解析几何（2）椭圆解答（4）

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（三）弦中点问题

（2008江西）过点$P(1,1)$作直线l，使得它被椭圆$\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$所截出的弦的中点恰为$P$，则直线$l$的方程为____.

$$key:\begin{cases}\dfrac{x_A^2}{9}+\dfrac{y_A^2}{4}=1\\ \dfrac{x_B^2}{9}+\dfrac{y_B^2}{4}=1\\ x_A+x_B=2\\ y_A+y_B=2\end{cases},\therefore \frac{(x_A-x_B)\cdot 2}{9}+\frac{(y_A-y_B)\cdot 2}{4}=0,\therefore y-1=-\frac{4}{9}(x-1)\text{即}4x+9y=13$$

（2013 湖北） 设$P(x_0,y_0)$为椭圆$\dfrac{x^2}{4}+y^2=1$内一定点（不在坐标轴上），过点P的两条直线分别与椭圆交于A、C和B、D，若$AB//CD$.（1）证明：直线AB的斜率为定值；

（2）过点P作AB的平行线，与椭圆交于E、F两点，证明：点P平分线段EF.

证明（1）$\because AB//CD,\therefore \dfrac{|AP|}{|PC|}=\dfrac{|PB|}{|PD|}=\lambda>0$

$$\text{则}\begin{cases}x_0-x_A=\lambda(x_C-x_0)\\ y_0-y_A=\lambda(y_C-y_0)\end{cases},\therefore\begin{cases}\dfrac{x_A^2}{4}+y_A^2=\dfrac{[(1+\lambda)x_0-\lambda x_C]^2}{4}+[(1+\lambda)y_0-\lambda y_C]^2=1\\ \dfrac{x_C^2}{4}+y_C^2=1\text{即}\dfrac{(\lambda x_C)^2}{4}+(\lambda y_C)^2=\lambda^2\end{cases}\quad\text{得}2-\frac{x_0}{2}x_C-2y_0y_C=0$$

同理$2-\dfrac{x_0}{2}x_D-2y_0y_D=0,\therefore \dfrac{x_0}{2}(x_C-x_D)+2y_0(y_C-y_D)=0,\therefore \dfrac{y_C-y_D}{x_C-x_D}=-\dfrac{x_0}{4y_0}$

（2）由（1）得$l_{EF}:y-y_0=-\dfrac{x_0}{4y_0}(x-x_0)$代入椭圆方程得:$(1+\dfrac{x_0^2}{4y_0^2})x^2-\dfrac{x_0(x_0^2+4y_0^2)}{2y_0^2}x+\dfrac{(x_0^2+4y_0^2)^2}{4y_0^2}-4=0$

$$\therefore \frac{x_E+x_F}{2}=\frac{\dfrac{x_0(x_0^2+4y_0^2)}{2y_0^2}}{2(1+\dfrac{x_0^2}{4y_0^2})}=x_0,\therefore P\text{是}EF\text{的中点}$$

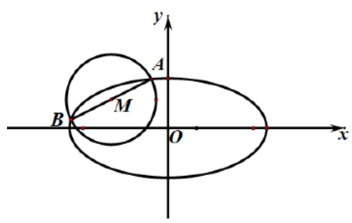
$$key2:(\text{同一法})\frac{(x_E-x_F)\cdot 2x}{4}+(y_E-y_F)\cdot 2y=0,\therefore \frac{x}{4y}=-\frac{y_E-y_F}{x_E-x_F}=\frac{x_0}{4y_0},\therefore (x,y)=(x_0,y_0),\text{得证}$$

（2015陕西）已知椭圆$E:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的半焦距为$c$，原点$O$到经过两点$(c,0),(0,b)$的直线的距离为$\dfrac{1}{2}c$.

①则椭圆E的离心率为___________；

②如图，AB是圆$M:(x+2)^2+(y-1)^2=\dfrac{5}{2}$的一条直径，若椭圆$E$经过$A$、$B$两点，则椭圆$E$的方程为_____.

① $\dfrac{bc}{a}=\dfrac{1}{2}c$得$a=2b,\therefore e=\dfrac{\sqrt{3}}{2}$



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②$key1:\begin{cases}x_A^2+4y_A^2=a^2\cdots①\\x_B^2+4y_B^2=a^2\cdots②\\x_A+x_B=-4\\y_A+y_B=2\\(x_A-x_B)^2+(y_A-y_B)^2=10\end{cases}$，$\therefore$ ①-②得：$-4(x_A-x_B)+8(y_A-y_B)=0$即$x_A-x_B=2(y_A-y_B)$

$\therefore (y_A-y_B)^2=2,(x_A-x_B)^2=8,\therefore (x_A+x_B)^2+(x_A-x_B)^2+4[(y_A+y_B)^2+(y_A-y_B)^2]$

$=2(x_A^2+x_B^2)+8(y_A^2+y_B^2)=4a^2=16+8+4(4+2)=48$得$a^2=12,\therefore E$方程为：$\dfrac{x^2}{12}+\dfrac{y^2}{3}=1$

$key2$：$A(-2+\sqrt{\dfrac{5}{2}}\cos\alpha,1+\sqrt{\dfrac{5}{2}}\sin\alpha),B(-2-\sqrt{\dfrac{5}{2}}\cos\alpha,1-\sqrt{\dfrac{5}{2}}\sin\alpha)$

则$\begin{cases}(-2+\sqrt{\dfrac{5}{2}}\cos\alpha)^2+4(1+\sqrt{\dfrac{5}{2}}\sin\alpha)^2=a^2\\(-2-\sqrt{\dfrac{5}{2}}\cos\alpha)^2+4(1-\sqrt{\dfrac{5}{2}}\sin\alpha)^2=a^2\end{cases}$，$\therefore \tan\alpha=\dfrac{1}{2}$，且$2a^2=21+15\sin^2\alpha=24,\therefore a^2=12,b^2=3$

（2006陕西）若a,b,c成等差数列，则直线$ax+by+c=0$被椭圆$\dfrac{x^2}{2}+\dfrac{y^2}{8}=1$截得线段的中点的轨迹方程为________.

$key:l:a(2x+y)+c(y+2)=0$经过定点$(1,-2)$

$\therefore\begin{cases}\dfrac{x_1^2}{2}+\dfrac{y_1^2}{8}=1\\\dfrac{x_2^2}{2}+\dfrac{y_2^2}{8}=1\end{cases}$得$\dfrac{2x\cdot(x_1-x_2)}{2}+\dfrac{2y\cdot(y_1-y_2)}{8}=0$即$\dfrac{y_1-y_2}{x_1-x_2}=-\dfrac{4x}{y}$

由$\dfrac{y_1-y_2}{x_1-x_2}=\dfrac{y+2}{x-1}=-\dfrac{4x}{y}$即$4x^2+y^2-4x+2y=0$即$2(x-\dfrac{1}{2})^2+\dfrac{(y+1)^2}{2}=1$（除去点$(1,-2)$）

（2010江苏）直角坐标系xOy中，设A、B、M是椭圆$C:\dfrac{x^2}{4}+y^2=1$上三点.若$\overrightarrow{OM}=\dfrac{3}{5}\overrightarrow{OA}+\dfrac{4}{5}\overrightarrow{OB}$,

证明：线段AB的中点在椭圆$\dfrac{x^2}{2}+2y^2=1$上.

2013广东解：设$N(x,y)$,由已知得$\begin{cases}x_A^2+4y_B^2=4\\x_B^2+4y_B^2=4\\x_M^2+4y_M^2=4\\x_M=\dfrac{3}{5}x_A+\dfrac{4}{5}x_B\cdots①\\y_M=\dfrac{3}{5}y_A+\dfrac{4}{5}y_B\cdots②\\x_A+x_B=2x\cdots③\\y_A+y_B=2y\cdots④\end{cases}$

由①②得$1=\frac{1}{25}(9x_A^2+24x_Ax_B+16x_B^2)+\frac{4}{25}(9y_A^2+24y_Ay_B+16y_B^2)=1+\frac{24}{25}(x_Ax_B+4y_Ay_B)$即$x_Ax_B+4y_Ay_B=0$

由③④得$4x^2+16y^2=(x_A+x_B)^2+4(y_A+y_B)^2=8+2(x_Ax_B+4y_Ay_B)=8$即$\frac{x^2}{2}+2y^2=1$, 证毕

（2005Ⅰ）已知椭圆的中心为坐标原点O，焦点在x轴上，斜率为1且过椭圆右焦点F的直线交椭圆于A、B两点，$\overrightarrow{OA}+\overrightarrow{OB}$与$\vec{a}=(3,-1)$共线.⑴ 求椭圆的离心率；

（2）设M为椭圆上任意一点，且$\overrightarrow{OM}=\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}(\lambda,\mu\in R)$,证明：$\lambda^2+\mu^2$为定值.

2005Ⅰ ⑴ 设椭圆方程为$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$, 右焦点$F(c,0)$

设$l_{AB}:x=y+c$代入椭圆方程得:$(a^2+b^2)y^2+2b^2cy-b^4=0$

$\therefore\overrightarrow{OA}+\overrightarrow{OB}=(x_A+x_B,y_A+y_B)=(-\frac{2b^2c}{a^2+b^2}+2c,-\frac{2b^2c}{a^2+b^2})//\vec{a}=(3,-1)$

$\therefore a^2=3b^2$得$e=\frac{\sqrt6}{3}$

（2）由 ⑴ 得$\begin{cases}y_A+y_B=-\frac{\sqrt2}{2}b\\ y_Ay_B=-\frac{b^2}{4}\end{cases}$, 且$a=\sqrt3b,c=\sqrt2b,$

且$\begin{cases}x_A^2+3y_A^2=a^2\\ x_B^2+3y_B^2=a^2\\ (\lambda x_A+\mu x_B)^2+3(\lambda y_A+\mu y_B)^2=a^2=\lambda^2+\mu^2+\lambda\mu(x_Ax_B+3y_Ay_B)=a^2\end{cases}$

而$x_Ax_B+3y_Ay_B=(y_A+c)(y_B+c)+3y_Ay_B=4\cdot\frac{-b^2}{4}+\sqrt2b\cdot\frac{-\sqrt2b}{2}+2b^2=0,$

$\therefore\lambda^2+\mu^2=a^2$为定值.

（2015Ⅱ）已知椭圆$C:9x^2+y^2=m^2(m>0)$，直线l不过原点O且不平行于坐标轴，l与C有两个交点A、B，线段AB的中点为M.⑴ 证明:直线OM的斜率与l的斜率的乘积为定值；

（2）若l过点$(\frac{m}{3},m)$，延长线段OM与C交于点P，四边形$OAPB$能否为平行四边形？若能，求此时l的斜率;若不能，说明理由.

⑴ 证明：设$l:y=kx+t(kt\neq0)$,代入C方程得:$(9+k^2)x^2+2ktx+t^2-m^2=0$

$\therefore\begin{cases}x_A+x_B=-\frac{2kt}{k^2+9}\\ x_Ax_B=\frac{t^2-m^2}{k^2+9}\end{cases}$, 且$\Delta=4(-9t^2+9m^2+k^2m^2)>0,\therefore M(-\frac{kt}{k^2+9},\frac{9t}{k^2+9}),$

$\therefore k_{OM}\cdot k=-\frac{9}{k}\cdot k=-9$为定值，证毕

（2）假设能，则OP的中点也为M，

$\therefore$由 ⑴ 得:$m=\frac{km}{3}+t,$且$9(-\frac{2kt}{k^2+9})^2+(\frac{18t}{k^2+9})^2=m^2$

$\Leftrightarrow 36t^2(k^2+9)\cdot\dfrac{1}{(k^2+9)^2}=36m^2(1-\dfrac{k}{3})^2\cdot\dfrac{1}{k^2+9}=m^2\Leftrightarrow k=4\pm\sqrt{7}$

$\therefore OAPB$能为平行四边形，此时l的斜率为$4\pm\sqrt{7}$

（2017贵州）如图，已知$\triangle ABC$的三个顶点在椭圆$\dfrac{x^2}{12}+\dfrac{y^2}{4}=1$上，坐标原点$O$为$\triangle ABC$的重心，则$\triangle ABC$的面积为________.9

key:当$AB\not\perp x$轴时，设$l_{AB}:y=kx+m$代入椭圆方程得:$(1+3k^2)x^2+6kmx+3m^2-12=0$

$\therefore AB$的中点$M(\dfrac{-3km}{1+3k^2},\dfrac{m}{1+3k^2})$, 且$\Delta=12(4+12k^2-m^2)>0$,

$\therefore C(\dfrac{6km}{1+3k^2},-\dfrac{2m}{1+3k^2}),\therefore\dfrac{3k^2m^2}{(1+3k^2)^2}+\dfrac{m^2}{(1+3k^2)^2}=\dfrac{m^2}{1+3k^2}=1$

$\therefore S_{\triangle ABC}=3S_{\triangle ABO}=3\cdot\dfrac{1}{2}\sqrt{1+k^2}\cdot\dfrac{2\sqrt{3}\sqrt{4+12k^2-m^2}}{1+3k^2}\cdot\dfrac{|m|}{\sqrt{1+k^2}}=9$

当$AB\perp x$轴时, $S_{\triangle ABC}=9$

（2018山东）若直线$6x-5y-28=0$交椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$于点$A$、$C$.设$B(0,b)$为椭圆的上顶点，而$\triangle ABC$的重心为椭圆的右焦点，则椭圆的方程为_____. $\dfrac{x^2}{20}+\dfrac{y^2}{16}=1$

key:由已知的AB的中点$M(\dfrac{3c}{2},-\dfrac{b}{2}),\therefore 9c+\dfrac{5b}{2}-28=0,$

且$\dfrac{(x_A-x_B)\cdot 3c}{a^2}+\dfrac{(y_A-y_B)\cdot(-b)}{b^2}=0$即$\dfrac{3bc}{a^2}=\dfrac{y_A-y_B}{x_A-x_B}=\dfrac{6}{5}$得$b=2c=4$

（2019重庆）已知$\triangle ABC$为椭圆$\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$的内接三角形，且$AB$过点$P(1,0)$，则$\triangle ABC$的面积的最大值为_____.

key:设$l_{AB}:x=ty+1$代入椭圆方程得$(4t^2+9)y^2+8ty-32=0,\therefore\Delta=64(9t^2+18)$

由$x=ty+n$代入椭圆方程得$(4t^2+9)y^2+8tny+4n^2-36=0,\therefore\Delta_1=144(4t^2+9-n^2)=0$

$\therefore S_{\triangle ABC}\le\dfrac{1}{2}\cdot\sqrt{1+t^2}\cdot\dfrac{24\sqrt{t^2+2}}{4t^2+9}\cdot\dfrac{\sqrt{4t^2+9}+1}{\sqrt{1+t^2}}=12\cdot\dfrac{\sqrt{t^2+2}(\sqrt{4t^2+9}+1)}{4t^2+9}(u=\sqrt{4t^2+9}\ge 3)$

$=12\cdot\dfrac{\dfrac{\sqrt{u^2-1}}{2}\cdot(u+1)}{u^2}=6\cdot\dfrac{\sqrt{(u-1)(u+1)^3}}{u^2}=2\sqrt{3}\cdot\dfrac{\sqrt{3(u-1)\cdot(u+1)(u+1)(u+1)}}{u^2}$

$\le 2\sqrt{3}\cdot\dfrac{\sqrt{(\dfrac{6u}{4})^4}}{u^2}$（等号不成立），用导数得$\dfrac{16\sqrt{2}}{3}$

$key2$:（仿射变换）令$x'=x,y'=\dfrac{3}{2}y$得$x'^2+y'^2=9$

则$P'(1,0),\therefore S_{\triangle A'B'C'}\le\dfrac{1}{2}\cdot 4\sqrt{2}\cdot 4=8\sqrt{2},\therefore(S_{\triangle ABC})_{\max}=\dfrac{2}{3}\cdot 8\sqrt{2}=\dfrac{16\sqrt{2}}{3}$

（公式：$S_{\triangle ABC}=\dfrac{1}{2}\left\|\begin{matrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{matrix}\right\|=\dfrac{1}{2}\left\|\begin{matrix}\lambda x_1' & \mu y_1' & 1\\ \lambda x_2' & \mu y_2' & 1\\ \lambda x_3' & \mu y_3' & 1\end{matrix}\right\|=\lambda\mu S_{\triangle A'B'C'}$）

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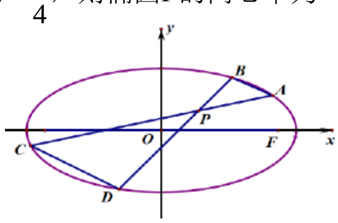
变式 1（1）已知椭圆$\Gamma:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$内有一定点$P(1,1)$，过点$P$的两条直线$l_1,l_2$分别与椭圆$\Gamma$交于

A、C和B、D且满足$\overrightarrow{AP}=\lambda\overrightarrow{PC},\overrightarrow{BP}=\lambda\overrightarrow{PD}$，若$\lambda$变化时，直线$CD$的斜率总为$-\dfrac{1}{4}$，则椭圆$\Gamma$的离心率为（ ）

$A.\dfrac{\sqrt{3}}{2}$ $B.\dfrac{1}{2}$ $C.\dfrac{\sqrt{2}}{2}$ $D.\dfrac{\sqrt{5}}{5}$



$key1$:由已知得$\triangle PAB\sim\triangle PCD,\therefore AB//CD,$

$\therefore$（平行弦的中点轨迹）AB,CD的中点轨迹经过点P,

$$\frac{x_A^2-x_B^2}{a^2}+\frac{y_A^2-y_B^2}{b^2}=\frac{(x_A-x_B)\cdot 2x}{a^2}+\frac{(y_A-y_B)\cdot 2y}{b^2}=0\text{即}\frac{x}{a^2}+(-\frac{1}{4})\frac{y}{b^2}=0,\therefore\frac{1}{a^2}-\frac{1}{4b^2}=0,\therefore e=\frac{\sqrt{3}}{2}$$

（2）已知椭圆$C:\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$,则椭圆$C$的长为2的弦的中点$M$的轨迹方程为____.

$key1$: 设$M(x,y)$,弦AB的端点$A(x_1,y_1),B(x_2,y_2)$,倾角为α,则弦端点$A(x+\cos\alpha,y+\sin\alpha),B(x+\cos\alpha,y+\sin\alpha)$

$$\text{则}\begin{cases}3(x+\cos\alpha)^2+4(y+\sin\alpha)^2=12\cdots①\\3(x-\cos\alpha)^2+4(y-\sin\alpha)^2=12\cdots②\end{cases}$$

①−②得:$3x\cos\alpha+4y\sin\alpha=0$即$\tan\alpha=-\dfrac{3x}{4y}$;①+②得:$3x^2+4y^2+3\cos^2\alpha+4\sin^2\alpha=12$

$$\therefore 3x^2+4y^2+\frac{9x^2}{9x^2+16y^2}=9$$

$$key2:\begin{cases}3x_P^2+4y_P^2=12\cdots①\\3x_Q^2+4y_Q^2=12\cdots②\\(x_P-x_Q)^2+(y_P-y_Q)^2=4\cdots③\text{，①−②得：}y_P-y_Q=-\dfrac{3x}{4y}(x_P-x_Q)\\x_P+x_Q=2x\\y_P+y_Q=2y\end{cases}$$

代入③得:$(x_P-x_Q)^2=\dfrac{64y^2}{9x^2+16y^2},(y_P-y_Q)^2=\dfrac{36x^2}{9x^2+16y^2}$,

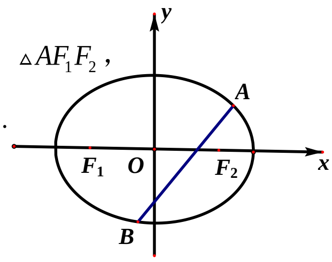
$$\therefore x_P^2+x_Q^2=2x^2+\frac{32y^2}{9x^2+16y^2},y_P^2+y_Q^2=2y^2+\frac{18x^2}{9x^2+16y^2}$$

①+②得:$24=3(x_P^2+x_Q^2)+4(y_P^2+y_Q^2)=6x^2+8y^2+\dfrac{96y^2}{9x^2+16y^2}+\dfrac{72x^2}{9x^2+16y^2}=6x^2+8y^2+6+\dfrac{18x^2}{9x^2+16y^2}$

即$3x^2+4y^2+\dfrac{9x^2}{9x^2+16y^2}=9$

（四）垂直、角及斜率问题

（201021）已知$m>1$，直线$l:x-my-\dfrac{m^2}{2}=0$，椭圆$C:\dfrac{x^2}{m^2}+y^2=1$，F_1,F_2分别为椭圆C的左、右焦点.



（I）当直线l过右焦点F_2时，求直线l的方程；（II）设直线l与椭圆C交于A,B两点，$\triangle AF_1F_2$，$\triangle BF_1F_2$的重心分别为G,H.若原点O在以线段GH为直径的圆内，求实数m的取值范围.

（Ⅰ）$x-\sqrt{2}y-1=0$;（Ⅱ）$(1,2)$

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（2016山西）设直线$y=x+\sqrt{2}$与椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$交于点$M$、$N$，且$OM\perp ON$（$O$为坐标原点），若$|MN|=\sqrt{6}$,则椭圆方程为______. $\frac{x^2}{4+2\sqrt{2}}+\frac{y^2}{4-2\sqrt{2}}=1$

（2022 乙）20. 已知椭圆$C:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\left(a>b>0\right)$的右顶点为$A$，上顶点为$B$，直线$AB$的斜率为$-\frac{\sqrt{3}}{2}$，原点$O$到直线$AB$的距离为$\frac{2\sqrt{21}}{7}$.（1）求$C$的方程；（2）直线$l$交$C$于$M$，$N$两点，$\angle MBN=90°$，证明：$l$恒过定点.

(1) 解：由已知得$\begin{cases}\frac{b}{a}=\frac{\sqrt{3}}{2}\\ \frac{ab}{\sqrt{a^2+b^2}}=\frac{2\sqrt{21}}{7}\end{cases}$得$a=2,b=\sqrt{3},\therefore C$的方程为$\frac{x^2}{4}+\frac{y^2}{3}=1$

(2) 证明：设$l_{MN}:y=kx+m$代入C得:$(3+4k^2)x^2+8kmx+4m^2-12=0$

$\therefore\begin{cases}x_M+x_N=-\frac{8km}{3+4k^2}\\ x_Mx_N=\frac{4m^2-12}{3+4k^2}\end{cases}$,且$\Delta=48(3+4k^2-m^2)>0$

$\because\angle MBN=90°,\therefore\overrightarrow{BM}\cdot\overrightarrow{BN}=x_Mx_N+(kx_M+m-\sqrt{3})(kx_N+m-\sqrt{3})=(1+k^2)x_Mx_N+k(m-\sqrt{3})(x_M+x_N)+m^2$

$=\frac{(4m^2-12)(1+k^2)}{3+4k^2}+\frac{-8k^2m(m-\sqrt{3})}{3+4k^2}+\frac{(m-\sqrt{3})^2(3+4k^2)}{3+4k^2}=0$

$\Leftrightarrow m=-\frac{\sqrt{3}}{7},or,m=\sqrt{3}$(舍去），$\therefore l$经过定点$(0,-\frac{\sqrt{3}}{7})$

变式 1（1）已知点P、Q在椭圆$C:\frac{x^2}{3}+\frac{y^2}{2}$上，且点$A(0,2)$，若$AP\perp AQ$，则$AP$的斜率的取值范围为_______.

key:设$l_{AP}:y=kx+2$代入C方程得$(2+3k^2)x^2+12kx+6=0$,

$\therefore\Delta=144k^2-24(2+3k^2)\geq 0$得$3k^2-2\geq 0$,且$\frac{3}{k^2}-2\geq 0$得$k\in[-\frac{\sqrt{6}}{2},-\frac{\sqrt{6}}{3}]\cup[\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{2}]$

（ 2 ） 已知椭圆的中心为原点O,焦点在坐标轴上，直线$y=x+1$与此椭圆交于点P、Q，且$OP\perp OQ$，$|PQ|=\frac{\sqrt{10}}{2}$,则此椭圆的方程为___. $\frac{1}{2}x^2+\frac{3}{2}y^2=1,or,\frac{3}{2}x^2+\frac{1}{2}y^2=1$

（3）已知椭圆$E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$的右焦点$F$，上顶点$B$，$PQ$为椭圆$E$的弦，$O$为坐标原点.

若弦PQ过F，且$OP\perp OQ$.则椭圆E的离心率的取值范围为_____; $[\frac{\sqrt{5}-1}{2},1)$

$key1$:当P,Q为长短轴端点时,$|OP|\cdot|OQ|=ab$;

当P,Q均不为长短轴端点时，设$l_{OP}:y=kx$代入C得$x_P^2=\frac{a^2b^2}{a^2k^2+b^2}$

$\therefore|OP|=\sqrt{1+k^2}\cdot\frac{ab}{\sqrt{a^2k^2+b^2}}$,同理$|OQ|=\sqrt{1+\frac{1}{k^2}}\cdot\frac{ab}{\sqrt{\frac{a^2}{k^2}+b^2}}=\frac{ab\sqrt{1+k^2}}{\sqrt{a^2+b^2k^2}}$

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$\therefore \dfrac{1}{|OP|^2}+\dfrac{1}{OQ|^2}=\dfrac{a^2k^2+b^2}{a^2b^2(1+k^2)}+\dfrac{a^2+b^2k^2}{a^2b^2(1+k^2)}=\dfrac{1}{a^2}+\dfrac{1}{b^2}$,

$key2$: 设$P(s,t)$，则$Q(\lambda t,-\lambda s),\therefore \begin{cases}\dfrac{s^2}{a^2}+\dfrac{t^2}{b^2}=1\\ \dfrac{\lambda^2t^2}{a^2}+\dfrac{\lambda^2s^2}{b^2}=1即\dfrac{t^2}{a^2}+\dfrac{s^2}{b^2}=\dfrac{1}{\lambda^2}\end{cases},\therefore (s^2+t^2)(\dfrac{1}{a^2}+\dfrac{1}{b^2})=1+\dfrac{1}{\lambda^2}$

$\therefore\therefore \dfrac{1}{|OP|^2}+\dfrac{1}{OQ|^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}$

$key3: |OP|=p,|OQ|=q$, 则$P(p\cos\theta,p\sin\theta),Q(q\cos(\dfrac{\pi}{2}+\theta),q\sin(\dfrac{\pi}{2}+\theta))$即$(-q\sin\theta,q\cos\theta)$

$\therefore \begin{cases}\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}=\dfrac{1}{p^2}\\ \dfrac{\sin^2\theta}{a^2}+\dfrac{\cos^2\theta}{b^2}=\dfrac{1}{q^2}\end{cases},\therefore \dfrac{1}{p^2}+\dfrac{1}{q^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}$

$\therefore d_{O\to PQ}=\dfrac{ab}{\sqrt{a^2+b^2}}\le c\Leftrightarrow a^2(a^2-c^2)\le c^2(2a^2-c^2)\Leftrightarrow 1-e^2\le e^2(2-e^2)$得$e\in[\dfrac{\sqrt5-1}{2},1)$

若$\triangle BPQ$是以B为直角顶点的等腰直角三角形，且PQ与y轴不垂直.则椭圆E的离心率 的取值范围为__；

key: 设$BP: y=kx+b(k>0)$代入E得:$(a^2k^2+b^2)x^2+2a^2bkx=0,\therefore |BP|=\sqrt{1+k^2}\cdot\dfrac{2a^2bk}{a^2k^2+b^2}$,

同理$|BQ|=\sqrt{1+\dfrac{1}{k^2}}\cdot\dfrac{2a^2b\cdot\dfrac{1}{k}}{a^2\cdot\dfrac{1}{k^2}+b^2}=\sqrt{1+k^2}\cdot\dfrac{2a^2b}{a^2+b^2k^2}=\sqrt{1+k^2}\cdot\dfrac{2a^2bk}{a^2k^2+b^2}$即$\dfrac{a^2}{b^2}=k+\dfrac{1}{k}+1>3$得$e\in(\dfrac{\sqrt6}{3},1)$