三角函数解答（4）解三角形（2）

2023-04-09

（4）（2021 江西）$\triangle ABC$中，$AB=c,BC=a,AC=b,$ 且 $a^4+b^4+c^4=2c^2(a^2+b^2)$，

若$A=72°$，则$B=$______

解：$\cos^2 C=\left(\dfrac{a^2+b^2-c^2}{2ab}\right)^2=\dfrac{a^4+b^4+c^4+2a^2b^2-2c^2(a^2+b^2)}{4a^2b^2}=\dfrac{2a^2b^2}{4a^2b^2}=\dfrac{1}{2}$，$\cos C=\pm\dfrac{\sqrt{2}}{2}$，但 $C<180°$

$-A=108°$，$\cos C=\dfrac{\sqrt{2}}{2}$，$C=45°$，因此 $B=180°-A-C=63°$.

（5）①（2015 甘肃）已知$\triangle ABC$ 的外接圆半径为 R，且 $2R(\sin^2 A-\sin^2 C)=(\sqrt{2}a-b)\sin B$，其中

a,b是$\angle A,\angle B$的对边，则$\angle C=$____．$\dfrac{\pi}{4}$

②（2018 重庆）在$\triangle ABC$中，$\sin^2 A+\sin^2 C=2018\sin^2 B$，则$\dfrac{(\tan A+\tan C)\tan^2 B}{\tan A+\tan B+\tan C}=$______．$\dfrac{2}{2017}$

③（2018 辽宁）在$\triangle ABC$ 中，角 A、B、C 的对边分别为 a、b、c.若 $a^2+b^2=2019c^2$，则

$\dfrac{\tan A\tan B}{\tan A\tan C+\tan B\tan C}=$___ .1009

3（1）①若$2B=A+C$，且$\sin A-\sin C+\dfrac{\sqrt{2}}{2}\cos(A-C)=\dfrac{\sqrt{2}}{2}$，则$A=$______，$C=$________．

$key:B=\dfrac{\pi}{3},A+C=\dfrac{2\pi}{3},$

$\sin A-\sin C+\dfrac{\sqrt{2}}{2}\cos(A-C)=2\cos\dfrac{A+C}{2}\sin\dfrac{A-C}{2}+\dfrac{\sqrt{2}}{2}(1-2\sin^2\dfrac{A-C}{2})$

$=\sin\dfrac{A-C}{2}-\sqrt{2}\sin^2\dfrac{A-C}{2}+\dfrac{\sqrt{2}}{2}=\dfrac{\sqrt{2}}{2}$得$A=C=\dfrac{\pi}{3}$，或$A=\dfrac{7\pi}{12},C=\dfrac{\pi}{6}$

②$\sin A(\sin B+\cos B)-\sin C=0$，$\sin B+\cos 2C=0$，则角$A,B,C$的大小依次为_______．

$key:\sin A\sin B+\sin A\cos B-\sin A\cos B-\cos A\sin B=\sin A\sin B-\cos\sin B=0,\therefore A=\dfrac{\pi}{4}$

$\sin B+\cos 2C=\sin(C+\dfrac{\pi}{4})+\sin(\dfrac{\pi}{2}+2C)=\sin(C+\dfrac{\pi}{4})(1+2\cos(C+\dfrac{\pi}{4}))=0$得$C=\dfrac{5\pi}{12},B=\dfrac{\pi}{3}$

③（2017 内蒙古）锐角三角形的内角 A,B满足 $\tan A-\dfrac{1}{\sin 2A}=\tan B$，且$\cos^2\dfrac{B}{2}=\dfrac{\sqrt{6}}{3}$，

则$\sin 2A=$______．$\dfrac{2\sqrt{6}-3}{3}$

$key:\cos^2\dfrac{B}{2}=\dfrac{1+\cos B}{2}=\dfrac{\sqrt{6}}{3}(B\in(0,\dfrac{\pi}{2}))$得$\cos B=\dfrac{2\sqrt{6}}{3}-1,$

$\tan A-\dfrac{1}{\sin 2A}=\dfrac{\sin A}{\cos A}-\dfrac{1}{\sin 2A}=\dfrac{2\sin^2 A-1}{\sin 2A}=-\dfrac{\cos 2A}{\sin 2A}=\dfrac{\sin B}{\cos B}$得$\cos(2A-B)=0$

而$2A-B\in(-\dfrac{\pi}{2},\pi),\therefore 2A-B=\dfrac{\pi}{2},\therefore\sin 2A=\sin(\dfrac{\pi}{2}+B)=\cos B=\dfrac{2\sqrt{6}-3}{3}$

④（2019 江西）$\triangle ABC$ 的三个内角 A,B,C 满足 $A=3B=9C$，则 $\cos A\cos B+\cos B\cos C+\cos C\cos A=$____．

$key: C=\frac{\pi}{13}, B=\frac{3\pi}{13}, A=\frac{9\pi}{13}, \therefore$ 原式 $=\cos\frac{9\pi}{13}\cos\frac{3\pi}{13}+\cos\frac{3\pi}{13}\cos\frac{\pi}{13}+\cos\frac{9\pi}{13}\cos\frac{\pi}{13}$

$=\frac{1}{2}(\cos\frac{12\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{4\pi}{13}+\cos\frac{2\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{10\pi}{13})$

$=\frac{1}{4\sin\frac{\pi}{13}}(\cos\frac{12\pi}{13}\cdot 2\sin\frac{\pi}{13}+\cos\frac{6\pi}{13}\cdot 2\sin\frac{\pi}{13}+\cos\frac{4\pi}{13}\cdot 2\sin\frac{\pi}{13}+\cos\frac{2\pi}{13}\cdot 2\sin\frac{\pi}{13}+\cos\frac{8\pi}{13}\cdot 2\sin\frac{\pi}{13}+\cos\frac{10\pi}{13}\cdot 2\sin\frac{\pi}{13})$

$=\frac{1}{4\sin\frac{\pi}{13}}(\sin\frac{3\pi}{13}-\sin\frac{\pi}{13}+\sin\frac{5\pi}{13}-\sin\frac{3\pi}{13}+\cdots+\sin\frac{13\pi}{13}-\sin\frac{11\pi}{13})=-\frac{1}{4}$

⑤（2015 山东）在 $\triangle ABC$ 中，$\angle A<\angle B<\angle C, \frac{\sin A+\sin B+\sin C}{\cos A+\cos B+\cos C}=\sqrt{3}$.则 $\angle B=$____． $\frac{\pi}{3}$

$key:$ 由已知得 $\sin A-\sqrt{3}\cos A+\sin B-\sqrt{3}\cos B+\sin C-\sqrt{3}\cos C$

$=2\sin(A-\frac{\pi}{3})+2\sin(B-\frac{\pi}{3})+2\sin(C-\frac{\pi}{3})=4\sin\frac{A+C-\frac{2\pi}{3}}{2}\cos\frac{A-C}{2}+2\sin(B-\frac{\pi}{3})$

$=-4\sin\frac{B-\frac{\pi}{3}}{2}\cos\frac{A-C}{2}+4\sin\frac{B-\frac{\pi}{3}}{2}\cos\frac{B-\frac{\pi}{3}}{2}=0(\because A<B<C), \therefore B=\frac{\pi}{3}$

（2） ①已知角 A 为锐角，且 $\sin^2 A=4\sin B\sin C=(\frac{\sin B+\sin C}{m})^2$，则实数 m 范围为_____

$(-\sqrt{2},-\frac{\sqrt{6}}{2})\cup(\frac{\sqrt{6}}{2},\sqrt{2})$

②（2017 新疆）已知在 $\triangle ABC$ 中，$\tan A+\tan C=2(1+\sqrt{2})\tan B$.则 $\angle B$ 的最小值为___________. $\frac{\pi}{4}$

$key: \tan A\tan B\tan C=\tan A+\tan B+\tan C=(3+2\sqrt{2})\tan B,$

$\therefore\frac{3+2\sqrt{2}}{1}=\tan A\tan C=\frac{\sin A\sin C}{\cos A\cos C}\Leftrightarrow\frac{4+2\sqrt{2}}{2+2\sqrt{2}}=\frac{\cos(A-C)}{-\cos(A+C)}, \therefore\cos B=\frac{\sqrt{2}}{2}\cos(A-C)\le\frac{\sqrt{2}}{2}, \therefore B\ge\frac{\pi}{4}$

③（2018 四川）在 $\triangle ABC$ 中，$\cos B=\frac{1}{4}$，则 $\frac{1}{\tan A}+\frac{1}{\tan C}$ 的最小值为______. $\frac{2\sqrt{15}}{5}$

$key: \frac{1}{\tan A}+\frac{1}{\tan C}=\frac{\cos A}{\sin A}+\frac{\cos C}{\sin C}=\frac{\sin B}{\sin A\sin C}=\frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}(\cos(A-C)-\cos(A+C))}$

$=\frac{\sqrt{15}}{2}\cdot\frac{1}{\cos(A-C)+\frac{1}{4}}\ge\frac{2\sqrt{15}}{5}$

④（2015 陕西）在 $\triangle ABC$ 中，若 $\tan\frac{A}{2}+\tan\frac{B}{2}=1$，则 $\tan\frac{C}{2}$ 的最小值为______． $\frac{3}{4}$

$key:\tan\dfrac{A}{2}+\tan\dfrac{B}{2}=\dfrac{\sin\dfrac{A+B}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}}=\dfrac{\cos\dfrac{C}{2}}{\dfrac{1}{2}(\cos\dfrac{A+B}{2}+\cos\dfrac{A-B}{2})}=\dfrac{2\cos\dfrac{C}{2}}{\sin\dfrac{C}{2}+\cos\dfrac{A-B}{2}}=1$

得$2\cos\dfrac{C}{2}-\sin\dfrac{C}{2}=\cos\dfrac{A-B}{2}\le 1,\therefore \sin\dfrac{C}{2}\ge\dfrac{3}{5},\therefore \tan\dfrac{C}{2}\ge\dfrac{3}{4}$

$key2:1=\tan\dfrac{A+B}{2}(1-\tan\dfrac{A}{2}\tan\dfrac{B}{2})=\dfrac{1-\tan\dfrac{A}{2}\tan\dfrac{B}{2}}{\tan\dfrac{C}{2}}$,

$\therefore \tan\dfrac{C}{2}=1-\tan\dfrac{A}{2}(1-\tan\dfrac{A}{2})=(\tan\dfrac{A}{2}-1)^2+\dfrac{3}{4}\ge\dfrac{3}{4}$

⑤（2021山东）设A,B,C是$\triangle ABC$的三个内角，则使得$\dfrac{1}{\sin A}+\dfrac{1}{\sin B}\ge\dfrac{\lambda}{3+2\cos C}$恒成立的实数$\lambda$的最大值是______.

$key:\Leftrightarrow \lambda\le\dfrac{(3+2\cos C)(\sin A+\sin B)}{\sin A\sin B}=\dfrac{(3+2\cos C)\cdot 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}{\dfrac{1}{2}(\cos(A-B)-\cos(A+B))}$

$=4(3+2\cos C)\cdot\dfrac{\cos\dfrac{C}{2}\cos\dfrac{A-B}{2}}{2\cos^2\dfrac{A-B}{2}-1+\cos C}=4(3+2\cos C)\cdot\dfrac{\cos\dfrac{C}{2}}{2\cos\dfrac{A-B}{2}-\dfrac{2\sin^2\dfrac{C}{2}}{\cos\dfrac{A-B}{2}}}(\because 0<\cos\dfrac{A-B}{2}\le 1)$

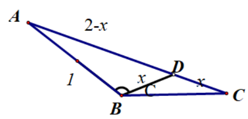
$\ge\dfrac{4(3+2\cos C)\cos\dfrac{C}{2}}{2\cos^2\dfrac{C}{2}}=\dfrac{2(4\cos^2\dfrac{C}{2}+1)}{\cos\dfrac{C}{2}}\ge 8,\therefore \lambda_{\max}=8$

⑥（2021贵州）在斜$\triangle ABC$中，$\cos^2 A+\cos^2 B+\cos^2 C=\sin^2 B$，则$\tan A\tan C=$____.

$key:\dfrac{1+\cos 2A}{2}+\dfrac{1+\cos 2C}{2}+\cos 2B=1+\cos(A+C)\cos(A-C)+\cos 2(A+C)=0$

即$\cos(A+C)(2\cos(A+C)+\cos(A-C))=0(\because A+C\ne\dfrac{\pi}{2})$

$\therefore 2(\cos A\cos C-\sin A\sin C)+\cos A\cos C+\sin A\sin C=0,\therefore \tan A\tan C=3$



（2021A）在$\triangle ABC$中，$AB=1,AC=2,B-C=\dfrac{2\pi}{3}$，则$\triangle ABC$的面积为______.

$key:$如图，有$1+x^2+x=(2-x)^2$得$x=\dfrac{3}{5},\therefore\dfrac{\dfrac{3}{5}}{\sin A}=\dfrac{\dfrac{7}{5}}{\dfrac{\sqrt{3}}{2}}$得$\sin A=\dfrac{3\sqrt{3}}{14},\therefore S_{\triangle ABC}=\dfrac{3\sqrt{3}}{14}$