

(2) ①若实数 x, y 满足 $3x^2 + 4xy + 5y^2 = 1$, 则 $5x^2 - 4xy + 3y^2$ 的最小值为_____, 最大值为_____.

key: 令 $t = 5x^2 - 4xy + 3y^2$, 则 $t + 1 = 8(x^2 + y^2)$

$$\text{而 } 1 = 3x^2 + 5y^2 + 2 \cdot 2\lambda x \cdot \frac{1}{\lambda} y \leq 3x^2 + 5y^2 + 4\lambda^2 x^2 + \frac{1}{\lambda^2} y^2 = (3 + 4\lambda^2)x^2 + (5 + \frac{1}{\lambda^2})y^2$$

$$(\text{其中 } 3 + 4\lambda^2 = 5 + \frac{1}{\lambda^2} \text{ 即 } \lambda^2 = \frac{1 + \sqrt{5}}{4}) = (4 + \sqrt{5})(x^2 + y^2)$$

$$1 = 3x^2 + 5y^2 - 2 \cdot (-2\lambda x) \cdot \frac{1}{\lambda} y \geq 3x^2 + 5y^2 - 4\lambda^2 x^2 - \frac{1}{\lambda^2} y^2 = (3 - 4\lambda^2)x^2 + (5 - \frac{1}{\lambda^2})y^2$$

$$(\text{其中 } 3 - 4\lambda^2 = 5 - \frac{1}{\lambda^2} \text{ 即 } \lambda^2 = \frac{-1 + \sqrt{5}}{4}) = (4 - \sqrt{5})(x^2 + y^2)$$

$$\therefore t + 1 = 8(x^2 + y^2) \in [\frac{8}{4 + \sqrt{5}}, \frac{8}{4 - \sqrt{5}}], \therefore t \in [\frac{21 - 8\sqrt{5}}{11}, \frac{21 + 8\sqrt{5}}{11}].$$

②已知 $x > 0$, 设 $t = \frac{xy + 1}{x^2 + y^2 - 3y + 4}$. 当 $y = 1$ 时, t 的最大值为____; 当 $y > 0$ 时, t 的最大值为____. $\frac{\sqrt{3} + 1}{4}, 1$

$$\text{key1: } 0 = tx^2 - yx + t(y^2 - 3y + 4) - 1 = (\sqrt{t}x - \frac{y}{2\sqrt{t}})^2 - \frac{y^2}{4t} + t(y^2 - 3y + 4) - 1$$

$$\geq -\frac{y^2}{4t} + t(y^2 - 3y + 4) - 1 = (t - \frac{1}{4t})y^2 - 3ty + 4t - 1$$

$$\therefore t - \frac{1}{4t} \leq 0, \text{ 且 } \Delta = 9t^2 - 4(t - \frac{1}{4t})(4t - 1) \leq 0 \text{ 即 } 7t^3 - 4t^2 - 4t + 1 \geq 0, \therefore 0 < t \leq 1$$

$$\text{key2: } x^2 + y^2 - 3y + 4 = x^2 + \lambda^2 y^2 + (1 - \lambda^2)y^2 - 3y + 4 \geq 2\lambda xy + \frac{16(1 - \lambda^2) - 9}{4(1 - \lambda^2)} \quad (\text{其中 } 2\lambda = \frac{16(1 - \lambda^2) - 9}{4(1 - \lambda^2)} \text{ 得 } \lambda = \frac{1}{2})$$

(3) 点 M 为圆 $C: x^2 + y^2 = 20$ 上任意一点, 则点 M 到直线 $x = -8$ 与直线 $y = -1$ 的距离之积的最大值为()

A.50 B.54 C.56 D.58

$$\text{key2: } (x + 8)(y + 1) = xy + x + 8y + 8 = 2 \cdot x \cdot \frac{1}{2} y + 2 \cdot \frac{1}{2} x \cdot 1 + 2 \cdot y \cdot 4 + 8$$

$$\leq x^2 + \frac{1}{4} y^2 + \frac{1}{4} x^2 + 1 + y^2 + 16 + 8 = 50$$

练习 1. 已知 $x^2 - 2\sqrt{3}xy + 5y^2 = 1, x, y \in R$, 则 $x^2 + y^2$ 的最小值为_____. $\frac{3 - \sqrt{7}}{2}$

2. 对于 $c > 0$, 当非零实数 a, b 满足 $4a^2 - 2ab + 4b^2 - c = 0$ 且使 $|2a + b|$ 最大时, $\frac{3}{a} - \frac{4}{b} + \frac{5}{c}$

的最小值为_____. -2

$$\text{key1: 设 } t = 2a + b, \text{ 则 } b = t - 2a, \therefore 4a^2 - 2a(t - 2a) + 4(t - 2a)^2 - c = 24a^2 - 18ta + 4t^2 - c = 0$$

$$\therefore \Delta = 18^2 t^2 - 4 \times 24(4t^2 - c) \geq 0 \text{ 得 } |t| \leq \sqrt{\frac{8c}{5}} \quad (\text{当且仅当 } a = \frac{3t}{8}, b = \frac{1}{4}t \text{ 时, 取 } =)$$

$$\therefore \frac{3}{a} - \frac{4}{b} + \frac{5}{c} = \frac{8}{t} - \frac{16}{t} + \frac{5}{c} = -\frac{8}{t} + \frac{5}{c} \geq -\frac{8}{\sqrt{\frac{8c}{5}}} + \frac{5}{c} = 5(\sqrt{\frac{1}{c}})^2 - 2\sqrt{10}(\frac{1}{\sqrt{c}}) \geq \frac{-4 \times 10}{4 \times 5} = -2$$

$$\text{key2: 由已知得 } c = 4a^2 - 2ab + 4b^2 = \frac{5}{2}a^2 + \frac{5}{8}b^2 - 2ab + \frac{3}{2}(a^2 + \frac{9}{4}b^2) \geq \frac{5}{2}a^2 + \frac{5}{8}b^2 - 2ab + \frac{9}{2}ab = \frac{5}{8}(2a+b)^2$$

$$(\text{当且仅当 } a = \frac{3}{2}b \text{ 时取}) , \therefore |2a+b| \leq \sqrt{\frac{8c}{5}}, \therefore |2a+b| \text{ 取最大值时 } a = \frac{3}{2}b = \pm \frac{3}{2}\sqrt{\frac{c}{10}}$$

$$\therefore \frac{3}{a} - \frac{4}{b} + \frac{5}{c} = -\frac{2}{b} + \frac{5}{c} \geq -\frac{2}{\sqrt{\frac{c}{10}}} + \frac{5}{c} = 5(\frac{1}{\sqrt{c}} - \sqrt{\frac{2}{5}})^2 - 2 \geq -2$$

3. 已知正实数 x, y, z 满足 $x^2 + y^2 + z^2 = 1$, 则 $\frac{2-3xy}{z}$ 的最小值是_____.

$$\text{key: } 1 + \lambda = x^2 + y^2 + z^2 + \lambda \geq 2xy + 2\sqrt{\lambda}z \text{ 得 } (1 + \lambda) - 2xy \geq 2\sqrt{\lambda}z \text{ 其中 } \frac{1+\lambda}{2} = \frac{-2}{-3} \text{ 即 } \lambda = \frac{1}{3},$$

$$\therefore \frac{4}{3} - 2xy \geq \frac{2}{\sqrt{3}}z, \therefore \frac{2-3xy}{z} \geq \sqrt{3}$$

4. 实数 x, y 满足 $x^2 - xy + y^2 = 1$, 设 $S = x^2 + y^2 + xy$, 则 $\frac{1}{S_{\max}} + \frac{1}{S_{\min}} = \frac{10}{3}$.

$$\text{key: } \because 1 = x^2 + y^2 - xy \geq x^2 + y^2 - \frac{x^2 + y^2}{2} = \frac{x^2 + y^2}{2}; 1 = x^2 + y^2 + (-x)y \leq x^2 + y^2 + \frac{x^2 + y^2}{2} = \frac{3(x^2 + y^2)}{2},$$

$$\therefore \frac{2}{3} \leq x^2 + y^2 \leq 2, \therefore S + 1 = 2(x^2 + y^2) \in [\frac{4}{3}, 4], \therefore S \in [\frac{1}{3}, 3]$$

5. 已知 $a, b \in \mathbb{R}$, 且 $a^2 + b^2 - ab = 1$, 则 $a + b + ab$ 的取值范围为_____.

$$\text{key: } 1 = a^2 + b^2 - ab = (a+b)^2 - 3ab \geq (a+b)^2 - 3(\frac{a+b}{2})^2 = \frac{1}{4}(a+b)^2,$$

$$\therefore a+b \in [-2, 2], \text{ 且 } ab = \frac{(a+b)^2 - 1}{3}$$

$$\therefore a+b+ab = a+b + \frac{(a+b)^2 - 1}{3} = \frac{1}{3}(a+b + \frac{3}{2})^2 - \frac{13}{12} \in [-\frac{13}{12}, 3]$$

(2014 文科) 16. 已知实数 a, b, c 满足 $a + b + c = 0$, $a^2 + b^2 + c^2 = 1$, 则 a 的最大值为_____.

$$\text{若 } c < b < a, \text{ 则 } a \text{ 的取值范围为 } (\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3})$$

(18 竞赛 9) 设 $x, y \in \mathbb{R}$ 满足 $x - 6\sqrt{y} - 4\sqrt{x-y} + 12 = 0$, 则 x 的取值范围为_____.

$$\text{key: 柯西不等式: } (a_1a_2 + b_1b_2)^2 \leq (a_1^2 + b_1^2)(a_2^2 + b_2^2)$$

$$x + 12 = 6\sqrt{y} + 4\sqrt{x-y} \leq \sqrt{(36+16)(y+x-y)} \text{ 即得}$$

$$(2021 \text{ 浙江}) \text{ 已知 } x = u, y = v, z = \frac{2u+v-2}{\sqrt{5}}, \text{ 则 } (x^2 + y^2 + z^2)_{\min} = \text{_____}.$$

$$2021 \text{ key: } 2 = 2 \cdot x + 1 \cdot y - \sqrt{5} \cdot z \leq \sqrt{4+1+5} \cdot \sqrt{x^2 + y^2 + z^2}, \therefore x^2 + y^2 + z^2 \geq \frac{2}{5}$$

(2018 江西) 设 $x, y, z \in \mathbb{R}^+$, 满足 $x + y + z = xyz$, 则函数 $f(x, y, z) = x^2(yz-1) + y^2(zx-1) + z^2(xy-1)$ 的最小值是_____.

(2018江西) 由 $x + y + z = xyz$ 得 $yz - 1 = \frac{y+z}{x}$, $zx - 1 = \frac{x+z}{y}$, $xy - 1 = \frac{x+y}{z}$

$$\text{且 } \frac{1}{yz} + \frac{1}{xy} + \frac{1}{xz} = 1$$

$$\therefore f(x, y, z) = 2(xy + yz + zx) = 2\left(\frac{1}{\frac{1}{xy}} + \frac{1}{\frac{1}{yz}} + \frac{1}{\frac{1}{zx}}\right) \geq 2 \cdot \frac{(1+1+1)^2}{\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}} = 18$$

变式 1 (1) 若 $a, b, c \in \mathbb{R}$, 且满足 $\begin{cases} a+b+c=0, \\ a^2+3b^2+6c^2=6, \end{cases}$ 令 $M = \max\{|a|, |b|, |c|\}$, 则 M 的最大值为 $\sqrt{2}$.

(2) 已知实数 a, b, c 满足: $a+b+c=-2, abc=-4$. 则 $|a|+|b|+|c|$ 的最小值为 6

$$\text{key: } \begin{cases} a+b=-2-c \\ ab=-\frac{4}{c} \end{cases}, \text{ 则 } a, b \text{ 是关于 } x \text{ 的方程 } x^2 + (2+c)x - \frac{4}{c} = 0 \text{ 的两根,}$$

$$\text{则 } \Delta = (c+2)^2 + \frac{16}{c} \geq 0 \text{ 得 } c \leq -4, \text{ or } c > 0. \text{ 当 } c \leq -4 \text{ 时, } |a|+|b|+|c| = |a+b|+|c| = -2-2c \geq 6$$

$$\text{当 } c > 0 \text{ 时, } |a|+|b|+|c| = |a-b|+c = \sqrt{(c+2)^2 + \frac{16}{c}} + c = \sqrt{c^2 + 4c + \frac{16}{c}} + 4 + c > 6$$

$$\Leftrightarrow c \leq 6, \text{ or } c^2 + 4c + \frac{16}{c} + 4 > (c-6)^2 \text{ 成立}$$

变式 2 (1) 已知实数 a, b, c, d 满足 $a+2b+3c+6d=3, a^2+2b^2+3c^2+6d^2=5$, 则 a 的取值范围为 $\left[\frac{3-\sqrt{101}}{12}, \frac{3+\sqrt{101}}{12}\right]$.

$$\text{key: } |3-a| = |2b+3c+6d| = |\sqrt{2} \cdot \sqrt{2}b + \sqrt{3} \cdot \sqrt{3}c + \sqrt{6} \cdot \sqrt{6}d| \leq \sqrt{2+3+6} \cdot \sqrt{2b^2+3c^2+6d^2} \\ = \sqrt{11} \cdot \sqrt{5-a^2} \text{ 得 } a \in \left[\frac{3-\sqrt{101}}{12}, \frac{3+\sqrt{101}}{12}\right]$$

(2) ① 已知 $2x+3y-z=1$, 则 $(x-1)^2+2(y-2)^2+3(z+1)^2$ 的最小值为 $\frac{27}{10}$;

$$\text{key: } (x-1)^2+2(y-2)^2+3(z+1)^2 = [(x-1)^2+2(y-2)^2+3(z+1)^2] \cdot \left(1+2+\frac{1}{3}\right) \cdot \frac{3}{10} \\ \geq \frac{3}{10} (x-1+2(y-2)+z+1)^2 = \frac{27}{10}$$

② 已知 $a, b, c, d \in \mathbb{R}$, 且 $a^2+2b^2+3c^2=\frac{3}{2}$, 则 $a+2b+3c$ 的取值范围为 $[-\sqrt{3}, \sqrt{3}]$

$$\text{key: } (a+2b+3c)^2 = (1 \cdot a + \sqrt{2} \cdot \sqrt{2}b + \sqrt{3} \cdot \sqrt{3}c)^2 \leq \sqrt{1+2+3} \cdot \sqrt{a^2+2b^2+3c^2} = 3$$

③ 已知 $(x-1)^2+2(y-2)^2+3(z-3)^2 \leq 1$, 则 $x+y+z$ 的最大值为 $6+\sqrt{\frac{11}{6}}$.

$$\text{key: } \frac{11}{6} \geq [(x-1)^2+2(y-2)^2+3(z-3)^2] \cdot \left(1+\frac{1}{2}+\frac{1}{3}\right) \geq (x-1+y-2+z-3)^2, \therefore x+y+z \in \left[6-\sqrt{\frac{11}{6}}, 6+\sqrt{\frac{11}{6}}\right]$$

变式 3 (1) 已知实数 a, b 满足 $a^2-ab+b^2=3$, 则 $\frac{(1+ab)^2}{a^2+b^2+1}$ 的值域为 $\left[0, \frac{16}{7}\right]$

$$\text{key: } a^2 + b^2 = ab + 3 \geq 2ab, ab + 3 \geq -2ab, \therefore -1 \leq ab \leq 3$$

(2) 已知 x, y 均为非负实数, 且 $x + y \leq 1$, 则 $4x^2 + 4y^2 + (1 - x - y)^2$ 的取值范围为 (A)

- A. $[\frac{2}{3}, 4]$ B. $[1, 4]$ C. $[2, 4]$ D. $[2, 9]$

$$\text{key: } 4x^2 + 4y^2 + (1 - x - y)^2 = (2 + 2)(x^2 + y^2) + 1 - 2(x + y) + (x + y)^2 \geq 3(x + y)^2 - 2(x + y) + 1 \geq \frac{2}{3},$$

$$4x^2 + 4y^2 + (1 - x - y)^2 = 5x^2 - 2(1 - y)x + 4y^2 + (1 - y)^2 \quad (0 \leq x \leq 1 - y)$$

$$= \max\{5y^2 - 2y + 1, 4(1 - y)^2 + 4y^2\} \leq 4$$

(3) 若对任意正数 x, y 都有 $x + 2\sqrt{xy} \leq a(x + y)$, 则实数 a 的最小值为 _____ $\frac{\sqrt{5} + 1}{2}$

$$\text{key: } a \geq \frac{x + 2\sqrt{xy}}{x + y} = \frac{x + 2 \cdot \lambda \sqrt{x} \cdot \frac{1}{\lambda} \sqrt{y}}{x + y} \leq \frac{x + \lambda^2 x + \frac{1}{\lambda^2} y^2}{x + y} \quad (\text{其中 } 1 + \lambda^2 = \frac{1}{\lambda^2})$$

(4) 已知 $x, y, z \in \mathbb{R}^+, x + y + z = 1$, 则 $\sqrt{xy} + \sqrt{xz} - y - z$ 的最大值为 _____ $\frac{\sqrt{3} - 1}{2}$

$$\text{key: } \sqrt{xy} + \sqrt{xz} - y - z = 2 \cdot \sqrt{\lambda x} \cdot \frac{1}{2\sqrt{\lambda}} \sqrt{y} + 2 \cdot \sqrt{\lambda x} \cdot \frac{1}{2\sqrt{\lambda}} \sqrt{z} - y - z$$

$$\leq 2\lambda x + (\frac{1}{4\lambda} - 1)y + (\frac{1}{4\lambda} - 1)z = \frac{\sqrt{3} - 1}{2}(x + y + z) = \frac{\sqrt{3} - 1}{2} \quad (\text{其中 } 2\lambda = \frac{1}{4\lambda} - 1 \text{ 即 } \lambda = \frac{-1 + \sqrt{3}}{4})$$

(2017A) 已知 $a, b, c, d \in \mathbb{R}, a > 0, c > 0$, 且 $a^2 - b^2 = 2, c^2 - d^2 = 2$, 则 $ac - bd$ 的取值范围为 _____ $[2, +\infty)$

$$\text{key: 由 } a^2 - b^2 = (a + b)(a - b) = 2 \text{ 令 } \begin{cases} a + b = t \\ a - b = \frac{2}{t} \end{cases} \therefore \begin{cases} a = \frac{1}{2}(t + \frac{2}{t}) > 0 \text{ 得 } t > 0 \\ b = \frac{1}{2}(t - \frac{2}{t}) \end{cases},$$

$$\text{同理令 } \begin{cases} c = \frac{1}{2}(s + \frac{2}{s}) > 0 \text{ 得 } s > 0 \\ d = \frac{1}{2}(s - \frac{2}{s}) \end{cases}, \therefore ac - bd = \frac{s}{t} + \frac{t}{s} \geq 2$$

变式 4: 已知实数 x, y 满足 $x^2 + 2xy - 3y^2 = 1$, 则 $2x - y \in$ _____; 可以用判断式法

$$x^2 + xy \in \text{_____}; \quad x^2 + y^2 \text{ 的最小值为 } \text{_____}.$$

$$\text{key: (可因式分解代数换元)} \quad x^2 + 2xy - 3y^2 = (x + 3y)(x - y) = 1 \text{ 令 } \begin{cases} x + 3y = t \\ x - y = \frac{1}{t} \end{cases} \text{ 即 } \begin{cases} x = \frac{1}{4}(t + \frac{3}{t}) \\ y = \frac{1}{4}(t - \frac{1}{t}) \end{cases}$$

$$2x - y = \frac{t}{2} + \frac{7}{4t} \in (-\infty, -\sqrt{\frac{7}{2}}] \cup [\sqrt{\frac{7}{2}}, +\infty); \quad x^2 + xy = \frac{1}{8}(t^2 + \frac{3}{t^2} + 4) \in [\frac{\sqrt{3} + 2}{4}, +\infty); \quad x^2 + y^2 = \frac{1}{8}(t^2 + \frac{5}{t^2} + 2) \geq \frac{\sqrt{5} + 1}{4}$$

练习 1. 已知实数 x, y 满足 $x^2 - y^2 = 4$, 则 $\frac{1}{x^2} - \frac{y}{x}$ 的取值范围是 () A

A. $(-1, 1)$ B. $(-\infty, -1]$ C. $(-\infty, -1] \cup [\frac{1}{4}, +\infty)$ D. $(-1, \frac{1}{4}]$

$$\text{key: } (x-y)(x+y) = 4, \text{ 令 } \begin{cases} x-y=2t \\ x+y=\frac{2}{t} \end{cases}, \therefore \begin{cases} x=t+\frac{1}{t} \\ y=\frac{1}{t}-t \end{cases}$$

$$\therefore \frac{1}{x^2} - \frac{y}{x} = \frac{1-xy}{x^2} = \frac{t^4+t^2-1}{t^4+2t^2+1} = -u^2 - u + 1 \text{ (令 } u = \frac{1}{t^2+1} \in (0, 1)) = -(u + \frac{1}{2})^2 + \frac{5}{4} \in (-1, 1)$$

2. 已知实数 a, b 满足 $2b^2 - a^2 = 4$, 则 $|a - 2b|$ 的最小值为 _____. 2

3. 设实数 x, y 满足 $\frac{x^2}{4} - y^2 = 1$, 则 $3x^2 - 2xy$ 的最小值是 _____. $6 + 4\sqrt{2}$

4. 若实数 x, y 满足 $2x^2 + xy - y^2 = 1$, 则 $\frac{x-2y}{5x^2-2xy+2y^2}$ 的最大值为 ____.

$$\text{key: } (2x-y)(x+y) = 1 \text{ 令 } 2x-y=t, x+y=\frac{1}{t}, \text{ 则 } x=\frac{1}{3}(t+\frac{1}{t}), y=\frac{1}{3}(\frac{2}{t}-t)$$

$$\therefore \frac{x-2y}{5x^2-2xy+2y^2} = \frac{t-\frac{1}{t}}{t^2+\frac{1}{t^2}} = \frac{u}{u^2+2} \leq \frac{\sqrt{2}}{4}$$

变式 5 (1) 已知 $a+b=4 (a, b \in \mathbb{R})$, 则 $(\frac{1}{a^2+1} + \frac{1}{b^2+1})_{\max} = \frac{2+\sqrt{5}}{4}$. (均值换元)

(2) 若实数 x, y 满足 $x+y=6$, 则 $f(x, y) = (x^2+4)(y^2+4)$ 的最小值为 _____ 144 (均值换元)

$$\text{key1: } f(x, y) = x^2 y^2 + 16(x^2 + y^2) + 16 = x^2 y^2 + 4[(x+y)^2 - 2xy] + 16 = x^2 y^2 - 8xy + 160$$

$$= (xy-4)^2 + 144 \geq 144 (\because xy \leq (\frac{x+y}{2})^2 = 9)$$

$$\text{key2: 由 } x+y=6 \text{ 令 } x=3+t, y=3-t, \text{ 则 } f(x, y) = [(3+t)^2+4][(3-t)^2+4] = (t^2+6t+13)(t^2-6t+13)$$

$$= (t^2+13)^2 - 36t^2 = t^4 - 10t^2 + 169 = (t^2-5)^2 + 144 \geq 144$$

(3) 已知实数 x, y 满足 $\frac{x}{y} + \frac{y}{x} = x^2 - y^2$, 则 $x^2 + y^2$ 的最小值是 _____ 4 (比值换元)

$$\text{key: 令 } t = \frac{y}{x}, \text{ 则 } t + \frac{1}{t} = x^2 - t^2 x^2 \text{ 得 } x^2 = \frac{1+t^2}{t(1-t^2)} > 0$$

$$\therefore x^2 + y^2 = \frac{(1+t^2)^2}{t(1-t^2)} = \frac{(t+\frac{1}{t})^2}{\frac{1}{t}-t} = \frac{u^2+2u+1}{u} = u + \frac{1}{u} + 2 \geq 4 (u = \frac{1}{t} - t > 0)$$

(201501 会考) 已知 $a \in \mathbb{R}, b > 0$, 且 $(a+b)b=1$, 则 $a + \frac{2}{a+b}$ 的最小值为 _____. 2

(1710)16. 正实数 x, y 满足 $x+y=1$, 则 $\frac{1+y}{x} + \frac{1}{y}$ 的最小值是 () A. $3+\sqrt{2}$ B. $2+2\sqrt{2}$ C. 5 D. $\frac{11}{2}$ B

(1811 学考) 若实数 a, b 满足 $ab > 0$, 则 $a^2 + 4b^2 + \frac{1}{ab}$ 的最小值为 () A.8 B.6 C.4 D.2 C

(2020 天津) 14. 已知 $a > 0, b > 0$, 且 $ab = 1$, 则 $\frac{1}{2a} + \frac{1}{2b} + \frac{8}{a+b}$ 的最小值为 _____. 4

(2020 江苏) 12. 已知 $5x^2y^2 + y^4 = 1 (x, y \in \mathbb{R})$, 则 $x^2 + y^2$ 的最小值是 _____. $\frac{4}{5}$

$$\text{key1: 由已知得 } y^2(x^2 + \frac{1}{5}y^2) = \frac{1}{5}, \therefore x^2 + y^2 = x^2 + \frac{1}{5}y^2 + \frac{4}{5}y^2 \geq 2\sqrt{(x^2 + \frac{1}{5}y^2) \cdot \frac{4}{5}y^2} = \frac{4}{5}$$

$$\text{key2: 由已知得 } x^2 = \frac{1-y^4}{5y^2} \geq 0, \therefore x^2 + y^2 = \frac{1-y^4}{5y^2} + y^2 = \frac{1}{5y^2} + \frac{4}{5}y^2 \geq \frac{4}{5}$$

(2021 天津) 若 $a > 0, b > 0$, 则 $\frac{1}{a} + \frac{a}{b^2} + b$ 的最小值为 _____. $2\sqrt{2}$

(2021 上海) 已知正实数 a, b 满足 $a(a+b) = 27$, 求 a^2b 的最大值.

$$(2021 \text{ 上海}) \text{ key1: } \because a(a+b) = a^2 + ab = 27, a, b > 0, \therefore a^2b = \sqrt{a^2 \cdot \frac{1}{2}ab \cdot \frac{1}{2}ab \cdot 4} \leq \sqrt{4(\frac{a^2+ab}{3})^3} = 54$$

$$\text{key2: 由已知得 } b = \frac{27}{a} - a, \text{ 则 } a^2b = a^2(\frac{27}{a} - a) = a(27 - a^2) = \sqrt{\frac{1}{2} \cdot 2a^2(27 - a^2)(27 - a^2)} \leq 54$$

(2018 辽宁) 若正实数 x, y 满足 $x^3 + y^3 = (4x - 5y)y$, 则 y 的最大值为 ____.

$$(2018 \text{ 辽宁}) \text{ key: } y^3 + 5y^2 = 4xy - x^3 = x(4y - x^2) = \sqrt{\frac{1}{2} \cdot 2x^2(4y - x^2)(4y - x^2)}$$

$$\leq \sqrt{\frac{1}{2}(\frac{8y}{3})^3} = \frac{16y\sqrt{y}}{3\sqrt{3}} \Leftrightarrow (y+5)\sqrt{y} \leq \frac{16}{3\sqrt{3}} \Leftrightarrow y \leq \frac{1}{3}$$

变式 1 (1) ① 已知 $a > 0, b > 0, a + 2b = 1$, 则 $\frac{1}{3a+4b} + \frac{1}{a+3b}$ 取得最小值时的 $a =$ ____.

$$\text{key: 令 } 3a + 4b = x, a + 3b = y, \therefore x + 2y = 3a + 4b + 2(a + 3b) = 5$$

$$\therefore \frac{1}{3a+4b} + \frac{1}{a+3b} = \frac{1}{x} + \frac{2}{2y} \geq \frac{(1+\sqrt{2})^2}{x+2y} \geq \frac{3+2\sqrt{2}}{5} \text{ (当且仅当 } \frac{x}{x} = \frac{y}{2y} \text{ 即 } y = \frac{5}{2+\sqrt{2}} \text{ 时取=)}$$

② 已知 $a > 0, b > 0$, 且 $\frac{1}{2a+b} + \frac{1}{b+1} = 1$, 则 $a + 2b$ 的最小值为 ____.

$$\text{key: } a + 2b = \frac{1}{2}(2a+b) + \frac{3}{2}(b+1) - \frac{3}{2} = \frac{\frac{1}{2}}{\frac{1}{2a+b}} + \frac{\frac{3}{2}}{\frac{1}{b+1}} \geq \frac{(\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}})^2}{\frac{1}{2a+b} + \frac{1}{b+1}}$$

$$\geq \frac{1}{2} + \sqrt{3} \text{ (当且仅当 } \frac{(2a+b)^2}{2} = \frac{3(b+1)^2}{2} \text{ 时, 取=)}$$

③ 若正实数 a, b, c 满足 $ab = a + 2b$, $abc = a + 2b + c$, 则 c 的最大值为 _____. $\frac{8}{7}$

$$\text{key}: c = \frac{a+2b}{ab-1} = \frac{a+2b}{a+2b-1} = 1 + \frac{1}{ab-1} \leq 1 + \frac{1}{7} = \frac{8}{7} \text{ (而 } ab = a+2b \geq 2\sqrt{2ab} \text{ 得 } ab \geq 8)$$

④ 已知实数 x, y 满足 $x > y > 0$, 且 $x + y \leq 2$, 则 $\frac{2}{x+3y} + \frac{1}{x-y}$ 的最小值为_____.

$$\text{key}: \text{令 } a = x + 3y, b = x - y > 0, \text{ 则 } a + b = 2(x + y) \leq 4$$

$$\therefore \frac{2}{x+3y} + \frac{1}{x-y} = \frac{2}{a} + \frac{1}{b} \geq \frac{(\sqrt{2}+1)^2}{a+b} \geq \frac{3+2\sqrt{2}}{4} \text{ (当且仅当 } x = \frac{\sqrt{2}+3}{4}, y = \frac{\sqrt{2}-1}{4} \text{ 时取=)}$$

⑤ 已知正实数 x, y 满足 $xy + 2x + 3y = 42$, 则 $xy + 5x + 4y$ 的最小值为_____.55

$$\text{key}: (x+3)(y+2) = 48 \text{ 令 } a = x+3 > 3, b = y+2 > 2, \text{ 则 } ab = 48$$

$$\text{则 } xy + 5x + 4y = (a-3)(b-2) + 5(a-3) + 4(b-2) = 3a + b + 31 \geq 2\sqrt{3ab} + 31 = 55$$

⑥ 已知 $a, b > 0$, 且 $(a+b)(a+2b) + a + b = 9$, 则 $(3a+4b)_{\min} =$ _____.

$$\text{key}: (a+b)(a+2b) + a + b = (a+b)(a+2b+1) = 9$$

$$\text{令 } \begin{cases} a+b=x > 0 \\ a+2b+1=y > 0 \end{cases} \text{ 得 } \begin{cases} a=2x-y+1 \\ b=-x+y-1 \end{cases}, \text{ 且 } xy=9, \therefore 3a+4b=2x+y-1 \geq 6\sqrt{2}-1$$

(2) ① 函数 $y = \frac{1}{x} + \frac{2}{1-x}$ ($0 < x < 1$) 的值域为_____.

$$\text{key}: \frac{1}{x} + \frac{2}{1-x} = \frac{x+1-x}{x} + \frac{2(x+1-x)}{1-x} = 3 + \frac{1-x}{x} + \frac{2x}{1-x} \text{ (} t = \frac{1-x}{x} = \frac{1}{x} - 1 \in (0, +\infty) \text{)} \in [3 + 2\sqrt{2}, +\infty)$$

函数 $y = \frac{2}{x+1} + \frac{x}{2x+1}$ ($0 \leq x \leq 2$) 的值域为_____.

$$\text{key}: y = 2 \cdot \frac{2(x+1) - (2x+1)}{x+1} + \frac{2x+1 - (x+1)}{2x+1} = 5 - (2 \cdot \frac{2x+1}{x+1} + \frac{x+1}{2x+1}) \text{ (} t = \frac{2x+1}{x+1} = 2 - \frac{1}{x+1} \in [1, \frac{5}{3}] \text{)}$$

$$= 5 - (2t + \frac{1}{t}) \in [\frac{16}{15}, 2]$$

② 函数 $f(x) = \frac{(2x+1)^2}{x(2-x)}$ ($0 < x < 2$) 的最小值为_____;

$$\text{key1: (分母不动, 分子用分母表示, 齐次)} f(x) = \frac{(2x + \frac{2-x+x}{2})^2}{x(2-x)} = \frac{1}{4} (\frac{25x}{2-x} + \frac{2-x}{x} + 10) \geq 5 \text{ (当且仅当 } x = \frac{1}{3} \text{ 时, 取=)}$$

$$\text{key2: } f(x) = \lambda \cdot \frac{(2x+1)^2}{\lambda x(2-x)} \geq \lambda \cdot \frac{(2x+1)^2}{(\frac{\lambda x + 2-x}{2})^2} = 5 \text{ (其中 } \lambda - 1 = 4 \text{ 即 } \lambda = 5)$$

函数 $f(x) = \frac{2x^2+1}{x(2-x)}$ ($0 < x < 2$) 的最小值为_____.

$$\text{key: (分母不动, 分子用分母表示, 齐次)} f(x) = \frac{2x^2 + (\frac{x+2-x}{2})^2}{x(2-x)}$$

$$= \frac{1}{4} (\frac{9x}{2-x} + \frac{2-x}{x}) + \frac{1}{2} \geq 2 \text{ (当且仅当 } x = \frac{2}{3} \text{ 时, 取=)}$$