集合、代数运算、不等式性质及基本不等式解答（3）

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2（1）已知函数$f(x)=ax^2+bx+c(a\neq 0)$,若$-1\le f(-1)\le 1,-1\le f(0)\le 1,-1\le f(1)\le 1$,则$f(2)$的取值范围为__________.

$$key:由\begin{cases}f(-1)=a-b+c\\ f(0)=c\\ f(1)=a+b+c\end{cases}得\begin{cases}a=\dfrac{f(-1)+f(1)}{2}-f(0)\\ b=\dfrac{f(1)-f(-1)}{2}\\ c=f(0)\end{cases},$$

$\therefore f(2)=4a+2b+c=3f(1)+f(-1)-3f(0)\in[-7,7]$

（2）已知正数a,b满足$ab^2\in[1,2],\dfrac{a^2}{b}\in[1,2]$,则$ab$的取值范围为________.

$$令\begin{cases}x=ab^2\\ y=\dfrac{a^2}{b}\end{cases}得\begin{cases}a=x^{\frac{1}{5}}\cdot y^{\frac{2}{5}}\\ b=x^{\frac{2}{5}}\cdot y^{-\frac{1}{5}}\end{cases},\therefore ab=x^{\frac{2}{5}}\cdot y^{\frac{1}{5}}\in[1,2^{\frac{3}{5}}]$$

（2016 文科）20.设函数 $f(x)=x^3+\dfrac{1}{1+x}$， $x\in[0,1]$.证明：（I） $f(x)\ge 1-x+x^2$；（II） $\dfrac{3}{4}<f(x)\le\dfrac{3}{2}$.

(1) （作差因式分解）$f(x)-(1-x+x^2)=x^3-x^2+x+\dfrac{1}{x+1}-1$

$=x^3-x^2+x-\dfrac{x}{x+1}=x(x^2-x+1-\dfrac{1}{x+1})=x(x^2-x-\dfrac{x}{x+1})=x^2(x-1-\dfrac{1}{x+1})$

$=x^4\cdot\dfrac{1}{x+1}\ge(\because x\in[0,1])$

(II) $\dfrac{3}{2}-f(x)=1-x^3+\dfrac{1}{2}-\dfrac{1}{1+x}=(1-x)(1+x+x^2)+\dfrac{x-1}{2(1+x)}=(1-x)(x^2+x+\dfrac{2x+1}{2x+2})\ge 0$

3(1) ①若$a,b>0$,试比较$\dfrac{a^3}{b^2}+\dfrac{b^3}{a^2}$与$a+b$的大小；

$key:\because a,b>0,\therefore\dfrac{a^3}{b^2}-b+\dfrac{b^3}{a^2}-a=\dfrac{a^3-b^3}{b^2}-\dfrac{b^3-a^3}{a^2}=\dfrac{(a-b)^2(a+b)(a^2+ab+b^2)}{a^2b^2}\ge 0$

$\therefore$当$a=b$时,$\dfrac{a^3}{b^2}+\dfrac{b^3}{a^2}=a+b$;当$a\neq b$时,$\dfrac{a^3}{b^2}+\dfrac{b^3}{a^2}>a+b$.

②已知$a,b>0$,求证:$\dfrac{a+b}{2}\cdot\dfrac{a^2+b^2}{2}\cdot\dfrac{a^3+b^3}{2}\le\dfrac{a^6+b^6}{2}$.

证明:$\because a>0,b>0,\therefore 2(a^3+b^3)-(a+b)(a^2+b^2)=(a-b)^2(a+b)\ge 0$

$\therefore\dfrac{a^3+b^3}{2}\ge\dfrac{a+b}{2}\cdot\dfrac{a^2+b^2}{2}>0$

$\because 2(a^6+b^6)-(a^3+b^3)^2=(a^3-b^3)\ge 0,\therefore\dfrac{a^6+b^6}{2}\ge\dfrac{a^3+b^3}{2}\cdot\dfrac{a^3+b^3}{2}\ge\dfrac{a+b}{2}\cdot\dfrac{a^2+b^2}{2}\cdot\dfrac{a^3+b^3}{2}$

（2）若$a,b,c>0$,求证：①$a^ab^b\ge a^{\frac{a+b}{2}}b^{\frac{a+b}{2}}\ge a^bb^a$；②$a^ab^bc^c\ge a^{\frac{a+b+c}{3}}b^{\frac{a+b+c}{3}}c^{\frac{a+b+c}{3}}\ge a^cb^bc^a$.

（Ⅰ）证明：由对称性不妨设$a\ge b$,则$\dfrac{(ab)^{\frac{a+b}{2}}}{a^b b^a}=a^{\frac{a-b}{2}}b^{\frac{b-a}{2}}=(\dfrac{a}{b})^{\frac{a-b}{2}}\ge 1$

（Ⅱ）$\dfrac{a^a b^b c^c}{(abc)^{\frac{a+b+c}{3}}}=a^{\frac{2a-b-c}{3}}b^{\frac{2b-a-c}{3}}c^{\frac{2c-a-b}{3}}=(\dfrac{a}{b})^{\frac{a-b}{3}}\cdot(\dfrac{a}{c})^{\frac{a-c}{3}}\cdot(\dfrac{b}{c})^{\frac{b-c}{3}}\ge 1$

$\dfrac{(abc)^{\frac{a+b+c}{3}}}{a^c b^b c^a}=a^{\frac{a+b-2c}{3}}b^{\frac{a+c-2b}{3}}c^{\frac{b+c-2a}{3}}=(\dfrac{a}{c})^{\frac{a-c}{3}}\cdot(\dfrac{a}{b})^{\frac{b-c}{3}}\cdot(\dfrac{b}{c})^{\frac{a-b}{3}}\ge 1$

若$f(a,b,c)=f(b,a,c)$,则称f关于a,b对称；若$f(b,c,a)=f(a,b,c)$,则称f关于a,b,c轮换对称

(3) 若$a>b>0$,求证：①$\sqrt[3]{a-b}>\sqrt[3]{a}-\sqrt[3]{b}$；②$\sqrt{a+1}-\sqrt{a}<\sqrt{b+1}-\sqrt{b}$.

①证明:$\because a>b>0,\therefore(\sqrt[3]{a-b}+\sqrt[3]{b})^3=a-b+3\sqrt[3]{(a-b)^2b}+3\sqrt[3]{(a-b)b^2}+b>a$

$\therefore\sqrt[3]{a-b}+\sqrt[3]{b}>\sqrt[3]{a},\therefore\sqrt[3]{a-b}>\sqrt[3]{a}-\sqrt[3]{b}$得证

②$key1$:$\because a>b>0,\therefore\sqrt{a+1}>\sqrt{b+1},\sqrt{a}>\sqrt{b},\therefore\sqrt{a+1}-\sqrt{a}=\dfrac{1}{\sqrt{a+1}+\sqrt{a}}<\dfrac{1}{\sqrt{b+1}+\sqrt{b}}=\sqrt{b+1}-\sqrt{b}$得证

$key2:\sqrt{a+1}-\sqrt{a}-(\sqrt{b+1}-\sqrt{b})=\dfrac{a+1-(b+1)}{\sqrt{a+1}+\sqrt{b+1}}+\dfrac{b-a}{\sqrt{b}+\sqrt{a}}=(a-b)\cdot(\dfrac{1}{\sqrt{a+1}+\sqrt{b+1}}-\dfrac{1}{\sqrt{a}+\sqrt{b}})$

$=(a-b)\cdot\dfrac{(\sqrt{a}-\sqrt{a+1})+(\sqrt{b}-\sqrt{b+1})}{(\sqrt{a+1}+\sqrt{b+1})(\sqrt{a}+\sqrt{b})}<0$

（4）①若$a,b>0$,求证:$\sqrt[3]{a^3+b^3}>\sqrt[4]{a^4+b^4}$；②若$a,b>0,n,m\in N^*,n<m$,求证:$\sqrt[n]{a^n+b^n}>\sqrt[m]{a^m+b^m}$.

①$key1$:$\because(\sqrt[3]{a^3+b^3})^{12}-(\sqrt[4]{a^4+b^4})^{12}=(a^3+b^3)^4-(a^4+b^4)^3$

$=4a^9b^3+6a^6b^6+4a^3b^6-3a^8b^4-3a^4b^8=a^3b^3(4a^6+6a^3b^3+4b^6-3a^5b-3ab^5)$

$>a^3b^3(3a^6+3b^6-3a^5b-3ab^5)=3a^3b^3(a-b)(a^5-b^5)\ge 0$

②$key2$:$\because a,b>0,\therefore(\sqrt[n]{a^n+b^n})^m=(a^n+b^n)^{\frac{m}{n}}=(a^n+b^n)(a^n+b^n)^{\frac{m-n}{n}}$

$=a^n(a^n+b^n)^{\frac{m-n}{n}}+b^n(a^n+b^n)^{\frac{m-n}{n}}>a^n(a^n)^{\frac{m-n}{n}}+b^n(b^n)^{\frac{m-n}{n}}=a^m+b^m,\therefore\sqrt[n]{a^n+b^n}>\sqrt[m]{a^m+b^m}$得证

三、基本不等式：若$a,b\in R$，则$\dfrac{a^2+b^2}{2}\ge(\dfrac{a+b}{2})^2\ge ab;a^2+b^2\ge -2ab$

和$\ge$积

结论：$a^2+b^2+c^2\ge\dfrac{(a+b+c)^2}{3}\ge ab+bc+ca$

若$a,b\in R_+$,则$\sqrt{\dfrac{a^2+b^2}{2}}\ge\dfrac{a+b}{2}\ge\sqrt{ab}\ge\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$

推广：若$a_i>0(i=1,2,\cdots,n)$，则$\sqrt{\dfrac{a_1^2+a_2^2+\cdots+a_n^2}{n}}\ge\dfrac{a_1+a_2+\cdots+a_n}{n}\ge\sqrt[n]{a_1a_2\cdots a_n}$

2022-10-16

两维柯西不等式：若$a_1,a_2,b_1,b_2\in R$，则$|a_1a_2+b_1b_2|\le\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}$.

三维柯西不等式：若$a_1,a_2,a_3\in R,b_1,b_2,b_3\in R$，且$b_1b_2b_3\ne0$，则 平方和的积$\ge$积的和的平方

$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)\ge(a_1b_1+a_2b_2+a_3b_3)^2$（当且仅当$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$时，等号成立）

变形：（权方和不等式）若$a_1,a_2,a_3,b_1,b_2,b_3>0$，则$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}\ge\frac{(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3})^2}{b_1+b_2+b_3}$

（2009 竞赛） 设实数x,y,z满足$x^2+y^2+z^2=1$,则$\sqrt{2}xy+yz$的最大值为_______. $\frac{\sqrt{3}}{2}$

$key:1=x^2+\frac{2}{3}y^2+\frac{1}{3}y^2+z^2\ge2\cdot x\cdot\sqrt{\frac{2}{3}}y+2\cdot\sqrt{\frac{1}{3}}y\cdot z=\frac{2}{\sqrt{3}}(\sqrt{2}xy+yz)$

变式 1.实数x,y,z满足$x^2+y^2+z^2=1$.则$(xy+yz)_{\max}=$_______,$(xy+yz)_{\min}=$_______； $\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}$

$key:1=x^2+y^2+z^2=x^2+\frac{1}{2}y^2+\frac{1}{2}y^2+z^2=2\cdot x\cdot\frac{1}{\sqrt{2}}y+2\cdot\frac{1}{\sqrt{2}}y\cdot z=\sqrt{2}(xy+yz)$

$1=x^2+y^2+z^2=x^2+\frac{1}{2}y^2+\frac{1}{2}y^2+z^2=2\cdot x\cdot(-\frac{1}{\sqrt{2}}y)+2\cdot(-\frac{1}{\sqrt{2}}y)\cdot z=-\sqrt{2}(xy+yz)$

$(xy-2yz)_{\max}=$_______,$(xy-2yz)_{\min}=$_______；

$key:1=x^2+y^2+z^2=x^2+\frac{1}{5}y^2+\frac{4}{5}y^2+z^2=2\cdot x\cdot\frac{1}{\sqrt{5}}y+2\cdot\frac{2}{\sqrt{5}}y\cdot(-z)=\frac{2}{\sqrt{5}}(xy-2yz)$

$1=x^2+y^2+z^2=x^2+\frac{1}{5}y^2+\frac{4}{5}y^2+z^2=2\cdot(-x)\cdot\frac{1}{\sqrt{5}}y+2\cdot\frac{2}{\sqrt{5}}y\cdot z=\frac{2}{\sqrt{5}}(-xy+2yz)$

$(xy+yz+zx)\in$_______. $[-\frac{1}{2},1]$

$key:1=x^2+y^2+z^2=\frac{x^2+y^2}{2}+\frac{y^2+z^2}{2}+\frac{z^2+x^2}{2}\ge xy+yz+zx$

$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)\ge01,\therefore xy+yz+zx\ge-\frac{1}{2}$

变式 2（1）已知实数a,b,c满足$a^2+b^2+c^2=1$,则$ab+c$的最小值为（　）$A.-2$ $B.-\frac{3}{2}$ $C.-1$ $D.-\frac{1}{2}$ C

$key:2=a^2+b^2+c^2+1\ge-2ab-2c,\therefore ab+c\ge-1$

（2）已知正实数 x,y,z 满足 $x^2+y^2+z^2=1$，则 $\frac{5-8xy}{z}$ 的最小值是（　　）C

A．6　　B．5　　C．4　　D．3

$key:1+\lambda=x^2+y^2+z^2+\lambda\ge2xy+2\sqrt{\lambda}z$得$(1+\lambda)-2xy\ge2\sqrt{\lambda}z$其中$\frac{1+\lambda}{5}=\frac{-2}{-8}$即$\lambda=\frac{1}{4}$

$\therefore\frac{5}{4}-2xy\ge z,\therefore\frac{5-8xy}{z}\ge4$

（3）已知实数a,b,c满足$\frac{1}{4}a^2+\frac{1}{4}b^2+c^2=1$，则$ab+2bc+2ca$的取值范围是(　C　)

A．$(-\infty,4]$　　B．$[-4,4]$　　C．$[-2,4]$　　D．$[-1,4]$

$key1:由(a+b+2c)^2=a^2+b^2+4c^2+2(ab+2ac+2bc)\ge 0,\therefore ab+2bc+2ca\ge -2$

$4=a^2+b^2+4c^2=\frac{1}{2}a^2+\frac{1}{2}b^2+\frac{1}{2}a^2+2c^2+\frac{1}{2}b^2+2c^2\ge ab+2ac+2bc$

$\therefore -2\le ab+2bc+2ca\le 4$

$key2:ab+2bc+2ca=a\cdot b+2\cdot\lambda b\cdot\frac{1}{\lambda}c+2\cdot\lambda a\cdot\frac{1}{\lambda}c\le\frac{a^2+b^2}{2}+\lambda^2b^2+\frac{1}{\lambda^2}c^2+\lambda^2a^2+\frac{1}{\lambda^2}c^2$

$=(\frac{1}{2}+\lambda^2)a^2+(\frac{1}{2}+\lambda^2)b^2+\frac{2}{\lambda^2}c^2=4(其中\frac{1}{2}+\lambda^2=\frac{1}{4}\cdot\frac{2}{\lambda^2}即\lambda^2=\frac{1}{2})$

（4）已知 $x,y\in R^+,x+y+z=1,则\sqrt{xy}+\sqrt{xz}-y-z的最大值是$（　）A

$A.\frac{\sqrt{3}-1}{2}$　　$B.\frac{1}{2}$　　$C.0$　　$D.\frac{\sqrt{2}-1}{2}$

$key:\sqrt{xy}+\sqrt{yz}-y-z=2\cdot\sqrt{\lambda x}\cdot\frac{1}{2\sqrt{\lambda}}\sqrt{y}+2\cdot\sqrt{\lambda x}\cdot\frac{1}{2\sqrt{\lambda}}\sqrt{z}-y-z$

$\le 2\lambda x+(\frac{1}{4\lambda}-1)y+(\frac{1}{4\lambda}-1)z=\frac{\sqrt{3}-1}{2}(x+y+z)=\frac{\sqrt{3}-1}{2}$（$其中2\lambda=\frac{1}{4\lambda}-1即\lambda=\frac{-1+\sqrt{3}}{4}$）

（5）设$a,b,c是不全为0的实数,则(\frac{ab+ac+bc+c^2}{a^2+b^2+2c^2})_{\max}=$___;$(\frac{ab+ac+bc+c^2}{a^2+b^2+2c^2})_{\min}=$____.

$key:由\frac{ab+ac+bc+c^2}{a^2+b^2+2c^2}=\frac{xy+x+y+1}{x^2+y^2+2}(x=\frac{a}{c},y=\frac{b}{c}\in R)$

$key1:x^2+y^2+2=\lambda x^2+\lambda y^2+(1-\lambda)x^2+\mu+(1-\lambda)y^2+\mu+2-2\mu$

$\ge 2\lambda xy+2\sqrt{(1-\lambda)\mu}x+2\sqrt{(1-\lambda)\mu}y+2-2\mu$

$(其中\lambda,\mu>0,且2\lambda=2\sqrt{(1-\lambda)\mu}=2-2\mu即\lambda=\mu=\frac{1}{2}),\therefore\frac{xy+x+y+1}{x^2+y^2+2}\le 1$

$key2:xy+x+y+1=xy+\lambda x\cdot\frac{1}{\lambda}+\lambda y\cdot\frac{1}{\lambda}+1\le\frac{x^2+y^2}{2}+\frac{1}{2}(\lambda^2x^2+\frac{1}{\lambda^2})+\frac{1}{2}(\lambda^2y^2+\frac{1}{\lambda^2})+1$

$=\frac{1+\lambda^2}{2}x^2+\frac{1+\lambda^2}{2}y^2+1+\frac{1}{\lambda^2}=x^2+y^2+2(其中2\cdot\frac{1+\lambda^2}{2}=\frac{1+\lambda^2}{\lambda^2}即\lambda^2=1,且当且仅当x=y=1时取=)$

$key1:x^2+y^2+2=\lambda x^2+\lambda y^2+(1-\lambda)x^2+\mu+(1-\lambda)y^2+\mu+2-2\mu(其中\lambda,\mu>0)$

$\ge -2\lambda xy-2\sqrt{(1-\lambda)\mu}x-2\sqrt{(1-\lambda)\mu}y+2-2\mu=-\sqrt{2}(xy+x+y+1)$

$(其中-\lambda=-\sqrt{(1-\lambda)\mu}=1-\mu即\lambda=\frac{\sqrt{2}}{2},\mu=\frac{2+\sqrt{2}}{2}),$

$key2:xy+x+y+1=-x\cdot(-y)-\lambda x\cdot(-\frac{1}{\lambda})-(\lambda y)\cdot\frac{1}{\lambda}+1\ge-\frac{x^2+y^2}{2}-\frac{\lambda^2x^2+\frac{1}{\lambda^2}}{2}-\frac{\lambda^2y^2+\frac{1}{\lambda^2}}{2}+1(\lambda>0)$

$=-\frac{1+\lambda^2}{2}x^2-\frac{1+\lambda^2}{2}y^2+1-\frac{1}{\lambda^2}=-\frac{\sqrt{2}}{2}(x^2+y^2+2)(其中2\cdot(-\frac{1+\lambda^2}{2})=\frac{\lambda^2-1}{\lambda^2}即\lambda^2=-1+\sqrt{2})$

$\therefore\frac{xy+x+y+1}{x^2+y^2+2}\ge-\frac{\sqrt{2}}{2}$

（2011 高考）16．设 x, y 为实数，若 $4x^2+y^2+xy=1$, 则 $2x+y$ 的最大值是___________. $\underline{\dfrac{2\sqrt{10}}{5}}$

$key1$:（判别式法）令$t=2x+y$，则$4x^2+(t-2x)^2+x(t-2x)=6x^2-3tx+t^2=1$

$\therefore \Delta=9t^2-24(t^2-1)=3(-5t^2+8)\ge 0, \therefore t^2\le \frac{8}{5}$

$key2: 1=(2x+y)^2-\frac{3}{2}\cdot 2x\cdot y\ge (2x+y)^2-\frac{3}{2}(\frac{2x+y}{2})^2=\frac{5}{8}(2x+y)^2, \therefore (2x+y)^2\le \frac{8}{5}$

（2013 山东）12. 设正实数 x,y,z 满足 $x^2-3xy+4y^2-z=0$, 则当 $\frac{xy}{z}$ 取得最大值时, $\frac{2}{x}+\frac{1}{y}-\frac{2}{z}$ 的最大值为（　）

A.0　　　B.1　　　C. $\frac{9}{4}$　　　D.3

$key: z=x^2-3xy+4y^2\ge 2\cdot x\cdot 2y-3xy=xy, \therefore \frac{xy}{z}\le 1$（当且仅当$x=2y$，$z=2y^2$时取$=$）

$\therefore \frac{2}{x}+\frac{1}{y}-\frac{2}{z}=\frac{2}{y}-\frac{1}{y^2}=-(\frac{1}{y}-1)^2+1\le 1$

（2018天津）实数x, y满足$x^2+y^2=20$，则$xy+8x+y$的最大值为_____.

（2018天津）$key1: xy+8x+y=2\cdot\frac{1}{2}x\cdot y+2\cdot x\cdot 4+2\cdot\frac{1}{2}y\cdot 1\le \frac{1}{4}x^2+y^2+x^2+16+\frac{1}{4}y^2+1$

$=42$（当且仅当$\begin{cases} x=2y \\ x=4 \\ y=2 \end{cases}$时，取$=$）

$key2: (xy+8+y)^2=(x\cdot y+8\cdot x+y\cdot 1)^2\le (x^2+64+y^2)(y^2+x^2+1)$

$=84\cdot 21$（当且仅当$\frac{x}{y}=\frac{8}{x}=\frac{y}{1}$时，取$=$）

（2022II）12．若实数 x，y 满足 $x^2+y^2-xy=1$，则（　BC　）

A． $x+y<1$　　B． $x+y\ge -2$　　C． $x^2+y^2\le 2$　　D． $x^2+y^2\ge 1$

变式 1（1）设 x, y 为实数，若 $4x^2+y^2+xy=1$. 则 $(x+y)\in$________; $[-\frac{4}{\sqrt{15}}, \frac{4}{\sqrt{15}}]$

$key1$: 判别式法

$key2: 1=4x^2+y^2+xy=\lambda x^2+\lambda y^2+(4-\lambda)x^2+(1-\lambda)y^2+xy$

$\ge \lambda x^2+\lambda y^2+(2\sqrt{(4-\lambda)(1-\lambda)}+1)xy$（其中$2\lambda=2\sqrt{(4-\lambda)(1-\lambda)}+1$即$\lambda=\frac{15}{16}$）

$=\frac{15}{16}(x+y)^2, \therefore (x+y)^2\le \frac{16}{15}$

$(x^2-xy)_{\max}=$________, $(x^2-xy)_{\min}=$________; $\frac{3+2\sqrt{6}}{15}, \frac{3-2\sqrt{6}}{15}$（基本方法：消掉 xy 项）

$key1: 1=4x^2+y^2+xy=\lambda x^2+(4-\lambda)x^2+y^2+xy\ge \lambda x^2+(2\sqrt{4-\lambda}+1)xy$（其中$\lambda+2\sqrt{4-\lambda}+1=0$即$\lambda=-3-2\sqrt{6}$）

$=(-3-2\sqrt{6})(x^2-xy),\therefore x^2-xy\ge \dfrac{1}{-3-2\sqrt{6}}$

$1=4x^2+y^2+xy=\lambda x^2+(4-\lambda)x^2+y^2+xy\ge \lambda x^2+(-2\sqrt{4-\lambda}+1)xy$（其中$\lambda-2\sqrt{4-\lambda}+1=0$即$\lambda=-3+2\sqrt{6}$）

$=(-3+2\sqrt{6})(x^2-xy),\therefore x^2-xy\le \dfrac{1}{-3+2\sqrt{6}}$

$key2:$令$t=x^2-xy$，则$t+1=5x^2+y^2$

由$1=4x^2+y^2+xy=4x^2+y^2+2\cdot\lambda x\cdot\dfrac{1}{2\lambda}y\le(4+\lambda^2)x^2+(1+\dfrac{1}{4\lambda^2})y^2$

$=\dfrac{9+\sqrt{6}}{10}(5x^2+y^2)$（其中$4+\lambda^2=5(1+\dfrac{1}{4\lambda^2})$即$\lambda^2=\dfrac{1+\sqrt{6}}{2}$）$\therefore t+1=5x^2+y^2\ge\dfrac{10}{9+\sqrt{6}}$

由$1=4x^2+y^2+xy=4x^2+y^2-2\cdot(-\lambda x)\cdot\dfrac{1}{2\lambda}y\ge(4-\lambda^2)x^2+(1-\dfrac{1}{4\lambda^2})y^2$

$=\dfrac{9-\sqrt{6}}{10}(5x^2+y^2)$（其中$4-\lambda^2=5(1-\dfrac{1}{4\lambda^2})$即$\lambda^2=\dfrac{-1+\sqrt{6}}{2}$），$\therefore t+1=5x^2+y^2\le\dfrac{10}{9-\sqrt{6}}$

$(x^2+2y^2)_{\max}=$______，$(x^2+2y^2)_{\min}=$______；（基本型：缺 xy 项） $\dfrac{4}{9+\sqrt{51}},\dfrac{4}{9-\sqrt{51}}$

$key: 1=4x^2+y^2+xy=4x^2+y^2+2\cdot\lambda x\cdot\dfrac{1}{2\lambda}y\le 4x^2+y^2+\lambda^2x^2+\dfrac{1}{4\lambda^2}y^2=(4+\lambda^2)x^2+(1+\dfrac{1}{4\lambda^2})y^2$

（其中$2(4+\lambda^2)=1+\dfrac{1}{4\lambda^2}$即$\lambda^2=\dfrac{-7+\sqrt{51}}{4}$）$=\dfrac{9+\sqrt{51}}{4}(x^2+2y^2),\therefore x^2+2y^2\le\dfrac{4}{9+\sqrt{51}}$

$1=4x^2+y^2+xy=4x^2+y^2-2\cdot(-\lambda x)\cdot\dfrac{1}{2\lambda}y\ge 4x^2+y^2-(\lambda^2x^2+\dfrac{1}{4\lambda^2}y^2)=(4-\lambda^2)x^2+(1-\dfrac{1}{4\lambda^2})y^2$

（其中$2(4-\lambda^2)=1-\dfrac{1}{4\lambda^2}$即$\lambda^2=\dfrac{7-\sqrt{51}}{4}$）$=\dfrac{-9+\sqrt{51}}{4}(x^2+2y^2),\therefore x^2+2y^2\le\dfrac{4}{-9+\sqrt{51}}$

$(x^2-y^2)_{\max}=$________，$(x^2-y^2)_{\min}=$________． $\dfrac{2}{3+2\sqrt{6}},\dfrac{2}{3-2\sqrt{6}}$ （基本型：缺 xy 项）

$key: 1=4x^2+y^2-2\cdot(-\lambda x)\cdot(\dfrac{1}{2\lambda}y)\ge 4x^2+y^2-(\lambda^2x^2+\dfrac{1}{4\lambda^2}y^2)$

$=(4-\lambda^2)x^2+(1-\dfrac{1}{4\lambda^2})y^2$（其中$4-\lambda^2=-(1-\dfrac{1}{4\lambda^2})$即$\lambda^2=\dfrac{5\pm2\sqrt{6}}{2}$）

$=\dfrac{3\pm2\sqrt{6}}{2}(x^2-y^2),\therefore\dfrac{2}{3-2\sqrt{6}}\le x^2-y^2\le\dfrac{2}{3+2\sqrt{6}}$，

$(2x+y+xy)\in$____________．

$\because 1=4x^2+xy+y^2=(2x+y)^2-3xy,\therefore xy=\dfrac{1}{3}[1-(2x+y)^2]$

$\therefore 2x+y+xy=(2x+y)+\dfrac{1}{3}[1-(2x+y)^2]=-\dfrac{1}{3}(2x+y-3)^2+\dfrac{10}{3}$

$\in[-\dfrac{1+2\sqrt{10}}{5},\dfrac{-1+2\sqrt{10}}{5}]$（$\because 2x+y\in[-\sqrt{\dfrac{8}{5}},\sqrt{\dfrac{8}{5}}]$）