（2）①若实数x,y满足$3x^2+4xy+5y^2=1$，则$5x^2-4xy+3y^2$的最小值为______，最大值为_______.

$key:$令$t=5x^2-4xy+3y^2$，则$t+1=8(x^2+y^2)$

而$1=3x^2+5y^2+2\cdot 2\lambda x\cdot\frac{1}{\lambda}y\le 3x^2+5y^2+4\lambda^2x^2+\frac{1}{\lambda^2}y^2=(3+4\lambda^2)x^2+(5+\frac{1}{\lambda^2})y^2$

（其中$3+4\lambda^2=5+\frac{1}{\lambda^2}$即$\lambda^2=\frac{1+\sqrt{5}}{4}$）$=(4+\sqrt{5})(x^2+y^2)$

$1=3x^2+5y^2-2\cdot(-2\lambda x)\cdot\frac{1}{\lambda}y\ge 3x^2+5y^2-4\lambda^2x^2-\frac{1}{\lambda^2}y^2=(3-4\lambda^2)x^2+(5-\frac{1}{\lambda^2})y^2$

（其中$3-4\lambda^2=5-\frac{1}{\lambda^2}$即$\lambda^2=\frac{-1+\sqrt{5}}{4}$）$=(4-\sqrt{5})(x^2+y^2)$

$\therefore t+1=8(x^2+y^2)\in[\frac{8}{4+\sqrt{5}},\frac{8}{4-\sqrt{5}}],\therefore t\in[\frac{21-8\sqrt{5}}{11},\frac{21+8\sqrt{5}}{11}].$

②已知$x>0$，设$t=\frac{xy+1}{x^2+y^2-3y+4}$.当$y=1$时，$t$的最大值为___；当$y>0$时，$t$的最大值为___.　$\frac{\sqrt{3}+1}{4}$,1

$key1:0=tx^2-yx+t(y^2-3y+4)-1=(\sqrt{t}x-\frac{y}{2\sqrt{t}})^2-\frac{y^2}{4t}+t(y^2-3y+4)-1$

$\ge-\frac{y^2}{4t}+t(y^2-3y+4)-1=(t-\frac{1}{4t})y^2-3ty+4t-1$

$\therefore t-\frac{1}{4t}\le 0$，且$\Delta=9t^2-4(t-\frac{1}{4t})(4t-1)\le 0$即$7t^3-4t^2-4t+1\ge 0,\therefore 0<t\le 1$

$key2:x^2+y^2-3y+4=x^2+\lambda^2y^2+(1-\lambda^2)y^2-3y+4\ge 2\lambda xy+\frac{16(1-\lambda^2)-9}{4(1-\lambda^2)}$（其中$2\lambda=\frac{16(1-\lambda^2)-9}{4(1-\lambda^2)}$得$\lambda=\frac{1}{2}$）

（3）点M为圆$C:x^2+y^2=20$上任意一点，则点M到直线$x=-8$与直线$y=-1$的距离之积 的最大值为（　）

$A.50$　　　$B.54$　　　$C.56$　　　$D.58$

$key2:(x+8)(y+1)=xy+x+8y+8=2\cdot x\cdot\frac{1}{2}y+2\cdot\frac{1}{2}x\cdot 1+2\cdot y\cdot 4+8$

$\le x^2+\frac{1}{4}y^2+\frac{1}{4}x^2+1+y^2+16+8=50$

练习 1.已知$x^2-2\sqrt{3}xy+5y^2=1,x,y\in R$，则$x^2+y^2$的最小值为______.　$\frac{3-\sqrt{7}}{2}$

2.对于$c>0$，当非零实数a,b满足$4a^2-2ab+4b^2-c=0$且使$|2a+b|$最大时，$\frac{3}{a}-\frac{4}{b}+\frac{5}{c}$的最小值为______. -2

$key1:$设$t=2a+b$，则$b=t-2a,\therefore 4a^2-2a(t-2a)+4(t-2a)^2-c=24a^2-18ta+4t^2-c=0$

$\therefore\Delta=18^2t^2-4\times 24(4t^2-c)\ge 0$得$|t|\le\sqrt{\frac{8c}{5}}$（当且仅当$a=\frac{3t}{8},b=\frac{1}{4}t$时，取$=$）

$\therefore\frac{3}{a}-\frac{4}{b}+\frac{5}{c}=\frac{8}{t}-\frac{16}{t}+\frac{5}{c}=-\frac{8}{t}+\frac{5}{c}\ge-\frac{8}{\sqrt{\frac{8c}{5}}}+\frac{5}{c}=5(\sqrt{\frac{1}{c}})^2-2\sqrt{10}(\frac{1}{\sqrt{c}})\ge\frac{-4\times 10}{4\times 5}=-2$

$key2:$由已知得$c=4a^2-2ab+4b^2=\frac{5}{2}a^2+\frac{5}{8}b^2-2ab+\frac{3}{2}(a^2+\frac{9}{4}b^2)\geq\frac{5}{2}a^2+\frac{5}{8}b^2-2ab+\frac{9}{2}ab=\frac{5}{8}(2a+b)^2$

（当且仅当$a=\frac{3}{2}b$时取$=$），$\therefore|2a+b|\leq\sqrt{\frac{8c}{5}},\therefore|2a+b|$取最大值时$a=\frac{3}{2}b=\pm\frac{3}{2}\sqrt{\frac{c}{10}}$

$\therefore\frac{3}{a}-\frac{4}{b}+\frac{5}{c}=-\frac{2}{b}+\frac{5}{c}\geq-\frac{2}{\sqrt{\frac{c}{10}}}+\frac{5}{c}=5(\frac{1}{\sqrt{c}}-\sqrt{\frac{2}{5}})^2-2\geq-2$

3.已知正实数 x,y,z 满足 $x^2+y^2+z^2=1$，则 $\frac{2-3xy}{z}$ 的最小值是_______.

$key:1+\lambda=x^2+y^2+z^2+\lambda\geq2xy+2\sqrt{\lambda}z$得$(1+\lambda)-2xy\geq2\sqrt{\lambda}z$其中$\frac{1+\lambda}{2}=\frac{-2}{-3}$即$\lambda=\frac{1}{3}$,

$\therefore\frac{4}{3}-2xy\geq\frac{2}{\sqrt{3}}z,\therefore\frac{2-3xy}{z}\geq\sqrt{3}$

4.实数 x,y 满足 $x^2-xy+y^2=1$，设 $S=x^2+y^2+xy$，则 $\frac{1}{S_{\max}}+\frac{1}{S_{\min}}=$______. $\frac{10}{3}$

$key:\because1=x^2+y^2-xy\geq x^2+y^2-\frac{x^2+y^2}{2}=\frac{x^2+y^2}{2};1=x^2+y^2+(-x)y\leq x^2+y^2+\frac{x^2+y^2}{2}=\frac{3(x^2+y^2)}{2},$

$\therefore\frac{2}{3}\leq x^2+y^2\leq2,\therefore S+1=2(x^2+y^2)\in[\frac{4}{3},4],\therefore S\in[\frac{1}{3},3]$

5.已知$a,b\in R,$且$a^2+b^2-ab=1$，则$a+b+ab$的取值范围为_______.

$key:1=a^2+b^2-ab=(a+b)^2-3ab\geq(a+b)^2-3(\frac{a+b}{2})^2=\frac{1}{4}(a+b)^2,$

$\therefore a+b\in[-2,2]$，且$ab=\frac{(a+b)^2-1}{3}$

$\therefore a+b+ab=a+b+\frac{(a+b)^2-1}{3}=\frac{1}{3}(a+b+\frac{3}{2})^2-\frac{13}{12}\in[-\frac{13}{12},3]$

（2014 文科）16.已知实数 a,b,c 满足 $a+b+c=0$，$a^2+b^2+c^2=1$，则 a 的最大值为____. $\frac{\sqrt{6}}{3}$

若$c<b<a$，则a的取值范围为______ $(\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3})$

（18 竞赛 9）设$x,y\in R$满足$x-6\sqrt{y}-4\sqrt{x-y}+12=0$，则$x$的取值范围为_____. $[14-2\sqrt{13},14+2\sqrt{13}]$

$key:$柯西不等式:$(a_1a_2+b_1b_2)^2\leq(a_1^2+b_1^2)(a_2^2+b_2^2)$
$x+12=6\sqrt{y}+4\sqrt{x-y}\leq\sqrt{(36+16)(y+x-y)}$即得

（2021浙江）已知$x=u,y=v,z=\frac{2u+v-2}{\sqrt{5}}$，则$(x^2+y^2+z^2)_{\min}=$_____.

$2021key:2=2\cdot x+1\cdot y-\sqrt{5}\cdot z\leq\sqrt{4+1+5}\cdot\sqrt{x^2+y^2+z^2},\therefore x^2+y^2+z^2\geq\frac{2}{5}$

（2018江西）设$x,y,z\in R^+$，满足$x+y+z=xyz$，则函数$f(x,y,z)=x^2(yz-1)+y^2(zx-1)+z^2(xy-1)$的最小值是_______.

（2018江西）由$x+y+z=xyz$得$yz-1=\dfrac{y+z}{x},zx-1=\dfrac{x+z}{y},xy-1=\dfrac{x+y}{z}$

且$\dfrac{1}{yz}+\dfrac{1}{xy}+\dfrac{1}{xy}=1$

$\therefore f(x,y,z)=2(xy+yz+zx)=2(\dfrac{1}{\frac{1}{xy}}+\dfrac{1}{\frac{1}{yz}}+\dfrac{1}{\frac{1}{zx}})\ge 2\cdot\dfrac{(1+1+1)^2}{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}}=18$

变式 1（1）若$a,b,c\in R$,且满足$\begin{cases}a+b+c=0,\\a^2+3b^2+6c^2=6,\end{cases}$令$M=\max\{|a|,|b|,|c|\}$，则$M$的最大值为_____. $\sqrt{2}$

（2） 已知实数a,b,c满足：$a+b+c=-2$，$abc=-4$.则$|a|+|b|+|c|$的最小值为____6

$key:\begin{cases}a+b=-2-c\\ab=-\dfrac{4}{c}\end{cases}$,则$a,b$是关于$x$的方程$x^2+(2+c)x-\dfrac{4}{c}=0$的两根，

则$\Delta=(c+2)^2+\dfrac{16}{c}\ge 0$得$c\le -4,or,c>0$.当$c\le -4$时,$|a|+|b|+|c|=|a+b|+|c|=-2-2c\ge 6$

当$c>0$时,$|a|+|b|+|c|=|a-b|+c=\sqrt{(c+2)^2+\dfrac{16}{c}}+c=\sqrt{c^2+4c+\dfrac{16}{c}+4}+c>6$

$\Leftrightarrow c\le 6,or,c^2+4c+\dfrac{16}{c}+4>(c-6)^2$成立

变式 2（1）已知实数a,b,c,d满足$a+2b+3c+6d=3,a^2+2b^2+3c^2+6d^2=5$，则$a$的取值范围为____.

$key:|3-a|=|2b+3c+6d|=|\sqrt{2}\cdot\sqrt{2}b+\sqrt{3}\cdot\sqrt{3}c+\sqrt{6}\cdot\sqrt{6}d|\le\sqrt{2+3+6}\cdot\sqrt{2b^2+3c^2+6d^2}$
$=\sqrt{11}\cdot\sqrt{5-a^2}$得$a\in[\dfrac{3-\sqrt{101}}{12},\dfrac{3+\sqrt{101}}{12}]$

（2）①已知$2x+3y-z=1$，则$(x-1)^2+2(y-2)^2+3(z+1)^2$的最小值为_______；

$key:(x-1)^2+2(y-2)^2+3(z+1)^2=[(x-1)^2+2(y-2)^2+3(z+1)^2]\cdot(1+2+\dfrac{1}{3})\cdot\dfrac{3}{10}$
$\ge\dfrac{3}{10}(x-1+2(y-2)+z+1)^2=\dfrac{27}{10}$

②已知$a,b,c,d\in R$，且$a^2+2b^2+3c^2=\dfrac{3}{2}$,则$a+2b+3c$的取值范围为_____，$[-\sqrt{3},\sqrt{3}]$

$key:(a+2b+3c)^2=(1\cdot a+\sqrt{2}\cdot\sqrt{2}b+\sqrt{3}\cdot\sqrt{3}c)^2\le\sqrt{1+2+3}\cdot\sqrt{a^2+2b^2+3c^2}=3$

③已知$(x-1)^2+2(y-2)^2+3(z-3)^2\le 1$，则$x+y+z$的最大值为____.

$key:\dfrac{11}{6}\ge[(x-1)^2+2(y-2)^2+3(z-3)^2]\cdot(1+\dfrac{1}{2}+\dfrac{1}{3})\ge(x-1+y-2+z-3)^2,\therefore x+y+z\in[6-\sqrt{\dfrac{11}{6}},6+\sqrt{\dfrac{11}{6}}]$

变式 3（1）已知实数a,b满足$a^2-ab+b^2=3$，则$\dfrac{(1+ab)^2}{a^2+b^2+1}$的值域为_________. $[0,\dfrac{16}{7}]$

$key: a^2+b^2=ab+3\ge 2ab, ab+3\ge -2ab, \therefore -1\le ab\le 3$

（2）已知 x, y 均为非负实数，且 $x+y\le 1$, 则$4x^2+4y^2+(1-x-y)^2$ 的取值范围为 （A ）

A. $[\frac{2}{3},4]$　　B. $[1,4]$　　C. $[2,4]$　　D. $[2,9]$

$key: 4x^2+4y^2+(1-x-y)^2=(2+2)(x^2+y^2)+1-2(x+y)+(x+y)^2\ge 3(x+y)^2-2(x+y)+1\ge\frac{2}{3}$,

$4x^2+4y^2+(1-x-y)^2=5x^2-2(1-y)x+4y^2+(1-y)^2(0\le x\le 1-y)$

$=\max\{5y^2-2y+1, 4(1-y)^2+4y^2\}\le 4$

（3）若对任意正数 x, y 都有 $x+2\sqrt{xy}\le a(x+y)$,则实数 a 的最小值为______ $\frac{\sqrt{5}+1}{2}$

$key: a\ge\frac{x+2\sqrt{xy}}{x+y}=\frac{x+2\cdot\lambda\sqrt{x}\cdot\frac{1}{\lambda}\sqrt{y}}{x+y}\le\frac{x+\lambda^2x+\frac{1}{\lambda^2}y^2}{x+y}$(其中$1+\lambda^2=\frac{1}{\lambda^2}$)

（4）已知$x, y, z\in R^+, x+y+z=1$, 则$\sqrt{xy}+\sqrt{xz}-y-z$的最大值为______. $\frac{\sqrt{3}-1}{2}$

$key: \sqrt{xy}+\sqrt{yz}-y-z=2\cdot\sqrt{\lambda x}\cdot\frac{1}{2\sqrt{\lambda}}\sqrt{y}+2\cdot\sqrt{\lambda x}\cdot\frac{1}{2\sqrt{\lambda}}\sqrt{z}-y-z$

$\le 2\lambda x+(\frac{1}{4\lambda}-1)y+(\frac{1}{4\lambda}-1)z=\frac{\sqrt{3}-1}{2}(x+y+z)=\frac{\sqrt{3}-1}{2}$（其中$2\lambda=\frac{1}{4\lambda}-1$即$\lambda=\frac{-1+\sqrt{3}}{4}$）

（2017A）已知$a, b, c, d\in R, a>0, c>0$, 且$a^2-b^2=2, c^2-d^2=2$, 则$ac-bd$的取值范围为______. $[2,+\infty)$

$key:$由$a^2-b^2=(a+b)(a-b)=2$令$\begin{cases}a+b=t\\ a-b=\frac{2}{t}\end{cases}, \therefore\begin{cases}a=\frac{1}{2}(t+\frac{2}{t})>0\text{得}t>0\\ b=\frac{1}{2}(t-\frac{2}{t})\end{cases}$,

同理令$\begin{cases}c=\frac{1}{2}(s+\frac{2}{s})>0\text{得}s>0\\ d=\frac{1}{2}(s-\frac{2}{s})\end{cases}, \therefore ac-bd=\frac{s}{t}+\frac{t}{s}\ge 2$

变式 4：已知实数x, y满足$x^2+2xy-3y^2=1$，则 $2x-y\in$______；可以用判断式法

$x^2+xy\in$________；x^2+y^2的最小值为______.

$key:$(可因式分解代数换元）$x^2+2xy-3y^2=(x+3y)(x-y)=1$令$\begin{cases}x+3y=t\\ x-y=\frac{1}{t}\end{cases}$即$\begin{cases}x=\frac{1}{4}(t+\frac{3}{t})\\ y=\frac{1}{4}(t-\frac{1}{t})\end{cases}$

$2x-y=\frac{t}{2}+\frac{7}{4t}\in(-\infty,-\sqrt{\frac{7}{2}}]\cup[\sqrt{\frac{7}{2}},+\infty); x^2+xy=\frac{1}{8}(t^2+\frac{3}{t^2}+4)\in[\frac{\sqrt{3}+2}{4},+\infty); x^2+y^2=\frac{1}{8}(t^2+\frac{5}{t^2}+2)\ge\frac{\sqrt{5}+1}{4}$

练习 1.已知实数x,y满足$x^2-y^2=4$，则$\dfrac{1}{x^2}-\dfrac{y}{x}$的取值范围是（ ） A

$A.(-1,1)$ $B.(-\infty,-1]$ $C.(-\infty,-1]\cup[\dfrac{1}{4},+\infty)$ $D.(-1,\dfrac{1}{4}]$

$key:(x-y)(x+y)=4$, 令$\begin{cases}x-y=2t\\ x+y=\dfrac{2}{t}\end{cases}$, $\therefore\begin{cases}x=t+\dfrac{1}{t}\\ y=\dfrac{1}{t}-t\end{cases}$

$\therefore\dfrac{1}{x^2}-\dfrac{y}{x}=\dfrac{1-xy}{x^2}=\dfrac{t^4+t^2-1}{t^4+2t^2+1}=-u^2-u+1($令$u=\dfrac{1}{t^2+1}\in(0,1))=-(u+\dfrac{1}{2})^2+\dfrac{5}{4}\in(-1,1)$

2. 已知实数a,b满足$2b^2-a^2=4$,则$|a-2b|$的最小值为_____. 2

3.设实数x,y满足$\dfrac{x^2}{4}-y^2=1$，则$3x^2-2xy$的最小值是_______. $6+4\sqrt{2}$

4.若实数x,y满足$2x^2+xy-y^2=1$,则$\dfrac{x-2y}{5x^2-2xy+2y^2}$的最大值为_____.

$key:(2x-y)(x+y)=1$令$2x-y=t,x+y=\dfrac{1}{t}$,则$x=\dfrac{1}{3}(t+\dfrac{1}{t}),y=\dfrac{1}{3}(\dfrac{2}{t}-t)$

$\therefore\dfrac{x-2y}{5x^2-2xy+2y^2}=\dfrac{t-\dfrac{1}{t}}{t^2+\dfrac{1}{t^2}}=\dfrac{u}{u^2+2}\le\dfrac{\sqrt{2}}{4}$

变式 5（1）已知$a+b=4(a,b\in R)$.则$(\dfrac{1}{a^2+1}+\dfrac{1}{b^2+1})_{\max}=$______.（均值换元）$\dfrac{2+\sqrt{5}}{4}$

（2）若实数x，y满足$x+y=6$，则$f(x,y)=(x^2+4)(y^2+4)$的最小值为________144（均值换元）

$key1:f(x,y)=x^2y^2+16(x^2+y^2)+16=x^2y^2+4[(x+y)^2-2xy]+16=x^2y^2-8xy+160$

$=(xy-4)^2+144\ge144(\because xy\le(\dfrac{x+y}{2})^2=9)$

$key2:$由$x+y=6$令$x=3+t,y=3-t$,则$f(x,y)=[(3+t)^2+4][(3-t)^2+4]=(t^2+6t+13)(t^2-6t+13)$

$=(t^2+13)^2-36t^2=t^4-10t^2+169=(t^2-5)^2+144\ge144$

（3）已知实数x,y满足$\dfrac{x}{y}+\dfrac{y}{x}=x^2-y^2$，则$x^2+y^2$的最小值是___________.4（比值换元）

$key:$令$t=\dfrac{y}{x}$，则$t+\dfrac{1}{t}=x^2-t^2x^2$得$x^2=\dfrac{1+t^2}{t(1-t^2)}>0$

$\therefore x^2+y^2=\dfrac{(1+t^2)^2}{t(1-t^2)}=\dfrac{(t+\dfrac{1}{t})^2}{\dfrac{1}{t}-t}=\dfrac{u^2+2u+1}{u}=u+\dfrac{1}{u}+2\ge4(u=\dfrac{1}{t}-t>0)$

（201501 会考）已知$a\in R,b>0$，且$(a+b)b=1$，则$a+\dfrac{2}{a+b}$的最小值为_______.2

(1710)16.正实数x,y满足$x+y=1$，则$\dfrac{1+y}{x}+\dfrac{1}{y}$的最小值是（ ）$A.3+\sqrt{2}$ $B.2+2\sqrt{2}$ $C.5$ $D.\dfrac{11}{2}$ B

（1811 学考）若实数a,b满足$ab>0$，则$a^2+4b^2+\dfrac{1}{ab}$的最小值为（ ）$A.8$ $B.6$ $C.4$ $D.2$　　C

（2020 天津）14.已知 $a>0,b>0$，且$ab=1$，则 $\dfrac{1}{2a}+\dfrac{1}{2b}+\dfrac{8}{a+b}$ 的最小值为_________．4

（2020 江苏）12.已知$5x^2y^2+y^4=1(x,y\in R)$，则 x^2+y^2 的最小值是_______．$\dfrac{4}{5}$

$key1$:由已知得$y^2(x^2+\dfrac{1}{5}y^2)=\dfrac{1}{5},\therefore x^2+y^2=x^2+\dfrac{1}{5}y^2+\dfrac{4}{5}y^2\ge 2\sqrt{(x^2+\dfrac{1}{5}y^2)\cdot\dfrac{4}{5}y^2}=\dfrac{4}{5}$

$key2$:由已知得$x^2=\dfrac{1-y^4}{5y^2}\ge 0,\therefore x^2+y^2=\dfrac{1-y^4}{5y^2}+y^2=\dfrac{1}{5y^2}+\dfrac{4}{5}y^2\ge\dfrac{4}{5}$

（2021 天津）若 $a>0,b>0$，则 $\dfrac{1}{a}+\dfrac{a}{b^2}+b$ 的最小值为____________．$2\sqrt{2}$

（2021上海）已知正实数a,b满足$a(a+b)=27$,求a^2b的最大值.

（2021上海）$key1$:$\because a(a+b)=a^2+ab=27,a,b>0,\therefore a^2b=\sqrt{a^2\cdot\dfrac{1}{2}ab\cdot\dfrac{1}{2}ab\cdot 4}\le\sqrt{4(\dfrac{a^2+ab}{3})^3}=54$

$key2$:由已知得$b=\dfrac{27}{a}-a$,则$a^2b=a^2(\dfrac{27}{a}-a)=a(27-a^2)=\sqrt{\dfrac{1}{2}\cdot 2a^2(27-a^2)(27-a^2)}\le 54$

（2018辽宁）若正实数x,y满足$x^3+y^3=(4x-5y)y$，则y的最大值为____.

(2018辽宁）$key:y^3+5y^2=4xy-x^3=x(4y-x^2)=\sqrt{\dfrac{1}{2}\cdot 2x^2(4y-x^2)(4y-x^2)}$

$\le\sqrt{\dfrac{1}{2}(\dfrac{8y}{3})^3}=\dfrac{16y\sqrt{y}}{3\sqrt{3}}\Leftrightarrow(y+5)\sqrt{y}\le\dfrac{16}{3\sqrt{3}}\Leftrightarrow y\le\dfrac{1}{3}$

变式 1（1）①已知 $a>0,b>0,a+2b=1$，则$\dfrac{1}{3a+4b}+\dfrac{1}{a+3b}$取得最小值时的$a=$______.

key:令$3a+4b=x,a+3b=y,\therefore x+2y=3a+4b+2(a+3b)=5$

$\therefore\dfrac{1}{3a+4b}+\dfrac{1}{a+3b}=\dfrac{1}{x}+\dfrac{2}{2y}\ge\dfrac{(1+\sqrt{2})^2}{x+2y}\ge\dfrac{3+2\sqrt{2}}{5}$(当且仅当$\dfrac{\frac{1}{x}}{x}=\dfrac{\frac{1}{y}}{2y}$即$y=\dfrac{5}{2+\sqrt{2}}$时取$=$)

② 已知 $a>0$, $b>0$，且 $\dfrac{1}{2a+b}+\dfrac{1}{b+1}=1$，则 $a+2b$ 的最小值为________．

$key:a+2b=\dfrac{1}{2}(2a+b)+\dfrac{3}{2}(b+1)-\dfrac{3}{2}=\dfrac{\frac{1}{2}}{\frac{1}{2a+b}}+\dfrac{\frac{3}{2}}{\frac{1}{b+1}}\ge\dfrac{(\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{2}})^2}{\frac{1}{2a+b}+\frac{1}{b+1}}$

$\ge\dfrac{1}{2}+\sqrt{3}$(当且仅当$\dfrac{(2a+b)^2}{2}=\dfrac{3(b+1)^2}{2}$时,取$=$)

③若正实数 a、b、c 满足 $ab=a+2b$，$abc=a+2b+c$，则 c 的最大值为________．$\dfrac{8}{7}$

$key: c=\frac{a+2b}{ab-1}=\frac{a+2b}{a+2b-1}=1+\frac{1}{ab-1}\le 1+\frac{1}{7}=\frac{8}{7}$（而$ab=a+2b\ge 2\sqrt{2ab}$得$ab\ge 8$）

④ 已知实数x,y满足$x>y>0$,且$x+y\le 2$，则$\frac{2}{x+3y}+\frac{1}{x-y}$的最小值为______.

key:令$a=x+3y, b=x-y>0$,则$a+b=2(x+y)\le 4$

$\therefore \frac{2}{x+3y}+\frac{1}{x-y}=\frac{2}{a}+\frac{1}{b}\ge\frac{(\sqrt{2}+1)^2}{a+b}\ge\frac{3+2\sqrt{2}}{4}$(当且仅当$x=\frac{\sqrt{2}+3}{4}, y=\frac{\sqrt{2}-1}{4}$时取$=$)

⑤已知正实数x,y满足$xy+2x+3y=42$，则$xy+5x+4y$的最小值为______.55

$key:(x+3)(y+2)=48$令$a=x+3>3, b=y+2>2$,则$ab=48$

则$xy+5x+4y=(a-3)(b-2)+5(a-3)+4(b-2)=3a+b+31\ge 2\sqrt{3ab}+31=55$

⑥已知$a,b>0$,且$(a+b)(a+2b)+a+b=9$,则$(3a+4b)_{\min}=$______.

$key:(a+b)(a+2b)+a+b=(a+b)(a+2b+1)=9$

令$\begin{cases}a+b=x>0\\a+2b+1=y>0\end{cases}$得$\begin{cases}a=2x-y+1\\b=-x+y-1\end{cases}$,且$xy=9, \therefore 3a+4b=2x+y-1\ge 6\sqrt{2}-1$

（2）①函数$y=\frac{1}{x}+\frac{2}{1-x}(0<x<1)$的值域为_______,

$key:\frac{1}{x}+\frac{2}{1-x}=\frac{x+1-x}{x}+\frac{2(x+1-x)}{1-x}=3+\frac{1-x}{x}+\frac{2x}{1-x}(t=\frac{1-x}{x}=\frac{1}{x}-1\in(0,+\infty))\in[3+2\sqrt{2},+\infty)$

函数$y=\frac{2}{x+1}+\frac{x}{2x+1}(0\le x\le 2)$的值域为_______,

$key: y=2\cdot\frac{2(x+1)-(2x+1)}{x+1}+\frac{2x+1-(x+1)}{2x+1}=5-(2\cdot\frac{2x+1}{x+1}+\frac{x+1}{2x+1})(t=\frac{2x+1}{x+1}=2-\frac{1}{x+1}\in[1,\frac{5}{3}])$

$=5-(2t+\frac{1}{t})\in[\frac{16}{15},2]$

②函数$f(x)=\frac{(2x+1)^2}{x(2-x)}(0<x<2)$的最小值为________；

$key1$:（分母不动，分子用分母表示，齐次）$f(x)=\frac{(2x+\frac{2-x+x}{2})^2}{x(2-x)}=\frac{1}{4}(\frac{25x}{2-x}+\frac{2-x}{x}+10)\ge 5$（当且仅当$x=\frac{1}{3}$时，取$=$）

$key2: f(x)=\lambda\cdot\frac{(2x+1)^2}{\lambda x(2-x)}\ge\lambda\cdot\frac{(2x+1)^2}{(\frac{\lambda x+2-x}{2})^2}=5$（其中$\lambda-1=4$即$\lambda=5$）

函数$f(x)=\frac{2x^2+1}{x(2-x)}(0<x<2)$的最小值为________.

key:（分母不动，分子用分母表示，齐次）$f(x)=\frac{2x^2+(\frac{x+2-x}{2})^2}{x(2-x)}$

$=\frac{1}{4}(\frac{9x}{2-x}+\frac{2-x}{x})+\frac{1}{2}\ge 2$(当且仅当$x=\frac{2}{3}$时，取$=$)