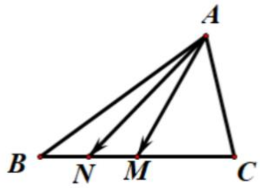
平面向量解答（3）

2023-02-26

（2017A）在△ABC中, M是边BC的中点, N是线段BM的中点.若$\angle A=\frac{\pi}{3}$, △ABC的面积为$\sqrt{3}$, 则$\overrightarrow{AM}\cdot\overrightarrow{AN}$的最小值为______.

2017$Akey$:（基向量思想）$\frac{1}{2}bc\cdot\frac{\sqrt{3}}{2}=\sqrt{3}$即$bc=4$,



$\overrightarrow{AM}\cdot\overrightarrow{AN}=(\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AC})\cdot(\frac{3}{4}\overrightarrow{AB}+\frac{1}{4}\overrightarrow{AC})=\frac{3}{8}c^2+\frac{1}{8}b^2+\frac{1}{2}bc\cdot\frac{1}{2}$

$\ge 2\cdot\sqrt{\frac{3}{8}}c\cdot\sqrt{\frac{1}{8}}b+\frac{1}{4}bc=\sqrt{3}+1$

（202207）18.已知平面向量$\vec{a},\vec{b}$满足$|2\vec{a}-\vec{b}|=1$, $\vec{a}$在$\vec{b}$上的投影向量模长为1, 则$(4\vec{a}-\vec{b})\cdot\vec{b}$的取值范围为__.

key:由已知得:$\begin{cases}4a^2-4ab\cos\theta+b^2=1\\ a|\cos\theta|=1\end{cases}$得$\begin{cases}\cos\theta=\frac{1}{a}\le 1\\ 4a^2-4b+b^2=1=4a^2+(b-2)^2-4=1\end{cases}$得$1\le a\le\sqrt{\frac{5}{4}}$

$\therefore(4\vec{a}-\vec{b})\cdot\vec{b}=4ab\cos\theta-b^2=4a^2-1\in[3,4]$

变式1（1）设平面向量$\vec{\alpha},\vec{\beta}$满足$|2\vec{\alpha}-\vec{\beta}|=1,|2\vec{\alpha}+\vec{\beta}|=2$, 则$\vec{\alpha}\cdot\vec{\beta}=$_____, $|\vec{\alpha}|+|\vec{\beta}|\in$____.

key:设$\vec{p}=2\vec{\alpha}-\vec{\beta},\vec{q}=2\vec{\alpha}+\vec{\beta}$, 则$|\vec{p}|=1,|\vec{q}|=2$, 且$\vec{\alpha}=\frac{1}{4}(\vec{p}+\vec{q}),\vec{\beta}=\frac{1}{2}(\vec{q}-\vec{p}),\therefore\vec{\alpha}\cdot\vec{\beta}=\frac{1}{8}(\vec{q}^2-\vec{p}^2)=\frac{3}{8}$,

$|\vec{\alpha}|+|\vec{\beta}|=\frac{1}{4}\sqrt{5+4\cos\theta}+\frac{1}{2}\sqrt{5-4\cos\theta}=(\frac{1}{4},\frac{1}{2})\cdot(\sqrt{5+4\cos\theta},\sqrt{5-4\cos\theta})\in[\frac{5}{4},\frac{5\sqrt{2}}{4}]$

（2）①设向量$\vec{a},\vec{b},\vec{c},\vec{e}$是单位向量且$\vec{a}+\vec{b}+\vec{c}=\vec{0}$, 则$(\vec{a}-\vec{e})\cdot(\vec{b}-\vec{e})+(\vec{b}-\vec{e})\cdot(\vec{c}-\vec{e})+(\vec{c}-\vec{e})\cdot(\vec{a}-\vec{e})=$__.

key:原式$=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+3-2\vec{e}\cdot(\vec{a}+\vec{b}+\vec{c})=3+\frac{(\vec{a}+\vec{b}+\vec{c})^2-\vec{a}^2-\vec{b}^2-\vec{c}^2}{2}=\frac{3}{2}$

②若$2\le|\vec{a}|,|\vec{b}|,|\vec{a}+\vec{b}|\le 5$, 则$\vec{a}\cdot\vec{b}$的取值范围为_____.

①$\vec{a}\cdot\vec{b}=\frac{(\vec{a}+\vec{b})^2-\vec{a}^2-\vec{b}^2}{2}\le\frac{25-4-4}{2}=\frac{21}{2}$（等号不成立）

$\vec{a}\cdot\vec{b}=\frac{(\vec{a}+\vec{b})^2-(\vec{a}-\vec{b})^2}{4}\le\frac{25}{4}$（等号能成立）

$\vec{a}\cdot\vec{b}=\frac{(\vec{a}+\vec{b})^2-\vec{a}^2-\vec{b}^2}{2}\ge\frac{4-25-25}{2}=-23$（等号能成立）

③已知平面向量$\vec{a},\vec{b}$满足$1\le|\vec{a}|\le 2,1\le|\vec{a}+\vec{b}|\le 3,1\le\vec{a}\cdot\vec{b}\le 2$, 则$|\vec{b}|$的取值范围为______.

$key1$:$\vec{b}^2=(\vec{a}+\vec{b})^2-2\vec{a}\cdot\vec{b}-\vec{a}^2\le 6$（等号能成立）

$key2$:$|\vec{a}|=a\in[1,2],|\vec{b}|=b$,

则$\vec{a}\cdot\vec{b}=ab\cos\theta\in[1,2],\therefore\frac{2}{a\cos\theta}\ge b\ge\frac{1}{a\cos\theta}\ge\frac{1}{2}$（等号能成立）

$|\vec{a}+\vec{b}|^2=a^2+b^2+2ab\cos\theta\in[1,9],\therefore a^2+b^2\le 7,\therefore b\le\sqrt{6}$（等号能成立）

（3）①已知平面向量$\vec{e_1},\vec{e_2},\vec{e_3},\vec{p}$满足$|\vec{e_1}|=|\vec{e_2}|=|\vec{e_3}|=1,\vec{e_1}\cdot\vec{e_2}=0,|\vec{p}|\le 1$, 则$(\vec{p}-\vec{e_1})\cdot(\vec{p}-\vec{e_2})+(\vec{p}-\vec{e_2})\cdot(\vec{p}-\vec{e_3})+(\vec{p}-\vec{e_3})\cdot(\vec{p}-\vec{e_1})$的最小值为________, 最大值为______.

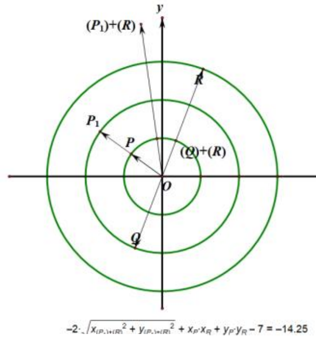
key:原式$=3\vec{p}^2-2(\vec{e_1}+\vec{e_2}+\vec{e_3})\cdot\vec{p}+(\vec{e_1}+\vec{e_2})\cdot\vec{e_3}\le 3+2|\vec{e_1}+\vec{e_2}+\vec{e_3}|+(\vec{e_1}+\vec{e_2})\cdot\vec{e_3}$

$=3+2\sqrt{3+2t}+t\le 3+2\sqrt{3+2\sqrt{2}}+\sqrt{2}=5+3\sqrt{2}$（设$t=(\vec{e_1}+\vec{e_2})\cdot\vec{e_3}$）

$3\vec{p}^2-2(\vec{e_1}+\vec{e_2}+\vec{e_3})\cdot\vec{p}+(\vec{e_1}+\vec{e_2})\cdot\vec{e_3}=3(\vec{p}-\frac{\vec{e_1}+\vec{e_2}+\vec{e_3}}{3})^2-\frac{1}{3}(\vec{e_1}+\vec{e_2}+\vec{e_3})^2+(\vec{e_1}+\vec{e_2})\cdot\vec{e_3}$

$\geq -\frac{1}{3}(\overrightarrow{e_1}+\overrightarrow{e_2}+\overrightarrow{e_3})^2+(\overrightarrow{e_1}+\overrightarrow{e_2})\cdot\overrightarrow{e_3}=\frac{1}{3}(\overrightarrow{e_1}+\overrightarrow{e_2})\cdot\overrightarrow{e_3}-\frac{1}{3}\geq -\frac{\sqrt{2}+1}{3}$

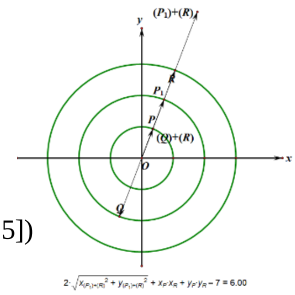
②已知平面向量 $\overrightarrow{a_1},\overrightarrow{a_2},\overrightarrow{a_3}$ 满足 $\frac{|\overrightarrow{a_1}|}{1}=\frac{|\overrightarrow{a_1}+\overrightarrow{a_2}|}{2}=\frac{|\overrightarrow{a_1}+\overrightarrow{a_2}+\overrightarrow{a_3}|}{3}=1$，则 $3\overrightarrow{a_1}\cdot\overrightarrow{a_2}+2\overrightarrow{a_1}\cdot\overrightarrow{a_3}+\overrightarrow{a_2}\cdot\overrightarrow{a_3}$ 的最小值是_____；最大值是_________. $-\frac{57}{4},6$



$key:$ 设 $\vec{p}=\overrightarrow{a_1},\vec{q}=\overrightarrow{a_1}+\overrightarrow{a_2},\vec{r}=\overrightarrow{a_1}+\overrightarrow{a_2}+\overrightarrow{a_3}$, 则 $|\vec{p}|=1,|\vec{q}|=2,|\vec{r}|=3$, 且 $\begin{cases}\overrightarrow{a_1}=\vec{p}\\ \overrightarrow{a_2}=\vec{q}-\vec{p}\\ \overrightarrow{a_3}=\vec{r}-\vec{q}\end{cases}$

$\therefore 3\overrightarrow{a_1}\cdot\overrightarrow{a_2}+2\overrightarrow{a_1}\cdot\overrightarrow{a_3}+\overrightarrow{a_2}\cdot\overrightarrow{a_3}=3\vec{p}\cdot(\vec{q}-\vec{p})+2\vec{p}\cdot(\vec{r}-\vec{q})+(\vec{q}-\vec{p})\cdot(\vec{r}-\vec{q})$

$=-7+2\vec{p}\cdot\vec{q}+\vec{r}\cdot\vec{p}+\vec{q}\cdot\vec{r}=-7+\vec{q}\cdot(2\vec{p}+\vec{r})+\vec{p}\cdot\vec{r}$



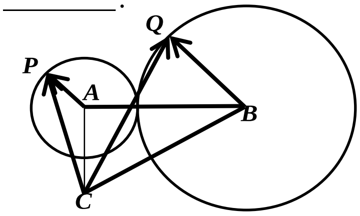
$\geq -7-2\sqrt{13+4\vec{p}\cdot\vec{r}}+\vec{p}\cdot\vec{r}$（令 $t=\sqrt{13+4\vec{p}\cdot\vec{r}}\in[1,5]$）

$=-7-2t+\frac{t^2-13}{4}=\frac{1}{4}(t-4)^2-\frac{57}{4}\geq -\frac{57}{4}$

$-7+\vec{q}\cdot(2\vec{p}+\vec{r})+\vec{p}\cdot\vec{r}\leq -7+2\sqrt{13+4\vec{p}\cdot\vec{r}}+\vec{p}\cdot\vec{r}$（令 $t=\sqrt{13+4\vec{p}\cdot\vec{r}}\in[1,5]$）

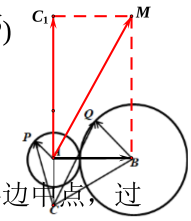
$=-7+2t+\frac{t^2-13}{4}\leq 6$

（201501学考）如图，已知 $AB\perp AC, AB=3, AC=\sqrt{3}$，圆 A 是以 A 为圆心半径为1的圆，圆 B 是以 B 为圆心的圆.设点 P、Q 分别为 $\odot A$、$\odot B$ 上的动点,且 $\overrightarrow{AP}=\frac{1}{2}\overrightarrow{BQ}$,则 $\overrightarrow{CP}\cdot\overrightarrow{CQ}$ 的取值范围为________.

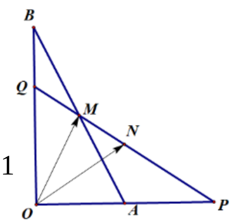


key:（基向量思想）$\overrightarrow{CP}\cdot\overrightarrow{CQ}=(\overrightarrow{CA}+\overrightarrow{AP})\cdot(\overrightarrow{CA}+\overrightarrow{AB}+2\overrightarrow{AP})$

$=5+\overrightarrow{AP}\cdot(3\overrightarrow{CA}+\overrightarrow{AB})\in[-1,11]$



（1507学考）如图，在 $Rt\triangle AOB$ 中，$OA=1, OB=2, M$ 是斜边中点，过 M 的直线分别交射线 OA、OB 于 P、Q 两点，N 是线段 PQ 的中点，则 $\overrightarrow{OM}\cdot\overrightarrow{ON}$ 的最小值为________________.



$key:$ 设 $\overrightarrow{OP}=\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}$, 则 $\overrightarrow{OM}=\frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\overrightarrow{OB}=\frac{1}{2\lambda}\overrightarrow{OA}+\frac{1}{2\mu}\overrightarrow{OB},\therefore \frac{1}{2\lambda}+\frac{1}{2\mu}=1$

$\therefore \overrightarrow{OM}\cdot\overrightarrow{ON}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})\cdot\frac{1}{2}(\overrightarrow{OP}+\overrightarrow{OQ})=\frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB})\cdot(\lambda\overrightarrow{OA}+\mu\overrightarrow{OB})$

$=\frac{1}{4}(\lambda+4\mu)(\frac{1}{2\lambda}+\frac{1}{2\mu})=\frac{1}{4}(\frac{5}{2}+\frac{2\mu}{\lambda}+\frac{\lambda}{2\mu})\geq\frac{1}{4}(\frac{5}{2}+2)=\frac{9}{8}$

（05A）四点 A,B,C,D 满足 $|\overrightarrow{AB}|=3,|\overrightarrow{BC}|=7,|\overrightarrow{CD}|=11,|\overrightarrow{DA}|=9$, 则 $\overrightarrow{AC}\cdot\overrightarrow{BD}$ 的值（　　）A

A.只有一个　B.有二个　C.有四个　D.有无穷多个

$05Akey:\overrightarrow{AC}\cdot\overrightarrow{BD}=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{BA}+\overrightarrow{AD})=-\overrightarrow{AB}^2+\overrightarrow{BC}\cdot\overrightarrow{BA}+\overrightarrow{AB}\cdot\overrightarrow{AD}+\overrightarrow{BC}\cdot\overrightarrow{AD}$

$\overrightarrow{AC}\cdot\overrightarrow{BD}=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{BC}+\overrightarrow{CD})=\overrightarrow{BC}^2+\overrightarrow{AB}\cdot\overrightarrow{BC}+\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{BC}\cdot\overrightarrow{CD}$

$\therefore 2\overrightarrow{AC}\cdot\overrightarrow{BD}=-\overrightarrow{AB}^2+\overrightarrow{BC}^2+\overrightarrow{CD}\cdot(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{DA}-\overrightarrow{DA})+\overrightarrow{DA}\cdot(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}-\overrightarrow{CD})$

$=-\overrightarrow{AB}^2+\overrightarrow{BC}^2-\overrightarrow{CD}^2+\overrightarrow{AD}^2=0$

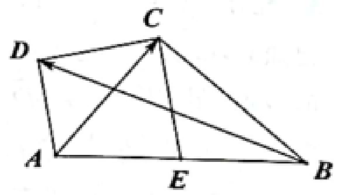
变式 1 （1）①在四边形 $ABCD$ 中，$|\overrightarrow{AC}|=|\overrightarrow{BD}|=2$, 则 $(\overrightarrow{AB}+\overrightarrow{DC})\cdot(\overrightarrow{CB}+\overrightarrow{DA})=$_____.

$key:(\overrightarrow{AB}+\overrightarrow{DC})\cdot(\overrightarrow{CB}+\overrightarrow{DA})=(\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}-\overrightarrow{BD})\cdot(\overrightarrow{CB}+\overrightarrow{BD}+\overrightarrow{DA}-\overrightarrow{BD})$

$=(\overrightarrow{AC}-\overrightarrow{BD})\cdot(\overrightarrow{CA}-\overrightarrow{BD})=-\overrightarrow{AC}^2+\overrightarrow{BD}^2=0$

②如图，已知四边形 $ABCD$， $AD\perp CD$， $AC\perp BC$， E 是 AB 的中点， $CE=1$，

若 $AD//CE$， 则 $\overrightarrow{AC}\cdot\overrightarrow{BD}$ 的最小值为______.



key:（基向量思想）设$\overrightarrow{DA}=\lambda\overrightarrow{CE}(\lambda>0)$,则

$\overrightarrow{DA}=\frac{\lambda}{2}(\overrightarrow{CA}+\overrightarrow{CB})$,且$a^2+b^2=4(b^2=\overrightarrow{CA}^2,a^2=\overrightarrow{CB}^2)$

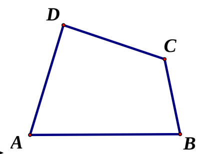
$\therefore\overrightarrow{DA}\cdot\overrightarrow{CD}=\frac{\lambda}{2}(\overrightarrow{CA}+\overrightarrow{CB})\cdot(\overrightarrow{CA}+\frac{\lambda}{2}(\overrightarrow{CA}+\overrightarrow{CB}))=\frac{\lambda(2+\lambda)}{4}b^2+\frac{\lambda^2}{4}a^2=0$得$\lambda=-\frac{b^2}{2}$,

$\therefore\overrightarrow{AC}\cdot\overrightarrow{BD}=-\overrightarrow{CA}\cdot(\overrightarrow{CA}+\frac{\lambda}{2}(\overrightarrow{CA}+\overrightarrow{CB})-\overrightarrow{CB})=-\frac{2+\lambda}{2}b^2=\frac{1}{4}b^2(b^2-4)=\frac{1}{4}(b^2-2)^2-1\ge-1$

（2） 在凸四边形$ABCD$中，$AD\perp DC,AB=2,AD=1.$

①若$AB\perp BC$,则$\overrightarrow{AC}\cdot\overrightarrow{BD}=$_________.

$key1:\overrightarrow{AC}\cdot\overrightarrow{BD}=\overrightarrow{AC}\cdot(\overrightarrow{AD}-\overrightarrow{AB})=\overrightarrow{AC}\cdot\overrightarrow{AD}-\overrightarrow{AC}\cdot\overrightarrow{AB}=\frac{\overrightarrow{AC}\cdot\overrightarrow{AD}}{|\overrightarrow{AD}|}\cdot|\overrightarrow{AD}|-\frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|\overrightarrow{AB}|}\cdot|\overrightarrow{AB}|=\overrightarrow{AD}^2-\overrightarrow{AB}^2=-3$

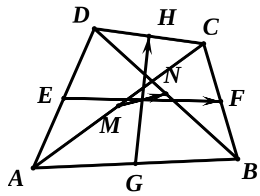


$key2:\overrightarrow{AC}\cdot\overrightarrow{BD}=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{AD}-\overrightarrow{AB})=-\overrightarrow{AB}^2+\overrightarrow{AB}\cdot\overrightarrow{AD}+\overrightarrow{BC}\cdot\overrightarrow{AD}$

$=-\overrightarrow{AB}^2+(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}-\overrightarrow{CD})\cdot\overrightarrow{AD}=-\overrightarrow{AB}^2+\overrightarrow{AD}^2=-3$

② 设E,F,G,H,M,N分别为AD,BC,AB,CD,AC,BD的中点. 若$BC=2,GH=\frac{3}{2}$,且$\overrightarrow{AB}\cdot\overrightarrow{DC}=2$,则

$\overrightarrow{AC}\cdot\overrightarrow{BD}=$_____,$\overrightarrow{EF}\cdot\overrightarrow{GH}=$_______.



$key:\overrightarrow{GH}^2=\frac{1}{4}(\overrightarrow{AD}+\overrightarrow{BC})^2=\frac{1}{4}(1+4+2\overrightarrow{AD}\cdot\overrightarrow{BC})=\frac{9}{4},\therefore\overrightarrow{AD}\cdot\overrightarrow{BC}=2$

$\therefore\overrightarrow{AC}\cdot\overrightarrow{BD}=(\overrightarrow{AD}+\overrightarrow{DC})\cdot(\overrightarrow{BA}+\overrightarrow{AD})=\overrightarrow{AD}\cdot\overrightarrow{BA}+\overrightarrow{DC}\cdot\overrightarrow{BA}+\overrightarrow{AD}\cdot\overrightarrow{AD}+\overrightarrow{DC}\cdot\overrightarrow{AD}$

$=\overrightarrow{AD}\cdot(\overrightarrow{BA}+\overrightarrow{AD}+\overrightarrow{DC})+\overrightarrow{AB}\cdot\overrightarrow{CD}=\overrightarrow{AD}\cdot\overrightarrow{BC}+\overrightarrow{AB}\cdot\overrightarrow{CD}=0$

$\because\overrightarrow{AC}\cdot\overrightarrow{AD}=\overrightarrow{AD}^2=1,\overrightarrow{BC}^2=(\overrightarrow{AC}-\overrightarrow{AB})^2=4$得$\overrightarrow{AC}^2-2\overrightarrow{AC}\cdot\overrightarrow{AB}=0$

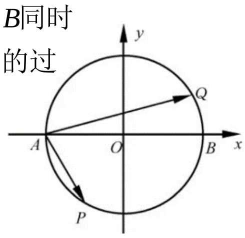
$\overrightarrow{AB}\cdot\overrightarrow{DC}=\overrightarrow{AB}\cdot(\overrightarrow{AC}-\overrightarrow{AD})=\overrightarrow{AB}\cdot\overrightarrow{AC}-\overrightarrow{AB}\cdot\overrightarrow{AD}=2,$

$\overrightarrow{AD}\cdot\overrightarrow{BC}=\overrightarrow{AD}\cdot\overrightarrow{AC}-\overrightarrow{AD}\cdot\overrightarrow{AB}=1-\overrightarrow{AD}\cdot\overrightarrow{AB}=2,\therefore\overrightarrow{AD}\cdot\overrightarrow{AB}=-1,\overrightarrow{AB}\cdot\overrightarrow{AC}=1,\overrightarrow{AC}^2=2,$

$\therefore\overrightarrow{BD}^2=\overrightarrow{AD}^2-2\overrightarrow{AD}\cdot\overrightarrow{AB}+\overrightarrow{AB}^2=1+2+4=7,\overrightarrow{CD}^2=(\overrightarrow{AD}-\overrightarrow{AC})^2=1-1+2=2,$

$\therefore\overrightarrow{EF}\cdot\overrightarrow{GH}=\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{DB})\cdot\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{BD})=\frac{1}{4}(\overrightarrow{AC}^2-\overrightarrow{BD}^2)=-\frac{5}{4}$

(1811学考）如图，O是坐标原点，圆O的半径为1，点$A(-1,0),B(1,0)$.点P,Q分别从点A,B同时出发，在圆O上按逆时针方向运动，若点P的速度大小是点Q的两倍.则在点P运动一周的过程中,$\overrightarrow{AP}\cdot\overrightarrow{AQ}$的最大值为______.



$key:Q(\cos\theta,\sin\theta)$,则$P(\cos(\pi+2\theta),\sin(\pi+2\theta))$即$(-\cos2\theta,-\sin2\theta)$

$\therefore\overrightarrow{AP}\cdot\overrightarrow{AQ}=(\cos\theta+1)((-\cos2\theta+1)+\sin\theta\cdot(-\sin2\theta)=1-\cos2\theta\in[0,2]$

(202205浙江初赛）平面向量$\vec{a},\vec{b},\vec{c}$满足$|\vec{a}|=1,\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=1,|\vec{a}-\vec{b}+\vec{c}|\le2\sqrt{2}$，则$\vec{a}\cdot\vec{c}$的最大值为____.

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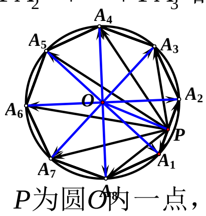
202205浙江key: 设$\vec{a}=(1,0),\vec{b}=(1,b),\vec{c}=(x,y)$, 则$x+by=1$即$by=1-x$

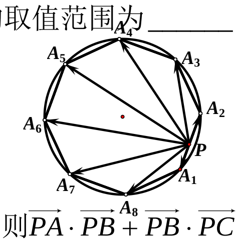
且$|\vec{a}-\vec{b}+\vec{c}|^2=|(x,-b+y)|^2=x^2+(b-y)^2\le 8,\therefore$

$\begin{cases}8\ge x^2\\ 8\ge x^2+b^2-2by+y^2\ge x^2-4by=x^2-4(1-x)\end{cases}$ 得$-2\sqrt{2}\le x\le 2,\therefore \vec{a}\cdot\vec{c}=x\le 2$

（2021I）10. 已知O为坐标原点，点$P_1(\cos\alpha,\sin\alpha)$, $P_2(\cos\beta,-\sin\beta)$, $P_3(\cos(\alpha+\beta),\sin(\alpha+\beta))$，$A(1,0)$，则（　AC　）A. $|\overrightarrow{OP_1}|=|\overrightarrow{OP_2}|$　B. $|\overrightarrow{AP_1}|=|\overrightarrow{AP_2}|$　C. $\overrightarrow{OA}\cdot\overrightarrow{OP_3}=\overrightarrow{OP_1}\cdot\overrightarrow{OP_2}$　D. $\overrightarrow{OA}\cdot\overrightarrow{OP_1}=\overrightarrow{OP_2}\cdot\overrightarrow{OP_3}$

(2022)17.设点P在单位圆内接正八边形$A_1A_2\cdots A_8$边A_1A_2上，则$\overrightarrow{PA_1}^2+\overrightarrow{PA_2}^2+\cdots+\overrightarrow{PA_8}^2$的取值范围为______.





202206浙江key: 设圆心为O，则$\overrightarrow{PA_1}^2+\overrightarrow{PA_2}^2+\cdots+\overrightarrow{PA_8}^2=8\overrightarrow{OP}^2+8$

$\in[12+2\sqrt{2},16](\because|\overrightarrow{OP}|^2\in[\cos^2\frac{\pi}{8},1]=[\frac{2+\sqrt{2}}{4},1])$

变式1（1）①已知A,B,C是半径为1的圆上三点，AB是圆O的直径，P为圆O内一点，则$\overrightarrow{PA}\cdot\overrightarrow{PB}+\overrightarrow{PB}\cdot\overrightarrow{PC}+\overrightarrow{PC}\cdot\overrightarrow{PA}$的取值范围为_______.

key: $\overrightarrow{PA}\cdot\overrightarrow{PB}+\overrightarrow{PB}\cdot\overrightarrow{PC}+\overrightarrow{PC}\cdot\overrightarrow{PA}=\overrightarrow{PO}^2-1+\overrightarrow{PC}\cdot 2\overrightarrow{PO}=\overrightarrow{PO}^2-1+2(\overrightarrow{PE}^2-\frac{1}{4})$（$E$为$CO$的中点）

$=\overrightarrow{PO}^2+2\overrightarrow{PE}^2-\frac{3}{2}=\frac{2}{3}(1+\frac{1}{2})(\overrightarrow{PO}^2+2\overrightarrow{PE}^2)-\frac{3}{2}\ge\frac{2}{3}(|\overrightarrow{PO}|+|\overrightarrow{PE}|)^2-\frac{3}{2}\ge\frac{2}{3}\cdot\overrightarrow{OE}^2-\frac{3}{2}=-\frac{4}{3}$

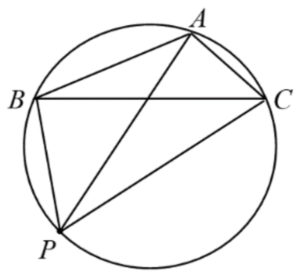
$\overrightarrow{PO}^2+2\overrightarrow{PE}^2-\frac{3}{2}\le 1+2(|\overrightarrow{OE}|+|\overrightarrow{OP}|)^2-\frac{3}{2}=4,\therefore$ 所求的取值范围为$[-\frac{4}{3},4]$

②在$\triangle ABC$中，内角A,B,C所对的边为a,b,c, 点P是其外接圆O上的任意一点，若$a=2\sqrt{3},b=c=\sqrt{7}$,
$\overrightarrow{PA}^2+\overrightarrow{PB}^2+\overrightarrow{PC}^2$的取值范围为____.

key: $\overrightarrow{PA}^2+\overrightarrow{PB}^2+\overrightarrow{PC}^2=(\overrightarrow{OA}-\overrightarrow{OP})^2+(\overrightarrow{OB}-\overrightarrow{OP})^2+\overrightarrow{OC}-\overrightarrow{OP})^2$

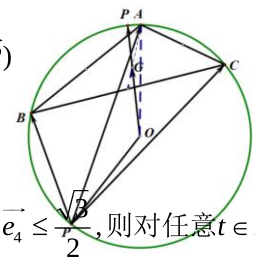
$=-6\overrightarrow{OG}\cdot\overrightarrow{OP}+6R^2\in[14,\frac{91}{4}]$（$G$为$\triangle ABC$的重心, 外接圆半径$R=\frac{7}{4},|\overrightarrow{OG}|=\frac{5}{12}$）

③如图，已知$\triangle ABC$为钝角三角形，$AC<AB<BC$, 点P是$\triangle ABC$外接圆上的点，
则当$\overrightarrow{PA}\cdot\overrightarrow{PB}+\overrightarrow{PB}\cdot\overrightarrow{PC}+\overrightarrow{PC}\cdot\overrightarrow{PA}$取最小值时，点$P$在（　）C



A.$\angle BAC$所对弧上（不包括弧的端点）B.$\angle ABC$所对弧上（不包括弧的端点）
C.$\angle ACB$所对弧上（不包括弧的端点）D.$\triangle ABC$的顶点

key: 原式$=(\overrightarrow{OA}-\overrightarrow{OP})\cdot(\overrightarrow{OB}-\overrightarrow{OP})+(\overrightarrow{OB}-\overrightarrow{OP})\cdot(\overrightarrow{OC}-\overrightarrow{OP})$
$+(\overrightarrow{OA}-\overrightarrow{OP})\cdot(\overrightarrow{OC}-\overrightarrow{OP})$
$=3R^2-6\overrightarrow{OG}\cdot\overrightarrow{OP}+\overrightarrow{OA}\cdot\overrightarrow{OB}+\overrightarrow{OB}\cdot\overrightarrow{OC}+\overrightarrow{OC}\cdot\overrightarrow{OA}$



（2）①已知单位向量$\vec{e_1},\vec{e_2},\vec{e_3},\vec{e_4}$满足$\vec{e_1}\cdot\vec{e_2}=-\frac{31}{32},0<\vec{e_3}\cdot\vec{e_4}\le\frac{\sqrt{3}}{2}$, 则对任意$t\in R,|\vec{e_1}+\vec{e_2}+\vec{e_3}+t\vec{e_4}|$的最小值为________.

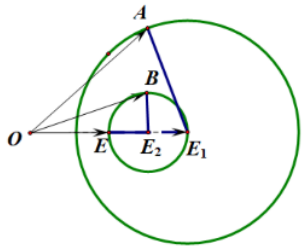
key: 设$\vec{e_4}=(1,0),\vec{e_3}=(\cos\theta,\sin\theta),\theta\in[\frac{\pi}{6},\frac{\pi}{2}),\vec{e_1}=(\cos\alpha,\sin\alpha),\vec{e_2}=(\cos\beta,\sin\beta),\alpha-\beta=\pi-\arccos\frac{31}{32}$,

$\therefore|\vec{e_1}+\vec{e_2}+\vec{e_3}+t\vec{e_4}|=|(\cos\theta+\cos\alpha+\cos\beta+t,\sin\theta+\sin\alpha+\sin\beta)|$

$=\sqrt{(\cos\theta+\cos\alpha+\cos\beta+t)^2+(\sin\theta+\sin\alpha+\sin\beta)^2}$

$\ge|\sin\theta+2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}|=|\sin\theta+\frac{1}{4}\sin\frac{\alpha+\beta}{2}|\ge\frac{1}{4}(\because\sin\theta\in[\frac{1}{2},1),\frac{1}{4}\sin\frac{\alpha+\beta}{2}\in[-\frac{1}{4},\frac{1}{4}])$

②已知$\vec a,\vec b,\vec e$是平面向量,$\vec e$是单位向量.若$\vec a^2-4\vec a\cdot\vec e+2\vec e^2=0,\vec b^2-3\vec b\cdot\vec e+2\vec e^2=0$,则$\vec a^2-2\vec a\cdot\vec b+2\vec b^2$的最大值为________.



key:由$\vec a^2-4\vec a\cdot\vec e+2\vec e^2=(\vec a-2\vec e)^2-2=0$得$|\vec a-2\vec e|=\sqrt2$;

由$\vec b^2-3\vec b\cdot\vec e+2\vec e^2=(\vec b-\dfrac32\vec e)^2-\dfrac14=0$得$|\vec b-\dfrac32\vec e|=\dfrac12$

令$\vec e=(1,0),\vec a=(2+\sqrt2\cos\alpha,\sqrt2\sin\alpha),\vec b=(\dfrac32+\dfrac12\cos\beta,\dfrac12\sin\beta)$

则$\vec a^2-2\vec a\cdot\vec b+2\vec b^2=6+4\sqrt2\cos\alpha-2[(2+\sqrt2\cos\alpha)(\dfrac32+\dfrac12\cos\beta)+\dfrac{\sqrt2}{2}\sin\alpha\sin\beta]+2(\dfrac52+\dfrac32\cos\beta)$

$=\sqrt2\cos\alpha+\cos\beta-\sqrt2\cos\alpha\cos\beta-\sqrt2\sin\alpha\sin\beta+5$

$=\sqrt2(1-\cos\beta)\cos\alpha-\sqrt2\sin\beta\sin\alpha+\cos\beta+5\le\sqrt2\cdot\sqrt{2-2\cos\beta}+\cos\beta+5$

$=2\sqrt2|\sin\dfrac\beta2|-2\sin^2\dfrac\beta2+6=-2(|\sin\dfrac\beta2|-\dfrac{\sqrt2}{2})^2+7\le7$

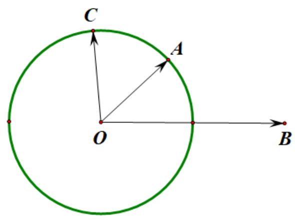
③已知$|\vec a|=|\vec b|=2,|\vec c|=1,\vec a\cdot\vec b=(\vec a+\vec b)\cdot\vec c$，则$\vec a\cdot\vec c$的最小值是______.

key:$\vec c=(1,0),\vec a=(2\cos\alpha,2\sin\alpha),\vec b=(2\cos\beta,2\sin\beta)$,

$\therefore4(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=2\cos\alpha+2\cos\beta$即$(2\cos\alpha-1)\cos\beta+2\sin\alpha\sin\beta=\cos\alpha$

$\therefore\dfrac{|\cos\alpha|}{\sqrt{5-4\cos\alpha}}\le1$即$-1\le\cos\alpha\le1,\therefore\vec a\cdot\vec c=2\cos\alpha\ge-2$

④已知平面向量$\vec a,\vec b,\vec c$满足$|\vec a|=|\vec c|=\dfrac12|\vec b|=1$，$|\vec a\cdot\vec b|\le1$.若$\vec d=\vec b+\vec c$，则$|\vec a\cdot\vec c|+|\vec b\cdot\vec d|$的最大值是____.



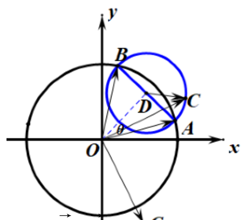
key:$\vec b=(2,0),\vec a=(\cos\alpha,\sin\alpha)(\dfrac\pi3\le\alpha\le\dfrac{2\pi}{3}),\vec c=(\cos\beta,\sin\beta)$

则$|\vec a\cdot\vec c|+|\vec b\cdot\vec d|=|\cos\alpha\cos\beta+\sin\alpha\sin\beta|+4+2\cos\beta$

$=\max\{(2+\cos\alpha)\cos\beta+\sin\alpha\sin\beta+4,(2-\cos\alpha)\cos\beta-\sin\alpha\sin\beta+4\}$

$\le\max\{\sqrt{5+4\cos\alpha}+4,\sqrt{5-4\cos\alpha}+4\}=\sqrt7+4$

⑤设$\vec a=(a_1,b_1),\vec b=(a_2,b_2),\vec c=(a_3,b_3)$,且$\vec a,\vec b$是平面内两个不共线的单位向量,若向量$\vec c$满足$(\vec a-\vec c)\cdot(\vec b-\vec c)=0$,则$(a_1-a_2)b_3+(b_2-b_1)a_3$的最大值为________.



key:$(a_1-a_2)b_3+(b_2-b_1)a_3=(a_1-a_2)b_3+(b_1-b_2)\cdot(-a_3)=(a_1-a_2,b_1-b_2)\cdot(b_3,-a_3)$

（其中$\overrightarrow{OA}=(a_1,b_1),\overrightarrow{OB}=(a_2,b_2),\overrightarrow{OC}=(a_3,b_3),\overrightarrow{OC_1}=(b_3,-a_3),\angle AOB=2\theta$）

$=\overrightarrow{BA}\cdot\overrightarrow{OC_1}\le2\sin\theta\cdot(\cos\theta+\sin\theta)=\sin2\theta+1-\cos2\theta\le1+\sqrt2$

⑥已知平面向量$\vec{e_1},\vec{e_2}$满足$|\vec{e_1}|=|\vec{e_2}|=1,\vec{e_1}\perp\vec{e_2}$.若对任意平面向量$\vec a,\vec b$都有$|\vec a-\vec b|^2\ge(t-2)\vec a\cdot\vec b+t(\vec a\cdot\vec{e_2})(\vec b\cdot\vec{e_1})$成立，则实数$t$的最大值是（C　）A.$\sqrt3-1$　　B.1　　C.$\sqrt5-1$　　D.2

key:设$\vec{e_1}=(1,0),\vec{e_2}=(0,1),\vec a=(x_1,y_1),\vec b=(x_2,y_2)$，则$|\vec a-\vec b|^2=(x_1-x_2)^2+(y_1-y_2)^2\ge(t-2)(x_1x_2+y_1y_2)+tx_2y_1$

$\therefore x_1^2+y_1^2+x_2^2+y_2^2\ge t(x_1x_2+y_1y_2+x_2y_1)$

$\because x_1^2+y_1^2+x_2^2+y_2^2=x_1^2+\lambda x_2^2+y_2^2+\lambda y_1^2+(1-\lambda)x_2^2+(1-\lambda)y_1^2\ge2\sqrt\lambda x_1x_2+2\sqrt\lambda y_1y_2+2(1-\lambda)x_1x_2$

（其中$\sqrt\lambda=1-\lambda$即$\sqrt\lambda=\dfrac{-1+\sqrt5}{2}$）$=(\sqrt5-1)(x_1x_2+y_1y_2+x_2y_1),\therefore t\le\sqrt5-1$