Problemas 4.5

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73. The authors of the article "A Probabilistic Insulation Life Model for Combined Thermal Electrical Stresses" (IEEE Trans. on Elect. Insulation, 1985: 519–522) state that "the Weibull distribution is widely used in statistical problems relating to aging of solid insulating materials subjected to aging and stress." They propose the use of the distribution as a model for time (in hours) to failure of solid insulating specimens subjected to AC voltage. The values of the parameters depend on the voltage and temperature; suppose α = 2.5 and β = 200 (values suggested by data in the article).

$$f(x;\alpha,\beta)=\begin{cases}\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-(x/\beta)^\alpha} & x\geq 0\\ 0 & x<0\end{cases}$$

$$F(x;\alpha,\beta)=\begin{cases}0 & x<0\\ 1-e^{-(x/\beta)^\alpha} & x\geq 0\end{cases}$$

a. What is the probability that a specimen's lifetime is at most 250? Less than 250? More than 300?

$$P(x\leq 250)=1-e^{-\left(\frac{x}{\beta}\right)^\alpha}=1-e^{-(1.25)^{2.5}}=1-e^{-1.7469}=0.8256$$

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$$P(x\geq 300)=e^{-\left(\frac{x}{\beta}\right)^\alpha}=e^{-(1.5)^{2.5}}=e^{-2.7556}=0.06357$$

b. What is the probability that a specimen's lifetime is between 100 and 250?

$$P(100\leq x\leq 250)=\left(1-e^{-(1.25)^{2.5}}\right)-\left(1-e^{-(0.5)^{2.5}}\right)=-e^{-(1.25)^{2.5}}+e^{-(0.5)^{2.5}}=-e^{-1.7469}+e^{-0.1767}=-0.1743+.8380=0.6637$$

c. What value is such that exactly 50% of all specimens have lifetimes exceeding that value?

$$F(x;2.5,200)=.5=1-e^{-(x/200)^{2.5}}$$

$$e^{-\left(\frac{x}{200}\right)^{2.5}}=.5$$

$$-\left(\frac{x}{200}\right)^{2.5}=\ln(.5)$$

$$\left(\frac{x}{200}\right)^{2.5}=-\ln(.5)$$

$$\frac{x}{200}=\ln(2)^{\frac{2}{5}}$$

$$x=200\ln(2)^{\frac{2}{5}}$$

x = 172.7269

77. The authors of the article from which the data in Exercise 1.27 was extracted suggested that a reasonable probability model for drill lifetime was a lognormal distribution with μ = 4.5 and σ = .8.

$$f(x;\mu,\sigma)=\begin{cases}\frac{1}{\sqrt{2\pi}\sigma x}e^{-[\ln(x)-\mu]^2/(2\sigma^2)} & x\geq 0\\ 0 & x<0\end{cases}$$

$$F(x;\mu,\sigma)=P(X\leq x)=P[\ln(X)\leq\ln(x)]=P\left(Z\leq\frac{\ln(x)-\mu}{\sigma}\right)=\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)\qquad x\geq 0$$

$$E(X)=e^{\mu+\sigma^2/2}\qquad V(X)=e^{2\mu+\sigma^2}\cdot(e^{\sigma^2}-1)$$

a. What are the mean value and standard deviation of lifetime?

$$E(X)=e^{4.5+\frac{0.8^2}{2}}=e^{4.82}=123.96509$$

$$V(X)=e^{2*4.5+0.8^2}\left(e^{0.8^2}-1\right)=e^{9.64}\left(e^{0.64}-1\right)=13776.52$$

$$\sqrt{V(X)}=\sqrt{13776.52}=117.3734$$

b. What is the probability that lifetime is at most 100?

$$F(100;4.5,.8)=\Phi\left(\frac{\ln(100)-4.5}{.8}\right)=\Phi(0.13146)=.5517$$

c. What is the probability that lifetime is at least 200? Greater than 200?

$$1-F(200;4.5,.8)=1-\Phi\left(\frac{\ln(200)-4.5}{.8}\right)=1-\Phi(0.9978)=1-.8389=0.1611$$

81. Sales delay is the elapsed time between the manufacture of a product and its sale. According to the article "Warranty Claims Data Analysis Considering Sales Delay" (Quality and Reliability Engr. Intl., 2013: 113–123), it is quite common for investigators to model sales delay using a lognormal distribution. For a particular product, the cited article proposes this distribution with parameter values μ = 2.05 and σ^2 = .06 (here the unit for delay is months).

a. What are the variance and standard deviation of delay time?

$V(X) = e^{2*2.05+0.06}\left(e^{0.06} - 1\right) = e^{4.16}\left(e^{0.06} - 1\right) = 3.9619$

$\sqrt{V(X)} = \sqrt{3.9619} = 1.9904$

b. What is the probability that delay time exceeds 12 months?

$$1 - F(12; 2.05, .2449) = 1 - \Phi\left(\frac{\ln(12) - 2.05}{.2449}\right) = 1 - \Phi(1.77) = 1 - .9616 = 0.0384$$

c. What is the probability that delay time is within one standard deviation of its mean value?

$E(X) = e^{2.05+\frac{.06}{2}} = 8.0044$

$$F(9.9948; 2.05, .2449) - F(6.014; 2.05, .2449) = \Phi\left(\frac{\ln(9.9948) - 2.05}{.2449}\right) - \Phi\left(\frac{\ln(6.014) - 2.05}{.2449}\right) = \Phi(1.02) - \Phi(-1.04) = .8461 - .1492 = .6969$$

d. What is the median of the delay time distribution?

$E(X) = e^{2.05+\frac{.06}{2}} = 8.0044$

e. What is the 99th percentile of the delay time distribution?

$$P(X \geq x) = .99 = \Phi\left(\frac{\ln(X) - 2.05}{.2449}\right) = \Phi(2.33)$$

$$\frac{\ln(X) - 2.05}{.2449} = 2.33$$

$\ln(X) = 2.33 * .2449 + 2.05$

$\mathrm{X} = e^{2.33*.2449+2.05}$

$\mathrm{X} = 13.7442$

f. Among 10 randomly selected such items, how many would you expect to have a delay time exceeding 8 months?

$$P(8 \leq x) = 1 - \Phi\left(\frac{\ln(8) - 2.05}{.2449}\right) = 1 - \Phi(0.1202) = 1 - 0.5478 = 0.4522$$

4.52 se esperan que excedan 8 meses

85. Let X have a standard beta density with parameters α and β.

A random variable X is said to have a **beta distribution** with parameters α, β (both positive), A, and B if the pdf of X is

$$f(x; \alpha, \beta, A, B) = \begin{cases} \dfrac{1}{B-A} \cdot \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\left(\dfrac{x-A}{B-A}\right)^{\alpha-1}\left(\dfrac{B-x}{B-A}\right)^{\beta-1} & A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$

The case $A = 0$, $B = 1$ gives the **standard beta distribution.**

$$\mu = A + (B - A) \cdot \frac{\alpha}{\alpha + \beta}$$

a. Verify the formula for E(X) given in the section.

$$E(x) = \int_{-\infty}^{\infty} x f(x; \alpha, \beta, 0,1) dx$$

$$E(x) = \int_0^1 x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$E(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1} dx$$

$$E(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1} dx$$

$$E(x) = \frac{\Gamma(\alpha+1) * \Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\alpha+\beta+1)} \int_0^1 \frac{\Gamma(\alpha+\beta+1)\, x^{(\alpha+1)-1}(1-x)^{\beta-1}}{\Gamma(\alpha+1) * \Gamma(\beta)} dx$$

$$E(x) = \frac{\Gamma(\alpha+1) * \Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\alpha+\beta+1)} \int_0^1 \frac{\Gamma(\alpha+\beta+1)\, x^{(\alpha+1)-1}(1-x)^{\beta-1}}{\Gamma(\alpha+1) * \Gamma(\beta)} dx$$

$$E(x) = \frac{\alpha\,\Gamma(\alpha) * \Gamma(\alpha+\beta)}{\Gamma(\alpha) * (\alpha+\beta)\Gamma(\alpha+\beta)} \int_0^1 \frac{\Gamma(\alpha+\beta+1)\, x^{(\alpha+1)-1}(1-x)^{\beta-1}}{\Gamma(\alpha+1) * \Gamma(\beta)} dx$$

$$E(x) = \frac{\alpha}{(\alpha+\beta)} \int_0^1 \frac{\Gamma(\alpha+\beta+1)\, x^{(\alpha+1)-1}(1-x)^{\beta-1}}{\Gamma(\alpha+1) * \Gamma(\beta)} dx$$

$\gamma = \alpha + 1$

$$E(x) = \frac{\alpha}{(\alpha+\beta)} \int_0^1 \frac{\Gamma(\gamma+\beta)\, x^{\gamma-1}(1-x)^{\beta-1}}{\Gamma(\gamma) * \Gamma(\beta)} dx$$

$$E(x) = \frac{\alpha}{(\alpha+\beta)} \int_0^1 \frac{\Gamma(\gamma+\beta)}{\Gamma(\gamma) * \Gamma(\beta)} x^{\gamma-1}(1-x)^{\beta-1} dx$$

Vemos que obtenemos la funcion beta pero con $\gamma = \alpha + 1$, dando como resultado 1

$$E(x) = \frac{\alpha}{(\alpha+\beta)}$$

b. Compute $E[(1 - X)^m]$. If X represents the proportion of a substance consisting of a particular ingredient, what is the expected proportion that does not consist of this ingredient?

$$E(x) = \int_{-\infty}^{\infty} (1-x)^m f(x; \alpha, \beta, 0{,}1) dx$$

$$E(x) = \int_0^1 (1-x)^m \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$E(x) = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta+m-1} dx$$

$$E(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta+m-1} dx$$

$$E(x) = \frac{\Gamma(\alpha+\beta) * \Gamma(\beta+m)}{\Gamma(\alpha+\beta+m) * \Gamma(\beta)} \int_0^1 \frac{\Gamma(\alpha+\beta+m)}{\Gamma(\alpha) * \Gamma(\beta+m)} x^{\alpha-1}(1-x)^{\beta+m-1} dx$$

$$E(x) = \frac{\Gamma(\alpha+\beta) * \Gamma(\beta+m)}{\Gamma(\alpha+\beta+m) * \Gamma(\beta)}$$