Hill Climbing Approach to Solving Quadratic Equations

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1 Introduction

Quadratic equations of the form $f(x) = ax^2 + bx + c$ are central to mathematics, with roots defined where $f(x) = 0$. The quadratic formula provides exact solutions, but numerical methods like hill climbing offer an iterative perspective, ideal for visualization and exploring optimization techniques. This report details the application of a hill climbing algorithm to find the roots of an example quadratic equation $f(x) = -x^2 + 4x + 2$, implemented in Python with visualization using Matplotlib.

2 Problem Statement

The objective is to numerically determine the x-values where $f(x) = -x^2 + 4x + 2 = 0$. The quadratic has:

- Coefficients: $a = -1$, $b = 4$, $c = 2$.
- Discriminant:
$$\Delta = b^2 - 4ac = 4^2 - 4 \cdot (-1) \cdot 2 = 16 + 8 = 24$$
- Exact roots:
$$x = \frac{-4 \pm \sqrt{24}}{2 \cdot (-1)} = \frac{-4 \pm 2\sqrt{6}}{-2} = 2 \pm \sqrt{6} \approx 4.449, -0.449$$

The hill climbing algorithm approximates these roots by minimizing $|f(x)|$, with a plot illustrating the convergence paths.

3 Methodology

3.1 Hill Climbing Algorithm

Hill climbing is a local search method that iteratively improves a solution by evaluating neighboring points. For root-finding:

- **Objective**: Minimize $|f(x)|$, since $|f(x)| = 0$ at a root.

- **Steps**:
  1. Start at an initial x-value.
  2. Compute $f(x)$ at the current point and neighbors ($x \pm$ step_size).
  3. Move to the neighbor with the smallest $|f(x)|$.
  4. Reduce step size if no improvement.
  5. Stop when $|f(x)| <$ tolerance or after a maximum number of iterations.

3.2 Discriminant Check

The discriminant determines the existence of real roots:

- $\Delta = b^2 - 4ac$.
- If $\Delta < 0$, no real roots exist; skip the search.
- If $\Delta \geq 0$, proceed to find roots.

3.3 Visualization

Matplotlib is used to plot:

- The quadratic curve.
- Search paths from initial points to roots.
- Root locations, marked with stars.

4 Implementation

The Python implementation includes:

4.1 Quadratic Function

```python
def quadratic(x, a, b, c):
    return a * x**2 + b * x + c
```

Evaluates $f(x) = -x^2 + 4x + 2$ for a given x.

4.2 Hill Climbing

```python
def hill_climbing(start, step_size, max_iterations, tolerance, a,
    b, c):
    current_x = start
    path = [current_x]
    for _ in range(max_iterations):
        current_value = quadratic(current_x, a, b, c)
        if abs(current_value) < tolerance:
            break
```

```python
        left_x = current_x - step_size
        right_x = current_x + step_size
        left_value = quadratic(left_x, a, b, c)
        right_value = quadratic(right_x, a, b, c)
        if abs(left_value) < abs(current_value):
            current_x = left_x
        elif abs(right_value) < abs(current_value):
            current_x = right_x
        else:
            step_size *= 0.5
        path.append(current_x)
    return current_x, path
```

Parameters:

- `step_size = 0.1`.
- `max_iterations = 1000`.
- `tolerance = 1e-6`.

Returns the root and the path taken.

4.3 Main Program

- **Discriminant**: $\Delta = 24 \geq 0$, confirming two real roots.
- **Start Points**:
  - Vertex: $x = -\frac{b}{2a} = \frac{-4}{2\cdot(-1)} = 2$.
  - Starts: $[1, 3]$, offset from the vertex.
- **Plot**:
  - Range: $[0, 4]$, centered on the vertex.
  - Displays quadratic, paths, and roots.
  - Saved as `hill_climbing_quadratic_with_discriminant.png`.

5 Results

5.1 Roots

- **Start** $x = 1$:
  - Root: $x \approx -0.449$.
  - Exact: $2 - \sqrt{6} \approx -0.449$.
- **Start** $x = 3$:
  - Root: $x \approx 4.449$.
  - Exact: $2 + \sqrt{6} \approx 4.449$.
- **Precision**: Within 10^{-6} of exact values.

5.2 Visualization

- **Curve**: Downward parabola, crossing the x-axis at the roots.
- **Paths**:
  - Red: From $x = 1$ to $x \approx -0.449$.
  - Green: From $x = 3$ to $x \approx 4.449$.
- **Roots**: Marked with star markers at convergence points.

6 Analysis

6.1 Advantages

- **Educational**: Demonstrates iterative root-finding visually.
- **Flexible**: Adapts to any quadratic via coefficients.
- **Robust**: Discriminant check prevents invalid searches.
- **Clear Output**: Plot enhances understanding.

6.2 Limitations

- **Start Dependency**: Convergence depends on initial points.
- **Single Root**: For $\Delta = 0$, paths may converge redundantly.
- **Speed**: Slower than the quadratic formula.

7 Conclusion

The hill climbing algorithm successfully identifies the roots of $f(x) = -x^2 + 4x + 2$ at $x \approx -0.449, 4.449$, aligning with exact solutions. The discriminant check and vertex-based start points ensure reliability, while the visualization clarifies the iterative process. Although less efficient than analytical methods, the approach is valuable for educational purposes and numerical exploration.