SYMMETRIES OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. — The article sums up some results concerning the transformation and invariance properties of a local, stochastic differential equations of the form

$$\mathrm{dX_t^i} = \sigma_j^i(\mathbf{Z_t})\mathrm{dW_t^j} + \mu^i(\mathbf{Z_t})\mathrm{dt} \tag{1}$$

In general, the coefficients μ^i of the drift in 1 do not share the transformation law of a contravariant tensor of rank one due to second order differentials arising from Ito′s formula. We present conditions concerning the local coordinates and the transformation itself under which the usual transformation laws hold.

Beside, we briefly examine invariance properties of differential system like under certain Lie groups G preserving the covariance structure $\sigma_{i,j}$.

Contents

1. Introduction

Throughout this article let $\mathbf{U}$ be an open subset of $\mathbb{R}^n$ with the standard coordinates x^i, $1 \leq i \leq n$ induced by the ambient space $\mathbb{R}^n$ and consider an $n-$dimensional stochastic process $\mathbf{X}_t$ with components x_t^i, $1 \leq i \leq n$ in $\mathbf{U}$ for $t \in [0, S]$ solving the following stochastic differential equation

$$(2) \qquad d\mathbf{X}_t = \sigma\left(\mathbf{X}_t\right) d\mathbf{W}_t + \mu\left(\mathbf{X}_t\right) dt$$

Here, $\sigma \in \mathscr{C}^\infty\left(\mathbf{U}, \mathbb{R}^n \times \mathbb{R}^n\right)$ denotes a smooth, matrix-valued function on $\mathbf{U}$ with coefficient functions σ_j^i i.e. $\sigma = (\sigma_j^i)_{ij}$. We further assume that $\sigma(x)$ is invertible for all $x \in \mathbf{U}$. Besides, let $\mu \in \mathscr{C}^\infty\left(\mathbf{U}, \mathbb{R}^n\right)$ be a smooth vector field on $\mathbf{U}$ with components μ^i so that $\mu = (\mu^1, \ldots, \mu^n)$. Instead of (2), we will frequently use its index based version given by

$$(3) \qquad dX_t^i = \sigma_j^i\left(\mathbf{X}_t\right) dW_t^j + \mu^i\left(\mathbf{X}_t\right) dt$$

where we agree to use the usual convention to perform summation over twice appearing indices. t

2. Contravariance of the drift

Let $\mathbf{X}_t$ in $\mathbf{U}$ be as defined in section (1) and fix diffeomorphism $\phi \in \mathscr{C}^\infty\left(\mathbf{U}, \mathbf{U}\right)$. In this setting, Itô's lemma states that the transformed process $\mathbf{Y}_t := \phi \circ \mathbf{X}_t$ on $\mathbf{U}$ solves the following equation

$$(4) \qquad dY_t = \frac{d\phi}{dx}\left(\mathbf{X}_t\right)\left[\sigma\left(\mathbf{X}_t\right) d\mathbf{W}_t + \mu\left(\mathbf{X}_t\right) dt\right] + \mathbf{R}\left(\phi, \sigma\right)\left(\mathbf{X}_t\right)$$

with

$$(5) \qquad \frac{d\phi}{dx}\left(\mathbf{X}_t\right) = \left(\frac{\partial \phi^i}{\partial x^j}\left(\mathbf{X}_t\right)\right)_{ij}$$

and a smooth vector valued function $\mathbf{R}\left(\phi, \sigma\right) \in \mathscr{C}^\infty(\mathbf{U}, \mathbb{R}^n)$ on $\mathbf{U}$ with components

$$(6) \qquad R^i\left(\phi, \sigma\right)\left(\mathbf{X}_t\right) = \frac{1}{2}\frac{\partial^2 \phi^i}{\partial x^k \partial x^\ell}(\mathbf{X}_t)\sigma_j^k(\mathbf{X}_t)\sigma_j^\ell(\mathbf{X}_t)$$

Note that the term $\mathbf{R}$ in equation (4) is the obstruction for the coefficients of equation (2) to transform contravariantly with respect to the change of coordinates ϕ on $\mathbf{U}$. To be more precise, if $\mathbf{R} \equiv 0$ on $\mathbf{U}$ the following transformation law would hold:

$$(7) \qquad dY_t^i = \frac{\partial \phi^i}{\partial x^j}dX_t^i$$

Before we proceed to specify the conditions, which have to be imposed on the map ϕ and the coordinates x^i, $1 \leq i \leq n$ on $\mathbf{U}$ so that the dY_t^i transform like the coefficients of a tensor of rank 1, we have to introduce and recall some definitions:

Definition 2.1. — Let σ be a defined as above, i.e. $\sigma \in \mathscr{C}^\infty\left(\mathbf{U}, \mathbb{R}^n \times \mathbb{R}^n\right)$ with $\sigma(x)$ invertible for all $x \in U$, then define a Riemannian metric $\mathbf{g}$ on $\mathbf{U}$ by setting

$$(8) \qquad \mathbf{g} := \left(\sigma^\mathsf{T}\right)^{-1}\sigma^{-1}$$

Note that by assumption, σ is invertible on $\mathbf{U}$ and hence $\mathbf{g}$ is well defined. Now, recall the following definitions from Riemannian geometry.

***Definition 2.2* (Christoffel symbols).** — Let $(\mathbf{M}, \mathbf{g})$ be a Riemannian manifold and $\varphi_{\mathbf{V}} \in \mathscr{C}^{\infty}(\mathbf{V}, \mathbf{U})$ be a chart from the open subset $\mathbf{V} \subset \mathbf{M}$ to $\mathbf{U} \subset \mathbb{R}^n$ with local coordinates x^i, $1 \leq i \leq n$ on $\mathbf{U}$ then the Christoffel symbol $\Gamma_{ij}^k \in \mathscr{C}^{\infty}(\mathbf{U})$ of the second kind for $1 \leq k \leq n$ is given by

$$(9) \qquad \Gamma_{ij}^k := \frac{1}{2} g^{i\ell} \left(-\frac{\partial g_{ij}}{\partial x^\ell} + \frac{\partial g_{\ell j}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} \right)$$

***Definition 2.3* (Laplacian operator).** — Let $(\mathbf{M}, \mathbf{g})$, $\mathbf{V} \subset \mathbf{M}$, $\mathbf{U}$ and $\varphi_{\mathbf{V}}$ be as in defintion 2.2), then the second order partial derivate operator on $\mathbf{U}$ given by

$$(10) \qquad \triangle^{\mathbf{g}} := g^{ik} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ik}^j \frac{\partial}{\partial x^k} \right)$$

is called the Laplacian operator on $\mathbf{U}$ with respect to $\mathbf{g}$.

***Definition 2.4* (Harmonic functions).** — Let $(\mathbf{M}, \mathbf{g})$ be a Riemannian manifold and $f \in \mathscr{C}^{\infty}(\mathbf{V}, \mathbb{R})$ with $\mathbf{V} \subset \mathbf{M}$, then f is called harmonic on $\mathbf{V}$ if locally with respect to the chart $\mathbf{U}$ (as in defintion 2.2), f solves

$$(11) \qquad \triangle^{\mathbf{g}} f = 0$$

We are now able to state the under which assumption we have $\mathbf{R} \equiv 0$ on $\mathbf{U}$ so that equation (7) holds.

***Lemma 2.5* (Contravariance of drift).** — *Let $\mathbf{X}_t$ be a process in $\mathbf{U}$ solving equation 2, then the image $\mathbf{Y}_t = \phi \circ \mathbf{X}_t$ under a diffeomorphism $\phi \in \mathscr{C}^{\infty}(\mathbf{U}, \mathbf{U})$ solves (7) if the following two conditions hold:*

1. *The local coordinates x^i are harmonic with respect to $\mathbf{g}$, i.e. $g^{ij} \Gamma_{ij}^k = 0$ for all $1 \leq i, j \leq n$.*
2. *The map ϕ is harmonic, i.e. each component ϕ^i is again a harmonic function with respect to x^i, $1 \leq i \leq n$.*

Proof. — By applying the lemma of Itô, the drift term of the transformed process $Z'^k = \phi^k(\mathbf{Z}_t)$ is given by the following formula

$$(12) \qquad dZ_t'^k = \cdots + \frac{\partial x'^k}{\partial x^\ell} \mu^\ell + g^{ij} \frac{\partial^2 x'^k}{\partial x^i \partial x^j}$$

In particular, as the proof is completed as soon as we have shown that the second term on the right hand side of the above equaiton vanishes, i.e.:

$$(13) \qquad g^{ij} \frac{\partial^2 x'^k}{\partial x^i \partial x^j} = 0$$

for all k.

For this, recall that condition is equivalent to

$$(14) \qquad \triangle x'^i = 0$$

for all $1 \leq i \leq n$, where $\triangle$ denotes the Laplace-Beltrami operator associated to the metric g which g is locally given by

$$(15) \qquad \triangle = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial}{\partial x^k}\right)$$

As 1) is equivalent to $g^{ij}\Gamma_{ij}^k = 0$ for all $1 \leq k \leq n$, it follows from condition 2) that

$$(16) \qquad 0 = g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$$

for all k, which in conjunction with step yields the result. $\qquad\square$

***Example 2.6* (Standard complex space).** — Let $U \subset \mathbb{C}^n$ and $\sigma^{ij} = \delta^{ij}$ so that g is given by the flat Euclidean metric. By function theroy, it is known that the real part $x^k = \mathfrak{Re}z^k$ as well as the imaginary part $y^k = \mathfrak{Im}z^k$ of the complex standard coordinates f $z^k = x^k + \sqrt{-1}y^k$ are harmonic function with respect to the standard metric structure. Furthermore, any holomorphic function f on U are harmonic. Hence by lemma, every

Consider the system

$$(17) \qquad dU =$$

defined on

***Example 2.7* (Local Kählerain space).** — Let $U \subset \mathbb{C}^n$ and

3. Symmetries of certain stochastic differential equations

In the sequel, let $(\mathbf{M}, g)$ be a Riemannain manifold with a smooth Lie group action $\phi : G \times \mathbf{M} \to \mathbf{M}$ which preserves the metric g so that $\sigma^* g = g$ for all $\sigma \in G$. For convenience, we will assume that $\mathbf{M}$ is given by an open subset $\mathbf{U}$ of $\mathbb{R}^m$ whit local coordinates $(x_1, \ldots, x_m)$.

Furthermore, we will fix a system of stochastic differential of the following form:

$$(18) \qquad d\mathbf{Z}_t = \sigma(\mathbf{Z}_t)d\mathbf{W}_t + \mathbf{v}(\mathbf{Z}_t)dt$$

Here σ denotes m×m matrix with entries σ_{ij}, $1 \leq i,j \leq m$ in the set $\mathscr{C}^\infty(\mathbf{U})$ of smooth, $\mathbb{R}$-valued functions on $\mathbf{U}$ and $\mathbf{W}_t = (W_{1,t}, \ldots, W_{m,t})$ is a random vector of standard, uncorrelated Brownian motions with respect to a probability space $(\sigma, \mathscr{A}, \mathbb{P})$. The drift term is supposed to be given by a smooth vector field $\mathbf{v} : \mathbf{M} \to T\mathbf{M}$ whose locally trivialization in the aforementioned coordinates is represented by the smooth vector $(v_1, \ldots, v_m)$.

Recall that each vector ξ of the Lie algebra $\text{Lie}(G) = \mathfrak{g}$ generates a smooth flow $\Phi_\xi : \mathbb{R} \times \mathbf{M} \to \mathbf{M}$ given by

$$(19) \qquad \Phi_\xi(t, x) = \mathbf{exp}(t\xi).x$$

which is a solution of the differential equation $\dot{\Phi}(t,x) = \mathbf{X}_\xi(\Phi(t,x))$. Here, $\mathbf{X}_\xi$ denotes the smooth vector field generated by the one-parameter flow Φ. To abbreviate the notation, we will frequently just set $\mathbf{X}_\xi = \xi$.

Recall that

Lemma 3.1. — *Let $\phi : U \to U$ be a local change of coordinates $x'^k = \phi^k(x)$ where x^j and ϕ^k are satisfying the conditions of lemma [?], then the transformation Z'^k under ϕ^k of any process solving*

$$\mathrm{dX}_t^i = \sigma_j^i(\mathbf{Z}_t)\,\mathrm{dW}_t^j \tag{20}$$

is again a solution of the original equation, i.e.

$$\mathrm{dX}_t'^k = \sigma_\ell^k(\mathbf{Z}_t')\,\mathrm{dW}_t^\ell \tag{21}$$

Proof. — As the starting process is driftless and the local coordintes as well as the change of local coordinates both meet the conditions of lemma, the transformed drift will have zero drift by the contravariance lemma. Hence, the lemma is proven as soon as we can verify that the quadratic variation $d[\mathbf{Z}_t, \mathbf{Z}_t]$ of $\mathbf{Z}_t$ and the corresponding quadratic variation $d[\widetilde{\mathbf{Z}}_t, \widetilde{\mathbf{Z}}_t]$ of $\widetilde{\mathbf{Z}}_t$ coincides.

Since ϕ is assumed to be an isometry with respect to g_{ij}, we have $g'_{k\ell} = g_{k\ell}$ or equivalently

$$g_{rs}\frac{\partial x'^r}{\partial x^k}\frac{\partial x^s}{\partial x^\ell} = g_{k\ell} \tag{22}$$

Using the definition of g, we get

$$\sigma_r^i\sigma_s^i\frac{\partial x'^r}{\partial x^k}\frac{\partial x'^s}{\partial x^\ell} = \sigma_k^i\sigma_\ell^i \tag{23}$$

which in turn by multiplying with $\frac{\partial x}{}$ and

$\square$

Lemma 3.2. — *Let $\phi : U \to U$ be an isometry with respect to the metric $\mathbf{g}$ given by the local matrix representation $\mathbf{g} = \left(\sigma\sigma^\mathsf{T}\right)^{-1}$ and let $\mathbf{Z}_t$ be a solution of*

$$d\mathbf{Z}_t = \sigma(\mathbf{Z}_t)\,d\mathbf{W}_t, \tag{24}$$

then $\widetilde{\mathbf{Z}}_t = \Phi \circ \mathbf{Z}_t$ is again a solution of 23, i.e.

$$d\widetilde{\mathbf{Z}}_t = \sigma(\widetilde{\mathbf{Z}}_t)\,d\mathbf{W}_t. \tag{25}$$

Proof. — Applying the lemma of ITÔ directly yields

$$\mathrm{d}\widetilde{\mathbf{Z}}_t = \mathrm{d}\Phi(\mathbf{Z}_t)\,\sigma(\mathbf{Z}_t)\mathrm{d}\mathbf{W}_t \tag{26}$$

As booth processes $\mathbf{Z}_t$ as well as $\widetilde{\mathbf{Z}}_t$ are drifless, the lemma is proven Here $d[\mathbf{Z}_t, \mathbf{Z}_t]$ and so $d[\widetilde{\mathbf{Z}}_t, \widetilde{\mathbf{Z}}_t]$ are understood to be matrices given by

$$\mathrm{d}[\mathbf{Z}_t, \mathbf{Z}_t] = ([\mathrm{d}Z_i, \mathrm{d}Z_j])_{1 \leq i,j \leq m} = \mathrm{d}\mathbf{Z}_t \cdot \mathrm{d}\mathbf{Z}_t^\mathsf{T} \tag{27}$$

and analog for $\widetilde{\mathbf{Z}}_t$.

Since Φ is an isometry with respect to $\mathbf{g} = \left(\sigma\sigma^\mathsf{T}\right)^{-1}$, it follows that

$$\mathbf{g}_{\Phi(z)}\left(\mathrm{d}\Phi(z)\cdot, \mathrm{d}\Phi(z)\cdot\right) = \mathbf{g}_z(\cdot, \cdot) \tag{28}$$

for all z in $\mathbf{M}$ or equivalently, written as matrix equation

$$\mathrm{d}\Phi^\mathsf{T}(z)\left(\sigma(\Phi(z))\sigma^\mathsf{T}(\Phi(z))\right)^{-1}\mathrm{d}\Phi(z) = \left(\sigma(z)\sigma^\mathsf{T}(z)\right)^{-1} \tag{29}$$

From equation 28 we infere by multiplying from the left with the inverse of $d\Phi^{\mathsf{T}}(z)$ and from the right with the inverse of $d\Phi(z)$ respectively the equation

$$(30) \qquad \left(\sigma(\Phi\,(z))\sigma^{\mathsf{T}}(\Phi\,(z))\right)^{-1} = \left(d\Phi^{\mathsf{T}}(z)\right)^{-1}\left(\sigma(z)\sigma^{\mathsf{T}}(z)\right)^{-1}\left(d\Phi(z)\right)^{-1}$$

which is equivalent to

$$(31) \qquad \sigma(\Phi\,(z))\sigma^{\mathsf{T}}(\Phi\,(z)) = d\Phi(z)\sigma(z)\sigma^{\mathsf{T}}(z)d\Phi^{\mathsf{T}}(z)$$

Using this equality and the property $d[\mathbf{W}_t,\mathbf{W}_t] = (\delta_{ij})_{1\leq,i,j\leq m}\,dt$, we compute

$$
\begin{aligned}
(32) \qquad d[\widetilde{\mathbf{Z}}_t,\widetilde{\mathbf{Z}}_t] &= d\Phi\,(\mathbf{Z}_t)\,\sigma(\mathbf{Z}_t)d\mathbf{W}_t \cdot d\mathbf{W}_t^{\mathsf{T}}\sigma(\mathbf{Z}_t)^{\mathsf{T}}d\Phi\,(\mathbf{Z}_t)^{\mathsf{T}} \\
&= d\Phi\,(\mathbf{Z}_t)\,\sigma(\mathbf{Z}_t)\sigma(\mathbf{Z}_t)^{\mathsf{T}}d\Phi\,(\mathbf{Z}_t)^{\mathsf{T}}\,dt \\
&= \sigma(\Phi\,(\mathbf{Z}_t))\sigma^{\mathsf{T}}(\Phi\,(\mathbf{Z}_t))dt \\
&= \sigma(\widetilde{\mathbf{Z}}_t)\sigma^{\mathsf{T}}(\widetilde{\mathbf{Z}}_t)dt
\end{aligned}
$$

where we have used the equation 30 in the third step. $\qquad\square$

Theorem 3.3 (**Transfomration Law**). — *Let* $(\mathbf{M},g)$ *and* G *be as above and consider the following system of stochastic differnetial equations on* $\mathbf{M} = \mathbf{U}$

$$(33) \qquad d\mathbf{Z}_t = \sigma(\mathbf{Z}_t)d\mathbf{W}_t + \xi(\mathbf{Z}_t)dt,$$

then the transformed solution $\widetilde{\mathbf{Z}}_t = \Phi_\eta(t,\cdot)\circ\mathbf{Z}_t$ *solves the transformed system given by*

$$(34) \qquad d\widetilde{\mathbf{Z}}_t = \sigma(\widetilde{\mathbf{Z}}_t)d\widetilde{\mathbf{W}}_t + [\eta(\mathbf{Z}_t) + (Ad_\eta\xi)\,(\mathbf{Z}_t)]\,dt,$$

with $\widetilde{\mathbf{W}}_t$ *following the same distribution as* $\mathbf{W}_t$.

Proof. — First of all, by the multi-dimensional Lemma of ITÔ we deduce that

$$
\begin{aligned}
(35) \qquad d\widetilde{\mathbf{Z}}_t &= \frac{d\Phi_\eta}{dx}(t,\mathbf{Z}_t)\,(\sigma(\mathbf{Z}_t)d\mathbf{W}_t + \xi(\mathbf{Z}_t)) \\
&\quad + \left[\frac{d\Phi_\eta}{dt}(t,\mathbf{Z}_t) + \frac{1}{2}d\mathbf{W}_t^T\sigma(\mathbf{Z}_t)^T\mathbf{Hess}(\Phi_\eta)\sigma(\mathbf{Z}_t)d\mathbf{W}_t\right]dt
\end{aligned}
$$

As Π_η is assumed to be harmonic, $\mathbf{Hess} = 0$. As the vector field generated by η evaluated at point $\mathbf{Z}_t$ is given by

$$(36) \qquad \frac{d\Phi_\eta}{dt}(t,\mathbf{Z}_t) = \eta(\Phi_\eta(t,\mathbf{Z}_t)) = \eta(t,\widetilde{\mathbf{Z}}_t)$$

per definition, the second line of equation 34 reduces to $\eta(t,\widetilde{\mathbf{Z}}_t)dt$ and hence, it follows

$$(37) \qquad d\widetilde{\mathbf{Z}}_t = \frac{d\Phi_\eta}{dx}(t,\mathbf{Z}_t)\,(\sigma(\mathbf{Z}_t)d\mathbf{W}_t + \xi(\mathbf{Z}_t)) + \eta(t,\widetilde{\mathbf{Z}}_t)dt$$

Now as $\Phi_\eta(t,\cdot)$ is assumed to be a Riemannian isometry with respect to g, it follows by lemma 3.2 that

$$(38) \qquad \frac{d\Phi_\eta}{dx}(t,\mathbf{Z}_t)\,\sigma(\mathbf{Z}_t)d\mathbf{W}_t = \sigma(\widetilde{\mathbf{Z}}_t)d\mathbf{W}_t$$

and hence equation becomes

$$(39) \qquad d\widetilde{\mathbf{Z}}_t = \sigma(\widetilde{\mathbf{Z}}_t)d\mathbf{W}_t + \xi(\mathbf{Z}_t) + \eta(t, \widetilde{\mathbf{Z}}_t)dt$$

the derivative of the flow $\Phi(t, \grave{)}$ with redspect to t, the above equations reduces to the following form:

or written down in local coordinates $(x_1, \ldots, x_m)$ $\qquad\qquad\square$

In particular, for vector fields d

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```
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The Société mathématique de France (SMF) provides authors with the following files:

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  • `smfplain.bst` (for numerical citations) and
  • `smfalpha.bst` (for alphabetical citations),
− a supplementary package `smfthm.sty` described in § 9,
− a supplementary package `smfenum.sty` for enumerations in the French style,
− a supplementary package `bull.sty` for articles submitted to the *Bulletin*.

They may be obtained on the web site of the SMF:
  `http://smf.emath.fr/`
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These classes have been written to remain compatible with the `amsbook` and `amsart` classes developed by the American Mathematical Society (AMS). To use them, you need:

− LaTeX 2_ε, preferably some recent version. The class doesn't work with the old LaTeX2.09 version which has been obsolete for years;
− the various packages furnished by the American Mathematical Society; it is better to have the November 1996 version although it should work with the 1995 one.
− To typeset an index, it is better to have the `multicol.sty` *package* available.

The file `smfbook.cls` (*resp.* `smfart.cls`) is used instead of `amsbook.cls` (*resp.* `amsart.cls`) and has to be put in the directory containing TeX inputs. In order to use the package `smfthm` (see § 9) or `bull.sty`, one should put the files `smfthm.sty` or `bull.sty` in the same directory.

Many standard packages add capabilities to LaTeX 2_ε. In this respect, we suggest using

− `epsfig.sty`, [**7**], for the inclusion of (encapsulated) POSTSCRIPT pictures;
− `graphics.sty` or `graphicx.sty`, [**8, 9**], in order to include pictures drawn by LaTeX;
− `babel.sty`, [**6**], for a text written in various languages (hyphenation, . . .);
− `xypic.sty`, [**11**], for the diagrams;
− BibTeX, [**1**, Appendix B] or [**10**], for the bibliography.

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`\title[⟨`*short title*`⟩]{⟨`*title of text*`⟩}`
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`\frontmatter`

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   ⟨*Abstract in the main language of text*⟩
  `\end{abstract}`

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 ⟨*Abstract in the other language (French or English)*⟩
`\end{altabstract}`

`\subjclass{⟨`*AMS classification*`⟩}`
`\keywords{⟨`*Key words*`⟩}`
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  `\translator{⟨`*Firstname Lastname*`⟩}`
  `\thanks{⟨`*Grants*`⟩}`
  `\dedicatory{⟨`*dedication*`⟩}`

`\maketitle`
  `\tableofcontents` ⟨*if needed*⟩

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`\backmatter`
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`\end{document}`

Remarks

  − If there are many authors, or if an author has more than one address, one may type as many
      `\author{⟨`*author*`⟩}`
      `\address{⟨`*address*`⟩}`

> `\email{`⟨*email address*⟩`}`
> `\urladdr{`⟨*WWW address*⟩`}`

commands as needed, in the right order of course.
– All data introduced before the `\maketitle` command will be used for different purposes: back cover, advertisement, electronic abstracts, data banks. It is therefore important that no personal macro is used in the corresponding fields.
– Do not hesitate to be prolix when filling the field `\subjclass`. One may consult for instance the web site

> `http://www.ams.org/msc/`

6. Class options

These options are entered the following way:

> `\documentclass[`⟨*option1,option2,...*⟩`]{smfbook or smfart}`

Default options are indicated with a star.

6.1. Usual options

– (⋆) `a4paper`, A4 printing
– `letterpaper`, US Letter printing, to make easier the typesetting of documents in the United States
– `draft`, preliminary draft, *overfull hbox*es are shown by black rules;
– (⋆) `leqno`, equation numbers on the left
– `reqno`, equation numbers on the right
– (⋆) `10pt`, normal character size = 10 points
– `11pt`, normal character size = 11 points
– `12pt`, normal character size = 12 points

6.2. Language of the text

– (⋆) `francais`, if the main language of the text is French
– `english`, if it is English.

6.3. Remark. — Do not mix up the `francais` or `english` options of the SMF class with the `francais` or `english` options of `babel`: the latter has to be entered as indicated in the example of § 5.

7. Sectioning commands

As in any LaTeX 2_ε class, some commands are devoted to the sectioning of the document:

```
\part
\chapter          smfbook only
\section
\subsection
\subsubsection
\paragraph
\subparagraph
```

The table of contents is inserted automatically with `\tableofcontents`.
The macro

   `\appendix`

starts the appendix.

The bibliography is entered as usual,

> `\begin{thebibliography}{`⟨*longest label*⟩`}`
> ⟨*Bibliography entries*⟩
> `\end{thebibliography}`

The use of BIBTEX is recommended, see for example [**1**, Appendix B] and [**10**] for an introduction. The BIBTEX styles `smfplain.bst` and `smfalpha.bst` may be obtained on the web site `http://smf.emath.fr/` of the SMF. The bibliography is then entered as follows

> `\bibliographystyle{smfplain or smfalpha}`
> `\bibliography{myfile.bib}`

if `myfile.bib` is the BIBTEX data file.

8. Presentation of theorems

Theorems are typeset thanks to the package `amsthm`. For details, we refer to its documentation [**5**]. One should use such environments in a *systematic* way for statements and proofs.

8.1. Theorem styles. — Three styles of theorems are defined: `plain`, `definition` and `remark`. The two last are identical and only differ from the first one in that the text of the statement is in straight letters instead of italics. All `\newtheorem(∗)` commands should be introduced clearly in the preamble.

The `\newtheorem` command creates or uses some counter in order to define the numbering of the corresponding environment.

Use the `\newtheorem*` command to get nonnumbered theoremlike environments, e.g.

`\newtheorem*{curveselectionlemma}{Curve Selection Lemma}`

Different kinds of numberings may also be introduced in the preamble, e.g. for propositions numbered alphabetically:

`\newtheorem{theoremalph}{Proposition}`
`\def\thetheoremalph{\Alph{theoremalph}}`.

8.2. Proof environment. — The proof environment
```
\begin{proof} ...\end{proof}
```
allows a standard presentation of proofs, beginning with "Proof" and ending with the traditional small box □.

It is possible to change the word "Proof" as in:
```
\begin{proof}[Idea of proof] ...\end{proof}
```
which shows

Idea of proof. — Exercise for the interested reader.   □

9. The `smfthm.sty` package

This section describes the `smfthm.sty` package. Its use is not mandatory.

9.1. Theoremlike environments. — Some theoremlike environments are defined. They use one and the same counter.

Style	Macro LaTeX	Nom français	English name
plain	theo	Théorème	*Theorem*
	prop	Proposition	*Proposition*
	conj	Conjecture	*Conjecture*
	coro	Corollaire	*Corollary*
	lemm	Lemme	*Lemma*
definition	defi	Définition	*Definition*
remark	rema	Remarque	*Remark*
	exem	Exemple	*Example*

One uses them e.g. as follows:
```
\begin{theo}[Wiles]
If $n\geq 3$ and if $x$, $y$, $z$ are integers
such that $x^n+y^n=z^n$, then $xyz=0$.
\end{theo}
```

Theorem 9.1 **(Wiles).** — *If $n \geq 3$ and if x, y, z are integers such that $x^n + y^n = z^n$, then $xyz = 0$.*

9.2. Fixing the choice of the numbering. — The way of numbering the statements is defined by the following commands, which have to been entered *before* the `\begin{document}`:

- `\NumberTheoremsIn{`⟨*counter name*⟩`}`, indicates the level at which the statement numbers are reset to zero, (`section` for instance); the counter `smfthm` is then defined;
- `\NumberTheoremsAs{`⟨*counter name*⟩`}`, allows the statement counter to be one of the usual sectioning counters (e.g. `section`, `subsection`, `paragraph`, etc.);

- `\SwapTheoremNumbers`, to put the statement number before the statement name, as in "1.4. Theorem"
- `\NoSwapTheoremNumbers`, the converse, e.g. "Theorem 3.1"

The default options of the package are

`\NumberTheoremsIn{section}\NoSwapTheoremNumbers`

which means that the counter `smfthm` is defined and reset at the beginning of every section and that the statement numbers, which take the form

`section number.value of the counter smfthm`

are written after the statement name.

9.3. Generic statement. — The `enonce` environment allows one to typeset a generic theorem whose name changes on demand, e.g.:

`\begin{enonce}{Assumption}`
$\langle\ldots\rangle$
`\end{enonce}`

typesets an 'Assumption', numbered as it should be.

The `enonce` environment uses the *plain* theorem style, but one can change this style by putting another style inside brakets, e.g.:

`\begin{enonce}[remark]{Key remark}`
$\langle\ldots\rangle$
`\end{enonce}`

Finally, there exists a corresponding `enonce*` environment.

9.4. Other statements. — The author may introduce other kinds of theoremlike environments as explained in §8.1. Notice, however, that in order to introduce environments numbered as the ones of `smfthm.sty`, one uses `enonce`:

`\newenvironment{scholie}{\begin{enonce}{Scholie}}{\end{enonce}}`

which should be entered *after* `\begin{document}`.

10. Adapting a manuscript from another dialect

If you already have typed your manuscript in Plain TeX, or in LaTeX2.09, or in LaTeX 2$_\varepsilon$, but with another class, and if you want to adapt it to the SMF classes, this paragraph will give you some hints.

10.1. From another LaTeX 2$_\varepsilon$ class. — If it is an AMS class, you'll have very little to do: for an article written in English for instance, replace

`\documentstyle[12pt,leqno]{amsart}`

with

`\documentstyle[leqno,english]{smfart}`

You'll need to enter another abstract (`altabstract`) and another title (`alttitle`), in French if your text is in English and in English otherwise.

The inverse transformation (SMF → AMS) can be done in a similar way.

If it is a standard class (`article` ou `book`), things are a bit more complicated. Be careful to type the abstracts *before* the `\maketitle`; some mathematical formulas might not work properly, but the AMS packages offer such a variety of uses, that it should not be very difficult to do.

10.2. From LATEX2.09. — In this case, you'll have to make the adjustments described in the previous paragraph, and also those needed by the LATEX2.09–LATEX 2$_\varepsilon$ mutation. A priori, it should mostly concern the font faces commands and the conforming to the *New Font Selection Scheme* (NFSS).

10.3. From PLAIN TEX. — In this case, you have to take up your manuscript again, and replace title, theorems, sectioning and bibliographical commands, by the adequate ones, refering to the LATEX 2$_\varepsilon$ user's guide and the recommendations above. We bring your attention to the automatic numbering of paragraphs and theoremlike environments: it might differ from the original one. Pay similar attention to your references. The macros PLAIN TEX uses to change the typefaces are most often ineffective in LATEX 2$_\varepsilon$, so you'll have to adapt them too. Concerning mathematics, few changes are needed, except for aligned equations and matrices.

Literature and sources

[1] L. LAMPORT. — *LaTeX: A Document Preparation System.* Second edition. Addison-Wesley, 1994.

[2] M. GOOSSENS, F. MITTELBACH, A. SAMARIN. — *The LaTeX Companion.* Addison-Wesley, 1993.

[3] M. GOOSSENS, S. RAHTZ AND F. MITTELBACH. — *The LaTeX Graphics Companion: Illustrating Documents With TeX and Postscript.* Tools and Techniques for Computer Typesetting Series, Addison-Wesley, 1996.

[4] *The Not So Short Introduction to LaTeX2e,* T. OETIKER, H. PARTL, I. HYNA, E. SCHLEGL, `http://www.loria.fr/tex/general/flshort2e.dvi`

[5] *AMS-LaTeX version 1.2 User's Guide,* `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/amslatex/amsldoc.dvi`

[6] *Babel, a multilingual package for use with LATEX's standard document classes,* J. BRAAMS, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/babel/babel.dvi`

[7] *The epsfig package,* S. RAHTZ, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/epsfig.dvi`

[8] *The graphics package,* D. CARLISLE, S. RAHTZ, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/graphics.dvi`

[9] *The graphicx package,* D. CARLISLE, S. RAHTZ, `http://www.loria.fr/tex/ctan-doc/macros/latex/packages/graphics/graphicx.dvi`

[10] *Hypatia's Guide to BibTeX,* `http://hypatia.dcs.qmw.ac.uk/html/bibliography.html`

[11] *Xy-pic User's Guide,* K. ROSE, R. MOORE, `http://www.loria.fr/tex/graph-pack/doc-xypic/xyguide.dvi`

The most recent versions of macros files and of their documentations are also available by anonymous `ftp` on the CTAN sites (*Comprehensive TeX Archive Network*) In the United States, one may use the address `ftp.shsu.edu`; the sites `ftp.loria.fr` or `ftp.jussieu.fr` in France, `ftp.tex.ac.uk` in England, and `ftp.dante.de` in Germany also hold the archive.

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