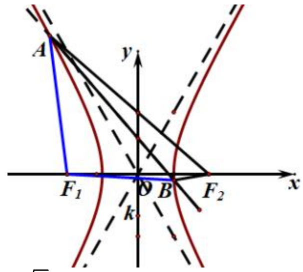
解析几何（3）双曲线解答（3）

2023-12-09

（2015 年福建）已知过点 $P(0,1)$、斜率为 k 的直线 l 与双曲线 $C: x^2-\dfrac{y^2}{3}=1$ 交于 A、B 两点.



则①k 的取值范围为________；$(-2,-\sqrt{3})\cup(-\sqrt{3},\sqrt{3})\cup(\sqrt{3},2)$

②若 F_2 为双曲线 C 的右焦点，且 $|AF_2|+|BF_2|=6$，则k 的值为______. ±1

key: 设$l: y=kx+1$，代入C方程得: $(3-k^2)x^2-2kx-4=0$

$\therefore \begin{cases} x_A+x_B=\dfrac{2k}{3-k^2} \\ x_Ax_B=\dfrac{-4}{3-k^2} \end{cases}$, 且$\Delta=12(4-k^2)>0$, 且$3-k^2\neq0$得$k\in(-2,-\sqrt{3})\cup(-\sqrt{3},\sqrt{3})\cup(\sqrt{3},2)$

$|AF_2|+|BF_2|=|-2x_A+1|+|2x_B-1|=\begin{cases} 2|x_A-x_B|, |k|<\sqrt{3}, \\ 2|x_A+x_B-1|, |k|>\sqrt{3} \end{cases}$

当$k^2<3$时，$\dfrac{4\sqrt{3}\sqrt{4-k^2}}{|3-k^2|}=6$得$k^2=1,\dfrac{11}{3}$（舍去）；

当$k^2>3$时，$2|\dfrac{2k}{3-k^2}-1|=6$得$k=-2,\dfrac{3}{2},\dfrac{1\pm\sqrt{13}}{2}$舍去，$\therefore k=\pm1$

变式1（1）已知双曲线$C: \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1(a>0,b>0)$.则

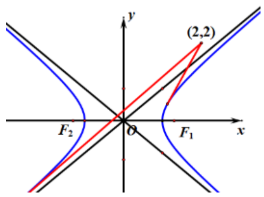
①过右焦点F的弦长的最小值为__________；$\min\{2a,\dfrac{2b^2}{a}\}$

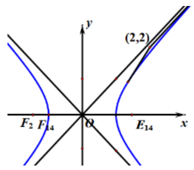
②过一点可作__________条直线与双曲线只有一个公共点.0,1,2,3,4

（2）已知双曲线 $x^2-\dfrac{y^2}{a^2}=1$，若过点 $(2,2)$ 能作该双曲线的两条切线，则该双曲线离心率 e 取值范围为

（ D ） A. $(\dfrac{\sqrt{21}}{3},+\infty)$ B. $(1,\dfrac{\sqrt{21}}{3})$ C. $(1,\sqrt{2})$ D. 以上选项均不正确

key: 由双曲线的一条渐近线为$y=ax(a>0)$, 如图，





$\therefore 2>2a$, 或$\begin{cases} 2<2a \\ 4-\dfrac{4}{a^2}<1 \end{cases}$得$0<a<1, or, 1<a<\dfrac{2}{\sqrt{3}}$

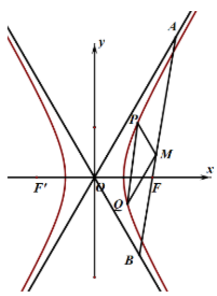
$\therefore e=\sqrt{1+a^2}\in(1,\sqrt{2})\cup(\sqrt{2},\dfrac{\sqrt{21}}{3})$

（2022II）设双曲线$C: \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1(a>0,b>0)$的右焦点为$F(2,0)$，渐近线方程为$y=\pm\sqrt{3}x$.(1) 求$C$的方程；

(2) 过F的直线与C的两条渐近线分别交于A,B两点，点$P(x_1,y_1),Q(x_2,y_2)$在C上，且$x_1>x_2>0,y_1>0$.

过P且斜率为$-\sqrt{3}$的直线与过Q且斜率为$\sqrt{3}$的直线交于点M.从下面①②③中选取两个作为条件，证明另一个成立.①M在AB上；②$PQ//AB$；③$|MA|=|MB|$.

（2022II）解: (1) 由已知得$c=2$, 且$\dfrac{b}{a}=\sqrt{3}$, $\therefore a=1,b=\sqrt{3}$, $\therefore C: x^2-\dfrac{y^2}{3}=1$



(2) 设$l_{AB}: x=ty+2(t>0)$代入$x^2-\dfrac{y^2}{3}=0$得$(3t^2-1)y^2+12ty+12=0$, $\therefore \begin{cases} y_A+y_B=\dfrac{-12t}{3t^2-1} \\ y_Ay_B=\dfrac{12}{3t^2-1} \end{cases}$,

$\therefore AB$的中点坐标为$(\dfrac{-2}{3t^2-1},\dfrac{-6t}{3t^2-1})$,

选①③为条件，②为结论，而① $\Leftrightarrow x_M=ty_M+2$

由①③得：M是AB的中点，$\therefore x_M=\dfrac{2}{1-3t^2},y_M=\dfrac{6t}{1-3t^2}$

设$P(\dfrac{1}{2}(p+\dfrac{1}{p}),\dfrac{\sqrt{3}}{2}(p-\dfrac{1}{p})),Q(\dfrac{1}{2}(q+\dfrac{1}{q}),\dfrac{\sqrt{3}}{2}(q-\dfrac{1}{q})),(p>q>0,p>1)$

则$l_{PM}:y-\dfrac{\sqrt{3}}{2}(p-\dfrac{1}{p})=-\sqrt{3}(x-\dfrac{1}{2}(p+\dfrac{1}{p})),\therefore \dfrac{6t}{1-3t^2}-\dfrac{\sqrt{3}}{2}(p-\dfrac{1}{p})=-\sqrt{3}(\dfrac{2}{1-3t^2}-\dfrac{1}{2}(p+\dfrac{1}{p}))$得$p=\dfrac{2\sqrt{3}t+2}{1-3t^2}$

同理：$\dfrac{6t}{1-3t^2}-\dfrac{\sqrt{3}}{2}(q-\dfrac{1}{q})=\sqrt{3}(\dfrac{2}{1-3t^2}-\dfrac{1}{2}(q+\dfrac{1}{q}))$得$q=\dfrac{1-3t^2}{2-2\sqrt{3}t}$

$\therefore k_{PQ}=\dfrac{\dfrac{\sqrt{3}}{2}(p-q-\dfrac{1}{p}+\dfrac{1}{q})}{\dfrac{1}{2}(p-q+\dfrac{1}{p}-\dfrac{1}{q})}=\sqrt{3}\cdot\dfrac{pq+1}{pq-1}=\sqrt{3}\cdot\dfrac{\dfrac{2\sqrt{3}t+2}{2-2\sqrt{3}t}+1}{\dfrac{2\sqrt{3}t+2}{2-2\sqrt{3}t}-1}=\sqrt{3}\cdot\dfrac{4}{4\sqrt{3}t}=\dfrac{1}{t}=k_{AB},\therefore PQ//AB$，证毕

$key2$:①②为条件，③为结论

由②设$l_{PQ}:x=ty+n$代入C方程得:$(3t^2-1)y^2+6tny+3n^2-3=0,$

$$\therefore\begin{cases}y_1+y_2=\dfrac{-6tn}{3t^2-1}\\ y_1y_2=\dfrac{3n^2-3}{3t^2-1}\end{cases},且\Delta=12(3t^2-1+n^2)>0$$

联立$l_{MP}:y-y_1=-\sqrt{3}(x-x_1)$与$l_{QM}:y-y_2=\sqrt{3}(x-x_2)$得

$$x_M=\dfrac{y_1-y_2}{2\sqrt{3}}+\dfrac{t(y_1+y_2)}{2}+n,y_M=\dfrac{\sqrt{3}t(y_1-y_2)+y_1+y_2}{2}$$

$\therefore\dfrac{y_1-y_2}{2\sqrt{3}}+\dfrac{t(y_1+y_2)}{2}+n=\dfrac{\sqrt{3}t^2(y_1-y_2)+t(y_1+y_2)}{2}+2$得$n=\sqrt{3t^2-1+n^2}+2$得$n=\dfrac{5-3t^2}{4}$

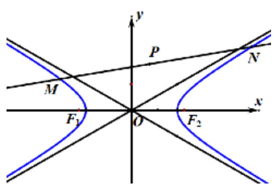
$\therefore y_M=\dfrac{\sqrt{3}t(y_1-y_2)+y_1+y_2}{2}=\dfrac{\sqrt{3}t}{2}\cdot2\sqrt{3}\cdot\dfrac{\dfrac{5-3t^2}{4}-2}{3t^2-1}-\dfrac{3t\cdot\dfrac{5-3t^2}{4}}{3t^2-1}=\dfrac{-6t}{3t^2-1}$即为$AB$的中点的纵坐标,

$\therefore|MA|=|MB|$,即②成立，证毕

（2022A）在平面直角坐标系中，双曲线$\Gamma:\dfrac{x^2}{3}-y^2=1$.对平面内不在$\Gamma$上的任意一点$P$，记$\Omega_P$为过点$P$且与$\Gamma$有两个交点的直线的全体.对任意直线$l\in\Omega_P$，记$M,N$为$l$与$\Gamma$的两个交点，定义$f_P(l)=|PM|\cdot|PN|$.若存在一条直线$l_0\in\Omega_P$满足：$l_0$与$\Gamma$的两个交点位于$y$轴异侧，且对任意直线$l\in\Omega_P,l\neq l_0$，均有$f_P(l)>f_P(l_0)$，则称$P$为“好点”.求所有好点所构成的区域的面积.

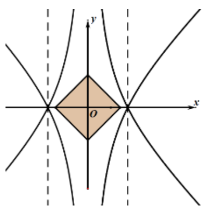
解：由题意设$P(s,t)$,直线$l:y=kx+m(t=ks+m$,且$\dfrac{s^2}{3}-t^2\neq1)$代入Γ得:$(1-3k^2)x^2-6kmx-3m^2-3=0,$

$$\therefore\begin{cases}x_M+x_N=\dfrac{6km}{1-3k^2}\\ x_Mx_N=\dfrac{-3m^2-3}{1-3k^2}\end{cases},且\Delta=12(m^2+1-3k^2)>0$$



$\therefore f_P(l)=(1+k^2)|(x_N-s)(s-x_M)|=(1+k^2)|-x_Mx_N+s(x_M+x_N)-s^2|$

$=(1+k^2)|\dfrac{3m^2+3}{1-3k^2}+\dfrac{6skm}{1-3k^2}-s^2|=(1+k^2)\cdot|\dfrac{3(t-ks)^2+3+6sk(t-ks)-s^2(1-3k^2)}{1-3k^2}|$



$=\dfrac{1+k^2}{|1-3k^2|}\cdot|3t^2-s^2+3|(k^2\neq\dfrac{1}{3})$

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当$|k|<\dfrac{1}{\sqrt{3}}$时，$\dfrac{1+k^2}{|1-3k^2|}=-1+\dfrac{4}{1-3k^2}\in[3,+\infty),\therefore f_P(l)_{\min}=|3t^2-s^2+3|$

当$|k|>\dfrac{1}{\sqrt{3}}$时，$f_P(l)=\dfrac{1+k^2}{3k^2-1}\cdot|3t^2-s^2+3|\ge|3t^2-s^2+3|$得$|k|\le1$

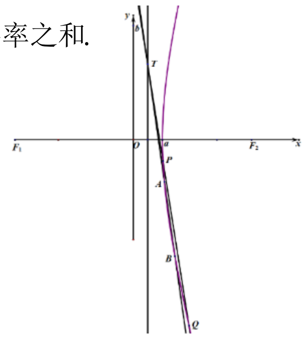
$\therefore$当$|k|>1$时，直线l与Γ不相交，

即$\Delta<0$即$(t-ks)^2+1-3k^2=(s^2-3)k^2-2stk+t^2+1\le0$对$|k|>1$恒成立，

$\therefore\begin{cases}s^2-3\le0\\ |\dfrac{st}{s^2-3}|<1\\ s^2-2st+t^2-2\le0\\ s^2+2st+t^2-2\le0\end{cases}$ 即$\begin{cases}|s|<\sqrt{3}\\ |t|<|s-\dfrac{3}{s}|,\\ |s-t|\le\sqrt{2}\\ |s+t|\le\sqrt{2}\end{cases}$，$or,\begin{cases}s^2-3<0\\ |\dfrac{st}{s^2-3}|\ge1\\ 4s^2t^2-4(s^2-3)(t^2+1)<0\text{即}3t^2-s^2+3<0\end{cases}$（无解），

$\therefore$所有好点构成的区域的面积为4.

（2021Ⅰ）在平面直角坐标系xOy中，已知点$F_1(-\sqrt{17},0),F_2(\sqrt{17},0)$，点$M$满足$|MF_1|-|MF_2|=2$，记$M$的轨迹为$C$.（Ⅰ）求$C$的方程；（Ⅱ）设点$T$在直线$x=\dfrac{1}{2}$上，过$T$的两条直线分别交$C$于$A,B$两点和$P,Q$两点，且$|TA|\cdot|TB|=|TP|\cdot|TQ|$，求直线$AB$的斜率与直线$PQ$的斜率之和.



解：（Ⅰ）由已知得轨迹C为双曲线的右支，且$c=\sqrt{17},a=1,\because b=4$，

$\therefore C$的方程为$x^2-\dfrac{y^2}{16}=1(x\ge1)$.

（Ⅱ）设$T(\dfrac{1}{2},t)$，直线AB的方程为$y-t=k_1(x-\dfrac{1}{2})$即$k_1(x-\dfrac{1}{2})-y+t=0$

直线PQ的方程为$y-t=k_2(x-\dfrac{1}{2})$即$k_2(x-\dfrac{1}{2})-y+t=0$

$\therefore$过A,B,P,Q四点的曲线系方程为$[k_1(x-\dfrac{1}{2})-y+t]\cdot[k_2(x-\dfrac{1}{2})-y+t]+\lambda(x^2-\dfrac{y^2}{16}-1)=0\cdots(*)$

由$|TA|\cdot|TB|=|TP|\cdot|TQ|$得$A,B,P,Q$四点共圆

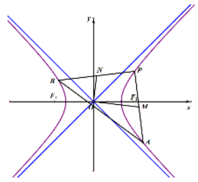
若(*)是圆方程，则$\begin{cases}k_1k_2+\lambda=1-\dfrac{\lambda}{16}\ne0\\ -k_1-k_2=0\end{cases}$，$\therefore$直线$AB$与直线$PQ$的斜率之和为0

变式1（1）已知直线l_1,l_2是过点$P(-\sqrt{2},0)$的两条互相垂直的直线，且l_1,l_2与双曲线$y^2-x^2=1$各有两个交点，分别为A_1、B_1和A_2、B_2.⑴ 求直线l_1的斜率k_1的取值范围；（2）若$|A_1B_1|=\sqrt{5}|A_2B_2|$，求$l_1$、$l_2$的方程.

$key(1)(-\sqrt{3},-1)\cup(-1,-\dfrac{\sqrt{3}}{3})\cup(\dfrac{\sqrt{3}}{3},1)\cup(1,\sqrt{3})$

$(2)y=\pm\sqrt{2}(x+\sqrt{2}),y=\pm\dfrac{\sqrt{2}}{2}(x+\sqrt{2})$

（2）等轴双曲线$x^2-y^2=a^2(a>0)$上的定点$P(x_0,y_0)$及两个动点A、B满足$\overrightarrow{PA}\cdot\overrightarrow{PB}=0$，$M$、$N$分别是$PA$、$PB$的中点则$\angle MON=$_______;$|AB|$的最小值为__________.



变式1.key:由已知得$\begin{cases}x_A^2-y_A^2=a^2\\ x_B^2-y_B^2=a^2\\ x_0^2-y_0^2=a^2\end{cases}\Rightarrow\begin{cases}(x_A-x_0)\cdot2x_M-(y_A-y_0)\cdot2y_M=0\\ (x_B-x_0)\cdot2x_N-(y_B-y_0)\cdot2y_N=0\end{cases}$

$\therefore(x_A-x_0)(x_B-x_0)\cdot x_Mx_N=(y_A-y_0)(y_B-y_0)y_My_N$

$\because \overrightarrow{PA}\cdot\overrightarrow{PB}=(x_A-x_0,y_A-y_0)\cdot(x_B-x_0,y_B-y_0)=(x_A-x_0)(x_B-x_0)+(y_A-y_0)(y_B-y_0)=0$

$\Leftrightarrow (x_A-x_0)(x_B-x_0)=-(y_A-y_0)(y_B-y_0),\therefore x_Mx_N=-y_My_N$,

$\therefore \overrightarrow{OM}\cdot\overrightarrow{ON}=0,\therefore \angle MON=90°$,

$\therefore |AB|=2|MN|=2\cdot\dfrac{|OP|}{\sin\angle ONP}=\dfrac{2\sqrt{x_0^2+y_0^2}}{\sin\angle ONP}\ge 2\sqrt{x_0^2+y_0^2}$

（3）已知双曲线 $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1(a>0,b>0)$ 的左、右焦点分别是 $F_1,F_2,P(x_1,y_1),Q(x_2,y_2)$ 是双曲线右支上的两点，$x_1+y_1=x_2+y_2=3$,.记$\triangle PQF_1,\triangle PQF_2$的周长分别为$C_1,C_2$，若$C_1-C_2=8$，则双曲线的右顶点到直线$PQ$的距离为______.

$key:C_1-C_2=|PF_1|+|QF_1|-|PF_2|-|QF_2|=4a=8$得$a=2,\therefore l_{PQ}:x+y-3=0,\therefore d=\dfrac{|2-3|}{\sqrt2}=\dfrac{\sqrt2}{2}$

（4）已知点$N(1,2)$,过点N的直线交双曲线$x^2-\dfrac{y^2}{2}=1$于A、B两点.

（Ⅰ）若$\overrightarrow{ON}=\dfrac12(\overrightarrow{OA}+\overrightarrow{OB})$,（ⅰ）求直线$AB$的方程；

（ⅱ）若过N的直线l交双曲线于C、D两点,且$\overrightarrow{CD}\cdot\overrightarrow{AB}=0$,那么$A$、$B$、$C$、$D$四点是否共圆？为什么？

（Ⅱ）求弦AB的中点P的轨迹方程.

解：（Ⅰ）（ⅰ）（利用点差法得）:AB的方程:$(x-1)-(y-2)=0$

（ⅱ）$key1:A,B,C,D$四点共圆$\Leftrightarrow|NA|\cdot|NB|=|NC|\cdot|ND|$

$\Leftrightarrow\sqrt2(x_1-1)\cdot\sqrt2(1-x_2)=\sqrt2(x_3-1)\cdot\sqrt2(1-x_4)$

$key2$:由$\overrightarrow{CD}\cdot\overrightarrow{AB}=0$得$CD$的方程:$(x-1)+(y-2)=0$,则直线$AB$与$CD$的方程为$(x-1)^2-(y-2)^2=0$,

则过A,B,C,D四点的曲线系方程为:

$(x-1)^2-(y-2)^2+\lambda(2x^2-y^2-2)=0$即$(1+2\lambda)x^2+(-1-\lambda)y^2-2x+4y-3-2\lambda=0$

令$1+2\lambda=-1-\lambda$得$\lambda=-\dfrac23$,此时曲线系方程为$-\dfrac13x^2-\dfrac13y^2-2x+4y-\dfrac53=0$

即$(x+3)^2+(y-6)^2=40$表示过A,B,C,D四点的圆，故A,B,C,D四点共圆.

（Ⅱ）设$P(x,y)$,则$\begin{cases}x_A^2-\dfrac{y_A^2}{2}=1\cdots①\\ x_B^2-\dfrac{y_B^2}{2}=1\cdots②\\ x_A+x_B=2x,且y_A+y_B=2y\\ \dfrac{y-2}{x-1}=\dfrac{y_A-y_B}{x_A-x_B}\end{cases}$，①−②得:$(x_A-x_B)\cdot2x-\dfrac12(y_A-y_B)\cdot2y=0$

$\therefore\dfrac{y-2}{x-1}=\dfrac{2x}{y}$即$2x^2-y^2-2x+2y=0$即为$P$的轨迹方程

（2014 湖南）如图，O 为坐标原点，椭圆 $C_1:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(a>b>0)$ 的左、右焦点分别为 F_1,F_2，离心率为 e_1；双曲线 $C_2:\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ 的左、右焦点分别为 F_3,F_4，离心率为 e_2．已知 $e_1e_2=\dfrac{\sqrt3}{2}$ 且 $|F_2F_4|=\sqrt3-1$

（1）求 C_1,C_2 的方程；（2）过 F_1 作 C_1 的不垂直于 y 轴的弦 AB 的中点 M,当直线 OM 与 C_2 交于 P,Q 两点时，求四边形 $APBQ$ 面积的最小值.

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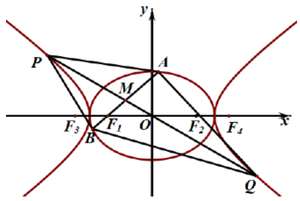


解：（1）由已知得 $\begin{cases} e_1e_2=\dfrac{\sqrt{a^2-b^2}}{a}\cdot\dfrac{\sqrt{a^2+b^2}}{a}=\dfrac{\sqrt{3}}{2} \\ \sqrt{a^2+b^2}-\sqrt{a^2-b^2}=\sqrt{3}-1 \end{cases}$ 得 $a=\sqrt{2},b=1,$

$\therefore C_1$的方程为$\dfrac{x^2}{2}+y^2=1,C_2$的方程为$\dfrac{x^2}{2}-y^2=1$

（2）设$l_{AB}:x=ty-1$代入C_1方程得$:(t^2+2)y^2-2ty-1=0,$

$\therefore \begin{cases} y_A+y_B=\dfrac{2t}{t^2+2} \\ y_Ay_B=-\dfrac{1}{t^2+2} \end{cases}$，且$\Delta=8(t^2+1)$，且$M(\dfrac{-2}{t^2+2},\dfrac{t}{t^2+2}),$

$\therefore l_{OM}:y=-\dfrac{t}{2}x$代入$C_2$方程得$x_Q^2=\dfrac{4}{2-t^2}>0$（得$0\le t^2<2$）

$\therefore S_{APBQ}=\dfrac{1}{2}\sqrt{1+t^2}\cdot\dfrac{2\sqrt{2}\sqrt{1+t^2}}{t^2+2}\cdot\dfrac{|x_Q-ty_Q-1-(-x_Q+ty_Q-1)|}{\sqrt{1+t^2}}=\dfrac{\sqrt{2}\sqrt{1+t^2}}{t^2+2}|2x_Q-2t\cdot(-\dfrac{t}{2})x_Q|$

$=\sqrt{2}\sqrt{1+t^2}\cdot\dfrac{2}{\sqrt{2-t^2}}=2\sqrt{2}\sqrt{-1+\dfrac{3}{2-t^2}}\ge 2$（当$t=0$时取$=$），$\therefore APBQ$面积的最小值为$2\sqrt{2}$

（2016福建）如图，F_1、F_2为双曲线$C:\dfrac{x^2}{4}-y^2=1$的左、右焦点，动点$P(x_0,y_0)(y_0\ge 1)$在双曲线C的右支上.

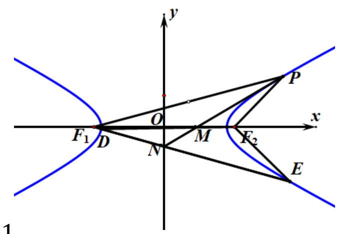
设$\angle F_1PF_2$的平分线与x轴、y轴分别交于点$M(m,0)$、N.（1）求m的取值范围；

（2）设过点F_1、N的直线l与双曲线C交于D、E两点，求$\triangle F_2DE$的面积的最大值.

2016福建key:（1）由$\angle F_1PF_2$的平分线方程为$\dfrac{x_0x}{4}-y_0y=1$得$m=\dfrac{4}{x_0},\because y_0\ge 1,\therefore x_0=2\sqrt{y_0^2+1}\ge 2,\therefore m\in(0,2]$

（2）由（1）得$N(0,\dfrac{1}{y_0}),$

由$l:\dfrac{x}{-\sqrt{5}}+\dfrac{y}{\frac{1}{y_0}}=1$即$x=-\sqrt{5}+\sqrt{5}y_0y$代入$C$得$:(5y_0^2-4)y^2-10y_0y+1=0$



$\therefore \begin{cases} y_D+y_E=\dfrac{10y_0}{5y_0^2-4} \\ y_Dy_E=\dfrac{1}{5y_0^2-4} \end{cases}$，且$\Delta=16(5y_0^2+1)>0$

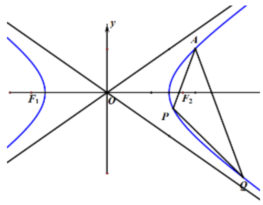
$\therefore S_{\triangle F_2DE}=\dfrac{1}{2}\cdot\sqrt{1+5y_0^2}\cdot\dfrac{4\sqrt{5y_0^2+1}}{|5y_0^2-4|}\cdot\dfrac{|\sqrt{5}+\sqrt{5}|}{\sqrt{1+5y_0^2}}=4\sqrt{5}\cdot\sqrt{\dfrac{5y_0^2+1}{(5y_0^2-4)^2}}$（令$t=\dfrac{1}{5y_0^2-4}\in(0,1]$）

$=4\sqrt{5}\cdot\sqrt{5t^2+t}\le 4\sqrt{30},\therefore\triangle F_2DE$的面积的最大值为$4\sqrt{30}$

（2022Ⅰ）已知点$A(2,1)$在双曲线$C:\dfrac{x^2}{a^2}-\dfrac{y^2}{a^2-1}=1(a>1)$上,直线$l$交$C$于$P,Q$两点,直线$AP,AQ$的斜率之和为0.

（1）求l的斜率；（2）若$\tan\angle PAQ=2\sqrt{2}$，求$\triangle PAQ$的面积.

2022Ⅰ 解:（1）由$\dfrac{4}{a^2}-\dfrac{1}{a^2-1}=1$得$a=\sqrt{2}$,设$l:y=kx+m$代入$C$得$:(1-2k^2)x^2-4kmx-2m^2-2=0$



$\therefore \begin{cases} x_P+x_Q=\dfrac{4km}{1-2k^2} \\ x_Px_Q=\dfrac{-2m^2-2}{1-2k^2} \end{cases}$，且$\Delta=8(m^2+1-2k^2)>0$

$\therefore k_{AP}+k_{AQ}=\dfrac{kx_P+m-1}{x_P-2}+\dfrac{kx_Q+m-1}{x_Q-2}=0\Leftrightarrow(kx_P+m-1)(x_Q-2)+(x_P-2)(kx_Q+m-1)$

$=2kx_Px_Q+(m-1-2k)(x_P+x_Q)-4m+4=\dfrac{2k(-2m^2-2)}{1-2k^2}+\dfrac{4km(m-1-2k)}{1-2k^2}-4m+4=0$

得$(k+1)(2k-m+1)=0,\therefore k=-1,or,2k-m+1=0$(此时$l$经过$A$，舍去)$,\therefore l$的斜率为$-1$

$key2:$设$l_{AP}:y-1=k(x-2)$代入C得$x_P=\dfrac{4k^2-4k+2}{2k^2-1},y_P=\dfrac{-4k^2+4k}{2k^2-1}+1$

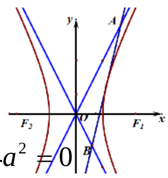
同理$x_Q=\dfrac{4k^2+4k+2}{2k^2-1},y_Q=\dfrac{-4k^2-4k}{2k^2-1}+1,\therefore k_l=\dfrac{\dfrac{-4k^2+4k}{2k^2-1}-\dfrac{-4k^2-4k}{2k^2-1}}{\dfrac{4k^2-4k+2}{2k^2-1}-\dfrac{4k^2+4k+2}{2k^2-1}}=-1$即为所求的

（2）由$k_{PA}=\tan\dfrac{\pi-\angle PAQ}{2}=\sqrt{2},\therefore x_P=\dfrac{10-4\sqrt{2}}{3},x_Q=\dfrac{10+4\sqrt{2}}{3}$

$\therefore S_{\triangle PAQ}=\dfrac{1}{2}\cdot\sqrt{3}\cdot|\dfrac{10-4\sqrt{2}}{3}-2|\cdot\sqrt{3}|\dfrac{10+4\sqrt{2}}{3}-2|\cdot\dfrac{2\sqrt{2}}{3}=\dfrac{16\sqrt{2}}{9}$

（2014 福建）已知双曲线$E:\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1(a>0,b>0)$的两条渐近线分别为$l_1:y=2x,l_2:y=-2x$.

（1）求双曲线E的离心率；（2）如图，O为坐标原点，动直线l分别交直线l_1,l_2于A,B两点（A,B分别在第一，四象限），且$\triangle OAB$的面积恒为 8，试探究：是否存在总与直线l有且只有一个公共点的双曲线E？若存在，求出双曲线E的方程；若不存在，说明理由.



解：⑴ 由已知得$\dfrac{b}{a}=2$得$e=\sqrt{5}$

(2) 假设存在，设$l_{AB}:y=kx+m,$代入E方程$4x^2-y^2=4a^2$得$:(4-k^2)x^2-2kmx-m^2-4a^2=0$

$\therefore\Delta=4k^2m^2+4(m^2+4a^2)(4-k^2)=16(m^2+4a^2-a^2k^2)=0$

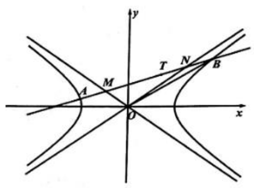
由$\begin{cases}y=2x\\y=kx+m\end{cases}$得$x_A=\dfrac{m}{2-k}>0,$由$\begin{cases}y=-2x\\y=kx+m\end{cases}$得$x=-\dfrac{m}{2+k}>0$

$\therefore S_{\triangle OAB}=\dfrac{1}{2}\sqrt{1+k^2}(\dfrac{m}{2-k}+\dfrac{m}{2+k})\cdot\dfrac{|m|}{\sqrt{1+k^2}}=\dfrac{2m^2}{k^2-4}=8$记$m^2=4(k^2-4)$

$\therefore m^2+a^2(4-k^2)=m^2-a^2\cdot\dfrac{m^2}{4}=0$得$a=2,\therefore$存在，且双曲线的方程为$\dfrac{x^2}{4}-\dfrac{y^2}{16}=1$

变式 1.如图，已知双曲线$C:\dfrac{x^2}{2}-y^2=1,$经过点$T(1,1)$且斜率为k的直线l与C交于A,B两点，与C的渐近线交于M,N两点（从左至右的顺序依次为A,M,N,B），其中$k\in(0,\dfrac{\sqrt{2}}{2})$.

（Ⅰ）若点T是MN的中点，求k的值；（Ⅱ）求$\triangle OBN$面积的最小值.



解：⑴ 由$\begin{cases}\dfrac{x^2}{2}-y^2=1\\y-1=k(x-1)\text{即}y=kx-k+1\end{cases}$得$(1-2k^2)x^2-4k(1-k)x-2(1-k)^2-2=0$

$\therefore x_A+x_B=\dfrac{4k(1-k)}{1-2k^2},$且$\Delta_1=8(-k^2-2k+2)>0$即$-1-\sqrt{3}<k<-1+\sqrt{3}$得$k\in(0,\dfrac{\sqrt{2}}{2})$

由$\begin{cases}\dfrac{x^2}{2}-y^2=0\\y-1=k(x-1)\end{cases}$得$(1-2k^2)x^2-4k(1-k)x-2(1-k)^2=0,\therefore\Delta_2=8(1-k)^2,$且$x_M+x_N=\dfrac{4k(1-k)}{1-2k^2}=2$得$k=\dfrac{1}{2}$

（2）由（1）得AB与MN的中点重合，

$$\therefore S_{\triangle OBN}=S_{\triangle OAB}-s_{\triangle OMN}=\frac{1}{2}\cdot\frac{1}{2}(|AB|-|MN|)\cdot d_{O\to AB}=\frac{1}{4}\sqrt{1+k^2}\cdot\frac{2\sqrt{2}(\sqrt{2-2k-k^2}-(1-k))}{|1-2k^2|}\cdot\frac{1-k}{\sqrt{1+k^2}}$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{[\sqrt{2-2k-k^2}-(1-k)](1-k)}{1-2k^2}=\frac{\sqrt{2}}{2}\cdot\frac{1-k}{\sqrt{2-2k-k^2}+1-k}=\frac{\sqrt{2}}{2}\cdot\frac{1}{1+\sqrt{\dfrac{2-2k-k^2}{(1-k)^2}}}$$

记$f(k)=\dfrac{2-2k-k^2}{(1-k)^2}(0<k<\dfrac{\sqrt{2}}{2})$，则$f'(k)=\dfrac{2-4k}{(1-k)^2}>0\Leftrightarrow 0<k<\dfrac{1}{2}$

$\therefore f(k)_{\max}=3,\therefore\triangle OBN$面积的最小值为$\dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{1+\sqrt{3}}=\dfrac{\sqrt{6}-\sqrt{2}}{4}$

变式 2. 已知双曲线$C:\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1(a>0,b>0)$的离心率为$\dfrac{2\sqrt{3}}{3}$，且点$(3,\sqrt{2})$在$C$上.（1）求双曲线$C$的方程；（2）试问：在双曲线$C$的右支上是否存在一点$P$，使得过点$P$作圆$x^2+y^2=1$的两条切线，切点分别为$A,B$，直线$AB$与双曲线$C$的两条渐近线分别交于点$M,N$，且$S_{\triangle MON}=\dfrac{\sqrt{3}}{33}$？若存在，求出点$P$；若不存在，请说明理由.

解：（1）由$\begin{cases}\dfrac{c}{a}=\dfrac{2}{\sqrt{3}}\\ \dfrac{9}{a^2}-\dfrac{2}{b^2}=1\end{cases}$得$a=\sqrt{3},c=2,b=1,\therefore$双曲线$C$的方程为:$\dfrac{x^2}{3}-y^2=1$

（2）假设存在，设$P(u,v)(\dfrac{u^2}{3}-v^2=1,u>0)$

则$l_{AB}:ux+vy=1$,由$\begin{cases}y=\dfrac{\sqrt{3}}{3}x\\ ux+vy=1\end{cases}$得$x_M=\dfrac{1}{u+\dfrac{\sqrt{3}}{3}v}$,

由$\begin{cases}y=-\sqrt{3}x\\ ux+vy=1\end{cases}$得$x_N=\dfrac{1}{u-\dfrac{\sqrt{3}}{3}v}$

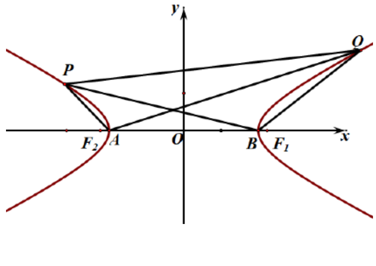
$$\therefore S_{\triangle MON}=\frac{1}{2}\cdot\sqrt{1+\frac{u^2}{v^2}}\cdot\left|\frac{1}{u+\frac{\sqrt{3}}{3}v}-\frac{1}{u-\frac{\sqrt{3}}{3}v}\right|\cdot\frac{1}{\sqrt{u^2+v^2}}=\frac{\sqrt{3}}{\frac{8}{3}u^2+1}=\frac{\sqrt{3}}{33}$$

得$u=2\sqrt{3},v=\pm\sqrt{3}.\therefore$存在,$P(2\sqrt{3},\pm\sqrt{3})$

变式 3. 已知双曲线$C:\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1(a>0,b>0)$的左、右顶点分别为$A$、$B$，渐近线方程为$y=\pm\dfrac{1}{2}x$，焦点到渐近线距离为 1，直线$l:y=kx+m$与$C$左右两支分别交于$P$，$Q$，且点$(\dfrac{2\sqrt{3}m}{3},\dfrac{2\sqrt{3}k}{3})$在双曲线$C$上.记$\triangle APQ$和$\triangle BPQ$面积分别为$S_1,S_2,AP,BQ$的斜率分别为$k_1$，$k_2$.（1）求双曲线$C$的方程；

（2）若$S_1S_2=432$，试问是否存在实数λ，使得$-k_1,\lambda k,k_2$成等比数列，若存在，求出λ的值，不存在说明理由.

解：（1）由已知得$\begin{cases}\dfrac{b}{a}=\dfrac{1}{2}\\ b=1\end{cases},\therefore a=2,\therefore C$的方程为:$\dfrac{x^2}{4}-y^2=1$



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（2）假设存在，由$\begin{cases} y=kx+m \\ x^2-4y^2=4 \end{cases}$ 消去y得：$(1-4k^2)x^2-8kmx-4m^2-4=0$

$$\therefore \begin{cases} x_P+x_Q=\dfrac{8km}{1-4k^2} \\ x_Px_Q=\dfrac{-4m^2-4}{1-4k^2} \end{cases}, \text{且}\Delta=16(m^2+1-4k^2)>0, \text{且}k^2<\frac{1}{4}$$

由$(\dfrac{2\sqrt{3}m}{3},\dfrac{2\sqrt{3}k}{3})$在$C$上得$\dfrac{m^2}{3}-\dfrac{4k^2}{3}=1$即$m^2-4k^2=3, \therefore \Delta=64,$

$$\therefore S_1S_2=\frac{1}{2}\cdot\sqrt{1+k^2}\cdot\frac{8}{1-4k^2}\cdot\frac{|-2k+m|}{\sqrt{1+k^2}}\cdot\frac{1}{2}\cdot\sqrt{1+k^2}\cdot\frac{8}{1-4k^2}\cdot\frac{|2k+m|}{\sqrt{1+k^2}}=\frac{48}{(1-4k^2)^2}=432$$

得$k^2=\dfrac{1}{6}, m^2=\dfrac{11}{3}$，且$\begin{cases} x_P+x_Q=24km \\ x_Px_Q=-56 \end{cases}$

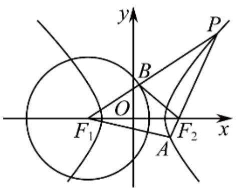
而$k_1=\dfrac{y_P}{x_P+2}, k_2=\dfrac{y_Q}{x_Q-2}$

由$-k_1,\lambda k,k_2$成等比数列，则$\lambda^2k^2=-k_1k_2$即$\lambda^2=-6k_1k_2=-6\cdot\dfrac{y_Py_Q}{(x_P+2)(x_Q-2)}$

$$=\frac{-6(kx_P+m)(kx_Q+m)}{x_Px_Q+2x_Q-2x_P-4}=\frac{-6(\frac{1}{6}\cdot(-56)+km\cdot 24km+m^2)}{-56+2\cdot\dfrac{8}{\frac{1}{3}}-4}=\frac{9}{2}, \therefore \text{存在，且}\lambda=\pm\frac{3\sqrt{2}}{2}$$

变式 4. 已知双曲线$C_1:\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$的离心率为$\sqrt{2}$，点$F_1(-c,0)$，$F_2(c,0)$分别是其左右焦点，过点$F_2$的直线交双曲线的右支于$P$，$A$两点，点$P$在第一象限.当直线$PA$的斜率不存在时，$|PA|=2\sqrt{2}$.

（1）求双曲线的标准方程；（2）线段PF_1交圆$C_2:(x+c)^2+y^2=4a^2$于点B，记$\triangle PF_2B$，$\triangle AF_2F_1$，$\triangle PAF_1$的面积分别为S_1,S_2,S，求$\dfrac{S}{S_1}+\dfrac{S}{S_2}$的最小值.



解：（1）由已知得$\begin{cases} \dfrac{c}{a}=\sqrt{2} \\ \dfrac{2b^2}{a}=2\sqrt{2} \end{cases}$ 得$a=b=\sqrt{2}, c=2, \therefore$双曲线的标准方程为$\dfrac{x^2}{2}-\dfrac{y^2}{2}=1$

（2）设$l_{PA}:x=ty+2$代入C_1方程得：$(t^2-1)y^2+4ty+2=0,$

$$\therefore \begin{cases} y_P+y_A=-\dfrac{4t}{t^2-1} \\ y_Py_A=\dfrac{2}{t^2-1} \end{cases}, \text{且}\Delta=8(t^2+1)(t^2-1\neq 0), \text{且}-2ty_Py_A=y_P+y_A$$

由$l_{PF_1}:y=\dfrac{y_P}{x_P+2}(x+2)$代入$C_2:(x+2)^2+y^2=8$得$y_B=\dfrac{2y_P}{x_P+1}=\dfrac{2y_P}{ty_P+3}$

$\therefore S=2(y_P-y_A), S_1=2(y_P-y_B), S_2=-2y_A,$

$$\therefore \frac{S}{S_1}+\frac{S}{S_2}=(y_P-y_A)\cdot(\frac{1}{y_P-y_B}+\frac{1}{-y_A})=(y_P-y_A)\cdot(\frac{ty_P+3}{y_P(ty_P+1)}-\frac{1}{y_A})$$

$$=(y_P-y_A)\cdot\frac{ty_Py_A+3y_A-y_P(ty_P+1)}{y_P(ty_Py_A+y_A)}=-\frac{2}{y_P}[(ty_P+3)\cdot\frac{-y_P}{2ty_P+1}-y_P(ty_P+1)]$$

$$=2(\frac{ty_P+3}{2ty_P+1}+ty_P+1)=\frac{5}{u}+u+2\geq 2\sqrt{5}+2(\text{令}u=2ty_P+1), \therefore \frac{S}{S_1}+\frac{S}{S_2}\text{的最小值为}2\sqrt{5}+2$$

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$key2:$ 设$\angle PF_2x=\theta(\theta\in(\frac{\pi}{4},\frac{3\pi}{4})),|PF_2|=r$，则$P(2+r\cos\theta,r\sin\theta)$

$\therefore 4+4r\cos\theta+r^2\cos^2\theta-r^2\sin^2\theta=2$即$(2\cos^2\theta-1)r^2+4r\cos\theta+2=0$得$r=|PF_2|=\frac{\sqrt{2}}{1-\sqrt{2}\cos\theta}$

同理$|AF_2|=\frac{\sqrt{2}}{1+\sqrt{2}\cos\theta},\therefore|PA|=|PF_2|+|AF_2|=\frac{2\sqrt{2}}{1-2\cos^2\theta},|PB|=|PF_1|-2\sqrt{2}=|PF_2|$

$$\therefore\frac{S}{S_1}+\frac{S}{S_2}=\frac{|PA|\cdot|PF_1|}{|PB|\cdot|PF_2|}+\frac{|PA|}{|AF_2|}=\frac{\sqrt{2}}{1+\sqrt{2}\cos\theta}\left(\frac{(3\sqrt{2}-4\cos\theta)(1-\sqrt{2}\cos\theta)}{2}+\frac{1+\sqrt{2}\cos\theta}{\sqrt{2}}\right)$$

$$=2\left(\frac{5}{1+\sqrt{2}\cos\theta}+\frac{1}{1-\sqrt{2}\cos\theta}-2\right)\geq 2\left(\frac{(\sqrt{5}+1)^2}{2}-2\right)=2+2\sqrt{5}$$

$key2':$ 设$\angle PF_2x=\theta(\theta\in(\frac{\pi}{4},\frac{3\pi}{4})),|PF_2|=r_P$，则$P(2+r_p\cos\theta,r_p\sin\theta)$

$\therefore 4+4r\cos\theta+r_P{}^2\cos^2\theta-r_P{}^2\sin^2\theta=2$即$(2\cos^2\theta-1)r_P{}^2+4r_P\cos\theta+2=0$得$r_P=\frac{\sqrt{2}}{1-\sqrt{2}\cos\theta}$

同理$r_A=|AF_2|=\frac{\sqrt{2}}{1+\sqrt{2}\cos\theta},\therefore\frac{1}{r_A}+\frac{1}{r_P}=\sqrt{2},|PA|=r_A+r_P,|PB|=|PF_1|-2\sqrt{2}=|PF_2|=r_P,$

$$\therefore\frac{S}{S_1}+\frac{S}{S_2}=\frac{|PA|\cdot|PF_1|}{|PB|\cdot|PF_2|}+\frac{|PA|}{|AF_2|}=(r_A+r_P)\left(\frac{r_P+2\sqrt{2}}{r_P^2}+\frac{1}{r_A}\right)=\sqrt{2}r_Ar_P\left(\sqrt{2}+\frac{2\sqrt{2}}{r_P^2}\right)$$

$$=\sqrt{2}(r_A+r_P)+\frac{4r_A}{r_P}=\sqrt{2}(r_A+r_P)+4r_A\left(\sqrt{2}-\frac{1}{r_A}\right)=\sqrt{2}(5r_A+r_P)-4$$

$$=\sqrt{2}\cdot\left(\frac{5}{\frac{1}{r_A}}+\frac{1}{\frac{1}{r_P}}\right)-4\geq\sqrt{2}\cdot\frac{(\sqrt{5}+1)^2}{\sqrt{2}}-4=2\sqrt{5}+2$$

$key3:$ 设$|PF_2|=m,|AF_2|=n,$则$|PF_1|=2\sqrt{2}+m,|AF_1|=2\sqrt{2}+n,$

$$\therefore\cos\angle PF_1F_2=\frac{(2\sqrt{2}+m)^2+16-m^2}{2(2\sqrt{2}+m)\cdot 4}=\frac{6+\sqrt{2}m}{2(2\sqrt{2}+m)},$$

且$\cos\angle F_1PF_2=\frac{(2\sqrt{2}+m)^2+m^2-16}{2m(2\sqrt{2}+m)}=\frac{(2\sqrt{2}+m)^2+(m+n)^2-(2\sqrt{2}+n)^2}{2(2\sqrt{2}+m)(m+n)}$得$n=\frac{m}{\sqrt{2}m-1}$

$$\therefore\frac{S}{S_1}+\frac{S}{S_2}=\frac{|PA|\cdot|PF_1|}{|PB|\cdot|PF_2|}+\frac{|PA|}{|PF_2|}=\frac{(m+n)(m+2\sqrt{2})}{(m-2\sqrt{2})m}+\frac{m+n}{m}$$

$key4:|PF_2|=\sqrt{2}x_P-\sqrt{2},|AF_2|=\sqrt{2}x_A-\sqrt{2},$设$\overrightarrow{PF_2}=\lambda\overrightarrow{F_2A}$则$\lambda=\frac{x_P-1}{x_A-1}=\frac{y_P}{-y_A}$

$$\therefore\begin{cases}x_P^2-y_P^2=2\\(\frac{x_P-1}{\lambda}+1)^2-(-\frac{y_P}{\lambda})^2=2\end{cases}\text{即}(x_P+\lambda-1)^2-y_P^2=2\lambda^2,\therefore x_P=\frac{\lambda+3}{2},|PF_2|=\frac{\sqrt{2}(\lambda+1)}{2},$$

而$\frac{S}{S_1}+\frac{S}{S_2}=\frac{|PA|\cdot|PF_1|}{|PB|\cdot|PF_2|}+\frac{|PA|}{|AF_2|}=\frac{(|PF_2|+|AF_2|)\cdot(|PF_2|+2\sqrt{2})}{|PF_2|\cdot|PF_2|}+\frac{|PF_2|+|AF_2|}{|AF_2|}$

$$=(1+\frac{1}{\lambda})(1+\frac{4}{\lambda+1})+1+\lambda=2+\frac{5}{\lambda}+\lambda\geq 5=2+2\sqrt{5}$$

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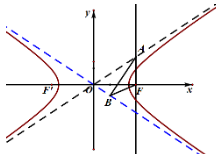
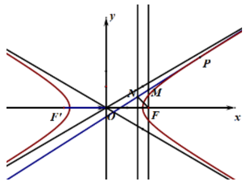


(2014江西）如图，已知双曲线$C:\dfrac{x^2}{a^2}-y^2=1(a>0)$的右焦点$F$，点$A,B$分别在$C$的两条渐近线上，$AF\perp x$轴，$AB\perp OB, BF//OA$（$O$为坐标原点）.(1) 求双曲线$C$的方程；



(2) 过C上一点$P(x_0,y_0)(y_0\neq 0)$的直线$l:\dfrac{x_0x}{a^2}-y_0y=1$与直线$AF$相交于点$M$，与直线$x=\dfrac{3}{2}$相交于点$N$，证明：点$P$在$C$上移动时，$\dfrac{|MF|}{|NF|}$恒为定值，并求此定值.

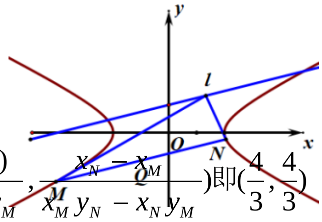
2014江西 (1) 由已知得$A(c,\dfrac{c}{a})$, 设$B(m,-\dfrac{m}{a})$, 则$\begin{cases}\dfrac{\frac{c}{a}+\frac{m}{a}}{c-m}\cdot(-\dfrac{1}{a})=-1\\ \dfrac{\frac{m}{a}}{c-m}=\dfrac{1}{a}\text{即}m=\dfrac{c}{2}\end{cases}$ 得$a=\sqrt{3}$, $\therefore C$的方程为$\dfrac{x^2}{3}-y^2=1$;

(2) 由已知得$M(2,\dfrac{2x_0-3}{3y_0}), N(\dfrac{3}{2},\dfrac{x_0-2}{2y_0})$,

$$\therefore\frac{|MF|}{|NF|}=\sqrt{\frac{(\frac{2x_0-3}{3y_0})^2}{\frac{1}{4}+\frac{(x_0-2)^2}{4y_0^2}}}=\frac{2}{3}\sqrt{\frac{4x_0^2-12x_0+9}{\frac{x_0^2}{3}-1+x_0^2-4x_0+4}}=\frac{2}{3}\sqrt{\frac{4x_0^2-12x_0+9}{\frac{4}{3}x_0^2-4x_0+3}}=\frac{2\sqrt{3}}{3}\text{为定值}$$

(2007 湖北）过点$Q(-1,-1)$作已知直线$l:y=\dfrac{1}{4}x+1$的平行线，交双曲线$\dfrac{x^2}{4}-y^2=1$于点M,N.

(1) 证明：点Q是线段MN的中点；(2) 分别过点M,N作双曲线的切线l_1,l_2，证明：三条直线l,l_1,l_2相交于同一点；(3) 设P为直线l上一动点，过点P作双曲线的切线PA,PB，切点分别为A,B.证明：点Q在直线AB上.

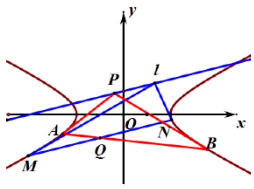


证明： (1) 由已知得$l_{MN}:y+1=\dfrac{1}{4}(x+1)$即$y=\dfrac{1}{4}x-\dfrac{3}{4}$代入双曲线方程得：

$3x^2+6x-25=0, \therefore x_M+x_N=-2=2x_Q, \therefore Q$为线段$MN$的中点

(2) 联立$l_1:\dfrac{x_Mx}{4}-y_My=1$与$\dfrac{x_Nx}{4}-y_Ny=1$得$l_1$与$l_2$的交点T的坐标为$(\dfrac{4(y_N-y_M)}{x_My_N-x_Ny_M},\dfrac{x_N-x_M}{x_My_N-x_Ny_M})$即$(\dfrac{4}{3},\dfrac{4}{3})$

而$\dfrac{4}{3}=\dfrac{1}{4}\cdot(\dfrac{4}{3})+1, \therefore T$在直线$l$上, $\therefore l,l_1,l_2$相交于同一点



(3) 设$P(t,\dfrac{1}{4}t+1)$, 而$l_{PA}:\dfrac{x_Ax}{4}-y_Ay=1, l_{PB}:\dfrac{x_Bx}{4}-y_By=1$

$\therefore l_{AB}:\dfrac{t}{4}x-(\dfrac{1}{4}t+1)y=1, \therefore \dfrac{t}{4}\cdot(-1)-(\dfrac{t}{4}+1)\cdot(-1)=1$即$Q$在$AB$上, 证毕