数列（1）等差与等比数列解答（3）

2024-02-23

（2011浙江竞赛）已知等差数列$\{a_n\}$前15项的和$S_{15}=30$,则$a_1+a_8+a_{15}=$______．6

（201610）设等差数列$\{a_n\}$的前n项和为S_n，$n\in N^*$,且$S_{2015}>0$，$S_{2016}<0$.若对于任意$n\in\{i|1\le i\le 2015,i\in N^*\}$,

均有$\dfrac{S_n}{a_n}\le\dfrac{S_k}{a_k}$,则正整数$k$的值为________.1008

$key:S_{2015}=\dfrac{2015(a_1+a_{2015})}{2}=2015a_{1008}>0,S_{2016}=\dfrac{2016(a_1+a_{2016})}{2}<0\Leftrightarrow a_{1008}+a_{1009}<0,$

$\therefore a_{1008}>0,a_{1009}<0,\therefore\{a_n\}$递减，且$a_n>0(n\le 1008),a_n<0(n\ge 1009),S_n>0(n\le 2015)$

变式 1（1）已知公差$d\ne 0$的等差数列$\{a_n\}$的前n项和为S_n，若$a_{2121}a_{2022}<0<a_{2021}+a_{2022}$，则（ ）

A. $a_1d>0$　　B. $|S_{2021}|<|S_{2022}|$　　C. $S_{4042}S_{4043}<0$　　D. $a_{2022}S_{4042}S_{4043}>0$

$key:$由$a_{2121}a_{2022}<0<a_{2021}+a_{2022}$得$\begin{cases}d>0\\a_{2021}<0<a_{2022}\\2a_{2021}+d>0\end{cases}$,或$\begin{cases}d<0\\a_{2021}>0>a_{2022}\\2a_{2021}+d>0\end{cases}$

$\therefore S_{4042}=2021(a_1+a_{4042})=2021(a_{2021}+a_{2022})>0,S_{4043}=4043a_{2022},\therefore$选$D$

变式 1（1）已知数列$\{a_n\}$为等差数列，$a_1^2+a_2^2=1,S_n$为$\{a_n\}$的前n项和，则S_5的取值范围是（ ）B

A．$[-\dfrac{15\sqrt2}{2},\dfrac{15\sqrt2}{2}]$　B．$[-5\sqrt5,5\sqrt5]$　C．$[-10,10]$　D．$[-5\sqrt3,5\sqrt3]$

$key:S_5=5a_1+\dfrac{5\cdot 4}{2}(a_2-a_1)=-5a_1+10a_2=(-5,10)\cdot(a_1,a_2)$

$\in[-\sqrt{5^2+10^2}\cdot\sqrt{a_1^2+a_2^2},\sqrt{5^2+10^2}\cdot\sqrt{a_1^2+a_2^2}]=[-5\sqrt5,5\sqrt5]$

（2）给定正整数n和正数M,对于满足条件$a_1^2+a_{n+1}^2\le M$的所有等差数列$a_1,a_2,a_3,\cdots$，则

$a_{n+1}+a_{n+2}+\cdots+a_{2n+1}$的最大值 ______．

$key:a_{n+1}+a_{n+2}+\cdots+a_{2n+1}=\dfrac{(n+1)(a_{n+1}+a_{2n+1})}{2}=\dfrac{n+1}{2}(3a_{n+1}-a_1)$

$=\dfrac{n+1}{2}(-1,3)\cdot(a_1,a_{n+1})\le\dfrac{n+1}{2}\cdot\sqrt{10M}$

（3）①若$S_n=m,S_m=n(m\ne n)$,则$S_{n+m}=$________．$-m-n$

②若$S_n=p,S_{3n}=q$,则$S_{4n}=$________,$S_{5n}=$______．$-2p+2q,-5p+\dfrac{10}{3}q$

③已知等差数列$\{a_n\}$中，$S_4=18,S_n=108,S_{n-4}=72$，则$n=$__________.16

（1996Ⅰ）等差数列$\{a_n\}$的前m项和为30，前$2m$项和为100,则它的前$3m$项和为（ ）

$A.130$　$B.170$　$C.210$　$D.260$

1996Ⅰ $key1:S_{a_{m+1}\to a_{2m}}=70=\dfrac{m(a_{m+1}+a_{2m})}{2},\therefore S_{3m}=\dfrac{3m(a_1+a_{3m})}{2}=210,$选$C$

$key2:S_m=30,S_{a_{m+1}\to a_{2m}}=70,\therefore S_{a_{2m+1}\to a_{3m}}=2\times 70-30=110,\therefore S_{3m}=210,$选$C$

$key3:$设$S_n=An^2+Bn,$则$\begin{cases}S_m=Am^2+Bm=30\\S_{2m}=4Am^2+2Bm=100\end{cases}$得$Am^2=20,Bm=10,\therefore S_{3m}=9Am^2+3Bm=210$

数列（1）等差与等比数列解答（3）

2024-02-23

(2006江西）已知等差数列$\{a_n\}$的前n项和为S_n，若$\overrightarrow{OB}=a_1\overrightarrow{OA}+a_{200}\overrightarrow{OC}$, 且$A,B,C$三点共线（该直线不过原点$O$），则$S_{200}=$（　）$A.100$　$B.101$　$C.200$　$D.201$

key: 2006江西$key: a_1+a_{200}=1, \therefore S_{200}=\dfrac{200\times 1}{2}=100$, 选$A$

(2009海南）等差数列$\{a_n\}$前n项和为S_n, 已知$a_{m-1}+a_{m+1}-a_m^2=0, S_{2m-1}=38$, 则$m=$ ____ .10

2009海南$key: 0=a_{m-1}+a_{m+1}-a_m^2=2a_m-a_m^2, S_{2m-1}=\dfrac{(2m-1)(a_1+a_{2m-1})}{2}=(2m-1)a_m=38, \therefore m=10$

(2013Ⅰ)7.设等差数列$\{a_n\}$的前n项和为$S_n, S_{m-1}=-2, S_m=0, S_{m+1}=3$, 则$m=$（　）$A.3$　$B.4$　$C.5$　$D.6$

2013Ⅰ $key: a_m=2, a_{m+1}=3, \therefore d=1, \therefore S_m=m(3-m)+\dfrac{m(m-1)}{2}=0$得$m=5$, 选$C$

(2013Ⅱ)16.等差数列$\{a_n\}$的前n项和为S_n，已知$S_{10}=0, S_{15}=25$，则nS_n的最小值为 ______ .

2013Ⅱ key: 设$S_n=An^2+Bn$，则$\begin{cases} S_{10}=100A+10B=0 \\ S_{15}=225A+15B=25 \end{cases}$得$A=\dfrac{5}{3}, B=-\dfrac{50}{3}$

$$\therefore nS_n=n(\frac{5}{3}n^2-\frac{50}{3}n)=\frac{5}{3}(n^3-10n^2)$$

$$\therefore (n+1)S_{n+1}-nS_n=\frac{5}{3}[(n+1)^2+n(n+1)+n^2-10(2n+1)]=\frac{5}{3}(3n^2-17n-9)=\frac{5}{3}[3(n-\frac{17}{6})^2-\frac{397}{12}]\ge -55$$

(2004 江苏）设无穷等差数列$\{a_n\}$的前n项和为S_n.

(1）若首项$a_1=\dfrac{3}{2}$，公差$d=1$，求满足$S_{k^2}=(S_k)^2$的正整数k;

(2）求所有的无穷等差数列$\{a_n\}$，使得对于一切正整数k都有$S_{k^2}=(S_k)^2$成立.

2004江苏解：（Ⅰ）由已知得$S_n=\dfrac{3}{2}n+\dfrac{n(n-1)}{2}=\dfrac{n^2}{2}+n$

$$\therefore S_k^2=(\frac{1}{2}k^2+k)^2=k^2(\frac{1}{4}k^2+k+1)=S_{k^2}=\frac{1}{2}k^4+k^2 \Leftrightarrow \frac{1}{4}k^2-k=0, \therefore k=4$$

（Ⅱ）由已知设$S_n=An^2+Bn$，则$S_k^2=k^2(A^2k^2+2ABk+B^2)=S_{k^2}=Ak^4+Bk^2$

$$\Leftrightarrow \begin{cases} A^2=A \\ 2AB=0 \\ B^2=B \end{cases}, \therefore \begin{cases} A=0 \\ B=0,or,1 \end{cases}, or, \begin{cases} A=1 \\ B=0 \end{cases}, \therefore S_n=0, or, n, or, n^2+n$$

$\therefore \{a_n\}=\{0\}$，或$\{1\}$，或$\{n\}$

(08竞赛）设非负等差数列$\{a_n\}$的公差$d\ne 0$, 记S_n为数列$\{a_n\}$的前n项和，证明：

（Ⅰ）若$m,n,p\in N^*$, 且$m+n=2p$，则$\dfrac{1}{S_m}+\dfrac{1}{S_n}\ge\dfrac{2}{S_p}$;（Ⅱ）若$a_{503}\le\dfrac{1}{1005}$，则$\sum\limits_{n=1}^{2007}\dfrac{1}{S_n}>2008$.

数列（1）等差与等比数列解答（3）

2024-02-23

key:①　设$S_n=An^2+Bn$(由已知得$A>0,a_1=A+B\ge 0$)

则$\dfrac{1}{S_n}+\dfrac{1}{S_m}=\dfrac{1}{An^2+Bn}+\dfrac{1}{Am^2+Bm}=\dfrac{A(m^2+n^2)+B(m+n)}{mn(Am+B)(An+B)}$

$\ge\dfrac{2A(\frac{m+n}{2})^2+2Bp}{(\frac{m+n}{2})^2(\frac{Am+B+An+B}{2})^2}=\dfrac{2Ap^2+2Bp}{p^2(Ap+B)^2}=\dfrac{2}{S_p}$

（2）由（1）得:$\dfrac{1}{S_i}+\dfrac{1}{S_{2008-i}}\ge\dfrac{2}{S_{1004}}=\dfrac{2}{502(a_1+a_{1004})}>\dfrac{2}{502(a_1+a_{1005})}=\dfrac{1}{502a_{503}}\ge\dfrac{1005}{502}$,

$\therefore\sum\limits_{n=1}^{2007}\dfrac{1}{S_n}>\dfrac{1005\cdot 2007}{1004}>2008$

$key2$:由已知得$d>0,\therefore 0\le a_1<a_2<a_3<a_4<\cdots<a_{503}\le\dfrac{1}{1005}$,

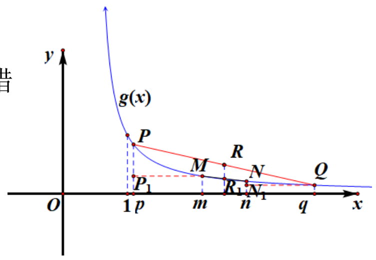
$\therefore\sum\limits_{n=1}^{2007}\dfrac{1}{S_n}>\dfrac{1}{S_1}+\dfrac{1}{S_2}+\dfrac{1}{S_3}+\dfrac{1}{S_4}>1005(1+\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4})>2010>2008$

变式 1（1）已知数列$\{a_n\}$是公差不为零且各项均为正数的无穷等差数列，其前n项和为S_n.若$p<m<n<q$,

且$p+q=m+n,p,q,m,n\in N^*$，则下列判断正确的是（　）D

$A.S_{2p}=2p\cdot a_p$　　$B.a_pa_q>a_ma_n$　$C.\dfrac{1}{a_p}+\dfrac{1}{a_q}<\dfrac{1}{a_m}+\dfrac{1}{a_n}$　$D.\dfrac{1}{S_p}+\dfrac{1}{S_q}>\dfrac{1}{S_m}+\dfrac{1}{S_n}$

key:（函数思想）设$a_n=dn+r$,则



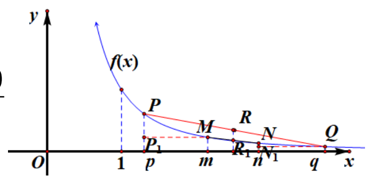
$a_pa_q-a_ma_n=(dp+r)(dq+r)-(dm+r)(dn+r)=d^2(pq-mn)<0,B$错

$\dfrac{1}{dn+r}+\dfrac{1}{dm+r}=\dfrac{d(n+m)+2r}{(dn+r)(dm+r)}\ge\dfrac{d(m+n)+2r}{(\frac{d(m+n)+2r}{2})^2}=\dfrac{2}{d(\frac{m+n}{2})+r}$,

$\therefore$函数$\dfrac{1}{dn+r}$下凸，得C错；

设$S_n=An^2+Bn(A>0,A+B>0),a_n=2An-A$，记$f(n)=\dfrac{1}{An^2+Bn},g(n)=\dfrac{1}{2An-A}$

$S_{2p}=\dfrac{2p(a_1+a_{2p})}{2}>2pa_p,A$错



$f(m)+f(n)=\dfrac{1}{S_n}+\dfrac{1}{S_m}=\dfrac{1}{An^2+Bn}+\dfrac{1}{Am^2+Bm}=\dfrac{A(m^2+n^2)+B(m+n)}{mn(Am+B)(An+B)}$

$\ge\dfrac{2A(\frac{m+n}{2})^2+2Bp}{(\frac{m+n}{2})^2(\frac{Am+B+An+B}{2})^2}=\dfrac{2Ap^2+2Bp}{p^2(Ap+B)^2}=\dfrac{2}{f(p)},\therefore f(n)$是下凸的,$\therefore\dfrac{1}{S_p}+\dfrac{1}{S_q}>\dfrac{1}{S_m}+\dfrac{1}{S_n}$

（2）数列$\{a_n\}$满足$a_n<a_{n+1}$，则下列说法错误的是（　）C

$A.$存在数列$\{a_n\}$使得对任意正整数p,q都满足$a_{pq}=q^2a_p+p^2a_q$

$B.$存在数列$\{a_n\}$使得对任意正整数p,q都满足$a_{pq}=pa_q+qa_p$

$C.$存在数列$\{a_n\}$使得对任意正整数p,q都满足$a_{p+q}=pa_q+qa_p$

数列（1）等差与等比数列解答（3）

2024-02-23

$D.$存在数列$\{a_n\}$使得对任意正整数p,q都满足$a_{p+q}=(\frac{1}{p}+\frac{1}{q})a_pa_q$

(1) $key:A:\Leftrightarrow \frac{a_{pq}}{(pq)^2}=\frac{a_p}{p^2}+\frac{a_q}{q^2}$，令$a_n=n^2\ln n$，则$\frac{a_{pq}}{(pq)^2}=\frac{(pq)^2\ln pq}{(pq)^2}=\ln p+\ln q=\frac{p^2\ln p}{p^2}+\frac{q^2\ln q}{q^2}=\frac{a_p}{p^2}+\frac{a_q}{q^2}$

$B:\Leftrightarrow \frac{a_{pq}}{pq}=\frac{a_p}{p}+\frac{a_q}{q}$，令$a_n=n\ln n$，则$\frac{a_{pq}}{pq}=\ln(pq)=\ln p+\ln q=\frac{p\ln p}{p}+\frac{q\ln q}{q}=\frac{a_p}{p}+\frac{a_q}{q}$

$C:$若C对，令$q=1$，则$a_{p+1}=pa_1+a_p$，$\therefore a_n=a_n-a_{n-1}+\cdots+a_2-a_1+a_1$

$=(n-1)a_1+(n-2)a_1+\cdots+1\cdot a_1+a_1=\frac{n(n-1)}{2}a_1+a_1=\frac{n^2-n+2}{2}a_1,n\in N^*$

此时$a_{p+q}=\frac{(p+q)^2-(p+q)+2}{2}a_1\neq pa_q+qa_p=\frac{p(q^2-q+2)}{2}a_1+\frac{q(p^2-p+2)}{2}a_1$

$D:a_n=n$，则$a_{p+q}=p+q=(\frac{1}{p}+\frac{1}{q})pq=(\frac{1}{p}+\frac{1}{q})a_pa_q$

或$\frac{a_{p+q}}{p+q}=\frac{a_p}{p}\cdot\frac{a_q}{q}$，取$a_n=ne^n$，则$\frac{a_{p+q}}{p+q}=e^{p+q}=\frac{a_p}{p}\cdot\frac{a_q}{q}$

（2010江苏）设各项均为正数的数列$\{a_n\}$的前n项和为S_n，已知$2a_2=a_1+a_3$，数列$\{\sqrt{S_n}\}$是公差为d的等差数列.(1）求数列$\{a_n\}$的通项公式（用n,d表示）；

(2）设c为实数，对满足$m+n=3k$且$m\neq n$的任意正整数m,n,k，不等式$S_m+S_n>cS_k$都成立，求证：c的最大值为$\frac{9}{2}$.

2010江苏（1）解：由已知设$\sqrt{S_n}=dn+q$即$S_n=d^2n^2+2dqn+q^2$

$\therefore a_n=\begin{cases}d^2+2dq+q^2,n=1\\ d(2dn-d+2q),n\geq 2\end{cases}$，$\therefore a_1+a_3=d^2+2dq+q^2+d(5d+2q)=6d^2+4dq+q^2$

$=2a_2=2d(3d+2q)\Leftrightarrow q^2=0\Leftrightarrow q=0,\therefore a_n=d^2(2n-1),n\in N^*$

(2）证明：由（1）得$:S_n=d^2n^2(d>0),\because m+n=3k(m\neq n)$

$\therefore S_m+S_n=d^2(m^2+n^2)>cS_k=cd^2k^2$，

$\therefore c<9\cdot\frac{m^2+n^2}{(m+n)^2}>9\cdot\frac{\frac{(m+n)^2}{2}}{(m+n)^2}=\frac{9}{2}$，$\therefore c$的最大值为$\frac{9}{2}$，证毕

（2013江苏）设$\{a_n\}$是首项为a，公差为d的等差数列$(d\neq 0)$，S_n是其前n项和，记$b_n=\frac{nS_n}{n^2+c},n\in N^*$，其中$c$为实数.(1) 若$c=0$，且$b_1,b_2,b_4$成等比数列，证明：$S_{nk}=n^2S_k(k,n\in N^*)$;

(2）若$\{b_n\}$是等差数列，证明$:c=0$.

2013江苏（1）证明：由已知得$S_n=na+\frac{n(n-1)}{2}d$

$\because c=0$，且b_1,b_2,b_4成等比数列，$\therefore b_n=a+\frac{d(n-1)}{2}=\frac{d}{2}n+a-\frac{d}{2}$

$\therefore b_2^2-b_1b_4=(a+\frac{d}{2})^2-a(\frac{3}{2}d+a)=-\frac{1}{2}ad+\frac{1}{4}d^2=0(\because d\neq 0),\therefore d=2a$

$\therefore S_n=an^2,\therefore S_{nk}=ak^2n^2=n^2S_k$，证毕

2024-02-23

（2）证明：由（1）得$S_n=na+\frac{n(n-1)}{2}d(d\neq0)$,

由$\{b_n\}$是等差数列，设$b_n=pn+q$，则$b_n=\frac{nS_n}{n^2+c}\Leftrightarrow n^2a+\frac{n^2(n-1)}{2}d=(pn+q)(n^2+c)$

$\Leftrightarrow\begin{cases}\frac{d}{2}=p\\ a-\frac{d}{2}=q\\ pc=0\\ qc=0\end{cases},\therefore\frac{d}{2}\cdot c=0,\because d\neq0,\therefore c=0$，证毕

(2019III)14.记S_n为等差数列$\{a_n\}$的前n项和，若$a_1\neq0,a_2=3a_1$，则$\frac{S_{10}}{S_5}=$______.

2019III$key:a_2=a_1+d=3a_1$得$d=2a_1,\therefore\frac{S_{10}}{S_5}=\frac{10a_1+\frac{9\times10}{2}\cdot2a_1}{5a_1+\frac{4\times5}{2}\cdot2a_1}=4$

（2019贵州）已知正项数列$\{a_n\}$的前n项和为S_n.若$\{a_n\},\{\sqrt{S_n}\}$均为公差为d的等差数列，则$S_n=$____. $\frac{n^2}{4}$

$key:a_n=a_1+(n-1)d,\sqrt{S_n}=\sqrt{na_1+\frac{n(n-1)}{2}d}=\sqrt{\frac{d}{2}n^2+(a_1-\frac{d}{2})n}$是等差数列，

$\therefore\begin{cases}a_1-\frac{d}{2}=0\\ \sqrt{\frac{d}{2}}=d\end{cases}$得$\begin{cases}d=\frac{1}{2}\\ a_1=\frac{1}{4}\end{cases},\therefore S_n=\frac{n^2}{4}$

（2019上海）设等差数列$\{a_n\}$的公差为$d(d\neq0)$，前n项和为S_n,若数列$\{\sqrt{8S_n+2n}\}$也是公差为d的等差数列，则$a_n=$____.

2019上海$key:a_n=dn+q$,则$S_n=\frac{n(dn+d+2q)}{2}=\frac{d}{2}n^2+\frac{d+2q}{2}n$

$\therefore\sqrt{8S_n+2n}=\sqrt{4dn^2+(4d+8q+2)n}=dn+q_1,\therefore\begin{cases}4d+8q+2=0\\ 2\sqrt{d}=d\end{cases}$得$\begin{cases}d=4\\ q=-\frac{9}{4}\end{cases},\therefore a_n=4n-\frac{9}{4}$

（2021 甲）18. 已知数列$\{a_n\}$的各项均为正数，记S_n为$\{a_n\}$的前n项和，从下面①②③中选取两个作为条件，证明另外一个成立．①数列$\{a_n\}$是等差数列：②数列$\{\sqrt{S_n}\}$是等差数列；③$a_2=3a_1$．

注：若选择不同的组合分别解答，则按第一个解答计分．

2021甲key:①③为条件，②为结论

由①设$a_n=a_1+(n-1)d(a_1>0$,公差$d>0)$

由③得$a_2=a_1+d=3a_1$得$d=2a_1,\therefore a_n=(2n-1)a_1$,

$\therefore S_n=\frac{n\cdot2na_1}{2}=a_1n^2,\therefore\sqrt{S_n}=\sqrt{a_1}\cdot n,\therefore\sqrt{S_n}-\sqrt{S_{n-1}}=\sqrt{a_1}$为常数,$\therefore\{\sqrt{S_n}\}$是等差数列，证毕

【详解】选①②作条件证明③：设$\sqrt{S_n}=an+b(a>0)$，则$S_n=(an+b)^2$，

当 $n=1$ 时，$a_1=S_1=(a+b)^2$；

当 $n\geq 2$ 时，$a_n=S_n-S_{n-1}=\left(an+b\right)^2-\left(an-a+b\right)^2=a(2an-a+2b)$；

因为 $\{a_n\}$ 也是等差数列，所以 $(a+b)^2=a(2a-a+2b)$，解得 $b=0$；

所以 $a_n=a^2(2n-1)$，所以 $a_2=3a_1$.

②③作条件证明①:

设 $\sqrt{S_n}=an+b(a>0)$，则 $S_n=(an+b)^2$，

当 $n=1$ 时，$a_1=S_1=(a+b)^2$；

当 $n\geq 2$ 时，$a_n=S_n-S_{n-1}=(an+b)^2-(an-a+b)^2=a(2an-a+2b)$；

因为 $a_2=3a_1$，所以 $a\left(3a+2b\right)=3\left(a+b\right)^2$，解得 $b=0$ 或 $b=-\dfrac{4a}{3}$；

当 $b=0$ 时，$a_1=a^2,a_n=a^2\left(2n-1\right)$，当 $n\geq 2$ 时，$a_n-a_{n-1}=2a^2$ 满足等差数列的定义，

此时 $\{a_n\}$ 为等差数列；

当 $b=-\dfrac{4a}{3}$ 时，$\sqrt{S_n}=an+b=an-\dfrac{4}{3}a$，$\sqrt{S_1}=-\dfrac{a}{3}<0$ 不合题意，舍去.

综上可知 $\{a_n\}$ 为等差数列.

(2023Ⅰ)7.记 S_n 为数列 $\{a_n\}$ 的前 n 项和，设甲:$\{a_n\}$ 为等差数列；乙:$\{\dfrac{S_n}{n}\}$ 为等差数列.则（C）

A.甲是乙充分不必要条件　B.甲是乙的必要不充分条件

C.甲是乙的充要条件　D.甲既不是乙的充分条件也不是乙的必要条件

$$2023\text{Ⅰ}\ key:\text{甲}\Rightarrow a_n=pn+q\Rightarrow\frac{S_n}{n}=\frac{\dfrac{n(pn+q+p+q)}{2}}{n}=\frac{pn+p+2q}{2}\text{是等差数列}$$

$$\text{乙}\Rightarrow\frac{S_n}{n}=pn+q\Leftrightarrow S_n=pn^2+qn,\therefore a_n=\begin{cases}p+q,n=1,\\ pn^2+qn-(p(n-1)^2+q(n-1))=p(2n-1)+q,n\geq 2,\end{cases}$$

$=p(2n-1)+q,\therefore\{a_n\}$ 是等差数列,∴选 C

(2015Ⅰ)10.设 $\{a_n\}$ 是公差为正数的等差数列，若 $a_1+a_2+a_3=15,a_1a_2a_3=80$,则 $a_{11}+a_{12}+a_{13}=$（　）

A.120　B.105　C.90　D.75

2015Ⅰ $key:a_1+a_2+a_3=3a_2=15$ 得 $a_2=5$,

$a_1a_2a_3=a_2(a_2^2-d^2)=5(25-d^2)=80(d>0)$ 得 $d=3$,

$\therefore a_{11}+a_{12}+a_{13}=3a_{12}=3(5+10\times 3)=105$, 选 B

(2015广东）在等差数列 $\{a_n\}$ 中，若 $a_3+a_4+a_5+a_6+a_7=25$，则 $a_2+a_8=$_______.

2015广东 $key:5a_5=25$ 得 $a_5=5,\therefore a_2+a_8=2a_5=10$

(2016Ⅰ)3.已知等差数列 $\{a_n\}$ 前9项的和为27, $a_{10}=8$，则 $a_{100}=$（　　）A.100　B.99　C.98　D.97

$$2016\text{Ⅰ}\ key:S_9=\frac{9(a_1+a_9)}{2}=9a_5=27\text{得}a_5=3,\therefore a_{100}=a_{10}+90\cdot\frac{a_{10}-a_5}{5}=98\text{选}C$$

(2017Ⅰ)4.记 S_n 为等差数列 $\{a_n\}$ 的前 n 项和，若 $a_4+a_5=24,S_6=48$，则 $\{a_n\}$ 的公差为（C）A.1　B.2　C.4 D.8

数列（1）等差与等比数列解答（3）

2024-02-23

2017Ⅰ $key:\begin{cases}a_4+a_5=2a_1+7d=24\\ S_6=6a_1+\dfrac{6\times5}{2}d=6a_1+15d=48\end{cases}$ 得$d=4$, 选C

变式.等差数列$\{a_n\}$中， 若$a_3+a_9+a_{15}=72$,则$a_{10}-\dfrac{1}{3}a_{12}=$__________. 16

（2020 山东）14.将数列$\{2n-1\}$与$\{3n-2\}$的公共项从小到大排列得到数列$\{a_n\}$，则$\{a_n\}$的前 n 项和为___.

$key:2n-1=3m-2$得$n=\dfrac{3(m-1)}{2}\in N^*,\therefore m=2k-1,k\in N^*,\therefore a_n=a_{2n-1}=3(2k-1)-2=6k-5,k\in N^*$

$\therefore\sum\limits_{i=1}^{n}a_i=\dfrac{n(1+6n-5)}{2}=3n^2-2n$

(1995Ⅰ）等差数列$\{a_n\},\{b_n\}$的前n项和分别为S_n与T_n，若$\dfrac{S_n}{T_n}=\dfrac{2n}{3n+1}$，则$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=$（ ）$A.1$ $B.\dfrac{\sqrt6}{3}$ $C.\dfrac{2}{3}$ $D.\dfrac{4}{9}$

1995Ⅰ $key1:S_n=k\cdot2n^2,T_n=kn(3n+1),\therefore a_n=S_n-S_{n-1}=2k(2n-1),b_n=k(3(2n-1)+1)=k(6n-2)$

$\therefore\dfrac{a_n}{b_n}=\dfrac{2n-1}{3n-1}$, 选$C$

(2007湖北）已知两个等差数列$\{a_n\},\{b_n\}$的前n项和分别为A_n与B_n，若$\dfrac{A_n}{B_n}=\dfrac{7n+45}{n+3}$,则使得$\dfrac{a_n}{b_n}$为整数的正整数$n$的个数是（ ）$A.2$ $B.3$ $C.4$ $D.5$

2007湖北$key2:\dfrac{a_n}{b_n}=\dfrac{2a_n}{2b_n}=\dfrac{a_1+a_{2n-1}}{b_1+b_{2n-1}}=\dfrac{A_{2n-1}}{B_{2n-1}}=\dfrac{7(2n-1)+45}{2n+2}=\dfrac{7n+19}{n+1}=7+\dfrac{12}{n+1},\therefore n+1=2,3,4,6,12,\therefore$ 选D

(2018四川）设S_n,T_n分别是等差数列$\{a_n\}$与$\{b_n\}$的前n项和，对任意正整数n，都有$\dfrac{S_n}{T_n}=\dfrac{2n+6}{n+1}$，若$\dfrac{a_m}{b_m}$为质数，则正整数$m$的值为（ ）$A.2$ $B.3$ $C.5$ $D.7$

2018四川key:由已知得$S_n=k(2n^2+6n),T_n=k(n^2+n)$,

$\therefore a_m=k(2m^2-2(m-1)^2+6)=4k(m+1),b_m=k(m^2-(m-1)^2+1)=2km,\therefore\dfrac{a_m}{b_m}=\dfrac{2(m+1)}{m}$为素数得$m=2$, 选$A$

变式1：已知等差数列$\{a_n\}$与$\{b_n\}$的前n项和分别为A_n,B_n.

（1）若$\dfrac{A_n}{A_m}=\dfrac{n^2-2n}{m^2-2m}$,则$\dfrac{a_{10}}{a_{20}}=$_______；若$\dfrac{A_n}{B_n}=\dfrac{3n+1}{7n+5}$,则$\dfrac{a_n}{b_n}=$__________.

①$A_n=k(n^2-2n),\therefore\dfrac{a_{10}}{a_{20}}=\dfrac{A_{10}-A_9}{A_{20}-A_{19}}=\dfrac{10^2-9^2-2}{20^2-19^2-2}=\dfrac{17}{37}$

②$key1:A_n=k(3n^2+n),B_n=k(7n^2+5n)$,则$a_n=k(3(2n-1)+1)=k(6n-2)$

$b_n=k(7(2n-1)+5)=k(14n-2),\therefore\dfrac{a_n}{b_n}=\dfrac{6n-2}{14n-2}=\dfrac{3n-1}{7n-1}$

$key2:\dfrac{a_n}{b_n}=\dfrac{2a_n}{2b_n}=\dfrac{a_1+a_{2n-1}}{b_1+b_{2n-1}}=\dfrac{A_{2n-1}}{B_{2n-1}}=\dfrac{3(2n-1)+1}{7(2n-1)+5}=\dfrac{3n-1}{7n-1}$

（2）若$\dfrac{a_n}{b_n}=\dfrac{3n+1}{7n+5}$,则$\dfrac{A_{2n+1}}{B_{2n+1}}=$__________,$\dfrac{A_n}{B_n}=$________.

2024-02-23

$\frac{A_{2n+1}}{B_{2n+1}}=\frac{a_1+a_{2n+1}}{b_1+b_{2n+1}}=\frac{a_{n+1}}{b_{n+1}}=\frac{3(n+1)+1}{7(n+1)+5}=\frac{3n+4}{7n+12}$

由已知得$a_n=k(3n+1),b_n=k(7n+5),\therefore\frac{A_n}{B_n}=\frac{a_1+a_n}{b_1+b_n}=\frac{4+3n+1}{12+7n+5}=\frac{3n+5}{7n+17}$

(1992Ⅰ）设等差数列$\{a_n\}$的前n项和为S_n,已知$a_3=12,S_{12}>0,S_{13}<0.$
（Ⅰ）求公差d的取值范围；（Ⅱ）指出$S_1,S_2,\cdots,S_{12}$中哪一个最大，并说明理由.

1992Ⅰ解：（Ⅰ）由已知得$\begin{cases}S_{12}=12(a_3-2d)+\frac{12\times11}{2}d=12(12+\frac{7}{2}d)>0\\S_{13}=13(a_3-2d)+\frac{13\times12}{2}d=13(12+4d)<0\end{cases},\therefore d\in(-\frac{24}{7}.-3)$

（Ⅱ）由$\begin{cases}S_{12}=6(a_1+a_{12})=6(a_6+a_7)>0\\S_{13}=\frac{13(a_1+a_{13})}{2}=13a_7<0\end{cases}$得$a_6>0,a_7<0,\therefore\{a_n\}$递减$,\therefore S_6$最大

(2004重庆）若数列$\{a_n\}$是等差数列，首项$a_1>0,a_{2003}+a_{2004}>0,a_{2003}\cdot a_{2004}<0$，则使前$n$项和$S_n>0$成立的最大自然数$n$是（ ）$A.4005$ $B.4006$ $C.4007$ $D.4008$

(2004重庆）$key:a_{2003}>0>a_{2004}$,

$\therefore S_{4006}=\frac{4006(a_1+a_{4006})}{2}>0,S_{4007}=\frac{4007(a_1+a_{4007})}{2}=4007a_{2004}<0,$选$B$

(1995A2009浙江）设等差数列$\{a_n\}$满足$3a_8=5a_{13}$,且$a_1>0,S_n$为其前n项之和.则S_n中最大的是（ ）
$A.S_{10}$ $B.S_{11}$ $C.S_{20}$ $D.S_{21}$

1995$A:a_1>0,3a_8=5a_{13},\therefore\{a_n\}$递减，且$a_{20}+a_{21}=0,\therefore a_{20}>0>a_{21},\therefore$选$C$

变式：若数列$\{a_n\}$是等差数列，数列$\{b_n\}$满足$b_n=a_na_{n+1}a_{n+2}(n\in N_+)$，其前$n$项和为$S_n$,若$4a_5=7a_{10}>0$,试问$n$多大时,$S_n$取得最大值？并证明你的结论.

key:由$4a_3=7a_{10}$得$2a_{19}+a_{20}=0,\because a_1>0,\therefore a_{19}>0,a_{20}<0$
$\therefore b_n>0(n\le17),b_{18}<0,b_{19}>0,b_n<0(n\ge20)$,
而$b_{18}+b_{19}=a_{18}a_{19}a_{20}+a_{19}a_{20}a_{21}=a_{19}a_{20}(a_{18}+a_{21})=a_{19}a_{20}(a_{19}+a_{20})>0$
$\therefore$当$n=19$时，S_n取得最大值

(2012浙江）（多选题）设S_n是公差为$d(d\neq0)$的无穷等差数列$\{a_n\}$的前n项和，则下列命题正确的是（ ）
$A.$若$d<0$，则数列$\{S_n\}$有最大项 $B.$若数列$\{S_n\}$有最大项，则$d<0$
$C.$若数列$\{S_n\}$是递增数列，则对任意的$n\in N^*$,均有$S_n>0$
$D.$若对任意的$n\in N^*$,均有$S_n>0$，则数列$\{S_n\}$是递增数列

2012:（ABD）D:由已知得$a_1>0,d>0,\therefore a_n>a_{n-1}>0,\therefore\{S_n\}$递增

(201610）设等差数列$\{a_n\}$的前n项和为S_n，$n\in N^*$,且$S_{2015}>0$，$S_{2016}<0$.若对于任意$n\in\{i\,|\,1\le i\le2015,i\in N^*\}$,均有$\frac{S_n}{a_n}\le\frac{S_k}{a_k}$,则正整数$k$的值为________.

(201610）$key:S_{2015}=\frac{2015(a_1+a_{2015})}{2}=2015a_{1008}>0,S_{2016}=1008(a_1+a_{2016})<0,\therefore a_{1008}>0,a_{1008}+a_{1009}<0,$

$\therefore a_1>\cdots>a_{1008}>0,a_n<0(n\ge1009),0<S_n<S_{n+1}(n\le1007),$

$\therefore$ 当$n\ge 1009$时$,\dfrac{S_n}{a_n}<0$；当$n\ge 1008$时$,\dfrac{S_n}{a_n}\le\dfrac{S_{1008}}{a_{1008}},\therefore k=1008$

（1998理）已知数列$\{b_n\}$是等差数列，$b_1=1,b_1+b_2+\cdots+b_{10}=145$.⑴ 求$b_n$;

（2）设数列$\{b_n\}$的通项$a_n=\log_a(1+\dfrac{1}{b_n})(a>0,$ 且$a\ne 1)$，记S_n是数列$\{a_n\}$的前n项和，试比较S_n与$\dfrac{1}{3}\log_a b_{n+1}$的大小，并证明你的结论.

1998⑴ 由$b_1+b_2+\cdots+b_{10}=10\times 1+\dfrac{10\times 9}{2}d=145$得$d=3,\therefore b_n=3n-2$

（2）由（1）得$a_n=\log_a\dfrac{3n-1}{3n-2},\therefore S_n=\log_a(\dfrac{2}{1}\cdot\dfrac{5}{4}\cdots\dfrac{3n-1}{3n-2})$, 而$\dfrac{1}{3}\log_a b_{n+1}=\log_a\sqrt[3]{3n+1}$

$key1:$ 设$f(n)=\dfrac{\frac{2}{1}\cdot\frac{5}{4}\cdots\frac{3n-1}{3n-2}}{\sqrt[3]{3n+1}}$，则$\dfrac{f(n+1)}{f(n)}=\dfrac{\sqrt[3]{3n+1}}{\sqrt[3]{3n+4}}\cdot\dfrac{3n+2}{3n+1}>1\Leftrightarrow(3n+2)^3>(3n+1)^2(3n+4)$

$\Leftrightarrow(3n+2)(1+\dfrac{1}{3n+1})^2=(3n+2)(1+\dfrac{2}{3n+1}+\dfrac{1}{(3n+1)^2})>3n+2+\dfrac{2(3n+2)}{3n+1}>3n+4$成立

$\therefore f(n)$递增$,\therefore f(n)\ge f(1)=\dfrac{2}{\sqrt[3]{4}}>1,\therefore\dfrac{2}{1}\cdot\dfrac{5}{4}\cdots\dfrac{3n-1}{3n-2}>\sqrt[3]{3n+1},$

$\therefore$ 当$a>1$时，$S_n>\dfrac{1}{3}\log_a b_{n+1}$;当$0<a<1$时，$S_n<\dfrac{1}{3}\log_a b_{n+1}$.

$key2:$ 设$A_n=\dfrac{2}{1}\cdot\dfrac{5}{4}\cdots\dfrac{3n-1}{3n-2},B_n=\dfrac{3}{2}\cdot\dfrac{6}{5}\cdots\dfrac{3n}{3n-1},C_n=\dfrac{4}{3}\cdot\dfrac{7}{6}\cdots\dfrac{3n+1}{3n},$

$\because\dfrac{2}{1}>\dfrac{3}{2}>\dfrac{4}{3}>\dfrac{5}{4}>\dfrac{6}{5}>\dfrac{7}{6}>\cdots>\dfrac{3n-1}{3n-2}>\dfrac{3n}{3n-1}>\dfrac{3n+1}{2n}>0,\therefore A_n^3>A_nB_nC_n=3n+1$得$A_n>\sqrt[3]{3n+1}$

$key3:$ 数学归纳法

一、等比数列

⑴ 定义$:\dfrac{a_n}{a_{n-1}}=q$（q为非零常数）

$\Rightarrow a_n^2=a_{n+1}a_{n-1}\Rightarrow a_n=a_1q^{n-1}=a_mq^{n-m}$

$$\Rightarrow S_n=\begin{cases}na_1,q=1\\ \dfrac{a_1(1-q^n)}{1-q},q\ne 1\end{cases}$$

（2）性质：若$\{a_n\}$是等比数列，则①若$\{k_n\}$是等差数列，且$k_n\in N^*$，则$\{a_{k_n}\}$是等比数列

②若$p_1+p_2+\cdots+p_m=q_1+q_2+\cdots+q_m,p_i,q_i\in N^*$,则$a_{p_1}\cdot a_{p_2}\cdots a_{p_m}=a_{q_1}\cdot a_{q_2}\cdots a_{q_m}$.

（2009北京）已知数集$A=\{a_1,a_2,\cdots,a_n\}(1\le a_1<a_2<\cdots<a_n,n\ge 2)$具有性质$P$：对任意的$i,j(1\le i\le j\le n)$, a_ia_j与$\dfrac{a_j}{a_i}$两个数中至少有一个属于A.⑴ 分别判断数集$\{1,3,4\}$与$\{1,2,3,6\}$是否具有性质P，并说明理由;

（2）证明：$a_1=1$, 且$\dfrac{a_1+a_2+\cdots+a_n}{a_1^{-1}+a_2^{-1}+\cdots+a_n^{-1}}=a_n$;

（3）证明：当$n=5$时，a_1,a_2,a_3,a_4,a_5成等比数列.

数列（1）等差与等比数列解答（3）

2024-02-23

2009北京（1）解：$A=\{1,3,4\}$有$3\times 4\notin A$, 且$\frac{4}{3}\notin A$, $\therefore\{1,3,4\}$不具有性质P

$A=\{1,2,3,6\}$有$1\times 2,1\times 3,1\times 6\in A; 2\times 3\in A,\frac{6}{2}=3\in A;\frac{6}{3}=2\in A,\therefore\{1,2,3,6\}$具有性质$P$

（2）证明：$\because 1\le a_1<a_2<\cdots<a_n,\therefore a_n\cdot a_n=a_n^2>a_n,\therefore a_n\cdot a_n\notin A,\therefore 1=\frac{a_n}{a_n}\in A,\therefore a_1=1$

$\therefore a_n=a_n\cdot a_1^{-1}>a_n\cdot a_2^{-1}>\cdots>a_n\cdot a_n^{-1}=1$, 且$a_na_i>a_n(i=2,3,\cdots,n-1)$

$\therefore a_na_i\notin A,\therefore a_n\cdot a_i^{-1}\in A(i=2,3,\cdots n-1)$

$\therefore\{a_2,a_3,\cdots,a_{n-1}\}=\{a_na_2^{-1},a_na_3^{-1},\cdots,a_na_{n-1}^{-1}\}$,

$\therefore a_1+a_2+\cdots+a_n=a_1^{-1}a_n+a_2^{-1}a_n+\cdots+a_n^{-1}a_n,\therefore\frac{a_1+a_2+\cdots+a_n}{a_1^{-1}+a_2^{-1}+\cdots+a_n^{-1}}=a_n$, 证毕

（3）证明：由（2）得$:a_1=1$, 且$\frac{a_5}{a_4}=a_2,\frac{a_5}{a_3}=a_3,\frac{a_5}{a_2}=a_4,\therefore a_3^2=a_2a_4=a_5a_1$

$\therefore a_1,a_3,a_5$与a_2,a_3,a_4都成等比数列，设$q=\frac{a_3}{a_2},\therefore a_3=a_2q,a_4=a_2q^2,a_5=a_2^2q^2$,

若$a_2a_3=a_2^2q\in A$，则$a_2^2q=a_2q^2,\therefore a_2=q$;若$\frac{a_3}{a_2}=q\in A$，而$a_3=a_2q>q,\therefore a_2=q$;

$\therefore a_1=1,a_2=q,a_3=q^2,a_4=q^3,a_5=q^4,\therefore a_1,a_2,a_3,a_4,a_5$成等比数列，证毕

（或$\therefore a_5=a_3^2=a_2a_4<a_3a_4,\therefore\frac{a_4}{a_3}=q$在数列$A$之中，而$1=a_1<q<a_3=a_2q,\therefore a_2=q$）