平面向量解答（1）

2023-02-12

一.向量线性运算：（1）加法:$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC},\overrightarrow{P_1P_2}+\overrightarrow{P_2P_3}+\cdots+\overrightarrow{P_nP_1}=\vec{0}$,

非零向量$\overrightarrow{a_1},\overrightarrow{a_2},\cdots,\overrightarrow{a_n}$组成封闭图形的充要条件是:$\overrightarrow{a_1}+\overrightarrow{a_2}+\cdots+\overrightarrow{a_n}=\vec{0}$.

（2）减法:$\overrightarrow{AB}-\overrightarrow{AC}=\overrightarrow{CB}$;（3）数乘:$\lambda\vec{a}$.共线充要条件：若$\vec{b}\neq\vec{0}$,则$\vec{a}//\vec{b}\Leftrightarrow\vec{a}=\lambda\vec{b}$

（4）平面向量基本定理:$\overrightarrow{OP}=\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}(\overrightarrow{OA}\not\parallel\overrightarrow{OB})$.①已知坐标:合成;

②求坐标：分解，点基向量、自身、平方;坐标的几何意义：长度之比

③（等和线）P、A、B共线$\Leftrightarrow\lambda+\mu=1$；④向量平行：坐标成比例.

（5）基本公式：中点$\overrightarrow{OP}=\frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\overrightarrow{OB}$,三等分点$\overrightarrow{OP}=\frac{1}{3}\overrightarrow{OA}+\frac{2}{3}\overrightarrow{OB}$,重心$\overrightarrow{OG}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$

（6）线性运算的坐标表示:$\vec{a}=(x_1,y_1),\vec{b}=(x_2,y_2)$.

（Ⅰ）运算法则：①加减法:$(x_1,y_1)\pm(x_2,y_2)=(x_1\pm x_2,y_1\pm y_2)$

②数乘：$\lambda(x_1,y_1)=(\lambda x_1,\lambda y_1)$;③数量积:$(x_1,y_1)\cdot(x_2,y_2)=x_1x_2+y_1y_2$

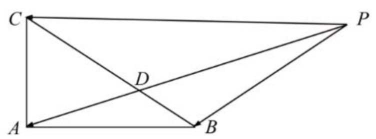
（Ⅱ）运算律:①交换律；②结合律；③分配律；

（2022Ⅰ）3. 在$\triangle ABC$中，点D在边AB上，$BD=2DA$．记$\overrightarrow{CA}=\vec{m},\overrightarrow{CD}=\vec{n}$，则$\overrightarrow{CB}=$（ B ）

A. $3\vec{m}-2\vec{n}$　　B. $-2\vec{m}+3\vec{n}$　　C. $3\vec{m}+2\vec{n}$　　D. $2\vec{m}+3\vec{n}$

(2020江苏)13.在$\triangle ABC$中，$AB=4,AC=3,\angle BAC=90°$,D在边BC上，延长AD到P，使得$AP=9$，

若$\overrightarrow{PA}=m\overrightarrow{PB}+(\frac{3}{2}-m)\overrightarrow{PC}$（$m$为常数），则$CD$的长度是______.



$key:\frac{2}{3}\overrightarrow{PA}=\frac{2}{3}m\overrightarrow{PB}+(1-\frac{2}{3}m)\overrightarrow{PC}=\overrightarrow{PD},\therefore|\overrightarrow{AD}|=3$

$\therefore|\overrightarrow{CD}|=0,or,|\overrightarrow{CD}|=2\cdot3\cdot\frac{3}{5}=\frac{18}{5}$

变式1（1）如图，已知$AD:DB=BE:EC=CF:FA=1:2$,则$\frac{S_{\triangle A_1B_1C_1}}{S_{\triangle ABC}}=$________.

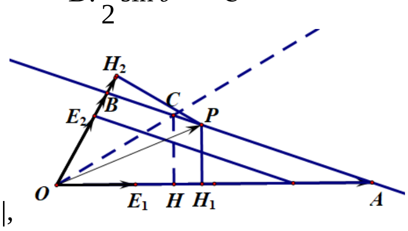
$key:\overrightarrow{AB_1}=x\overrightarrow{AE}=\frac{2x}{3}\overrightarrow{AB}+\frac{x}{3}\overrightarrow{AC},\therefore\overrightarrow{BB_1}=(\frac{2x}{3}-1)\overrightarrow{AB}+\frac{x}{3}\overrightarrow{AC}//\overrightarrow{BF}=-\overrightarrow{AB}+\frac{2}{3}\overrightarrow{AC}$

$\therefore1-\frac{2x}{3}=\frac{x}{2}$即$x=\frac{6}{7},\therefore\frac{S_{\triangle A_1B_1C_1}}{S_{\triangle ABC}}=1-3\cdot\frac{6}{7}\cdot\frac{1}{3}=\frac{1}{7}$

（2）若$\overrightarrow{e_1},\overrightarrow{e_2}$是两个所成角为$\theta(0<\theta<\frac{\pi}{2})$的单位向量，实数$x,y,\lambda,\mu$满足$xy=0,\lambda+3\mu=4$，则

$\min\{\max\{|(x+\lambda)\overrightarrow{e_1}+(y+\mu)\overrightarrow{e_2}|\}=$（ ）　$A.\cos\theta$　$B.\frac{1}{2}\cos\theta$　$C.\sin\theta$　$D.\frac{1}{2}\sin\theta$　C

$key:$如图,$\overrightarrow{OA}=4\overrightarrow{e_1},\overrightarrow{OB}=\frac{4}{3}\overrightarrow{e_2}$,点$P$在直线$AB$上，$PH_1\perp OA$于$H_1$，



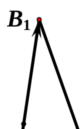
$PH_2\perp OB$于H_2,OC为$\angle AOB$的平分线，

$\therefore\min\{\max|(x+\lambda)\overrightarrow{e_1}+(y+\mu)\overrightarrow{e_2}|\}=\min\{\max\{|\frac{\lambda}{4}(4\overrightarrow{e_1})+\frac{3}{4}\mu(\frac{4}{3}\overrightarrow{e_2})-(-y)\overrightarrow{e_2}|,$

$|\frac{\lambda}{4}(4\overrightarrow{e_1})+\frac{3}{4}\mu(\frac{4}{3}\overrightarrow{e_2})-(-x)\overrightarrow{e_1}|\}\}=\min\{\max\{|\overrightarrow{PH_1}|,|\overrightarrow{PH_2}|\}\}$

$=|\overrightarrow{CH}|=|\overrightarrow{OC}|\sin\frac{\theta}{2}=\sin\theta$(由$\frac{1}{2}|\overrightarrow{OC}|\cdot\frac{4}{3}\sin\frac{\theta}{2}+\frac{1}{2}|\overrightarrow{OC}|\cdot4\sin\frac{\theta}{2}=\frac{1}{2}\cdot\frac{4}{3}\cdot4\cdot\sin\theta$得$|\overrightarrow{OC}|=2\cos\frac{\theta}{2}$)

（3）已知平面向量$\vec{a},\vec{b},\vec{c}$满足$|\vec{a}|=1,|\vec{b}|=2,|\vec{a}-\vec{c}|=|\vec{b}-\vec{c}|=3,\vec{c}=\lambda\vec{a}+\mu\vec{b}(\lambda,\mu>0)$,当$\lambda+\mu=4$，则$|\vec{c}|=$（ ）



$A.\dfrac{\sqrt{58}}{2}$ $B.\dfrac{\sqrt{62}}{2}$ $C.\dfrac{\sqrt{66}}{2}$ $D.\dfrac{\sqrt{70}}{2}$

$key:\vec{a}=\overrightarrow{OA},\overrightarrow{OA_1}=4\vec{a},\vec{b}=\overrightarrow{OB},\overrightarrow{OB_1}=4\vec{b},$

连AB交OC于M, $\because \lambda+\mu=4,\therefore \overrightarrow{OM}=\dfrac{1}{4}\overrightarrow{OC},$

设$\angle CAB=\theta$，则$\angle CAB=\angle BCB_1=\angle ACA_1=\angle AA_1C=\theta,$

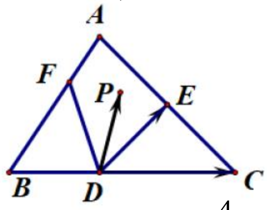
则$|\overrightarrow{AB}|=6\cos\theta,\therefore 1+36\cos^2\theta-2\cdot1\cdot6\cos\theta\cdot\cos\theta=4$得$\cos^2\theta=\dfrac{1}{8}$

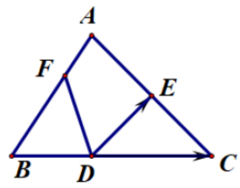
$\therefore|\overrightarrow{OC}|^2=1+9-2\cdot1\cdot3\cos2\theta=10-6(2\cdot\dfrac{1}{8}-1)=\dfrac{58}{4}$

(1905竞赛）如图，在$\triangle ABC$中，D,E,F分别为BC,CA,AB上的点，且$CD=\dfrac{3}{5}BC,EC=\dfrac{1}{2}AC,AF=\dfrac{1}{3}AB.$

设P为四边形$AEDF$内一点（P点不在边界上）.若$\overrightarrow{DP}=-\dfrac{1}{3}\overrightarrow{DC}+x\overrightarrow{DE},$

则实数x的取值范围为______ $(\dfrac{1}{2},\dfrac{4}{3})$



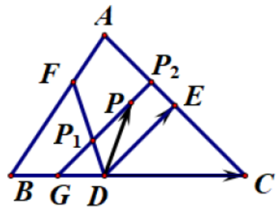


$key:\dfrac{|\overrightarrow{QP_2}|}{|\overrightarrow{DE}|}=\dfrac{|\overrightarrow{QC}|}{|\overrightarrow{DC}|}=\dfrac{4}{3},\therefore x_2=\dfrac{4}{3}$

（或$\overrightarrow{CP_2}=-\dfrac{1}{3}\overrightarrow{DC}+x_2\overrightarrow{DE}-\overrightarrow{DC}=-\dfrac{4}{3}\overrightarrow{DC}+x_2\overrightarrow{DE}//\overrightarrow{CA}=-2\overrightarrow{DC}+2\overrightarrow{DE}$得$x_2=\dfrac{4}{3}$）

$\overrightarrow{DF}=-\dfrac{2}{3}\overrightarrow{DC}+\dfrac{2}{3}\overrightarrow{BA},\overrightarrow{BA}=\overrightarrow{BC}+\overrightarrow{CA}=\dfrac{5}{3}\overrightarrow{DC}+2\overrightarrow{CE},\overrightarrow{CE}=\overrightarrow{DE}-\overrightarrow{DC}$

$\therefore\overrightarrow{DF}=\dfrac{4}{3}\overrightarrow{DE}-\dfrac{8}{9}\overrightarrow{DC},\therefore\overrightarrow{DP_1}=(-\dfrac{1}{3}\overrightarrow{DC}+x_1\overrightarrow{DE})//(-\dfrac{8}{9}\overrightarrow{DC}+\dfrac{4}{3}\overrightarrow{DE})\therefore x_1=\dfrac{1}{2}$



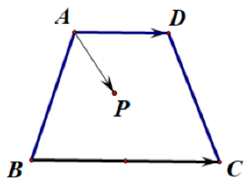
（20B） 7.在凸四边形$ABCD$中,$\overrightarrow{BC}=2\overrightarrow{AD}$,点$P$是四边形$ABCD$所在平面上一点，满足

$\overrightarrow{PA}+2020\overrightarrow{PB}+\overrightarrow{PC}+2020\overrightarrow{PD}=\vec{0}$. 设$s,t$分别为四边形$ABCD$与$\triangle PAB$的面积，则$\dfrac{t}{s}=$______. $\dfrac{337}{2021}$

$key:\overrightarrow{PA}+2020\overrightarrow{PB}+\overrightarrow{PC}+2020\overrightarrow{PD}=-\overrightarrow{AP}+2020(\overrightarrow{AB}-\overrightarrow{AP})+\overrightarrow{AC}-\overrightarrow{AP}+2020(\overrightarrow{AD}-\overrightarrow{AP})$

$=-4042\overrightarrow{AP}+2020\overrightarrow{AB}+\overrightarrow{AB}+2\overrightarrow{AD}+2020\overrightarrow{AD}=\vec{0},\therefore\overrightarrow{AP}=\dfrac{1}{2}\overrightarrow{AB}+\dfrac{1011}{2021}\overrightarrow{AD}$

$\therefore\dfrac{t}{s}=\dfrac{\frac{1011}{2021}S_{\triangle ABD}}{3S_{\triangle ABD}}=\dfrac{337}{2021}$

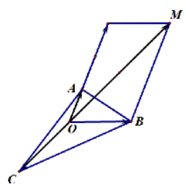


变式 2 （ 1 ） 设O为$\triangle ABC$内一点，且满足$\overrightarrow{OA}+2\overrightarrow{OB}+3\overrightarrow{OC}=3\overrightarrow{AB}+2\overrightarrow{BC}+\overrightarrow{CA}$, 则

$\dfrac{S_{\triangle AOB}+2S_{\triangle BOC}+3S_{\triangle COA}}{S_{\triangle ABC}}=$___.

$key1$:由$\overrightarrow{OA}+2\overrightarrow{OB}+3\overrightarrow{OC}=3\overrightarrow{AB}+2\overrightarrow{BC}+\overrightarrow{CA}=3(\overrightarrow{OB}-\overrightarrow{OA})+2(\overrightarrow{OC}-\overrightarrow{OB})+\overrightarrow{OA}-\overrightarrow{OC}$

得$3\overrightarrow{OA}+\overrightarrow{OB}+2\overrightarrow{OC}=\vec{0}$, 如图,

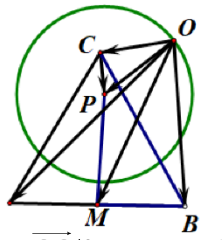


$\therefore S_{\triangle AOB}:S_{\triangle BOC}:S_{\triangle AOC}=\dfrac{1}{3}:\dfrac{1}{2}:\dfrac{1}{6},\therefore\dfrac{S_{\triangle AOB}+2S_{\triangle BOC}+3S_{\triangle COA}}{S_{\triangle ABC}}=\dfrac{\frac{1}{3}+1+\frac{1}{2}}{\frac{1}{3}+\frac{1}{2}+\frac{1}{6}}=\dfrac{11}{6}$

$key2:\overrightarrow{AB},\overrightarrow{AC}$为基向量

（2）已知$\triangle ABC$是边长为2的正三角形，平面上两动点O，P满足$\overrightarrow{OP}=\lambda_1\overrightarrow{OA}+\lambda_2\overrightarrow{OB}+\lambda_3\overrightarrow{OC}$

$(\lambda_1+\lambda_2+\lambda_3=1$且$\lambda_1,\lambda_2,\lambda_3\ge 0)$．若$|\overrightarrow{OP}|=1$，则$\overrightarrow{OA}\cdot\overrightarrow{OB}$的最大值为__________.



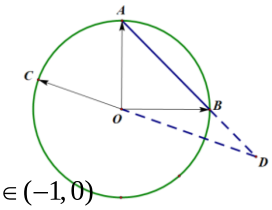
key:由$\overrightarrow{OP}=\lambda_1\overrightarrow{OA}+\lambda_2\overrightarrow{OB}+(1-\lambda_1-\lambda_2)\overrightarrow{OC}=\overrightarrow{OC}+\lambda_1\overrightarrow{CA}+\lambda_2\overrightarrow{CB}$

$(\lambda_1,\lambda_2\in[0,1]),1-\lambda_1-\lambda_2\ge 0$即$\lambda_1+\lambda_2\le 1),\therefore P$在$\triangle ABC$内的一点,

设M为AB的中点, 则$\overrightarrow{OA}\cdot\overrightarrow{OB}=\overrightarrow{OM}^2-1\le(|\overrightarrow{MP}|+1)^2-1\le(\sqrt{3}+1)^2-1=3+2\sqrt{3}$

变式3（1）若$\triangle ABC$的重心为G，$AB=3,AC=4,BC=5$,动点P满足$\overrightarrow{GP}=x\overrightarrow{GA}+y\overrightarrow{GB}+z\overrightarrow{GC}(0\le x,y,z\le 1)$,则点$P$的轨迹所覆盖的平面区域的面积为______.12

（2）①如图，A,B,C是圆O上的三点，CO的延长线与线段AB的延长线交于圆外的点D,若$\overrightarrow{OC}=m\overrightarrow{OA}+n\overrightarrow{OB}$, 且$\overrightarrow{OA}\perp\overrightarrow{OB}$.则$m\in$______,$m+n\in$______,$2m-n\in$_______.



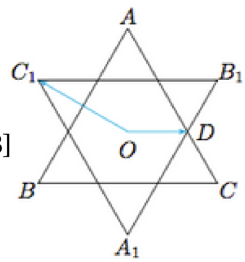
$key1$:（分解）过C作OA,OB的平行线;

$key2$:（点基向量）令$|\overrightarrow{OA}|=1$, 则$m=\overrightarrow{OC}\cdot\overrightarrow{OA}\in(0,\dfrac{\sqrt{2}}{2})$,

（提取$m+n)\overrightarrow{OC}=(m+n)(\dfrac{m}{m+n}\overrightarrow{OA}+\dfrac{n}{m+n}\overrightarrow{OB})=(m+n)\overrightarrow{OD},\therefore m+n=-\dfrac{1}{|\overrightarrow{OD}|}\in(-1,0)$

$2m-n=2\overrightarrow{OC}\cdot\overrightarrow{OA}-\overrightarrow{OC}\cdot\overrightarrow{OB}=\overrightarrow{OC}\cdot(2\overrightarrow{OA}-\overrightarrow{OB})=\overrightarrow{OC}\cdot\overrightarrow{OM}\in(-1,\dfrac{3\sqrt{2}}{2})$

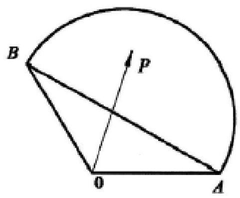
②如图，两个正三角形$ABC,A_1B_1C_1$组成“六芒星”，O为“六芒星”的中心，P为“六芒星”图案上一点（包括边界），且$\overrightarrow{OP}=x\overrightarrow{OD}+y\overrightarrow{OC_1}$.则$x\in$_____,$x+y\in$_______,$2x+y\in$_______.



$key:\overrightarrow{OP}\cdot\overrightarrow{OD}=x-\dfrac{3}{2}y,\overrightarrow{OP}\cdot\overrightarrow{OC_1}=-\dfrac{3}{2}x+3y,\therefore\begin{cases}x=2\overrightarrow{OP}\cdot(2\overrightarrow{OD}+\overrightarrow{OC_1})\in[-3,3]\\ y=\dfrac{4}{3}\overrightarrow{OP}\cdot(\dfrac{3}{2}\overrightarrow{OD}+\overrightarrow{OC_1})\end{cases}$

$x+y=\overrightarrow{OP}\cdot(6\overrightarrow{OD}+\dfrac{10}{3}\overrightarrow{OC_1})\in[-5,5],2x+y=\overrightarrow{OP}\cdot(10\overrightarrow{OD}+\dfrac{16}{3}\overrightarrow{OC_1})\in[-8,8]$

③如图,$\triangle ABO$是以$\angle O=120°$为顶点的等腰三角形，点P在以AB为直径的半圆内（包括边界），若$\overrightarrow{OP}=x\overrightarrow{OA}+y\overrightarrow{OB}(x,y\in R)$，则$x+y\in$_______,$x^2+y^2\in$_______.



key:令$|\overrightarrow{OA}|=1$, 则$|\overrightarrow{OP}|\le|\overrightarrow{OC}|+|\overrightarrow{CP}|\le\dfrac{1+\sqrt{3}}{2}$

$\overrightarrow{OP}=(x+y)(\dfrac{x}{x+y}\overrightarrow{OA}+\dfrac{y}{x+y}\overrightarrow{OB})=(x+y)\overrightarrow{OQ},\therefore x+y=\dfrac{|\overrightarrow{OP}|}{|\overrightarrow{OQ}|}\in[1,\sqrt{3}+1]$

$\dfrac{2+\sqrt{3}}{2}\ge\overrightarrow{OP}^2=x^2+y^2-xy\ge x^2+y^2-\dfrac{x^2+y^2}{2}=\dfrac{x^2+y^2}{2}\ge\dfrac{1}{2}\cdot\dfrac{(x+y)^2}{1+1}\ge\dfrac{1}{2}$,

$\therefore x^2+y^2\in[\dfrac{1}{2},2+\sqrt{3}]$