda(a+b)+(2-\lambda)a\ge 4\sqrt[4]{\frac{(1-\lambda)^2}{4}\cdot\lambda(2-\lambda)ab^2(a+b)}=2\sqrt{2}$

（其中$\begin{cases}\frac{1-\lambda}{2}b=\lambda(a+b)\\ (2-\lambda)a=\lambda(a+b)\end{cases}$ 即$\lambda=\frac{2-\sqrt{2}}{2}$)

$key3:4=ab^2(a+b)=(a^2+ab)b^2=(4a^2+4ab)b^2\le\frac{4a^2+4ab+b^2}{2},\therefore(2a+b)^2\ge 8$

⑤若x,y,z都是正实数，且$x^2+y^2+z^2=1$，则$\frac{yz}{x}+\frac{xz}{y}+\frac{xy}{z}$的最小值是__________.$\sqrt{3}$

$key:(\frac{yz}{x}+\frac{xz}{y}+\frac{xy}{z})^2=\frac{y^2z^2}{x^2}+\frac{x^2z^2}{y^2}+\frac{x^2y^2}{z^2}+2z^2+2x^2+2y^2\ge z^2+x^2+y^2+2=3$

⑥ 设a,b,c是不全为0的实数,则$(\frac{ab+ac+bc+c^2}{a^2+b^2+2c^2})_{\max}=$____;$(\frac{ab+ac+bc+c^2}{a^2+b^2+2c^2})_{\min}=$____ .

key:由$\frac{ab+ac+bc+c^2}{a^2+b^2+2c^2}=\frac{xy+x+y+1}{x^2+y^2+2}(x=\frac{a}{c},y=\frac{b}{c}\in R)$

$key1:x^2+y^2+2=\lambda x^2+\lambda y^2+(1-\lambda)x^2+\mu+(1-\lambda)y^2+\mu+2-2\mu$

$\ge 2\lambda xy+2\sqrt{(1-\lambda)\mu}x+2\sqrt{(1-\lambda)\mu}y+2-2\mu$

(其中$\lambda,\mu>0$，且$2\lambda=2\sqrt{(1-\lambda)\mu}=2-2\mu$即$\lambda=\mu=\frac{1}{2}),\therefore\frac{xy+x+y+1}{x^2+y^2+2}\le 1$

$key2:xy+x+y+1=xy+\lambda x\cdot\frac{1}{\lambda}+\lambda y\cdot\frac{1}{\lambda}+1\le\frac{x^2+y^2}{2}+\frac{1}{2}(\lambda^2x^2+\frac{1}{\lambda^2})+\frac{1}{2}(\lambda^2y^2+\frac{1}{\lambda^2})+1$

$=\frac{1+\lambda^2}{2}x^2+\frac{1+\lambda^2}{2}y^2+1+\frac{1}{\lambda^2}=x^2+y^2+2$（其中$2\cdot\frac{1+\lambda^2}{2}=\frac{1+\lambda^2}{\lambda^2}$即$\lambda^2=1$，且当且仅当$x=y=1$时取$=$）

$key1: x^2+y^2+2=\lambda x^2+\lambda y^2+(1-\lambda)x^2+\mu+(1-\lambda)y^2+\mu+2-2\mu$（其中$\lambda,\mu>0$）

$\geq -2\lambda xy-2\sqrt{(1-\lambda)\mu}x-2\sqrt{(1-\lambda)\mu}y+2-2\mu=-\sqrt{2}(xy+x+y+1)$

（其中$-\lambda=-\sqrt{(1-\lambda)\mu}=1-\mu$即$\lambda=\frac{\sqrt{2}}{2},\mu=\frac{2+\sqrt{2}}{2}$），

$key2: xy+x+y+1=-x\cdot(-y)-\lambda x\cdot(-\frac{1}{\lambda})-(\lambda y)\cdot\frac{1}{\lambda}+1\geq-\frac{x^2+y^2}{2}-\frac{\lambda^2x^2+\frac{1}{\lambda^2}}{2}-\frac{\lambda^2y^2+\frac{1}{\lambda^2}}{2}+1(\lambda>0)$

$=-\frac{1+\lambda^2}{2}x^2-\frac{1+\lambda^2}{2}y^2+1-\frac{1}{\lambda^2}=-\frac{\sqrt{2}}{2}(x^2+y^2+2)$

（当且仅当$\begin{cases}x=-y\\ \lambda x=-\frac{1}{\lambda}\\ \lambda y=\frac{1}{\lambda}\end{cases}$，且$2\cdot(-\frac{1+\lambda^2}{2})=\frac{\lambda^2-1}{\lambda^2}$即$\lambda^2=-1+\sqrt{2}$），$\therefore\frac{xy+x+y+1}{x^2+y^2+2}\geq-\frac{\sqrt{2}}{2}$

（5）①已知$a,b,c\in R^+$，且$a(3a+4b+2c)=4-\frac{8}{3}bc$，则$3a+2b+c$的最小值为(　　) C

A. $3\sqrt{2}$　　B. $2\sqrt{2}$　　C. $2\sqrt{3}$　　D. $4\sqrt{3}$

②已知$x,y,z>0$，且$xyz(x+y+z)=1$，则$(x+z)\cdot(x+y)_{\min}=$______________.2

（6）①若$a>0$，则$(a^4+3a^2+\frac{1}{a^2})_{\min}=$______.

$key1: a^4+3a^2+\frac{1}{a^2}=a^4-\lambda a^2+(3+\lambda)a^2+\frac{1}{a^2}\geq-\frac{\lambda^2}{4}+2\sqrt{3+\lambda}=\frac{15}{4}$（其中$\frac{\lambda}{2}=a^2=\frac{1}{\sqrt{3+\lambda}}$即$\lambda=1$）

$key2: a^4+3a^2+\frac{1}{a^2}=a^4+\underbrace{\frac{3}{n}a^2+\cdots+\frac{3}{n}a^2}_{n个}+\underbrace{\frac{1}{(n+2)a^2}+\cdots+\frac{1}{(n+2)a^2}}_{n+2个}$

$\geq(2n+3)\sqrt[2n+3]{(\frac{3}{n})^n(\frac{1}{n+2})^{n+2}}=\frac{15}{4}$（当且仅当$a^4=\frac{3a^2}{n}=\frac{1}{(n+2)a^2}$即$n=6$，且$a=\frac{1}{\sqrt{2}}$时，取$=$）

②已知$x,y\in R$，且$x+y=3$，则$\sqrt{x^2+1}+2\sqrt{y^2+4}$的最小值为___.

key：由$\sqrt{x^2+1}\cdot\sqrt{\lambda^2+1}\geq\lambda x+1$得$\sqrt{x^2+1}\geq\frac{\lambda}{\sqrt{\lambda^2+1}}x+\frac{1}{\sqrt{\lambda^2+1}}$；

由$\sqrt{y^2+4}\cdot\sqrt{\mu^2+1}\geq\mu y+2$得$2\sqrt{y^2+4}\geq\frac{2\mu}{\sqrt{\mu^2+1}}y+\frac{4}{\sqrt{\mu^2+1}}$

$\therefore\sqrt{x^2+1}+2\sqrt{y^2+4}\geq\frac{\lambda}{\sqrt{\lambda^2+1}}x+\frac{1}{\sqrt{\lambda^2+1}}+\frac{2\mu}{\sqrt{\mu^2+1}}y+\frac{4}{\sqrt{\mu^2+1}}$

$$=\frac{2}{\sqrt{5}}(x+y)+\frac{9}{\sqrt{5}}=3\sqrt{5}\text{（其中}\begin{cases}\dfrac{\lambda}{\sqrt{\lambda^2+1}}=\dfrac{2\mu}{\sqrt{\mu^2+1}}\\ \dfrac{x}{\lambda}=\dfrac{1}{1}\text{，且}\dfrac{y}{\mu}=2\quad\text{即}\mu=\dfrac{1}{2},\lambda=2\text{）}\\ x+y=\lambda+2\mu=3\end{cases}$$

③已知实数x,y满足$x^2+y^2=1$,且$x,y\in(0,1)$,则当$\frac{4}{x}+\frac{1}{y}$取最小值时,$\frac{x}{y}=$__

$key:\sqrt{1+\lambda^2}=\sqrt{(x^2+y^2)(1+\lambda^2)}\ge x+\lambda y$（当且仅当$\frac{x}{1}=\frac{y}{\lambda}$时取＝），

$$\therefore\sqrt{1+\lambda^2}\cdot(\frac{4}{x}+\frac{1}{y})\ge(x+\lambda y)(\frac{4}{x}+\frac{1}{y})=(x+\lambda y)(\frac{4}{x}+\frac{\lambda}{\lambda y})\ge(2+\sqrt{\lambda})^2$$

$$\therefore\frac{4}{x}+\frac{1}{y}\ge\frac{(2+\sqrt{\lambda})^2}{\sqrt{1+\lambda^2}}\text{（当且仅当}\begin{cases}\dfrac{4}{x^2}=\dfrac{1}{\lambda y^2}\\ \dfrac{x}{1}=\dfrac{y}{\lambda}\\ x^2+y^2=1\end{cases}\text{即}\begin{cases}\dfrac{y}{x}=\lambda=2^{-\frac{2}{3}}\\ y=\lambda x=\dfrac{\lambda}{\sqrt{1+\lambda^2}}\end{cases}\text{取＝）}$$

练习 1.若$a,b\in R$，下列等式不可能成立有（ C ）个.

①$\frac{a}{b}+\frac{b}{a}=1$;②$\sqrt{a^2+b^2}=|a|+|b|-1$;③$\frac{|a|^3+2}{|a|}=\sqrt{4-b^2}+b$.

A. 0　B. 1　C. 2　D. 3

2.设$a\in R$,若$a\sqrt{x}+\sqrt{1+x}\le1$对$x\ge0$恒成立,则a的最大值为 （ C）

A. -2　B. $-\frac{3}{2}$　C. -1　D. $-\frac{1}{2}$

3.若正实数x、y满足$x-2\sqrt{y}=\sqrt{2x-y}$, 则x的取值范围是 （C ）

A.$[4,20]$　B.$[16,20]$　C.$(2,10]$　*D.* $(2,2\sqrt{5}]$

4．已知$x>0$，则$(x+\frac{9}{x}-3)\cdot(x+\frac{25}{x}+5)$的最小值为（ B ）

A．$12\sqrt{15}$　B．14　C．$\frac{793}{16}$　D．45

$key:(x+\frac{9}{x}-3)\cdot(x+\frac{25}{x}+5)=x^2+\frac{225}{x^2}+2(x-\frac{15}{x})-15$（令$t=x-\frac{15}{x}\in R$）

$=t^2+2t+15=(t+1)^2+14\ge14$

5.设$a>b>c>0$，则$2a^2+\frac{1}{ab}+\frac{1}{a(a-b)}-10ac+25c^2$的最小值是（ B ）A.2　B.4　C. $2\sqrt{5}$　D.5

$$key:2a^2+\frac{1}{ab}+\frac{1}{a(a-b)}-10ac+25c^2=a^2+\frac{a-b+b}{ab(a-b)}+(a-5c)^2\ge a^2+\frac{1}{(\frac{b+a-b}{2})^2}=a^2+\frac{4}{a^2}\ge4$$

6.已知正实数a,b满足$a+b=1$，则$\dfrac{2a}{a^2+b}+\dfrac{b}{a+b^2}$的最大值为（　）$A.2$　$B.1+\sqrt{2}$　$C.1+\dfrac{2\sqrt{3}}{3}$　$D.1+\dfrac{3\sqrt{2}}{2}$

$key:\dfrac{2a}{a^2+b}+\dfrac{b}{a+b^2}=\dfrac{2a}{a^2-a+1}+\dfrac{1-a}{a^2-a+1}=\dfrac{1+a}{a^2-a+1}(t=a+1>1)$

$=\dfrac{t}{t^2-3t+3}=\dfrac{1}{t+\dfrac{3}{t}-3}\le\dfrac{1}{2\sqrt{3}-3}$

7.（多选题）已知$x,y>0,x+2y+xy-6=0$，则（　BC　）

A. xy的最大值为$\sqrt{2}$　　　　B. $x+2y$的最小值为4

C. $x+y$的最小值为$4\sqrt{2}-3$　　　D. $(x+2)^2+(y+1)^2$的最小值为1

8. 已知实数x,y满足$(2x-y)^2+4y^2=1$，则$2x+y$的最大值为____$\sqrt{2}$

9.已知实数a,b满足$a\ge\dfrac{1}{2}$，且$a+|b|\le1$，则$\dfrac{1}{2a}+b$的取值范围是__________.　$[\sqrt{2}-1,\dfrac{3}{2}]$

10.已知$a>0,b>0$，且$\dfrac{1}{a}+\dfrac{4}{b}=2$.则$(a+b)_{\min}=$_____;$(a+1)(b+4)_{\min}=$_____

$key:a+b=\dfrac{1}{\frac{1}{a}}+\dfrac{4}{\frac{4}{b}}\ge\dfrac{(1+2)^2}{\frac{1}{a}+\frac{4}{b}}=\dfrac{9}{2}$

$(a+1)(b+4)=ab+4a+b+4=\dfrac{4a+b}{2}+4a+b+4=\dfrac{3}{2}(\dfrac{4}{\frac{1}{a}}+\dfrac{4}{\frac{4}{b}})+4\ge\dfrac{3}{2}\cdot\dfrac{(2+2)^2}{2}+4=16$

11.已知$x,y\in R$，且$xy-3=x+y,x>1$，则$y(x+8)_{\min}=$_____.

$key:(x-1)(y-1)=4,\therefore y(x+8)=xy+8y=x+9y+3=(x-1)+9(y-1)+13\ge25$

12.已知$a>0,b>0$，且$a+3b=\dfrac{1}{b}-\dfrac{1}{a}$，则$b$的最大值为_________.

$key:\dfrac{1}{b}-3b=a+\dfrac{1}{a}\ge2$得$b\le\dfrac{1}{3}$

13.已知$x,y\in R$，且满足$4x+y+2xy+1=0$，则x^2+y^2+x+4y的最小值是_______.$-\dfrac{13}{4}$

14.已知$a,b\in R$，且满足$2ab-4a+3b-8=0$，则$a^2+2b^2+3a-8b$的最小值是_____.$2\sqrt{2}-\dfrac{41}{4}$

15.已知$x>0,y>-1$，且$x+y=1$，则$\dfrac{x^2+3}{x}+\dfrac{y^2}{y+1}$最小值为__________.　$2+\sqrt{3}$

16.已知$f(x)=ax^2+bx+c(0<2a<b),\forall x\in R,f(x)\ge0$恒成立，则$\dfrac{f(1)}{f(0)-f(-1)}$的最小值为_________.

$key:\Rightarrow\Delta=b^2-4ac\le0(b>2a>0),\therefore\dfrac{f(1)}{f(0)-f(-1)}=\dfrac{a+b+c}{c-(a-b+c)}=\dfrac{a+b+c}{b-a}$

$\ge\dfrac{a+b+\dfrac{b^2}{4a}}{b-a}=\dfrac{(2a+b)^2}{4a(b-a)}=\dfrac{(b-a+3a)^2}{4a(b-a)}=\dfrac{1}{4}(\dfrac{b-a}{a}+\dfrac{9a}{b-a}+6)\ge3$

17.已知$xy-z=0$,且$0<\frac{y}{z}<\frac{1}{2}$，则$\frac{x^2z^2+16y^2}{xz^2-4yz}$的最小值为____． $2\sqrt{2}$

18.已知正实数 x,y,z 满足 $2x(x+\frac{1}{y}+\frac{1}{z})=yz$，则 $(x+\frac{1}{y})(x+\frac{1}{z})$ 的最小值为_______ $\sqrt{2}$

$key:(x+\frac{1}{y})(x+\frac{1}{z})=(x+\frac{1}{y})x+(x+\frac{1}{y})\cdot\frac{1}{z}=x(x+\frac{1}{y}+\frac{1}{z})+\frac{1}{yz}\ge 2\sqrt{x(x+\frac{1}{y}+\frac{1}{z})\cdot\frac{1}{yz}}=\sqrt{2}$

19.已知$a>0,b>0$，则$(\frac{2a+b}{\sqrt{a^2+b^2}})_{\max}=$__________；$\sqrt{5}$

20.已知$x>0,y>0$，且$a(x+y)\ge x+\sqrt{\frac{1}{2}xy}$恒成立，则$a_{\min}=$____．$\frac{\sqrt{3}+1}{2}$

$key:a\ge\frac{x+\sqrt{\frac{1}{2}xy}}{x+y}\le\frac{x+2\cdot\sqrt{\frac{1}{2}\lambda x}\cdot\sqrt{\frac{y}{\lambda}}}{x+y}\le\frac{(1+\frac{1}{2}\lambda)x+\frac{1}{\lambda}y}{x+y}=\frac{\sqrt{3}+1}{2}$（其中$1+\frac{1}{2}\lambda=\frac{1}{\lambda}$即$\lambda=\sqrt{3}-1$)

21.设正数 a,b 满足,$a+\frac{1}{a}+3(b+\frac{1}{b})=16$,则 $a+3b$的最大值为______；$\frac{a}{b}+\frac{b}{a}$ 的最大值是______.$8+4\sqrt{5}$,18.

（2013 竞赛）若$a>0,b>0$,则$\min\{\max(a,b,\frac{1}{a^2}+\frac{1}{b^2})\}=$____________． $\sqrt[3]{2}$

（2006 年竞赛） $\max\limits_{a,b,c\in R^+}\min\{\frac{1}{a},\frac{1}{b^2},\frac{1}{c^3},a+b^2+c^3\}=$_________． $\sqrt{3}$

(2018山东)7.$\forall a,b\in R$,则$\min\{\max|a+b|,|a-b|,|1-a|\}=$_____.

key:设$M=\max\{|a+b|,|a-b|,|1-a|\}$,则$4M\ge|a+b|+|a-b|+2|1-a|$

$\ge|2a|+|2-2a|\ge 2,\therefore M_{\min}=\frac{1}{2}$

(2018山东)7.key:设$M=\max\{|a+b|,|a-b|,|1-a|\}$,

则$4M\ge|a+b|+|a-b|+2|1-a|\ge|a+b+a-b+2-2a|=2,\therefore M\ge\frac{1}{2}$

（2018 山东）13.实数a,b,c满足$a^2+b^2+c^2=\lambda(\lambda>0)$，试求$f=\min\{(a-b)^2,(b-c)^2,(c-a)^2\}$的最大值.

(2018山东13）不妨设$a\ge b\ge c$,则$a-c\ge a-b\ge 0,a-c\ge b-c\ge 0$,

且$(a-c)^2=(a-b+b-c)^2\ge(2\sqrt{(a-b)(b-c)})^2\ge 4f$

而$6f\le(a-b)^2+(b-c)^2+(c-a)^2=2(a^2+b^2+c^2)-(2ab+2bc+2ca)=2\lambda-[(a+b+c)^2-(a^2+b^2+c^2)]$

$=3\lambda-(a+b+c)^2\le 3\lambda,\therefore f\le\frac{1}{2}$

变式 1（1）已知$a,b,c>0$，则三个数$a+\frac{1}{b},b+\frac{1}{c},c+\frac{1}{a}$满足（ ）

A.都不大于2　B.都不小于2　C.至少有一个不大于2　D.至少有一个不小于2

$key:a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}=a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}\ge 6,\therefore$选$D$

（2）已知$a,b,c\in(0,1)$，记$M=\min\{(1-a)b,(1-b)c,(1-c)a\}$，则$M_{\max}=$______

$key1:\sqrt{M}\le\sqrt{(1-a)b}=\frac{1-a+b}{2},\sqrt{M}\le\sqrt{(1-b)c}\le\frac{1-a+b}{2},\sqrt{M}\le\sqrt{(1-c)a}\le\frac{1-c+a}{2},\therefore 3\sqrt{M}\le\frac{3}{2},\therefore M\le\frac{1}{4}$

$key2:M^3\le(1-a)a\cdot(1-b)b\cdot(1-c)c\le(\frac{1-a+a}{2})^2\cdot(\frac{1-b+b}{2})^2\cdot(\frac{1-c+c}{2})^2=\frac{1}{4^3},\therefore M\le\frac{1}{4}$

2（1）①已知$x,y>0,$则$\max\{\frac{3xy+y^2}{x^2},\frac{x^2+9xy}{9y^2}\}$的最小值为____________. $\sqrt{3}+\frac{1}{3}$

②已知 $x>0,y>0$，则 $\max\{\min\{2x,\frac{3y}{2x^2+3y^2}\}\}=$____________. $\sqrt[4]{\frac{3}{2}}$

$key:m=\min\{2x,\frac{3y}{2x^2+3y^2}\}\Rightarrow m^2\le\frac{6xy}{2x^2+3y^2}\le\frac{6xy}{2\sqrt{6}xy}=\sqrt{\frac{3}{2}}\Rightarrow\max[\min(2x,\frac{3y}{2x^2+3y^2})]\le\sqrt[4]{\frac{3}{2}}$

③记$\max\{x,y,z\}$表示x,y,z中的最大数，若$a>0,b>0,$则$\max\{a,b,\frac{1}{a}+\frac{3}{b}\}$的最小值为（ ） C

$A.\sqrt{2}$　　$B.\sqrt{3}$　　$C.2$　　$D.3$

$key:$设$M=\max\{a,b,\frac{1}{a}+\frac{3}{b}\},$（基于$a=b=2$时$\frac{1}{a}+\frac{3}{b}=2$的考量）

$\therefore 2M=\frac{1}{4}M+\frac{3}{4}M+M\ge\frac{1}{4}a+\frac{3}{4}b+\frac{1}{a}+\frac{3}{b}\ge1+3=4,\therefore M\ge2$

$key2:a\le M,b\le M,\therefore M\ge\frac{1}{a}+\frac{3}{b}\ge\frac{1}{M}+\frac{3}{M}$

（2）①已知 a,b,c 均为正实数，记 $M=\max\{\frac{1}{ac}+b,\frac{1}{a}+bc,\frac{a}{b}+c\}$，则 M 的最小值为__________.2

②已知实数x,y满足$x+y+z=1$，则$\max\{\min\{\frac{1}{2}x+y+\frac{2}{3}z,x+2y+\frac{1}{3}z,x-2y+2z\}\}=$________.

$key:$设$m=\min\{\frac{1}{2}x+y+\frac{2}{3}z,x+2y+\frac{1}{3}z,x-2y+2z\},$则

$(1+\lambda+\mu)m\le\frac{1}{2}x+y+\frac{2}{3}z+\lambda(x+2y+\frac{1}{3}z)+\mu(x-2y+2z)$

$=(\frac{1}{2}+\lambda+\mu)x+(1+2\lambda-2\mu)y+(\frac{2}{3}+\frac{1}{3}\lambda+2\mu)z$(其中$\frac{1}{2}+\lambda+\mu=1+2\lambda-2\mu=\frac{2}{3}+\frac{1}{3}\lambda+2\mu$即$\lambda=1,\mu=\frac{1}{2}$)

$=2(x+y+z)=2,\therefore\frac{5}{2}m\le2$即$m\le\frac{4}{5}$

（3） 设$a>0,b>0$，记m是$\frac{1}{a},\frac{1}{b},a^2+b^2-1$三者中的最大值，则$m$的最小值是______.

$$key1:m^2(m+1)\ge\frac{1}{ab}\cdot(a^2+b^2)\ge2\Rightarrow m\ge1;key2:\begin{cases}m\ge\frac{1}{a}\Rightarrow a\ge\frac{1}{m}\\ m\ge\frac{1}{b}\Rightarrow b\ge\frac{1}{m}\Rightarrow m\ge\frac{2}{m^2}-1\Rightarrow m\ge1\\ m\ge a^2+b^2-1\end{cases}$$