HW6

Sawyer Maloney

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Exercise 1. 4.13

(a) Prove that if f and g are both injective then so is $g \circ f$.

$g \circ f$ is injective if every input maps to a unique output. If we have $x_1 \& x_2 s.t. x_1 = x_2 \in A$, by injectivity, $f(x_1) = f(x_2) \in B$. Similarly by injectivity, $g(f(x_1)) = g(f(x_2)) \in C$. So, for every inputs of $g \circ f, x_1 \& x_2 s.t. x_1 = x_2 \implies g \circ f(x_1) = g \circ f(x_2)$. Thus $g \circ f$ is injective.

(b) Prove that if f and g are both surjective then so if $g \circ f$.

f surjective means that for every $f(x) \in B$, there exists $x s.t. x = f(x)$. Similarly, there is some $c \in C s.t. g(f(x)) = c$, by surjectivity. Thus, for every $c \in C$, there is some $x s.t. x = f(x), g(f(x)) = c$, thus $g \circ f$ is surjective.

(c) It follows from the previous two parts that if f and g are bijcetive then so if $g \circ f$. Is the converse true? Prove or give a counterexample.

It is not true. Take the following example: $A = 1, 2, B = 1, 2, 3, C = 1, 2$. Then, f maps $f(1) = 1, f(2) = 2$. g maps $g(1) = 1, g(2) = 2, g(3) = 2$. Thus, $g \circ f$ maps $1 \to 1, 2 \to 2$. It is bijective, but f is not surjective, and g is not injective.

Exercise 2. 4.17 Suppose we have functions $f : A \to B$ and $g : B \to C$ with inversese $f^{-1} : B \to A$ and $g^{-1} : C \to B$. Prove that $g \circ f$ is invertible, and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Solution. We know that f and g are both bijective from Theorem 4.8. From above, we thus know that $g \circ f$ is invertible as well.

$$g^{-1} \circ f^{-1}(f(g(x))) = g^{-1}(f^{-1}(f(g(x)))) = g^{-1}(g(x)) = x$$

Since the composition of the functions is just the identity on x, we know that this is the inverse of $g \circ f$. □

Exercise 3. 4.20 Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a surjection. Define a new function $g : \mathbb{C} \to \mathbb{C}$ yb the formula $g(x) = 2f(x + 1)$. Show that $g(x)$ is a surjection.

Solution. For any $y \in \mathbb{C}$, there is some $x \in \mathbb{C}$ such that $f(x+1) = y$, by surjection. Thus, similarly, for any $y_1 \in \mathbb{C}$, there is some $x_1 \in \mathbb{C}$ such that $f(x+1) \cdot 2 = y$, by surjection again. Thus, because this is the definition of $g(x)$, $g(x)$ is a surjection as well. $\square$

Exercise 4. 5.4 For each of the following statements, provide a proof or a counterexample

(a) If A, B are subsets of X then $f(A \cup B) = f(A) \cup f(B)$.

The left hand set is $f(A \cup B) = f(a) : a \in A \cup B, \implies a \in A$ or $a \in B$, so it is $f(a) : a \in A$ and $f(b) : b \in B \implies f(A) \cup f(B)$. The right hand side is

$$f(A) \cup f(B) \implies f(a) : a \in A \cup f(b) : b \in B$$
$$\implies f(a) : a \in A \cup B \implies f(A \cup B)$$

(b) If A, B are subsets of X then $f(A \cap B) = f(A) \cap (B)$.

Not true. If $A = 1$ and $B = 2$, but f maps $1, 2 \to 1$, then $f(A \cap B) =$ but $f(A) \cap f(B) = 1$.

(c) If C, D are subsets of Y then $f^{-1}(C \cup D) = f^{-1}C \cup f^{-1}(D)$.

Decompose the left hand side. $f^{-1}(C \cup D)$ means that it is the set of xs such that $f(x) \in C \cup D \implies f(x) \in C$ or $f(x) \in D$. Now the right hand side: $f^{-1}(C) = x : f(x) \in C$, and $f^{-1}(D) = x : f(x) \in D$, so their union is xs such that $f(x)$ is either in C or D, thus they are equal.

(d) If C, D are subsets of Y then $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

LHS: $f^{-1}(C \cap D)$ means all xx such that $f(x)$ is in C and D. RHS: $f^{-1}(C)$ means all xs such that $f(x) \in C$, and $f^{-1}(D)$ means all xs such that $f(x) \in D$. So, their intersection is all values x such that $f(x)$ is in both C and D. Thus they are equal.

Exercise 5. 5.9 Suppose $f^{-1}(f(A)) = A$ holds for *every* $A \subset X$. Prove that f is an injection.

Solution. Since the composition of the image and the preimage is the identity function, we know that it is a proper inverse function, which, by theorem 4.8, means f is bijective $\implies f$ is injective.

$\square$