数列（1）等差等比数列解答（5）

2024-03-09

变式 1.已知数列$\{a_n\}$的各项均为正数,给定正整数 k,若对任意的$n\in N^*$且$n>k$,都有$a_{n-k}a_{n-k+1}\cdots a_{n-1}a_{n+1}\cdots a_{n+k-1}a_{n+k}=a_n^{2k}$成立，则称数列$\{a_n\}$具有性质$T(k)$.

（1）若数列$\{a_n\}$具有性质$T(1)$，且$a_1=1$，$a_3=9$，求数列$\{a_n\}$的通项公式;

（2）若数列$\{a_n\}$既具有性质$T(2)$，又具有性质$T(3)$；证明：数列$\{a_n\}$是等比数列.

(1）解：$\because$ 数列$\{a_n\}$具有性质$T(1)$,且$a_1=1,a_3=9,a_n>0,\therefore a_{n-1}a_{n+1}=a_n^2,\therefore\{a_n\}$是等比数列，

$\therefore a_3=9=a_1q^2=q^2$得$q=3,\therefore a_n=3^{n-1}$

(2）证明：由数列$\{a_n\}$具有性质$T(2)$得$a_{n-2}a_{n-1}a_{n+1}a_{n+2}=a_n^4$

$\therefore a_{n-1}a_na_{n+2}a_{n+3}=a_{n+1}^4$,且$a_{n-3}a_{n-2}a_na_{n+1}=a_{n-1}^4,\therefore a_{n-1}^4a_{n+1}^4=a_{n-3}a_{n-2}a_{n-1}a_n^2a_{n+1}a_{n+2}a_{n+3}(n\ge4)$

由数列$\{a_n\}$具有性质$T(3)$得$a_{n-3}a_{n-2}a_{n-1}a_{n+1}a_{n+2}a_{n+3}=a_n^6$

$\therefore a_{n-1}^4a_{n+1}^4=a_n^8(\because a_n>0),\therefore a_{n-1}a_{n+1}=a_n^2(n\ge4)$

$\therefore \dfrac{a_{n+1}}{a_n}=\dfrac{a_n}{a_{n-1}}=\cdots=\dfrac{a_6}{a_5}=\dfrac{a_5}{a_4}=\dfrac{a_4}{a_3}$记为$q(q>0),\therefore a_1a_2a_4a_5=a_3^4\Leftrightarrow a_1a_2\cdot a_3q\cdot a_3q^2=a_3^4\Leftrightarrow a_1a_2q^3=a_3^2$

且$a_4^4=a_2a_3a_5a_6=a_2a_3\cdot a_3q^2\cdot a_3q^3=a_3^4q^4\Leftrightarrow a_2q=a_3$

$\therefore aq^2=a_3,\therefore\dfrac{a_3}{a_2}=\dfrac{a_2}{a_1}=q,\therefore\{a_n\}$是等比数列，证毕

变式 2. 若无穷数列$\{a_n\}$满足：$\exists m\in N^*$，对于$\forall n\ge n_0(n_0\in N^*)$，都有$\dfrac{a_{n+m}}{a_n}=q$（其中 q 为常数)，则称$\{a_n\}$具有性质“$Q(m,n_0,q)$”.（1）若$\{a_n\}$具有性质“$Q(4,2,3)$”，且$a_3=1$, $a_5=2$, $a_6+a_9+a_{11}=20$，求a_2；

（2）若无穷数列$\{b_n\}$是等差数列，无穷数列$\{c_n\}$是公比为 2 的等比数列，$b_2=c_3=4$, $b_1+c_1=c_2$, $a_n=b_n+c_n$，判断$\{a_n\}$是否具有性质“$Q(2,1,3)$”，并说明理由;

（3）设$\{a_n\}$既具有性质“$Q(i,1,q_1)$”，又具有性质“$Q(j,1,q_2)$”，其中$i,j\in N^*,i<j$，求证：$\{a_n\}$具有性质“$Q(j-i,i+1,q_2^{\frac{j-i}{j}})$”.

(1）解：由已知得$\begin{cases}a_6+a_9+a_{11}=3a_2+3a_5+9a_3=20\\a_3=1\\a_5=2\end{cases}$ 得$a_2=\dfrac{5}{3}\cdots4$分

(2）解: 由已知得$\begin{cases}b_1+d=4c_1=4\\b_1+c_1=2c_1\end{cases}$得$c_1=b_1=1,d=1,\therefore a_n=n+2^{n-1}$,

若$\{a_n\}$具有性质"$Q(2,1,3)$", 则$\dfrac{a_{n+2}}{a_n}=3\Leftrightarrow n+2+2^{n+1}=3(n+2^{n-1})\Leftrightarrow 0=3\cdot2^{n-1}-2n+2$对$n\in N^*$恒成立

设$p(n)=3\cdot2^{n-1}-2n+2$, 则$p(n+1)-p(n)=3\cdot2^{n-1}-2\ge1>0$

$\therefore p(n)$在$n\in N^*$上递增,$\therefore p(n)\ge p(1)=3>0,\therefore p(n)\ne0,\therefore\{a_n\}$不具有性质"$Q(2,1,3)$"

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（3）证明：$\because \{a_n\}$具有性质“$Q(i,1,q_1)$”，$\therefore \exists i\in N^*$，使得$\frac{a_{n+i}}{a_n}=q_1(n\in N^*)$即$a_{n+i}=a_nq_1$，

$\because \{a_n\}$也具有性质“$Q(j,1,q_2)$”，$\therefore \exists j\in N^*$，使得$\frac{a_{n+j}}{a_n}=q_2(n\in N^*)$即$a_{n+j}=a_nq_2$，

$\therefore a_{n+ij}=a_{n+i}q_1^{j-1}=a_nq_1^j$，且$a_{n+ij}=a_{n+j}q_2^{j-1}=a_nq_2^j$，$\therefore q_1^j=q_2^i$

$\{a_n\}$具有性质“$Q(j-i,i+1,q_2^{\frac{j-i}{j}})$”$\Leftrightarrow \frac{a_{n+j-i}}{a_n}=q_2^{\frac{j-i}{j}}(n\ge i+1)\cdots(*)$

而$\frac{a_{n+j-i}}{a_n}=\frac{a_{n-i+j}}{a_{n-i}}\cdot\frac{a_{n-i}}{a_n}=q_2\cdot\frac{1}{\frac{a_{n-i+i}}{a_{n-i}}}=\frac{q_2}{q_1}$(其中$n-i\ge 1$)

$\therefore (*)\Leftrightarrow \frac{q_2}{q_1}=q_2^{\frac{j-i}{j}}\Leftrightarrow q_2^{\frac{i}{j}}=q_2^{1-\frac{j-i}{j}}=q_1\Leftrightarrow q_2^i=q_1^j$成立，证毕

（1995全国）正项等比数列$\{a_n\}$的前n项和为S_n.⑴ 比较$\frac{\lg S_n+\lg S_{n+2}}{2}$与$\lg S_{n+1}$的大小;

（2）是否存在常数$c>0$，使得对任何$n\in N^*$，恒有$\frac{1}{2}[\lg(S_n-c)+\lg(S_{n+2}-c)]=\lg(S_{n+1}-c)$?

1995全国（1）由$S_nS_{n+2}-S_{n+1}^2=(S_{n+1}-a_{n+1})(S_{n+1}+a_{n+2})-S_{n+1}^2=a_{n+2}S_{n+1}-a_{n+1}S_{n+1}-a_{n+1}a_{n+2}$
$=a_{n+2}(a_1+a_2+\cdots+a_n+a_{n+1})-a_{n+1}(a_1+a_2+\cdots+a_{n+1})-a_{n+1}a_{n+2}=-a_{n+1}a_1<0(\because a_n>0)$
$\therefore S_nS_{n+2}<S_{n+1}^2$，$\therefore \frac{\lg S_n+\lg S_{n+2}}{2}<\lg S_{n+1}$

（2）假设存在，由$\frac{1}{2}[\lg(S_n-c)+\lg(S_{n+2}-c)]=\lg(S_{n+1}-c)$得

$(S_n-c)(S_{n+2}-c)-(S_{n+1}-c)^2=(S_{n+1}-c-a_{n+1})(S_{n+1}-c+a_{n+2})-(S_{n+1}-c)^2$
$=-a_{n+1}(a_1+a_2+\cdots+a_n+a_{n+1}-c)-a_{n+1}a_{n+2}+a_{n+2}(a_1+a_2+\cdots+a_n+a_{n+1})-ca_{n+2}$

$=-a_1a_{n+1}+ca_{n+1}-ca_{n+2}=a_{n+1}(-a_1+c-cq)=0$，$\therefore c=\frac{a_1}{1-q}>0$

此时$S_n-c=\frac{a_1(1-q^n)}{1-q}-\frac{a_1}{1-q}=\frac{-a_1q^n}{1-q}<0$，$\therefore$ 不存在

（1996Ⅰ）等比数列$\{a_n\}$的首项$a_1=-1$，前n项和为S_n，若$\frac{S_{10}}{S_5}=\frac{31}{32}$，则$\lim\limits_{n\to\infty}S_n=$（ ） $A.\frac{2}{3}$　$B.-\frac{2}{3}$　$C.2$　$D.-2$

1996Ⅰ key：由已知得$q\ne 1$，$\therefore \frac{S_{10}}{S_5}=q^5+1=\frac{31}{32}$得$q=-\frac{1}{2}$，$\therefore \lim\limits_{n\to\infty}S_n=\frac{-1}{1+\frac{1}{2}}=-\frac{2}{3}$，选$B$

（1998A）各项均为实数的等比数列$\{a_n\}$前n项和记为S_n，若$S_{10}=10$，$S_{30}=70$，则S_{40}等于（ ）
$A.150$　$B.-200$　$C.150$或-200　$D.400$或-50

1998$Akey$：由已知得$q\ne 1$，则$\begin{cases}S_{10}=\frac{a_1(1-q^{10})}{1-q}=10\\ S_{30}=\frac{a_1(1-q^{30})}{1-q}=70\end{cases}$，$\therefore 1+q^{10}+q^{20}=7$得$q^{10}=2$，$\therefore \frac{a_1}{1-q}=-10$

$\therefore S_{40}=\frac{a_1(1-q^{40})}{1-q}=-10\cdot(-15)=150$，选$A$

数列（1）等差等比数列解答（5）

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(2009辽宁)6.设等比数列$\{a_n\}$的前n项和为S_n，若$\dfrac{S_6}{S_3}=3$,则$\dfrac{S_9}{S_6}=$（　）$A.2$ $B.\dfrac{7}{3}$ $C.\dfrac{8}{3}$ $D.3$

2009辽宁key:$\dfrac{S_6}{S_3}=\dfrac{S_6-S_3+S_3}{S_3}=3$得$\dfrac{S_6-S_3}{S_3}=2,\therefore S_9-S_6=4S_3,\therefore\dfrac{S_9}{S_6}=\dfrac{S_3+2S_3+4S_3}{S_3+2S_3}=\dfrac{7}{3}$,选$B$

（2009山东2016四川）设等比数列$\{a_n\}$的前n项和为S_n，且$S_n=2^n+r$（r为常数），记$b_n=2(1+\log_2 a_n)(n\in N^*)$.(1) 求数列$\{a_nb_n\}$的前$n$项和$T_n$;

（2）若对于任意的正整数n，都有$\dfrac{1+b_1}{b_1}\cdot\dfrac{1+b_2}{b_2}\cdots\dfrac{1+b_n}{b_n}\ge k\sqrt{n+1}$成立，求实数$k$的最大值.

2009山东2016四川解：（1）由$a_n=\begin{cases}2+r,n=1,\\2^n-2^{n-1}=2^{n-1},n\ge 2\end{cases},\therefore 2+r=1=2^{1-1}=1$得$r=-1$

$\therefore b_n=2(1+n-1)=2n,\therefore a_nb_n=n\cdot 2^n$

$\therefore T_n=1\cdot 2^1+2\cdot 2^2+\cdots+n\cdot 2^n$,

$2T_n=1\cdot 2^2+\cdots+(n-1)\cdot 2^n+n\cdot 2^n,\therefore -T_n=2+2^2+\cdots+2^n-n\cdot 2^n=\dfrac{2(1-2^{n-1})}{1-2}-n\cdot 2^n,\therefore T_n=(n+1)\cdot 2^n-2$

（2）由（1）得$\dfrac{1+b_1}{b_1}\cdot\dfrac{1+b_2}{b_2}\cdot\cdots\cdot\dfrac{1+b_n}{b_n}=\dfrac{3\cdot 5\cdot\cdots\cdot(2n+1)}{2\cdot 4\cdot\cdots\cdot(2n)}\ge k\sqrt{n+1}\Leftrightarrow k\le\dfrac{3\cdot 5\cdot\cdots\cdot(2n+1)}{\sqrt{n+1}\cdot 2\cdot 4\cdot\cdots\cdot(2n)}$记为$f(n)$

则$\dfrac{f(n+1)}{f(n)}=\dfrac{\dfrac{3\cdot 5\cdot\cdots\cdot(2n+1)(2n+3)}{\sqrt{n+2}\cdot 2\cdot 4\cdot\cdots\cdot(2n)\cdot(2n+2)}}{\dfrac{3\cdot 5\cdot\cdots\cdot(2n+1)}{\sqrt{n+1}\cdot 2\cdot 4\cdot\cdots\cdot(2n)}}=\dfrac{\sqrt{n+1}(2n+3)}{\sqrt{n+2}\cdot(2n+2)}=\dfrac{2n+3}{2\sqrt{(n+1)(n+2)}}>1$

$\Leftrightarrow n\in N^*,\therefore f(n)_{\min}=f(1)=\dfrac{3}{2\sqrt{2}},\therefore k$的最大值为$\dfrac{3\sqrt{2}}{4}$

变式 1.已知正项等比数列$\{a_n\}$中，前n项和为S_n.

若$S_n=80,S_{2n}=6560$,且前n项中,最大项为54,则$n=$______.

key:由已知得$q\ne 1$，则$\begin{cases}S_n=\dfrac{a_1(1-q^n)}{1-q}=80\\S_{2n}=\dfrac{a_1(1-q^{2n})}{1-q}=6560\end{cases},\therefore 1+q^n=82$即$q^n=81,\because q>0,\therefore q>1$

$\therefore (a_n)_{\max}=a_1q^{n-1}=\dfrac{81a_1}{q}=54$即$3a_1=2q$,而$\dfrac{a_1(1-q^n)}{1-q}=\dfrac{80a_1}{q-1}=80,\therefore a_1=2,q=3,\therefore n=4$

（2005Ⅰ）设等比数列$\{a_n\}$的公比为q,前n项和$S_n>0(n\in N^*)$.(1）求q的取值范围;

（2）设$b_n=a_{n+2}-\dfrac{3}{2}a_{n+1}$,记$\{b_n\}$的前$n$项和为$T_n$,试比较$S_n$与$T_n$的大小.

（2005Ⅰ）key:$S_1=a_1>0,S_2=a_1(1+q)>0$得$q>-1$,

若$q>0$,则$S_n>0$；若$q\in(-1,0)$,则$S_n=\dfrac{a_1(1-q^n)}{1-q}>0,\therefore q\in(-1,0)\cup(0,+\infty)$

（2）由$b_n=a_n(q^2-\dfrac{3}{2}q)$得$T_n=(q^2-\dfrac{3}{2}q)S_n$

$\therefore T_n - S_n = S_n(q^2 - \frac{3}{2}q - 1) > 0 \Leftrightarrow -1 < q < -\frac{1}{2}, or, q > 2$

$\therefore$ 当$q \in (-1, -\frac{1}{2}) \cup (2, +\infty)$时，$T_n > S_n$；当$q \in \{-\frac{1}{2}, 2\}$时，$T_n = S_n$；当$q \in (-\frac{1}{2}, 0) \cup (0, 2)$时，$T_n < S_n$

（2020 II）6.数列$\{a_n\}$中，$a_1 = 2$，$a_{m+n} = a_m a_n$，若$a_{k+1} + a_{k+2} + \cdots + a_{k+10} = 2^{15} - 2^5$，则$k =$（　C　）

A. 2　　B. 3　　C. 4　　D. 5

（2020 山东）18.已知公比大于 1 的等比数列$\{a_n\}$满足$a_2 + a_4 = 20, a_3 = 8$．（1）求$\{a_n\}$的通项公式；

（2）记b_m为$\{a_n\}$在区间$(0, m](m \in N^*)$中的项的个数，求数列$\{b_m\}$的前 100 项和S_{100}．

解:（1）由于数列$\{a_n\}$是公比大于1的等比数列，设首项为a_1，公比为q，依题意有$\begin{cases} a_1 q + a_1 q^3 = 20 \\ a_1 q^2 = 8 \end{cases}$，

解得$a_1 = 2, q = 2$，或$a_1 = 32, q = \frac{1}{2}$(舍)，

所以$a_n = 2^n$，所以数列$\{a_n\}$的通项公式为$a_n = 2^n$．

（2）由$a_n = 2^n \le m \Leftrightarrow n \le \log_2 m$即$n \le [\log_2 m]$

$\therefore b_{2^k} = b_{2^k+1} = \cdots = b_{2^{k+1}-1} = k$

$\therefore S_{100} = 1 \times (2^2 - 2) + 2 \times (2^3 - 2^2) + 3 \times (2^4 - 2^3) + 4 \times (2^5 - 2^4) + 5 \times (2^6 - 2^5) + 6 \times (100 - 2^6 + 1) = 480$

（2021 甲）7. 等比数列$\{a_n\}$的公比为q，前n项和为S_n，设甲：$q > 0$，乙：$\{S_n\}$是递增数列，（ ）B

A. 甲是乙的充分条件但不是必要条件 B. 甲是乙的必要条件但不是充分条件

C. 甲是乙的充要条件　　　　　　　　D. 甲既不是乙的充分条件也不是乙的必要条件

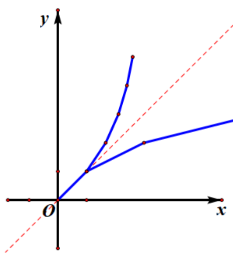
(2023II)8.记S_n为等比数列$\{a_n\}$的前n项和，若$S_4 = -5, S_6 = 21S_2$，则$S_8 =$(　) $A.120$　$B.85$　$C.-85$　$D.-120$

2023II key:由已知得$q \ne 1, \therefore \begin{cases} S_4 = \frac{a_1(1-q^4)}{1-q} = -5 \\ S_6 = \frac{a_1(1-q^6)}{1-q} = 21S_2 = \frac{21a_1(1-q^2)}{1-q} \end{cases}$得$q^2 = 4, \therefore S_8 = \frac{a_1(1-q^8)}{1-q} = -5 \cdot (1+16) = -85$, 选$C$

(1999) 已知函数$y = f(x)$的图象是自原点出发的一条折线，当$n \le y \le n+1 (n = 0, 1, 2, \cdots)$时，该图象是斜率为$b^n$的线段（其中正常数$b \ne 1$), 该数列$\{x_n\}$由$f(x_n) = n (n \in N^*)$定义.

（Ⅰ）求x_1, x_2和x_n的表达式；（Ⅱ）求$f(x)$的表达式，并写出其定义域；

(III) 求证：$y = f(x)$的图象与$y = x$的图象没有横坐标大于1的交点.



(1) 由已知得当$y = n$时, $x_n = n$，则$\frac{n+1-n}{x_{n+1} - x_n} = b^n$即$x_{n+1} - x_n = \frac{1}{b^n}$, 且$x_0 = 0, x_1 = 1, x_2 = 1 + \frac{1}{b}$,

$\therefore x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0) = \frac{1}{b^{n-1}} + \frac{1}{b^{n-2}} + \cdots + \frac{1}{b^0} = \frac{b}{1-b}(1 - \frac{1}{b^n}), n \in N^*$,

(2) 由 (1) 得$f(x) = b^n(x - x_n) + n(x_n \le x \le x_{n+1}$, 其中$x_n = \frac{b}{1-b}(1 - \frac{1}{b^n}))$, 定义域为$(0, +\infty)$.

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(3) 证明：当$b>1$时，只需证明：$x_n=\dfrac{b^0+b^1+\cdots+b^{n-1}}{b^{n-1}}<n(n\ge 2)\cdots(*)$

$\Leftrightarrow (\dfrac{1}{b})^{n-1}+(\dfrac{1}{b})^{n-2}+\cdots+(\dfrac{1}{b})^0<n(\because b>1,\therefore 0<\dfrac{1}{b}<1,\therefore (\dfrac{1}{b})^i<1)$

当$0<b<1$时，只需证明：$x_n=\dfrac{b^0+b^1+\cdots+b^{n-1}}{b^{n-1}}>n(n\ge 2)$

$\Leftrightarrow (\dfrac{1}{b})^{n-1}+(\dfrac{1}{b})^{n-2}+\cdots+(\dfrac{1}{b})^0>n(\because 0<b<1,\therefore \dfrac{1}{b}>1,\therefore (\dfrac{1}{b})^i>1),\therefore (*)$成立，$\therefore$ 得证

(2010竞赛)6.设$\{a_n\},\{b_n\}$分别为等差数列与等比数列，且$a_1=b_1=4,a_4=b_4=1$,则以下结论正确的是（　）

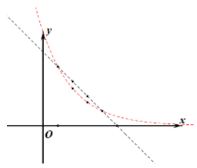
$A.a_2>b_2\quad B.a_3<b_3\quad C.a_5>b_5\quad D.a_6>b_6$　　A

变式 1：已知正项等比数列$\{a_n\}$与正项等差数列$\{b_n\}$满足$a_1=b_1,a_m=b_m(m>2,m\in N^*)$.试比较$a_n$与$b_n$的大小.

变式1：由已知得$q>0$,当公比$q=1$时，$a_n=b_n$;

当$q>1$时，$a_m=a_1+(m-1)d=b_m=a_1\cdot q^{m-1}$得$d=\dfrac{a_1(q^{m-1}-1)}{m-1}$,

当$1<n<m$时，$b_n-a_n=a_1q^{n-1}-(a_1+(n-1)\cdot\dfrac{a_1(q^{m-1}-1)}{m-1})=a_1[(q^{n-1}-1)-\dfrac{n-1}{m-1}(q^{m-1}-1)]$

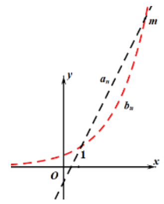


$=\dfrac{1}{m-1}a_1(q-1)[(m-1)(q^{n-2}+\cdots+q+1)-(n-1)(q^{m-2}+\cdots+q+1)]$

$=\dfrac{1}{m-1}a_1(q-1)[(m-n)(q^{n-2}+\cdots+q+1)-(n-1)(q^{m-2}+\cdots+q^{n-1})]$

$<\dfrac{1}{m-1}a_1(q-1)[(m-n)(n-1)q^{n-2}-(n-1)(m-n)q^{n-1}]=-\dfrac{(m-n)(n-1)}{m-1}a_1(q-1)^2q^{n-2}<0,\therefore a_n>b_n$

当$n>m$时，$b_n-a_n=a_1q^{n-1}-(a_1+(n-1)\cdot\dfrac{a_1(q^{m-1}-1)}{m-1})=a_1[(q^{n-1}-1)-\dfrac{n-1}{m-1}(q^{m-1}-1)]$



$=\dfrac{1}{m-1}a_1(q-1)[(m-1)(q^{n-2}+\cdots+q+1)-(n-1)(q^{m-2}+\cdots+q+1)]$

$=\dfrac{1}{m-1}a_1(q-1)[(m-1)(q^{n-2}+\cdots+q^{m-1})-(n-m)(q^{m-2}+\cdots+q+1)]$

$>\dfrac{1}{m-1}a_1(q-1)[(n-m)(m-1)q^{m-1}-(m-1)(n-m)q^{m-2}]=\dfrac{(n-m)(m-1)}{m-1}a_1(q-1)^2q^{m-2}>0,\therefore a_n<b_n$

(2022A)10.给定正整数$m(m\ge 3)$.设正项等差数列$\{a_n\}$与正项等比数列$\{b_n\}$满足:$\{a_n\}$的首项等于$\{b_n\}$的公比，$\{b_n\}$的首项等于$\{a_n\}$的公差，且$a_m=b_m$.求a_m的最小值，并确定当a_m取到最小值时a_1与b_1的比值.

解：设$\{a_n\}$的公差为d,$\{b_n\}$的公比为q,且$a_1=q>0,b_1=d>0$,

则$q+(m-1)d=d\cdot q^{m-1}$即$d=\dfrac{q}{q^{m-1}-m+1}>0$得$q>1$

$\therefore a_m=q+(m-1)d=q+(m-1)\cdot\dfrac{q}{q^{m-1}-m+1}=\dfrac{q^m}{q^{m-1}-m+1}$记为$f(q)$,

则$f'(q)=\dfrac{mq^{m-1}(q^{m-1}-m+1)-q^m\cdot(m-1)q^{m-2}}{(q^{m-1}-m+1)^2}=\dfrac{q^{m-1}(q^{m-1}-(m^2-m))}{(q^{m-1}-m+1)^2}>0\Leftrightarrow q>(m^2-m)^{\frac{1}{m-1}}(\because m\ge 3)$

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$\therefore a_m$的最小值为$f((m^2-m)^{\frac{1}{m-1}})=(\frac{m}{m-1})^{\frac{m}{m-1}}$，相应的$\frac{a_1}{b_1}=\frac{q}{d}=m(m-1)-(m-1)=(m-1)^2$

(2018Ⅰ)14.已知集合$A=\{x\mid x=2n-1,n\in N^*\},B=\{x\}x=2^n,n\in N^*\}$，将$A\cup B$的所有元素从小到大依次排列构成一个数列$\{a_n\}$的前$n$项和，则使得$S_n>12a_{n+1}$成立的$n$的最小值为______.

2018Ⅰ key：当$a_n=2^k$时，$S_n=(1+3+\cdots+(2^k-1))+(2^1+2^2+\cdots+2^k)=\frac{2^{k-1}\cdot 2^k}{2}+\frac{2(1-2^k)}{1-2}$

$=2^{2k-2}+2^{k+1}-2>12a_{n+1}=12\cdot(2^k+1)\Leftrightarrow 2^{2k-2}-20\cdot 2^{k-1}-14>0\Leftrightarrow 2^{k-1}>10+\sqrt{114},\therefore k\geq 6$

当$k=6$时，$a_n=2^6,n=2^5+6=38$

当$2^5<a_n<2^6-1$时，$S_n=(1+3+\cdots+(2^5-1))+(2^1+2^2+2^3+2^4+2^5)+(n-21)(2^5+1)+\frac{(n-21)(n-22)}{2}\cdot 2$

$=2^8+2^6-2+(n-21)(n+11)=n^2-10n+87>12a_{n+1}=12(2^5+1+(n-21)\cdot 2)=12(2n-9)$

$\Leftrightarrow n^2-34n+195>0$得$n>17+\sqrt{94},\therefore n\geq 27,\therefore n$的最小值为27

(2022Ⅱ)17.已知$\{a_n\}$为等差数列，$\{b_n\}$为公比为2的等比数列，且$a_2-b_2=a_3-b_3=b_4-a_4$.

(1) 证明：$a_1=b_1$；（2）求集合$\{k\mid b_k=a_m+a_1,1\leq m\leq 500\}$中元素个数.

2022Ⅱ解：（1）设$\{a_n\}$的公差为d，则由已知得$\begin{cases}2b_1=b_3-b_2=a_3-a_2=d\\2a_1+5d=a_3+a_4=b_3+b_4=12b_1\end{cases},\therefore d=2b_1,a_1=b_1$

(2) 由（1）得$b_k=b_1\cdot 2^{k-1}=a_m+a_1=2b_1+(m-1)\cdot 2b_1$

$\Leftrightarrow 2^{k-2}=m\in[1,500]\Leftrightarrow 2\leq k\leq 8,\therefore$已知集合中有7个元素

(2009天津）已知等差数列$\{a_n\}$的公差为$d(d\neq 0)$，等比数列$\{b_n\}$的公比为$q(q>1)$.

设$S_n=a_1b_1+a_2b_2+\cdots+a_nb_n,T_n=a_1b_1-a_2b_2+\cdots+(-1)^{n-1}a_nb_n,n\in N^*$.

(1) 若$a_1=b_1=1,d=2,q=3$，求S_3的值；（2）若$b_1=1$，证明:$(1-q)S_{2n}-(1+q)T_{2n}=\frac{2dq(1-q^{2n})}{1-q^2},n\in N^*$；

(3) 若正数n满足$2\leq n\leq q$，设$k_1,k_2,\cdots,k_n$和$l_1,l_2,\cdots,l_n$是$1,2,\cdots,n$的两个不同的排列，

$c_1=a_{k_1}b_1+a_{k_2}b_2+\cdots+a_{k_n}b_n,c_2=a_{l_1}b_1+a_{l_2}b_2+\cdots+a_{l_n}b_n$，证明:$c_1\neq c_2$.

2009天津（1）解：由已知得$a_n=1+2(n-1)=2n-1,b_n=3^{n-1},\therefore S_3=1+3\cdot 3+5\cdot 9=55$

(2) 证明：由已知得$a_n=a_1+(n-1)d,b_n=q^{n-1}$,

$\therefore[(1-q)a_{2k-1}b_{2k-1}-(1+q)(-1)^{2k-2}a_{2k-1}b_{2k-1}]+[(1-q)a_{2k}b_{2k}-(1+q)(-1)^{2k-1}a_{2k}b_{2k}]$

$=-2qa_{2k-1}b_{2k-1}+2a_{2k}b_{2k}=-2a_{2k-1}b_{2k}+2a_{2k}b_{2k}=2db_{2k}=2bq^{2k-1}$

$\therefore(1-q)S_{2n}-(1+q)T_{2n}=2d(b_2+b_4+\cdots+b_{2n})=2d\cdot\frac{q(1-q^{2n})}{1-q^2}=\frac{2dq(1-q^{2n})}{1-q^2}$，证毕

(3) 证明：由$c_1-c_2=db_1(k_1-l_1)+db_1(k_2-l_2)q^1+\cdots+db_1(k_n-l_n)q^{n-1}$

$\because db_1\neq 0,\therefore\frac{c_1-c_2}{db_1}=k_1-l_1+(k_2-l_2)q+\cdots+(k_n-l_n)q^{n-1}$

若$k_n\neq l_n$，取$i=n$；若$k_n=l_n$，取i满足$k_i\neq l_i$且$k_j=l_j(i+1\leq j\leq n),\therefore\frac{c_1-c_2}{db_1}=k_1-l_1+(k_2-l_2)q+\cdots+(k_i-l_i)q^{i-1}$

①当$k_i<l_i$时，得$k_i-l_i\leq -1,\because q\geq n\geq 2,\therefore k_j-l_j\leq q-1(j=1,2,\cdots,i-1)$

$\therefore\frac{c_1-c_2}{db_1}\leq q-1+(q-1)q+\cdots+(q-1)q^{i-1}-q^{i-1}=(q-1)\cdot\frac{1-q^{i-1}}{1-q}-q^{i-1}=-1<0$,

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②当$k_i > l_i$时，得$j_i - l_i \ge 1, \because q \ge n \ge 2, \therefore k_j - l_j \le q-1 (j=1,2,\cdots,i-1)$

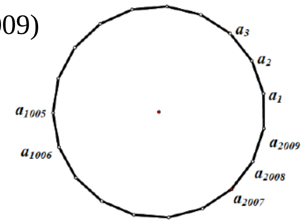
$\therefore \dfrac{c_1 - c_2}{db_1} \ge -(q-1)-(q-1)q-\cdots-(q-1)q^{i-1}+q^{i-1} = -(q-1)\cdot\dfrac{1-q^{i-1}}{1-q}+q^{i-1}=1>0$,综上：$c_1 \ne c_2$

(2009重庆)21.设m个不全相等的正数$a_1,a_2,\cdots,a_m (m\ge 7)$依次围成一个圆圈.

(1）若$m=2009$，且$a_1,a_2,\cdots,a_{1005}$是公差为d的等差数列，而$a_1,a_{2009},a_{2008},\cdots,a_{1006}$是公比为$q=d$的等比数列；数列$a_1,a_2,\cdots,a_m$的前$n$项和$S_n(n\le m)$满足：$S_3=15, S_{2009}=S_{2007}+12a_1$,求通项$a_n(n\le m)$;

(2）若每个数$a_n(n\le m)$是其左右相邻两数平方的等比中项，求证：$a_1+\cdots+a_6+a_7^2+\cdots+a_m^2 > ma_1a_2\cdots a_m$.

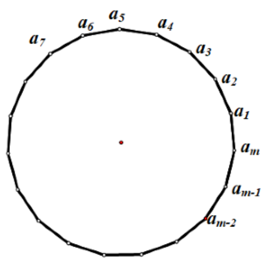
2009重庆（1）解：由已知得$a_i = a_1+(i-1)d (1\le i\le 1005), a_j = a_1\cdot d^{2010-j} (1006\le j\le 2009)$



$\therefore \begin{cases} S_3 = 3(a_1+d)=15 \\ S_{2009}-S_{2007}=a_{2008}+a_{2009}=a_1d(1+d)=12a_1 \end{cases}$ 得$a_1=2, d=3$,

$\therefore a_n = \begin{cases} 3i-1, 1\le i\le 1005, \\ 2\cdot 3^{2010-i}, 1006\le i\le 2009. \end{cases}$

(2）证明：由已知得$a_i = a_{i-1}a_{i+1} (2\le i\le m-1)$,且$a_1 = a_2a_m, a_m = a_1a_{m-1}$



$\therefore a_1a_2a_3\cdots a_{m-1}a_m = a_2a_m\cdot a_1a_3\cdot a_2a_4\cdot a_3a_5\cdots a_{m-3}a_{m-1}\cdot a_{m-2}a_m\cdot a_1a_{m-1}$

$= a_1^2a_2^2\cdots a_{m-1}^2a_m^2, \therefore a_1a_2\cdots a_m = 1$

由$a_2 = a_1a_3, a_3 = a_2a_4, a_4 = a_3a_5, a_5 = a_4a_6$, 得$a_2a_3 = a_1a_3a_2a_4 \Leftrightarrow a_1a_4 = 1, \therefore a_4 = \dfrac{1}{a_1}$,

$a_2a_3a_4 = a_1a_3a_2a_4a_3a_5 \Leftrightarrow 1 = a_1a_3a_5 = a_2a_5, \therefore a_5 = \dfrac{1}{a_2}, \therefore a_6 = \dfrac{a_5}{a_4} = \dfrac{1}{a_2a_4} = \dfrac{1}{a_3}, \therefore a_1a_2a_3a_4a_5a_6 = 1$

$\therefore$由平均值不等式得$a_1+\cdots+a_6+a_7^2+\cdots+a_m^2 > m(a_1\cdots a_6a_7^2\cdots a_m^2)^{\frac{1}{m}}$

$= m(\dfrac{1}{a_1\cdots a_6})^{\frac{1}{m}} = m = ma_1a_2\cdots a_m$，证毕

(2009湖南文）对于数列$\{u_n\}$，若存在常数$M>0$，对任意的$n\in N^*$，恒有$|u_{n+1}-u_n|+|u_n-u_{n-1}|+\cdots+|u_2-u_1|\le M$,则称数列$\{u_n\}$为$B-$数列.

(1) 首项为1，公比为$-\dfrac{1}{2}$的等比数列是否为$B-$数列？请说明理由；

(2）设S_n是数列$\{x_n\}$的前n项和，给出下列两组判断：A组：①数列$\{x_n\}$是$B-$数列，②数列$\{x_n\}$不是$B-$数列；B组：③数列$\{S_n\}$是$B-$数列，④数列$\{S_n\}$不是$B-$数列.请以其中一组中的一个论断为条件，另一组中的一个结论组成一个命题.判断所给命题的真假，并证明你的结论；

(3）若数列$\{a_n\}$是$B-$数列，证明：数列$\{a_n^2\}$也是$B-$数列.

2009湖南文（1）解：由已知得$u_n = (-\dfrac{1}{2})^{n-1}$

$\therefore |u_{n+1}-u_n|+|u_n-u_{n-1}|+\cdots+|u_2-u_1| = \dfrac{3}{2}[(\dfrac{1}{2})^{n-1}+(\dfrac{1}{2})^{n-2}+\cdots+1] = \dfrac{3}{2}\cdot\dfrac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 3-\dfrac{3}{2^n} < 3 = M$

$\therefore$数列$\{u_n\}$是$B-$数列

(2）解：若数列$\{x_n\}$是$B-$数列，则数列$\{S_n\}$不是$B-$数列，且是真命题

取$x_n=1$，则$x_{n+1}-x_n=0, \therefore |x_{n+1}-x_n|+|x_n-x_{n-1}|+\cdots+|x_2-x_1| = 0 < M$($M$为常数）

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此时$S_n=n,\therefore |S_{n+1}-S_n|+|S_n-S_{n-1}|+\cdots+|S_2-S_1|=n$

$\therefore$ 对于任意常数N,当$n>N$时,$|S_{n+1}-S_n|+|S_n-S_{n-1}|+\cdots+|S_2-S_1|>N,\therefore \{S_n\}$不是$B-$数列

(3) 证明:$\because \{a_n\}$是$B-$数列,$\therefore$ 存在常数$M>0$,对任意的$n\in N^*$，恒有$|a_{n+1}-a_n|+|a_n-a_{n-1}|+\cdots+|a_2-a_1|\le M$

$\therefore M\ge |a_i-a_{i-1}|+\cdots+|a_2-a_1|\ge |a_i-a_{i-1}+\cdots+a_2-a_1|=|a_i-a_1|.\therefore -M+a_1\le a_i\le M+a_1$

记$N=\max\{|-M+a_1|,|M+a_1|\},\therefore |a_i|\le N(i=1,2,\cdots,n)$，且$N$为常数,$\therefore |a_{i+1}+a_i|\le |a_{i+1}|+|a_i|\le 2N$

$\therefore |a_{n+1}^2-a_n^2|+|a_n^2-a_{n-1}^2|+\cdots+|a_2^2-a_1^2|=|a_{n+1}-a_n|\cdot|a_{n+1}+a_n|+|a_n-a_{n-1}|\cdot|a_n+a_{n-1}|+\cdots+|a_2-a_1|\cdot|a_2+a_1|$

$\le |a_{n+1}-a_n|\cdot 2N+|a_n-a_{n-1}|\cdot 2N+\cdots+|a_2-a_1|\cdot 2N=2N(|a_{n+1}-a_n|+|a_n-a_{n-1}|+\cdots+|a_2-a_1|)$

$\le 2MN$为常数,$\therefore \{a_n^2\}$是$B-$数列

(2012江苏)20.已知各项均为正数的两个数列$\{a_n\}$和$\{b_n\}$满足：$a_{n+1}=\dfrac{a_n+b_n}{\sqrt{{a_n}^2+{b_n}^2}}$，$n\in N^*$.

(1) 设$b_{n+1}=1+\dfrac{b_n}{a_n},n\in N^*$，求证：数列$\{(\dfrac{b_n}{a_n})^2\}$是等差数列；

(2) 设$b_{n+1}=\sqrt{2}\cdot\dfrac{b_n}{a_n}$，$n\in N^*$，且$\{a_n\}$是等比数列，求$a_1$和$b_1$的值.

(2012江苏) (1) 证明：令$c_n=(\dfrac{b_n}{a_n})^2$,

由$a_{n+1}=\dfrac{a_n+b_n}{\sqrt{a_n^2+b_n^2}}$得$a_{n+1}^2=\dfrac{(a_n+b_n)^2}{a_n^2+b_n^2}=\dfrac{(1+\dfrac{b_n}{a_n})^2}{1+\dfrac{b_n^2}{a_n^2}}$

$\therefore 1+c_n=\dfrac{b_{n+1}^2}{a_{n+1}^2}\cdot\dfrac{(1+\dfrac{b_n}{a_n})^2}{b_{n+1}^2}=c_{n+1}$即$c_{n+1}-c_n=1$为常数,$\therefore \{(\dfrac{b_n}{a_n})^2\}$是等差数列

(2) 解：设$a_n=a_1q^{n-1},\because b_{n+1}=\dfrac{\sqrt{2}b_n}{a_n}$,

$$\therefore b_n=\frac{b_n}{b_{n-1}}\cdot\frac{b_{n-1}}{b_{n-2}}\cdots\frac{b_2}{b_1}\cdot b_1=\frac{\sqrt{2}}{a_{n-1}}\cdot\frac{\sqrt{2}}{a_{n-2}}\cdots\frac{\sqrt{2}}{a_1}\cdot b_1=\frac{(\sqrt{2})^{n-1}b_1}{a_1^{n-1}\cdot q^{\frac{(n-1)(n-2)}{2}}}=b_1(\frac{\sqrt{2}}{a_1}q^{\frac{n-2}{2}})^{n-1}$$

$$\therefore a_{n+1}=a_1q^n=\sqrt{\frac{(a_1q^{n-1}+b_1(\frac{\sqrt{2}}{a_1}q^{\frac{n-2}{2}})^{n-1})^2}{a_1^2q^{2n-2}+b_1^2(\frac{\sqrt{2}}{a_1}q^{\frac{n-2}{2}})^{2n-2}}}\Leftrightarrow a_1^2q^{2n}(a_1^2q^{2n-2}+b_1^2(\frac{\sqrt{2}}{a_1}q^{\frac{n-2}{2}})^{2n-2})=(a_1q^{n-1}+b_1(\frac{\sqrt{2}}{a_1}q^{\frac{n-2}{2}})^{n-1})^2$$

$\therefore q=1$,且$a_1^2(a_1^2+b_1^2(\dfrac{\sqrt{2}}{a_1})^{2n-2})=(a_1+b_1(\dfrac{\sqrt{2}}{a_1})^{n-1})^2$得$\dfrac{\sqrt{2}}{a_1}=1,\therefore a_1=\sqrt{2},b_1=\sqrt{2}$

(2014浙江) 已知数列$\{a_n\}$和$\{b_n\}$满足$a_1a_2\cdots a_n=(\sqrt{2})^{b_n}(n\in N^*)$.若$\{a_n\}$为等比数列，且$a_1=2,b_3=6+b_2$.

(Ⅰ) 求a_n与b_n； (Ⅱ) $c_n=\dfrac{1}{a_n}-\dfrac{1}{b_n}$,记数列$\{c_n\}$的前$n$项和为$S_n$.

(ⅰ) 求S_n； (ⅱ) 求正整数k，使得对任意$n\in N^*$,均有$S_k\ge S_n$.

2014.(1) 由$a_1a_2\cdots a_n=2^n\cdot q^{\frac{n(n-1)}{2}}=2^{\frac{b_n}{2}},\therefore 8=2^{\frac{b_3-b_2}{2}}=\dfrac{2^3\cdot q^3}{2^2\cdot q^1}=2q^2$,且$q>0,\therefore q=2,\therefore a_n=2^n,b_n=n(n+1)$

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（2）由（1）得$c_n=\frac{1}{2^n}-\frac{1}{n(n+1)}=\frac{1}{2^n}+\frac{1}{n+1}-\frac{1}{n}$，∴（ⅰ）$S_n=\frac{\frac{1}{2}(1-\frac{1}{2^n})}{1-\frac{1}{2}}+\frac{1}{n+1}-\frac{1}{1}=\frac{1}{n+1}-\frac{1}{2^n}$;

（ⅱ）由$S_{n+1}-S_n=\frac{1}{n+2}-\frac{1}{2^{n+1}}-\frac{1}{n+1}+\frac{1}{2^n}=\frac{-1}{(n+1)(n+2)}+\frac{1}{2^{n+1}}>0$

$\Leftrightarrow 0>2^{n+1}-(n+1)(n+2)$记为$f(n)$

则$f(n+1)-f(n)=2^{n+2}-(n+2)(n+3)-2^{n+1}+(n+1)(n+2)=2^{n+1}-2n-4$记为$g(n)$

则$g(n+1)-g(n)=2^n-2>0\Leftrightarrow n>1$

$\therefore -2=g(1)=g(2)<6=g(3)<\cdots,\therefore f(n+1)-f(n)>0\Leftrightarrow g(n)>0\Leftrightarrow n\ge 3$

$\therefore f(1)>f(2)=-4=f(3)<2=f(4)<f(5)<\cdots$

$\therefore S_{n+1}-S_n>0\Leftrightarrow f(n)<0\Leftrightarrow n\le 3,\therefore S_1<S_2<S_3<S_4=\frac{13}{80}>S_5>\cdots,\therefore k=4$

$key2:$当$n\ge 4$时$,2^{n+1}=(1+1)^{n+1}=C_{n+1}^0+C_{n+1}^1+C_{n+1}^2+\cdots+C_{n+1}^{n+1}$

$\ge 2(1+n+1+\frac{n(n+1)}{2})=n^2+3n+4>(n+1)(n+2)$，此时$f(n)>0,$

而$f(1)=-4<0,\ f(2)=-4<0,\ f(3)=-4<0,\therefore S_1<S_2<S_3<S_4=\frac{13}{80}>\frac{13}{96}=S_5>\cdots,\therefore k=4$

（2016天津）已知$\{a_n\}$是各项均为正数的等差数列，公差为d，对任意的$n\in N^*$，b_n是a_n和a_{n+1}的等比中项.⑴ $c_n=b_{n+1}^2-b_n^2,n\in N^*$，求证：数列$\{c_n\}$是等差数列；

（2）设$a_1=d,T_n=\sum_{k=1}^{2n}(-1)^k b_k^2,n\in N^*$，求证:$\sum_{k=1}^{n}\frac{1}{T_k}<\frac{1}{2d^2}$.

2016天津证明：（1）由已知得$b_n^2=a_na_{n+1}$

$\therefore c_n=b_{n+1}^2-b_n^2=a_{n+1}a_{n+2}-a_na_{n+1}=a_{n+1}(a_{n+2}-a_n)=2da_{n+1}$

$\therefore c_{n+1}-c_n=2da_{n+2}-2da_{n+1}=2d^2$为常数$,\therefore\{c_n\}$是等差数列

(2)$\because a_1=d,\therefore a_n=nd(d>0),$

$\therefore b_n^2=a_na_{n+1}=d^2n(n+1),$

$\therefore (-1)^{2k-1}b_{2k-1}^2+(-1)^{2k}b_{2k}^2=-d^2(2n-1)\cdot 2n+2n(2n+1)d^2=4d^2n$

$\therefore T_n=\sum_{k=1}^{2n}(-1)^k b_k^2=\sum_{k=1}^{n}4d^2k=2d^2n(n+1),\therefore\frac{1}{T_k}=\frac{1}{2d^2}(\frac{1}{k}-\frac{1}{k+1})$

$\therefore\sum_{k=1}^{n}\frac{1}{T_k}=\frac{1}{2d^2}(\frac{1}{1}-\frac{1}{n+1})<\frac{1}{2d^2}$，证毕

(2018天津)18.设$\{a_n\}$是等比数列，公比大于0，其前n项和为$S_n(n\in N^*)$，$\{b_n\}$是等差数列，已知$a_1=1,a_3=a_2+2,a_4=b_3+b_5,a_5=b_4+2b_6$.⑴ 求$\{a_n\}$和$\{b_n\}$的通项公式；

（2）设数列$\{S_n\}$的前n项和为$T_n(n\in N^*)$.（ⅰ）求T_n;

（ⅱ）证明:$\sum_{k=1}^{n}\frac{(T_k+b_{k+2})b_k}{(k+1)(k+2)}=\frac{2^{n+2}}{n+2}-2(n\in N^*)$.

2018天津（1）解：由已知得$\begin{cases}q^2=q+2(q>0)\\q^3=2(b_1+3d)\\q^4=3b_1+13d\end{cases}$ 得$q=2,b_1=d=1,\therefore a_n=2^{n-1},b_n=n$

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（2）（ⅰ）由（1）得$S_n=\frac{1-2^n}{1-2}=2^n-1,\therefore T_n=\frac{2(1-2^n)}{1-2}-n=2^{n+1}-n-2$

（ⅱ）证明：由（ⅰ）得：$\frac{(T_k+b_{k+2})b_k}{(k+1)(k+2)}=\frac{2^{k+1}\cdot k}{(k+1)(k+2)}=\frac{2^{k+2}}{k+2}-\frac{2^{k+1}}{k+1}$

$\therefore \sum_{k=1}^{n}\frac{(T_k+b_{k+2})b_k}{(k+1)(k+2)}=\sum_{k=1}^{n}(\frac{2^{k+2}}{k+2}-\frac{2^{k+1}}{k+1})=\frac{2^{n+2}}{n+2}-\frac{2^2}{2}=\frac{2^{n+2}}{n+2}-2$，证毕

（2020天津）19.已知$\{a_n\}$为等差数列，$\{b_n\}$为等比数列，$a_1=b_1=1, a_5=5(a_4-a_3), b_5=4(b_4-b_3)$.

（1）求$\{a_n\}$和$\{b_n\}$的通项公式；（2）记$\{a_n\}$的前n项和为S_n，求证：$S_nS_{n+2}<S_{n+1}^2(n\in N^*)$；

（3）对任意的正整数n，设$c_n=\begin{cases}\frac{(3a_n-2)b_n}{a_na_{n+2}}, & n为奇数,\\ \frac{a_{n-1}}{b_{n+1}}, & n为偶数.\end{cases}$ 求数列$\{c_n\}$的前$2n$项和.

2020天津（1）解：由已知得$\begin{cases}1+4d=5d\\ q^4=4(q^3-q^2)\end{cases}$得$d=1, q=2,\therefore a_n=n, b_n=2^{n-1}$

（2）证明：由（1）得$S_n=\frac{n(n+1)}{2}$

$\therefore S_{n+1}^2-S_nS_{n+2}=\frac{1}{4}(n+1)^2(n+2)^2-\frac{1}{4}n(n+1)(n+2)(n+3)=\frac{1}{4}(n+1)(n+2)[(n+1)(n+2)-n(n+3)]$

$=\frac{1}{4}(n+1)(n+2)\cdot 2>0,\therefore S_{n+1}^2>S_nS_{n+2}$，证毕

（3）解：由（1）得$c_n=\begin{cases}\frac{(3n-2)\cdot 2^{n-1}}{n(n+2)}, n为奇数,\\ \frac{n-1}{2^n}, n为偶数,\end{cases}$

$\therefore c_{2k-1}=\frac{(6k-5)\cdot 2^{2k-2}}{(2k-1)(2k+1)}=\frac{2^{2k}}{2k+1}-\frac{2^{2k-2}}{2k-1}, c_{2k}=\frac{2k-1}{2^{2k}}=\frac{\frac{2}{3}k-\frac{1}{9}}{2^{2k-2}}-\frac{\frac{2}{3}k+\frac{5}{9}}{2^{2k}}$

$\therefore c_1+c_3+\cdots+c_{2n-1}+c_2+c_4+\cdots+c_{2n}$

$=\frac{2^{2n}}{2n+1}-1+\frac{5}{9}-\frac{\frac{2}{3}n+\frac{5}{9}}{2^{2n}}=\frac{4^n}{2n+1}-\frac{6n+5}{9\times 4^n}-\frac{4}{9}$

（2020浙江）已知数列$\{a_n\},\{b_n\},\{c_n\}$中，$a_1=b_1=c_1=1, c_n=a_{n+1}-a_n, c_{n+1}=\frac{b_n}{b_{n+2}}\cdot c_n(n\in N^*)$.

（Ⅰ）若数列$\{b_n\}$为等比数列，且公比$q>0$，且$b_1+b_2=6b_3$，求q与a_n的通项公式；

（Ⅱ）若数列$\{b_n\}$为等差数列，且公差$d>0$，证明：$c_1+c_2+\cdots+c_n<1+\frac{1}{d}$.

（Ⅰ）由已知得：$1+q=6q^2(q>0)$得$q=\frac{1}{2}$，

$\therefore c_{n+1}=4c_n,\therefore \{c_n\}$是首项为1，公比为4的等比数列，$\therefore c_n=4^{n-1}$，

$\therefore a_n=(a_n-a_{n-1})+\cdots+(a_2-a_1)+a_1=4^{n-2}+\cdots+4^0+1=\frac{4^{n-1}+2}{3}(n\geq 2)$且$a_1=1-\frac{4^{1-1}+2}{3},\therefore a_n=\frac{4^{n-1}+2}{3}$

（Ⅱ）由$c_n=\frac{c_n}{c_{n-1}}\cdot\frac{c_{n-1}}{c_{n-2}}\cdots\frac{c_2}{c_1}\cdot c_1=\frac{b_{n-1}}{b_{n+1}}\cdot\frac{b_{n-2}}{b_n}\cdots\frac{b_2}{b_4}\cdot\frac{b_1}{b_3}=\frac{1+d}{(1+nd)(1+(n-1)d)}=\frac{1+d}{d}(\frac{1}{1+(n-1)d}-\frac{1}{1+nd})(n\geq 3)$

$\because d>0,\therefore c_1=1<1+\dfrac{1}{d}$，且$c_1+c_2=1+\dfrac{1}{1+2d}<1+\dfrac{1}{d}$

当$n\geq 3$时，$c_1+c_2+\cdots+c_n=1+\dfrac{1}{1+2d}+\dfrac{1+d}{d}(\dfrac{1}{1+2d}-\dfrac{1}{1+nd})=1+\dfrac{1}{d}-\dfrac{1+d}{d(1+nd)}<1+\dfrac{1}{d}$ 得证

（2021 天津）19. 已知$\{a_n\}$是公差为 2 的等差数列，其前 8 项和为 64． $\{b_n\}$是公比大于 0 的等比数列，

$b_1=4,b_3-b_2=48$．（I）求$\{a_n\}$和$\{b_n\}$的通项公式；

（II）记$c_n=b_{2n}+\dfrac{1}{b_n},n\in N^*$．（i）证明$\{c_n^2-c_{2n}\}$是等比数列；（ii）证明$\sum\limits_{k=1}^{n}\sqrt{\dfrac{a_ka_{k+1}}{c_k^2-c_{2k}}}<2\sqrt{2}(n\in N^*)$.

2021天津（1）解：由已知得$\begin{cases}A_8=8a_1+\dfrac{8\times 7}{2}\cdot 2=64\\ b_3-b_2=4q^2-4q=48(q>0)\end{cases}$ 得$a_1=1,q=4$

$\therefore a_n=2n-1,b_n=4^n$

(2) 证明：（ i ）由（1）得$c_n=4^{2n}+\dfrac{1}{4^n}$

$\therefore c_n^2-c_{2n}=(4^{2n}+\dfrac{1}{4^n})^2-(4^{4n}+\dfrac{1}{4^{2n}})=2\cdot 4^n\neq 0$

$\therefore\dfrac{c_{n+1}^2-c_{2(n+1)}}{c_n^2-c_{2n}}=\dfrac{2\cdot 4^{n+1}}{2\cdot 4^n}=4$为常数，$\therefore\{c_n^2-c_{2n}\}$是等比数列

（ ii ）由上知$\sqrt{\dfrac{a_ka_{k+1}}{c_k^2-c_{2k}}}=\sqrt{\dfrac{(2k-1)(2k+1)}{2\cdot 4^k}}<\sqrt{\dfrac{4k^2}{2\cdot 4^k}}=\dfrac{\sqrt{2}k}{2^k}=\sqrt{2}(\dfrac{k+1}{2^{k-1}}-\dfrac{k+2}{2^k})$

$\therefore\sum\limits_{k=1}^{n}\sqrt{\dfrac{a_ka_{k+1}}{c_k^2-c_{2k}}}<\sqrt{2}\sum\limits_{k=1}^{n}(\dfrac{k+1}{2^{k-1}}-\dfrac{k+2}{2^k})=\sqrt{2}(2-\dfrac{n+2}{2^n})<2\sqrt{2}$, 证毕

(2022天津)18.设$\{a_n\}$是等差数列,$\{b_n\}$是等比数列，且$a_1=b_1=a_2-b_2=a_3-b_3=1$.

(1) 求$\{a_n\}$与$\{b_n\}$的通项公式；

(2) 设$\{a_n\}$的前n项和为S_n，求证:$(S_{n+1}+a_{n+1})b_n=S_{n+1}b_{n+1}-S_nb_n$;

(3) 求$\sum\limits_{k=1}^{2n}[a_{k+1}-(-1)^ka_k]b_k$.

$a_n=2n-1,b_n=2^{n-1},\dfrac{(6n-2)4^{n+1}+8}{9}$

2022天津（1）解：由已知得$\begin{cases}a_2-b_2=1+d-q=1\\ a_3-b_3=1+2d-q^2=1\end{cases}$得$d=q=2,\therefore a_n=2n-1,b_n=2^{n-1}$

(2) 证明：由（1）得:$S_n=\dfrac{n(1+2n-1)}{2}=n^2$,

$\therefore(S_{n+1}+a_{n+1})b_n=((n+1)^2+2n+1)\cdot 2^{n-1}=(n^2+4n+2)\cdot 2^{n-1}$

$S_{n+1}b_{n+1}-S_nb_n=(n+1)^2\cdot 2^n-n^2\cdot 2^{n-1}=(2n^2+4n+2-n^2)\cdot 2^{n-1}=(S_{n+1}+a_{n+1})b_n$, 证毕

(3) 由（1）得$[a_{2k}-(-1)^{2k-1}a_{2k-1}]b_{2k-1}+[a_{2k+1}-(-1)^{2k}a_{2k}]b_{2k}$

$=(8k-4)\cdot 2^{2k-2}+2\cdot 2^{2k-1}=k\cdot 2^{2k}=(\dfrac{2}{3}k-\dfrac{2}{9})\cdot 2^{2k+2}-(\dfrac{2}{3}k-\dfrac{8}{9})\cdot 2^{2k}$

$\therefore\sum\limits_{k=1}^{2n}[a_{k+1}-(-1)^ka_k]b_k=\sum\limits_{k=1}^{n}[(\dfrac{2}{3}k-\dfrac{2}{9})\cdot 2^{2k+2}-(\dfrac{2}{3}k-\dfrac{8}{9})\cdot 2^{2k}]=(\dfrac{2}{3}n-\dfrac{2}{9})\cdot 2^{2n+2}+\dfrac{8}{9}$