5118014 Programming Language Theory

Ch 10. Recursive Functions

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FAE with Named Local Function

$$e \ ::= \cdots \mid \mathrm{def}\, x(x) = e \ \mathrm{in}\ e$$

- $\mathrm{def}\, f(a) = e_1 \ \mathrm{in}\ e_2$ defines a function whose name is f and parameter is a and body is e_1, and to be used in e_2
- We can define a variable using the function definition feature
  - $\mathrm{val}\, x = e_1 \ \mathrm{in}\ e_2$
  - $\mathrm{def}\, f(x) = e_2 \ \mathrm{in}\ f(e_1)$

Semantics

$$\frac{\sigma[f \mapsto < \lambda a.e_1, \sigma >] \vdash e_2 \Rightarrow v}{\sigma \vdash \text{def } f(a) = e_1 \text{ in } e_2 \ \Rightarrow v}$$

$\text{def } f(a) \ = \ e_1 \text{ in } e_2$

$\text{val } f \ = \lambda(a).e_1 \text{ in } e_2$

$$\frac{\sigma \vdash e_1 \Rightarrow v_1 \qquad \sigma[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash \text{val } x{=}e_1 \text{ in } e_2 \Rightarrow v_2}$$

Defining Recursive Function

```
def sum(x: Int): Int =
  if (x == 0)
    0
  else
    x + sum(x - 1)
```

- add if-expression
- let a body of a function refer the function itself

If-Expression

$e ::= \cdots \mid \text{if0 } e\ e\ e$

$$\frac{\sigma \vdash e_1 \Rightarrow 0 \qquad \sigma \vdash e_2 \Rightarrow v}{\sigma \vdash \mathsf{if0}\ e_1\ e_2\ e_3 \Rightarrow v}\ [\text{If0-Z}] \qquad \frac{\sigma \vdash e_1 \Rightarrow v' \qquad v' \neq 0 \qquad \sigma \vdash e_3 \Rightarrow v}{\sigma \vdash \mathsf{if0}\ e_1\ e_2\ e_3 \Rightarrow v}\ [\text{If0-Nz}]$$

- $\text{if0 } e_1\ e_2\ e_3$ is a condition expression that is evaluated to the evaluation result of e_2 if e_1 is evaluated to zero; otherwise, that of e_3.

Recursive Function Definition: Problem

```
def sum(x: Int): Int =
  if (x == 0)
    0
  else
    x + sum(x - 1)

println(sum(10))
```

def sum(x)=if0 x 0 (x + sum(x-1)) in sum 10

$$\frac{\sigma[f \mapsto < \lambda a. e_1, \sigma >] \vdash e_2 \Rightarrow v}{\sigma \vdash \mathrm{def}\ f(a) = e_1\ \mathrm{in}\ e_2 \ \Rightarrow v}$$

Recursive Function Definition: Solution

FAE

$$\frac{\sigma[f \mapsto < \lambda a.e_1, \sigma >] \vdash e_2 \Rightarrow v}{\sigma \vdash \mathrm{def}\, f(a) = e_1 \,\mathrm{in}\, e_2 \;\Rightarrow v}$$

RFAE

$$\frac{\sigma' = \sigma[f \mapsto < \lambda a.e_1, \sigma' >] \qquad\qquad \sigma' \vdash e_2 \Rightarrow v}{\sigma \vdash \mathrm{def}\, f(a) = e_1 \,\mathrm{in}\, e_2 \;\Rightarrow v}$$

Interpreter

```
def interp(e: Expr, env: Env): Value = e match {
  ...
  case If0(c, t, f) =>
    interp(if (interp(c, env) == NumV(0)) t else f, env)
  case Rec(f, x, b, e) =>
    val cloV = CloV(x, b, env)
    val nenv = env + (f -> cloV)
    cloV.e = nenv
    interp(e, nenv)
}
```

Example

$$\frac{\sigma' = [f \mapsto <\lambda a.\, e_1, \sigma'>] \qquad \sigma' \vdash e_2 \Rightarrow v}{\sigma \vdash \text{def } f(a) = e_1 \text{ in } e_2 \Rightarrow v} \qquad \frac{\sigma \vdash e_1 \Rightarrow \langle \lambda x.e, \sigma' \rangle \qquad \sigma \vdash e_2 \Rightarrow v' \qquad \sigma'[x \mapsto v'] \vdash e \Rightarrow v}{\sigma \vdash e_1\ e_2 \Rightarrow v} \text{[App]}$$

- σ ⊢ def f(x)=if0 x 0 (x + f(x – 1)) in f 1 ⇒ 1
  - $\sigma' = [$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >]$
  - $\sigma' \vdash$ f 1 $\Rightarrow 1$
    - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >] \vdash$ f $\Rightarrow < \lambda x.$ if0 x 0 (x + f (x – 1))$, \sigma' >$
    - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >] \vdash 1 \Rightarrow 1$
    - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >, x \mapsto 1] \vdash$ if0 x 0 (x + f(x – 1)) $\Rightarrow 1$
      - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >, x \mapsto 1] \vdash$ 1 + f (1 – 1) $\Rightarrow 1$
        - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >, x \mapsto 1] \vdash$ f $0 \Rightarrow 0$
          - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >, x \mapsto 1] \vdash$ f $\Rightarrow < \lambda x.$ if0 x 0 (x+f(x – 1))$, \sigma' >$
          - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >, x \mapsto 1] \vdash 0 \Rightarrow 0$
          - $[$f $\mapsto< \lambda x.$ if0 x 0 (x + f(x – 1))$, \sigma' >, x \mapsto 0] \vdash$ if0 x 0 (x+f(x – 1)) $\Rightarrow$ 0