解析几何（2）椭圆解答（3）

2023-10-14

（二）面积问题

（2004福建）椭圆$x^2+4y^2=8$中，AB是长为$\frac{5}{2}$的动弦，O为坐标原点，求$\triangle AOB$面积的取值范围.

2004福建:当$l\perp x$轴时，$S_{\triangle OAB}=\frac{5\sqrt{7}}{8}$

当$l\not\perp x$轴时，设$l_{AB}:y=kx+m$代入椭圆方程得$(1+4k^2)x^2+8kmx+4m^2-8=0$

$$\therefore\begin{cases}x_1+x_2=\dfrac{-8km}{1+4k^2},\\ x_1x_2=\dfrac{4m^2-8}{1+4k^2}\end{cases}\text{且}\Delta=16(2+8k^2-m^2)>0$$

$$\therefore |AB|=\sqrt{1+k^2}\cdot\frac{4\sqrt{2+8k^2-m^2}}{1+4k^2}=\frac{5}{2}\text{得}|m|=\sqrt{2(1+4k^2)-\frac{25(1+4k^2)^2}{64(1+k^2)}}(\text{且}\frac{1+4k^2}{1+k^2}\le\frac{128}{25})$$

$$S=\frac{1}{2}\cdot\frac{5}{2}\cdot\frac{|m|}{\sqrt{1+k^2}}=\frac{5}{4}\sqrt{\frac{2(1+4k^2)}{1+k^2}-\frac{25}{64}(\frac{1+4k^2}{1+k^2})^2}\ (t=\frac{1+4k^2}{1+k^2}\in[1,4))$$

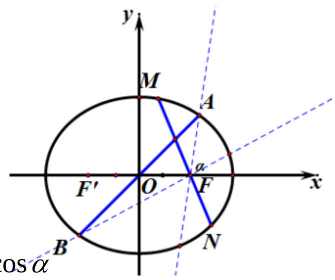
$$=\frac{5}{4}\sqrt{2t-\frac{25}{64}t^2}=\frac{5}{4}\sqrt{-\frac{25}{64}(t-\frac{64}{25})^2+\frac{64}{25}}\in[\frac{5}{32}\sqrt{103},2]\text{即为所求的}$$

（2010湖北）已知直线$y=x$与椭圆$C:\frac{x^2}{16}+\frac{y^2}{11}=1$交于$A,B$两点，过椭圆$C$的右焦点$F$倾斜角为$\alpha$的直线$l$交弦$AB$于点$P$，交椭$C$于点$M,N$.(1）用$\alpha$表示四边形$MANB$的面积；

（2）求四边形$MANB$的面积取得最大值时直线l的方程.

2010湖北：设$|MF|=r_1$，则$M(\sqrt{5}+r_1\cos\alpha,r_1\sin\alpha)$,$\therefore 11(5+2\sqrt{5}r_1\cos\alpha+r_1^2\cos^2\alpha)+16r_1^2\sin^2\alpha=176$

得$|MF|=r_1=\frac{11}{4+\sqrt{5}\cos\alpha}$,同理$|NF|=\frac{11}{4-\sqrt{5}\cos\alpha}$



$$\text{由}\begin{cases}y=x\\ \dfrac{x^2}{16}+\dfrac{y^2}{11}=1\end{cases}\text{得}x^2=\frac{16\times 11}{27},\therefore |AB|=\sqrt{2}\cdot\frac{8\sqrt{11}}{3\sqrt{3}}=\frac{8\sqrt{22}}{3\sqrt{3}}$$

$$\therefore S_{MANB}=\frac{1}{2}\cdot\frac{8\sqrt{22}}{3\sqrt{3}}\cdot(\frac{11}{4+\sqrt{5}\cos\alpha}+\frac{11}{4-\sqrt{5}\cos\alpha})\cdot\sin(\alpha-\frac{\pi}{4})=\frac{352\sqrt{33}}{9}\cdot\frac{\sin\alpha-\cos\alpha}{16-5\cos^2\alpha}$$

$$(2)\frac{\sin\alpha-\cos\alpha}{16-5\cos^2\alpha}=\frac{\sqrt{2}\sin(\alpha-\frac{\pi}{4})}{16-\frac{5(1+\cos 2\alpha)}{2}}=\frac{2\sqrt{2}\sin(\alpha-\frac{\pi}{4})}{27+5\sin(2\alpha-\frac{\pi}{2})}=\frac{2\sqrt{2}\sin\theta}{27+5\sin 2\theta}\text{记为}f(\theta)(\theta=\alpha-\frac{\pi}{4}>0)$$

$$\text{则}f'(\theta)=\frac{2\sqrt{2}(27\cos\theta+10\sin^3\theta)}{(27+5\sin 2\theta)^2}=\frac{2\sqrt{2}\cos^3\theta(\tan\theta+3)(10\tan^2\theta-3\tan\theta+9)}{(27+5\sin 2\theta)^2}=0\text{得}\tan\theta>-3$$

$$\therefore S_{MANB}\text{最大时},\tan\alpha=\tan(\theta+\frac{\pi}{4})=-\frac{1}{2},\therefore MANB\text{面积最大时，}l\text{的方程为}y=-\frac{1}{2}x+\frac{\sqrt{5}}{2}$$

（2011 山东）已知动直线l与椭圆$C:\frac{x^2}{3}+\frac{y^2}{2}=1$交于$P(x_1,y_1),Q(x_2,y_2)$两个不同点，且$\triangle OPQ$的面积为$S_{\triangle OPQ}=\frac{\sqrt{6}}{2}$,其中$O$为坐标原点.(1）证明：$x_1^2+x_2^2$和$y_1^2+y_2^2$为定值；

（2）椭圆C上是否存在点D、E、G，使得$S_{\triangle ODE}=S_{\triangle ODG}=S_{\triangle OEG}=\frac{\sqrt{6}}{2}$？若存在，判断$\triangle DEG$的形状；若不存在，请说明理由.

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（3）设线段PQ的中点为M.求（ⅰ）$|OM|\cdot|PQ|$的最大值；（ⅱ）求M的轨迹方程.

解:（1）当$l\not\perp x$轴时，设PQ方程为$y=kx+m$代入椭圆方程得:$(2+3k^2)x^2+6kmx+3m^2-6=0$

$$\therefore\begin{cases}x_1+x_2=\dfrac{-6km}{2+3k^2}\\[2mm] x_1x_2=\dfrac{3m^2-6}{2+3k^2}\end{cases},\text{且}\Delta=24(2+3k^2-m^2)>0$$

$$\therefore S_{\triangle OPQ}=\frac{1}{2}\sqrt{1+k^2}\cdot\sqrt{\frac{36k^2m^2}{(2+3k^2)^2}-4\cdot\frac{3m^2-6}{2+3k^2}}\cdot\frac{|m|}{\sqrt{1+k^2}}=\sqrt{\frac{6(2+3k^2-m^2)m^2}{(2+3k^2)^2}}=\frac{\sqrt{6}}{2}\text{得}2m^2=2+3k^2$$

$$\therefore\begin{cases}x_1+x_2=\dfrac{-3k}{m}\\[2mm] x_1x_2=\dfrac{3m^2-6}{2m^2}\end{cases},\text{且}\Delta=24m^2>0,$$

$$\therefore x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=\frac{9k^2}{m^2}-\frac{3m^2-6}{m^2}=3,y_1^2+y_2^2=2(1-\frac{x_1^2}{3})+2(1-\frac{x_2^2}{3})=2$$

当$l\perp x$轴时，有$\frac{1}{2}|x_1|\cdot\sqrt{2-\frac{2}{3}x_1^2}=\frac{\sqrt{6}}{2},\therefore x_1^2+x_2^2=3,y_1^2+y_2^2=2,\therefore x_1^2+x_2^2=3,y_1^2+y_2^2=2$为定值

（2）假设存在，由（Ⅰ）得$x_D^2+x_E^2=x_E^2+x_F^2=x_F^2+x_D^2=3,\therefore x_D^2=x_E^2=x_F^2$，

由椭圆的对称性的D,E,F中有两点与O共线，故不存在；

（3）（ⅰ）$key1:4|OM|^2+|PQ|^2=(x_1+x_2)^2+(y_1+y_2)^2+(x_1-x_2)^2+(y_1-y_2)^2$

$$=2(x_1^2+x_2^2)+2(y_1^2+y_2^2)=10\ge4|\overrightarrow{OM}|\cdot|\overrightarrow{OP}|,\therefore|\overrightarrow{OM}|\cdot|\overrightarrow{OP}|\le\frac{5}{2}$$

$$key2:|OM|\cdot|PQ|=\sqrt{\frac{9k^2}{4m^2}+\frac{1}{m^2}}\cdot\sqrt{1+k^2}\cdot\frac{2\sqrt{6}|m|}{2m^2}\le\frac{5}{2};$$

（ⅱ）设$M(x,y)$,由（Ⅰ）得:$\begin{cases}x=\dfrac{-3k}{2m}\text{即}m=-\dfrac{3k}{2x}\\[2mm] y=k\cdot\dfrac{-3k}{2m}=\dfrac{1}{m}\\[2mm] 2m^2=2+3k^2\end{cases},\therefore\begin{cases}k=\dfrac{-2x}{3y}\\[2mm] m=\dfrac{1}{y}\end{cases},\therefore\dfrac{2}{y^2}=2+\dfrac{4x^2}{3y^2}$即$\dfrac{2}{3}x^2+y^2=1$即为所求的

（2011湖北）已知椭圆$C:\dfrac{x^2}{4}+\dfrac{y^2}{2}=1$，过点$P(\dfrac{\sqrt{2}}{3},-\dfrac{1}{3})$而不过点$Q(\sqrt{2},1)$的动直线$l$交椭圆$C$于$A,B$两点.

（Ⅰ）求$\angle AQB$；（Ⅱ）记$\triangle QAB$的面积为S，证明:$S<3$.

（2011湖北）（Ⅰ）当$l\not\perp x$轴时，设$l:y+\dfrac{1}{3}=k(x-\dfrac{\sqrt{2}}{3})$即$y=kx+m(m=-\dfrac{1+\sqrt{2}k}{3},k\ne\sqrt{2})$

代入C得:$(1+2k^2)x^2+4kmx+2m^2-4=0,\therefore\begin{cases}x_A+x_B=-\dfrac{4km}{1+2k^2}\\[2mm] x_Ax_B=\dfrac{2m^2-4}{1+2k^2}\end{cases}$,且$\Delta=8(4k^2+2-m^2)>0$

$$\therefore\overrightarrow{QA}\cdot\overrightarrow{QB}=(x_A-\sqrt{2})(x_B-\sqrt{2})+(kx_A+m-1)(kx_B+m-1)=0\Leftrightarrow(1+k^2)x_Ax_B+((m-1)k-\sqrt{2})(x_A+x_B)+2+(m-1)^2$$

$$=\frac{(1+k^2)(2m^2-4)}{1+2k^2}-\frac{((m-1)k-\sqrt{2})\cdot4km}{1+2k^2}+\frac{(m^2-2m+3)(1+2k^2)}{1+2k^2}=0$$

$$\Leftrightarrow2k^2+4\sqrt{2}mk+3m^2-2m-1=2k^2-4\sqrt{2}k\cdot\frac{1+\sqrt{2}k}{3}+\frac{1+2\sqrt{2}k+2k^2}{3}+\frac{2+2\sqrt{2}k}{3}-1$$

$$=(2-\frac{8}{3}+\frac{2}{3})k^2+(-\frac{4\sqrt{2}}{3}+\frac{2\sqrt{2}}{3}+\frac{2\sqrt{2}}{3})k+\frac{1}{3}+\frac{2}{3}-1=0,$$

当$l\perp x$轴时,$\overrightarrow{OA}\cdot\overrightarrow{OB}=0,\therefore\angle AOB=90°$

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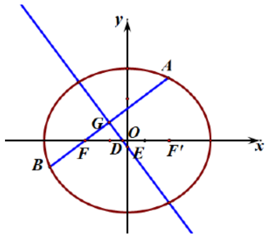
（Ⅱ）当l不$\perp x$轴时，由（Ⅰ）得$S=\frac{1}{2}\cdot\sqrt{1+k^2}\cdot\frac{2\sqrt{2}\sqrt{4k^2+2-\frac{1+2\sqrt{2}k+2k^2}{9}}}{1+2k^2}\cdot\frac{|\sqrt{2}k-1-\frac{1+\sqrt{2}k}{3}|}{\sqrt{1+k^2}}$

$$=\frac{4}{9}\sqrt{\frac{(34k^2-2\sqrt{2}k+17)(k-\sqrt{2})^2}{(2k^2+1)^2}}=\frac{4}{9}\sqrt{(17-\frac{2\sqrt{2}}{2k+\frac{1}{k}})\cdot\frac{1}{5t^2+4\sqrt{2}t+2}}\le\frac{4}{9}\sqrt{17\cdot\frac{1}{\frac{2}{5}}}=\sqrt{\frac{680}{81}}<3(t=\frac{1}{k-\sqrt{2}})$$

当$l\perp x$轴时，$S=\sqrt{\frac{17}{18}}\cdot\frac{2\sqrt{2}}{3}<3,\therefore S<3$得证

（2018河北）如图，椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$的左焦点为$F$，过点$F$的直线交椭圆于$A$、$B$两点.当直线$AB$经过椭圆的一个顶点时，其倾斜角为60°.(1) 求该椭圆的离心率；

(2) 设线段AB的中点为G, AB的中垂线与x轴、y轴分别交于D、E两点.记$\triangle GDF$的面积为S_1,$\triangle OED$（O为坐标原点）的面积为S_2，求$\frac{S_1}{S_2}$的取值范围.



解： (1) 由已知得$\frac{b}{c}=\tan 60°=\sqrt{3},\therefore e=\frac{c}{2c}=\frac{1}{2}$

(2) 设$l_{AB}:x=ty-c$代入椭圆$3x^2+4y^2=12c^2$得$(3t^2+4)y^2-6cty-9c^2=0$

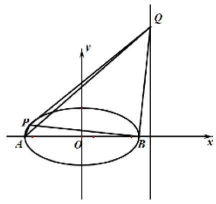
$$\therefore y_G=\frac{y_A+y_B}{2}=\frac{3ct}{3t^2+4},x_G=\frac{-4c}{3t^2+4}$$

$$\therefore\frac{\frac{3ct}{3t^2+4}}{\frac{-4c}{3t^2+4}-x_D}=-t得x_D=-\frac{c}{3t^2+4},且\frac{\frac{3ct}{3t^2+4}-y_E}{\frac{-4c}{3t^2+4}}=-t得y_E=\frac{-ct}{3t^2+4}$$

$$\therefore\frac{S_1}{S_2}=\frac{\frac{1}{2}(-\frac{c}{3t^2+4}+c)\cdot\frac{3ct}{3t^2+4}}{\frac{1}{2}\cdot\frac{c}{3t^2+4}\cdot\frac{ct}{3t^2+4}}=9(t^2+1)\in(9,+\infty)即为所求的$$

（2020Ⅲ）20. 已知椭圆$C:\frac{x^2}{25}+\frac{y^2}{m^2}=1(0<m<5)$的离心率为$\frac{\sqrt{15}}{4}$，$A$，$B$分别为$C$的左、右顶点.

（1）求C的方程；（2）若点P在C上，点Q在直线$x=6$上，且$|BP|=|BQ|$，$BP\perp BQ$，求$\triangle APQ$的面积.



解： (1) 由已知得$e=\frac{\sqrt{25-m^2}}{5}=\frac{\sqrt{15}}{4}$得$m=\frac{5}{4},\therefore C$的方程为$\frac{x^2}{25}+\frac{16y^2}{25}=1,$

(2) 设$Q(6,q)$(不妨设$q>0$),则$\overrightarrow{BQ}=(1,q),\overrightarrow{BP}=(-q,1),$

$$\therefore P(5-q,1),\therefore\frac{(5-q)^2}{25}+\frac{16}{25}=1即q=2,or,8,$$

当$q=2$时，$Q(6,2),P(3,1),l_{PQ}:x-3y=0,\therefore S_{\triangle APQ}=\frac{1}{2}\cdot\sqrt{1+\frac{1}{9}}\cdot 3\cdot\frac{5}{\sqrt{10}}=\frac{5}{2}$

$$(S_{\triangle APQ}=|\frac{1}{2}\begin{vmatrix}-5&0&1\\3&1&1\\6&2&1\end{vmatrix}|=\frac{5}{2})$$

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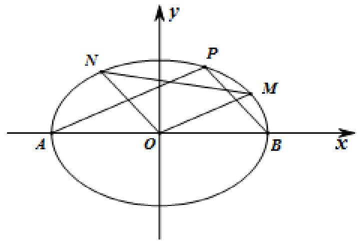
当$q=8$时，$Q(6,8),P(-3,1),l_{PQ}:7x-9y+30=0\therefore S_{\triangle APQ}=\frac{1}{2}\cdot\sqrt{1+(\frac{7}{9})^2}\cdot 9\cdot\frac{|-35+30|}{\sqrt{7^2+9^2}}=\frac{5}{2}$

$(S_{\triangle APQ}=|\frac{1}{2}\begin{vmatrix}-5&0&1\\-3&1&1\\6&8&1\end{vmatrix}|=\frac{5}{2}),\therefore\triangle APQ$的面积为$\frac{5}{2}$

变式 1（1）如图，已知椭圆$C:\frac{x^2}{4}+\frac{y^2}{2}=1$的左右顶点分别为$A$、$B$，$M$、$N$是椭圆$C$上非顶点的两点，且$\triangle MON$的面积为$\sqrt{2}$.过点$A$作$AP//OM$交椭圆$C$于点$P$，求证：$BP//ON$.



key:当$MN\perp x$轴时，${x_M}^2=2,y_M^2=1,\therefore x_P^2=0,\therefore OB//PB$

当$MN\not\perp x$轴时，设$MN:y=kx+m$代入C得$(1+2k^2)x^2+4kmx+2(m^2-2)=0$

$\therefore\begin{cases}x_M+x_N=\dfrac{-4km}{1+2k^2}\\x_Mx_N=\dfrac{2(m^2-2)}{1+2k^2}\end{cases}$，且$\Delta=8(2+4k^2-m^2)>0$

$\therefore S_{\triangle OMN}=\frac{1}{2}\cdot\sqrt{1+k^2}\cdot\frac{\sqrt{8(2+4k^2-m^2)}}{1+2k^2}\cdot\frac{|m|}{\sqrt{1+k^2}}=\frac{|m|\sqrt{2(2+4k^2-m^2)}}{1+2k^2}=\sqrt{2}$即$m^2=1+2k^2$

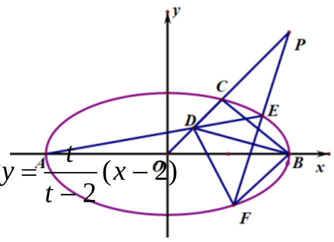
$AP:x=\frac{x_M}{y_M}y-2$代入C得：$P(x_M^2-2,x_My_M)$

$\therefore k_{PB}=\frac{x_My_M}{x_M^2-4}=\frac{x_My_M}{-2y_M^2}=\frac{x_M}{-2y_M},\therefore k_{PB}-k_{ON}=\frac{x_M}{-2y_M}-\frac{y_N}{x_N}=\frac{x_Mx_N+2y_My_N}{-2x_Ny_M}=0$

$\Leftrightarrow x_Mx_N+2y_My_N=x_Mx_N+2(kx_M+m)(kx_N+m)=(1+2k^2)x_Mx_N+2km(x_M+x_N)+2m^2$

$=\frac{(1+2k^2)\cdot 2(m^2-2)}{1+2k^2}+\frac{-8k^2m^2}{1+2k^2}+2m^2=2m^2-4-4(m^2-1)+2m^2=0$

（2）如图，椭圆$\frac{x^2}{4}+y^2=1$的左、右顶点分别为A、B，点P的坐标是$(2,2)$，线段OP交椭圆于点C,D在线段OC上（不包括端点），延长AD交椭圆于点E,延长PE交椭圆于点F.记S_1,S_2分别为$\triangle BCD$和$\triangle BDF$的面积.（1）求$|OC|$的值；（2）求$S_1\cdot S_2$的最大值.



解：（1）由$\begin{cases}y=x\\\dfrac{x^2}{4}+y^2=1\end{cases}$得$x_C=\frac{2}{\sqrt{5}},\therefore|OC|=\frac{2\sqrt{10}}{5}$

（2）设$D(t,t)(0<t<\frac{2}{\sqrt{5}})$，则$S_1=\frac{1}{2}\cdot\sqrt{2}\cdot(\frac{2}{\sqrt{5}}-t)\cdot\sqrt{2}=\frac{2}{\sqrt{5}}-t$，且$BD$方程为$y=\frac{t}{t-2}(x-2)$

AD方程为$x=\frac{t+2}{t}y-2$代入椭圆方程得$y_E=\frac{4t^2+8t}{5t^2+4t+4},x_E=\frac{-6t^2+8t+8}{5t^2+4t+4}$,

$\therefore k_{PE}=\frac{\frac{4t^2+8t}{5t^2+4t+4}-2}{\frac{-6t^2+8t+8}{5t^2+4t+4}-2}=\frac{3t^2+4}{8t^2},\therefore PE$方程为：$y-2=\frac{3t^2+4}{8t^2}(x-2)$即$y=\frac{3t^2+4}{8t^2}x+\frac{5t^2-4}{4t^2}$

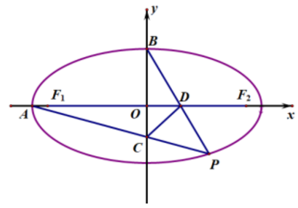
代入椭圆方程得$\frac{-6t^2+8t+8}{5t^2+4t+4}\cdot x_F=\frac{\frac{(5t^2-4)^2}{16t^4}-1}{\frac{1}{4}+\frac{(3t^2+4)^2}{64t^4}}=\frac{4(9t^2-4)(t^2-4)}{(5t^2-4t+4)(5t^2+4t+4)}$

得$x_F=\dfrac{-2(3t-2)(t+2)}{5t^2-4t+4}=\dfrac{-6t^2-8t+8}{5t^2-4t+4}$, $y_F=\dfrac{4t^2-8t}{5t^2-4t+4}$, $\therefore S_2=\dfrac{1}{2}\left\|\begin{matrix}2 & 0 & 1\\ t & t & 1\\ \dfrac{-6t^2-8t+8}{5t^2-4t+4} & \dfrac{4t^2-8t}{5t^2-4t+4} & 1\end{matrix}\right\|=2t$

$\therefore S_1S_2=(\dfrac{2}{\sqrt{5}}-t)\cdot 2t\le 2(\dfrac{\frac{2}{\sqrt{5}}-t+t}{2})^2=\dfrac{2}{5}$（当且仅当$t=\dfrac{1}{\sqrt{5}}$时，取＝），$\therefore$所求的最大值为$\dfrac{2}{5}$

（3）已知椭圆$C:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$的离心率为$\dfrac{\sqrt{3}}{2}$，且过点$(\sqrt{3},\dfrac{1}{2})$，点$P$在第四象限，$A$为左顶点，$B$为上顶点，$PA$交$y$轴于点$C$，$PB$交$x$轴于点$D$.（Ⅰ）求椭圆$C$的标准方程；（Ⅱ）求$\triangle PCD$的面积的最大值.

解:（Ⅰ）$\dfrac{x^2}{4}+y^2=1$



（Ⅱ）设$AP:x=ty-2(t<-2)$代入C得$y_P=\dfrac{4t}{t^2+4}$, $x_P=\dfrac{2t^2-8}{t^2+4}$;

由B,D,P共线得:$\dfrac{\frac{4t}{t^2+4}-1}{\frac{2t^2-8}{t^2+4}}=\dfrac{1}{-x_D}$得$x_D=\dfrac{2t^2-8}{t^2-4t+4}=\dfrac{2(t+2)}{t-2}$,

（或在AP方程中令$x=\dfrac{2t+4}{t-2}$得$y=\dfrac{4}{t-2}$, 则$S_{\triangle PCD}=\dfrac{1}{2}\cdot\dfrac{4}{2-t}\cdot\dfrac{2(t^2-4)}{t^2+4}=4\cdot\dfrac{-t-2}{t^2+4}$)

$S_{\triangle PAD}=\dfrac{1}{2}\sqrt{1+t^2}\cdot\left|\dfrac{4t}{t^2+4}-\dfrac{2}{t}\right|\cdot\dfrac{\left|\frac{2(t+2)}{t-2}+2\right|}{\sqrt{1+t^2}}=4\cdot\dfrac{-t-2}{t^2+4}=\dfrac{4}{u+\frac{8}{u}+4}\le\dfrac{4}{4\sqrt{2}+4}=\sqrt{2}-1(u=-t-2>0)$

$key2$: 设$P(2\cos\theta,\sin\theta)(\theta\in(-\dfrac{\pi}{2},0))$，由$P,D,B$三点共线得$\dfrac{\sin\theta-1}{2\cos\theta}=\dfrac{1}{-x_D}$得$x_D=\dfrac{2\cos\theta}{1-\sin\theta}$

由A,C,P三点共线得$\dfrac{\sin\theta}{2\cos\theta+2}=\dfrac{y_C}{2}$得$y_C=\dfrac{\sin\theta}{1+\cos\theta}$

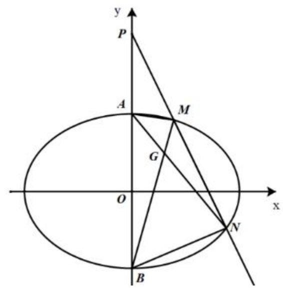
$\therefore S_{\triangle PCD}=\dfrac{1}{2}\left\|\begin{matrix}2\cos\theta & \sin\theta & 1\\ \dfrac{2\cos\theta}{1-\sin\theta} & 0 & 1\\ 0 & \dfrac{\sin\theta}{1+\cos\theta} & 1\end{matrix}\right\|=\dfrac{-\sin\theta\cos\theta(1+\cos\theta-\sin\theta)}{(1+\cos\theta)(1-\sin\theta)}=-\sin\theta+\cos\theta-1$

$=\sqrt{2}\cos(\theta+\dfrac{\pi}{4})-1\le\sqrt{2}-1$

（4）已知椭圆$C:\dfrac{x^2}{8}+\dfrac{y^2}{4}=1$的上下顶点分别为$A,B$，过点$P(0,4)$斜率为$-k(k>0)$的直线与椭圆$C$自上而下交于$M,N$两点.（I）证明：直线$BM$与$AN$的交点$G$在定直线$y=1$上.

（II）记$\triangle AGM$和$\triangle BGN$的面积分别为S_1和S_2，求$\dfrac{S_1}{S_2}$的取值范围.

（Ⅰ）证明：由$\begin{cases}y=-kx+4\\ x^2+2y^2=8\end{cases}$消去$y$得:$(1+2k^2)x^2-16kx+24=0$, $\therefore\begin{cases}x_M+x_N=\dfrac{16k}{1+2k^2}\\ x_Mx_N=\dfrac{24}{1+2k^2}\end{cases}$, 2023-10-14



且$\Delta=32(2k^2-3)>0$, 且$\frac{x_M+x_N}{x_Mx_N}=\frac{2k}{3}$即$2kx_Mx_N=3x_M+3x_N$

而AN方程为：$x=\frac{x_N}{y_N-2}(y-2)\cdots$①; 而$BM$方程为$x=\frac{x_M}{y_M+2}(y+2)\cdots$②

由①②得$\frac{y_G-2}{y_G+2}=\frac{x_M(y_N-2)}{x_N(y_M+2)}=\frac{2x_M-\frac{3}{2}(x_M+x_N)}{6x_N-\frac{3}{2}(x_M+x_N)}=-\frac{1}{3}$得$y_G=1$，得证

另解: $\frac{y_G-2}{y_G+2}=\frac{x_M(y_N-2)}{x_N(y_M+2)}=\frac{-x_Mx_N}{2(y_M+2)(y_N+2)}=\cdots=-\frac{1}{3}$

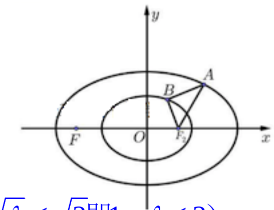
(II) $\frac{S_1}{S_2}=\frac{\frac{1}{2}|GA|\cdot|GM|\sin\angle AGM}{\frac{1}{2}|GN|\cdot|GB|\sin\angle BGN}=\frac{(y_A-y_G)\cdot(y_M-y_G)}{(y_G-y_N)(y_G-y_B)}=\frac{-kx_M+3}{3(kx_N-3)}=\frac{1}{3}\cdot\frac{-\frac{3x_M+3x_N}{2x_N}+3}{\frac{3x_M+3x_N}{2x_M}-3}=\frac{x_M}{3x_N}$

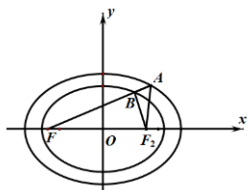
令$t=\frac{x_M}{x_N}<1$,则$t+\frac{1}{t}+2=\frac{32k^2}{3(1+2k^2)}=\frac{16}{3}(1-\frac{1}{1+2k^2})\in(4,\frac{16}{3})$得$\frac{1}{3}<t<1$,$\therefore\frac{S_1}{S_2}$的取值范围为$(\frac{1}{9},\frac{1}{3})$

（5）已知椭圆$C:\frac{x^2}{2}+y^2=1$右焦点为F_2，椭圆$E:\frac{x^2}{2}+y^2=\lambda(\lambda>1)$的左焦点为$F$，点$A$为椭圆$E$上一动点（不在$x$轴上），点$B$为线段$AF$与椭圆$C$的公共点（且靠近点$A$).

①若点F恰为椭圆C的左顶点，求椭圆E的方程；

②令$\triangle ABF_2$面积的最大值为$f(\lambda)$，求$f(\lambda)$的取值范围.





解：①由已知得$\sqrt{\lambda}=\sqrt{2}$得$\lambda=2$,$\therefore$椭圆E的方程为$\frac{x^2}{4}+\frac{y^2}{2}=1$

②设$A(\sqrt{2\lambda}\cos\alpha,\sqrt{\lambda}\sin\alpha)$, $B(\sqrt{2}\cos\beta,\sin\beta)$(其中$F$在$C$内即$1<\sqrt{\lambda}\le\sqrt{2}$即$1<\lambda\le2$)

由F,B,A共线得$\frac{\sqrt{\lambda}\sin\alpha}{\sqrt{2\lambda}\cos\alpha+\sqrt{\lambda}}=\frac{\sin\alpha}{\sqrt{2}\cos\alpha+1}=\frac{\sin\beta}{\sqrt{2}\cos\beta+\sqrt{\lambda}}$即$\sqrt{2}\sin(\alpha-\beta)+\sqrt{\lambda}\sin\alpha-\sin\beta=0$

$$\text{而}S_{\triangle ABF_2}=\frac{1}{2}\left\|\begin{matrix}1 & 0 & 1\\ \sqrt{2\lambda}\cos\alpha & \sqrt{\lambda}\sin\alpha & 1\\ \sqrt{2}\cos\beta & \sin\beta & 1\end{matrix}\right\|=\frac{1}{2}|\sqrt{\lambda}\sin\alpha+\sqrt{2\lambda}\cos\alpha\sin\beta-\sqrt{2\lambda}\sin\alpha\cos\beta-\sin\beta|$$

$$=\frac{1}{2}|\sqrt{2\lambda}\sin(\beta-\alpha)+\sqrt{2}\sin(\beta-\alpha)|=\frac{\sqrt{2}(\sqrt{\lambda}+1)}{2}\cdot|\sin(\alpha-\beta)|$$

令$\theta=\alpha-\beta$,则$\sqrt{2}\sin\theta+\sqrt{\lambda}\sin\alpha-\sin(\alpha-\theta)=(\sqrt{\lambda}-\cos\theta)\sin\alpha+\sin\theta\cos\alpha+\sqrt{2}\sin\theta=0$

$$\therefore\frac{|\sqrt{2}\sin\theta|}{\sqrt{(\sqrt{\lambda}-\cos\theta)^2+\sin^2\theta}}\le1\text{得}-1\le\cos\theta\le\frac{\sqrt{\lambda}-\sqrt{2+\lambda}}{2}$$

$$\therefore f(\lambda)=\frac{\sqrt{2}(\sqrt{\lambda}+1)}{2}\cdot\sqrt{1-(\frac{\sqrt{\lambda+2}-\sqrt{\lambda}}{2})^2}=\frac{1}{2}(\sqrt{\lambda}+1)\cdot\sqrt{1-\lambda+\sqrt{2\lambda+\lambda^2}}$$

$$=\frac{1+\sqrt{\lambda}}{2}\cdot\sqrt{1+\frac{2\lambda}{\sqrt{2\lambda+\lambda^2}+\lambda}}=\frac{1+\sqrt{\lambda}}{2}\cdot\sqrt{1+\frac{2}{\sqrt{\frac{2}{\lambda}+1}+1}}\in(0,\frac{1+\sqrt{2}}{2}](\text{在}1<\lambda\le2\text{上递增})$$