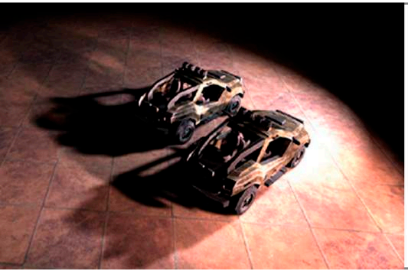
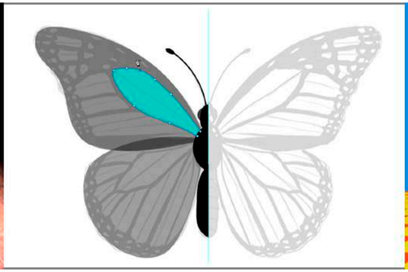
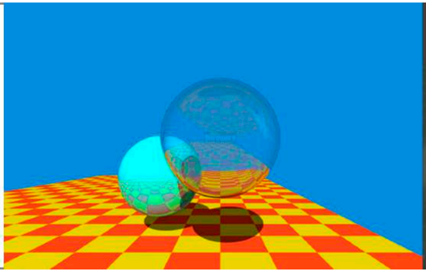
Introduction to Computer Graphics

GAMES101, Lingqi Yan, UC Santa Barbara

Lecture 2:
Review of Linear Algebra


http://www.cs.ucsb.edu/~lingqi/teaching/cs180.html

Announcements

- Slides and recordings of Lecture 1 now available
- (Pre)-reading materials will be out before lectures

	日期	主题
第 1 周	Feb 11	**计算机图形学概述** [课件][录像]
	Feb 14	**向量与线性代数** 阅读材料：第 2 章（Miscellaneous Math），第 5 章（Linear Algebra）

- Happy Valentine’s Day!

Last Lecture

- What is Computer Graphics?
- Why study Computer Graphics?
- Course Topics
- Course Logistics

迅速的 野蛮的

A Swift and Brutal Introduction to Linear Algebra!

(in fact it's relatively easy…)

Graphics' Dependencies

- Basic mathematics

  线性代数 微积分 统计.

  - Linear algebra, calculus, statistics
- Basic physics

  光学 力学

  - Optics, Mechanics
- Misc
  - Signal processing 信号处理
  - Numerical analysis 数值计算.
- And a bit of aesthetics

This Course

- More dependent on Linear Algebra

  向量 点积 叉积
  - Vectors (dot products, cross products, …)

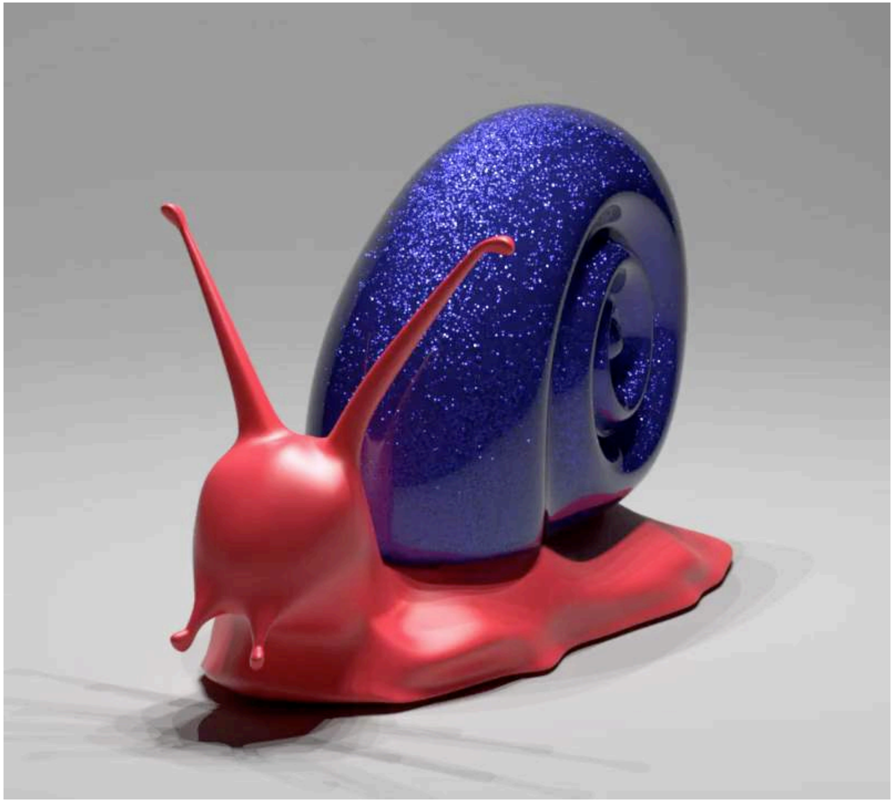
  矩阵
  - Matrices (matrix-matrix, matrix-vector mult., …)

- For example,
  - A point is a vector (?)

  平移 旋转
  - An operation like translating or rotating objects can be matrix-vector multiplication

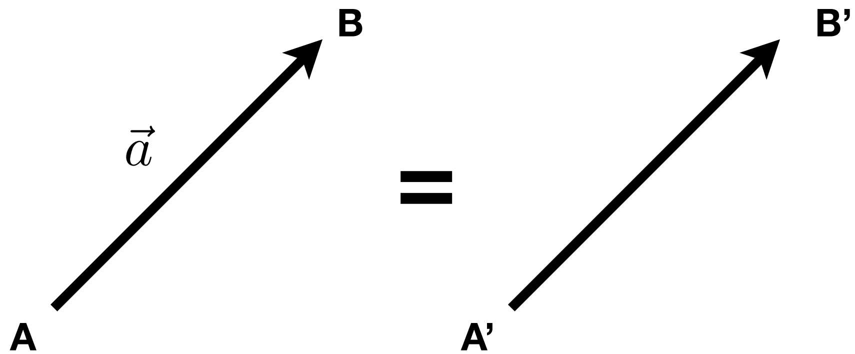
  矩阵-向量乘法

An Example of Rotation



Rendering Glints on High-Resolution Normal-Mapped Specular Surfaces, Lingqi Yan, 2014

向量 Vectors



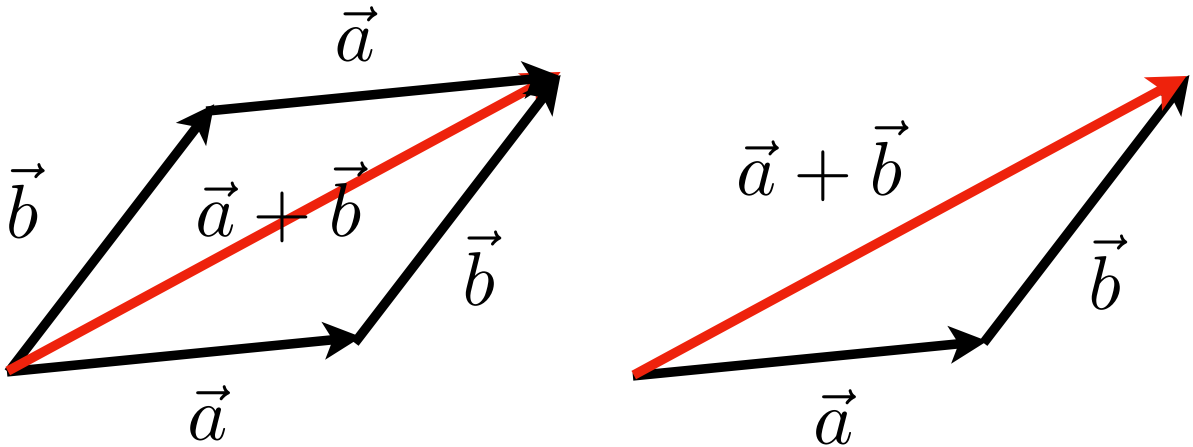
- Usually written as $\vec{a}$ or in bold $\boldsymbol{a}$
- Or using start and end points $\overrightarrow{AB} = B - A$
- Direction and length 方向、长度.
- No absolute starting position 无绝对起始位置.

Vector Normalization

- Magnitude (length) of a vector written as $\|\vec{a}\|$ 向量长度.
- Unit vector 单位向量
  - A vector with magnitude of 1
  - Finding the unit vector of a vector (normalization): $\hat{a} = \vec{a}/\|\vec{a}\|$
  - Used to represent directions 表示方向

向量加法.

Vector Addition



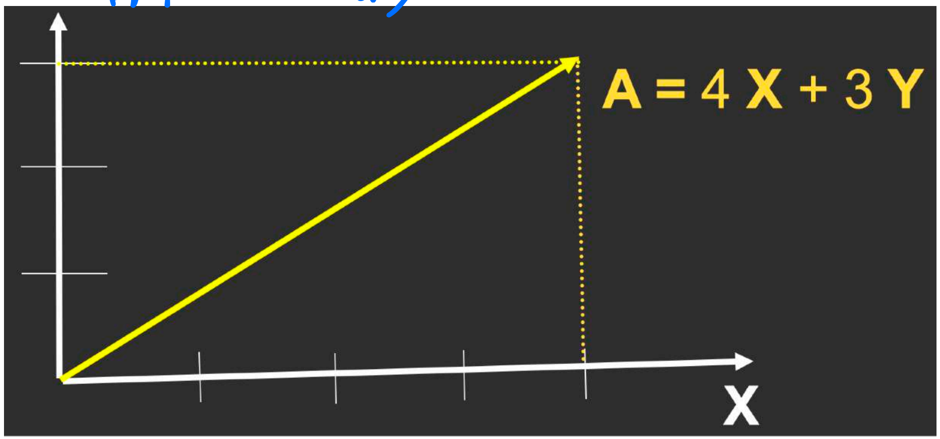
几何上:　　平行四边形法则　　三角形法则.

- Geometrically: Parallelogram law & Triangle law
- Algebraically: Simply add coordinates

代数上:　　坐标相加

Cartesian Coordinates

笛卡尔坐标系



正交

- X and Y can be any (usually orthogonal unit) vectors

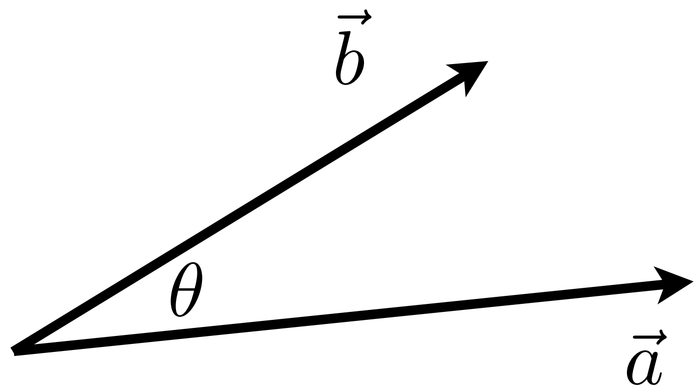
$$\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{A}^T = (x, y) \qquad \mathbf{A} = \sqrt{x^2 + y^2}$$

默认：列向量

Vector Multiplication

- Dot product 点乘
- Cross product 叉乘.
- Orthonormal bases and coordinate frames

Dot (scalar) Product



点乘: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

- For unit vectors

单位向量: $\cos\theta = \hat{a} \cdot \hat{b}$

Dot (scalar) Product

- Properties

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$$

Dot Product in Cartesian Coordinates

部分乘法　　　　相加.

- Component-wise multiplication, then adding up
  - In 2D

$\vec{a}^T \vec{b}$

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

  - In 3D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

Dot Product in Graphics

① 求两个向量间夹角.

- Find angle between two vectors
(e.g. cosine of angle between light source and surface)
光源.表面之间夹角余弦值

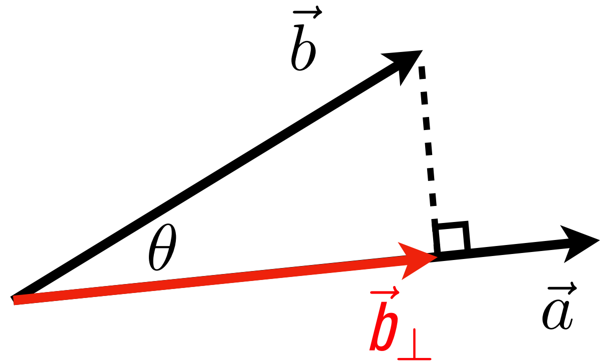
- Finding projection of one vector on another

② 求一个向量在另一向量的投影

Dot Product for Projection

$\vec{b}$ 投影在 $\vec{a}$ 上

- $\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$
  - $\vec{b}_\perp$ must be along $\vec{a}$ (or along $\hat{a}$)
    - $b_\perp = k\hat{a}$
  - What's its magnitude k? 长度.
    - $k = \|\vec{b}_\perp\| = \|\vec{b}\| \cos\theta$



Dot Product in Graphics

① 衡量两个方向多接近 { 1 完全相同方向 ; -1 完全相反方向 }（单位向量）

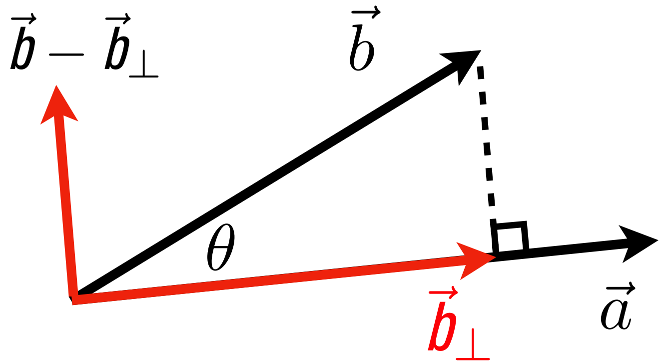
- Measure how close two directions are

② 分解一个向量

- Decompose a vector

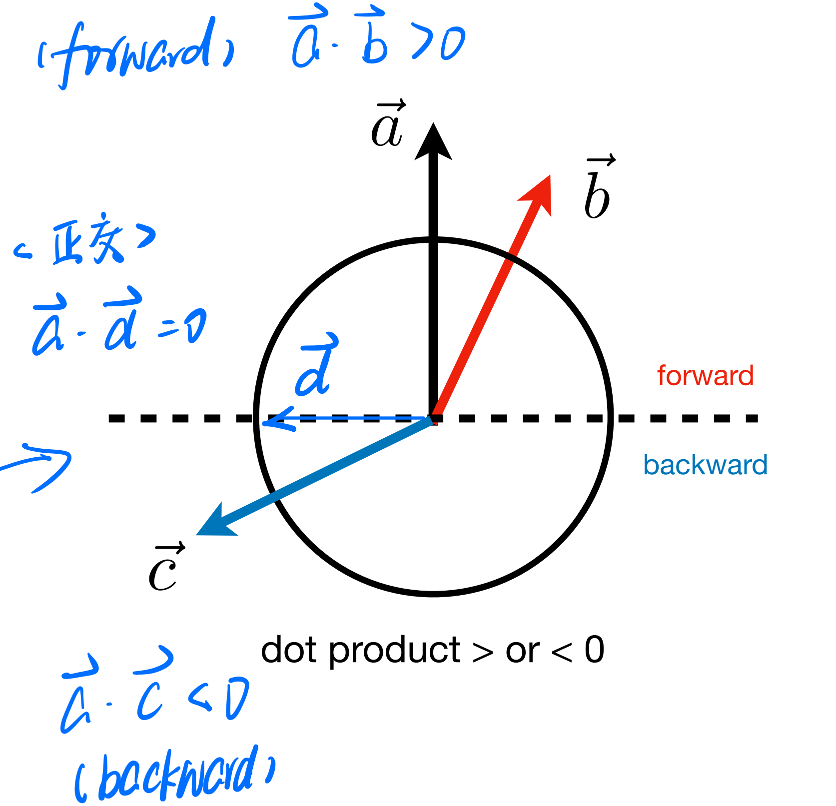
③ 确定 前/后信息（方向）

- Determine forward / backward



Dot Product in Graphics

- Measure how close two directions are
- Decompose a vector
- Determine forward / backward



Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

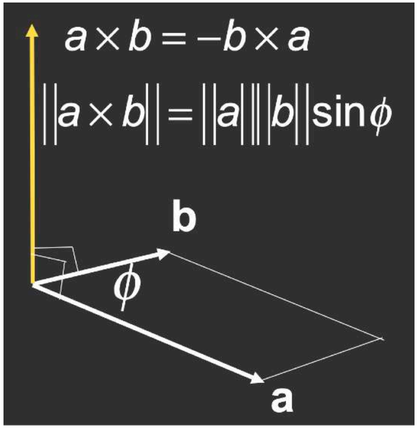
Cross (vector) Product

注：三维软件提供翻转法线的功能

叉乘求得的结果垂直于两个原始向量，常用于求法线.

注：OpenGL 右手系

DirectX 左手系



叉乘正交于两个初始向量

- Cross product is orthogonal to two initial vectors

右手

- Direction determined by right-hand rule

用于构建坐标系

- Useful in constructing coordinate systems (later)

Cross product: Properties

$$\vec{x} \times \vec{y} = +\vec{z}$$

右手坐标系

$$\vec{y} \times \vec{x} = -z$$

$$\vec{y} \times \vec{z} = +x$$

$$\vec{z} \times \vec{y} = -x$$

$$\vec{z} \times \vec{x} = +y$$

$$\vec{x} \times \vec{z} = -y$$

$$a \times b = -b \times a$$

$$\vec{a} \times \vec{a} = \vec{0}$$

长度为0的向量.

$$a \times (b + c) = a \times b + a \times c$$

$$a \times (kb) = k(a \times b)$$

Cross Product: Cartesian Formula?

$$a \times b = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

- Later in this lecture

矩阵相乘.

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

Cross Product in Graphics

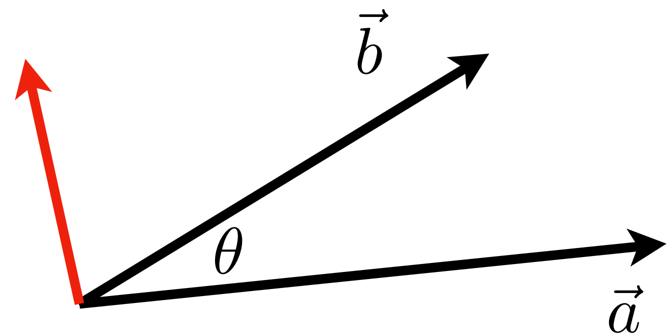
叉积：
作用：

① 判定左/右

- Determine left / right

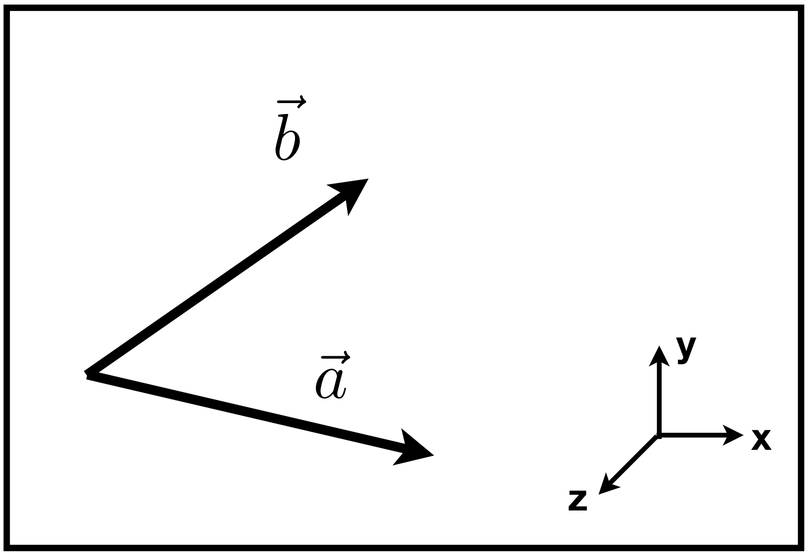
② 判定内/外

- Determine **inside / outside**

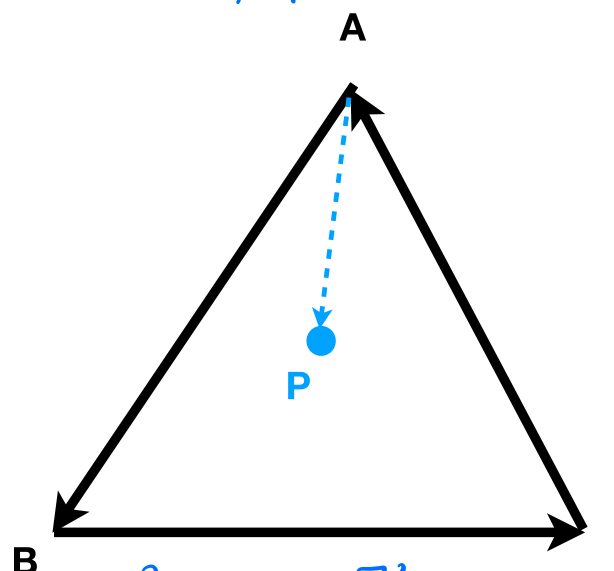


Cross Product in Graphics

左/右.



内/外



判断P是否在△ABC里面

① P都在 $\vec{AB}$ $\vec{BC}$ $\vec{CA}$ 左面 $\vec{AB}\times\vec{AP}$ $\vec{BC}\times\vec{BP}$ $\vec{CA}\times\vec{CP}$

② P都在 $\vec{AC}$ $\vec{CB}$ $\vec{BA}$ 右面 $\vec{AP}\times\vec{AC}$ $\vec{CP}\times\vec{CB}$ $\vec{BP}\times\vec{BA}$

Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

标准正交基/坐标系

Orthonormal Bases / Coordinate Frames

- Important for representing points, positions, locations

许多坐标系

- Often, many sets of coordinate systems
  - Global, local, world, model, parts of model (head, hands, …)

这些 system/base 之间的转换.

- Critical issue is transforming between these systems/ bases
  - A topic for next week

Orthonormal Coordinate Frames

- Any set of 3 vectors (in 3D) that

$$\vec{u} \quad = \quad \vec{v} \quad = \quad \vec{w} \quad = 1$$

$$u \cdot v = v \cdot w = u \cdot w = 0$$

$$w = u \times v \qquad \text{(right-handed)}$$

$\|p\|\cos\theta$

$$\vec{p} = \underbrace{(\vec{p} \cdot \vec{u})}_{\text{(projection)}}\vec{u} + \underbrace{(\vec{p} \cdot \vec{v})}\vec{v} + \underbrace{(\vec{p} \cdot \vec{w})}\vec{w}$$

投影(在该方向长度)

Questions?

Matrices

- Magical 2D arrays that haunt in every CS course

- In Graphics, pervasively used to represent transformations

  平移 旋转 剪切 缩放

  - Translation, rotation, shear, scale
    (more details in the next lecture)

What is a matrix

- Array of numbers ($m \times n$ = $\overset{3}{m}$ rows, $\overset{2}{n}$ columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition and multiplication by a scalar are trivial: element by element

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix}$$

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 9 & \overset{27}{?} & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & \underset{12}{?} \end{pmatrix}$$

- Element (i, j) in the product is
  the dot product of row i from A and column j from B

Matrix-Matrix Multiplication

- Properties
  - **Non-commutative** 无交换律. X.
    (AB and BA are different in general)
  - Associative and distributive
    - (AB)C=A(BC) 结合律 ✓
    - A(B+C) = AB + AC 分配律 ✓
    - (A+B)C = AC + BC

Matrix-Vector Multiplication

向量认为列矩阵.

- Treat vector as a column matrix (m×1)

- Key for transforming points (next lecture)

- Official spoiler: 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

<镜像>

Transpose of a Matrix

- Switch rows and columns (ij -> ji)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix}$$

- Property

$$\underline{(AB)^T = B^T A^T}$$

Identity Matrix and Inverses

单位矩阵.

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

对角线都是1.

矩阵的逆

$$\begin{cases} AA^{-1} = A^{-1}A = I \\ (AB)^{-1} = B^{-1}A^{-1} \end{cases}$$

矩阵形式的向量乘法

Vector multiplication in Matrix form

- Dot product?

$$\underline{\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}}$$

$$= \begin{pmatrix} x_a & y_a & z_a \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} x_a x_b + y_a y_b + z_a z_b \end{pmatrix}$$

- Cross product?

$$\vec{a} \times \vec{b} = \underline{\underline{A^*}} b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

An Example of General Transformation

The Sponza Scene, rendered by Lingqi Yan using Real-time Ray Tracing (RTRT)

Questions?

Next

- Transform!

Transformers: The Last Knight, 2017 movie

Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)