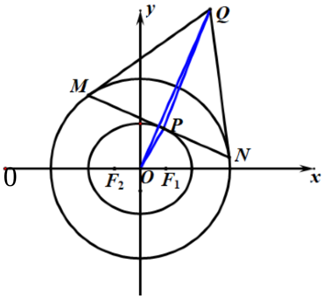
解析几何（2）椭圆解答（7）

2023-11-11

（2008湖北）设P为椭圆$\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$上的一个动点，过点$P$作椭圆的切线与$\odot O: x^2+y^2=12$相交于$M,N$两点，$\odot O$在$M,N$两点处的切线相交于点$Q$.（Ⅰ）求点$Q$的轨迹方程；

（Ⅱ）若P是第一象限的点，求$\triangle OPQ$的面积的最大值.



（2008湖北）（Ⅰ）设$Q(s,t)$，则$l_{MN}: sx+ty=12$,

由$\begin{cases} sx+ty=12\text{即}4t^2y^2=4(12-sx)^2 \\ 3x^2+4y^2=12\text{即}4t^2y^2=t^2(12-3x^2) \end{cases}$消去$y$得$(4s^2+3t^2)x^2-96sx+576-12t^2=0$

$\therefore \Delta=96^2s^2-48(48-t^2)(4s^2+3t^2)=0$即$4s^2-48\times3+3t^2=0$,

$\therefore Q$的轨迹方程为$\dfrac{x^2}{36}+\dfrac{y^2}{48}=1$

（Ⅱ）由（Ⅰ）得：$P(\dfrac{48s}{4s^2+3t^2},\dfrac{36t}{4s^2+3t^2})$（其中$\dfrac{s^2}{36}+\dfrac{t^2}{48}=1$，令$s=6\cos\theta, t=8\sin\theta, \theta\in(0,\dfrac{\pi}{2})$）

$$\therefore S_{\triangle OPQ}=\frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ s & t & 1 \\ \dfrac{48s}{4s^2+3t^2} & \dfrac{36t}{4s^2+3t^2} & 1 \end{vmatrix}\right|=\frac{1}{2}\left|\frac{36st}{4s^2+3t^2}-\frac{48st}{4s^2+3t^2}\right|=\frac{6st}{4s^2+3t^2}$$

$=\dfrac{6}{4\tan\theta+\dfrac{3}{\tan\theta}}\le\dfrac{6}{2\sqrt{12}}=\dfrac{\sqrt{3}}{2}$（当且仅当$\tan\theta=\dfrac{\sqrt{3}}{2}$时，取=），$\therefore\triangle OPQ$面积的最大值为$\dfrac{\sqrt{3}}{2}$

（2009新疆）从直线$l: \dfrac{x}{12}+\dfrac{y}{8}=1$上任意一点$P$向椭圆$C: \dfrac{x^2}{24}+\dfrac{y^2}{16}=1$引切线$PA$、$PB$，切点分别为$A$、$B$，试求线段$AB$的中点$M$的轨迹方程.

解：设$P(s,t)(2s+3t=24)$，则$l_{AB}: 2sx+3ty=48$

设$M(x_0,y_0)$，则$2(x_A^2-x_B^2)+3(y_A^2-y_B^2)=0$得$k_{AB}=-\dfrac{2x_0}{3y_0}$,

$\therefore l_{AB}: y-y_0=-\dfrac{2x_0}{3y_0}(x-x_0)$即$2x_0x+2y_0y=2x_0^2+3y_0^2$

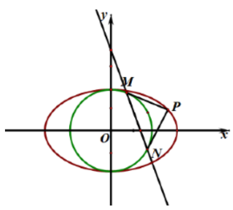
$\therefore\begin{cases} 2s+3t=24 \\ \dfrac{2s}{2x_0}=\dfrac{3t}{3y_0}=\dfrac{48}{2x_0^2+3y_0^2} \end{cases}$，$\therefore\dfrac{2x_0\cdot48}{2x_0^2+3y_0^2}+\dfrac{3y_0\cdot48}{2x_0^2+3y_0^2}=24$即$4x_0+6y_0=2x_0^2+3y_0^2$,

$\therefore M$的轨迹方程为$2x^2+3y^2-4x-6y=0$

（2010江西）给定椭圆$C: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$以及$\odot O: x^2+y^2=b^2$，自椭圆上异于其顶点的任意一点$P$，作$\odot O$的两条切线，切点为$M$、$N$，若直线$MN$在$x,y$轴上的截距分别为$m$、$n$.证明：$\dfrac{a^2}{n^2}+\dfrac{b^2}{m^2}=\dfrac{a^2}{b^2}$.

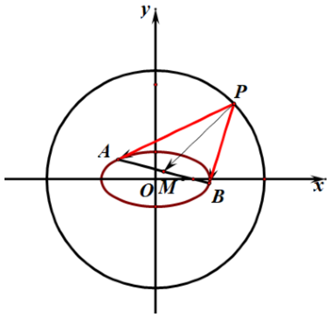
证明：设$P(a\cos\theta, b\sin\theta)(\theta\ne\dfrac{k\pi}{2}, k\in Z)$，则$l_{MN}: a\cos\theta\cdot x+b\sin\theta\cdot y=b^2$，$\therefore m=\dfrac{b^2}{a\cos\theta}, n=\dfrac{b}{\sin\theta}$

$\therefore\dfrac{a^2}{n^2}+\dfrac{b^2}{m^2}=\dfrac{a^2}{\dfrac{b^2}{\sin^2\theta}}+\dfrac{b^2}{\dfrac{b^4}{a^2\cos^2\theta}}=\dfrac{a^2\sin^2\theta}{b^2}+\dfrac{a^2\cos^2\theta}{b^2}=\dfrac{a^2}{b^2}$，证毕



2023-11-11

（2012河南）已知椭圆$\frac{x^2}{4}+y^2=1$，P是圆$x^2+y^2=16$上任意一点，过P点作椭圆的切线PA、PB，切点分别为A、B，求$\overrightarrow{PA}\cdot\overrightarrow{PB}$的最大值和最小值.



解：设$P(4\cos\theta,4\sin\theta)$，则$l_{AB}:x\cos\theta+4y\sin\theta=1$代入椭圆方程得：

$(4\cos^2\theta+16\sin^2\theta)y^2-8y\sin\theta+1-4\cos^2\theta=0$

$$\therefore\begin{cases}y_A+y_B=\dfrac{8\sin\theta}{4+12\sin^2\theta}\\ y_Ay_B=\dfrac{1-4\cos^2\theta}{4+12\sin^2\theta}\end{cases},\therefore AB\text{的中点}M\left(\frac{\cos\theta}{1+3\sin^2\theta},\frac{\sin\theta}{1+3\sin^2\theta}\right)\text{，且}\Delta=48\cos^2\theta,$$

$$\therefore|AB|=\sqrt{1+\frac{16\sin^2\theta}{\cos^2\theta}}\cdot\frac{\sqrt{3}|\cos\theta|}{1+3\sin^2\theta}=\frac{\sqrt{3}\sqrt{1+15\sin^2\theta}}{1+3\sin^2\theta}$$

$$\therefore\overrightarrow{PA}\cdot\overrightarrow{PB}=\overrightarrow{PM}^2-\frac{1}{4}\overrightarrow{AB}^2=\left(\frac{\cos\theta}{1+3\sin^2\theta}-4\cos\theta\right)^2+\left(\frac{\sin\theta}{1+3\sin^2\theta}-4\sin\theta\right)^2-\frac{3(1+15\sin^2\theta)}{(1+3\sin^2\theta)^2}$$

$$=\frac{6-9\sin^2\theta}{(1+3\sin^2\theta)^2}=9t^2-3t\in[-\frac{3}{16},6]\left(t=\frac{1}{1+3\sin^2\theta}\in[\frac{1}{4},1]\right)$$

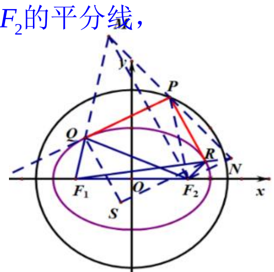
$\therefore$最大值为6，最小值为$-\frac{3}{16}$

（2018甘肃）已知点P为直线$x+2y=4$上一动点，过点P作椭圆$x^2+4y^2=4$的两条切线，切点分别为A、B.当点P运动时，直线AB过定点的坐标是________.$(1,\frac{1}{2})$

key：设$P(s,t)$（其中$s+2t=4$），则$l_{AB}:sx+4ty=sx+2y(4-s)=4\Leftrightarrow s(x-2y)+8y-4=0$经过定点$(1,\frac{1}{2})$

变式 1（1）（蒙日圆）由点P向椭圆$C:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$作出两条互相垂直的切线，则$P$的轨迹 方程为___ .

$key1$：作F_2关于PQ,PR的对称点，由椭圆的光学性质得SQ,SR为$\angle F_1QF_2,\angle F_1RF_2$的平分线，



$\therefore F_1,Q,M$共线，F_1,R,N共线，$\angle PMF_2=\angle F_2PR=\angle RPN,\therefore M,P,N$共线，

而$|F_1M|=2a=|F_1N|,\therefore P$是$MN$的中点，

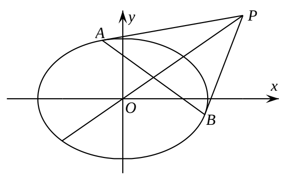
$\therefore 2PO^2+2c^2=PF_1^2+PF_2^2=PF_1^2+PN^2=F_1N^2=4a^2$，

$\therefore PO^2=2a^2-c^2=a^2+b^2$（蒙日圆）

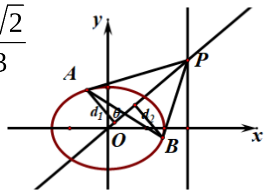
$key2$：设$P(u,v),l_{PQ}:y-v=k_1(x-u)$即$y=k_1x+v-k_1u$代入椭圆得：$(a^2-u^2)k_1^2+2uvk_1+b^2-v^2=0$

设$l_{PR}:y-v=k_2(x-u)$，同理得：$(a^2-u^2)k_2^2+2uvk_2+b^2-v^2=0,\therefore k_1k_2=\frac{b^2-v^2}{a^2-u^2}=-1$即$u^2+v^2=a^2+b^2$

（2）已知椭圆$C:\frac{x^2}{2}+y^2=1$和点$P(2,t)(t\in R)$，过点P作椭圆C的两条切线，切点是A，B，记点A，B到直线PO（O是坐标原点）的距离d_1，d_2．则$\frac{|AB|}{d_1+d_2}$的最大值为_______.



key：设AB与OP的夹角为θ，则$\frac{|AB|}{d_1+d_2}=\frac{|AB|}{|AB|\sin\theta}=\frac{1}{\sin\theta}\le\frac{2\sqrt{2}}{3}$



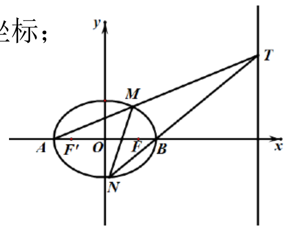
（由$l_{AB}:x+ty=1,\therefore\tan\theta=\left|\frac{\frac{t}{2}+\frac{1}{t}}{1+\frac{t}{2}\cdot(-\frac{1}{t})}\right|=|t+\frac{2}{t}|\ge2\sqrt{2}$）

解析几何（2）椭圆解答（7）

2023-11-11

九、定点、定值问题

（2010 江苏）在平面直角坐标系 xOy 中，如图，已知椭圆 $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$ 的左、右顶点为 A、B，右焦点为 F，设过点 $T(t,m)$ 的直线 TA、TB 与此椭圆分别交于点 $M(x_1,y_1),N(x_2,y_2)$，其中 $m>0,y_1>0,y_2<0$.



（1）设动点 P 满足 $PF^2-PB^2=4$，求点 P 的轨迹；（2）设 $x_1=2,x_2=\dfrac{1}{3}$，求点 T 的坐标；

（3）设 $t=9$，求证：直线 MN 必过 x 轴上的一定点（其坐标与 m 无关）.

(1) 解：$PF^2-PB^2=(x-2)^2+y^2-((x-3)^2+y^2)=2x-5=0$即为$P$的轨迹方程

(2) 由$M(2,\frac{5}{3}),N(\frac{1}{3},-\frac{4}{9})$得$l_{TA}:y=\dfrac{\frac{5}{3}}{5}(x+3)=\dfrac{1}{3}(x+3),l_{TB}:y=\dfrac{-\frac{4}{9}}{\frac{1}{3}-3}(x-3)=\dfrac{1}{6}(x-3)$

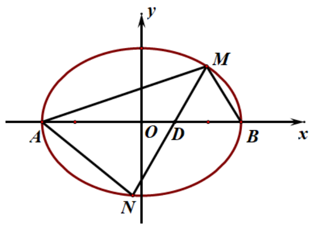
由$\begin{cases}y=\dfrac{1}{3}(x+3)\\ y=\dfrac{1}{6}(x-3)\end{cases}$得$T(9,4)$

(3) 证明：由$l_{TA}:y=\dfrac{m}{12}(x+3)$代入椭圆方程得$M(\dfrac{240-3m^2}{m^2+80},\dfrac{40m}{m^2+80})$

$l_{TB}:y=\dfrac{m}{6}(x-3)$代入椭圆方程得$N(\dfrac{3m^2-60}{m^2+20},\dfrac{-20m}{m^2+20})$，$\therefore k_{MN}=\dfrac{\frac{40m}{m^2+80}-\frac{-20m}{m^2+20}}{\frac{240-3m^2}{m^2+80}-\frac{3m^2-60}{m^2+20}}=\dfrac{-10m}{m^2-40}$

$\therefore l_{MN}:y+\dfrac{20m}{m^2+20}=\dfrac{-10m}{m^2-40}(x-\dfrac{3m^2-60}{m^2+20})$即$y=\dfrac{-10m}{m^2-40}x+\dfrac{10m}{m^2-40}=\dfrac{10m}{m^2-40}(-x+1)$过定点$(1,0)$

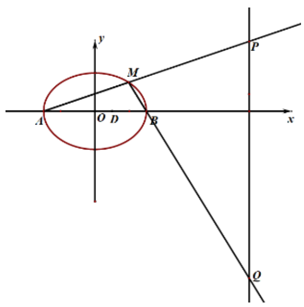
变式1.如图，椭圆$C:\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$的左右顶点分别为$A$、$B$，过点$D(1,0)$的直线$MN$与椭圆$C$分别交于点$M$、$N$.（1）设直线$AM$、$BM$、$AN$的斜率分别为$k_1$、$k_2$、$k_3$.



（ i ）求k_1k_2的值；（ ii ）求k_1k_3的值；（iii）求$\dfrac{k_2}{k_3}$的值.

（2）设直线AM,BM与直线$x=9$分别交于点P、Q，求P与Q纵坐标之积的值.

（3）设直线AM与BN交于一点T，求点T横坐标的值.



$(I)(i)k_1k_2=\dfrac{y_M}{x_M+a}\cdot\dfrac{y_M}{x_M-a}=\dfrac{y_M^2}{x_M^2-a^2}=-\dfrac{b^2}{a^2}=-\dfrac{5}{9}$

$(ii)k_1k_3=\dfrac{y_M}{x_M+a}\cdot\dfrac{y_N}{x_N+a}=\cdots=\dfrac{5}{18}$

$(iii)key1:\dfrac{k_2}{k_3}=\dfrac{y_M}{x_M-a}\cdot\dfrac{x_N+a}{y_N}=\dfrac{b^2}{a^2}\cdot\dfrac{x_M+a}{y_M}\cdot\dfrac{x_N+a}{y_N}=2$

$key2:MN:x=ty+1\Rightarrow(5t^2+9)y^2+10ty-40=0,\therefore\begin{cases}y_M+y_N=-\dfrac{10t}{5t^2+9}\\ y_My_N=\dfrac{-40}{5t^2+9}\end{cases},\therefore\dfrac{y_M+y_N}{y_My_N}=\dfrac{t}{4}$即$ty_My_N=4y_M+4y_N$

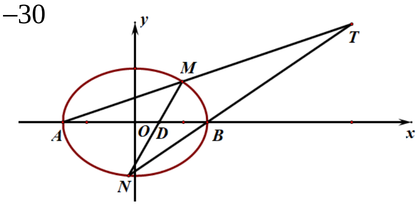
$\therefore\dfrac{k_2}{k_3}=\dfrac{y_M}{x_M-a}\cdot\dfrac{x_N+a}{y_N}=\dfrac{y_M(ty_N+4)}{y_N(ty_M-2)}=\dfrac{ty_My_N+4y_M}{ty_My_N-2y_N}=\dfrac{8y_M+4y_N}{4y_M+2y_N}=2$

解析几何（2）椭圆解答（7）

2023-11-11

（2）$key1:$ 设$M(s,t)(\frac{s^2}{9}+\frac{t^2}{5}=1)$，由$A,M,P$共线得$\frac{y_P}{9}=\frac{t}{s+3}$即$y_P=\frac{9t}{s+3}$

由B,M,Q共线得$\frac{y_Q}{6}=\frac{t}{s-3}$即$y_Q=\frac{6t}{s-3}$，$\therefore y_Py_Q=\frac{54t^2}{s^2-9}=\frac{54\cdot5(1-\frac{s^2}{9})}{s^2-9}=-30$

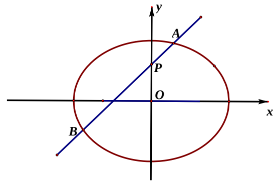


（3）$key1:MN:x=ty+1\Rightarrow(5t^2+9)y^2+10ty-40=0$

$\therefore\begin{cases}y_M+y_N=-\frac{10t}{5t^2+9}\\ y_My_N=\frac{-40}{5t^2+9}\end{cases}$，$\therefore\frac{y_M+y_N}{y_My_N}=\frac{t}{4}$即$ty_My_N=4y_M+4y_N$

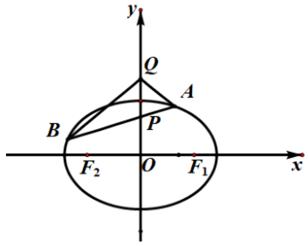
$AM:y=\frac{y_M}{ty_M+4}(x+3);BN:y=\frac{y_N}{ty_N-2}(x-3),\therefore\frac{x+3}{x-3}=\frac{(ty_M+4)y_N}{(ty_N-2)y_M}=\frac{4y_M+8y_N}{4y_N+2y_M}=2,\therefore x_T=9$

（2015 四川）已知椭圆$\frac{x^2}{4}+\frac{y^2}{2}=1$，过点$P(0,1)$的动直线$l$与椭圆相交于$A$、$B$两点.在平面直角坐标系$xOy$中，是否存在与点$P$不同的定点$Q$，使得$\frac{|QA|}{|QB|}=\frac{|PA|}{|PB|}$恒成立？若存在，求出点$Q$的坐标；若不存在，请说明理由.



2015II:（先特殊化求出Q，再证明）当$AB\perp y$轴时，$\frac{|PA|}{|PB|}=1=\frac{|QA|}{|QB|}$，得$Q$在$y$轴上；

当$l\perp x$轴时，$\frac{|QA|}{|QB|}=\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{y_Q-\sqrt{2}}{y_Q+\sqrt{2}}$得$y_Q=2$，得$Q(0,2)$，



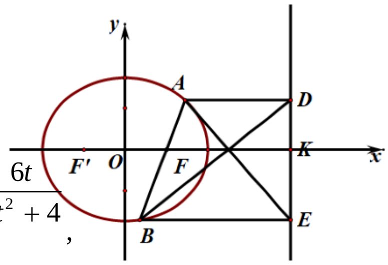
$key1:$ 设$l:y=kx+1$代入椭圆方程得:$(1+2k^2)x^2+4kx-2=0$，$\therefore\begin{cases}x_A+x_B=-\frac{4k}{1+2k^2}\\ x_Ax_B=\frac{-2}{1+2k^2}\end{cases}$

$\therefore k_{QA}+k_{QB}=\frac{y_A-2}{x_A}+\frac{y_B-2}{x_B}=0\Leftrightarrow x_B(y_A-2)+x_A(y_B-2)=x_B(kx_A-1)+x_A(kx_B-1)$

$=2kx_Ax_B-k(x_A+x_B)=\frac{-4k}{1+2k^2}+\frac{4k}{1+2k^2}=0$，$\therefore QP$是$\angle AQB$的平分线，$\therefore\frac{|QA|}{|QB|}=\frac{|PA|}{|PB|}$，$\therefore$存在，点$Q(0,2)$

（2016湖南）设椭圆$C:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$经过点$(0,\sqrt{3})$，离心率为$\frac{1}{2}$，直线$l$经过椭圆$C$的右焦点$F$，与椭圆$C$交于点$A$、$B$，$A$、$F$、$B$在直线$x=4$上的射影依次为$D$、$K$、$E$.(1) 求椭圆$C$的方程；

(2) 联结AE、BD，当直线l的倾斜角变化时，直线AE与BD是否交于定点？若是，求出定点的坐标并给予证明；否则，说明理由.



解：（1）$\frac{x^2}{4}+\frac{y^2}{3}=1$，

（2）设$l:x=ty+1$代入C方程得:$(3t^2+4)y^2+6ty-9=0$，$\therefore\begin{cases}y_A+y_B=-\frac{6t}{3t^2+4}\\ y_Ay_B=\frac{-9}{3t^2+4}\end{cases}$，

则$l_{AE}:y-y_B=\frac{y_B-y_A}{4-x_A}(x-4)$，$l_{BD}:y-y_A=\frac{y_A-y_B}{4-x_B}(x-4)$

2023-11-11

$\therefore \dfrac{y-y_B}{y-y_A}=-\dfrac{4-x_B}{4-x_A}=\dfrac{-3+ty_B}{3-ty_A},\therefore \dfrac{2y-(y_A+y_B)}{y_B-y_A}=\dfrac{t(y_B-y_A)}{6-t(y_A+y_B)}$

$\therefore 2y=(y_A+y_B)+\dfrac{t(y_B-y_A)^2}{6-t(y_A+y_B)}=\dfrac{6(y_A+y_B)-4ty_Ay_B}{6-t(y_A+y_B)}=\dfrac{\dfrac{-36t}{3t^2+4}-\dfrac{-36t}{3t^2+4}}{6-t(y_A+y_B)}=0$

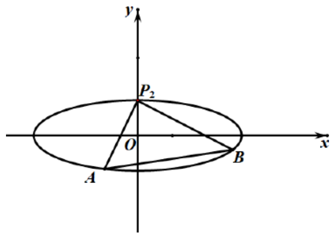
且$y_A-y_B=(x-4)(\dfrac{y_B-y_A}{4-x_A}-\dfrac{y_A-y_B}{4-x_B})$即$x-4=-\dfrac{(3-ty_A)(3-ty_B)}{(3-ty_A)+(3-ty_B)}=-\dfrac{9-3t\cdot\dfrac{-6t}{3t^2+4}+\dfrac{-9t^2}{3t^2+4}}{6+\dfrac{6t^2}{3t^2+4}}=-\dfrac{3}{2}$

即$x=\dfrac{5}{2},\therefore AE$与$BD$交于定点$(\dfrac{5}{2},0)$

（2017I）已知椭圆 $C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$,四点$P_1(1,1),P_2(0,1),P_3(-1,\dfrac{\sqrt{3}}{2}),P_4(1,\dfrac{\sqrt{3}}{2})$ 中恰有三点在椭圆 C 上.

（1）求 C 的方程；（2）设直线 l 不经过 P_2 点且与 C 相交于 A，B 两点.若直线 P_2A 与直线 P_2B 的斜率的和为 -1，证明：l 过定点.

⑴ 解：由题意得$\begin{cases}\dfrac{1}{a^2}+\dfrac{3}{4b^2}=1\\ b=1\end{cases}$得$a=2,b=1,$或$\begin{cases}\dfrac{1}{a^2}+\dfrac{3}{4b^2}=1\\ \dfrac{1}{a^2}+\dfrac{1}{b^2}=1\end{cases}$（无解），$\therefore C:\dfrac{x^2}{4}+y^2=1$



（2）证明：设$l_{AP_2}:y=k_1x+1$代入C得$A(-\dfrac{8k_1}{4k_1^2+1},1-\dfrac{8k_1^2}{4k_1^2+1})$

设$l_{P_2B}:y=k_2x+1(k_1+k_2=-1)$，同理$B(-\dfrac{8k_2}{4k_2^2+1},1-\dfrac{8k_2^2}{4k_2^2+1})$

$\therefore k_{AB}=\dfrac{-\dfrac{8k_1^2}{4k_1^2+1}+\dfrac{8k_2^2}{4k_2^2+1}}{-\dfrac{8k_1}{4k_1^2+1}+\dfrac{8k_2}{4k_2^2+1}}=\dfrac{1}{4k_1k_2-1}=-\dfrac{1}{(2k_1+1)^2}$

$\therefore l_{AB}:y-1+\dfrac{8k_1^2}{4k_1^2+1}=-\dfrac{1}{(2k_1+1)^2}(x+\dfrac{8k_1}{4k_1^2+1})$令$x=m$得$y-1=\dfrac{-1}{(2k_1+1)^2}\cdot\dfrac{4mk_1^2+8k_1+m}{4k_1^2+1}-\dfrac{8k_1^2}{4k_1^2+1}$

$=\dfrac{-32k_1^4-32k_1^3-(4m+8)k_1^2-8k_1-m}{16k_1^4+16k_1^3+8k_1^2+4k_1+1}$为常数，得$m=2,y=-1,\therefore AB$经过定点$(2,-1)$

$key2:$设$A(2\cos\alpha,\sin\alpha),B(2\cos\beta,\sin\beta),$则$k_{P_2A}+k_{P_2B}=\dfrac{\sin\alpha-1}{2\cos\alpha}+\dfrac{\sin\beta-1}{2\cos\beta}=-1$

$\Leftrightarrow 2=\dfrac{1-\sin\alpha}{\cos\alpha}+\dfrac{1-\sin\beta}{\cos\beta}=\dfrac{1-\tan\dfrac{\alpha}{2}}{1+\tan\dfrac{\alpha}{2}}+\dfrac{1-\tan\dfrac{\beta}{2}}{1+\tan\dfrac{\beta}{2}}\Leftrightarrow 2\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}+\tan\dfrac{\alpha}{2}+\tan\dfrac{\beta}{2}=0$

$\therefore \tan\dfrac{\beta}{2}=\dfrac{-t}{2t+1}$(其中$t=\tan\dfrac{\alpha}{2}$)$,\therefore \cos\alpha=\dfrac{1-t^2}{1+t^2},\sin\alpha=\dfrac{2t}{1+t^2},\cos\beta=\dfrac{3t^2+4t+1}{5t^2+4t+1},\sin\beta=\dfrac{-4t^2-2t}{5t^2+4t+1}$

而$l_{AB}:y-\sin\alpha=\dfrac{\sin\beta-\sin\alpha}{2\cos\beta-2\cos\alpha}(x-2\cos\alpha)$即$y-\dfrac{2t}{1+t^2}=\dfrac{-t^2-2t-1}{4t^2}(x-\dfrac{2-2t^2}{1+t^2})$

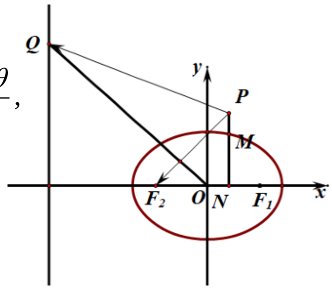
即$y=-\dfrac{(t+1)^2}{4t^2}x+\dfrac{-t^2+2t+1}{2t^2}$令$x=m$得$y=\dfrac{(-m-2)t^2+(4-2m)+2-m}{4t^2}$为常数，则$m=2,y=-1$

$\therefore$直线AB经过定点$(2,-1)$.

（2017Ⅱ）设O为坐标原点，动点M在椭圆$C:\dfrac{x^2}{2}+y^2=1$上，过M作x轴的垂线，垂足为N，点P满足$\overrightarrow{NP}=\sqrt{2}\,\overrightarrow{NM}$.(1) 求点$P$的轨迹方程；（2）设点$Q$在直线$x=-3$上，且$\overrightarrow{OP}\cdot\overrightarrow{PQ}=1$，证明：过点$P$且垂直于$OQ$的直线$l$过$C$的左焦点.

(1) 解：设$P(x,y)$,由$\overrightarrow{NP}=\sqrt{2}\,\overrightarrow{NM}$得$M(x,\dfrac{1}{\sqrt{2}}y)$,$\therefore\dfrac{x^2}{2}+\dfrac{y^2}{2}=1$即$x^2+y^2=2$即为$P$的轨迹方程

(2) 证明：设$P(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta),Q(-3,q)$，则



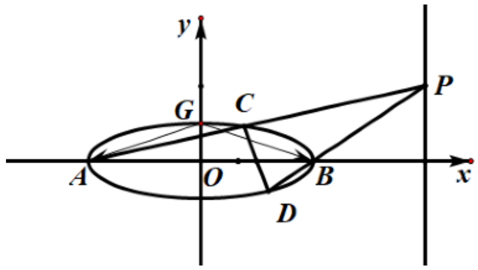
$\overrightarrow{OP}\cdot\overrightarrow{PQ}=\sqrt{2}\cos\theta(-3-\sqrt{2}\cos\theta)+\sqrt{2}\sin\theta(q-\sqrt{2}\sin\theta)=1$得$q=\dfrac{3+3\sqrt{2}\cos\theta}{\sqrt{2}\sin\theta}$,

$\therefore k_{OQ}k_{PF_2}=\dfrac{q}{-3}\cdot\dfrac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+1}=\dfrac{3(1+\sqrt{2}\cos\theta)}{-3\sqrt{2}\sin\theta}\cdot\dfrac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+1}=-1,\therefore OQ\perp PF_2$

$\therefore$过点P且垂直于OQ的直线l过C的左焦点F_2

（2020Ⅰ）已知A,B分别为椭圆$E:\dfrac{x^2}{a^2}+y^2=1(a>1)$的左、右顶点，$G$为$E$的上顶点,$\overrightarrow{AG}\cdot\overrightarrow{GB}=8$，$P$为直线$x=6$上的动点，$PA$与$E$的另一交点为$C$，$PB$为$E$的另一交点为$D$.(1) 求$E$的方程；（2）证明：直线$CD$过定点.

(1) $\dfrac{x^2}{9}+y^2=1$,



(2) 设$P(6,t)$，则$l_{PA}:y=\dfrac{t}{9}(x+3)$代入$E$得：$C(\dfrac{27-3t^2}{9+t^2},\dfrac{6t}{9+t^2})$

$l_{PB}:y=\dfrac{t}{3}(x-3)$代入$E$得：$D(\dfrac{3t^2-3}{t^2+1},\dfrac{-2t}{t^2+1})$,

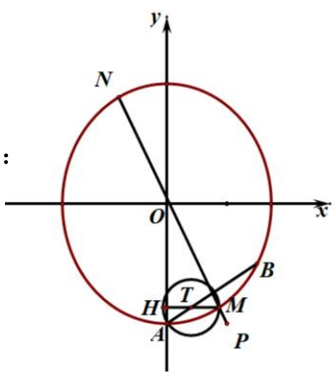
$$key1:l_{CD}:y+\frac{2t}{t^2+1}=\frac{\dfrac{6t}{9+t^2}+\dfrac{2t}{t^2+1}}{\dfrac{27-3t^2}{9+t^2}-\dfrac{3t^2-3}{t^2+1}}(x-\frac{3t^2-3}{t^2+1})=-\frac{4t}{3(t^2-3)}(x-\frac{3t^2-3}{t^2+1})$$

即$y=-\dfrac{4t}{3(t^2-3)}x+\dfrac{2t}{t^2-3}=\dfrac{2t}{t^2-3}(-\dfrac{2}{3}x+1)$经过定点$(\dfrac{3}{2},0)$

（2022乙）已知椭圆E的中心为坐标原点，对称轴为z轴、y轴，且过$A(0,-2)$、$B(\dfrac{3}{2},-1)$两点.

(1) 求E的方程；（2）设过点$P(1,-2)$的直线交E于M、N两点，过M且平行于x轴的直线与线段AB交于点T，点H满足$\overrightarrow{MT}=\overrightarrow{TH}$.证明：直线$HN$过定点.

(1) 解:$\dfrac{y^2}{4}+\dfrac{x^2}{3}=1$;



(2) 证明：当$MN\not\perp x$轴时，设$l_{MN}:y+2=k(x-1)$即$y=kx-k-2$代入E的方程得：

$(3k^2+4)x^2-6k(k+2)x+3(k+2)^2-12=0$

$\therefore\begin{cases}x_M+x_N=\dfrac{6k(k+2)}{3k^2+4}\\x_Mx_N=\dfrac{3k^2+12k}{3k^2+4}\end{cases}$,且$\Delta=96(k^2+2k)>0$,且$\dfrac{x_M+x_N}{x_Mx_N}=\dfrac{2k+4}{k+4}$

而$l_{AB}: y=\frac{2}{3}x-2, \therefore T(\frac{3}{2}(y_M+2), y_M)$（且$-2\le y_M\le -1$）

由$\overrightarrow{MT}=\overrightarrow{TH}$得$H(3y_M+6-x_M, y_M)$

$$\therefore l_{HN}: y-y_N=\frac{y_M-y_N}{3y_M+6-x_M-x_N}(x-x_N)=\frac{k(x_M-x_N)}{(3k-1)x_M-x_N-3k}(x-x_N)$$

$$\text{即} y=\frac{k(x_M-x_N)}{(3k-1)x_M-x_N-3k}x-\frac{k(x_M-x_N)x_N}{(3k-1)x_M-x_N-3k}+kx_N-k-2$$

$$=\frac{k(x_M-x_N)}{(3k-1)x_M-x_N-3k}x+\frac{k[(3k-2)x_Mx_N-(3k-1)(x_M+x_N)+3k]}{(3k-1)x_M-x_N-3k}-2$$

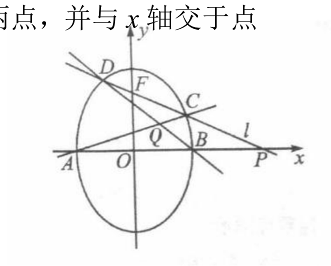
$$=\frac{k(x_M-x_N)}{(3k-1)x_M-x_N-3k}x+\frac{k[\frac{(3k-2)(3k^2+12k)}{3k^2+4}+\frac{(1-3k)(6k^2+12k)}{3k^2+4}+\frac{3k(3k^2+4)}{3k^2+4}]}{(3k-1)x_M-x_N-3k}-2$$

$$=\frac{k(x_M-x_N)}{(3k-1)x_M-x_N-3k}x-2 \text{经过定点}(0,-2),$$

当$MN\perp x$轴时，$M(1,-\frac{2\sqrt{6}}{3}), N(1,\frac{2\sqrt{6}}{3}), T(3-\sqrt{6},-\frac{2\sqrt{6}}{3}), H(5-2\sqrt{6},-\frac{2\sqrt{6}}{3})$，此时$N,H,A$共线

$\therefore NH$经过定点$A(0,-2)$，证毕

（2011II）椭圆有两顶点$A(-1,0), B(1,0)$，过其焦点$F(0,1)$的直线l与椭圆交于C、D两点，并与x轴交于点P．直线AC与直线BD交于点Q．（1）当$|CD|=\frac{3\sqrt{2}}{2}$时，求直线l的方程；



（2）当点P异于A、B两点时，求证：$\overrightarrow{OP}\cdot\overrightarrow{OQ}$ 为定值.

（1）解：由已知得$\begin{cases}b=1\\c=1\end{cases}$，$\therefore$椭圆方程为$\frac{y^2}{2}+x^2=1$

设$l_{CD}: y=kx+1$代入椭圆方程得：$(k^2+2)x^2+2kx-1=0, \therefore \begin{cases}x_C+x_D=-\frac{2k}{k^2+2}\\x_Cx_D=\frac{-1}{k^2+2}\end{cases}$，且$\Delta=8(k^2+1)$

$\therefore |CD|=\sqrt{1+k^2}\cdot\frac{2\sqrt{2}\sqrt{k^2+1}}{k^2+2}=\frac{2\sqrt{2}(k^2+1)}{k^2+2}=\frac{3\sqrt{2}}{2}$得$k=\pm\sqrt{2}$，$\therefore l$的方程为$y=\pm\sqrt{2}x+1$

（2）证明：由（1）得$\overrightarrow{OP}=(-\frac{1}{k},0)(k\ne\pm1)$

联立$l_{AC}: y=\frac{y_C}{x_C+1}(x+1)$与$l_{DB}: y=\frac{y_D}{x_D-1}(x-1)$得$\frac{x_Q+1}{x_Q-1}=\frac{y_D(x_C+1)}{y_C(x_D-1)}$

$key1$（不对称韦达定理）：$\frac{x_Q+1}{x_Q-1}=\frac{(kx_D+1)(x_C+1)}{(kx_C+1)(x_D-1)}=\frac{kx_Cx_D+x_C+kx_D+1}{kx_Cx_D-kx_C+x_D-1}$（由$\frac{x_C+x_D}{x_Cx_D}=2k$）

$$\therefore x_Q=\frac{2x_Q}{2}=\frac{2kx_Cx_D+(1-k)x_C+(k+1)x_D}{(1+k)x_C+(k-1)x_D+2}$$

$$\therefore \overrightarrow{OP}\cdot\overrightarrow{OQ}=-\frac{1}{k}\cdot\frac{x_C+x_D+(1-k)x_C+(k+1)x_D}{(1+k)x_C+(k-1)x_D+2}=-\frac{1}{k}\cdot\frac{(2-k)x_C+(k+2)x_D}{(1+k)x_C+(k-1)x_D+2}$$

$$=-\frac{1}{k}\cdot\frac{2(x_C+x_D)-k(x_C-x_D)}{k(x_C+x_D)+x_C-x_D+2}=-\frac{1}{k}\cdot\frac{\frac{-4k}{k^2+2}+k\cdot\frac{2\sqrt{2}\sqrt{k^2+1}}{k^2+2}}{\frac{-2k^2}{k^2+2}-\frac{2\sqrt{2}\sqrt{k^2+1}}{k^2+2}+2}=\frac{4-2\sqrt{2}\sqrt{k^2+1}}{4-2\sqrt{2}\sqrt{k^2+1}}=1\text{为定值，}$$

$key2$(椭圆上的点纵横坐标转换方法：利用直线或第三定义)：

$$\frac{x_Q+1}{x_Q-1}=-\frac{2(x_D+1)(x_C+1)}{y_Cy_D}\left(\text{由}\frac{y_D^2}{2}+x_D^2=1\text{得}\frac{y_D}{x_D-1}=-\frac{2(1+x_D)}{y_D}\right)$$

$$=-2\cdot\frac{x_Cx_D+x_C+x_D+1}{k^2x_Cx_D+k(x_C+x_D)+1}=-2\cdot\frac{\frac{-1}{k^2+2}+\frac{-2k}{k^2+2}+1}{\frac{-k^2}{k^2+2}+\frac{-2k^2}{k^2+2}+1}=\frac{k-1}{k+1},\therefore x_Q=\frac{2x_Q}{2}=\frac{2k}{-2}=-k,\therefore \overrightarrow{OP}\cdot\overrightarrow{OQ}=-\frac{1}{k}\cdot(-k)=1\text{为定值}$$

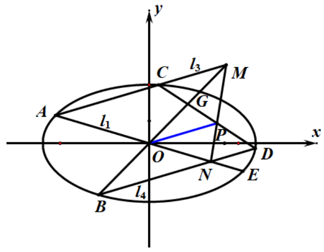
(2015河南）如图，过椭圆$ax^2+by^2=1(b>a>0)$中心O的直线l_1、l_2分别与椭圆交于点A、E、B、G，且直线l_1、l_2的斜率之积为$-\frac{a}{b}$，过点A、B作两条平行线l_3、l_4，设l_2与l_3、l_1与l_4、CD与MN分别交于点M,N,P.证明：$OP//l_3$.

证明：设$A(\frac{1}{\sqrt{a}}\cos\alpha,\frac{1}{\sqrt{b}}\sin\alpha),B(\frac{1}{\sqrt{a}}\cos\beta,\frac{1}{\sqrt{b}}\sin\beta)$,则$k_{OA}k_{OB}=\frac{a\sin\alpha\sin\beta}{b\cos\alpha\cos\beta}=-\frac{a}{b}$即$\cos(\alpha-\beta)=0$,

$\therefore\beta=\alpha+\frac{\pi}{2},\therefore B(-\frac{1}{\sqrt{a}}\sin\alpha,\frac{1}{\sqrt{b}}\cos\alpha)$,

设$C(\frac{1}{\sqrt{a}}\cos\theta,\frac{1}{\sqrt{b}}\sin\theta),D(\frac{1}{\sqrt{a}}\cos\delta,\frac{1}{\sqrt{b}}\sin\delta),\because AC//BD$,

$$\therefore k_{AC}=\frac{\frac{1}{\sqrt{b}}\sin\theta-\frac{1}{\sqrt{b}}\sin\alpha}{\frac{1}{\sqrt{a}}\cos\theta-\frac{1}{\sqrt{a}}\cos\alpha}=k_{BD}=\frac{\frac{1}{\sqrt{b}}\sin\delta-\frac{1}{\sqrt{b}}\cos\alpha}{\frac{1}{\sqrt{a}}\cos\delta+\frac{1}{\sqrt{a}}\sin\alpha}\text{即}\sin(\theta-\delta)+\cos(\theta-\alpha)+\sin(\delta-\alpha)=1$$



即$\cos\frac{\theta+\alpha-2\delta}{2}=\sin\frac{\theta-\alpha}{2},\therefore\delta=\theta-\frac{\pi}{2}$得$D(\frac{1}{\sqrt{a}}\sin\theta,-\frac{1}{\sqrt{b}}\cos\theta)$

$\because l_3//l_4,\therefore\frac{|ON|}{|OA|}=\frac{|OB|}{|OM|}$即为$t$,且$\frac{|NP|}{|PM|}=\frac{|ND|}{|CM|}=\frac{|PD|}{|PC|}$

$\therefore M(\frac{t\sin\alpha}{\sqrt{a}},-\frac{t\cos\alpha}{\sqrt{b}}),N(-\frac{\cos\alpha}{t\sqrt{a}},-\frac{\sin\alpha}{t\sqrt{b}})$,

$$\text{设}\frac{|MC|}{|MA|}=\lambda_1=\frac{\frac{t\sin\alpha}{\sqrt{a}}-\frac{1}{\sqrt{a}}\cos\theta}{\frac{t\sin\alpha}{\sqrt{a}}-\frac{\cos\alpha}{\sqrt{a}}}=\frac{t\sin\alpha-\cos\theta}{t\sin\alpha-\cos\alpha}=\frac{-\frac{t\cos\alpha}{\sqrt{b}}-\frac{\sin\theta}{\sqrt{b}}}{-\frac{t\cos\alpha}{\sqrt{b}}-\frac{\sin\alpha}{\sqrt{b}}}=\frac{t\cos\alpha+\sin\theta}{t\cos\alpha+\sin\alpha}\text{得}\lambda_1=\frac{t^2-1}{t^2+1}$$

设$\frac{|ND|}{|BN|}=\lambda_2$,同理得$\lambda_2=\frac{t^2-1}{t^2+1}=\lambda_1$

$\therefore\frac{|NP|}{|PM|}=\frac{|ND|}{|CM|}=\frac{\lambda_2|NB|}{\lambda_1|MA|}=\frac{|NB|}{|MA|}=\frac{|ON|}{|OA|},\therefore OP//l_3$

证明二：$\because l_3//l_4,\therefore\frac{|ON|}{|AO|}=\frac{|BO|}{|OM|}$记为$t$,且$\frac{|NP|}{|PM|}=\frac{|ND|}{|CM|},\therefore M(-tx_B,-ty_B),N(-\frac{1}{t}x_A,-\frac{1}{t}y_A)$,

设$\frac{|MC|}{|MA|}=\lambda_1,\frac{|ND|}{|BN|}=\lambda_2$,则$\overrightarrow{MC}=(x_C+tx_B,y_C+ty_B)=\lambda_1\overrightarrow{MA}=\lambda_1(x_A+tx_B,y_A+ty_B)$

2023-11-11

得$\begin{cases}x_C=\lambda_1x_A+t(\lambda_1-1)x_B\\y_C=\lambda_1y_A+t(\lambda_1-1)y_B\end{cases}$，

代入椭圆方程得：$\lambda_1^2(ax_A^2+by_A^2)+2t\lambda_1(\lambda_1-1)(ax_Ax_B+by_Ay_B)+(\lambda_1-1)^2t^2(ax_B^2+by_B^2)=1$

$\because k_{l_1}\cdot k_{l_2}=\dfrac{y_Ay_B}{x_Ax_B}=-\dfrac{a}{b},\therefore ax_Ax_B+by_Ay_B=0,\therefore\lambda_1^2+(1-\lambda_1)^2t^2=1$即$\lambda_1=\dfrac{t^2-1}{t^2+1}$，

由$\overrightarrow{ND}=\lambda_2\overrightarrow{BN}$得$\overrightarrow{ND}=(x_D+\dfrac{1}{t}x_A,y_D+\dfrac{1}{t}y_A)=\lambda_2(-\dfrac{1}{t}x_A-x_B,-\dfrac{1}{t}y_A-y_B)$得$\begin{cases}x_D=-\lambda_2x_B-\dfrac{1+\lambda_2}{t}x_A\\y_D=-\lambda_2y_B-\dfrac{1+\lambda_2}{t}y_A\end{cases}$

代入椭圆方程得：$\lambda_2^2(ax_B^2+by_B^2)+\dfrac{2\lambda_2(1+\lambda_2)}{t}(ax_Ax_B+by_Ay_B)+\dfrac{(1+\lambda_2)^2}{t^2}(ax_A^2+by_B^2)=1$

即$\lambda_2^2+\dfrac{(1+\lambda_2)^2}{t^2}=1$得$\lambda_2=\dfrac{t^2-1}{t^2+1}=\lambda_1,\therefore\dfrac{NP}{PM}=\dfrac{ND}{CM}=\dfrac{\lambda_2NB}{\lambda_1MA}=\dfrac{NB}{MA}=\dfrac{NO}{OA},\therefore OP//l_3$

（2021Ⅱ）已知椭圆C的方程为$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$，右焦点$F(\sqrt{2},0)$，且离心率为$\dfrac{\sqrt{6}}{3}$.

（1）求椭圆C的方程；$\dfrac{x^2}{3}+y^2=1$

（2）设M、N是椭圆C上的两点，直线MN与曲线$x^2+y^2=b^2(x>0)$相切，证明：M、N、F三点共线的充要条件是$|MN|=\sqrt{3}$.

（1）由$\begin{cases}c=\sqrt{2}\\\dfrac{c}{a}=\dfrac{\sqrt{6}}{3}\end{cases}$得$a=\sqrt{3},b=1,\therefore C:\dfrac{x^2}{3}+y^2=1$

（2）设$l_{MN}:x=ty+n$代入C得：$(t^2+3)y^2+2tny+n^2-3=0,\therefore\begin{cases}y_M+y_N=-\dfrac{2tn}{t^2+3}\\y_My_N=\dfrac{n^2-3}{t^2+3}\end{cases}$，且$\Delta=12(t^2+3-n^2)>0$

由MN与曲线$x^2+y^2=b^2=1$相切$\Leftrightarrow\dfrac{|n|}{\sqrt{1+t^2}}=1$即$n^2=t^2+1(n>0)$

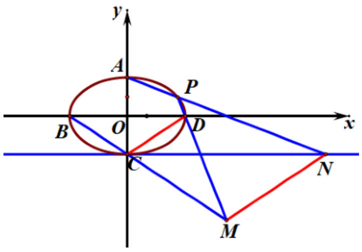
$|MN|=\sqrt{1+t^2}\cdot\dfrac{2\sqrt{3}\sqrt{t^2+3-n^2}}{t^2+3}=n\cdot\dfrac{2\sqrt{6}}{n^2+2}=\sqrt{3}\Leftrightarrow n^2-2\sqrt{2}n+2=0\Leftrightarrow n=\sqrt{2}$

M,N,F三点共线$\Leftrightarrow n=\sqrt{2},\therefore M,N,F$三点共线的充要条件为$|MN|=\sqrt{3}$

（2023北京）已知椭圆$E:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的离心率为$\dfrac{\sqrt{5}}{3}$，$A$、$C$分别是$E$的上、下顶点，$B$、$D$分别是$E$的左、右顶点，$|AC|=4$.（1）求$E$的方程；

（2）设P为第一象限内E上的动点，直线PD与直线BC交于点M，直线PA与直线$y=-2$交于点N，求证：$MN//CD$.

（1）解：由已知得$\begin{cases}\dfrac{c}{a}=\dfrac{\sqrt{5}}{3}\\2b=4\end{cases}$得$b=2,a=3,c=\sqrt{5},\therefore E$的方程为$\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$



（2）证明：设$P(3\cos\theta,2\sin\theta)(\theta\in(0,\dfrac{\pi}{2}))$,

则$l_{PA}:y=\dfrac{2\sin\theta-2}{3\cos\theta}x+2$令$y=-2$得$M(\dfrac{6\cos\theta}{1-\sin\theta},-2)$

$l_{PD}:y=\dfrac{2\sin\theta}{3\cos\theta-3}(x-3),\ l_{BC}:\dfrac{x}{-3}+\dfrac{y}{-2}=1$即$2x+3y+6=0$

$\therefore l_{MN}:2\sin\theta\cdot(x-3)-(3\cos\theta-3)y+\lambda(2x+3y+6)=0$

（其中$2\sin\theta\cdot(\dfrac{6\cos\theta}{1-\sin\theta}-3)+2(3\cos\theta-3)+\lambda\cdot\dfrac{12\cos\theta}{1-\sin\theta}=0$即$\lambda=\dfrac{\cos\theta-\sin\theta-1}{2}$）

$$\therefore k_{Mn}=-\frac{2\sin\theta+2\lambda}{3-3\cos\theta+3\lambda}=-\frac{1}{3}\cdot\frac{2\sin\theta+\cos\theta-\sin\theta-1}{1-\cos\theta+\dfrac{\cos\theta-\sin\theta-1}{2}}=-\frac{2}{3}\cdot\frac{\sin\theta+\cos\theta-1}{1-\cos\theta-\sin\theta}=\frac{2}{3}=k_{CD}$$

$\therefore MN//CD$