数列（1）等差等比数列解答（1）

2024-01-13

一、等差数列

⑴ 定义：$a_n - a_{n-1} = d$（d为常数）

$\Leftrightarrow 2a_n = a_{n+1} + a_{n-1}$

$\Leftrightarrow a_n = a_1 + (n-1)d = pn + q \Leftrightarrow a_n = a_m + (n-m)d$

$\Leftrightarrow S_n = An^2 + Bn \Leftrightarrow S_n = \dfrac{n(a_1 + a_n)}{2}$

⑵ 性质：若$\{a_n\}$是等差数列，则

①若$\{k_n\}$是等差数列，且$k_n \in N^*$，则$\{a_{k_n}\}$是等差数列

②若$p_1 + p_2 + \cdots + p_m = q_1 + q_2 + \cdots + q_m, p_i, q_i \in N^*$,则$a_{p_1} + a_{p_2} + \cdots + a_{p_m} = a_{q_1} + a_{q_2} + \cdots + a_{q_m}$.

(1993I ）已知等差数列$\{a_n\}$的公差$d > 0$，首项$a_1 > 0, S_n = \sum\limits_{i=1}^{n} \dfrac{1}{a_i a_{i+1}}$，则$\lim\limits_{n\to\infty} S_n =$____.

1993I $key: S_n = \dfrac{1}{d}(\dfrac{1}{a_1} - \dfrac{1}{a_1 + nd}), \therefore \lim\limits_{n\to\infty} S_n = \dfrac{1}{a_1 d}$

(2005II)11.如果$a_1, a_2, \cdots, a_8$为各项都大于零的等差数列，公差$d \neq 0$，则（ ）

$A.a_1a_8 > a_4a_5$ $B.a_1a_8 < a_4a_5$ $C.a_1 + a_8 > a_4 + a_5$ $D.a_1a_8 = a_4a_5$

2005II $key: a_1a_8 - a_4a_5 = a_1(a_1 + 7d) - (a_1 + 3d)(a_1 + 4d) = -12d^2 < 0$, 选$B$

(2006江苏）设数列$\{a_n\}, \{b_n\}, \{c_n\}$满足：$b_n = a_n - a_{n+2}, c_n = a_n + 2a_{n+1} + 3a_{n+2} (n = 1, 2, 3, \cdots)$,
证明:$\{a_n\}$为等差数列的充要条件是$\{c_n\}$为等差数列且$b_n \le b_{n+1} (n = 1, 2, 3 \cdots)$.

2006江苏证明：①必要性:$\because \{a\}$是等差数列，设其公差为d_a,

则$c_{n+1} - c_n = (a_{n+1} + 2a_{n+2} + 3a_{n+3}) - (a_n + 2a_{n+1} + 3a_{n+2})$

$= a_{n+1} - a_n + 2(a_{n+2} - a_{n+1}) + 3(a_{n+3} - a_{n+2}) = d_a + 2d_a + 3d_a = 6d_a$为常数,$\therefore \{c_n\}$是等差数列，

且$b_n = -2d_a = b_{n+1}, \therefore b_n \le b_{n+1}$

②充分性:$\because \{c_n\}$为等差数列(设其公差为d_c),

$\therefore 3a_{n+3} - a_{n+2} - a_{n+1} - a_n = (a_{n+1} + 2a_{n+2} + 3a_{n+3}) - (a_n + 2a_{n+1} + 3a_{n+2}) = c_{n+1} - c_n = d_c$

$\therefore 0 = (3a_{n+4} - a_{n+3} - a_{n+2} - a_{n+1}) - (3a_{n+3} - a_{n+2} - a_{n+1} - a_n) = 3a_{n+4} - 4a_{n+3} + a_n$

$(\because b_n = a_n - a_{n+2} \le b_{n+1} = a_{n+1} - a_{n+3}, \therefore a_n \le a_{n+1} + a_{n+2} - a_{n+3})$

$\le 3a_{n+4} - 4a_{n+3} + a_{n+1} + a_{n+2} - a_{n+3} = 3a_{n+4} - 5a_{n+3} + a_{n+1} + a_{n+2}$

$\le 3a_{n+4} - 5a_{n+3} + a_{n+2} + a_{n+2} + a_{n+3} - a_{n+4} = 2a_{n+4} - 4a_{n+3} + 2a_{n+2}$

即$a_{n+4} - 2a_{n+3} + a_{n+2} \ge 0$

而$0 = 3a_{n+4} - 4a_{n+3} + a_n = 3(a_{n+4} - 2a_{n+3} + a_{n+2}) + 2(a_{n+3} - 2a_{n+2} + a_{n+1}) + (a_{n+2} - 2a_{n+1} + a_n) = 0$

$\therefore a_{n+2} - 2a_{n+1} + a_n = 0 (n \ge 3), \therefore a_5 - 2a_4 + a_3 = 0$

$$\text{而}\begin{cases} c_1 = a_1 + 2a_2 + 3a_3, \\ c_1 + d_c = a_2 + 2a_3 + 3a_4 \\ c_1 + 2d_c = a_3 + 2a_4 + 3a_5 \\ b_1 = a_1 - a_3 \le b_2 = a_2 - a_4 \\ b_2 = a_2 - a_4 \le b_3 = a_3 - a_5 \end{cases}, \therefore \begin{cases} c_1 = a_1 + 2a_2 + 3a_3, \\ d_c = -a_1 - a_2 - a_3 + 3a_4 \\ d_c = -a_2 - a_3 - a_4 + 3a_5, \\ a_1 - a_2 \le a_3 - a_4 \\ a_2 - a_4 \le a_3 - a_5 \end{cases}$$

$$\therefore \begin{cases} 0 = a_1 - 4a_4 + 3a_5 = a_1 - 4a_4 + 3(2a_4 - a_3) = 2a_4 - 3a_3 + a_1 \\ a_1 - a_2 \le a_3 - a_4 \\ a_2 - a_4 \le a_3 - a_5 \end{cases}$$

2024-01-13

$\therefore \begin{cases} a_4 = \frac{3}{2}a_3 - \frac{1}{2}a_1, \\ a_5 = 2a_4 - a_3 = 2a_3 - a_1 \\ a_1 - a_2 \le a_3 - (\frac{3}{2}a_3 - \frac{1}{2}a_1) = -\frac{1}{2}a_3 + \frac{1}{2}a_1 \text{得} a_1 + a_3 \le 2a_2 \\ a_2 - a_4 \le a_3 - a_5 \text{即} a_2 - \frac{3}{2}a_3 + \frac{1}{2}a_1 \le a_3 - 2a_3 + a_1 \text{即} 2a_2 \le a_1 + a_3 \end{cases}$ $, \therefore a_1 + a_3 = 2a_2, \text{且} a_2 + a_4 = 2a_3$

$\therefore a_n - 2a_{n+1} + a_{n+2} = 0 \text{即} a_{n+2} - a_{n+1} = a_{n+1} - a_n, \therefore \{a_n\}$是等差数列.由①②得证

（2009 江苏）设$\{a_n\}$是公差不为零的等差数列，S_n为其前n项和，满足$a_2{}^2 + a_3{}^2 = a_4{}^2 + a_5{}^2, S_7 = 7.$

（1）求数列$\{a_n\}$的通项公式及前n项和S_n；（2）试求所有的正整数m，使得$\frac{a_m a_{m+1}}{a_{m+2}}$为数列$\{a_n\}$中的项.

解：（1）由$\begin{cases} a_4^2 - a_2^2 + a_5^2 - a_3^2 = 2d(2a_1 + 4d) + 2d(2a_1 + 6d) = 0 (d \ne 0) \\ S_7 = 7a_1 + \frac{6 \times 7}{2} d = 7 \end{cases}$ 得$a_1 = -5, d = 2$

$\therefore a_n = 2n - 7, S_n = n^2 - 6n$

（2）由（1）得：$\frac{a_m a_{m+1}}{a_{2m+3}} = \frac{(2m-7)(2m-5)}{2m-3} = 2m - 3 - 6 + \frac{8}{2m-3} \in \{a_n\}$

$\therefore \frac{8}{2m-3} = \pm 1, \text{得} m = 2$