SYMMETRIES OF STOCHASTIC DIFFERENTIAL EQUATIONS

Daniel Berger

Abstract. — The article sums up some results concerning the transformation and invariance properties of a local, stochastic differential equations of the form

$$\mathrm{dX}^{\mathrm{i}}_{\mathrm{t}} = \sigma^{\mathrm{i}}_{\mathrm{j}}(\mathbf{Z})\mathrm{dW}^{\mathrm{j}}_{\mathrm{t}} + \mu^{\mathrm{i}}(\mathbf{Z})\mathrm{dt} \tag{1}$$

In general, the coefficients μ^{i} of the drift in **??** do not share the transformation law of a contravariant tensor of rank one due to second order differentials arising from ITO$'s$ formula. We present conditions concerning the local coordinates and the transformation itself under which the usual transformation laws hold.

Beside, we briefly examine invariance properties of differential system like under certain Lie groups G preserving the covariance structure $\sigma_{\mathrm{i,j}}$.

Contents

Key words and phrases. — Lie group symmetries, invariance properties, Riemannian manifold, stochastic process, reflection principle.

1. Introduction

Throughout this article let $\mathbf{U}$ be an open subset of $\mathbb{R}^{\mathrm{n}}$ with the standard coordinates x^{i}, $1 \leq \mathrm{i} \leq \mathrm{n}$ induced by the ambiant space $\mathbb{R}^{\mathrm{n}}$ and fix a Brownian motion $\mathbf{W}_{\mathrm{t}} = \left(\mathrm{W}^1_{\mathrm{t}}, \cdots, \mathrm{W}^{\mathrm{n}}_{\mathrm{t}}\right)$ of dimension n with respect to a probability space $(\Sigma, \mathscr{A}, \mathbb{Q})$ where we assume the $\mathrm{W}^{\mathrm{i}}_{\mathrm{t}}$ to be uncorrelated, i.e. $\mathrm{d}[\mathrm{W}^{\mathrm{i}}_{\mathrm{t}}, \mathrm{W}^{\mathrm{j}}_{\mathrm{t}}] = 0$ for all $1 \leq \mathrm{i} < \mathrm{j} \leq \mathrm{n}$.

Our object of interest is an n−dimensional stochastic process $\mathbf{X}_{\mathrm{t}}$ with components $\mathrm{X}^{\mathrm{i}}_{\mathrm{t}}$, $1 \leq \mathrm{i} \leq \mathrm{n}$ in $\mathbf{U}$ for $\mathrm{t} \in [0, \mathrm{S}]$ solving the following stochastic differential equation

$$\mathrm{d}\mathbf{X}_{\mathrm{t}} = \sigma\left(\mathbf{X}_{\mathrm{t}}\right) \mathrm{d}\mathbf{W}_{\mathrm{t}} + \mu\left(\mathbf{X}_{\mathrm{t}}\right) \mathrm{dt} \tag{2}$$

Here, $\sigma \in \mathscr{C}^{\infty}\left(\mathbf{U}, \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}\right)$ denotes a smooth, matrix-valued function on $\mathbf{U}$ with coefficient functions $\sigma^{\mathrm{i}}_{\mathrm{j}}$ i.e. $\sigma = (\sigma^{\mathrm{i}}_{\mathrm{j}})_{\mathrm{ij}}$. We further assume that $\sigma(x)$ is invertible for all $x \in \mathbf{U}$. Besides, let $\mu \in \mathscr{C}^{\infty}\left(\mathbf{U}, \mathbb{R}^{\mathrm{n}}\right)$ be a smooth vector field on $\mathbf{U}$ with components μ^{i} so that $\mu = (\mu^1, \ldots, \mu^{\mathrm{n}})$. Instead of (**??**), we will frequently use its index based version given by

$$\mathrm{dX}^{\mathrm{i}}_{\mathrm{t}} = \sigma^{\mathrm{i}}_{\mathrm{j}}\left(\mathbf{X}_{\mathrm{t}}\right) \mathrm{dW}^{\mathrm{i}}_{\mathrm{t}} + \mu^{\mathrm{i}}\left(\mathbf{X}_{\mathrm{t}}\right) \mathrm{dt} \tag{3}$$

where we agree to use the usual convention to perform summation over twice appearing indices. In the next sections, we will assume that the paths $\mathbf{X}_{\mathrm{t}}$ do not leave $\mathbf{U}$ for all $\mathrm{t} \geq 0$.

2. Contravariance of the drift

Let $\mathbf{X}_{\mathrm{t}}$ in $\mathbf{U}$ be as defined in section (**??**) and fix diffeomorphism $\phi \in \mathscr{C}^{\infty}\left(\mathbf{U}, \mathbf{U}\right)$. In this setting, Itô's lemma states that the transformed process $\mathbf{Y}_{\mathrm{t}} := \phi \circ \mathbf{X}_{\mathrm{t}}$ on $\mathbf{U}$ solves the following equation

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \frac{\mathrm{d}\phi}{\mathrm{dx}}\left(\mathbf{X}_{\mathrm{t}}\right)\left[\sigma\left(\mathbf{X}_{\mathrm{t}}\right) \mathrm{d}\mathbf{W}_{\mathrm{t}} + \mu\left(\mathbf{X}_{\mathrm{t}}\right) \mathrm{dt}\right] + \mathbf{R}\left(\phi, \sigma\right)\left(\mathbf{X}_{\mathrm{t}}\right)\mathrm{dt} \tag{4}$$

with

$$\frac{\mathrm{d}\phi}{\mathrm{dx}}\left(\mathbf{X}_{\mathrm{t}}\right) = \left(\frac{\partial \phi^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{j}}}\left(\mathbf{X}_{\mathrm{t}}\right)\right)_{\mathrm{ij}} \tag{5}$$

and a smooth vector valued function $\mathbf{R}\left(\phi, \sigma\right) \in \mathscr{C}^{\infty}(\mathbf{U}, \mathbb{R}^{\mathrm{n}})$ on $\mathbf{U}$ with components

$$\mathrm{R}^{\mathrm{i}}\left(\phi, \sigma\right)\left(\mathbf{X}_{\mathrm{t}}\right) = \frac{1}{2}\frac{\partial^2 \phi^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{k}} \partial \mathrm{x}^{\ell}}(\mathbf{X}_{\mathrm{t}})\sigma^{\mathrm{k}}_{\mathrm{j}}(\mathbf{X}_{\mathrm{t}})\sigma^{\ell}_{\mathrm{j}}(\mathbf{X}_{\mathrm{t}}) \tag{6}$$

Note that the term $\mathbf{R}$ in equation (**??**) is the obstruction for the coefficients of equation (**??**) to transform contravariantly with respect to the change of coordinates ϕ on $\mathbf{U}$. To be more precise, if $\mathbf{R} \equiv 0$ on $\mathbf{U}$ the following transformation law would hold:

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \frac{\mathrm{d}\phi}{\mathrm{dx}}\left(\mathbf{X}_{\mathrm{t}}\right)\left[\sigma\left(\mathbf{X}_{\mathrm{t}}\right) \mathrm{d}\mathbf{W}_{\mathrm{t}} + \left(\phi_* \mu\right)\left(\mathbf{Y}_{\mathrm{t}}\right) \mathrm{dt}\right] \tag{7}$$

where $\phi_* : \mathrm{T}\mathbf{U} \to \mathrm{T}\mathbf{U}$ denotes the *push forward* associated to ϕ which is defined by

$$(\phi_* \mu)(\mathrm{x}) := \frac{\mathrm{d}\phi}{\mathrm{dx}}_{|\phi^{-1}(\mathrm{x})} \mu\left(\phi^{-1}(\mathrm{x})\right) \tag{8}$$

for all $\mathrm{x} \in \mathbf{U}$. Before we proceed to specify the conditions, which have to be imposed on the map ϕ and the coordinates x^{i}, $1 \leq \mathrm{i} \leq \mathrm{n}$ on $\mathbf{U}$ so that the $\mathrm{dY}^{\mathrm{i}}_{\mathrm{t}}$ transform like the coefficients of a tensor of rank 1, we have to introduce and recall some definitions:

Definition 2.1. — Let σ be a defined as above, i.e. $\sigma \in \mathscr{C}^{\infty}(\mathbf{U}, \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}})$ with $\sigma(x)$ invertible for all $\mathrm{x} \in \mathrm{U}$, then define a Riemannian metric $\mathbf{g}$ on $\mathbf{U}$ by setting

$$\mathbf{g}^{\sigma} := \left(\sigma^{\mathsf{T}}\right)^{-1} \sigma^{-1} \tag{9}$$

Note that by assumption, σ is invertible on $\mathbf{U}$ and hence $\mathbf{g}$ is well defined. Now, recall the following definitions from Riemannian geometry.

Definition 2.2 **(Christoffel symbols).** — Let $(\mathbf{M}, \mathbf{g})$ be a Riemannian manifold and $\varphi_{\mathbf{V}} \in \mathscr{C}^{\infty}(\mathbf{V}, \mathbf{U})$ be a chart from the open subset $\mathbf{V} \subset \mathbf{M}$ to $\mathbf{U} \subset \mathbb{R}^{\mathrm{n}}$ with local coordinates x^{i}, $1 \leq \mathrm{i} \leq \mathrm{n}$ on $\mathbf{U}$ then the Christoffel symbol $\Gamma^{\mathrm{k}}_{\mathrm{ij}} \in \mathscr{C}^{\infty}(\mathbf{U})$ of the second kind for $1 \leq \mathrm{k} \leq \mathrm{n}$ is given by

$$\Gamma^{\mathrm{k}}_{\mathrm{ij}} := \frac{1}{2} \mathrm{g}^{\mathrm{i}\ell} \left(-\frac{\partial \mathrm{g}_{\mathrm{ij}}}{\partial \mathrm{x}^{\ell}} + \frac{\partial \mathrm{g}_{\ell \mathrm{j}}}{\partial \mathrm{x}^{\mathrm{i}}} + \frac{\partial \mathrm{g}_{\mathrm{i}\ell}}{\partial \mathrm{x}^{\mathrm{j}}} \right) \tag{10}$$

Definition 2.3 **(Laplacian operator).** — Let $(\mathbf{M}, \mathbf{g})$, $\mathbf{V} \subset \mathbf{M}$, $\mathbf{U}$ and $\varphi_{\mathbf{V}}$ be as in defintion **??**), then the second order partial derivate operator on $\mathbf{U}$ given by

$$\triangle^{\mathbf{g}} := \mathrm{g}^{\mathrm{ik}} \left(\frac{\partial^2}{\partial \mathrm{x}^{\mathrm{i}} \partial \mathrm{x}^{\mathrm{j}}} - \Gamma^{\mathrm{j}}_{\mathrm{ik}} \frac{\partial}{\partial \mathrm{x}^{\mathrm{k}}} \right) \tag{11}$$

is called the Laplacian operator on $\mathbf{U}$ with respect to $\mathbf{g}$. Here, g^{ij} are the coefficients of the inverse $\mathbf{g}^{-1}$.

Definition 2.4 **(Harmonic functions).** — Let $(\mathbf{M}, \mathbf{g})$ be a Riemannian manifold and $\mathrm{f} \in \mathscr{C}^{\infty}(\mathbf{V}, \mathbb{R})$ with $\mathbf{V} \subset \mathbf{M}$, then f is called harmonic on $\mathbf{V}$ if locally with respect to the chart $\mathbf{U}$ (as in defintion **??**), f solves

$$\triangle^{\mathbf{g}} \mathrm{f} = 0 \tag{12}$$

We are now able to state the under which assumption we have $\mathbf{R} \equiv 0$ on $\mathbf{U}$ so that equation (**??**) holds.

Lemma 2.5 **(Contravariance of drift).** — *Let $\mathbf{X}_{\mathrm{t}}$ be a process in $\mathbf{U}$ solving equation (**??**), then the image $\mathbf{Y}_t = \phi \circ \mathbf{X}_t$ under a diffeomorphism $\phi \in \mathscr{C}^{\infty}(\mathbf{U}, \mathbf{U})$ solves (**??**) if the following two conditions hold:*

1. *The local coordinates x^{i} are harmonic with respect to $\mathbf{g}^{\sigma}$, i.e. $\mathrm{g}^{\mathrm{ij}} \Gamma^{\mathrm{k}}_{\mathrm{ij}} = 0$ for all $1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n}$.*
2. *The map ϕ is harmonic, i.e. each component ϕ^{i} is again a harmonic function with respect to x^{i}, $1 \leq \mathrm{i} \leq \mathrm{n}$.*

Proof. — As the coordinates x^{i} are harmonic by assumption, we have

$$\triangle^{\mathbf{g}} \mathrm{f} = \mathrm{g}^{\mathrm{ik}} \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^{\mathrm{i}} \partial \mathrm{x}^{\mathrm{j}}} \tag{13}$$

for arbitrary, smooth functions $f \in \mathscr{C}^\infty(\mathbf{U}, \mathbb{R})$. In particular, using the second assumption, it then follows for $f = \phi^k$, $1 \leq k \leq n$

$$\triangle^{\mathbf{g}} \phi^k = g^{ik} \frac{\partial^2 \phi^k}{\partial x^i \partial x^j} = 0 \tag{14}$$

Therefore, the claim of the lemma which is equivalent to

$$0 = R^i(\phi, \sigma)(x) = \frac{1}{2} \frac{\partial^2 \phi^i}{\partial x^k \partial x^\ell}(x) \sigma^k_j(x) \sigma^\ell_j(x) \tag{15}$$

for all $x \in \mathbf{U}$ is proven, as soon as we have shown that

$$g^{ij} = \left(\sigma^\intercal \sigma\right)_{ij} \tag{16}$$

However, equation (**??**) is an immediate consequence of $g^{ij} = \left(\mathbf{g}^{-1}\right)_{ij}$ and definition **??**. □

3. Invariance of the volatility structure

As we have seen in the preceding section **??**, the volatility structure σ on $\mathbf{U}$ gives rise to Riemannian metric $\mathbf{g}$. In this section, we will find a necessary condition for ϕ in terms of $\mathbf{g}$ so that the transformation $\mathbf{Y}_t = \phi \circ \mathbf{X}_t$ of $\mathbf{X}_t$ is solves the original equation (**??**) module a drift term. To be more precise, we will prove that if ϕ is compatible with the metric $\mathbf{g}$, there exists an n−dimensional Brownian motion $\widetilde{\mathbf{W}}_t$ which itself is a version of $\mathbf{W}_t$, i.e. $\widetilde{\mathbf{W}}_t \sim \mathbf{W}_t$ for all $t \leq 0$ so that

$$d\mathbf{Y}_t = \sigma(\mathbf{Y}_t) d\widetilde{\mathbf{W}}_t + \alpha(\mathbf{Y}_t) dt \tag{17}$$

holds. Before we proceed to establish a geometric sufficient condition for equation (**??**) to be valid, we prove the following helpful lemma.

Lemma 3.1. — *Let $\mathbf{X}_t$ be to stochastic processes on $\mathbf{U}$ solving*

$$d\mathbf{X}_t = \sigma(\mathbf{X}_t) d\mathbf{W}_t + \alpha(\mathbf{X}_t) dt \tag{18}$$

where $\sigma \in \mathscr{C}^\infty(\mathbf{U}, \mathbb{R}^n \times \mathbb{R}^n)$ is a smooth, matrix valued function on $\mathbf{U}$ with $\sigma(x)$ invertible for all $x \in \mathbf{U}$ and let ρ be a second matrix valued function unction sharing the same properties as σ such that

$$\sigma \sigma^\intercal = \rho \rho^\intercal \tag{19}$$

on $\mathbf{U}$. Then, it is

$$d\mathbf{X}_t = \rho(\mathbf{X}_t) d\widetilde{\mathbf{W}}_t + \alpha(\mathbf{X}_t) \tag{20}$$

for $\widetilde{\mathbf{W}}_t \sim \mathbf{W}_t$.

Proof. — Observe that by (**??**), we conclude

$$\rho^{-1} \sigma = \rho^\intercal \left(\sigma^{-1}\right)^\intercal \Leftrightarrow \left(\sigma^{-1} \rho\right)^{-1} = \left(\sigma^{-1} \rho\right)^\intercal, \tag{21}$$

so $\sigma^{-1}\rho$ is orthogonal. Hence, for each $x \in \mathbf{U}$, there exists $\mathbf{o} \in O(\mathbb{R}^n)$ so that

$$\sigma(x) = \mathbf{o}(x) \rho(x) \tag{22}$$

Since $\mathbf{o}(\mathrm{x})\mathbf{W}_\mathrm{t} \sim \mathbf{W}_\mathrm{t}$ for all $0 \leq \mathrm{t}$, it follows that

$$\mathrm{d}\mathbf{X}_\mathrm{t} = \sigma(\mathbf{X}_\mathrm{t})\mathrm{d}\mathbf{W}_\mathrm{t} + \alpha(\mathbf{X}_\mathrm{t}) = \rho(\mathbf{X}_\mathrm{t})\mathbf{o}(\mathbf{X}_\mathrm{t})\mathrm{d}\mathbf{W}_\mathrm{t} + \alpha(\mathbf{X}_\mathrm{t}) = \rho(\mathbf{X}_\mathrm{t})\mathrm{d}\mathbf{U}_\mathrm{t} + \alpha(\mathbf{X}_\mathrm{t}) \tag{23}$$

with $\mathbf{U}_\mathrm{t} := \mathbf{o}(\mathrm{t})\mathbf{W}_\mathrm{t}$ and similar for $\mathrm{d}\mathbf{Y}_\mathrm{t}$. □

Recall that a diffeomorphism φ of $\mathbf{U}$ is called metric if it satisfies the following compatibility condition:

Definition 3.2. — A diffeomorphism φ of $\mathbf{U}$ is called metric if

$$\mathbf{g}_{|\phi(\mathrm{x})}\left(\frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{x})\mathfrak{v}, \frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{x})\mathfrak{w}\right) = \mathbf{g}_{|\mathrm{x}}\left(\mathfrak{v}, \mathfrak{w}\right) \tag{24}$$

for all $\mathrm{x} \in \mathbf{U}$ and all $\mathfrak{v}, \mathfrak{w} \in \mathrm{T}_\mathrm{x}\mathbf{U}$.

***Lemma 3.3* (Invariance modulo drift).** — *Let $\mathbf{X}_\mathrm{t}$ be a process in $\mathbf{U}$ solving equation (**??**) and let $\mathbf{Y}_\mathrm{t} = \phi \circ \mathbf{X}_\mathrm{t}$ be its transformation under a metric diffeomorphism, then there exists $\widetilde{\mathbf{W}}_t$ with $\widetilde{\mathbf{W}}_t \sim \mathbf{W}_\mathrm{t}$ so that $\mathbf{Y}_\mathrm{t}$ solves*

$$\mathrm{d}\mathbf{Y}_\mathrm{t} = \sigma\left(\mathbf{Y}_\mathrm{t}\right)\mathrm{d}\widetilde{\mathbf{W}}_\mathrm{t} + \alpha(\mathbf{Y}_\mathrm{t})\mathrm{dt} \tag{25}$$

for α apropriately.

Proof. — First of all note, that condition (**??**) is equivalent to the matrix equation

$$\frac{\mathrm{d}\phi}{\mathrm{dx}}^{\mathsf{T}}(\mathrm{x}) \cdot \mathbf{g}^\sigma_{|\phi(\mathrm{x})} \cdot \frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{x}) = \mathbf{g}^\sigma_{|\mathrm{x}} \tag{26}$$

which, using the definition of $\mathbf{g}^\sigma$, translate immediately to

$$\frac{\mathrm{d}\phi}{\mathrm{dx}}^{\mathsf{T}}(\mathrm{x}) \cdot \left(\sigma(\phi(\mathrm{x}))\sigma^{\mathsf{T}}(\phi(\mathrm{x}))\right)^{-1} \cdot \frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{x}) = \left(\sigma(\mathrm{x})\sigma^{\mathsf{T}}(\mathrm{x})\right)^{-1} \tag{27}$$

By multiplying (**??**) from the left with the inverse of $\frac{\mathrm{d}\phi}{\mathrm{dx}}^{\mathsf{T}}(\mathrm{x})$ and from the right with the inverse of $\frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{x})$ respectively, we infer

$$\left(\sigma(\phi(\mathrm{x}))\sigma^{\mathsf{T}}(\phi(\mathrm{x}))\right)^{-1} = \left(\frac{\mathrm{d}\phi}{\mathrm{dx}}^{\mathsf{T}}(\mathrm{x})\right)^{-1} \left(\sigma(\mathrm{x})\sigma^{\mathsf{T}}(\mathrm{x})\right)^{-1} \left(\frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{x})\right)^{-1} \tag{28}$$

which in turn by inversion is equivalent to

$$\sigma(\phi(\mathrm{x}))\sigma^{\mathsf{T}}(\phi(\mathrm{x})) = \frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{x})\sigma(\mathrm{x})\sigma^{\mathsf{T}}(\mathrm{x})\frac{\mathrm{d}\phi}{\mathrm{dx}}^{\mathsf{T}}(\mathrm{x}) \tag{29}$$

Now, define

$$\rho(\mathrm{x}) = \frac{\mathrm{d}\phi}{\mathrm{dx}}(\phi^{-1}(\mathrm{x}))\sigma(\phi^{-1}(\mathrm{x})) \tag{30}$$

on $\mathbf{U}$, then equation (**??**) we deduce

$$\rho\rho^{\mathsf{T}} = \sigma\sigma^{\mathsf{T}} \tag{31}$$

Using ρ, equation

$$\mathrm{d}\mathbf{Y}_\mathrm{t} = \frac{\mathrm{d}\phi}{\mathrm{dx}}\left(\mathbf{X}_\mathrm{t}\right)\sigma\left(\mathbf{X}_\mathrm{t}\right)\mathrm{d}\mathbf{W}_\mathrm{t} + \alpha(\mathbf{Y}_\mathrm{t})\mathrm{dt} \tag{32}$$

becomes

$$\mathrm{d}\mathbf{Y}_t = \rho(\mathbf{Y}_t)\mathrm{d}\mathbf{W}_t + \alpha(\mathbf{Y}_t)\mathrm{d}t \tag{33}$$

Applying lemma **??**, which is possible because of (**??**), then completes the proof. □

4. Local Kähler spaces and their transformation properties

In this section, we will consider a special configuration of our starting data $(\mathbf{U}, \mathbf{X}_t, \phi)$ so that the stochastic differential equation of the transformed process $\mathbf{Y}_t = \phi \circ \mathbf{X}_t$ automatically results in a very pleasant from where basically the vola structure is stabilized in the sense of section **??** whereas the drift is transformed like a vectorfield as described in section **??**. It turns out that the natural transformation laws are valid, if the pair $(\mathbf{U}, \mathbf{g}^\sigma)$ induced by $(\mathbf{U}, \mathbf{X}_t, \phi)$ is a Kähler manifold.

Before we proceed to formulate the exact result, we will first recall some definitions about Kähler manifolds.

4.1. Kähler manifolds. — Let $\mathbf{M}$ be a complex manifold and $\mathbf{g}$ a Riemannian metric.

As a starting point for this, let now $\mathbf{U} \subset \mathbb{C}^n$ with holomorphic standard coordinates $z^k = x^k + iy^k$, $1 \leq k \leq n$.

Lemma 4.1. — *Let* f *be a holormphic function defined on a Kähler domain* $(\mathbf{U}, \mathbf{g})$*, then* $\mathfrak{Re}\, f$ *and* $\mathfrak{Im}\, f$ *are harmonic functions.*

Proof. — First of all note that if $v, w \in T_p\mathbf{U}$ and

Let $p \in \mathbf{U}$. As $(\mathbf{U}, \mathbf{g})$ is assumed to be Kähler, there exists an orthogonal base of $T_p\mathbf{U}$ given by $\{X_i, Y_i\}$ where $Y_i = J\, X_i$ for all $1 \leq i \leq n$. With respect to this base, it the Laplacian of $u := \mathfrak{Re}\, f$ takes the following form

$$\triangle^{\mathbf{g}} u = \sum_{i=1}^{n} \mathbf{g}\left(\nabla_{X_i} \mathbf{grad}\, u, X_i\right) + \mathbf{g}\left(\nabla_{J\, X_i} \mathbf{grad}\, u, J\, X_i\right) \tag{34}$$

Since $\mathbf{g}$ is $J-$invariant, we deduce

$$\begin{aligned} \triangle^{\mathbf{g}} u &= \sum_{i=1}^{n} \mathbf{g}\left(J\, \nabla_{X_i} \mathbf{grad}\, u, J\, X_i\right) + \mathbf{g}\left(J\, \nabla_{J\, X_i} \mathbf{grad}\, u, J^2\, X_i\right) \\ \triangle^{\mathbf{g}} u &= \sum_{i=1}^{n} \mathbf{g}\left(J\, \nabla_{X_i} \mathbf{grad}\, u, J\, X_i\right) - \mathbf{g}\left(J\, \nabla_{J\, X_i} \mathbf{grad}\, u, X_i\right) \end{aligned} \tag{35}$$

where we used $J^2 = -\mathbf{Id}$. Since J commutes with the Levi-Civita connection which follows from the assumption that $(\mathbf{U}, \mathbf{g})$ is a Kähler domain, equation (**??**) yields

$$\triangle^{\mathbf{g}} u = \sum_{i=1}^{n} \mathbf{g}\left(\nabla_{X_i} J\, \mathbf{grad}\, u, J\, X_i\right) - \mathbf{g}\left(\nabla_{J\, X_i} J\, \mathbf{grad}\, u, X_i\right) \tag{36}$$

As f is holomorphic, we have $\mathrm{J}\,\mathbf{grad}\,\mathrm{u} = \mathrm{v} := \mathfrak{Im}\,\mathrm{f}$ and hence

$$\triangle^{\mathbf{g}}\mathrm{u} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{g}\left(\nabla_{\mathrm{X_i}}\mathbf{grad}\,\mathrm{v}, \mathrm{J\,X_i}\right) - \mathbf{g}\left(\nabla_{\mathrm{J\,X_i}}\mathbf{grad}\,\mathrm{v}, \mathrm{X_i}\right) \tag{37}$$

Now, recall that in general, we have

$$\mathbf{Hess}_{|\mathrm{p}}(\mathrm{f})(\mathrm{v},\mathrm{w}) := \mathbf{g}_{|\mathrm{p}}\left(\nabla_{\mathrm{A}}\,\mathbf{grad}\,\mathrm{f}, \mathrm{B}\right) = \mathbf{g}_{|\mathrm{p}}\left(\nabla_{\mathrm{B}}\,\mathbf{grad}\,\mathrm{f}, \mathrm{A}\right) \tag{38}$$

for all $\mathrm{A}, \mathrm{B} \in \mathrm{T_p}\mathbf{U}$, and therefore equation (**??**) turns into

$$\triangle^{\mathbf{g}}\mathrm{u} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{g}\left(\nabla_{\mathrm{J\,X_i}}\mathbf{grad}\,\mathrm{u}, \mathrm{X_i}\right) - \mathbf{g}\left(\nabla_{\mathrm{J\,X_i}}\mathbf{grad}\,\mathrm{u}, \mathrm{X_i}\right) = 0 \tag{39}$$

□

For further abbreviation, we define:

Definition 4.2. — Let $\mathbf{X}_\mathrm{t}$ be a stochastic process in $\mathbf{U} \subset \mathbb{C}^\mathrm{n}$ where

$$\mathrm{d}\mathbf{X}_\mathrm{t} = \sigma\left(\mathbf{X}_\mathrm{t}\right)\mathrm{d}\mathbf{W}_\mathrm{t} + \mu\left(\mathbf{X}_\mathrm{t}\right)\mathrm{dt} \tag{40}$$

and let $\phi : \mathbf{U} \to \mathbf{U}$ be a biholomorphic map. If

1. $(\mathbf{U}, \mathbf{g}^\sigma)$ is a Kähler domain and
2. if ϕ is metric with respect to $\mathbf{g}^\sigma$,

then we say that the tuple $(\mathbf{U}, \mathbf{X}_\mathrm{t}, \phi)$ is a *σ-compatible transformation* of $\mathbf{X}_\mathrm{t}$.

We proceed by introducing some examples of local Kähler spaces with natural, associated transformations compatible with their Kähler structure.

***Example 4.3* (Upper Half Plane).** — Let $\mathbb{H} := \{\mathrm{z} : \mathfrak{Im}\,\mathrm{z} = \mathrm{y} > 0\} \subset \mathbb{C}^2$ where $\mathrm{z} = \mathrm{x} + \sqrt{-1}\mathrm{y}$ and let $\mathbf{X}_\mathrm{t}$ be the process in $\mathbb{H}$ defined by the system of equations

$$\begin{aligned} \mathrm{dX_t} &= \mathrm{Y_t dW_t^0} + \mu^0\left(\mathbf{Z}_\mathrm{t}\right)\mathrm{dt} \\ \mathrm{dY_t} &= \mathrm{Y_t dW_t^1} + \mu^1\left(\mathbf{Z}_\mathrm{t}\right)\mathrm{dt} \end{aligned} \tag{41}$$

or equivalently $\mathrm{d}\mathbf{Z}_\mathrm{t} = \sigma\left(\mathbf{Z}_\mathrm{t}\right)\mathrm{d}\mathbf{W}_\mathrm{t} + \mu\left(\mathbf{Z}_\mathrm{t}\right)\mathrm{dt}$ with

$$\sigma(\mathrm{z}) = \mathrm{y}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{42}$$

so that the induced Riemannian metric $\mathbf{g}^\sigma$ on $\mathbb{H}$ is given by the Poincare metric

$$\mathrm{g} = \mathrm{y}^{-2}\left(\mathrm{dx}^2 + \mathrm{dy}^2\right) \tag{43}$$

It is known that the upper half plane model given by $(\mathbb{H}, \mathbf{g}^\sigma)$ is a Kähler manifold with harmonic coordinates given by x, y. In particular, the last statement can be verfied directly by having a look at the explicit formulas of $\Gamma_{\mathrm{ij}}^{\mathrm{k}}$ which are given by

$$\Gamma_{11}^1 = \Gamma_{22}^1 = 0, \quad -\Gamma_{12}^1 = -\Gamma_{21}^1 = -\Gamma_{22}^2 = \Gamma_{11}^2 = \frac{1}{\mathrm{y}} \tag{44}$$

This in turn yields $\Delta^{\mathbf{g}} = \mathrm{y}^2\left(\frac{\partial^2}{\partial\mathrm{x}^2} + \frac{\partial^2}{\partial\mathrm{y}^2}\right)$.

4.2. The transformation law. —

***Theorem 4.4* (Transformation law).** — *If* $(\mathbf{U}, \mathbf{X}_{\mathrm{t}}, \phi)$ *is a* σ*-compatible transformation of* $\mathbf{X}_{\mathrm{t}}$*, it follows*

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \left(\phi_{*}\mu\right)\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{dt} \tag{45}$$

for $\widetilde{\mathbf{W}}_{t} \sim \mathbf{W}_{\mathrm{t}}$.

Proof. — By lemma **??** and lemma **??** the claim follows as soon we have verified that the coordinates $\mathfrak{Re}\,\mathrm{z}^{\mathrm{i}} = \mathrm{x}^{\mathrm{i}}, \mathfrak{Im}\,\mathrm{z}^{\mathrm{i}} = \mathrm{y}^{\mathrm{i}},\ 1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n}$ of $\mathbf{U} \subset \mathbb{C}^{\mathrm{n}} = \mathbb{R}^{2\mathrm{n}}$ as well as the components $\mathfrak{Re}\,\phi^{\mathrm{i}}, \mathfrak{Im}\,\phi^{\mathrm{i}}$ of $\phi = \left(\phi^{1}, \ldots, \phi^{\mathrm{n}}\right)$ are harmonic with respect to $\mathbf{g}^{\sigma}$. However, this is direct consequence of lemma **??** and the fact that z^{i} and ϕ^{i} are holomorphic. □

In particular, if μ is invariant under ϕ, it follows:

***Corollary 4.5* (Invariant processes).** — *Let* $(\mathbf{U}, \mathbf{X}_{\mathrm{t}}, \phi)$ *be a* σ*-compatible transformation of* $\mathbf{X}_{\mathrm{t}}$ *so that* $\phi_{*}\mu = \mu$*, then*

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \mu\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{dt} \tag{46}$$

for $\widetilde{\mathbf{W}}_{t} \sim \mathbf{W}_{\mathrm{t}}$.

In particular, if $\phi(\mathbf{X}_0) = \mathbf{X}_0$*, the transformation* $\mathbf{Y}_t$ *is modification of* $\mathbf{X}_t$*, i.e.*

$$\mathbb{P}(\mathbf{X}_{\mathrm{t}} = \mathbf{Y}_{\mathrm{t}}) = 1 \tag{47}$$

for all $\mathrm{t} \leq 0$.

Proof. — Since invariance of μ is equivalent to $\phi_{*}\mu = \mu$ by definition, theorem **??** yields

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \mu\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{dt} \tag{48}$$

and hence the claim. □

We conclude this section by reformulating the transfomation theorem **??** in terms of the distribution function $\nu_{\mathbf{X}} : \mathbb{R}^{\leq 0} \times \mathbf{U} \to \mathbb{R}$

$$\mathbb{P}\left(\mathbf{X}_{\mathrm{t}} \in \mathbf{A}\right) = \int_{\mathbf{A}} \nu_{\mathbf{X}}\left(\mathrm{t}, \mathrm{x}\right)\left|\mathrm{dx}^{1} \wedge \cdots \wedge \mathrm{dx}^{\mathrm{n}}\right| \tag{49}$$

for $\mathbf{A} \in \mathscr{A}$ where $(\mathbf{U}, \mathbf{X}_{\mathrm{t}}, \phi)$ is a σ*-compatible transformation* of $\mathbf{X}_{\mathrm{t}}$ as before. Before, recall that according to the Feynman-Kac formula, we have

$$-\frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{t}} + \frac{1}{2}\sigma^{\mathrm{i}}_{\mathrm{k}}\sigma^{\mathrm{j}}_{\mathrm{k}}\frac{\partial^{2}\nu_{\mathbf{X}}}{\partial \mathrm{x}^{\mathrm{i}}\partial \mathrm{x}^{\mathrm{j}}} + \mu^{\mathrm{i}}\frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{x}^{\mathrm{i}}} = 0 \tag{50}$$

or equivalently

$$-\frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{t}} + \frac{1}{2}\left(\mathbf{g}^{\sigma}\right)^{\mathrm{ij}}\frac{\partial^{2}\nu_{\mathbf{X}}}{\partial \mathrm{x}^{\mathrm{i}}\partial \mathrm{x}^{\mathrm{j}}} + \mu^{\mathrm{i}}\frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{x}^{\mathrm{i}}} = 0 \tag{51}$$

by definition **??**. Note that by equation (**??**) of lemma **??** we even have

$$\left(\mathbf{g}^{\sigma}\right)^{\mathrm{ij}}\frac{\partial^{2}\nu_{\mathbf{X}}}{\partial \mathrm{x}^{\mathrm{i}}\partial \mathrm{x}^{\mathrm{j}}} = \triangle^{\mathbf{g}}\nu_{\xi} \tag{52}$$

Using the notation $\mu = \mu^{\mathrm{i}} \frac{\partial}{\partial \mathrm{x}^{\mathrm{i}}}$, equation (**??**) turns into

$$-\frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{t}} + \frac{1}{2} \triangle^{\mathbf{g}} \nu_{\mathbf{X}} + \mu\, \nu_{\mathbf{Y}} = 0 \tag{53}$$

Now if $\nu_{\mathbf{Y}}$ denotes the density distribution function of the transformed process $\mathbf{Y}_{\mathrm{t}} = \phi \circ \mathbf{X}_{\mathrm{t}}$, one can ask for the corresponding FEYNMAN-KAC formula. Note, that by the transformation law for integrals under diffeomorphism such as ϕ, it follows that

$$\begin{aligned} \mathbb{P}\left(\mathbf{Y}_{\mathrm{t}} \in \mathbf{A}\right) &= \mathbb{P}\left(\mathbf{X}_{\mathrm{t}} \in \phi^{-1}(\mathrm{A})\right) \\ &= \int_{\phi^{-1}(\mathbf{A})} \nu_{\mathbf{X}}(\mathrm{t}, \mathrm{x}) \left|\mathrm{dx}^1 \wedge \cdots \wedge \mathrm{dx}^{\mathrm{n}}\right| \\ &= \int_{\mathbf{A}} \left(\nu_{\mathbf{X}} \circ \phi^{-1}\right)(\mathrm{t}, \mathrm{y}) \cdot \left|\mathbf{det}\left(\frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{y})\right)\right|^{-1} \left|\mathrm{dy}^1 \wedge \cdots \wedge \mathrm{dy}^{\mathrm{n}}\right| \end{aligned} \tag{54}$$

where $\mathrm{y}^{\mathrm{i}} = \phi^{\mathrm{i}}(\mathrm{x})$ are the new coordinates on $\mathbf{U}$ induced by ϕ. As

$$\left|\mathbf{det}\left(\frac{\mathrm{d}\phi}{\mathrm{dx}}(\mathrm{y})\right)\right|^{-1} \left|\mathrm{dy}^1 \wedge \cdots \wedge \mathrm{dy}^{\mathrm{n}}\right| = \left|\mathrm{dx}^1 \wedge \cdots \wedge \mathrm{dx}^{\mathrm{n}}\right| \tag{55}$$

it follows

$$\nu_{\mathbf{Y}} = \nu_{\mathbf{X}} \circ \phi^{-1} \tag{56}$$

***Theorem 4.6* (Transformation law of densities).** — *If $(\mathbf{U}, \mathbf{X}_{\mathrm{t}}, \phi)$ is a σ-compatible transformation of $\mathbf{X}_{\mathrm{t}}$, then the distribution function $\nu_{\mathbf{Y}}$ of the transformation $\mathbf{Y}_{\mathrm{t}}$ solves the following version of the* FEYNMAN-KAC *formula:*

$$-\frac{\partial \nu_{\mathbf{Y}}}{\partial \mathrm{t}} + \frac{1}{2} \triangle^{\mathbf{g}} \nu_{\mathbf{Y}} + \left(\phi_* \mathrm{X}_{\mu}\right) \nu_{\mathbf{Y}} = 0 \tag{57}$$

Proof. — First of all note, that the claim already follows by theorem **??**. An alternative argument however can be provided by running along the following lines.

First of all, note that we have

$$\frac{\partial \nu_{\mathbf{Y}}}{\partial \mathrm{t}}(\mathrm{t}, \mathrm{x}) = \left(\frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{t}}\right)\left(\mathrm{t}, \phi^{-1}(\mathrm{x})\right) = \frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{t}} \circ \phi^{-1}(\mathrm{t}, \mathrm{x}) \tag{58}$$

Furthermore, as

$$\left(\phi_* \mathrm{X}_{\mu}\right)\left(\mathrm{f} \circ \phi^{-1}\right)(\mathrm{x}) = \left(\mathrm{X}_{\mu} \mathrm{f}\right)\left(\phi^{-1}(\mathrm{x})\right) \tag{59}$$

for $\mathrm{x} \in \mathbf{U}$ and all smooth functions $\mathrm{f} \in \mathscr{C}^{\infty}(\mathbf{U}, \mathbb{R})$ on $\mathbf{U}$, we infer that

$$\left(\phi_* \mathrm{X}_{\mu}\right) \nu_{\mathbf{Y}} = \left(\mathrm{X}_{\mu} \nu_{\mathbf{X}}\right) \circ \phi^{-1} \tag{60}$$

In particular, the theorem is proven as soon as we have shown that

$$\triangle^{\mathbf{g}} \nu_{\mathbf{Y}} = \triangle^{\mathbf{g}} \nu_{\mathbf{X}} \circ \phi^{-1} \tag{61}$$

because then it would follow that

$$0 = \left(-\frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{t}} + \frac{1}{2} \triangle^{\mathbf{g}} \nu_{\mathbf{X}} + \mu^{\mathrm{i}} \frac{\partial \nu_{\mathbf{X}}}{\partial \mathrm{x}^{\mathrm{i}}}\right) \circ \phi^{-1} = -\frac{\partial \nu_{\mathbf{Y}}}{\partial \mathrm{t}} + \frac{1}{2} \triangle^{\mathbf{g}} \nu_{\mathbf{Y}} + \mu^{\mathrm{i}} \frac{\partial \nu_{\mathbf{Y}}}{\partial \mathrm{x}^{\mathrm{i}}} \tag{62}$$

However, equation **??** is a direct consequence of the fact that the Laplace-Beltrami operator is invariant under metric transfomrations such as ϕ ☐

In particular, if μ commutates with ϕ, i.e. $\phi_*\mu = \mu$, we deduce:

***Corollary 4.7* (Invariant processes).** — *If* $(\mathbf{U}, \mathbf{X}_\mathrm{t}, \phi)$ *is a* σ*-compatible transformation of* $\mathbf{X}_\mathrm{t}$*, then the distribution function* $\nu_\mathbf{Y}$ *of the transformation* $\mathbf{Y}_\mathrm{t}$ *solves the original* FEYNMAN-KAC *formula (of* $\mathbf{X}_\mathrm{t}$*)*

$$-\frac{\partial \nu_\mathbf{Y}}{\partial \mathrm{t}} + \frac{1}{2}\triangle^\mathbf{g}\nu_\mathbf{Y} + \mathrm{X}_\mu \nu_\mathbf{Y} = 0 \tag{63}$$

Proof. — The proof is a direct application of theorem **??**. ☐

4.3. Gluinig local data. — Let $\mathbf{M}$ be a complex manifold and choose a holomorphic atlas $\{(\varphi_\alpha, \mathbf{V}_\alpha)\}$ of holomorphic charts $\varphi_\alpha : \mathbf{V}_\alpha \to \mathbf{U}_\alpha := \varphi(\mathbf{V}_\alpha)$. Furthermore, let $\{(\sigma_\alpha, \mu_\alpha)\}_\alpha$ be a collection of tuples where each $\sigma_\alpha \in \mathscr{C}^\infty(\mathbf{U}_\alpha, \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n})$ denotes a smooth, matrix-valued function with $\sigma_\alpha(\mathrm{x})$ invertible for all $\mathrm{x} \in \mathbf{U}_\alpha$ and $\mu_\alpha \in \mathscr{C}^\infty(\mathbf{U}_\alpha, \mathbb{R}^\mathrm{n})$. For each α we assume that $\mathbf{g}^{\sigma_\alpha}$ turns $\mathbf{U}_\alpha$ into a Kähler domain. Now, if $\mathbf{X}_{\alpha,\mathrm{t}}$ denotes a solution of

$$\mathrm{d}\mathbf{X}_{\alpha,\mathrm{t}} = \sigma_\alpha(\mathbf{X}_{\alpha,\mathrm{t}})\,\mathrm{d}\mathbf{W}_\mathrm{t} + \mu_\alpha(\mathbf{X}_{\alpha,\mathrm{t}})\,\mathrm{dt} \tag{64}$$

in $\mathbf{U}_\alpha$, then by a slight extension of theorem **??** the transformation

$$\mathbf{X}_{\beta\alpha,\mathrm{t}} := \left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)(\mathbf{X}_{\alpha,\mathrm{t}}) \tag{65}$$

where $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathbf{V}_\alpha \cap \mathbf{V}_\beta) \to \varphi_\beta(\mathbf{V}_\alpha \cap \mathbf{V}_\beta)$ solves

$$\mathrm{d}\mathbf{X}_{\beta,\mathrm{t}} = \sigma_\beta(\mathbf{X}_{\beta,\mathrm{t}})\,\mathrm{d}\widetilde{\mathbf{W}}_\mathrm{t} + \mu_\beta(\mathbf{X}_{\beta,\mathrm{t}})\,\mathrm{dt} \tag{66}$$

if

1. the local data $\{(\mathbf{U}_\alpha, \mathbf{g}^{\sigma_\alpha})\}_\alpha$ could be patch together to a Kähler manifold via the holomorphic change of coordinates induced by $\varphi_\beta \circ \varphi_\alpha^{-1}$
2. and if $\mu_\beta = \left(\varphi_\beta \circ \varphi_\alpha^{-1}\right)_* \mu_\alpha$.

***Example 4.8* (Random walk on sphere).** — Let $\mathbf{U}_0 = \mathbf{U}_1 = \mathbb{C}$ and let $\mathrm{z} = \mathrm{x} + \mathrm{iy}$ be the holomorphic coordinate on $\mathbf{U}_0$ and $\mathrm{w} = \mathrm{u} + \mathrm{iv} =$ on U_1. Furthermore, fix $\varphi_{10}(\mathrm{z}) = \mathrm{w} = \frac{1}{\mathrm{z}}$ and consider the local data $\{(\sigma_\alpha, \mu_\alpha)\}_{\alpha=0,1}$ given by

$$\sigma_0(\mathrm{z}) = \left(1 + |\mathrm{z}|^2\right)\mathrm{dz}\,\mathrm{d\bar{z}}, \quad \mu_0(\mathrm{z}) = 0 \quad \text{and} \quad \sigma_1(\mathrm{w}) = \left(1 + |\mathrm{w}|^2\right)\mathrm{dw}\,\mathrm{d\overline{w}}, \quad \mu_1(\mathrm{w}) = 0 \tag{67}$$

The corresponding local stochastic differential equation $\mathrm{d}\mathbf{Z} = \mathrm{d}\mathbf{X}_\mathrm{t} + \mathrm{i}\mathbf{Y}_\mathrm{t}$ on $\mathbf{U}_0$ is given by

$$\begin{aligned} \mathrm{d}\mathbf{X}_\mathrm{t} &= \left(1 + \mathbf{X}_\mathrm{t}^2 + \mathbf{Y}_\mathrm{t}^2\right)\mathrm{dW}_\mathrm{t}^0 \\ \mathrm{d}\mathbf{Y}_\mathrm{t} &= \left(1 + \mathbf{X}_\mathrm{t}^2 + \mathbf{Y}_\mathrm{t}^2\right)\mathrm{dW}_\mathrm{t}^1 \end{aligned} \tag{68}$$

on

5. Symmetries under group actions

In this section, we will examine the transformation properties of equation (ref-stochdiff) under a Lie group G acting holomorphically on the domain $\mathbf{U}$. Before we start our considerations, recall that each vector ξ of the Lie algebra $\mathrm{Lie(G)} = \mathfrak{g}$ generates a smooth flow $\Phi^\xi : \mathbb{R} \times \mathbf{U} \to \mathbf{U}$ defined by

$$\Phi^\xi(\mathrm{t}, \mathrm{x}) = \exp(\mathrm{t}\xi).\mathrm{x} \tag{69}$$

where $\exp : \mathfrak{g} \to \mathrm{G}$ denotes the exponential map of G. Moreover, each $\eta \in \mathfrak{g}$ gives rise to a smooth vector field $\mathbf{X}_\xi$ on $\mathbf{U}$ by

$$\mathbf{X}_\xi(\mathrm{x}) = \frac{\mathrm{d}}{\mathrm{dt}}_{|\mathrm{t}=0} \Phi^\eta(\mathrm{t}, \mathrm{x}) = \frac{\mathrm{d}}{\mathrm{dt}}_{|\mathrm{t}=0} \exp(\mathrm{t}\xi).\mathrm{x} \tag{70}$$

To streamline the notation, we will simply set $\mathbf{X}_\xi = \xi$ where it will be clear from the context to distinguish between $\mathbf{X}_{\mathrm{xi}}$ and the vector $\xi \in \mathfrak{g}$ itself. As G stabilizes the neutral element e of G and hence $\mathrm{T_eG} = \mathfrak{g}$ as well, there exists a natural group action of G on $\mathfrak{g}$. In particular, it is possible to conjugate elements in $\mathbf{g}$ by G. This action is usuallly called the *adjoint action* of G and denoted by $\mathrm{Ad} : \mathrm{G} \times \mathfrak{g} \times \mathfrak{g}$, i.e.

$$\mathrm{Ad}(\mathrm{g}, \xi) = \mathrm{Ad_g}\xi := \mathrm{g}^{-1}\xi\mathrm{g} \tag{71}$$

The adjoint action of the group G is known to be compatible with the push forward of its vector fields in the sense as stated in the following lemma.

Lemma 5.1. — *Let G be a Lie group acting on a smooth manifold* $\mathbf{M}$ *and let* $\mathbf{X}_\eta = \eta$ *be the vector field on* $\mathbf{M}$ *associated to* $\eta \in \mathfrak{g}$*, then*

$$\left(\left(\Phi^\xi_{\mathrm{t}}\right)_* \mathbf{X}_\eta\right)(\mathrm{x}) = \mathrm{Ad}_{\exp(\mathrm{t}\xi)}\eta(\mathrm{x}) \tag{72}$$

for all $\mathrm{x} \in \mathbf{M}$ *and* $\xi \in \mathfrak{g}$.

In this setting, we will introduce the following notion analogous to definition **??**.

Definition 5.2. — Let $\mathbf{X}_\mathrm{t}$ be a stochastic process in $\mathbf{U} \subset \mathbb{C}^\mathrm{n}$ where

$$\mathrm{d}\mathbf{X}_\mathrm{t} = \sigma(\mathbf{X}_\mathrm{t})\,\mathrm{d}\mathbf{W}_\mathrm{t} + \mu(\mathbf{X}_\mathrm{t})\,\mathrm{dt} \tag{73}$$

and let *mathrm*G be a Lie group acting holomorphically on the domain $\mathbf{U}$. If

1. $(\mathbf{U}, \mathbf{g}^\sigma)$ is a Kähler domain and
2. G acts isometrically with respect to $\mathbf{g}^\sigma$,

then we the tuple $(\mathbf{U}, \mathbf{X}_\mathrm{t}, \mathrm{G})$ is said to be a *σ-compatible group of transformations* for $\mathbf{X}_\mathrm{t}$.

Theorem 5.3. — *Let* $(\mathbf{U}, \mathbf{X}_\mathrm{t}, G)$ *be a σ-compatible group of transformations where* $\mathbf{X}_\mathrm{t}$ *solves*

$$\mathrm{d}\mathbf{X}_\mathrm{t} = \sigma(\mathbf{X}_\mathrm{t})\,\mathrm{d}\mathbf{W}_\mathrm{t} + \eta(\mathbf{X}_\mathrm{t})\,\mathrm{dt} \tag{74}$$

for some $\eta \in \mathfrak{g}$*, then the transformation* $\mathbf{Y}^\xi_\mathrm{t} = \Phi^\xi_\mathrm{t} \circ \mathbf{X}_\mathrm{t}$ *is a solution of*

$$\mathrm{d}\mathbf{Y}_\mathrm{t} = \sigma(\mathbf{Y}_\mathrm{t})\,\mathrm{d}\widetilde{\mathbf{W}}_\mathrm{t} + \left[\left(\mathrm{Ad}_{\exp(\mathrm{t}\xi)}\eta\right)(\mathbf{Y}_\mathrm{t}) + \xi(\mathbf{Y}_\mathrm{t})\right]\mathrm{dt} \tag{75}$$

for $\widetilde{\mathbf{W}}_t \sim \mathbf{W}_\mathrm{t}$ *and all* $s \geq 0$.

Proof. — Similar to theorem (**??**), the proof follows mainly by lemma **??** and lemma **??**. For this note that by Itô's lemma, the transformation $\mathbf{Y}_{\mathrm{t}} := \phi \circ \mathbf{X}_{\mathrm{t}}$ on $\mathbf{U}$ solves

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \frac{\mathrm{d}\Phi^{\xi}}{\mathrm{dx}}\left(\mathbf{X}_{\mathrm{t}}\right)\left[\sigma\left(\mathbf{X}_{\mathrm{t}}\right)\mathrm{d}\mathbf{W}_{\mathrm{t}} + \eta\left(\mathbf{X}_{\mathrm{t}}\right)\right]\mathrm{dt} + \frac{\partial\Phi^{\xi}}{\partial\mathrm{t}}(\mathbf{X}_{\mathrm{t}})\mathrm{dt} + \mathbf{R}\left(\Phi^{\xi},\sigma\right)(\mathbf{X}_{\mathrm{t}})\mathrm{dt} \tag{76}$$

As the standard real coordinates $\mathrm{x}^{\mathrm{i}}, \mathrm{y}^{\mathrm{i}}$, $1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n}$ of $\mathbf{U} \subset \mathbb{C}^{\mathrm{n}} = \mathbb{R}^{2\mathrm{n}}$ are harmonic and since each transformation Φ_{t}^{ξ} of $\mathbf{U}$ is holomorphic and inparticular harmonic as well, it follows by lemma **??** in conjunction with lemma **??** that

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \left[\left(\left(\Phi_{\mathrm{t}}^{\xi}\right)_{*}\eta\right)\left(\mathbf{Y}_{\mathrm{t}}\right) + \frac{\partial\Phi^{\xi}}{\partial\mathrm{t}}(\mathbf{X}_{\mathrm{t}})\right]\mathrm{dt} \tag{77}$$

for $\widetilde{\mathbf{W}}_{\mathrm{t}} \sim \mathbf{W}_{\mathrm{t}}$ and all $s \geq 0$. Using lemma **??**, this can be transformed into

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \left[\left(\mathrm{Ad}_{\exp(\mathrm{t}\xi)}\eta\right)\left(\mathbf{Y}_{\mathrm{t}}\right) + \frac{\partial\Phi^{\xi}}{\partial\mathrm{t}}(\mathbf{X}_{\mathrm{t}})\right]\mathrm{dt} \tag{78}$$

As

$$\frac{\partial\Phi^{\xi}}{\partial\mathrm{t}}(\mathbf{X}_{\mathrm{t}}) = \frac{\partial\Phi^{\xi}}{\partial\mathrm{s}}_{|\mathrm{s}=0}\Phi\left(\mathrm{t},\mathbf{X}_{\mathrm{t}}\right) = \xi\left(\mathbf{Y}_{\mathrm{t}}\right) \tag{79}$$

this is equivalent to

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \left[\left(\mathrm{Ad}_{\exp(\mathrm{t}\xi)}\eta\right)\left(\mathbf{Y}_{\mathrm{t}}\right) + \xi(\mathbf{Y}_{\mathrm{t}})\right]\mathrm{dt} \tag{80}$$

□

Corollary 5.4. — *Let $(\mathbf{U}, \mathbf{X}_{\mathrm{t}}, G)$ be a σ-compatible group of transformations where $\mathbf{X}_{\mathrm{t}}$ solves*

$$\mathrm{d}\mathbf{X}_{\mathrm{t}} = \sigma\left(\mathbf{X}_{\mathrm{t}}\right)\mathrm{d}\mathbf{W}_{\mathrm{t}} + \eta\left(\mathbf{X}_{\mathrm{t}}\right)\mathrm{dt} \tag{81}$$

for some $\eta \in \mathfrak{g}$ and assume that

$$\Phi_{\mathrm{t}}^{\xi} \circ \Phi_{\mathrm{t}}^{\eta} = \Phi_{\mathrm{t}}^{\eta} \circ \Phi_{\mathrm{t}}^{\xi} \tag{82}$$

then the transformation $\mathbf{Y}_{\mathrm{t}}^{\xi} = \Phi_{\mathrm{t}}^{\xi} \circ \mathbf{X}_{\mathrm{t}}$ solves

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \left[\eta\left(\mathbf{Y}_{\mathrm{t}}\right) + \xi(\mathbf{Y}_{\mathrm{t}})\right]\mathrm{dt} \tag{83}$$

for $\widetilde{\mathbf{W}}_{t} \sim \mathbf{W}_{\mathrm{t}}$ and all $s \geq 0$. In particular, if $\Phi^{\xi_{\mathrm{t}}} = \phi$

Proof. — Since invariance of μ is equivalent to $\phi_{*}\mu = \mu$ by definition, theorem **??** yields

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \mu\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{dt} \tag{84}$$

and hence the claim. □

***Corollary 5.5* (Generating drifts).** — *Let $\mathbf{X}_{\mathrm{t}}$ be a driftless process*

$$\mathrm{d}\mathbf{X}_{\mathrm{t}} = \sigma\left(\mathbf{X}_{\mathrm{t}}\right)\mathrm{d}\mathbf{W}_{\mathrm{t}} \tag{85}$$

and assume $(\mathbf{U}, \mathbf{X}_{\mathrm{t}}, G)$ to be a σ-compatible group of transformations of $\mathbf{X}_{\mathrm{t}}$, then

$$\mathrm{d}\mathbf{Y}_{\mathrm{t}} = \sigma\left(\mathbf{Y}_{\mathrm{t}}\right)\mathrm{d}\widetilde{\mathbf{W}}_{\mathrm{t}} + \xi(\mathbf{Y}_{\mathrm{t}})dt \tag{86}$$

for $\widetilde{\mathbf{W}}_{t} \sim \mathbf{W}_{\mathrm{t}}$.

Proof. — The claim is a direct consequence of the aforementioned theorem. □

Example 5.6 **(Periodic Process on the Upper Half Plane)**
Let $\left(\mathbb{H}, \mathbf{g} = \mathrm{y}^{-2}\left(\mathrm{dx}^2 + \mathrm{dy}^2\right)\right)$ by the Upper Half Plane Model as introduced in example (**??**). Recall that a the projective special linear group of dimension 2 over the real numbers

$$\mathrm{PSL}\left(2, \mathbb{R}\right) := \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) : \alpha\delta - \beta\gamma = 1 \right\} \Big/ \left\{\pm\mathbb{E}\right\} \tag{87}$$

acts on $\mathbb{H}$ by holomorphic isometries with respect to $\mathbf{g}$ by

$$\mathrm{g.z} = \frac{\alpha \mathrm{z} + \beta}{\gamma \mathrm{z} + \delta} \tag{88}$$

6. The generalized reflection principle

As before, let $(\mathbf{U}, \mathbf{X}_\mathrm{t}, \phi)$ be a *σ-compatible transformation* of $\mathbf{X}_\mathrm{t}$ where $\mathbf{X}_\mathrm{t}$ is a solution of equation (**??**).

In this section, we will furthermore assume that ϕ acts as a reflection on $\mathbf{U}$, i.e. $\phi^2 = \phi$. Now, let

$$\mathbf{C}(\phi) := \{\mathrm{x} \in \mathbf{U} : \phi(\mathrm{x}) = \mathrm{x}\} \subset \mathbf{U} \tag{89}$$

be the reflection center of ϕ. Throughout this section, we will assume that $\mathbf{C}(\phi)$ separates $\mathbf{U}$ into two connected, disjoint sets $\mathbf{U}_0$ and $\mathbf{U}_1$ such that

$$\mathbf{U} = \mathbf{U}_0 \cup \mathbf{C}(\phi) \cup \mathbf{U}_1 \tag{90}$$

Having introduced the above partition $\{\mathbf{U}_{0,1}, \mathbf{C}(\phi)\}$ of $\mathbf{U}$, we define the refelcted process $\mathbf{X}^\phi$ of $\mathbf{X}$ in $\mathbf{U}$ by setting

$$\mathbf{X}^\phi_\mathrm{t} := \begin{cases} \mathbf{X}_\mathrm{t} & \text{for } \mathrm{t} \leq \tau^\phi \\ \phi\left(\mathbf{X}_\mathrm{t}\right) & \text{for } \mathrm{t} > \tau^\phi \end{cases}$$

Theorem 6.1. — *Let $\{\mathbf{U}_{0,1}, \mathbf{C}(\phi)\}$ be a disjoint partition of $\mathbf{U}$ into connected subsets $\mathbf{U}_0$, $\mathbf{U}_1$ and consider a stochastic process $\mathbf{X}_\mathrm{t}$ solving*

$$\mathrm{d}\mathbf{X}_\mathrm{t} = \sigma\left(\mathbf{X}_\mathrm{t}\right)\mathrm{d}\mathbf{W}_\mathrm{t} \tag{91}$$

with the initial value condition $\mathbf{X}_0 \in \mathrm{U}_0$, then if $(\mathbf{U}, \mathbf{X}_\mathrm{t}, \phi)$ is a σ-compatible transformation of $\mathbf{X}_\mathrm{t}$, it follows

$$\mathbb{P}\left(\tau^\phi \leq \mathrm{t}\right) = 2\mathbb{P}\left(\mathbf{X}_\mathrm{t} \in \mathbf{U}_1\right) \tag{92}$$

The file sent by the author will be adapted to the style of the journal where it will be published by the editorial board of the Société mathématique de France. It is therefore *important* that the LaTeX 2$_\varepsilon$ file is prepared in a very standard way, in particular by a *systematic* use of theorem- and proof-like environments (see § **??**), of `\label` and `\ref` commands for referring to the corresponding numbers, and of `\cite` for bibliographical references. Moreover, “home” macros must be clearly written in the preamble. *No* “home” macro will be used in the title, the address, the abstracts (French and English), the keywords.

6.1. Horizontal and vertical spacing

- Delete all spacing commands like `\,` or `\;` or `\!` *before or after* mathematical symbols, parentheses, punctuation marks, etc. Horizontal spaces (in mathematical mode in particular) are handled automatically by TEX, the author should not add any.
- On the other hand, the author may impose indivisible spaces in places where she/he does not want a carriage return, e.g. `Tintin~\cite{RG3}` instead of `Tintin \cite{RG3}`.
- The author should not type any space or carriage return *before* punctuation marks. However, such a space or carriage return *always* comes after punctuation marks
- No space *before* a closing parentheses or bracket, as well as *after* an opening parentheses or bracket.
- Do not use any `\linebreak`, `\\`, `\pagebreak`, `\newpage`, etc. in the text.
- Avoid commands as `\hskip`, `\hspace` or `\vskip`, `\vspace` in the text.

6.2. Punctuation marks

- Do not put *any* punctuation marks at the end of any title:
  - `\section{Introduction}` and not `\section{Introduction.}`
  - `\begin{remark}` and not `\begin{remark.}`
  - etc.
- In text mode, punctuation marks are typed *outside of* the mathematical mode. Write for example:
  “... `the level $\eta_0$:  $$ A=B.$$`”
  and not
  “... `the level $\eta_0:$ $$A=B.$$`”
- Concerning points of suspension:
  - replace `...` with `\ldots\ ` in the text (in English);
  - replace `...` or `\ldots` with `\cdots` between operators (as in, for instance, $A < \cdots < B$, $A + \cdots + B$ or $A = \cdots = B$) and with `\dots` or `\ldots` for mathematical punctuation (for instance $i = 1, \ldots, n$);
  - suppress `...` after “etc.”.
- For a product, use `\cdot` and not `.`; In the same way, rewrite formulas like $h(.)$ or $(.,.)$ as $h(\cdot)$ or $(\cdot,\cdot)$.
- Replace explicit hyphenation (as in `presenta-tion`) with optional hyphenation `\-` (as in `presenta\-tion`). Of course, ordinary hyphens are kept for compound words.

6.3. Titles. — Titles begin with an upper case letter and are typed in *lower case letters*. When necessary, LATEX will produce an upper case output. No punctuation marks should be inserted at the end of titles (see above).

6.4. Language. — The author should follow the rules of the language she/he uses, in particular when typing numbers: in French, one should write “deux nombres égaux à 2” and in the file one should type `deux nombres \'egaux \`a 2.`

On the other hand, recall that French upper case letters take accents as do lower case letters.

6.5. Numbering

- Use as much as possible the automatic numbering and the corresponding LaTeX 2ε commands `\label`, `\ref`. To this end, keep *a consistent numbering convention.* Do not "ask" commands such as `\section` or `\begin{theoreme}` to produce a complicated output. Recall that the final output will be done by the editorial board of the Société mathématique de France: please, try to help the secretary in her/his task.
- Use a simple logic for internal references:
  - `\label{sec:1}` for the first section,
  - `\label{th:invfunct}` for the inverse function theorem,
  - `\label{rem:stupid}` for an interesting remark.
- Do not number equations which are not referred to in the text.

6.6. The mathematical mode

- Do not put pieces of text between `$ $` to change their style. The mathematical mode should only be used for writing mathematical formulas.
- The numbers written as digits should be typed in mathematical mode, even if this does not appear to be necessary.
- Do not add horizontal spaces in mathematical formulas. When necessary, the editorial board will do it.
- Use the right mathematical TeX or LaTeX symbol at the right place: for instance, the symbols `<` and `>` should not be used for making a bracket $\langle,\rangle$; this bracket is obtained with `$\langle,\rangle$`.
- Please, before using your own solution, check all available AMS-LaTeX capabilities to place and cut mathematical formulas in display style (see [?]).

6.7. The bibliography

- Make a uniform bibliography and do not change the convention according to the entry (use BibTeX for instance).
- *Systematically* use the `\cite` command to cite the entries of the bibliography.

7. The environment

The Société mathématique de France (SMF) provides authors with the following files:

- two class files `smfbook.cls` (for monographs) and `smfart.cls` (for articles),
- two BibTeX style files:
  - `smfplain.bst` (for numerical citations) and
  - `smfalpha.bst` (for alphabetical citations),
- a supplementary package `smfthm.sty` described in § ??,

– a supplementary package `smfenum.sty` for enumerations in the French style,
– a supplementary package `bull.sty` for articles submitted to the *Bulletin*.

They may be obtained on the web site of the SMF:

`http://smf.emath.fr/`

under the heading `Publications/Formats`.

These classes have been written to remain compatible with the `amsbook` and `amsart` classes developped by the American Mathematical Society (AMS). To use them, you need:

– LaTeX 2_ε, preferebly some recent version. The class doesn't work with the old LaTeX2.09 version which has been obsolete for years;
– the various packages furnished by the American Mathematical Society; it is better to have the November 1996 version although it should work with the 1995 one.
– To typeset an index, it is better to have the `multicol.sty` *package* available.

The file `smfbook.cls` (*resp.* `smfart.cls`) is used instead of `amsbook.cls` (*resp.* `amsart.cls`) and has to be put in the directory containing TeX inputs. In order to use the package `smfthm` (see § **??**) or `bull.sty`, one should put the files `smfthm.sty` or `bull.sty` in the same directory.

Many standard packages add capabilities to LaTeX 2_ε. In this respect, we suggest using

– `epsfig.sty`, [**?**], for the inclusion of (encapsulated) PostScript pictures;
– `graphics.sty` or `graphicx.sty`, [**?**, **?**], in order to include pictures drawn by LaTeX;
– `babel.sty`, [**?**], for a text written in various languages (hyphenation, ...);
– `xypic.sty`, [**?**], for the diagrams;
– BibTeX, [**?**, Appendix B] or [**?**], for the bibliography.

8. Structure of the document

A document typeset with one of the classes `smfbook` or `smfart` has the following structure. Fields within brackets are optional.

`\documentclass[`⟨*options*⟩`]{smfbook` or `smfart}`
Preamble (packages, macros, theoremlike environments, ...) e.g.
`\usepackage[francais,english]{babel}`
`\usepackage{smfthm}`
`\usepackage{bull}` (for articles submitted to the *Bulletin*)
`\theoremstyle{plain} \newtheorem{scholie}{Scholie}`

`\author[`⟨*short name*⟩`]{`⟨*Firstname Lastname*⟩`}`
`\address{`⟨*line 1*⟩`\\` ⟨*line 2*⟩`\\` ...⟨*line n*⟩`}`
`\email{`⟨*email address*⟩`}`
`\urladdr{`⟨*WWW address*⟩`}`

`\title[`⟨*short title*⟩`]{`⟨*title of text*⟩`}`
`\alttitle{`⟨*title in the other language (French or English)*⟩`}`

\begin{document}
\frontmatter

\begin{abstract}
⟨*Abstract in the main language of text*⟩
\end{abstract}

\begin{altabstract}
⟨*Abstract in the other language (French or English)*⟩
\end{altabstract}

\subjclass{⟨*AMS classification*⟩}
\keywords{⟨*Key words*⟩}
\altkeywords{⟨*Mots-clefs in the other language (French or English)*⟩}

\translator{⟨*Firstname Lastname*⟩}
\thanks{⟨*Grants*⟩}
\dedicatory{⟨*dedication*⟩}

\maketitle
\tableofcontents ⟨*if needed*⟩

\mainmatter
Main body of the text

\backmatter
Bibliography, index, etc.
\end{document}

Remarks

– If there are many authors, or if an author has more than one address, one may type as many
  \author{⟨*author*⟩}
  \address{⟨*address*⟩}
  \email{⟨*email address*⟩}
  \urladdr{⟨*WWW address*⟩}
  commands as needed, in the right order of course.
– All data introduced before the \maketitle command will be used for different purposes: back cover, advertisement, electronic abstracts, data banks. It is therefore important that no personal macro is used in the corresponding fields.
– Do not hesitate to be prolix when filling the field \subjclass. One may consult for instance the web site
  http://www.ams.org/msc/

9. Class options

These options are entered the following way:

\documentclass[⟨*option1,option2,...*⟩]{smfbook or smfart}

Default options are indicated with a star.

9.1. Usual options

– (⋆) `a4paper`, A4 printing
– `letterpaper`, US Letter printing, to make easier the typesetting of documents in the United States
– `draft`, preliminary draft, *overfull hbox*es are shown by black rules;
– (⋆) `leqno`, equation numbers on the left
– `reqno`, equation numbers on the right
– (⋆) `10pt`, normal character size = 10 points
– `11pt`, normal character size = 11 points
– `12pt`, normal character size = 12 points

9.2. Language of the text

– (⋆) `francais`, if the main language of the text is French
– `english`, if it is English.

9.3. Remark. — Do not mix up the `francais` or `english` options of the SMF class with the `francais` or `english` options of `babel`: the latter has to be entered as indicated in the example of § **??**.

10. Sectioning commands

As in any LaTeX 2ε class, some commands are devoted to the sectioning of the document:

`\part`	
`\chapter`	`smfbook` only
`\section`	
`\subsection`	
`\subsubsection`	
`\paragraph`	
`\subparagraph`	

The table of contents is inserted automatically with `\tableofcontents`.
The macro
`\appendix`
starts the appendix.

The bibliography is entered as usual,

`\begin{thebibliography}{`⟨*longest label*⟩`}`
⟨*Bibliography entries*⟩
`\end{thebibliography}`

The use of BibTeX is recommended, see for example [**?**, Appendix B] and [**?**] for an introduction. The BibTeX styles `smfplain.bst` and `smfalpha.bst` may be obtained on the web site `http://smf.emath.fr/` of the SMF. The bibliography is then entered as follows

`\bibliographystyle{smfplain` or `smfalpha}`
`\bibliography{myfile.bib}`

if `myfile.bib` is the BibTeX data file.

11. Presentation of theorems

Theorems are typeset thanks to the package `amsthm`. For details, we refer to its documentation [**?**]. One should use such environments in a *systematic* way for statements and proofs.

11.1. Theorem styles. — Three styles of theorems are defined: `plain`, `definition` and `remark`. The two last are identical and only differ from the first one in that the text of the statement is in straight letters instead of italics. All `\newtheorem`(∗) commands should be introduced clearly in the preamble.

The `\newtheorem` command creates or uses some counter in order to define the numbering of the corresponding environment.

Use the `\newtheorem*` command to get nonnumbered theoremlike environments, e.g.

```
\newtheorem*{curveselectionlemma}{Curve Selection Lemma}
```

Different kinds of numberings may also be introduced in the preamble, e.g. for propositions numbered alphabetically:

```
\newtheorem{theoremalph}{Proposition}
\def\thetheoremalph{\Alph{theoremalph}}.
```

11.2. Proof environment. — The proof environment

`\begin{proof}` ... `\end{proof}`

allows a standard presentation of proofs, beginning with "Proof" and ending with the traditional small box □.

It is possible to change the word "Proof" as in:

`\begin{proof}[Idea of proof]` ... `\end{proof}`

which shows

Idea of proof. — Exercise for the interested reader. □

12. The `smfthm.sty` package

This section describes the `smfthm.sty` package. Its use is not mandatory.

12.1. Theoremlike environments. — Some theoremlike environments are defined. They use one and the same counter.

Style	Macro LaTeX	Nom français	English name
plain	`theo`	Théorème	*Theorem*
	`prop`	Proposition	*Proposition*
	`conj`	Conjecture	*Conjecture*
	`coro`	Corollaire	*Corollary*
	`lemm`	Lemme	*Lemma*
definition	`defi`	Définition	*Definition*
remark	`rema`	Remarque	*Remark*
	`exem`	Exemple	*Example*

One uses them e.g. as follows:

```
\begin{theo}[Wiles]
If $n\geq 3$ and if $x$, $y$, $z$ are integers
such that $x^n+y^n=z^n$, then $xyz=0$.
\end{theo}
```

***Theorem 12.1* (Wiles).** — *If $n \geq 3$ and if x, y, z are integers such that $x^n + y^n = z^n$, then $xyz = 0$.*

12.2. Fixing the choice of the numbering. — The way of numbering the statements is defined by the following commands, which have to been entered *before* the `\begin{document}`:

- `\NumberTheoremsIn{`⟨*counter name*⟩`}`, indicates the level at which the statement numbers are reset to zero, (`section` for instance); the counter `smfthm` is then defined;
- `\NumberTheoremsAs{`⟨*counter name*⟩`}`, allows the statement counter to be one of the usual sectioning counters (e.g. `section`, `subsection`, `paragraph`, etc.);
- `\SwapTheoremNumbers`, to put the statement number before the statement name, as in “1.4. Theorem”
- `\NoSwapTheoremNumbers`, the converse, e.g. “Theorem 3.1”

The default options of the package are
`\NumberTheoremsIn{section}\NoSwapTheoremNumbers`
which means that the counter `smfthm` is defined and reset at the beginning of every section and that the statement numbers, which take the form
`section number.value of the counter smfthm`
are written after the statement name.

12.3. Generic statement. — The `enonce` environment allows one to typeset a generic theorem whose name changes on demand, e.g.:

```
\begin{enonce}{Assumption}
⟨...⟩
\end{enonce}
```

typesets an ‘Assumption’, numbered as it should be.

The `enonce` environment uses the *plain* theorem style, but one can change this style by putting another style inside brakets, e.g.:

```
\begin{enonce}[remark]{Key remark}
⟨...⟩
\end{enonce}
```

Finally, there exists a corresponding `enonce*` environment.

he SMF classes, this paragraph will give you some hints.

12.4. From another LaTeX 2ε class. — If it is an AMS class, you'll have very little to do: for an article written in English for instance, replace

```
\documentstyle[12pt,leqno]{amsart}
```

with

```
\documentstyle[leqno,english]{smfart}
```

You'll need to enter another abstract (`altabstract`) and another title (`alttitle`), in French if your text is in English and in English otherwise.

Literature and sources

[1] L. Lamport. — *LaTeX: A Document Preparation System.* Second edition. Addison-Wesley, 1994.

[2] M. Goossens, F. Mittelbach, A. Samarin. — *The LaTeX Companion.* Addison-Wesley, 1993.

[3] M. Goossens, S. Rahtz and F. Mittelbach. — *The LaTeX Graphics Companion: Illustrating Documents With TeX and Postscript.* Tools and Techniques for Computer Typesetting Series, Addison-Wesley, 1996.

[4] *The Not So Short Introduction to LaTeX2e,* T. Oetiker, H. Partl, I. Hyna, E. Schlegl, `http://www.loria.fr/tex/general/flshort2e.dvi`

[5] *AMS-LaTeX version 1.2 User's Guide*, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/amslatex/amsldoc.dvi`

[6] *Babel, a multilingual package for use with LaTeX's standard document classes,* J. Braams, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/babel/babel.dvi`

[7] *The* `epsfig` *package,* S. Rahtz, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/epsfig.dvi`

[8] *The* `graphics` *package,* D. Carlisle, S. Rahtz, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/graphics.dvi`

[9] *The* `graphicx` *package,* D. Carlisle, S. Rahtz, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/graphicx.dvi`

[10] *Hypatia's Guide to BibTeX,* `http://hypatia.dcs.qmw.ac.uk/html/bibliography.html`

[11] *Xy-pic User's Guide,* K. Rose, R. Moore, `http://www.loria.fr/tex/graph-pack/doc-xypic/xyguide.dvi`

The most recent versions of macros files and of their documentations are also available by anonymous `ftp` on the CTAN sites (*Comprehensive TeX Archive Network*) In the United States, one may use the address `ftp.shsu.edu`; the sites `ftp.loria.fr` or `ftp.jussieu.fr` in France, `ftp.tex.ac.uk` in England, and `ftp.dante.de` in Germany also hold the archive.

English version 5, dated November 2019

Daniel Berger, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, F-75231 Paris cedex 05
E-mail : `christia@dma.ens.fr` • *Url :* `http://smf.emath.fr/`