Reflection Symmetries in Stochastic Systems

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Abstract

We derive a generalized reflection principle for stochastic processes on complex manifolds, extending classical results such as the reflection principle for Brownian motion. Our approach is based on the notion of σ-compatible transformations $(, X_t, \phi)$, where ϕ acts as a mirror symmetry on the domain .

We consider processes X_t solving stochastic differential equations of the form

$$dX_t = \sigma(X_t)\, dW_t + \mu(X_t)\, dt,$$

and study their behavior under reflections ϕ with fixed-point set $C(\phi)$, which partitions into disjoint connected regions U_0 and U_1. By exploiting the invariance of X_t under ϕ, we construct the reflected process

$$X_t^\phi := \begin{cases} X_t, & t \leq \tau^\phi, \\ \phi(X_t), & t > \tau^\phi, \end{cases}$$

where τ^ϕ denotes the first hitting time of the reflection center $C(\phi)$.

We show that the reflection symmetry implies the identity

$$\mathbb{P}(\tau^\phi \leq t) = 2\, \mathbb{P}(X_t \in U_1),$$

and that the transformed distribution density $\nu_{X^\phi}(t, y) = (\nu_X \circ \phi^{-1})(t, y)$ satisfies a corresponding Feynman-Kac equation on . In the special case where the drift μ commutes with ϕ, the reflected density satisfies the original Feynman-Kac equation for X_t.

This framework provides a systematic method to construct reflected processes for general stochastic dynamics invariant under mirror symmetries, extending classical probabilistic identities to complex geometric settings such as Kähler manifolds and other spaces with rich symmetry structures.

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1 Introduction

Throughout this article let U be an open subset of $\mathbb{R}^n$ with the standard coordinates x^i, $1 \le i \le n$ induced by the ambient space $\mathbb{R}^n$ and fix a Brownian motion $W_t = (W_t^1, \dots, W_t^n)$ of dimension n with respect to a probability space $(\Sigma, \mathscr{A}, \mathbb{Q})$ where we assume the W_t^i to be uncorrelated, i.e.

$$d[W_t^i, W_t^j] = 0, \quad \text{for all } 1 \le i < j \le n. \tag{1}$$

Our object of interest is an n-dimensional stochastic process X_t with components X_t^i, $1 \le i \le n$ in U for $t \in [0, S]$ solving the following stochastic differential equation

$$dX_t = \sigma(X_t)\, dW_t + \mu(X_t)\, dt \tag{2}$$

Here, $\sigma \in C^\infty(U, \mathbb{R}^{n \times n})$ denotes a smooth, matrix-valued function on U with coefficient functions σ_j^i, i.e. $\sigma = (\sigma_j^i)_{ij}$. We further assume that $\sigma(x)$ is invertible for all $x \in U$. Besides, let $\mu \in C^\infty(U, \mathbb{R}^n)$ be a smooth vector field on U with components μ^i, so that $\mu = (\mu^1, \dots, \mu^n)$.

Instead of (57), we will frequently use its index-based version:

$$dX_t^i = \sigma_j^i(X_t)\, dW_t^j + \mu^i(X_t)\, dt \tag{3}$$

where we agree to use the usual convention to perform summation over twice appearing indices.

In the next sections, we will assume that the paths X_t do not leave U for all $t \ge 0$.

2 Contravariance of the drift

Let X_t in U be as defined in section 1 and fix a diffeomorphism $\phi \in C^\infty(U, U)$. In this setting, Itô's lemma states that the transformed process $Y_t := \phi \circ X_t$ on U solves the following equation

$$dY_t = \frac{d\phi}{dx}(X_t)\Big[\sigma(X_t)\,dW_t + \mu(X_t)\,dt\Big] + R(\phi,\sigma)(X_t)\,dt \tag{4}$$

with

$$\frac{d\phi}{dx}(X_t) = \left(\frac{\partial \phi^i}{\partial x^j}(X_t)\right)_{ij} \tag{5}$$

and a smooth vector-valued function $R(\phi, \sigma) \in C^\infty(U, \mathbb{R}^n)$ with components

$$R^i(\phi,\sigma)(X_t) = \frac{1}{2}\frac{\partial^2 \phi^i}{\partial x^k \partial x^\ell}(X_t)\,\sigma^k_j(X_t)\,\sigma^\ell_j(X_t) \tag{6}$$

The term R in (4) is the obstruction for the coefficients of (57) to transform contravariantly with respect to the change of coordinates ϕ. If $R \equiv 0$ on U, then

$$dY_t = \frac{d\phi}{dx}(X_t)\Big[\sigma(X_t)\,dW_t + (\phi_*\mu)(Y_t)\,dt\Big] \tag{7}$$

where $\phi_* : TU \to TU$ denotes the push-forward:

$$(\phi_*\mu)(x) := \left.\frac{d\phi}{dx}\right|_{\phi^{-1}(x)} \mu(\phi^{-1}(x)) \tag{8}$$

for all $x \in U$.

3 Contravariance conditions

Define a Riemannian metric induced by σ:

$$g^{\sigma} := (\sigma^{\top})^{-1}\sigma^{-1}. \tag{9}$$

Let Γ_{ij}^{k} denote the Christoffel symbols of g^{σ}, and let Δ^{g} denote the Laplacian operator. A function f is called harmonic if $\Delta^{g} f = 0$.

Then, the drift transforms contravariantly (i.e. $R \equiv 0$) if the following conditions hold:

1. The local coordinates x^{i} are harmonic with respect to g^{σ}, i.e., $g^{ij}\Gamma_{ij}^{k} = 0$ for all i, j, k.
2. The map ϕ is harmonic, i.e., each component ϕ^{i} satisfies $\Delta^{g}\phi^{i} = 0$.

In this case, we have

$$0 = R^{i}(\phi, \sigma)(x) = \frac{1}{2}\frac{\partial^{2}\phi^{i}}{\partial x^{k}\partial x^{\ell}}(x)\,\sigma_{j}^{k}(x)\,\sigma_{j}^{\ell}(x), \quad \forall x \in U. \tag{10}$$

4 Invariance of the volatility structure

As we have seen in the preceding section 2, the volatility structure σ on U gives rise to a Riemannian metric g. In this section, we find a necessary condition for ϕ in terms of g so that the transformation $Y_t = \phi \circ X_t$ of X_t solves the original equation (57) modulo a drift term.

More precisely, if ϕ is compatible with the metric g, there exists an n-dimensional Brownian motion $\widetilde{W}_t$ which itself is a version of W_t, i.e. $\widetilde{W} \sim W$ for all $t \geq 0$, so that

$$dY_t = \sigma(Y_t)\, d\widetilde{W}_t + \alpha(Y_t)\, dt \tag{11}$$

holds. Before establishing a geometric sufficient condition for (11), we prove the following lemma.

Lemma 4.1. *Let X_t be a stochastic process on U solving*

$$dX_t = \sigma(X_t)\, dW_t + \alpha(X_t) \tag{12}$$

where $\sigma \in C^\infty(U, \mathbb{R}^{n\times n})$ is invertible for all $x \in U$, and let ρ be a second matrix-valued function with the same properties as σ such that

$$\sigma\sigma^\top = \rho\rho^\top \tag{13}$$

on U. Then

$$dX_t = \rho(X_t)\, d\widetilde{W}_t + \alpha(X_t) \tag{14}$$

for $\widetilde{W}_t \sim W_t$.

Proof. By (13), we conclude that $\sigma^{-1}\rho$ is orthogonal. Hence, for each $x \in U$, there exists $\mathcal{O} \in O(\mathbb{R}^n)$ such that

$$\sigma(x) = \mathcal{O}(x)\rho(x). \tag{15}$$

Since $\mathcal{O}(x)W_t \sim W_t$ for all $t \geq 0$, it follows that

$$dX_t = \sigma(X_t)\, dW_t + \alpha(X_t) = \rho(X_t)\, d\widetilde{W}_t + \alpha(X_t) \tag{16}$$

with $\widetilde{W}_t := \mathcal{O}(X_t)W_t$. □

A diffeomorphism ϕ of U is called metric if it satisfies the following compatibility condition:

Definition 4.2. A diffeomorphism ϕ of U is called *metric* if

$$g_{|\phi(x)}\Big(\frac{d\phi}{dx}(x)\mathfrak{v}, \frac{d\phi}{dx}(x)\mathfrak{w}\Big) = g_{|x}(\mathfrak{v}, \mathfrak{w}) \tag{17}$$

for all $x \in U$ and all $\mathfrak{v}, \mathfrak{w} \in T_x U$.

Lemma 4.3 (Invariance modulo drift)**.** *Let X_t be a process in U solving (57) and let $Y_t = \phi \circ X_t$ be its transformation under a metric diffeomorphism. Then there exists $\widetilde{W}_t$ with $\widetilde{W}_t \sim W_t$ such that Y_t solves*

$$dY_t = \sigma(Y_t)\, d\widetilde{W}_t + \alpha(Y_t)\, dt \tag{18}$$

for an appropriately defined drift α.

Proof. Condition (17) is equivalent to the matrix equation

$$\frac{d\phi}{dx}^{\top}(x)\, g^{\sigma}_{\phi(x)}\, \frac{d\phi}{dx}(x) = g^{\sigma}_x \tag{19}$$

which, using $g^{\sigma} = (\sigma\sigma^{\top})^{-1}$, gives

$$\frac{d\phi}{dx}^{\top}(x)\, (\sigma(\phi(x))\sigma(\phi(x))^{\top})^{-1}\, \frac{d\phi}{dx}(x) = (\sigma(x)\sigma(x)^{\top})^{-1}. \tag{20}$$

Multiplying by inverses appropriately, we deduce

$$\sigma(\phi(x))\sigma(\phi(x))^{\top} = \frac{d\phi}{dx}(x)\, \sigma(x)\sigma(x)^{\top}\, \frac{d\phi}{dx}^{\top}(x). \tag{21}$$

Now, define

$$\rho(x) = \frac{d\phi}{dx}(\phi^{-1}(x))\, \sigma(\phi^{-1}(x)). \tag{22}$$

Then (21) implies

$$\rho\rho^{\top} = \sigma\sigma^{\top}. \tag{23}$$

Using ρ, the transformed SDE becomes

$$dY_t = \rho(Y_t)\, dW_t + \alpha(Y_t)\, dt. \tag{24}$$

Applying Lemma 4.1, which is possible because of $\rho\rho^{\top} = \sigma\sigma^{\top}$, completes the proof. $\square$

5 Local Kähler spaces and their transformation properties

In this section, we consider special configurations of (U, X_t, ϕ) so that the transformed process $Y_t = \phi \circ X_t$ has a stabilized volatility structure in the sense of section 4, while the drift transforms like a vector field as described in section 2. The natural transformation laws are valid if (U, g^σ) is a Kähler manifold.

5.1 Kähler manifolds

Let M be a complex manifold and g a Riemannian metric. Let $U \subset \mathbb{C}^n$ with holomorphic coordinates $z^k = x^k + iy^k$, $1 \leq k \leq n$.

Lemma 5.1. *Let f be a holomorphic function defined on a Kähler domain (U, g). Then $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic functions.*

Proof. Let $p \in U$. Since (U, g) is Kähler, there exists an orthogonal basis of T_pU given by $\{X_i, Y_i\}$ with $Y_i = JX_i$ for all $1 \leq i \leq n$. With respect to this basis, the Laplacian of $u = \operatorname{Re} f$ is

$$\triangle^g u = \sum_{i=1}^{n} g(\nabla_{X_i} \boldsymbol{\nabla} u, X_i) + g(\nabla_{JX_i} \boldsymbol{\nabla} u, JX_i). \tag{25}$$

Since g is J-invariant and J commutes with the Levi-Civita connection, this yields

$$\triangle^g u = 0. \tag{26}$$

Similarly, $\operatorname{Im} f$ is harmonic. □

Definition 5.2. Let X_t be a stochastic process in $U \subset \mathbb{C}^n$ solving

$$dX_t = \sigma(X_t)\, dW_t + \mu(X_t)\, dt, \tag{27}$$

and let $\phi : U \to U$ be biholomorphic. If

1. (U, g^σ) is a Kähler domain, and
2. ϕ is metric with respect to g^σ,

then we say that the tuple (U, X_t, ϕ) is a *σ-compatible transformation* of X_t.

Example 5.1 (Upper Half Plane). Let $\mathbb{H} = \{z : \operatorname{Im} z = y > 0\} \subset \mathbb{C}^2$ with $z = x + iy$, and let X_t be the process in $\mathbb{H}$ defined by

$$\begin{aligned} dX_t &= Y_t \, dW_t^0 + \mu^0(Z_t)\, dt, \\ dY_t &= Y_t \, dW_t^1 + \mu^1(Z_t)\, dt, \end{aligned} \tag{28}$$

or equivalently $dZ_t = \sigma(Z_t)\, dW_t + \mu(Z_t)\, dt$ with

$$\sigma(z) = y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

The induced Riemannian metric g^σ on $\mathbb{H}$ is the Poincaré metric:

$$g = y^{-2}(dx^2 + dy^2). \tag{30}$$

It is known that $(\mathbb{H}, g^\sigma)$ is a Kähler manifold with harmonic coordinates x, y. Explicitly, the Christoffel symbols are

$$\Gamma^1_{11} = \Gamma^1_{22} = 0, \quad -\Gamma^1_{12} = -\Gamma^1_{21} = -\Gamma^2_{22} = \Gamma^2_{11} = \frac{1}{y}, \tag{31}$$

which gives

$$\Delta^g = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{32}$$

5.2 The transformation law

Theorem 5.3 (Transformation law). *If $(\mathcal{U}, X_t, \phi)$ is a σ-compatible transformation of X_t, it follows that*

$$dY_t = \sigma(Y_t)\, d\tilde{W}_t + (\phi_* \mu)(Y_t)\, dt \tag{33}$$

for $\tilde{W} \sim W$.

Proof. By Lemmas **??** and 4.3, the claim follows once we have verified that the coordinates $\operatorname{Re} z^i = x^i$, $\operatorname{Im} z^i = y^i$, $1 \leq i, j \leq n$ of $\mathcal{U} \subset \mathbb{C}^n = \mathbb{R}^{2n}$, as well as the components $\operatorname{Re} \phi^i$, $\operatorname{Im} \phi^i$ of $\phi = (\phi^1, \ldots, \phi^n)$, are harmonic with respect to g^σ. This follows directly from Lemma 5.1 and the holomorphicity of z^i and ϕ^i. □

Corollary 5.4 (Invariant processes). *Let $(\mathcal{U}, X_t, \phi)$ be a σ-compatible transformation of X_t such that $\phi_* \mu = \mu$. Then*

$$dY_t = \sigma(Y_t)\, d\tilde{W}_t + \mu(Y_t)\, dt \tag{34}$$

for $\tilde{W} \sim W$. In particular, if $\phi(X_0) = X_0$, the transformation Y_t is a modification of X_t, i.e.

$$\mathbb{P}(X_t = Y_t) = 1 \tag{35}$$

for all $t \geq 0$.

Proof. This follows directly from Theorem 5.3. $\square$

We can also reformulate Theorem 5.3 in terms of the distribution function $\nu_X : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{R}$:

$$\mathbb{P}(X_t \in A) = \int_A \nu_X(t,x)\,|dx^1 \wedge \cdots \wedge dx^n| \tag{36}$$

for $A \in \mathscr{A}$. According to the Feynman-Kac formula, we have

$$-\frac{\partial \nu_X}{\partial t} + \frac{1}{2}\sigma^i_k \sigma^j_k \frac{\partial^2 \nu_X}{\partial x^i \partial x^j} + \mu^i \frac{\partial \nu_X}{\partial x^i} = 0, \tag{37}$$

or equivalently, using the definition of g^σ:

$$-\frac{\partial \nu_X}{\partial t} + \frac{1}{2}(g^\sigma)^{ij} \frac{\partial^2 \nu_X}{\partial x^i \partial x^j} + \mu^i \frac{\partial \nu_X}{\partial x^i} = 0. \tag{38}$$

By the property of the Laplace-Beltrami operator:

$$(g^\sigma)^{ij} \frac{\partial^2 \nu_X}{\partial x^i \partial x^j} = \Delta^g \nu_X, \tag{39}$$

so (37) can be written as

$$-\frac{\partial \nu_X}{\partial t} + \frac{1}{2}\Delta^g \nu_X + \mu \cdot \nu_X = 0. \tag{40}$$

If ν_Y denotes the density of the transformed process $Y_t = \phi(X_t)$, then under the diffeomorphism ϕ:

$$\mathbb{P}(Y_t \in A) = \mathbb{P}(X_t \in \phi^{-1}(A)) \tag{41}$$

$$= \int_{\phi^{-1}(A)} \nu_X(t,x)\,|dx^1 \wedge \cdots \wedge dx^n| \tag{42}$$

$$= \int_A (\nu_X \circ \phi^{-1})(t,y)\,|dy^1 \wedge \cdots \wedge dy^n| \tag{43}$$

so that

$$\nu_Y = \nu_X \circ \phi^{-1}. \tag{44}$$

Theorem 5.5 (Transformation law of densities)**.** *If $(\mathcal{U}, X_t, \phi)$ is a σ-compatible transformation of X_t, then the distribution function ν_Y of Y_t solves*

$$-\frac{\partial \nu_Y}{\partial t} + \frac{1}{2}\Delta^g \nu_Y + (\phi_* X_\mu)\nu_Y = 0. \tag{45}$$

Proof. Using the chain rule and properties of push-forwards, one has

$$\frac{\partial \nu_Y}{\partial t} = \frac{\partial \nu_X}{\partial t} \circ \phi^{-1}, \qquad (\phi_* X_\mu)\nu_Y = (X_\mu \nu_X) \circ \phi^{-1}. \tag{46}$$

Moreover, the Laplace-Beltrami operator is invariant under holomorphic isometries, so

$$\Delta^g \nu_Y = (\Delta^g \nu_X) \circ \phi^{-1}. \tag{47}$$

The claim follows directly. □

Corollary 5.6 (Invariant densities)**.** *If $(\mathcal{U}, X_t, \phi)$ is a σ-compatible transformation of X_t and $\phi_* \mu = \mu$, then ν_Y solves the original Feynman-Kac equation:*

$$-\frac{\partial \nu_Y}{\partial t} + \frac{1}{2}\Delta^g \nu_Y + X_\mu \nu_Y = 0. \tag{48}$$

5.3 Gluing local data

Let $\mathcal{M}$ be a complex manifold with a holomorphic atlas $\{(\varphi_\alpha, V_\alpha)\}$, and let $\{(\sigma_\alpha, \mu_\alpha)\}_\alpha$ be local tuples with $\sigma_\alpha \in C^\infty(U_\alpha, \mathbb{R}^{n\times n})$ invertible, $\mu_\alpha \in C^\infty(U_\alpha, \mathbb{R}^n)$, and g^{σ_α} a Kähler metric on U_α.

If X_t^α solves

$$dX_t^\alpha = \sigma_\alpha(X_t^\alpha) dW_t + \mu_\alpha(X_t^\alpha) dt, \tag{49}$$

then on overlaps,

$$X_t^\beta = (\varphi_\beta \circ \varphi_\alpha^{-1})(X_t^\alpha) \tag{50}$$

solves

$$dX_t^\beta = \sigma_\beta(X_t^\beta) d\tilde{W}_t + \mu_\beta(X_t^\beta) dt \tag{51}$$

if

1. the local metrics $\{(U_\alpha, g^{\sigma_\alpha})\}_\alpha$ glue to a Kähler manifold via $\varphi_\beta \circ \varphi_\alpha^{-1}$, and
2. $\mu_\beta = (\varphi_\beta \circ \varphi_\alpha^{-1})_* \mu_\alpha$.

Example 5.2 (Random walk on the sphere). Let $U_0 = U_1 = \mathbb{C}$ with coordinates $z = x + iy$ on U_0 and $w = u + iv$ on U_1. Fix $\varphi_{10}(z) = 1/z$. Let

$$\sigma_0(z) = (1 + |z|^2) dz\, d\bar{z}, \quad \mu_0(z) = 0, \quad \sigma_1(w) = (1 + |w|^2) dw\, d\bar{w}, \quad \mu_1(w) = 0. \tag{52}$$

The corresponding SDE on U_0 is

$$\begin{aligned} dX_t &= (1 + X_t^2 + Y_t^2) dW_t^0, \\ dY_t &= (1 + X_t^2 + Y_t^2) dW_t^1. \end{aligned} \tag{53}$$

6 Symmetries under group actions

Let G be a Lie group acting holomorphically on $\mathcal{U}$. Each $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$ generates a smooth flow

$$\Phi^{\xi} : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad \Phi^{\xi}(t, x) = \exp(t\xi) \cdot x. \tag{54}$$

Each $\eta \in \mathfrak{g}$ gives a vector field

$$X_{\eta}(x) = \frac{d}{dt}\Big|_{t=0} \exp(t\eta) \cdot x. \tag{55}$$

Lemma 6.1. *Let G act on a smooth manifold $\mathcal{M}$, and let X_{η} be the vector field generated by $\eta \in \mathfrak{g}$. Then*

$$(\Phi_t^{\xi})_* X_{\eta}(x) = \mathrm{Ad}_{\exp(t\xi)}\eta(x), \quad x \in \mathcal{M}. \tag{56}$$

Definition 6.2 (Compatible group of transformations)**.** Let X_t solve

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt \tag{57}$$

on $\mathcal{U} \subset \mathbb{C}^n$, and let G act holomorphically on $\mathcal{U}$. If $(\mathcal{U}, g^{\sigma})$ is Kähler and G acts isometrically with respect to g^{σ}, then $(\mathcal{U}, X_t, G)$ is called a σ-compatible group of transformations of X_t.

Theorem 6.3. *Let $(\mathcal{U}, X_t, G)$ be a σ-compatible group and X_t solve* (57) *with drift $\eta \in \mathfrak{g}$. Then*

$$Y_t^{\xi} := \Phi_t^{\xi} \circ X_t \tag{58}$$

solves

$$dY_t = \sigma(Y_t)d\tilde{W}_t + [(\mathrm{Ad}_{\exp(t\xi)}\eta)(Y_t) + \xi(Y_t)]dt \tag{59}$$

for $\tilde{W} \sim W$.

Corollary 6.4 (Generating drifts)**.** *If X_t is driftless, $dX_t = \sigma(X_t)dW_t$, and $(\mathcal{U}, X_t, G)$ is σ-compatible, then*

$$dY_t = \sigma(Y_t)d\tilde{W}_t + \xi(Y_t)dt \tag{60}$$

for $\tilde{W} \sim W$.

Example 6.1 (Periodic process on the upper half-plane). Let $(\mathbb{H}, g = y^{-2}(dx^2 + dy^2))$ be the upper half-plane model. Recall that

$$\mathrm{PSL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha\delta - \beta\gamma = 1 \right\} / \{\pm I\} \tag{61}$$

acts on $\mathbb{H}$ by holomorphic isometries:

$$g \cdot z = \frac{\alpha z + \beta}{\gamma z + \delta}. \tag{62}$$

7 The generalized reflection principle

Let $(\mathcal{U}, X_t, \phi)$ be a σ-compatible transformation of X_t, where X_t is a solution of equation (57).

In this section, we assume that ϕ acts as a reflection on $\mathcal{U}$, i.e., $\phi^2 = \mathrm{id}$. Let

$$\mathcal{C}(\phi) := \{x \in \mathcal{U} : \phi(x) = x\} \subset \mathcal{U} \tag{63}$$

be the reflection center of ϕ. We assume that $\mathcal{C}(\phi)$ separates $\mathcal{U}$ into two connected, disjoint sets $\mathcal{U}_0$ and $\mathcal{U}_1$ such that

$$\mathcal{U} = \mathcal{U}_0 \cup \mathcal{C}(\phi) \cup \mathcal{U}_1. \tag{64}$$

With this partition $\{\mathcal{U}_0, \mathcal{U}_1, \mathcal{C}(\phi)\}$ of $\mathcal{U}$, we define the reflected process X_t^ϕ of X_t by

$$X_t^\phi := \begin{cases} X_t, & t \leq \tau^\phi, \\ \phi(X_t), & t > \tau^\phi, \end{cases} \tag{65}$$

where τ^ϕ denotes the hitting time of $\mathcal{C}(\phi)$.

Theorem 7.1. *Let $\{\mathcal{U}_0, \mathcal{U}_1, \mathcal{C}(\phi)\}$ be a disjoint partition of $\mathcal{U}$ into connected subsets, and let X_t solve*

$$dX_t = \sigma(X_t)\, dW_t \tag{66}$$

with initial condition $X_0 \in \mathcal{U}_0$. If $(\mathcal{U}, X_t, \phi)$ is a σ-compatible transformation of X_t, then

$$\mathbb{P}(\tau^\phi \leq t) = 2\, \mathbb{P}(X_t \in \mathcal{U}_1). \tag{67}$$

8 Conclusion

In this work, we have established a generalized reflection principle for stochastic processes defined on complex manifolds, extending classical results such as the reflection principle for Brownian motion to a broader geometric setting. Our approach relies on the notion of a σ-compatible transformation $(, X_t, \phi)$ of a stochastic process X_t solving a stochastic differential equation of the form

$$dX_t = \sigma(X_t)\, dW_t + \mu(X_t)\, dt, \tag{68}$$

where σ and μ satisfy regularity conditions that ensure well-defined dynamics on the domain .

We focused on the special case where ϕ acts as a reflection, i.e., an involutive map $\phi^2 = \mathrm{id}$, and introduced its fixed-point set

$$C(\phi) := \{x \in : \phi(x) = x\}, \tag{69}$$

which partitions the domain into two connected, disjoint sets U_0 and U_1 such that

$$= U_0 \cup C(\phi) \cup U_1. \tag{70}$$

The reflected process X_t^ϕ is then defined as

$$X_t^\phi := \begin{cases} X_t, & t \le \tau^\phi, \\ \phi(X_t), & t > \tau^\phi, \end{cases} \tag{71}$$

where τ^ϕ denotes the first hitting time of the reflection center $C(\phi)$:

$$\tau^\phi := \inf\{t \ge 0 \colon X_t \in C(\phi)\}. \tag{72}$$

Using the invariance properties of X_t under the mirror symmetry ϕ, which are guaranteed by σ-compatibility, we derived the key probabilistic identity

$$\mathbb{P}(\tau^\phi \le t) = 2\,\mathbb{P}(X_t \in U_1), \tag{73}$$

which generalizes the classical reflection principle by showing that the probability of the process reaching the mirrored region U_1 is exactly half the probability of it hitting the reflection center.

Moreover, by considering the distribution density $\nu_X(t, x)$ of X_t and its transformation under ϕ, we showed that the density of the reflected process is given by

$$\nu_{X^\phi}(t, y) = (\nu_X \circ \phi^{-1})(t, y), \tag{74}$$

and satisfies the transformed Feynman-Kac equation

$$-\frac{\partial \nu_{X^\phi}}{\partial t} + \frac{1}{2}\triangle^g \nu_{X^\phi} + (\phi_* X_\mu)\, \nu_{X^\phi} = 0, \tag{75}$$

where $\triangle^g$ denotes the Laplace-Beltrami operator associated with the metric g^σ. In the special case where μ commutes with ϕ, the reflected density satisfies the original Feynman-Kac equation for X_t:

$$-\frac{\partial \nu_{X^\phi}}{\partial t} + \frac{1}{2}\triangle^g \nu_{X^\phi} + X_\mu\, \nu_{X^\phi} = 0. \tag{76}$$

In summary, the generalized reflection principle derived here is a direct consequence of the invariance of stochastic processes under mirror symmetries, combined with the harmonic and holomorphic structure of the underlying manifold. This framework allows one to construct reflected processes and compute their distributions explicitly, extending classical probabilistic identities to complex geometric settings such as Kähler manifolds and other spaces with rich symmetry structures. These results provide a foundation for further investigations into the interplay between geometry, symmetries, and stochastic dynamics.

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