# 解析几何（2）椭圆解答（5）

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（3）已知椭圆$E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$的右焦点$F$，上顶点$B$，$PQ$为椭圆$E$的弦，$O$为坐标原点.

若弦$PQ$过$F$，且$OP\perp OQ$.则椭圆$E$的离心率的取值范围为_____；$[\frac{\sqrt{5}-1}{2},1)$

$key1$:当$P,Q$为长短轴端点时，$|OP|\cdot|OQ|=ab$;

当$P,Q$均不为长短轴端点时，设$l_{OP}:y=kx$代入$C$得$x_P^2=\frac{a^2b^2}{a^2k^2+b^2}$

$\therefore|OP|=\sqrt{1+k^2}\cdot\frac{ab}{\sqrt{a^2k^2+b^2}}$,同理$|OQ|=\sqrt{1+\frac{1}{k^2}}\cdot\frac{ab}{\sqrt{\frac{a^2}{k^2}+b^2}}=\frac{ab\sqrt{1+k^2}}{\sqrt{a^2+b^2k^2}}$

$\therefore\frac{1}{|OP|^2}+\frac{1}{OQ|^2}=\frac{a^2k^2+b^2}{a^2b^2(1+k^2)}+\frac{a^2+b^2k^2}{a^2b^2(1+k^2)}=\frac{1}{a^2}+\frac{1}{b^2}$,

$key2$:设$P(s,t)$，则$Q(\lambda t,-\lambda s)$,$\therefore\begin{cases}\frac{s^2}{a^2}+\frac{t^2}{b^2}=1\\ \frac{\lambda^2t^2}{a^2}+\frac{\lambda^2s^2}{b^2}=1即\frac{t^2}{a^2}+\frac{s^2}{b^2}=\frac{1}{\lambda^2}\end{cases}$,$\therefore(s^2+t^2)(\frac{1}{a^2}+\frac{1}{b^2})=1+\frac{1}{\lambda^2}$

$\therefore\therefore\frac{1}{|OP|^2}+\frac{1}{OQ|^2}=\frac{1}{a^2}+\frac{1}{b^2}$

$key3$:$|OP|=p,|OQ|=q$,则$P(p\cos\theta,p\sin\theta),Q(q\cos(\frac{\pi}{2}+\theta),q\sin(\frac{\pi}{2}+\theta))$即$(-q\sin\theta,q\cos\theta)$

$\therefore\begin{cases}\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{p^2}\\ \frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}=\frac{1}{q^2}\end{cases}$,$\therefore\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{a^2}+\frac{1}{b^2}$

$\therefore d_{O\to PQ}=\frac{ab}{\sqrt{a^2+b^2}}\le c\Leftrightarrow a^2(a^2-c^2)\le c^2(2a^2-c^2)\Leftrightarrow 1-e^2\le e^2(2-e^2)$得$e\in[\frac{\sqrt{5}-1}{2},1)$

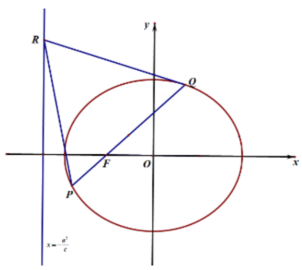
若$\triangle BPQ$是以$B$为直角顶点的等腰直角三角形，且$PQ$与$y$轴不垂直.则椭圆$E$的离心率 的取值范围为__；

$key$:设$BP:y=kx+b(k>0)$代入$E$得:$(a^2k^2+b^2)x^2+2a^2bkx=0$,$\therefore|BP|=\sqrt{1+k^2}\cdot\frac{2a^2bk}{a^2k^2+b^2}$,

同理$|BQ|=\sqrt{1+\frac{1}{k^2}}\cdot\frac{2a^2b\cdot\frac{1}{k}}{a^2\cdot\frac{1}{k^2}+b^2}=\sqrt{1+k^2}\cdot\frac{2a^2b}{a^2+b^2k^2}=\sqrt{1+k^2}\cdot\frac{2a^2bk}{a^2k^2+b^2}$即$\frac{a^2}{b^2}=k+\frac{1}{k}+1>3$得$e\in(\frac{\sqrt{6}}{3},1)$

变式 2（1）设椭圆$C:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$，线段$PQ$是过左焦点$F(-c,0)$且不与$x$轴垂直的焦点弦，

若在直线$x=-\frac{a^2}{c}$上存在点$R$，使$\triangle PQR$为正三角形，求椭圆的离心率$e$的取值范围.




2（1）设$l_{PQ}:x=ty-c$代入$C$得:$(b^2t^2+a^2)y^2-2b^2cty-b^4=0$,

$\therefore\begin{cases}y_P+y_Q=\frac{2b^2ct}{b^2t^2+a^2}\\ y_Py_Q=\frac{-b^4}{b^2t^2+a^2}\end{cases}$,且$\Delta=4a^2b^4(1+t^2)>0$

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$\therefore |PQ|=\sqrt{1+t^2}\cdot\dfrac{2ab^2\sqrt{1+t^2}}{b^2t^2+a^2}=\dfrac{2ab^2(1+t^2)}{b^2t^2+a^2}$，且$PQ$中点$M(\dfrac{-a^2c}{b^2t^2+a^2},\dfrac{b^2ct}{b^2t^2+a^2})$

$\therefore \triangle PQR$是正三角形，$\therefore |RM|=\sqrt{1+t^2}\cdot|\dfrac{-a^2c}{b^2t^2+a^2}+\dfrac{a^2}{c}|=\dfrac{a^2b^2(1+t^2)\sqrt{1+t^2}}{c(b^2t^2+a^2)}$

$=\dfrac{\sqrt{3}}{2}|PQ|=\dfrac{\sqrt{3}}{2}\cdot\dfrac{2ab^2(1+t^2)}{b^2t^2+a^2}\Leftrightarrow e=\dfrac{\sqrt{3}}{3}\sqrt{1+t^2}\in(\dfrac{\sqrt{3}}{3},1)$

若$BP\perp BQ$.则$B$在$PQ$上的射影的轨迹方程为________.

$key$：设$PQ: y=kx+m$，代入$E$得：$(a^2k^2+b^2)x^2+2a^2kmx+a^2m^2-a^2b^2=0$

$$\therefore\begin{cases}x_P+x_Q=-\dfrac{2a^2km}{a^2k^2+b^2}\\ x_Px_Q=\dfrac{a^2m^2-a^2b^2}{a^2k^2+b^2}\end{cases}，且\Delta>0$$

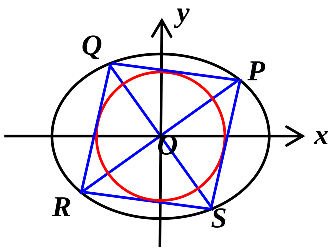
$$\therefore \overrightarrow{BP}\cdot\overrightarrow{BQ}=x_Px_Q+(kx_P+m-b)(kx_Q+m-b)=(1+k^2)\cdot\dfrac{a^2m^2-a^2b^2}{a^2k^2+b^2}+k(m-b)\cdot\dfrac{-2a^2km}{a^2k^2+b^2}+(m-b)^2=0$$

得$m=\dfrac{b(a^2-b^2)}{a^2+b^2}$，或$m=b$（舍去），

$\therefore$射影轨迹方程为$x^2+(y-\dfrac{b(a^2-b^2)}{2(a^2+b^2)})^2=\dfrac{b^2(a^2-b^2)^2}{4(a^2+b^2)^2}$（除去点$B$）

（2000 全国竞赛）已知圆$C_1: x^2+y^2=1$，椭圆$C_2:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$，若对$C_2$上任意一点$P$均存在以$P$为顶点且与$C_1$外切与$C_2$内接的平行四边形，则$a$、$b$满足的条件为_______.

$key3$：如图，由$PQRS$为圆外切平行四边形得$PQRS$是菱形



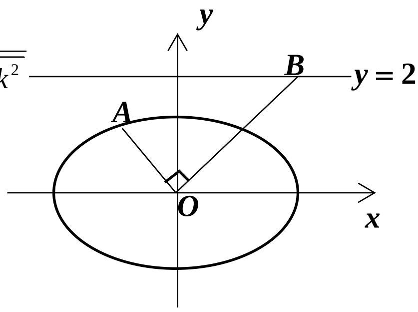

设$P(x_0,y_0)$，则$Q(\lambda y_0,-\lambda x_0)$，

$$\therefore\begin{cases}\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}=1\\ \dfrac{\lambda^2y_0^2}{a^2}+\dfrac{\lambda^2x_0^2}{b^2}=1\\ \dfrac{1}{OP^2}+\dfrac{1}{OQ^2}=\dfrac{1}{x_0^2+y_0^2}+\dfrac{1}{\lambda^2y_0^2+\lambda^2x_0^2}=1,\end{cases}\qquad\therefore\dfrac{1}{a^2}+\dfrac{1}{b^2}=1$$

（2014 北京）已知椭圆$C: x^2+2y^2=4$.（1）求椭圆 $C$ 的离心率.$\dfrac{\sqrt{2}}{2}$

（2）设$O$为原点，若点$A$在椭圆$C$上，点$B$在直线$y=2$上，且$OA\perp OB$ 试判断直线$AB$与圆$x^2+y^2=2$的位置关系.

$key$：由已知设$OA: y=kx$代入$C$得：$x_A^2=\dfrac{4}{1+2k^2}$，$\therefore |OA|=\sqrt{1+k^2}\cdot\dfrac{2}{\sqrt{1+2k^2}}$




$\because OA\perp OB$，$\therefore OB: y=-\dfrac{1}{k}x$即$x=-ky$，$\therefore |OB|=\sqrt{1+k^2}\cdot 2$

$\therefore\dfrac{1}{{d_{O\to AB}}^2}=\dfrac{1}{OA^2}+\dfrac{1}{OB^2}=\dfrac{1+2k^2}{4(1+k^2)}+\dfrac{1}{4(1+k^2)}=\dfrac{1}{2}$，$\therefore d_{O\to AB}=\sqrt{2}$

# 解析几何（2）椭圆解答（5）

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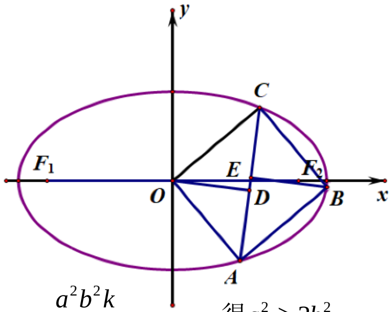
$key2:设A(s,t)，则B(\lambda t,-\lambda s)，且\begin{cases}\dfrac{s^2}{4}+\dfrac{t^2}{2}=1\\ -\lambda s=2\end{cases},\therefore \dfrac{1}{|OA|^2}+\dfrac{1}{|OB|^2}=\dfrac{1}{s^2+t^2}+\dfrac{1}{\frac{4t^2}{s^2}+4}=\dfrac{4+s^2}{4(s^2+2(1-\frac{s^2}{4}))}=\dfrac{1}{2}$

变式 3（1）设$A,B,C$是椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$上的三个点，若四边形$OABC$为矩形，

则该椭圆的离心率$e$的 最小值为______.




$key1:设OA:y=kx代入C得x_Ax_C=\dfrac{a^2b^2k}{\sqrt{a^2+b^2k^2}\cdot\sqrt{a^2k^2+b^2}}$

$由\begin{cases}\dfrac{x_A^2}{a^2}+\dfrac{y_A^2}{b^2}=1\\ \dfrac{x_B^2}{a^2}+\dfrac{y_B^2}{b^2}=1\\ \dfrac{(x_A+x_B)^2}{a^2}+\dfrac{(y_A+y_B)^2}{b^2}=1\\ x_Ax_B+y_Ay_B=0\end{cases},\therefore\begin{cases}\dfrac{x_Ax_B}{a^2}+\dfrac{y_Ay_B}{b^2}=-\dfrac{1}{2}\\ x_Ax_B+y_Ay_B=0\end{cases},\therefore x_Ax_B=\dfrac{a^2b^2}{2(a^2-b^2)}=\dfrac{a^2b^2k}{\sqrt{a^2+b^2k^2}\cdot\sqrt{a^2k^2+b^2}}得a^2\ge 3b^2$

$key2:\dfrac{1}{OA^2}+\dfrac{1}{OC^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2},\therefore 2d_{O\to AC}=\dfrac{2ab}{\sqrt{a^2+b^2}}\le a得e_{\min}=\dfrac{\sqrt{6}}{3}$

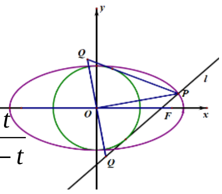
（2）已知椭圆$E:\dfrac{x^2}{6}+\dfrac{y^2}{3}=1$，设圆$O:x^2+y^2=2$的切线$l$交椭圆$E$于$P$、$Q$两点，求$|OP|\cdot|OQ|$的最大值.

$key:l:y=kx+m可得(1+2k^2)x^2+4kmx+2m^2-6=0,且\dfrac{|m|}{\sqrt{1+k^2}}=\sqrt{2}，且OP\perp OQ,$

$\therefore|OP|\cdot|OQ|=\sqrt{2}|PQ|=\sqrt{2}\sqrt{1+k^2}\cdot\dfrac{2\sqrt{2}\sqrt{3+6k^2-m^2}}{1+2k^2}=2\sqrt{2}\cdot\dfrac{\sqrt{2+2k^2}\cdot\sqrt{1+4k^2}}{1+2k^2}\le 3\sqrt{2}$

（3）若点$P$在椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$上，过$P$的直线$l$与圆$E:x^2+y^2=b^2$相切，点$Q$在直线$l$上，

若$OP\perp OQ$，求点$Q$的轨迹方程.



$key:设Q(s,t),\because\overrightarrow{OP}\perp\overrightarrow{OQ},则P(\lambda t,-\lambda s),且\dfrac{\lambda^2t^2}{a^2}+\dfrac{\lambda^2s^2}{b^2}=1\cdots(*),而PQ方程为:\dfrac{x-s}{\lambda t-s}=\dfrac{y-t}{-\lambda s-t}$

$即(\lambda s+t)(x-s)+(\lambda t-s)(y-t)=0,$

$\therefore\dfrac{|-s(\lambda s+t)-t(\lambda t-s)|}{\sqrt{(\lambda s+t)^2+(\lambda t-s)^2}}=\dfrac{|\lambda|(s^2+t^2)|}{\sqrt{1+\lambda^2}\cdot\sqrt{s^2+t^2}}=\dfrac{|\lambda|\sqrt{s^2+t^2}}{\sqrt{1+\lambda^2}}=b即\lambda^2=\dfrac{b^2}{s^2+t^2-b^2}$

$代入(*)得:\dfrac{b^2t^2+a^2s^2}{a^2b^2}\cdot\dfrac{b^2}{s^2+t^2-b^2}=1即t=\pm\dfrac{ab}{c}$

$(\dfrac{1}{|OP|^2}+\dfrac{1}{|OQ|^2}=\dfrac{1}{\lambda^2(s^2+t^2)}+\dfrac{1}{s^2+t^2}=\dfrac{1}{b^2}得\dfrac{1}{\lambda^2}=\dfrac{s^2+t^2}{b^2}-1,\therefore\dfrac{t^2}{a^2}+\dfrac{s^2}{b^2}=\dfrac{s^2+t^2}{b^2}-1得t=\pm\dfrac{ab}{c})$

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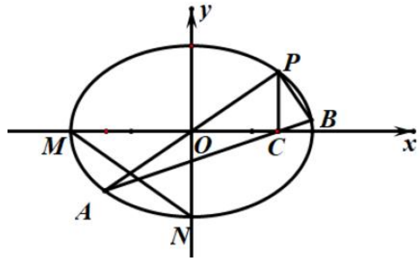
（2011江苏）如图，在平面直角坐标系$xOy$中，$M$、$N$分别是椭圆$\frac{x^2}{4}+\frac{y^2}{2}=1$的顶点，过坐标原点的直线交椭圆于$P$、$A$两点，其中$P$在第一象限，过$P$作$x$轴的垂线，垂足为$C$，连接$AC$,并延长交椭圆于点$B$，设直线$PA$的斜率为$k$.(1) 当直线$PA$平分线段$MN$时，求$k$的值；




（2）当$k=2$时，求点$P$到直线$AB$的距离$d$；

（3）（2014山东）对任意$k>0$，求证：$PA\perp PB$.

解：(1) 由已知得$M(-2,0),N(0,-\sqrt{2})$, 则$k=\frac{-\frac{\sqrt{2}}{2}}{-1}=\frac{\sqrt{2}}{2}$

（2）由$\begin{cases}y=2x\\ \frac{x^2}{4}+\frac{y^2}{2}=1\end{cases}$得P$(\frac{2}{3},\frac{4}{3}),A(-\frac{2}{3},-\frac{4}{3}),C(\frac{2}{3},0),\therefore l_{AB}:y=\frac{\frac{4}{3}}{\frac{4}{3}}(x-\frac{2}{3})=x-\frac{2}{3}$

$\therefore d=\frac{|\frac{2}{3}-\frac{4}{3}-\frac{2}{3}|}{\sqrt{2}}=\frac{2\sqrt{2}}{3}$

（3）证明：设$P(s,t)(\frac{s^2}{4}+\frac{t^2}{2}=1,s,t>0)$，则$A(-s,-t),C(s,0)$,

$key1:\because l_{AB}:y=\frac{t}{2s}(x-s)$即$x=\frac{2s}{t}y+s$代入椭圆方程得$:B(\frac{2s(4-s^2)}{4+3s^2}+s,\frac{(4-s^2)t}{4+3s^2})$

$\therefore k_{PA}\cdot k_{PB}=\frac{t}{s}\cdot\frac{\frac{(4-s^2)t}{4+3s^2}-t}{\frac{2s(4-s^2)}{4+3s^2}}=\frac{-2t^2}{4-s^2}=\frac{-2\cdot 2(1-\frac{s^2}{4})}{4-s^2}=-1,\therefore PA\perp PB$

$key2:k_{BP}\cdot k_{BA}=\frac{y_B-t}{x_B-s}\cdot\frac{y_B+t}{x_B+s}=\frac{y_B^2-t^2}{x_B^2-s^2}=\frac{2(1-\frac{x_B^2}{4})-2(1-\frac{s^2}{4})}{x_B^2-s^2}=-\frac{1}{2}$,

由$A,C,B$三点共线得$k_{AB}=k_{AC}=\frac{t}{2s}=\frac{1}{2}k_{PA},\therefore k_{PB}\cdot\frac{1}{2}k_{PA}=-\frac{1}{2}$

$\therefore k_{PB}\cdot k_{PA}=-1,\therefore PA\perp PB,$

（2012湖北）设$A$是单位圆$x^2+y^2=1$上的任意一点，$l$是过点$A$与$x$轴垂直的直线，$D$是直线$l$与$x$轴的交点，点$M$在直线$l$上，且满足$|MD|=m|DA|(m>0,$且$m\neq 1)$. 当点$A$在圆上运动时，记点$M$的轨迹为曲线$C$.

（1）求曲线$C$的方程，判断曲线$C$为何种圆锥曲线，并求焦点坐标；

（2）过原点且斜率为$k$的直线交曲线$C$于$P$、$Q$两点，其中$P$在第一象限，它在$y$轴上的射影为点$N$，直线$QN$交曲线$C$于另一点$H$，是否存在$m$，使得对任意的$k>0$，都有$PQ\perp PH$？若存在，求$m$的值；若不存在，请说明理由.

解：(1) $x^2+\frac{y^2}{m^2}=1(m>0,m\neq 1)$

（2）$key1:P(\frac{m}{\sqrt{k^2+m^2}},\frac{km}{\sqrt{k^2+m^2}}),Q(-\frac{m}{\sqrt{k^2+m^2}},-\frac{km}{\sqrt{k^2+m^2}}),N(0,\frac{km}{\sqrt{k^2+m^2}})$

$QN:y=2kx+\frac{km}{\sqrt{k^2+m^2}}\Rightarrow(m^2+4k^2)x^2+\frac{4k^2m}{\sqrt{k^2+m^2}}x-\frac{m^4}{k^2+m^2}=0$

$\therefore x_H=\frac{m^3}{\sqrt{k^2+m^2}(m^2+4k^2)},y_H=\frac{km(3m^2+4k^2)}{\sqrt{k^2+m^2}(m^2+4k^2)},\therefore k_{PH}\cdot k_{PQ}=\frac{-m^2}{2k}\cdot k=-\frac{m^2}{2}=-1\Leftrightarrow m=\sqrt{2}$

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$$key2:\because Q(-x_P,-y_P),\therefore\begin{cases}x_P^2+\dfrac{y_P^2}{m^2}=1\cdots①\\x_H^2+\dfrac{y_H^2}{m^2}=1\cdots②\\\dfrac{y_H+y_P}{x_H+x_P}=\dfrac{2y_P}{x_P}\cdots③\\\dfrac{y_H-y_P}{x_H-x_P}\cdot\dfrac{y_P}{x_P}=-1\cdots④\end{cases}$$

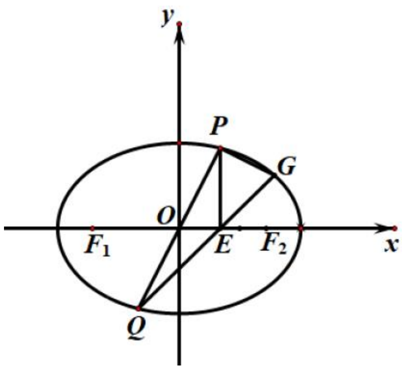
①-②得$(x_P-x_H)(x_P+x_H)+\dfrac{1}{m^2}(y_H-y_P)(y_H+y_P)=0$即$\dfrac{y_H+y_P}{x_H+x_P}\cdot\dfrac{y_H-y_P}{x_H-x_P}=-m^2$

由③④得$\dfrac{y_H+y_P}{x_H+x_P}\cdot\dfrac{y_H-y_P}{x_H-x_P}=-2,\therefore m^2=2$即$m=\sqrt{2}$

（2019Ⅱ）已知点$A(-2,0),B(2,0)$，动点$M(x,y)$满足直线$AM$与$BM$的斜率之积为$-\dfrac{1}{2}$.记$M$的轨迹为曲线$C$.

⑴ 求$C$的方程，并说明$C$是什么曲线；（2）过坐标原点$O$的直线交$C$于$P,Q$两点，点$P$在第一象限，$PE\perp x$轴，垂足为$E$，连接$QE$并延长交$C$于点$G$.

（Ⅰ）证明:$\triangle PQG$是直角三角形；（Ⅱ）求$\triangle PQG$面积的最大值.




解：（1）$\dfrac{y}{x+2}\cdot\dfrac{y}{x-2}=-\dfrac{1}{2}$即$\dfrac{x^2}{4}+\dfrac{y^2}{2}=1(x\neq\pm2)$即为所求的,$C$是椭圆

(2)（Ⅰ）$key1$:设$P(s,t)(s,t>0,\dfrac{s^2}{4}+\dfrac{t^2}{2}=1)$,则$Q(-s,-t),E(s,0)$,

有$k_{GP}\cdot k_{GQ}=\dfrac{y_G-t}{x_G-s}\cdot\dfrac{y_G+t}{x_G+s}=\dfrac{y_G^2-t^2}{x_G^2-s^2}=-\dfrac{1}{2}$,且$k_{GQ}=k_{EQ}=\dfrac{t}{2s}=\dfrac{1}{2}k_{PQ}$

$\therefore k_{PQ}\cdot k_{PG}=\dfrac{2t}{2s}\cdot(-\dfrac{1}{2})\cdot\dfrac{2s}{t}=-1,\therefore PG\perp PQ,\therefore\triangle PQG$是直角三角形

$key2:l_{QG}:y=\dfrac{t}{2s}(x-s)$即$x=\dfrac{2s}{t}y+s$代入椭圆方程得:$G(\dfrac{2s(4-s^2)}{4+3s^2}+s,\dfrac{(4-s^2)t}{4+3s^2})$

$$\therefore k_{PQ}\cdot k_{PG}=\frac{t}{s}\cdot\frac{\dfrac{(4-s^2)t}{4+3s^2}-t}{\dfrac{2s(4-s^2)}{4+3s^2}}=\frac{-2t^2}{4-s^2}=\frac{-2\cdot2(1-\dfrac{s^2}{4})}{4-s^2}=-1,\therefore PQ\perp PG$$

$$（Ⅱ）由（Ⅰ）得:S_{\triangle PQG}=\frac{1}{2}\left\|\begin{matrix}s&t&1\\-s&-t&1\\\dfrac{s(12+s^2)}{4+3s^2}&\dfrac{(4-s^2)t}{4+3s^2}&1\end{matrix}\right\|=\frac{2st(4+s^2)}{4+3s^2}=\frac{2st(2s^2+2t^2)}{4s^2+2t^2}=\frac{2st(s^2+t^2)}{2s^2+t^2}$$

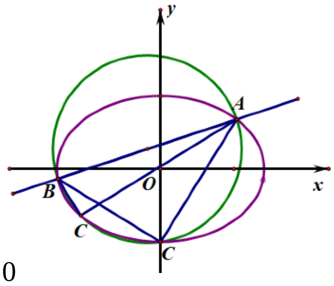
$$=\frac{8st(s^2+t^2)}{(2s^2+t^2)(s^2+2t^2)}=\frac{8k(1+k^2)}{(2+k^2)(1+2k^2)}=\frac{8(\dfrac{1}{k}+k)}{(\dfrac{2}{k}+k)(\dfrac{1}{k}+2k)}=\frac{8(k+\dfrac{1}{k})}{2(k+\dfrac{1}{k})^2+1}$$

$$=\frac{8}{2u+\dfrac{1}{u}}\leq\frac{16}{9}(u=k+\frac{1}{k}=\frac{t}{s}+\frac{t}{s}\geq2)$$

# 解析几何（2）椭圆解答（5）

2023-10-28

变式：已知直线 $x-3y+1=0$ 与椭圆 $\Gamma:\dfrac{x^2}{2}+y^2=1$ 相交于与 $A,B$ 两点，若椭圆上存在点 $C$，使得 $\angle ACB=90^\circ$，则点 $C$ 的坐标为____________.




$$key:由\begin{cases}x-3y+1=0\\x^2+2y^2=2\end{cases}消去x得11y^2-6y-1=0,\therefore\begin{cases}y_A+y_B=\dfrac{6}{11}\\y_Ay_B=-\dfrac{1}{11}\end{cases},且\Delta=80$$

$$\therefore以AB为直径的圆方程为:(x+\frac{2}{11})^2+(y-\frac{3}{11})^2=\frac{200}{121}即x^2+y^2+\frac{4}{11}x-\frac{6}{11}y-\frac{17}{11}=0$$

$$代入\Gamma得:2-y^2+\frac{4}{11}x-\frac{6}{11}y-\frac{17}{11}=-y^2+\frac{4}{11}x-\frac{6}{11}y+\frac{5}{11}=0即x=\frac{1}{4}(11y^2+6y-5)$$

$$代入\Gamma得:\frac{1}{32}(11y^2+6y-5)^2+y^2=1\Leftrightarrow121y^4+132y^3-42y^2-60y-7=(11y^2-6y-1)(11y^2+18y+7)=0$$

$$\therefore y=-1,or,-\frac{7}{11},\therefore C(0,-1),or,(-\frac{12}{11},-\frac{7}{11})$$

$$key2:\Leftrightarrow\frac{1}{32}(11y-5)^2(y+1)^2=1-y^2=(1-y)(1+y)$$

$$\therefore y=-1,or,(11y-5)^2(y+1)=32(1-y)\Leftrightarrow121y^3+11y^2-53y-7=(11y^2-6y-1)(11y+7)=0$$

（2017陕西）如图，已知椭圆$E:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$，圆$O:x^2+y^2=a^2$与$y$轴正半轴交于点$B$，过点$B$的直线与椭圆$E$相切，且与圆$O$交于另一点$A$.若$\angle AOB=60^\circ$,则椭圆$E$的离心率为（$D$）

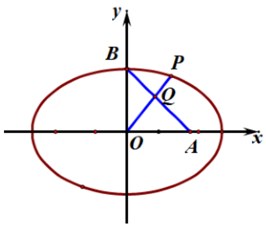
$A.\dfrac{1}{2}\quad B.\dfrac{1}{3}\quad C.\dfrac{\sqrt{2}}{2}\quad D.\dfrac{\sqrt{3}}{3}$

（2005 福建）过点$E(-2,0)$的直线$l$与椭圆$C:\dfrac{x^2}{6}+\dfrac{y^2}{2}=1$交于点$M$、$N$，且$\overrightarrow{OM}\cdot\overrightarrow{ON}\tan\angle MON=\dfrac{4\sqrt{6}}{3}$，则直线$l$的方程为_______. $x=\pm\sqrt{3}y-2$

$$key:\overrightarrow{OM}\cdot\overrightarrow{ON}\tan\angle MON=|\overrightarrow{OM}|\cdot|\overrightarrow{ON}|\cos\angle MON\cdot\frac{\sin\angle MON}{\cos\angle MON}=2S_{\triangle MON}=\frac{4\sqrt{6}}{3},$$

（2018天津）设椭圆$\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$的上顶点为$B$，点$A(2,0)$，设直线$l:y=kx(k>0)$与椭圆在第一象限的焦点为$P$，且$l$与直线$AB$交于点$Q$,若$\dfrac{|AQ|}{|PQ|}=\dfrac{5\sqrt{2}}{4}\sin\angle AOQ$（$O$为原点），则$k$的值为_________.

$$2018天津key:l_{AB}:x+y=2联立y=kx得x_Q=\frac{2}{k+1},$$




$$由\begin{cases}y=kx\\4x^2+9y^2=36\end{cases}得x_P=\frac{6}{\sqrt{4+9k^2}}$$

$$\therefore\frac{|AQ|}{|PQ|}=\frac{\sqrt{2}(2-\dfrac{2}{k+1})}{\sqrt{1+k^2}(\dfrac{6}{\sqrt{4+9k^2}}-\dfrac{2}{k+1})}=\frac{5\sqrt{2}}{4}\cdot\frac{k}{\sqrt{1+k^2}}得k=1,or,\frac{11}{28}$$

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（2018I）19.设椭圆 $C:\dfrac{x^2}{2}+y^2=1$ 的右焦点为 $F$，过 $F$ 的直线 $l$ 与 $C$ 交于 $A,B$ 两点，点 $M$ 的坐标为 $(2,0)$.

（1）当 $l$ 与 $x$ 轴垂直时，求直线 $AM$ 的方程；（2）设 $O$ 为坐标原点，证明：$\angle OMA=\angle OMB$.

⑴ 解：由已知得$A(1,\pm\dfrac{1}{2}),\therefore AM$的方程为$y=\pm\dfrac{\sqrt{2}}{2}(x-2)$

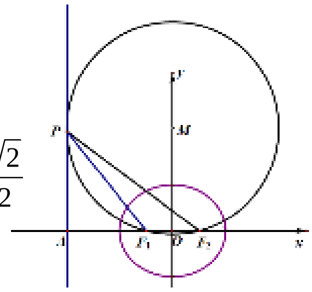
（2）证明：由题意设$l:x=ty+1$代入$C$得:$(t^2+2)y^2+2ty-1=0,\therefore\begin{cases}y_A+y_B=-\dfrac{2t}{t^2+2}\\ y_Ay_B=\dfrac{-1}{t^2+2}\end{cases}$

$\therefore k_{MA}+k_{MB}=\dfrac{y_M}{ty_M-1}+\dfrac{y_N}{ty_N-1}=0\Leftrightarrow y_M(ty_N-1)+y_N(ty_M-1)=2t\cdot\dfrac{-1}{t^2+2}-\dfrac{-2t}{t^2+2}=0$

$\therefore\angle OMA=\angle OMB$

变式 1（1）已知椭圆 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\left(a>b>0\right)$ 的左，右焦点分别为 $F_1,F_2$，点 $P$ 为直线 $x=-\dfrac{a^2}{c}$ 上的一个动点（不在坐标轴上），则当 $\angle F_1PF_2$ 的最大值为 $\dfrac{\pi}{6}$ 时，椭圆的离心率是_________.

$key$:（切割线定理）$t^2=AP^2=|AF_1|\cdot|AF_2|=(-c+\dfrac{a^2}{c})(c+\dfrac{a^2}{c})=\dfrac{a^4}{c^2}-c^2(P(-\dfrac{a^2}{c},t)$

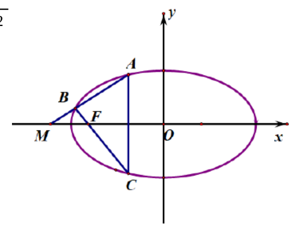


且$\tan\angle F_1PF_2=\dfrac{\dfrac{\frac{a^2}{c}+c}{t}-\dfrac{\frac{a^2}{c}-c}{t}}{1+\dfrac{\frac{a^4}{c^2}-c^2}{t^2}}=\dfrac{2ct}{t^2+\dfrac{a^4-c^4}{c^2}}=\dfrac{2c\sqrt{\dfrac{a^4-c^4}{c^2}}}{2\cdot\dfrac{a^4-c^4}{c^2}}\dfrac{c^2}{\sqrt{a^4-c^4}}=\dfrac{\sqrt{3}}{3}$得$e=\dfrac{\sqrt{2}}{2}$

（2）如图，已知椭圆$E:\dfrac{x^2}{6}+\dfrac{y^2}{2}=1$的左焦点为$F$，直线$l:x=ty-3(t>0)$与椭圆$E$交于$A$、$B$两点，

延长$BF$交椭圆$E$于点$C$.若$\angle MAC=60°$,则$t=$________.

$key$:设直线$l$的方程为$x=ty-3(t>0)$代入$E$得:$(3+t^2)y^2-6ty+3=0,\therefore\begin{cases}y_A+y_B=\dfrac{6t}{3+t^2}\\ y_Ay_B=\dfrac{3}{3+t^2}\end{cases}$，且$\Delta=12(2t^2-3)>0$




而$l_{BF}:x=\dfrac{x_B+2}{y_B}y-2$代入椭圆$E$得$(\dfrac{(x_B+2)^2}{y_B^2}+3)y^2-\dfrac{4(x_B+2)}{y_B}y-2=0,$

$\therefore y_By_C=\dfrac{-2y_B^2}{(x_B+2)^2+3y_B^2}=\dfrac{-y_B^2}{2ty_B-1}$得$C(\dfrac{-5ty_B+3}{2ty_B-1},\dfrac{-y_B}{2ty_B-1}),$

$\therefore x_A-x_C=ty_A-3-\dfrac{-5ty_B+3}{2ty_B-1}=\dfrac{2t^2y_Ay_B-ty_A-ty_B}{2ty_B-1}=\dfrac{\dfrac{6t^2}{t^2+2}-\dfrac{6t^2}{t^2+2}}{2ty_B-1}=0,\therefore AC\perp x$轴$,\therefore$直线$l$的斜率为$\dfrac{\sqrt{3}}{3},\therefore t=\sqrt{3}$

（3）椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的两焦点为$F_1(-c,0),F_2$，直线$l_1:x=-\dfrac{a^2}{c},l_2:x=\dfrac{a^2}{c}$,过椭圆上的一点$P$,作平行于$F_1F_2$的直线，分别交$l_1,l_2$于$M_1,M_2$，直线$M_1F_1$与$M_2F_2$交于点$Q$，证明:$P,F_1,Q,F_2$四点共圆.

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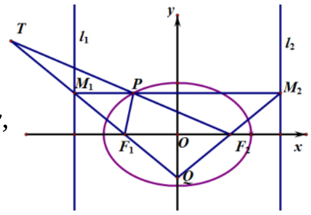
$key:$ 设$P(u,v)$(其中$\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$), 则$M_1(-\frac{a^2}{c},v),M_2(\frac{a^2}{c},v)$,




由对称性得$Q$在$y$轴上，由$M_1,F_1,Q$三点共线得: $\frac{v}{-\frac{a^2}{c}+c}=\frac{-y_Q}{-c}$即$y_Q=-\frac{c^2}{b^2}v$,

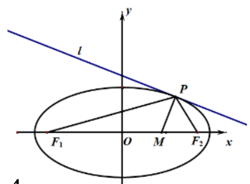
$k_{PF_1}=\frac{v}{u+c},k_{QF_1}=k_{M_{11}F_1}=-\frac{cv}{b^2},k_{PF_2}=\frac{v}{u-c},k_{QF_2}=\frac{cv}{b^2}$

$\therefore \tan\angle PF_1Q=\frac{\frac{v}{u+c}+\frac{cv}{b^2}}{1-\frac{v}{u+c}\cdot\frac{cv}{b^2}}=\frac{v(a^2+cu)}{b^2(u+c)-cb^2(1-\frac{u^2}{a^2})}=\frac{a^2v}{b^2u}$

$\tan\angle PF_2Q=\frac{\frac{-v}{u-c}+\frac{cv}{b^2}}{1-\frac{-v}{u-c}\cdot\frac{cv}{b^2}}=\frac{v(-a^2+cu)}{b^2(u-c)+cb^2(1-\frac{u^2}{a^2})}=-\frac{a^2v}{b^2u}$,$\therefore\angle PF_1Q+\angle PF_2Q=\pi$,$\therefore P,F_1,Q,F_2$四点共圆

（2013 山东）椭圆$C:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$的左、右焦点分别是$F_1,F_2$，离心率为$\frac{\sqrt3}{2}$，过$F_1$且垂直于$x$的直线被椭圆$C$截得的线段长为 1.（1 求椭圆$C$的方程；

（2）点$P$是椭圆$C$上除长轴端点外的任一点，连接$PF_1,PF_2$，设$\angle F_1PF_2$的角平分线$PM$交$C$的长轴于点$M(m,0)$，求$m$的取值范围;（3）在（2）的条件下，过$P$点作斜率为$k$的直线$l$，使得$l$与椭圆$C$有且只有一个公共点，设直线$PF_1,PF_2$的斜率分别为$k_1,k_2$,若$k\neq0$,试证明:$\frac{1}{kk_1}+\frac{1}{kk_2}$为定值，并求出这个定值.




（1）解：由已知得$\begin{cases}\frac{c}{a}=\frac{\sqrt3}{2}\\ \frac{b^2}{a}=\frac12\end{cases}$得$a=2,b=1,\therefore C$的方程为$\frac{x^2}{4}+y^2=1$

（2）由椭圆在P处的切线方程为$\frac{x_Px}{4}+y_Py=1,\therefore\angle F_1PF_2$的角平分线方程为$y-y_P=\frac{4y_P}{x_P}(x-x_P)$

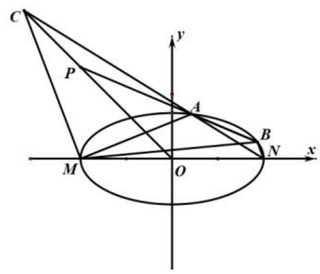
$\therefore m=\frac34x_P\in(-\frac32,\frac32)$即为所求的

（3）证明：由（2）得$k=-\frac{x_P}{4y_P},k_1=\frac{y_P}{x_P+\sqrt3},k_2=\frac{y_P}{x_P-\sqrt3}$

$\therefore\frac{1}{kk_1}+\frac{1}{kk_2}=-\frac{4y_P}{x_P}\cdot\frac{2x_P}{y_P}=-8$为定值

(17吉林）已知椭圆$E:x^2+4y^2=4$的左、右顶点分别为$M$、$N$，过点$P(-2,2)$作直线与椭圆$E$交于$A$、$B$两点，且$A$、$B$位于第一象限，$A$在线段$BP$上，直线$OP$与直线$NA$相交于$C$点，连结$MB$、$MC$、$AM$.直线$AM$、$AC$、$MB$、$MC$的斜率分别记为$k_{AM},k_{AC},k_{MB},k_{MC}$.求证:$\frac{k_{MB}}{k_{AM}}=\frac{k_{AC}}{k_{MC}}$.

$key1:$ 设$l_{AB}:y-2=k(x+2)$即$y=kx+2k+2$代入$E$得:$(1+4k^2)+16(k^2+k)x+16(k+1)^2-4=0$




$\therefore\begin{cases}x_A+x_B=-\frac{16(k^2+k)}{1+4k^2}\\ x_Ax_B=\frac{16k^2+32k+12}{1+4k^2}\end{cases}$,且$\Delta=-16(8k+3)>0$

联立$l_{OP}: y=-x$与$l_{AN}: y=\dfrac{y_A}{x_A-2}(x-2)$得$C(\dfrac{-2y_A}{2-x_A-y_A},\dfrac{2y_A}{2-x_A-y_A})$

$$\therefore k_{MB}k_{MC}-k_{AC}k_{AM}=\frac{y_B}{x_B+2}\cdot\frac{\dfrac{2y_A}{2-x_A-y_A}}{\dfrac{-2y_A}{2-x_A-y_A}+2}-\frac{\dfrac{2y_A}{2-x_A-y_A}-y_A}{\dfrac{-2y_A}{2-x_A-y_A}-x_A}\cdot\frac{y_A}{x_A+2}$$

$$=\frac{y_B}{x_B+2}\cdot\frac{y_A}{-(2k+1)(x_A+2)}-\frac{y_A}{x_A-2}\cdot\frac{y_A}{x_A+2}=0\Leftrightarrow(kx_B+2k+2)(x_A-2)+(2k+1)(kx_A+2k+2)(x_B+2)$$

$$=2k(k+1)x_Ax_B+(4k^2+4k+2)(x_A+x_B)+8k(k+1)$$

$$=(2k^2+2k)\cdot\frac{16k^2+32k+12}{1+4k^2}-(4k^2+4k+2)\cdot\frac{16(k^2+k)}{1+4k^2}+\frac{8k(k+1)(1+4k^2)}{1+4k^2}=0$$

$key2$: 设$A(2\cos\alpha,\sin\alpha),B(2\cos\beta,\sin\beta)$

由$P,A,B$共线得$\dfrac{\sin\beta-\sin\alpha}{2\cos\beta-2\cos\alpha}=\dfrac{2-\sin\alpha}{-2-2\cos\alpha}\Leftrightarrow(\sin\alpha-\sin\beta)(1+\cos\alpha)=(\cos\beta-\cos\alpha)(2-\sin\alpha)$

$$\Leftrightarrow\sin(\alpha-\beta)+\sin\alpha-\sin\beta+2(\cos\alpha-\cos\beta)$$

$$=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}-4\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=0\Leftrightarrow\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}=1$$

联立$l_{OP}: y=-x$与$l_{NA}: y=\dfrac{\sin\alpha}{2\cos\alpha-2}(x-2)=\dfrac{1}{-2\tan\dfrac{\alpha}{2}}(x-2)$得$C(\dfrac{-2}{2\tan\dfrac{\alpha}{2}-1},\dfrac{2}{2\tan\dfrac{\alpha}{2}-1})$

$$\therefore k_{MB}\cdot k_{MC}=\frac{\sin\beta}{2\cos\beta+2}\cdot\frac{\dfrac{2}{2\tan\dfrac{\alpha}{2}-1}}{\dfrac{-2}{2\tan\dfrac{\alpha}{2}-1}+2}=\frac{2\sin\dfrac{\beta}{2}\cos\dfrac{\beta}{2}}{4\cos^2\dfrac{\beta}{2}}\cdot\frac{1}{2\tan\dfrac{\alpha}{2}-2}=\frac{\tan\dfrac{\beta}{2}}{-4\tan\dfrac{\beta}{2}}=-\frac{1}{4}$$

$$k_{AM}\cdot k_{AC}=\frac{\sin\alpha}{2\cos\alpha+2}\cdot\frac{\sin\alpha-\dfrac{2}{2\tan\dfrac{\alpha}{2}-1}}{2\cos\alpha-\dfrac{-2}{2\tan\dfrac{\alpha}{2}-1}}=\frac{1}{2}\tan\frac{\alpha}{2}\cdot\frac{\dfrac{(2\tan\dfrac{\alpha}{2}-1)\cdot 2\tan\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}}-2}{\dfrac{2(1-\tan^2\dfrac{\alpha}{2})(2\tan\dfrac{\alpha}{2}-1)}{1+\tan^2\dfrac{\alpha}{2}}+2}=-\frac{1}{4},\therefore\frac{k_{MB}}{k_{AM}}=\frac{k_{AC}}{k_{MC}}$$