数列（1）等差数列与等比数列解答（2）

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（2010安徽）设数列$a_1,a_2,\cdots,a_n,\cdots$中的每一项都不为0.证明:$\{a_n\}$为等差数列的充分必要条件是：

对任何$n\in N^*$，都有$\dfrac{1}{a_1a_2}+\dfrac{1}{a_2a_3}+\cdots+\dfrac{1}{a_na_{n+1}}=\dfrac{n}{a_1a_{n+1}}$.

2010安徽证明：①必要性:$\because\{a_n\}$是等差数列，设其公差为d

$\therefore\dfrac{1}{a_ka_{k+1}}=\dfrac{1}{d}(\dfrac{1}{a_k}-\dfrac{1}{a_{k+1}}),\therefore\dfrac{1}{a_1a_2}+\dfrac{1}{a_2a_3}+\cdots+\dfrac{1}{a_na_{n+1}}=\dfrac{1}{d}(\dfrac{1}{a_1}-\dfrac{1}{a_2}+\dfrac{1}{a_2}-\dfrac{1}{a_3}+\cdots+\dfrac{1}{a_n}-\dfrac{1}{a_{n+1}})$

$=\dfrac{1}{d}(\dfrac{1}{a_1}-\dfrac{1}{a_{n+1}})=\dfrac{a_{n+1}-a_1}{da_1a_{n+1}}=\dfrac{n}{a_1a_{n+1}}$,

②充分性:$\because\dfrac{1}{a_1a_2}+\dfrac{1}{a_2a_3}+\cdots+\dfrac{1}{a_na_{n+1}}=\dfrac{n}{a_1a_{n+1}}$对任意$n\in N^*$恒成立

$\therefore\dfrac{1}{a_1a_2}+\dfrac{1}{a_2a_3}+\cdots+\dfrac{1}{a_na_{n+1}}+\dfrac{1}{a_{n+1}a_{n+2}}=\dfrac{n+1}{a_1a_{n+2}}$

$\therefore\dfrac{1}{a_{n+1}a_{n+2}}=\dfrac{n+1}{a_1a_{n+2}}-\dfrac{n}{a_1a_{n+1}}\Leftrightarrow a_1=(n+1)a_{n+1}-na_{n+2}$

$\therefore n(a_{n+2}-a_1)=(n+1)(a_{n+1}-a_1),\therefore\dfrac{a_{n+2}-a_1}{n+1}=\dfrac{a_{n+1}-a_1}{n}=\cdots=\dfrac{a_2-a_1}{1}$

$\therefore a_{n+1}=a_1+n(a_2-a_1)$,而$a_1=a_1+0(a_2-a_1)$,

$\therefore a_n=a_1+(n-1)(a_2-a_1),\therefore a_{n+1}-a_n=a_2-a_1,\therefore\{a_n\}$是等差数列.

由①②可知,$\{a_n\}$为等差数列的充要条件为$\dfrac{1}{a_1a_2}+\dfrac{1}{a_2a_3}+\cdots+\dfrac{1}{a_na_{n+1}}=\dfrac{n}{a_1a_{n+1}}$.

（2013上海）给定常数$c>0$,定义函数$f(x)=2|x+c+4|-|x+c|$,数列$a_1,a_2,a_3,\cdots$,满足$a_{n+1}=f(a_n),n\in N^*$.

(1) 若$a_1=-c-2$,求a_2及a_3；（2）求证：对任意$n\in N^*,a_{n+1}-a_n\geq c$;

(3) 是否存在a_1,使得$a_1,a_2,\cdots,a_n,\cdots$成等差数列？若存在，求出所有这样的$a_1$，若不存在，说明理由.

2013上海 (1) $a_1=-c-2,a_2=2,a_3=c+10$

(2) 当$x\geq -c$时，$f(x)-(x+c)=2(x+c+4)-(x+c)-(x+c)=8\geq 0$

当$-c-4\leq x\leq -c$时，$f(x)-(x+c)=2(x+c+4)+x+c-(x+c)=2(x+c+4)\geq 0$

当$x\leq -c-4$时, $f(x)-(x+c)=-2(x+c+4)\geq 0$,综上: $f(x)\geq x+c$

$\therefore a_{n+1}=f(a_n)\geq a_n+c$即$a_{n+1}-a_n\geq c$,

(3) 由（2）知：公差$d\geq c>0$,且存在$N_0\in N^*$,当$n>N_0$时，$a_n>0$,

此时$a_{n+1}=f(a_n)=2(a_n+c+4)-(a_n+c)=a_n+c+8,\therefore d=c+8$,

$$\text{故}a_2=f(a_1)=2|a_1+c+4|-|a_1+c|=\begin{cases}a_1+c+8,a_1\geq -c,\\3a_1+3c+8,-c-4\leq a_1<-c,=a_1+c+8\\-a_1-c-8,a_1<-c-4\end{cases}$$

得$a_1\geq -c,or,3a_1+3c+8=a_1+c+8$即$a_1=-c,or,-a_1-c-8=a_1+c+8$即$a_1=-c-8$

$\therefore a_1$的取值范围为$[-c,+\infty)\cup\{-c-8\}$

变式 1（1）已知等差数列$\{a_n\}$满足:$|a_1|+|a_2|+\cdots+|a_n|=|a_1+1|+|a_2+1|+\cdots+|a_n+1|=|a_1+2|+|a_2+2|+\cdots$

$+|a_n+2|=|a_1+3|+|a_2+3|+\cdots+|a_n+3|=2010$,则（ ）

$A.n$的最大值为50 $B.n$的最小值为50　$C.n$的最大值是51　$D.n$的最小值为51

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$key:$由$4020=\sum_{i=1}^{n}|a_i|+\sum_{i=1}^{n}|a_i+3|\ge\sum_{i=1}^{n}|2a_i+4|=4020,\therefore a_i(a_i+3)\ge 0$

由$4020=\sum_{i=1}^{n}|a_i+3|+\sum_{i=1}^{n}|a_i+1|\ge\sum_{i=1}^{n}|2a_i+4|=4020,\therefore (a_i+1)(a_i+3)\ge 0,\therefore a_i,a_i+1,a_i+2,a_i+3$同号，

不妨设公差$d>0,$则$d\ge 3,$

设$a_k<0\le a_{k+1},$则$a_k+1<0,a_k+2<0,a_k+3\le 0,a_{k+1}+1>0,a_{k+1}+2>0,a_{k+1}+3>0$

$\therefore -(a_1+\cdots+a_k)+(a_{k+1}+\cdots+a_n)=-(a_1+\cdots+a_k)-k+(a_{k+1}+\cdots+a_n)+n-k$

$=-(a_1+\cdots+a_k)-2k+(a_{k+1}+\cdots+a_n)+2(n-k)=-(a_1+\cdots+a_k)-3k+(a_{k+1}+\cdots+a_n)+3(n-k)=2010$

$$\therefore\begin{cases}n-2k=0\\ k^2d=2010,\therefore k=\sqrt{\dfrac{2010}{d}}\le\sqrt{670},\therefore n_{\max}=(2k)_{mx}=50\\ d\ge 3\end{cases}$$

（2）设等差数列 $a_1,a_2,\cdots,a_n(n\ge 3,n\in N^*)$ 的公差为 d，满足$|a_1|+|a_2|+\cdots+|a_n|=|a_1-1|+|a_2-1|+\cdots$ $+|a_n-1|=|a_1+2|+|a_2+2|+\cdots+|a_n+2|=m$，则下列说法正确的是（　　A）

A. $|d|\ge 3$　　B.n 的值可能为奇数　C.存在 $i\in N^*$，满足 $-2<a_i<1$　　D.m 的可能取值为 11

（2014山东）已知等差数列$\{a_n\}$的公差为2，前n项和为S_n，且S_1,S_2,S_4成等比数列.

⑴ 求数列$\{a_n\}$的通项公式；（2）令$b_n=(-1)^{n-1}\dfrac{4n}{a_na_{n+1}}$,求数列$\{b_n\}$的前$n$项和$T_n$.

2014山东解：⑴ 由已知得$S_2^2-S_1S_4=(2a_1+d)^2-a_1(4a_1+\dfrac{3\times 4}{2}d)$

$=4(1-a_1)=0$得$a_1=1,\therefore a_n=1+2(n-1)=2n-1$

(2) 由 ⑴ 得$b_n=(-1)^{n-1}\cdot\dfrac{4n}{(2n-1)(2n+1)}=(-1)^{n-1}(\dfrac{1}{2n-1}+\dfrac{1}{2n+1})=\dfrac{(-1)^{n-1}}{2n-1}-\dfrac{(-1)^n}{2n+1},\therefore T_n=1-\dfrac{(-1)^n}{2n+1}$

(2015湖北)5.设$a_1,a_2,\cdots,a_n\in R,n\ge 3.$若$p:a_1,a_2,\cdots,a_n$成等比数列；

$q:(a_1^2+a_2^2+\cdots+a_{n-1}^2)(a_2^2+a_3^2+\cdots+a_n^2)=(a_1a_2+a_2a_3+\cdots+a_{n-1}a_n)^2$，则（$A$）

$A.p$是q的充分条件，当不是必要条件　　$B.p$是q的必要条件，当不是q的充分条件

$C.p$是a的充分必要条件　　$D.p$既不是q的充分条件，也不是q的必要条件

2015湖北$key:$若p，则$a_n=a_1q^{n-1},\therefore (a_1^2+\cdots+a_{n-1}^2)(a_2^2+\cdots+a_n^2)=q^2(a_1^2+\cdots+a_{n-1}^2)^2$

$=(a_1^2q+\cdots+a_{n-1}^2q)^2=(a_1a_2+a_2a_3+\cdots+a_{n-1}a_n)^2,\therefore q$成立；

若q，则$(a_1^2+a_2^2)(a_2^2+a_3^2)=(a_1a_2+a_2a_3)^2\Leftrightarrow a_1^2a_3^2+a_2^4-2a_1a_2^2a_3=(a_1a_3-a_2^2)^2=0,\therefore a_1a_3=a_2^2$

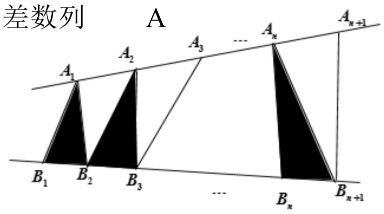
当$a_1=0$时，a_1,a_2,a_3不成等比数列$,\therefore$选A

（2016浙江）如图,$\{A_n\},\{B_n\}$分别在某锐角的两边上，且$|A_nA_{n+1}|=|A_{n+1}A_{n+2}|,A_n\ne A_{n+1},n\in N^*,$

$|B_nB_{n+1}|=|B_{n+1}B_{n+2}|,B_n\ne B_{n+1},n\in N^*,(P\ne Q$表示点$P$与$Q$不重合).若$d_n=|A_nB_n|,S_n$为$\triangle A_nB_nB_{n+1}$的面积，

则（　　）$A.\{S_n\}$是等差数列 $B.\{S_n^2\}$是等差数列 $C.\{d_n\}$是等差数列 $D.\{d_n^2\}$是等差数列　　A

2016浙江$key:S_n=\dfrac{1}{2}|B_nB_{n+1}|\cdot|A_0A_n|\sin\theta$是等差数列，$A$对



数列（1）等差数列与等比数列解答（2）

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（2017B） 设数列$\{a_n\}$是等差数列，数列$\{b_n\}$满足$b_n=a_{n+1}a_{n+2}-a_n^2, n=1,2,\cdots$.

（Ⅰ）证明：数列$\{b_n\}$也是等差数列；

（Ⅱ）设数列$\{a_n\},\{b_n\}$的公差均是$d\neq 0$，并且存在正整数s,t，使得a_s+b_t是整数，求$|a_1|$的最小值.

2017B(1) 证明：由已知设$a_n=pn+q$(p,q为实常数），

则$b_n=(p(n+1)+q)(p(n+2)+q)-(pn+q)^2$

$=p^2n^2+p(3p+2q)n+(p+q)(2p+q)-(p^2n^2+2pqn+q^2)=3p^2n+2p^2+3pq,$

$\therefore b_{n+1}-b_n=3p^2(n+1)-3p^2n=3p^2$为常数,$\therefore \{b_n\}$也是等差数列

(2) 解：由 (1) 得$p=3p^2(p\neq 0),\therefore d=p=\dfrac{1}{3},$

$\therefore a_s+b_t=\dfrac{1}{3}s+q+\dfrac{1}{3}t+\dfrac{2}{9}+q=\dfrac{s+t}{3}+\dfrac{2}{9}+2q=k+2q+\dfrac{2}{9},\dfrac{5}{9},\dfrac{8}{9}\in Z,$(而$\dfrac{s+t}{3}=k+0,\dfrac{1}{3},\dfrac{2}{3},k\in N^*$)

$\therefore$要使$a_1|=|\dfrac{1}{3}+q|$最小，只要$q\in\{-\dfrac{1}{9},\dfrac{7}{18},\dfrac{2}{9},-\dfrac{5}{18},-\dfrac{4}{9},\dfrac{1}{18}\},\therefore|a_1|$的最小值为$\dfrac{1}{18}$

(2021A) 等差数列$\{a_n\}$满足$a_{2021}=a_{20}+a_{21}=1$，则a_1的值为______. $\dfrac{1981}{4001}$

$2021Akey: a_{20}+a_{21}=a_{2021}-2001d+a_{2020}-2000d=2-4001d=1$得$d=\dfrac{1}{4001}$

$\therefore a_1=a_{2021}-2020d=1-\dfrac{2020}{4001}=\dfrac{1981}{4001}$

(2023乙)10.已知等差数列$\{a_n\}$的公差为$\dfrac{2\pi}{3}$,集合$S=\{\cos a_n\mid n\in N^*\}$,若$S=\{a,b\}$，则$ab=$（B）

$A.-1\quad B.-\dfrac{1}{2}\quad C.0\quad D.\dfrac{1}{2}$

2023乙$key: a_n=a_1+\dfrac{2(n-1)\pi}{3},\therefore\cos a_n=\cos(a_1-\dfrac{2\pi}{3}+\dfrac{2n}{3}\pi)$的周期$T=3$,且只有两个值$a,b$

而$\{a_n\}$的前3项为$\cos a_1,\cos(a_1+\dfrac{2\pi}{3}),\cos(a_1+\dfrac{4\pi}{3})$

当$\cos a_1=\cos(a_1+\dfrac{2\pi}{3})=-\dfrac{1}{2}\cos a_1-\dfrac{\sqrt{3}}{2}\sin a_1$时,$\tan a_1=-\sqrt{3},\therefore ab=-\dfrac{1}{2}$

当$\cos a_1=\cos(a_1+\dfrac{4\pi}{3})=-\dfrac{1}{2}\cos a_1+\dfrac{\sqrt{3}}{2}\sin a_1$时,$\tan a_1=\sqrt{3},\therefore ab=-\dfrac{1}{2}$

当$\cos(a_1+\dfrac{2\pi}{3})=-\dfrac{1}{2}\cos a_1-\dfrac{\sqrt{3}}{2}\sin a_1=\cos(a_1+\dfrac{4\pi}{3})=-\dfrac{1}{2}\cos a_1+\dfrac{\sqrt{3}}{2}\sin a_1$时,$\sin a_1=0,ab=-\dfrac{1}{2}$,选$B$

(2023Ⅰ)20.设等差数列$\{a_n\}$的公差为d，且$d>1$.令$b_n=\dfrac{n^2+n}{a_n}$,记S_n,T_n分别为数列$\{a_n\},\{b_n\}$的前n项和.

(1) 若$3a_2=3a_1+a_3,S_3+T_3=21$，求$\{a_n\}$的通项公式； (2) 若$\{b_n\}$为等差数列，且$S_{99}-T_{99}=99$，求d.

2023Ⅰ 解： (1) 由$\begin{cases}3(a_1+d)=3a_1+a_1+2d\text{即}a_1=d\\3a_1+3d+\dfrac{2}{a_1}+\dfrac{6}{a_1+d}+\dfrac{12}{a_1+2d}=21\end{cases}(d>1)$得$a_1=d=\dfrac{3}{2}.\therefore a_n=\dfrac{3}{2}n$

(2) 由已知设$b_n=pn+q,a_n=dn+r,$则$b_n=\dfrac{n^2+n}{a_n}=\dfrac{n^2+n}{dn+r},$

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当$r=d$时，$a_n=dn+d,b_n=\frac{1}{d}n,\therefore a_n-b_n=\frac{d^2-1}{d}n+d,\therefore S_{99}-T_{99}=\frac{99(100\cdot\frac{d^2-1}{d}+2d)}{2}$

$=99(\frac{50(d^2-1)}{d}+d)=99$得$d=\pm1$舍去

当$r=0$时，$a_n-b_n=dn-\frac{1}{d}(n+1)=\frac{d^2-1}{d}n-\frac{1}{d}$，

$\therefore S_{99}-T_{99}=\frac{99(100\cdot\frac{d^2-1}{d}-\frac{2}{d})}{2}=99\cdot\frac{50d^2-51}{d}=99$得$d=\frac{51}{50}$

变式 1（1）①已知$a_1=\frac{1}{25}$，从第10项开始大于1，则公差d的取值范围为________________．$(\frac{8}{75},\frac{3}{25}]$

②设公差为d（d 为奇数，且$d>1$）的等差数列$\{a_n\}$的前 n 项和为S_n，若$S_{m-1}=-9,S_m=0(m>3,m\in N^*)$，

则$a_n=$_____. $3n-12$

（2）① 已知等差数列$\{a_n\}$的首项为a_1，公差$d\neq0$，记S_n为数列$\{(-1)^n\cdot a_n\}$的前n项和，且存在

$k\in N^*$，使得$S_{k+1}=0$成立，则（ B ）A. $a_1d>0$　　B. $a_1d<0$　　C. $|a_1|>|d|$　　D. $|a_1|<|d|$

$key:\because d\neq0,\therefore S_{2n}=(-a_1+a_2)+(-a_3+a_4)+\cdots+(-a_{2n-1}+a_{2n})=nd\neq0,$

$\because S_{k+1}=0,\therefore k=2m,m\in N*,\therefore S_{k+1}=S_{2m+1}=-a_1+(a_2-a_3)+\cdots+(-a_{2m}+a_{2m+1})=-a_1-md=0$即$a_1=-md$

$\therefore|a_1|=m|d|\geq|d|,a_1d=-md^2<0$

②已知首项为a_1，公差为$d(d\neq0)$的等差数列$\{a_n\}$的前n项和为S_n，若存在$m\geq4,m\in N^*$，使得：

$|S_m|=a_m$，$S_{m-1}\neq0$，则下列说法不正确的是（ C ）

A．$d>0$　　B．$a_1d<0$　　C．$a_{m-1}<0$　　D．$S_{m-1}<0$

$key:|S_m|=|ma_1+\frac{m(m-1)}{2}d|=m|a_1+\frac{m-1}{2}d|=a_1+(m-1)d\geq0,$

当$a_1+\frac{m-1}{2}d\geq0$时$,(m-1)a_1+(m-1)d\cdot(\frac{m}{2}-1)=0$即$a_1+(\frac{m}{2}-1)d=0,\therefore S_{m-1}=(m-1)(a_1+\frac{m-2}{2}d=0$,矛盾

$\therefore a_1+\frac{m-1}{2}d<0$,且$-ma_1-\frac{m(m-1)}{2}d=a_1+(m-1)d$即$\frac{(m-1)d}{2}=-\frac{m+1}{m+2}a_1$

$\therefore a_1+\frac{m-1}{2}d=a_1-\frac{m+1}{m+2}a_1=\frac{1}{m+2}a_1<0$即$a_1<0,$

$\therefore a_{m-1}=a_1+(m-2)d=a_1+(m-2)\cdot\frac{-2(m+1)}{(m-1)(m+2)}a_1=\frac{-m^2+3m+2}{(m-1)(m+2)}a_1>0(\because m\geq4)$

③已知等差数列$\{a_n\}$满足$a_n>0,a_1=1$,公差为d，数列$\{b_n\}$满足$b_n=e^{a_n-2}+e^{2-a_n}$，若$\forall n\in N^*$,都有$b_n\geq b_5$,

则公差d的取值范围是（ ）$A.[\frac{2}{11},\frac{2}{9}]$　$B.[\frac{2}{9},\frac{2}{7}]$　$C.[\frac{2}{11},\frac{2}{7}]$　$D.[\frac{2}{9},\frac{2}{5}]$

key:由已知得$b_n=t+\frac{1}{t}(t=e^{dn-d-1},d>0)$

$\because b_n\geq b_5,\therefore\begin{cases}b_6\geq b_5\\b_4\geq b_5\end{cases}\Leftrightarrow\begin{cases}e^{5d-1}+e^{1-5d}\geq e^{4d-1}+e^{1-4d}\\e^{3d-1}+e^{1-3d}\geq e^{4d-1}+e^{1-4d}\end{cases}\Leftrightarrow\begin{cases}1-\frac{1}{e^{9d-2}}\geq0\\-1+\frac{1}{e^{7d-2}}\geq0\end{cases},\therefore\frac{2}{9}\leq d\leq\frac{2}{7}$

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（2005江苏）设数列$\{a_n\}$的前n项和为S_n，已知$a_1=1, a_2=6, a_3=11$，且$(5n-8)S_{n+1}-(5n+2)S_n=An+B$，$n=1,2,3,\cdots$，其中$A,B$为常数.⑴ 求$A$与$B$的值；（2）证明：数列$\{a_n\}$为等差数列；

(3) 证明：不等式$\sqrt{5a_{mn}}-\sqrt{a_m a_n}>1$对任何正整数$m,n$都成立.

2005江苏（1）$\begin{cases}-3S_2-7S_1=-28=A+B\\ 2S_3-12S_2=-48=2A+B\end{cases}$ 得$A=-20, B=-8$

（2）证明：由（1）得:$(5n-8)S_{n+1}-(5n+2)S_n=(5n-8)a_{n+1}-10S_n=-20n-8$

$\therefore 10S_n=(5n-8)a_{n+1}+20n+8, \therefore 10S_{n+1}=(5n-3)a_{n+2}+20(n+1)+8$

$\therefore 10a_{n+1}=(5n-3)a_{n+2}-(5n-8)a_{n+1}+20$即$(5n-3)a_{n+2}=(5n+2)a_n-20$

即$(5n-3)(a_{n+2}-4)=(5n+2)(a_n-4), \therefore \dfrac{a_{n+2}-4}{5n+2}=\dfrac{a_{n+1}-4}{5n-3}$

$\therefore \{\dfrac{a_{n+1}-4}{5n-3}\}$是常数数列，且$\dfrac{a_{1+1}-4}{5\times1-3}=1, \therefore a_{n+1}=5n+1$

而$a_1=5\times0+1, \therefore a_n=5(n-1)+1=5n-4, \therefore a_{n+1}-a_n=5$为常数,$\therefore \{a_n\}$是等差数列

（3）证明：由（2）得$\sqrt{5a_{mn}}-\sqrt{a_m a_n}=\sqrt{25mn-20}-\sqrt{(5m-4)(5n-4)}>1$

$\Leftrightarrow 25mn-20>(5m-4)(5n-4)+2\sqrt{(5m-4)(5n-4)}+1 \Leftrightarrow 10(m+n)-\dfrac{37}{2}>\sqrt{(5m-4)(5n-4)}$

而$\sqrt{(5m-4)(5n-4)}\le\dfrac{5m-4+5n-4}{2}=\dfrac{5(m+n)}{2}-4<10(m+n)-\dfrac{37}{2}$

$\Leftrightarrow 15(m+n)>29\cdots(*), \therefore m,n\in N^*, \therefore m+n>2, \therefore (*)$成立,$\therefore$得证

$key2:\sqrt{5a_{mn}}-\sqrt{a_m a_n}=\sqrt{25mn-20}-\sqrt{(5m-4)(5n-4)}=\dfrac{20m+20n-4}{\sqrt{25mn-20}+\sqrt{(5m-4)(5n-4)}}>1$

而$\sqrt{25mn-20}+\sqrt{(5m-4)(5n-4)}\le\sqrt{(1+1)(50mn-20(m+n)-4)}$

$\le\sqrt{100(\dfrac{m+n}{2})^2-40(m+n)-8}<\sqrt{25(m+n)^2-40(m+n)+16}=5(m+n)-4$

（2011浙江高考）已知公差不为0的等差数列$\{a_n\}$的首项a_1为$a(a\in R)$，设数列$\{a_n\}$的前n项和为S_n，且

$\dfrac{1}{a_1},\dfrac{1}{a_2},\dfrac{1}{a_4}$成等比数列.（Ⅰ）求$a_n$及$S_n$;

（Ⅱ）记$A_n=\dfrac{1}{S_1}+\dfrac{1}{S_2}+\dfrac{1}{S_3}+\cdots+\dfrac{1}{S_n}$，$B_n=\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_{2^2}}+\cdots+\dfrac{1}{a_{2^n}}$，当$n\ge2$时，试比较$A_n$与$B_n$的大小.

（Ⅰ）$a_n=na, S_n=\dfrac{an(n+1)}{2}$;

（Ⅱ）$A_n=\dfrac{1}{S_1}+\dfrac{1}{S_2}+\dfrac{1}{S_3}+\cdots+\dfrac{1}{S_n}=\dfrac{2}{a}(1-\dfrac{1}{n+1})$，$B_n=\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_{2^2}}+\cdots+\dfrac{1}{a_{2^{n-1}}}=\dfrac{1}{a}\cdot\dfrac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}=\dfrac{2}{a}(1-\dfrac{1}{2^n})$.

$\because n\ge2, \therefore 2^n=C_n^0+C_n^1+\cdots+C_n^n\ge n+2>n+1$,(或数学归纳法）

$\therefore \dfrac{1}{n+1}>\dfrac{1}{2^n}, \therefore 1-\dfrac{1}{n+1}<1-\dfrac{1}{2^n}$

当$a>0$时，$A_n<B_n$；当$a<0$时，$A_n>B_n$

数列（1）等差数列与等比数列解答（2）

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（2012浙江高考）设S_n是公差为$d(d\neq 0)$的无穷等差数列$\{a_n\}$的前n项和，则下列命题错误的是（　）C

$A.$若$d<0$，则数列$\{S_n\}$有最大项$B.$若数列$\{S_n\}$有最大项，则$d<0$

$C.$若数列$\{S_n\}$是递增数列，则对任意的$n\in N^*$，均有$S_n>0$

$D.$若对任意的$n\in N^*$，均有$S_n>0$，则数列$\{S_n\}$是递增数列

（2015浙江高考）已知$\{a_n\}$是等差数列，公差d不为零，前n项和是S_n，若a_3,a_4,a_8成等比数列，则（　）B

$A.a_1d>0,dS_4>0$　$B.a_1d<0,dS_4<0$　$C.a_1d>0,dS_4<0$　$D.a_1d<0,dS_4>0$

（2015浙江竞赛）8.若集合$A=\{(m,n)\mid(m+1)+(m+2)+\cdots+(m+n)=10^{2015},m\in Z,n\in N^*\}$，则集合$A$中的元素个数为（　）$A.4030$　$B.4032$　$C.2015^2$　$D.2016^2$　　B

$key:(m+1)+\cdots+(m+n)=\dfrac{n(2m+n+1)}{2}=10^{2015}$即$n\cdot(2m+n+1)=2^{2016}\cdot 5^{2015}$

由$n+2m+n+1=2m+2n+1$是奇数得n与$2m+n+1$是一奇一偶数，

即$\begin{cases}n=1,5,\cdots,5^{2015}\\2m+n+1=2^{2016}\cdot(5^{2015},\cdots,5,1)\end{cases}$或$\begin{cases}n=2^{2016}\cdot(1,5,\cdots,5^{2015})\\2m+n+1=5^{2015},\cdots,5,1\end{cases}$，$\therefore$选$B$

（2017II）15.等差数列$\{a_n\}$的前n项和为S_n，$a_3=3,S_4=10$，则$\sum\limits_{k=1}^{n}\dfrac{1}{S_k}=$_____.$\dfrac{2n}{n+1}$

（2018I）4.设S_n为等差数列$\{a_n\}$的前n项和，若$3S_3=S_2+S_4,a_1=2$，则$a_5=$（　）

$A.-12$　$B.-10$　$C.10$　$D.12$

$key:3S_3=9(2+d)=2a_1+d+4a_1+6d=12+7d,\therefore d=-3,a_5=2+4\cdot(-3)=-10$

（2019I）9.记S_n为等差数列$\{a_n\}$的前n项和，已知$S_4=0,a_5=5$，则（　）

$A.a_n=2n-5$　$B.a_n=3n-10$　$C.S_n=2n^2-8n$　$D.S_n=\dfrac{1}{2}n^2-2n$

$key:\begin{cases}S_4=4a_1+6d=0\\a_5=a_1+4d=5\end{cases}$得$d=2,a_1=-3,\therefore a_n=2n-5$，选$A$

（2020浙江）已知等差数列$\{a_n\}$的前n项和为S_n，公差$d\neq 0,\dfrac{a_1}{d}\le 1$，记$b_1=S_2,b_{n+1}=S_{2n+2}-S_{2n},n\in N^*$，

下列等式不可能成立的是（）$A.2a_4=a_2+a_6$　$B.2b_4=b_2+b_6$　$C.a_4^2=a_2a_8$　$D.b_4^2=b_2b_8$

（2020学考$key:S_n=na_1+\dfrac{n(n-1)}{2}d,b_1=2a_1+d$，

$b_{n+1}=a_{2n+2}+a_{2n+1}=2a_1+(4n+1)d,\therefore A,B$都对；

$b_4^2-b_2b_8=(2a_1+13d)^2-(2a_1+5d)(2a_1+29d)=-16a_1d+24d^2=24d^2(1-\dfrac{2a_1}{3d})>0,\therefore$选$D$

（2020江苏）已知递增数列$\{a_n\}$的前n项和为S_n满足$2S_n-na_n=n$.（Ⅰ）求证：数列$\{a_n\}$是等差数列；

（Ⅱ）设$b_n=\dfrac{S_{n+1}}{n}$，求证：存在唯一的正整数n，使得$a_{n+1}\le b_n<a_{n+2}$.

2020江苏证明：（1）由$2S_n-na_n=n$得$2S_{n+1}-(n+1)a_{n+1}=n+1$

$\therefore 2a_{n+1}-(n+1)a_{n+1}+na_n=1$即$(n-1)a_{n+1}=na_n-1$，且$a_1=1,a_2>1$

$\therefore (n-1)(a_{n+1}-1)=n(a_n-1),\therefore \dfrac{a_{n+1}-1}{n}=\dfrac{a_n-1}{n-1}(n\geq 2)$

$\therefore \{\dfrac{a_n-1}{n-1}\}(n\geq 2)$是常数列，$\therefore \dfrac{a_n-1}{n-1}=\dfrac{a_2-1}{1}$即$a_n=(a_2-1)(n-1)+1$

而$a_1=(a_2-1)(1-1)+1,\therefore a_{n+1}-a_n=a_2-1$为常数，$\therefore \{a_n\}$是等差数列

（2）由（1）得$b_n=\dfrac{1}{n}\cdot(n+1+\dfrac{n(n+1)}{2}(a_2-1))=\dfrac{(n+1)((a_2-1)n+2)}{2n}$

$a_{n+1}\leq b_n<a_{n+2}\Leftrightarrow 1+nd\leq \dfrac{(n+1)(dn+2)}{2n}<1+(n+1)d$（其中$d=a_2-1>0$）

$$\Leftrightarrow \begin{cases} dn^2-dn-2\leq 0 \\ dn^2+dn-2>0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{2}(1-\sqrt{1+\dfrac{8}{d}})<n\leq \dfrac{1}{2}(1+\sqrt{1+\dfrac{8}{d}}) \\ n>\dfrac{1}{2}(-1+\sqrt{1+\dfrac{8}{d}}) \end{cases} \Leftrightarrow \dfrac{1}{2}(-1+\sqrt{1+\dfrac{8}{d}})<n\leq \dfrac{1}{2}(1+\sqrt{1+\dfrac{8}{d}})$$

而$\dfrac{1}{2}(1+\sqrt{1+\dfrac{8}{d}})-\dfrac{1}{2}(-1+\sqrt{1+\dfrac{8}{d}})=1,$

$\therefore$在区间$(\dfrac{1}{2}(-1+\sqrt{1+\dfrac{8}{d}}),\dfrac{1}{2}(1+\sqrt{1+\dfrac{8}{d}})]$有唯一整数，得证

（2021 乙）19．记S_n为数列$\{a_n\}$的前n项和，b_n为数列$\{S_n\}$的前n项积，已知$\dfrac{2}{S_n}+\dfrac{1}{b_n}=2.$

（1）证明：数列$\{b_n\}$是等差数列；（2）求数列$\{a_n\}$的通项公式．

【详解】（1）由已知$\dfrac{2}{S_n}+\dfrac{1}{b_n}=2$得$S_n=\dfrac{2b_n}{2b_n-1}$，且$b_n\neq 0$，$b_n\neq \dfrac{1}{2}$，

取$n=1$，由$S_1=b_1$得$b_1=\dfrac{3}{2}$，由于b_n为数列$\{S_n\}$的前n项积，

所以$\dfrac{2b_1}{2b_1-1}\cdot\dfrac{2b_2}{2b_2-1}\cdots\dfrac{2b_n}{2b_n-1}=b_n$，所以$\dfrac{2b_1}{2b_1-1}\cdot\dfrac{2b_2}{2b_2-1}\cdots\dfrac{2b_{n+1}}{2b_{n+1}-1}=b_{n+1}$，所以$\dfrac{2b_{n+1}}{2b_{n+1}-1}=\dfrac{b_{n+1}}{b_n}$，

由于$b_{n+1}\neq 0$，所以$\dfrac{2}{2b_{n+1}-1}=\dfrac{1}{b_n}$，即$b_{n+1}-b_n=\dfrac{1}{2}$，其中$n\in N^*$

所以数列$\{b_n\}$是以$b_1=\dfrac{3}{2}$为首项，以$d=\dfrac{1}{2}$为公差等差数列；

$key2$：由已知得$b_n(\dfrac{2}{S_n}+\dfrac{1}{b_n})=2b_{n-1}+1=2b_n,\therefore b_n-b_{n-1}=\dfrac{1}{2},\therefore \{b_n\}$是公差为$\dfrac{1}{2}$的等差数列

（2）由（1）可得，数列$\{b_n\}$是以$b_1=\dfrac{3}{2}$为首项，以$d=\dfrac{1}{2}$为公差的等差数列，

$\therefore b_n=\dfrac{3}{2}+(n-1)\times\dfrac{1}{2}=1+\dfrac{n}{2}$，$S_n=\dfrac{2b_n}{2b_n-1}=\dfrac{2+n}{1+n}$，

当n=1 时，$a_1=S_1=\dfrac{3}{2}$，

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当 $n\geqslant 2$ 时，$a_n=S_n-S_{n-1}=\frac{2+n}{1+n}-\frac{1+n}{n}=-\frac{1}{n(n+1)}$，显然对于 $n=1$ 不成立，∴ $a_n=\begin{cases}\frac{3}{2}, n=1\\ -\frac{1}{n(n+1)}, n\geq 2\end{cases}$.

变式 1：已知数列$\{a_n\}$，其前n项和为S_n.（Ⅰ）若$\{a_n\}$是公差为$d(d>0)$的等差数列，且$\{\sqrt{S_n+n}\}$也是公差为d的等差数列，求a_n；（Ⅱ）若数列$\{a_n\}$对任意$m,n\in N^*$，都有$\frac{2S_{m+n}}{m+n}=a_m+a_n+\frac{a_m-a_n}{m-n}$，求证：数列$\{a_n\}$是等差数列.

（Ⅰ）$a_n=\frac{1}{2}n-\frac{5}{4}$

（Ⅱ）$\frac{2S_{m+n}}{m+n}=a_m+a_n+\frac{a_m-a_n}{m-n}$.令$m=n+1$得$\frac{2S_{2n+1}}{2n+1}=a_{n+1}+a_n+\frac{a_{n+1}-a_n}{1}=2a_{n+1}$

$\frac{2S_{2n+1}}{2n+1}=a_{n+2}+a_{n-1}+\frac{a_{n+2}-a_{n-1}}{3}=\frac{4a_{n+2}+2a_{n-1}}{3}$

$\therefore 2a_{n+1}=\frac{4a_{n+2}+2a_{n-1}}{3}$即$2a_{n+2}-3a_{n+1}+a_{n-1}=0$即$a_{n+2}+a_n-2a_{n+1}=-\frac{1}{2}(a_{n+1}+a_{n-1}-2a_n)$,

由$S_3=3a_2$得$a_3+a_1-2a_2=0,\therefore a_{n+1}+a_{n-1}-2a_n=0$即$a_{n+1}-a_n=a_n-a_{n-1}=\cdots=a_2-a_1,\therefore\{a_n\}$是等差数列