SYMMETRIES OF STOCHASTIC DIFFERENTIAL EQUATIONS

Daniel Berger

Abstract. — The classes `smfbook` and `smfart` are intended to help the preparation in LaTeX of the monographs and articles to be published by the Société mathématique de France. They require LaTeX 2_ε and the AMS-LaTeX packages.

This paper exhibits the main features of these classes.

Résumé. — Les classes `smfbook` et `smfart` sont destinées à la composition en LaTeX des monographies et articles édités par la Société mathématique de France. Elles nécessitent LaTeX 2_ε ainsi que les macros LaTeX de l'AMS.

Ce document en présente l'utilisation.

Contents

Key words and phrases. — Lie group symmetries, invariance properties, Riemannian manifold, stochastic process, reflection principle.

1. Introduction

The Société mathématique de France provides the authors of its publications with LaTeX 2_ε "document classes" (`smfbook` for monographs and `smfart` for articles). The authors should submit their articles as LaTeX 2_ε files prepared using these classes or with the $\mathcal{AMS}$-LaTeX classes (`amsbook` or `amsart`) provided by the AMS (see § 11 for the compatibility between these classes).

This text contains a user's guide to these classes as well as some elementary typographical rules that the authors should read thoroughly before preparing their manuscript as a LaTeX 2_ε file: sending a file prepared with the SMF classes following these rules will reduce the number of errors introduced by the editing process and will save the author proofreading time. This will hence minimize the time needed for the article to reach its printed form (and reduce the publishing costs).

2. Preserving the drift

Throughout this section we will fix a system of stochastic differential equations on the open subset $\mathbf{U} \subset \mathbb{R}^n$ as introduced above

$$\mathrm{dX}^{\mathrm{i}}_{\mathrm{t}} = \sigma_{\mathrm{i,j}}(\mathbf{Z}_{\mathrm{t}})\mathrm{dW}^{\mathrm{j}}_{\mathrm{t}} + \mu_{\mathrm{i}}(\mathbf{Z}_{\mathrm{t}})\mathrm{dt} \tag{1}$$

Furthermore, let phi : $\mathbf{U} \to V$ denote a diffeoemorphism into an open subset V of $\mathbb{R}^m$. To be more specific, we are interested in ϕ being a coordinate transfomation from chart to chart. In this setting, the following question naturally arises: When do the coefficients μ_{i} of the drift term in the above equation transforms like the coefficients of a covariant tensor of rank one, i.e. under which assumptions do we have

$$\mathrm{dZ}'^{,\mathrm{j}}_{\mathrm{t}} = [\dots] + \mu'_{\mathrm{j}}(\mathbf{Z}_{\mathrm{t}})\mathrm{dt} \tag{2}$$

where

$$\mu'_{\mathrm{j}} = \frac{\partial \mathrm{x}^{\mathrm{k}}}{\partial \mathrm{x}'^{,\mathrm{j}}} \mu_{\mathrm{k}} \tag{3}$$

The following result specifies two conditions under which the transfomation law in fact holds:

Lemma 2.1. — *Equation transforms like a covariatn tensor of first degree if the following two conditions holds:*

1. *The local coordinates* x^{i} *are harmonic with respect to g, i.e.* $\mathrm{g}^{\mathrm{ij}}\Gamma^{\mathrm{k}}_{\mathrm{ij}} = 0$ *for all* $1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n}$.

2. *The change of coordinates is harmonic, i.e. each component $x'^{,j}$ is again harmonic functions with respect to* $\{\mathrm{x}^{\mathrm{i}}\}_{1\leq \mathrm{i}\leq \mathrm{n}}$

Proof. — By applying the lemma of Itô, the drift term of the transformed process $Z'^{k} = \phi^{\mathrm{k}}(\mathbf{Z}_{\mathrm{t}})$ is given by the following formula

$$\mathrm{d}\mathbf{Z}_{\mathrm{t}_{\mathrm{t}}}'^{\mathrm{k}} = [\dots] + \frac{\partial \mathrm{x}'^{\mathrm{k}}}{\partial \mathrm{x}^{\mathrm{l}}}\mu^{\mathrm{l}} + \mathrm{g}^{\mathrm{ij}}\frac{\partial^2 \mathrm{x}'^{\mathrm{k}}}{\partial \mathrm{x}_{\mathrm{i}}\partial \mathrm{x}_{\mathrm{j}}} \tag{4}$$

In particular, as the proof is completed as soon as we have shown that the second term on the right hand side of the above equaiton vanishes, i.e.:

$$\mathrm{g}^{\mathrm{ij}}\frac{\partial^2 \mathrm{x}'^{\mathrm{k}}}{\partial \mathrm{x}_{\mathrm{i}}\partial \mathrm{x}_{\mathrm{j}}} = 0 \tag{5}$$

for all k.

For this, recall that condition is equivalent to

$$\triangle \mathrm{x}'^{,\mathrm{i}} = 0 \tag{6}$$

for all $1 \leq \mathrm{i} \leq \mathrm{n}$, where $\triangle$ denotes the Laplace-Beltrami operator associated to the metric g which g is locally given by

$$\triangle = \mathrm{g}^{\mathrm{ij}}\left(\frac{\partial^2}{\partial \mathrm{x}^{\mathrm{i}}\partial \mathrm{x}^{\mathrm{j}}} - \Gamma^{\mathrm{k}}_{\mathrm{ij}}\frac{\partial}{\partial \mathrm{x}^{\mathrm{k}}}\right) \tag{7}$$

As 1) is equivalent to $\mathrm{g}^{\mathrm{ij}}\Gamma^{\mathrm{k}}_{\mathrm{ij}} = 0$ for all $1 \leq \mathrm{k} \leq \mathrm{n}$, it follows from conditions 2) that

$$0 = \mathrm{g}^{\mathrm{ij}}\frac{\partial^2}{\partial \mathrm{x}'^{\mathrm{i}}\partial \mathrm{x}^{\mathrm{j}}} \tag{8}$$

for all k , which in conjunction with step yields the result. □

***Example 2.2* (Standard complex space).** — Let $\mathrm{U} \subset \mathbb{C}^{\mathrm{n}}$ and $\sigma^{\mathrm{ij}} = \delta^{\mathrm{ij}}$ so that g is given by the flat Euclidean metric. By function theroy, it is known that the real part $\mathrm{x}^{\mathrm{k}} = \mathfrak{Re}\mathrm{z}^{\mathrm{k}}$ as well as the imaginary part $\mathrm{y}^{\mathrm{k}} = \mathfrak{Im}\mathrm{z}^{\mathrm{k}}$ of the complex standard coordinates f $\mathrm{z}^{\mathrm{k}} = \mathrm{x}^{\mathrm{k}} + \sqrt{-1}\mathrm{y}^{\mathrm{k}}$ are harmonic function with respect to the standard metric structure. Furthermore, any holomorphic function f on U are harmonic. Hence by lemma, every

Consider the system

$$\mathrm{dU} = \tag{9}$$

defined on

***Example 2.3* (Local Kählerain space).** — Let $\mathrm{U} \subset \mathbb{C}^{\mathrm{n}}$ and

3. Preserving the covariance

As in the section, we will consider a system of stochastic differential equations on the open subset $\mathbf{U} \subset \mathbb{R}^n$ as introduced above

In this setting, the following question naturally arises: When do the coefficients σ_{ij} defining the covariance structure transform like the coefficients of a covariant tensor of rank two, i.e. under which assumptions do we have

$$\mathrm{dX}'^{\mathrm{i}}_{\mathrm{t}} = \sigma'_{\mathrm{i,j}}(\mathbf{Z}_{\mathrm{t}})\mathrm{dW}^{\mathrm{j}}_{\mathrm{t}} + [\ldots]\mathrm{dt} \tag{10}$$

where

$$\sigma'_{k,l} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}\sigma'_{i,j} \tag{11}$$

4. Symmetries of certain stochastic differential equations

In the sequel, let $(\mathbf{M}, g)$ be a Riemannain manifold with a smooth Lie group action $\phi : \mathrm{G}\times\mathbf{M} \to \mathbf{M}$ which preserves the metric g so that $\sigma^* g = g$ for all $\sigma \in \mathrm{G}$. For convenience, we will assume that $\mathbf{M}$ is given by an open subset $\mathbf{U}$ of $\mathbb{R}^m$ whit local coordinates $(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{m})$.

Furthermore, we will fix a system of stochastic differential of the following form:

$$\mathrm{d}\mathbf{Z}_{\mathrm{t}} = \sigma(\mathbf{Z}_{\mathrm{t}})\mathrm{d}\mathbf{W}_{\mathrm{t}} + \mathbf{v}(\mathbf{Z}_{\mathrm{t}})\mathrm{dt} \tag{12}$$

Here σ denotes m×m matrix with entries $\sigma_{\mathrm{ij}}, 1 \leq$i,j$\leq$m in the set $\mathscr{C}^\infty(\mathbf{U})$ of smooth, $\mathbb{R}$-valued functions on $\mathbf{U}$ and $\mathbf{W}_{\mathrm{t}} = (\mathrm{W}_{1,\mathrm{t}}, \ldots, \mathrm{W}_{\mathrm{m,t}})$ is a random vector of standard, uncorrelated Brownian motions with respect to a probability space $(\sigma, \mathscr{A}, \mathbb{P})$. The drift term is supposed to be given by a smooth vector field $\mathbf{v} : \mathbf{M} \to \mathrm{T}\mathbf{M}$ whose locally trivialization in the aforementioned coordinates is represented by the smooth vector $(\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{m})$.

Recall that each vector ξ of the Lie algebra $\mathrm{Lie(G)} = \mathfrak{g}$ generates a smooth flow $\Phi_\xi : \mathbb{R}\times\mathbf{M} \to \mathbf{M}$ given by

$$\Phi_\xi(\mathrm{t, x}) = \mathbf{exp}(\mathrm{t}\xi).\mathrm{x} \tag{13}$$

which is a solution of the differential equation $\dot{\Phi}(\mathrm{t,x}) = \mathbf{X}_\xi(\Phi(\mathrm{t,x}))$. Here, $\mathbf{X}_\xi$ denotes the smooth vector field generated by the one-parameter flow Φ. To abbreviate the notation, we will frequently just set $\mathbf{X}_\xi = \xi$.

Recall that

Lemma 4.1. — *Let $\Phi : \mathbf{M} \to \mathbf{M}$ be an isometry with respect to the metric g given by the local matrix representation $g = \left(\sigma\sigma^{\mathsf{T}}\right)^{-1}$ and let $\mathbf{Z}_t$ be a solution of*

$$d\mathbf{Z}_t = \sigma(\mathbf{Z}_t)d\mathbf{W}_t, \tag{14}$$

then $\widetilde{\mathbf{Z}}_t = \Phi \circ \mathbf{Z}_t$ *is again a solution of 14, i.e.*

$$d\widetilde{\mathbf{Z}}_t = \sigma(\widetilde{\mathbf{Z}}_t) d\mathbf{W}_t. \tag{15}$$

Proof. — Applying the lemma of Itô directly yields

$$\mathrm{d}\widetilde{\mathbf{Z}}_{\mathrm{t}} = \mathrm{d}\Phi\left(\mathbf{Z}_{\mathrm{t}}\right) \sigma(\mathbf{Z}_{\mathrm{t}}) \mathrm{d}\mathbf{W}_{\mathrm{t}} \tag{16}$$

As booth processes $\mathbf{Z}_{\mathrm{t}}$ as well as $\widetilde{\mathbf{Z}}_{\mathrm{t}}$ are drifless, the lemma is proven as soon as we can verify that the quadratic variation $d[\mathbf{Z}_{\mathrm{t}}, \mathbf{Z}_{\mathrm{t}}]$ of $\mathbf{Z}_{\mathrm{t}}$ and the corresponding quadratic variation $d[\widetilde{\mathbf{Z}}_{\mathrm{t}}, \widetilde{\mathbf{Z}}_{\mathrm{t}}]$ of $\widetilde{\mathbf{Z}}_{\mathrm{t}}$ coincides. Here $d[\mathbf{Z}_{\mathrm{t}}, \mathbf{Z}_{\mathrm{t}}]$ and so $d[\widetilde{\mathbf{Z}}_{\mathrm{t}}, \widetilde{\mathbf{Z}}_{\mathrm{t}}]$ are understood to be matrices given by

$$\mathrm{d}[\mathbf{Z}_{\mathrm{t}}, \mathbf{Z}_{\mathrm{t}}] = ([\mathrm{dZ}_i, \mathrm{dZ}_j])_{1 \leq, \mathrm{i}, \mathrm{j} \leq \mathrm{m}} = \mathrm{d}\mathbf{Z}_{\mathrm{t}} \cdot \mathrm{d}\mathbf{Z}_{\mathrm{t}}{}^{\mathsf{T}} \tag{17}$$

and analog for $\widetilde{\mathbf{Z}}_{\mathrm{t}}$.

Since Φ is an isometry with respect to $\mathrm{g} = \left(\sigma\sigma^{\mathsf{T}}\right)^{-1}$, it follows that

$$\mathrm{g}_{\Phi(z)}\left(\mathrm{d}\Phi(\mathrm{z})\cdot, \mathrm{d}\Phi(\mathrm{z})\cdot\right) = \mathrm{g}_{\mathrm{z}}(\cdot, \cdot) \tag{18}$$

for all z in $\mathbf{M}$ or equivalently, written as matrix equation

$$\mathrm{d}\Phi^{\mathsf{T}}(\mathrm{z}) \left(\sigma(\Phi\left(z\right))\sigma^{\mathsf{T}}(\Phi\left(z\right))\right)^{-1} \mathrm{d}\Phi(\mathrm{z}) = \left(\sigma(\mathrm{z})\sigma^{\mathsf{T}}(\mathrm{z})\right)^{-1} \tag{19}$$

From equation 19 we infere by multiplying from the left with the inverse of $\mathrm{d}\Phi^{\mathsf{T}}(\mathrm{z})$ and from the right with the inverse of $\mathrm{d}\Phi(\mathrm{z})$ respectively the equation

$$\left(\sigma(\Phi\left(z\right))\sigma^{\mathsf{T}}(\Phi\left(z\right))\right)^{-1} = \left(\mathrm{d}\Phi^{\mathsf{T}}(\mathrm{z})\right)^{-1} \left(\sigma(\mathrm{z})\sigma^{\mathsf{T}}(\mathrm{z})\right)^{-1} (\mathrm{d}\Phi(\mathrm{z}))^{-1} \tag{20}$$

which is equivalent to

$$\sigma(\Phi\left(z\right))\sigma^{\mathsf{T}}(\Phi\left(z\right)) = \mathrm{d}\Phi(\mathrm{z})\sigma(\mathrm{z})\sigma^{\mathsf{T}}(\mathrm{z})\mathrm{d}\Phi^{\mathsf{T}}(\mathrm{z}) \tag{21}$$

Using this equality and the property $d[\mathbf{W}_{\mathrm{t}}, \mathbf{W}_{\mathrm{t}}] = (\delta_{ij})_{1 \leq, \mathrm{i}, \mathrm{j} \leq \mathrm{m}} \mathrm{dt}$, we compute

$$\begin{aligned} d[\widetilde{\mathbf{Z}}_{\mathrm{t}}, \widetilde{\mathbf{Z}}_{\mathrm{t}}] &= \mathrm{d}\Phi\left(\mathbf{Z}_{\mathrm{t}}\right) \sigma(\mathbf{Z}_{\mathrm{t}}) \mathrm{d}\mathbf{W}_{\mathrm{t}} \cdot \mathrm{d}\mathbf{W}_{\mathrm{t}}^{\mathsf{T}} \sigma(\mathbf{Z}_{\mathrm{t}})^{\mathsf{T}} \mathrm{d}\Phi\left(\mathbf{Z}_{\mathrm{t}}\right)^{\mathsf{T}} \\ &= \mathrm{d}\Phi\left(\mathbf{Z}_{\mathrm{t}}\right) \sigma(\mathbf{Z}_{\mathrm{t}}) \sigma(\mathbf{Z}_{\mathrm{t}})^{\mathsf{T}} \mathrm{d}\Phi\left(\mathbf{Z}_{\mathrm{t}}\right)^{\mathsf{T}} \mathrm{dt} \\ &= \sigma(\Phi\left(\mathbf{Z}_{\mathrm{t}}\right))\sigma^{\mathsf{T}}(\Phi\left(\mathbf{Z}_{\mathrm{t}}\right))\mathrm{dt} \\ &= \sigma(\widetilde{\mathbf{Z}}_{\mathrm{t}})\sigma^{\mathsf{T}}(\widetilde{\mathbf{Z}}_{\mathrm{t}})\mathrm{dt} \end{aligned} \tag{22}$$

where we have used the equation 21 in the third step. □

***Theorem 4.2* (Transfomration Law).** — *Let* $(\mathbf{M}, g)$ *and* G *be as above and consider the following system of stochastic differnetial equations on* $\mathbf{M} = \mathbf{U}$

$$d\mathbf{Z}_t = \sigma(\mathbf{Z}_t) d\mathbf{W}_t + \xi(\mathbf{Z}_t) dt, \tag{23}$$

then the transformed solution $\widetilde{\mathbf{Z}}_t = \Phi_\eta(t,\cdot) \circ \mathbf{Z}_t$ *solves the transformed system given by*

$$d\widetilde{\mathbf{Z}}_t = \sigma(\widetilde{\mathbf{Z}}_t) d\widetilde{\mathbf{W}}_t + \left[\eta(\mathbf{Z}_t) + (A\, d_\eta \xi)\,(\mathbf{Z}_t)\right] dt, \tag{24}$$

with $\widetilde{\mathbf{W}}_t$ *following the same distribution as* $\mathbf{W}_t$*.*

Proof. — First of all, by the multi-dimensional Lemma of Itô we deduce that

$$\begin{aligned} \mathrm{d}\widetilde{\mathbf{Z}}_{\mathrm{t}} &= \frac{\mathrm{d}\Phi_\eta}{\mathrm{dx}}(t, \mathbf{Z}_{\mathrm{t}})\,(\sigma(\mathbf{Z}_{\mathrm{t}})\mathrm{d}\mathbf{W}_{\mathrm{t}} + \xi(\mathbf{Z}_{\mathrm{t}})) \\ &+ \left[\frac{\mathrm{d}\Phi_\eta}{\mathrm{dt}}(t, \mathbf{Z}_{\mathrm{t}}) + \frac{1}{2}\mathrm{d}\mathbf{W}_{\mathrm{t}}^T \sigma(\mathbf{Z}_{\mathrm{t}})^T \mathbf{Hess}(\Phi_\eta)\sigma(\mathbf{Z}_{\mathrm{t}})\mathrm{d}\mathbf{W}_{\mathrm{t}}\right] \mathrm{dt} \end{aligned} \tag{25}$$

As Π_η is assumed to be harmonic, $\mathbf{Hess} = 0$. As the vector field generated by η evaluated at point $\mathbf{Z}_{\mathrm{t}}$ is given by

$$\frac{\mathrm{d}\Phi_\eta}{\mathrm{dt}}(t, \mathbf{Z}_{\mathrm{t}}) = \eta(\Phi_\eta(\mathrm{t}, \mathbf{Z}_{\mathrm{t}})) = \eta(\mathrm{t}, \widetilde{\mathbf{Z}}_{\mathrm{t}}) \tag{26}$$

per definition, the second line of equation 25 reduces to $\eta(\mathrm{t}, \widetilde{\mathbf{Z}}_{\mathrm{t}})\mathrm{dt}$ and hence, it follows

$$\mathrm{d}\widetilde{\mathbf{Z}}_{\mathrm{t}} = \frac{\mathrm{d}\Phi_\eta}{\mathrm{dx}}(t, \mathbf{Z}_{\mathrm{t}})\,(\sigma(\mathbf{Z}_{\mathrm{t}})\mathrm{d}\mathbf{W}_{\mathrm{t}} + \xi(\mathbf{Z}_{\mathrm{t}})) + \eta(\mathrm{t}, \widetilde{\mathbf{Z}}_{\mathrm{t}})\mathrm{dt} \tag{27}$$

Now, recall from that

the derivative of the flow $\Phi(t, \dot{)}$ with redspect to t, the above equations reduces to the following form:

or written down in local coordinates $(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{m})$ □

In particular, for vector fields d

The file sent by the author will be adapted to the style of the journal where it will be published by the editorial board of the Société mathématique de France. It is therefore *important* that the LaTeX 2$_\varepsilon$ file is prepared in a very standard way, in particular by a *systematic* use of theorem- and proof-like environments (see § 9), of `\label` and `\ref` commands for referring to the corresponding numbers, and of `\cite` for bibliographical references. Moreover, “home” macros must be clearly written in the preamble. *No* “home” macro will be used in the title, the address, the abstracts (French and English), the keywords.

4.1. Horizontal and vertical spacing

— Delete all spacing commands like `\,` or `\;` or `\!` *before or after* mathematical symbols, parentheses, punctuation marks, etc. Horizontal spaces (in mathematical mode in particular) are handled automatically by TeX, the author should not add any.

— On the other hand, the author may impose indivisible spaces in places where she/he does not want a carriage return, e.g. `Tintin~\cite{RG3}` instead of `Tintin \cite{RG3}`.
— The author should not type any space or carriage return *before* punctuation marks. However, such a space or carriage return *always* comes after punctuation marks
— No space *before* a closing parentheses or bracket, as well as *after* an opening parentheses or bracket.
— Do not use any `\linebreak`, `\\`, `\pagebreak`, `\newpage`, etc. in the text.
— Avoid commands as `\hskip`, `\hspace` or `\vskip`, `\vspace` in the text.

4.2. Punctuation marks

— Do not put *any* punctuation marks at the end of any title:
  — `\section{Introduction}` and not `\section{Introduction.}`
  — `\begin{remark}` and not `\begin{remark.}`
  — etc.
— In text mode, punctuation marks are typed *outside of* the mathematical mode. Write for example:
  "... `the level $\eta_0$:  $$ A=B.$$`"
and not
  "... `the level $\eta_0:$ $$A=B.$$`"
— Concerning points of suspension:
  — replace `...` with `\ldots\ ` in the text (in English);
  — replace `...` or `\ldots` with `\cdots` between operators (as in, for instance, $A < \cdots < B$, $A + \cdots + B$ or $A = \cdots = B$) and with `\dots` or `\ldots` for mathematical punctuation (for instance $i = 1, \ldots, n$);
  — suppress `...` after "etc.".
— For a product, use `\cdot` and not `.`; In the same way, rewrite formulas like $h(.)$ or $(.,.)$ as $h(\cdot)$ or $(\cdot,\cdot)$.
— Replace explicit hyphenation (as in `presenta-tion`) with optional hyphenation `\-` (as in `presenta\-tion`). Of course, ordinary hyphens are kept for compound words.

4.3. Titles. — Titles begin with an upper case letter and are typed in *lower case letters*. When necessary, LaTeX will produce an upper case output. No punctuation marks should be inserted at the end of titles (see above).

4.4. Language. — The author should follow the rules of the language she/he uses, in particular when typing numbers: in French, one should write "deux nombres égaux à 2" and in the file one should type

```
deux nombres \'egaux \`a $2$.
```

On the other hand, recall that French upper case letters take accents as do lower case letters.

4.5. Numbering

- — Use as much as possible the automatic numbering and the corresponding LaTeX 2_ε commands `\label`, `\ref`. To this end, keep *a consistent numbering convention.* Do not "ask" commands such as `\section` or `\begin{theoreme}` to produce a complicated output. Recall that the final output will be done by the editorial board of the Société mathématique de France: please, try to help the secretary in her/his task.
- — Use a simple logic for internal references:
  - — `\label{sec:1}` for the first section,
  - — `\label{th:invfunct}` for the inverse function theorem,
  - — `\label{rem:stupid}` for an interesting remark.
- — Do not number equations which are not referred to in the text.

4.6. The mathematical mode

- — Do not put pieces of text between `$ $` to change their style. The mathematical mode should only be used for writing mathematical formulas.
- — The numbers written as digits should be typed in mathematical mode, even if this does not appear to be necessary.
- — Do not add horizontal spaces in mathematical formulas. When necessary, the editorial board will do it.
- — Use the right mathematical TeX or LaTeX symbol at the right place: for instance, the symbols `<` and `>` should not be used for making a bracket $\langle,\rangle$; this bracket is obtained with `$\langle,\rangle$`.
- — Please, before using your own solution, check all available AMS-LaTeX capabilities to place and cut mathematical formulas in display style (see [**5**]).

4.7. The bibliography

- — Make a uniform bibliography and do not change the convention according to the entry (use BibTeX for instance).
- — *Systematically* use the `\cite` command to cite the entries of the bibliography.

5. The environment

The Société mathématique de France (SMF) provides authors with the following files:

- — two class files `smfbook.cls` (for monographs) and `smfart.cls` (for articles),

— two BibTeX style files:
  — `smfplain.bst` (for numerical citations) and
  — `smfalpha.bst` (for alphabetical citations),
— a supplementary package `smfthm.sty` described in § 10,
— a supplementary package `smfenum.sty` for enumerations in the French style,
— a supplementary package `bull.sty` for articles submitted to the *Bulletin*.

They may be obtained on the web site of the SMF:

`http://smf.emath.fr/`

under the heading `Publications/Formats`.

These classes have been written to remain compatible with the `amsbook` and `amsart` classes developped by the American Mathematical Society (AMS). To use them, you need:

— LaTeX 2_ε, preferebly some recent version. The class doesn't work with the old LaTeX2.09 version which has been obsolete for years;
— the various packages furnished by the American Mathematical Society; it is better to have the November 1996 version although it should work with the 1995 one.
— To typeset an index, it is better to have the `multicol.sty` *package* available.

The file `smfbook.cls` (*resp.* `smfart.cls`) is used instead of `amsbook.cls` (*resp.* `amsart.cls`) and has to be put in the directory containing TeX inputs. In order to use the package `smfthm` (see § 10) or `bull.sty`, one should put the files `smfthm.sty` or `bull.sty` in the same directory.

Many standard packages add capabilities to LaTeX 2_ε. In this respect, we suggest using

— `epsfig.sty`, [**7**], for the inclusion of (encapsulated) PostScript pictures;
— `graphics.sty` or `graphicx.sty`, [**8, 9**], in order to include pictures drawn by LaTeX;
— `babel.sty`, [**6**], for a text written in various languages (hyphenation, ...);
— `xypic.sty`, [**11**], for the diagrams;
— BibTeX, [**1**, Appendix B] or [**10**], for the bibliography.

6. Structure of the document

A document typeset with one of the classes `smfbook` or `smfart` has the following structure. Fields within brackets are optional.

`\documentclass[`⟨*options*⟩`]{smfbook` or `smfart}`

Preamble (packages, macros, theoremlike environments, . . .) e.g.

```
\usepackage[francais,english]{babel}
\usepackage{smfthm}
\usepackage{bull}   (for articles submitted to the Bulletin)
\theoremstyle{plain} \newtheorem{scholie}{Scholie}
\author[⟨short name⟩]{⟨Firstname Lastname⟩}
\address{⟨line 1⟩\\ ⟨line 2⟩\\ ...⟨line n⟩}
\email{⟨email address⟩}
\urladdr{⟨WWW address⟩}
\title[⟨short title⟩]{⟨title of text⟩}
\alttitle{⟨title in the other language (French or English)⟩}
\begin{document}
\frontmatter
\begin{abstract}
    ⟨Abstract in the main language of text⟩
  \end{abstract}
\begin{altabstract}
  ⟨Abstract in the other language (French or English)⟩
\end{altabstract}
\subjclass{⟨AMS classification⟩}
\keywords{⟨Key words⟩}
\altkeywords{⟨Mots-clefs in the other language (French or
  English)⟩}
  \translator{⟨Firstname Lastname⟩}
  \thanks{⟨Grants⟩}
  \dedicatory{⟨dedication⟩}
\maketitle
  \tableofcontents ⟨if needed⟩
\mainmatter
Main body of the text
\backmatter
Bibliography, index, etc.
\end{document}
```

Remarks

— If there are many authors, or if an author has more than one address, one may type as many

```
\author{⟨author⟩}
\address{⟨address⟩}
```

`\email{`$\langle$*email address*$\rangle$`}`
`\urladdr{`$\langle$*WWW address*$\rangle$`}`

commands as needed, in the right order of course.

— All data introduced before the `\maketitle` command will be used for different purposes: back cover, advertisement, electronic abstracts, data banks. It is therefore important that no personal macro is used in the corresponding fields.
— Do not hesitate to be prolix when filling the field `\subjclass`. One may consult for instance the web site
`http://www.ams.org/msc/`

7. Class options

These options are entered the following way:

`\documentclass[`$\langle$*option1,option2,...*$\rangle$`]{smfbook` or `smfart}`

Default options are indicated with a star.

7.1. Usual options

— `(⋆) a4paper`, A4 printing
— `letterpaper`, US Letter printing, to make easier the typesetting of documents in the United States
— `draft`, preliminary draft, *overfull hbox*es are shown by black rules;
— `(⋆) leqno`, equation numbers on the left
— `reqno`, equation numbers on the right
— `(⋆) 10pt`, normal character size = 10 points
— `11pt`, normal character size = 11 points
— `12pt`, normal character size = 12 points

7.2. Language of the text

— `(⋆) francais`, if the main language of the text is French
— `english`, if it is English.

7.3. Remark. — Do not mix up the `francais` or `english` options of the SMF class with the `francais` or `english` options of `babel`: the latter has to be entered as indicated in the example of § 6.

8. Sectioning commands

As in any LaTeX 2_ε class, some commands are devoted to the sectioning of the document:

```
\part
\chapter          smfbook only
\section
\subsection
\subsubsection
\paragraph
\subparagraph
```

The table of contents is inserted automatically with `\tableofcontents`. The macro

`\appendix`

starts the appendix.

The bibliography is entered as usual,

\begin{thebibliography}{⟨*longest label*⟩}
⟨*Bibliography entries*⟩
\end{thebibliography}

The use of BibTeX is recommended, see for example [**1**, Appendix B] and [**10**] for an introduction. The BibTeX styles `smfplain.bst` and `smfalpha.bst` may be obtained on the web site `http://smf.emath.fr/` of the SMF. The bibliography is then entered as follows

\bibliographystyle{smfplain or smfalpha}
\bibliography{myfile.bib}

if `myfile.bib` is the BibTeX data file.

9. Presentation of theorems

Theorems are typeset thanks to the package `amsthm`. For details, we refer to its documentation [**5**]. One should use such environments in a *systematic* way for statements and proofs.

9.1. Theorem styles. — Three styles of theorems are defined: `plain`, `definition` and `remark`. The two last are identical and only differ from the first one in that the text of the statement is in straight letters instead of italics. All `\newtheorem`(∗) commands should be introduced clearly in the preamble.

The `\newtheorem` command creates or uses some counter in order to define the numbering of the corresponding environment.

Use the `\newtheorem*` command to get nonnumbered theoremlike environments, e.g.

`\newtheorem*{curveselectionlemma}{Curve Selection Lemma}`

Different kinds of numberings may also be introduced in the preamble, e.g. for propositions numbered alphabetically:

```
\newtheorem{theoremalph}{Proposition}
\def\thetheoremalph{\Alph{theoremalph}}.
```

9.2. Proof environment. — The proof environment
`\begin{proof} ...\end{proof}`
allows a standard presentation of proofs, beginning with "Proof" and ending with the traditional small box □.

It is possible to change the word "Proof" as in:
`\begin{proof}[Idea of proof] ...\end{proof}`
which shows

Idea of proof. — Exercise for the interested reader. □

10. The smfthm.sty package

This section describes the `smfthm.sty` package. Its use is not mandatory.

10.1. Theoremlike environments. — Some theoremlike environments are defined. They use one and the same counter.

Style	Macro LaTeX	Nom français	English name
plain	`theo`	Théorème	*Theorem*
	`prop`	Proposition	*Proposition*
	`conj`	Conjecture	*Conjecture*
	`coro`	Corollaire	*Corollary*
	`lemm`	Lemme	*Lemma*
definition	`defi`	Définition	*Definition*
remark	`rema`	Remarque	*Remark*
	`exem`	Exemple	*Example*

One uses them e.g. as follows:

```
\begin{theo}[Wiles]
If $n\geq 3$ and if $x$, $y$, $z$ are integers
such that $x^n+y^n=z^n$, then $xyz=0$.
\end{theo}
```

***Theorem 10.1* (Wiles).** — *If $n \geq 3$ and if x, y, z are integers such that $x^n + y^n = z^n$, then $xyz = 0$.*

10.2. Fixing the choice of the numbering. — The way of numbering the statements is defined by the following commands, which have to been entered *before* the `\begin{document}`:

— `\NumberTheoremsIn{`⟨*counter name*⟩`}`, indicates the level at which the statement numbers are reset to zero, (`section` for instance); the counter `smfthm` is then defined;

— `\NumberTheoremsAs{`⟨*counter name*⟩`}`, allows the statement counter to be one of the usual sectioning counters (e.g. `section`, `subsection`, `paragraph`, etc.);

— `\SwapTheoremNumbers`, to put the statement number before the statement name, as in "1.4. Theorem"

— `\NoSwapTheoremNumbers`, the converse, e.g. "Theorem 3.1"

The default options of the package are

`\NumberTheoremsIn{section}\NoSwapTheoremNumbers`

which means that the counter `smfthm` is defined and reset at the beginning of every section and that the statement numbers, which take the form

`section number.value of the counter smfthm`

are written after the statement name.

10.3. Generic statement. — The `enonce` environment allows one to typeset a generic theorem whose name changes on demand, e.g.:

```
\begin{enonce}{Assumption}
⟨...⟩
\end{enonce}
```

typesets an 'Assumption', numbered as it should be.

The `enonce` environment uses the *plain* theorem style, but one can change this style by putting another style inside brakets, e.g.:

```
\begin{enonce}[remark]{Key remark}
⟨...⟩
\end{enonce}
```

Finally, there exists a corresponding `enonce*` environment.

10.4. Other statements. — The author may introduce other kinds of theoremlike environments as explained in § 9.1. Notice, however, that in order to introduce environments numbered as the ones of `smfthm.sty`, one uses `enonce`:

`\newenvironment{scholie}{\begin{enonce}{Scholie}}{\end{enonce}}`

which should be entered *after* `\begin{document}`.

11. Adapting a manuscript from another dialect

If you already have typed your manuscript in PLAIN TEX, or in LATEX2.09, or in LATEX 2_ε, but with another class, and if you want to adapt it to the SMF classes, this paragraph will give you some hints.

11.1. From another LATEX 2_ε class. — If it is an AMS class, you'll have very little to do: for an article written in English for instance, replace

`\documentstyle[12pt,leqno]{amsart}`

with

`\documentstyle[leqno,english]{smfart}`

You'll need to enter another abstract (`altabstract`) and another title (`alttitle`), in French if your text is in English and in English otherwise.

The inverse transformation (SMF → AMS) can be done in a similar way.

If it is a standard class (`article` ou `book`), things are a bit more complicated. Be careful to type the abstracts *before* the `\maketitle`; some mathematical formulas might not work properly, but the AMS packages offer such a variety of uses, that it should not be very difficult to do.

11.2. From LATEX2.09. — In this case, you'll have to make the adjustments described in the previous paragraph, and also those needed by the LATEX2.09–LATEX 2_ε mutation. A priori, it should mostly concern the font faces commands and the conforming to the *New Font Selection Scheme* (NFSS).

11.3. From PLAIN TEX. — In this case, you have to take up your manuscript again, and replace title, theorems, sectioning and bibliographical commands, by the adequate ones, refering to the LATEX 2_ε user's guide and the recommendations above. We bring your attention to the automatic numbering of paragraphs and theoremlike environments: it might differ from the original one. Pay similar attention to your references. The macros PLAIN TEX uses to change the typefaces are most often ineffective in LATEX 2_ε, so you'll have to adapt them too. Concerning mathematics, few changes are needed, except for aligned equations and matrices.

Literature and sources

[1] L. LAMPORT. — *LaTeX: A Document Preparation System.* Second edition. Addison-Wesley, 1994.

[2] M. GOOSSENS, F. MITTELBACH, A. SAMARIN. — *The LaTeX Companion.* Addison-Wesley, 1993.

[3] M. GOOSSENS, S. RAHTZ AND F. MITTELBACH. — *The LaTeX Graphics Companion: Illustrating Documents With TeX and Postscript.* Tools and Techniques for Computer Typesetting Series, Addison-Wesley, 1996.

[4] *The Not So Short Introduction to LaTeX2e,* T. Oetiker, H. Partl, I. Hyna, E. Schlegl, `http://www.loria.fr/tex/general/flshort2e.dvi`

[5] *AMS-LaTeX version 1.2 User's Guide*, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/amslatex/amsldoc.dvi`

[6] *Babel, a multilingual package for use with LaTeX's standard document classes,* J. Braams, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/babel/babel.dvi`

[7] *The* `epsfig` *package,* S. Rahtz, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/epsfig.dvi`

[8] *The* `graphics` *package,* D. Carlisle, S. Rahtz, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/graphics.dvi`

[9] *The* `graphicx` *package,* D. Carlisle, S. Rahtz, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/graphicx.dvi`

[10] *Hypatia's Guide to BibTeX,* `http://hypatia.dcs.qmw.ac.uk/html/bibliography.html`

[11] *Xy-pic User's Guide,* K. Rose, R. Moore, `http://www.loria.fr/tex/graph-pack/doc-xypic/xyguide.dvi`

The most recent versions of macros files and of their documentations are also available by anonymous `ftp` on the CTAN sites (*Comprehensive TeX Archive Network*) In the United States, one may use the address `ftp.shsu.edu`; the sites `ftp.loria.fr` or `ftp.jussieu.fr` in France, `ftp.tex.ac.uk` in England, and `ftp.dante.de` in Germany also hold the archive.

English version 5, dated November 2019

Daniel Berger, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, F-75231 Paris cedex 05 • *E-mail :* `christia@dma.ens.fr` • *Url :* `http://smf.emath.fr/`