解析几何（2）椭圆解答（1）

2023-09-30

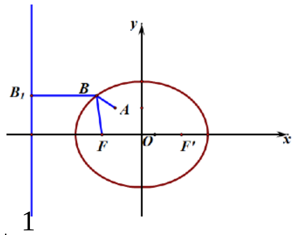
1°. 椭圆几何性质

1.定义 $:|PF_1|+|PF_2|=2a(2a>|F_1F_2|)$

$$2.\text{标准方程}\begin{cases}\text{焦点在}x\text{轴上}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)\\ \text{焦点在}y\text{轴上}:\dfrac{y^2}{a^2}+\dfrac{x^2}{b^2}=1(a>b>0)\end{cases}$$

$$3.\text{几何性质}\begin{cases}\text{范围}:x=\pm a,y=\pm b\,/\,x=\pm b,y=\pm a\text{围成的矩形内部}\\ \text{对称轴及对称中心：}x\text{、}y\text{轴，原点}O\\ \text{顶点：（对称轴与曲线的交点）}(\pm a,0),(0,\pm b)\,/\,(\pm b,0),(0,\pm a)\\ \text{焦点}:(\pm c,0)\,/\,(0,\pm c)\\ \text{离心率}:e=\dfrac{c}{a}\in(0,1)(a^2=b^2+c^2)\\ \text{焦半径}:|PF|=a-\dfrac{c}{a}x\text{或}|PF|=a-\dfrac{c}{a}y\end{cases}$$

（1999A）给定$A(-2,2)$，已知B是椭圆$\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$上的动点，$F$是左焦点，则当$|AB|+\dfrac{5}{3}|BF|$取最小值时，点$B$的坐标为______.



key: $\dfrac{5}{3}|BF|=\dfrac{5}{3}\sqrt{(x_B+3)^2+16(1-\dfrac{x_B^2}{25})}=\dfrac{5}{3}\sqrt{\dfrac{9}{25}x_B^2+6x_B+25}=x_B+\dfrac{25}{3}$

$\therefore|AB|+\dfrac{5}{3}|BF|\ge-2+\dfrac{25}{3}$, 此时$B(-\dfrac{5\sqrt{3}}{2},2)$

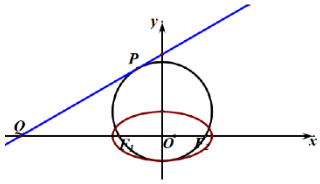
（2006湖南）过椭圆$\dfrac{x^2}{36}+\dfrac{y^2}{4}=1$的一个焦点$F$作弦$AB$，若$|AF|=m,|BF|=n$, 则$\dfrac{1}{m}+\dfrac{1}{n}=$______.

key: 设$\angle AFx=\theta$，F为左焦点，则$A(-c+m\cos\theta,m\sin\theta)$,

$\therefore b^2(c^2-2cm\cos\theta+m^2\cos^2\theta)+a^2m^2\sin^2\theta=a^2b^2$即$(a^2-c^2\cos^2\theta)m^2-2b^2c\cos\theta\cdot m-b^4=0$

$\therefore m=\dfrac{b^2}{a-c\cos\theta}$, 同理$n=\dfrac{b^2}{a+c\cos\theta}$, $\therefore\dfrac{1}{m}+\dfrac{1}{n}=\dfrac{2a}{b^2}=3$

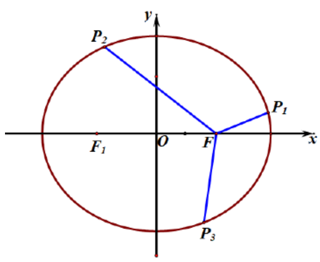
（2006A）已知椭圆$\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$的左、右焦点分别为$F_1$、$F_2$，点$P$在直线$l:x-\sqrt{3}y+8+2\sqrt{3}=0$上，当$\angle F_1PF_2$取最大值时, $\dfrac{|PF_1|}{|PF_2|}$的值为_______.



key: 如图$|QP|^2=|QF_1|\cdot|QF_2|=8\cdot(8+4\sqrt{3})$得$|QP|=4(\sqrt{3}+1)$

$\therefore\sqrt{1+\dfrac{1}{3}}(x_P+8+2\sqrt{3})=4(\sqrt{3}+1)$得$x_P=-2$, 此时$P(-2,2+2\sqrt{3})$, $\therefore\dfrac{|PF_1|}{|PF_2|}=\sqrt{3}-1$

（2007重庆）如图，在椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$上任取三个不同点$P_1,P_2,P_3$，使$\angle P_1FP_2=\angle P_2FP_3=\angle P_3FP_1$, F是椭圆C的右焦点，则$\dfrac{1}{|FP_1|}+\dfrac{1}{|FP_2|}+\dfrac{1}{|FP_3|}=$______.



key: 设$\angle P_1Fx=\theta$，$|P_1F|=m$，则$P_1(c+m\cos\theta,m\sin\theta)$,

$\therefore b^2(c^2+2c\cos\theta\cdot m+m^2\cos^2\theta)+a^2m^2\sin^2\theta=a^2b^2$得$|P_1F|=\dfrac{b^2}{a+c\cos\theta}$

同理 $|P_2F|=\dfrac{b^2}{a+c\cos(\theta+120°)},|P_3F|=\dfrac{b^2}{a+c\cos(\theta+240°)}$

$\therefore \dfrac{1}{|FP_1|}+\dfrac{1}{|FP_2|}+\dfrac{1}{|FP_3|}=\dfrac{3a}{b^2}+c(\cos\theta+\cos(\theta+120°)+\cos(\theta+240°))=\dfrac{3a}{b^2}$

(2009Ⅰ)12.已知椭圆$C:\dfrac{x^2}{2}+y^2=1$的右焦点F，直线$l:x=2$,点$A\in l$，线段AF交C于点B，若$\overrightarrow{FA}=3\overrightarrow{FB}$，则$|\overrightarrow{AF}|=$（ ）$A.\sqrt{2}$ $B.2$ $C.\sqrt{3}$ $D.3$ A

（2010Ⅰ)已知 F 是椭圆 C 的一个焦点，B 是短轴的一个端点，线段 BF 的延长线交 C 于点 D，且 $\overrightarrow{BF}=2\overrightarrow{FD}$，则 C 的离心率为______________.

key:设$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0),B(0,b),F(c,0)$，则$D(\dfrac{3c}{2},-\dfrac{b}{2}),\therefore \dfrac{9c^2}{4a^2}+\dfrac{1}{4}=1$得$e=\dfrac{\sqrt{3}}{3}$

（2011浙江）设F_1,F_2分别为椭圆$\dfrac{x^2}{3}+y^2=1$的左、右焦点，若$\overrightarrow{F_1A}=5\overrightarrow{F_2B}$，则点$A$的坐标为______.

$key1$:由$\overrightarrow{F_1A}=(x_A+\sqrt{2},y_A)=5\overrightarrow{F_2B}=5(x_B-\sqrt{2},y_B)$得$\begin{cases}x_B=\dfrac{x_A+6\sqrt{2}}{5}\\ y_B=\dfrac{y_A}{5}\end{cases}$

$\therefore\begin{cases}\dfrac{x_A^2}{3}+y_A^2=1\\ \dfrac{(x_A+6\sqrt{2})^2}{3}+y_A^2=25\end{cases}$ $\therefore \dfrac{6\sqrt{2}(2x_A+6\sqrt{2})}{3}=24$即$x_A=0,y_A=\pm1,\therefore A(0,\pm1)$

$key2$:设$AC:x=ty-\sqrt{2}$代入椭圆方程得$(t^2+3)y^2-2\sqrt{2}ty-1=0,\therefore\begin{cases}y_A+y_C=\dfrac{2\sqrt{2}t}{3+t^2}\\ y_Ay_C=\dfrac{-1}{3+t^2}\end{cases}$

而$y_A=5y_B=-5y_C,\therefore\begin{cases}-4y_C=\dfrac{2\sqrt{2}t}{3+t^2}\\ -5y_C^2=\dfrac{-1}{3+t^2}\end{cases}$解得$t=\pm\sqrt{2},\therefore y_A=-5y_C=\pm1,\therefore A(0,\pm1)$

（2012江苏）如图，在平面直角坐标系xOy中，椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$左、右焦点分别为$F_1(-c,0)$、$F_2(c,0)$.

已知$(1,e)$和$(e,\dfrac{\sqrt{3}}{2})$都在椭圆上，其中e为椭圆的离心率.⑴ 求椭圆的离心率；

（2）设A、B是椭圆上位于x轴上方的两点，且直线AF_1与直线BF_2平行，AF_2与BF_1交于点P.

（ i ）若$AF_1-BF_2=\dfrac{\sqrt{6}}{2}$,求$AF_1$的斜率；（ ii ）求证：$PF_1+PF_2$是定值.

⑴ 解：由已知得$\begin{cases}\dfrac{1}{a^2}+\dfrac{e^2}{b^2}=1\text{即}b^2=\dfrac{b^2}{a^2}+\dfrac{c^2}{a^2}=1\\ \dfrac{e^2}{a^2}+\dfrac{3}{4b^2}=1\text{即}b^2=\dfrac{e^2(a^2-c^2)}{a^2}+\dfrac{3}{4}=e^2(1-e^2)+\dfrac{3}{4}=1\end{cases}$得$e=\dfrac{\sqrt{2}}{2}$

（2）由 ⑴ 得椭圆方程为$\dfrac{x^2}{2}+y^2=1,$

延长AF_1交椭圆于C，由对称性得$|BF_2|=|CF_1|$,设AC方程为$x=ty-1$

代入椭圆方程得:$(t^2+2)y^2-2ty-1=0,\therefore\begin{cases}y_A+y_C=\dfrac{2t}{t^2+2}\\ y_Ay_C=\dfrac{-1}{t^2+2}\end{cases}$,且$\Delta=8(t^2+1)>0$

(Ⅰ)$\therefore|AF_1|-|BF_2|=|AF_1|-|CF_1|=\sqrt{1+t^2}\,|y_A|-\sqrt{1+t^2}\,|y_C|=\sqrt{1+t^2}\cdot|y_A+y_C|$

$=\sqrt{1+t^2}\cdot\dfrac{2|t|}{t^2+2}=\dfrac{\sqrt6}{2}(t>0)$得$t=\sqrt2,\therefore AF_1$的斜率为$\dfrac{\sqrt2}{2}$

(Ⅱ)$key1:\because AF_1//BF_2,\therefore\dfrac{|PB|}{|PF_1|}=\dfrac{|BF_2|}{|AF_1|}=\dfrac{|CF_1|}{|AF_1|}$而$\dfrac{|PB|}{|PF_1|}=\dfrac{|BF_1|-|PF_1|}{|PF_1|}=\dfrac{2\sqrt2-|CF_1|}{|PF_1|}-1$

得$|PF_1|=\dfrac{|AF_1|(2\sqrt2-|CF_1|)}{|AC|}$

$\dfrac{|PA|}{|PF_2|}=\dfrac{|AF_1|}{|BF_2|}=\dfrac{|AF_1|}{|CF_1|}$而$\dfrac{|PA|}{|PF_2|}=\dfrac{|AF_2|-|PF_2|}{|PF_2|}=\dfrac{2\sqrt2-|AF_1|}{|PF_2|}-1$得$|PF_2|=\dfrac{|CF_1|(2\sqrt2-|AF_1|)}{|AC|}$

$\therefore|PF_1|+|PF_2|=\dfrac{|AF_1|(2\sqrt2-|CF_1|)}{|AC|}+\dfrac{|CF_1|(2\sqrt2-|AF_1|)}{|AC|}=2\sqrt2-\dfrac{2|AF_1|\cdot|CF_1|}{|AC|}$

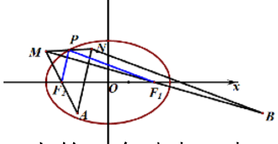
$=2\sqrt2-\dfrac{2(1+t^2)\cdot|y_Ay_C|}{\sqrt{1+t^2}\,|y_A-y_C|}=2\sqrt2-\dfrac{2\sqrt{1+t^2}\cdot\dfrac{1}{t^2+2}}{\dfrac{2\sqrt2\sqrt{1+t^2}}{t^2+2}}=\dfrac{3\sqrt2}{2}$

$key2:AF_2:x=\dfrac{x_A-c}{y_A}y+c=\dfrac{ty_A-2}{y_A}y+1$即$\dfrac{x-1}{y}=t-\dfrac{2}{y_A}\cdots①$,

$BF_1:x=\dfrac{x_B+c}{y_B}y-c=\dfrac{x_C-c}{y_C}y-c=\dfrac{ty_C-2}{y_C}y-1$即$\dfrac{x+1}{y}=t-\dfrac{2}{y_C}\cdots②$

①+②得:$\dfrac{x}{y}=t-\dfrac{y_A+y_C}{y_Ay_C}=3t$,①-②得:$\dfrac{1}{|y|}=|\dfrac{y_C-y_A}{y_Ay_C}|=2\sqrt2\cdot\sqrt{1+t^2}=2\sqrt2\sqrt{1+\dfrac{x^2}{9y^2}}$即$\dfrac89x^2+8y^2=1$

(2014辽宁)15.已知椭圆$C:\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$，点$M$与$C$的焦点不重合，若$M$关于$C$的焦点的对称点分别为$A$、$B$，线段$MN$的中点在$C$上，则$|AN|+|BN|=$______.

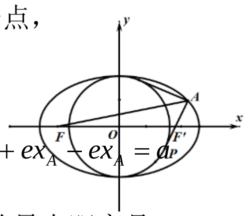


$key:|AN|+|BN|=2(|PF_2|+|PF_1|)=12$

(2015B)在平面直角坐标系xOy中，P是椭圆$\dfrac{y^2}{4}+\dfrac{x^2}{3}=1$上的一个动点，点$A(1,1),B(0,-1)$，则$|PA|+|PB|$的最大值为________.5

$key:|PA|+|PB|=|PA|+4-|PB'|\le4+|AB'|=5$($B$、$B'$是椭圆的上、下焦点)

(2017天津)设F是椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的左焦点，$A$是该椭圆上位于第一象限的一点，过$A$作圆$x^2+y^2=b^2$的切线为$P$.则$|AF|-|AP|=$_____.



2017天津$key:|AF|-|AP|=ex_A+a-\sqrt{x_A^2+y_A^2-b^2}=a+ex_A-\sqrt{x_A^2+b^2(1-\dfrac{x_A^2}{a^2})-b^2}=a+ex_A-ex_A=a$

(2014 福建)9.设P,Q分别为$x^2+(y-6)^2=2$和椭圆$\dfrac{x^2}{10}+y^2=1$上的点,则P,Q两点间的最大距离是(　D　)

A. $5\sqrt2$　　B. $\sqrt{46}+\sqrt2$　　C. $7+\sqrt2$　　D. $6\sqrt2$

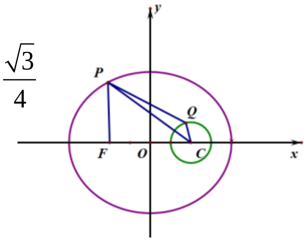
(2021 乙)11. 设B是椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的上顶点，若$C$上的任意一点$P$都满足$|PB|\le2b$，则

C 的离心率的取值范围是（ C ） A. $[\frac{\sqrt{2}}{2},1)$　　B. $[\frac{1}{2},1)$　　C. $(0,\frac{\sqrt{2}}{2}]$　　D. $(0,\frac{1}{2}]$

$key:|PB|=\sqrt{x^2+(y-b)^2}=\sqrt{a^2(1-\frac{y^2}{b^2})+y^2-2by+b^2}=\sqrt{-\frac{c^2}{b^2}y^2-2by+a^2+b^2}$

$=\sqrt{-\frac{c^2}{b^2}(y+\frac{b^3}{c^2})^2+a^2+b^2+\frac{b^4}{c^2}}\le 2b,\therefore -\frac{b^3}{c^2}\le -b$得$e\in(0,\frac{\sqrt{2}}{2}]$

变式 1（1）已知椭圆$C:\frac{x^2}{16}+\frac{y^2}{12}=1$的左焦点为 F，点 P 是椭圆 C 上的动点，点 Q 是圆$T:(x-2)^2+y^2=1$上的动点，则$\frac{|PF|}{|PQ|}$的最小值是（ B） A. $\frac{1}{2}$　　B. $\frac{2}{7}$　　C. $\frac{2}{3}$　　D. $\frac{\sqrt{3}}{4}$



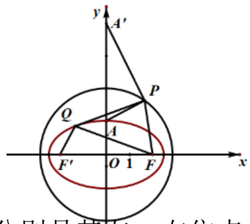
$key:\frac{|PF|}{|PQ|}\ge\frac{|PF|}{|PC|+1}=\frac{8-|PC|}{|PC|+1}=\frac{9}{|PC|+1}-1\ge\frac{9}{4+2+1}-1=\frac{2}{7}$

（2）已知椭圆$\frac{x^2}{6}+\frac{y^2}{2}=1$的右焦点为$F$，上顶点为$A$，点$P$在圆$x^2+y^2=8$上，点$Q$在椭圆上，则$2|PA|+|PQ|-|QF|$的最小值为__________.

变式1$key:2|PA|+|PQ|-|QF|=|PA'|+|PQ|-(2\sqrt{6}-|QF'|)$

$(A'(0,4\sqrt{2}),|PA'|=2|PA|$（阿波罗尼斯圆）

$=|PA'|+|PQ|+|QF'|-2\sqrt{6}\ge 6-2\sqrt{6}$



（2005湖南）设P是椭圆$\frac{x^2}{16}+\frac{y^2}{9}=1$上异于长轴端点的任意一点，$F_1,F_2$分别是其左、右焦点，$O$为中心，则$|PF_1|\cdot|PF_2|+|OP|^2=$________.25

$key:|PF_1|\cdot|PF_2|+|OP|^2=\sqrt{(x_P+c)^2+y_P{}^2}\cdot\sqrt{(x_P-c)^2+y_P^2}+(x_P^2+y_P^2)$

$=\sqrt{(x_P^2+y_P^2+c^2)^2-4c^2x_P^2}+(x_P^2+b^2(1-\frac{x_P^2}{a^2}))=\sqrt{(x_P^2+b^2(1-\frac{x_P^2}{a^2})+c^2)^2-4c^2x_P^2}+(b^2+\frac{c^2}{a^2}x_P^2)$

$=a^2-\frac{c^2}{a^2}x_P^2+b^2+\frac{c^2}{a^2}x_P^2=25$

（2006湖南）椭圆$\frac{x^2}{4}+\frac{y^2}{9}=1$在直线$2x+y-2=0$上的射影长为________.

key：与直线$2x+y-2=0$垂直的直线$x-2y+m=0$与椭圆相切，

则$9(2y-m)^2+4y^2=36$即$40y^2-36my+9m^2-36=0,\therefore\Delta=36^2m^2-36\cdot 40(m^2-4)=0$得$m=\pm 2\sqrt{10}$

$\therefore$射影长为$\frac{4\sqrt{10}}{\sqrt{5}}=4\sqrt{2}$

（2007吉林）已知椭圆$\frac{x^2}{4}+\frac{y^2}{3}=1$的左、右焦点分别为$F_1$、$F_2$，过椭圆的右焦点$F_2$作一条直线$l$交椭圆于$P$、$Q$两点，则$\triangle F_1PQ$内切圆面积的最大值是_________. $\frac{9\pi}{16}$

$key1:$设$l:x=ty+1$代入椭圆方程得$(3t^2+4)y^2+6ty-9=0,\therefore\begin{cases}y_P+y_Q=-\frac{6t}{3t^2+4}\\ y_Py_Q=-\frac{9}{3t^2+4}\end{cases}$，且$\Delta=144(t^2+1)$

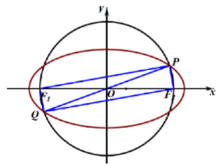
$\therefore\frac{1}{2}\cdot 8\cdot r=S_{\triangle F_1PQ}=\frac{1}{2}\sqrt{1+t^2}\cdot\frac{12\sqrt{t^2+1}}{3t^2+4}\cdot\frac{2}{\sqrt{t^2+1}}=\frac{12\sqrt{t^2+1}}{3t^2+4}$

$\therefore S=9\pi\cdot\dfrac{t^2+1}{(3t^2+4)^2}=\dfrac{9\pi}{9u+\dfrac{1}{u}+6}\le\dfrac{9\pi}{16}$

$key2:$设$\angle PF_2x=\theta$,则$\dfrac{1}{2}(\dfrac{b^2}{a+c\cos\theta}+\dfrac{b^2}{a-c\cos\theta})\cdot 2c\cdot\sin\theta=\dfrac{1}{2}\cdot 4a\cdot r$

得$r=\dfrac{3}{\dfrac{3}{\sin\theta}+\sin\theta}\ge\dfrac{3}{4}$

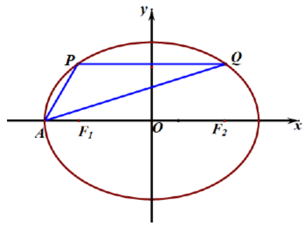
（2021 甲）15. 已知F_1,F_2为椭圆$C:\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$的两个焦点，$P$，$Q$为$C$上关于坐标原点对称的两点，且$|PQ|=|F_1F_2|$，则四边形$PF_1QF_2$的面积为________. 8



$key:$如图，F_1QF_2P是矩形，且$\begin{cases}PF_1^2+PF_2^2=4c^2=48\\|PF_1|+|PF_2|=8\end{cases}$，$\therefore S_{QF_1PF_2}=8$

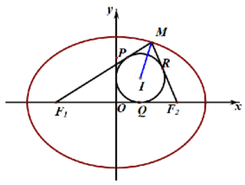
（2022 甲）10. 椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的左顶点为$A$，点$P$，$Q$均在$C$上，且关于$y$轴对称. 若直线$AP,AQ$的斜率之积为$\dfrac{1}{4}$，则$C$的离心率为（ A ）

A. $\dfrac{\sqrt{3}}{2}$　　B. $\dfrac{\sqrt{2}}{2}$　　C. $\dfrac{1}{2}$　　D. $\dfrac{1}{3}$



$key:k_{AP}k_{AQ}=\dfrac{y_P}{x_P+a}\cdot\dfrac{y_P}{-x_P+a}=\dfrac{b^2(1-\dfrac{x_P^2}{a^2})}{a^2-x_P^2}=\dfrac{b^2}{a^2}=\dfrac{1}{4},\therefore e=\dfrac{\sqrt{3}}{2}$

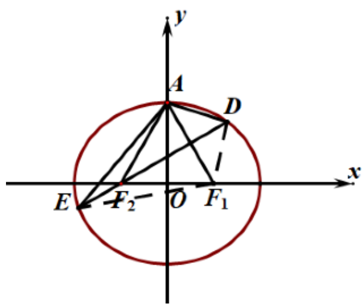
（2010湖北）设椭圆$\dfrac{x^2}{4}+y^2=1$的左、右焦点分别为F_1、F_2,M为椭圆上异于长轴端点的一点，$\angle F_1MF_2=2\theta$，$\triangle MF_1F_2$的内心为I，则$|MI|\cos\theta=$_______.$2-\sqrt{3}$



$key:$而$|F_1M|-|F_1F_2|=|MP|-|QF_1|=|MR|-|RF_2|,|MR|+|RF_2|=|MF_2|$

$\therefore 2|MP|=2|MR|=2a-2c$得$|MI|\cos\theta=|MP|=a-c=2-\sqrt{3}$

（2022I）16. 已知椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$，$C$的上顶点为$A$，两个焦点为$F_1$，$F_2$，离心率为$\dfrac{1}{2}$. 过$F_1$且垂直于$AF_2$的直线与$C$交于$D$，$E$两点，$|DE|=6$，则$\triangle ADE$的周长是__13______.



$key:$由$e=\dfrac{c}{a}=\dfrac{1}{2}$的$a=2c,\therefore\triangle AF_1F_2$是正三角形，

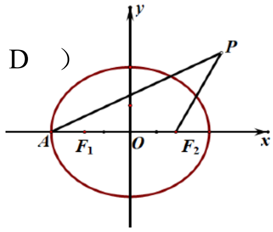
$\therefore ED$是AF_1的垂直平分线,$\therefore|AD|+|AE|=|F_1A|+|F_1D|=4a$

由$\begin{cases}y=\dfrac{\sqrt{3}}{3}(x+c)\\\dfrac{x^2}{4c^2}+\dfrac{y^2}{3c^2}=1\end{cases}$消去$y$得:$13x^2+8cx-32c^2=0,\therefore\Delta=64\times27c^2>0$

$key2:|DE|=|F_2E|+|F_2E|=\dfrac{b^2}{a-c\cdot\dfrac{\sqrt{3}}{2}}+\dfrac{b^2}{a+c\cdot\dfrac{\sqrt{3}}{2}}=6$得$c=\dfrac{13}{8}$

$\therefore|DE|=\sqrt{\dfrac{4}{3}}\cdot\dfrac{24\sqrt{3}c}{13}=6$得$c=\dfrac{13}{8},\therefore|AD|+|DE|=13$

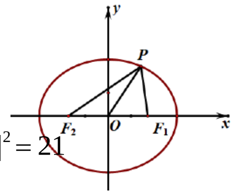
（2018II）12.已知 F_1,F_2是椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$ 的左，右焦点，A 是 C 的左顶点，点 P 在过 A 且斜率为 $\dfrac{\sqrt{3}}{6}$ 的直线上，$\triangle PF_1F_2$ 为等腰三角形，$\angle F_1F_2P=120°$，则 C 的离心率为（ D ）



$A.\dfrac{2}{3}$　　$B.\dfrac{1}{2}$　　$C.\dfrac{1}{3}$　　$D.\dfrac{1}{4}$

key:由已知得$P(c+2c\cos 60°,2c\sin 60°)$即$(2c,\sqrt{3}c)$,$\therefore k_{AP}=\dfrac{\sqrt{3}c}{a+2c}=\dfrac{\sqrt{3}}{6}$得$e=\dfrac{1}{4}$

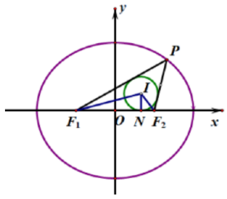
(2023甲)12.设O为坐标原点，F_1,F_2为椭圆$C:\dfrac{x^2}{9}+\dfrac{y^2}{6}=1$的两个焦点，点$P$在$C$上,$\cos\angle F_1PF_2=\dfrac{3}{5}$，则$|OP|=$（　）$A.\dfrac{13}{5}$ $B.\dfrac{\sqrt{30}}{2}$ $C.\dfrac{14}{5}$ $D.\dfrac{\sqrt{35}}{2}$



$key:\begin{cases}PF_1^2+PF_2^2-2|PF_1|\cdot|PF_2|\cdot\dfrac{3}{5}=12\\|PF_1|+|PF_2|=6\end{cases}$得$|PF_1|\cdot|PF_2|=\dfrac{15}{2}$,$\therefore|PF_1|^2+|PF_2|^2=21$

$\therefore|PF_1|^2+|PF_2|^2=2|PO|^2+2|OF_1|^2=2|PO|^2+6$得$|OP|=\dfrac{\sqrt{30}}{2}$,选$B$

变式 1（1）已知F_1,F_2是椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的左右两焦点，点$P$在椭圆上，$O$为坐标原点.



① 若$\angle F_1PF_2=\theta$，则$\triangle PF_1F_2$的面积为________；$b^2\tan\dfrac{\theta}{2}$

若$\angle PF_1F_2=\alpha,\angle PF_2F_1=\beta$，则$\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=$________.

$$key1:\frac{2a}{\sin\alpha+\sin\beta}=\frac{|PF_1|}{\sin\beta}=\frac{|PF_2|}{\sin\alpha}=\frac{2c}{\sin(\alpha+\beta)},\therefore e=\frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

$$=\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}=\frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}},\therefore\frac{e+1}{e-1}=\frac{2}{-2\tan\frac{\alpha}{2}\tan\frac{\beta}{2}},\therefore\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}$$

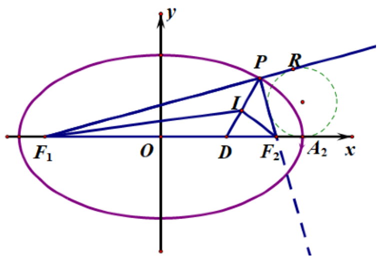
$$key2:由\begin{cases}2c=|F_1N|+|F_2N|\\|F_1N|-|F_2N|=|PF_1|-|PF_2|=2ex_P(由|PF_1|=a+ex_P)\end{cases},\therefore|F_1N|\cdot|F_2N|=c^2-e^2x_P^2=c^2(1-\frac{x_P^2}{a^2})=\frac{c^2y_P^2}{b^2}$$

$$\therefore\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{r}{|F_1N|}\cdot\frac{r}{|F_2N|}(\frac{1}{2}\cdot(2a+2c)\cdot r=\frac{1}{2}\cdot 2c\cdot y_P得r=\frac{cy_P}{a+c})=\frac{\frac{c^2y_P^2}{(a+c)^2}}{\frac{c^2y_P^2}{a^2-c^2}}=\frac{1-e}{1+e}$$

② 已知I为$\triangle F_1PF_2$的内心，设$\angle F_1PF_2$的平分线交x轴于D，则$\dfrac{|DI|}{|IP|}=$________；e

$$key1:\frac{|DI|}{|IP|}=\frac{y_I}{y_P-y_I}=e(\frac{1}{2}\cdot 2(a+c)\cdot y_I=\frac{1}{2}\cdot 2c\cdot y_P)$$

$$key2:\frac{|DI|}{|IP|}=\frac{|DF_2|}{|F_2P|}=\frac{|F_1D|}{|F_1P|}=\frac{|DF_2|+|F_1D|}{|F_2P|+|F_1P|}=\frac{2c}{2a}=e$$



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若点 P 在椭圆上运动，则内心 I 的轨迹方程为____________.

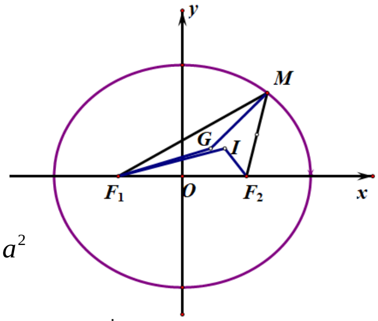
$key:$ 设 $I(x,y),P(x_0,y_0)$, 则 $\dfrac{|IP|}{|DI|}=\dfrac{y_0-y}{y}=\dfrac{1}{e}$ 得 $y_0=\dfrac{e+1}{e}y$,

由 $\dfrac{|F_1D|}{|F_2D|}=\dfrac{|PF_1|}{|PF_2|}\Leftrightarrow\dfrac{x_D+c}{c-x_D}=\dfrac{a+ex_0}{a-ex_0}$ 得 $x_D=e^2x_0$

由 $\dfrac{|IP|}{|DI|}=\dfrac{x_0-x}{x-x_D}=\dfrac{1}{e}$ 得 $x_D=(1+e)x-ex_0=e^2x_0$ 得 $x_0=\dfrac{1}{e}x,\therefore\dfrac{x^2}{a^2e^2}+\dfrac{(e+1)^2y^2}{b^2e^2}=1$ 即为所求的

③已知 $F_1(-1,0)$, $F_2(1,0)$, M 是第一象限内的点，且满足 $|MF_1|+|MF_2|=4$，若 I 是 $\triangle MF_1F_2$ 的内心，G 是 $\triangle MF_1F_2$ 的重心，记 $\triangle IF_1F_2$ 与 $\triangle GF_1M$ 的面积分别为 S_1，S_2，则（　B　）

A．$S_1>S_2$　　B．$S_1=S_2$　　C．$S_1<S_2$　　D．S_1 与 S_2 大小不确定

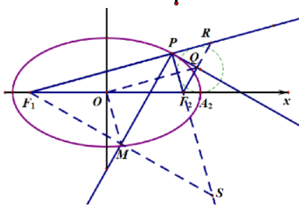


$key:$ 由 $S_2=\dfrac{1}{3}S_{\triangle MF_1F_2}=\dfrac{1}{3}\cdot\dfrac{1}{2}\cdot2\cdot y_M=\dfrac{1}{3}y_M$

$\dfrac{1}{2}(2a+2c)y_I=S_{\triangle MF_1F_2}=\dfrac{1}{2}\cdot2c\cdot y_M$ 得 $y_I=\dfrac{1}{3}y_M$, $\therefore S_1=\dfrac{1}{2}\cdot2\cdot y_I=S_2$

④（I）焦点 F_2 在 $\angle F_1PF_2$ 的外角平分线上的射影 Q 的轨迹方程为______. $x^2+y^2=a^2$

（II）若 l 是 $\angle F_1PF_2$ 的平分线，且 $F_1M\perp l$ 于点 M，则 $|OM|$ 的取值范围为______；



$key:|OM|=\dfrac{1}{2}|F_2N|=\dfrac{1}{2}||PF_1|-|PF_2||=|a-|PF_2||\in[0,c]$