Confounding-Robust Policy Evaluation
in Infinite-Horizon Reinforcement Learning

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Abstract

Off-policy evaluation of sequential decision policies from observational data is necessary in applications of batch reinforcement learning such as education and healthcare. In such settings, however, unobserved variables confound observed actions, rendering exact evaluation of new policies impossible, *i.e.*, unidentifiable. We develop a robust approach that estimates sharp bounds on the (unidentifiable) value of a given policy in an infinite-horizon problem given data from another policy with unobserved confounding, subject to a sensitivity model. We consider stationary or baseline unobserved confounding and compute bounds by optimizing over the set of all stationary state-occupancy ratios that agree with a new partially identified estimating equation and the sensitivity model. We prove convergence to the sharp bounds as we collect more confounded data. Although checking set membership is a linear program, the support function is given by a difficult nonconvex optimization problem. We develop approximations based on nonconvex projected gradient descent and demonstrate the resulting bounds empirically.

1 Introduction

Evaluation of sequential decision-making policies under uncertainty is a fundamental problem for learning sequential decision policies from observational data, as is necessarily the case in application areas such as education and healthcare Jiang & Li [11], Precup et al. [28], Thomas & Brunskill [36]. However, with a few exceptions, the literature on off-policy evaluation in reinforcement learning (RL) assumes (implicitly or otherwise) the *absence* of unobserved confounders, auxiliary state information that affects both the policy that generated the original data as well as transitions to the next state. Precisely in the same important domains where observational off-policy evaluation is *necessary* due to the cost of or ethical constraints on experimentation, such as in healthcare [27, 30] or operations, it is *also* generally the case that *unobserved confounders* are present. This contributes to fundamental challenges for advancing reinforcement learning in observational settings [7].

In this work, we study partial identification in RL off-policy evaluation under unobserved confounding, focusing specifically on the *infinite-horizon* setting. Recognizing that policy value cannot actually be point-identified from confounded observational data, we propose instead to compute the sharpest bounds on policy value that can be supported by the data

and any assumptions on confounding. This can then support credible conclusions about policy value from the data and can ensure safety in downstream policy learning.

Recent advancements [6, 8, 13, 17] improve variance reduction of *unconfounded* off-policy evaluation by estimating density ratios on the *stationary* occupancy distribution. But this assumes unconfounded data. Other advances [14] tackle partial identification of policy values from confounded data but in the logged bandit setting (single decision point) rather than the RL setting (many or infinite decision points). Our work can be framed as appropriately combining these perspectives.

Our contributions are as follows: we establish a partially identified estimating equation that allows for the estimation of sharp bounds. We provide tractable approximations of the resulting difficult non-convex program based on non-convex first order methods. We then demonstrate the approach on a gridworld task with unobserved confounding.

2   Problem setup

We assume data is generated from an infinite-horizon MDP with an augmented state space: $\mathcal{S}$ is the space of the *observed* portion of the state and $\mathcal{U}$ is the space of the *unobserved* (confounding) portion of the state. We assume the standard decision protocol for MDPs on the full-information state space $\mathcal{S} \times \mathcal{U}$: at each decision epoch, the system occupies state s_t, u_t, the decision-maker receives a reward $\Phi(s_t)$ for being in state s_t and chooses an action, a_t, from allowable actions. Then the system transitions to the next state on $\mathcal{S} \times \mathcal{U}$, with the (unknown) transition probability $p(s', u' \mid s, u, a)$. The full-information MDP is represented by the tuple $M = (\mathcal{S} \times \mathcal{U}, \mathcal{A}, P, \Phi)$. We let $\mathcal{H}_t = \{(s_0, u_0, a_0), \ldots, (s_t, u_t, a_t)\}$ denote the (inaccessible) full-information history up to time t. A policy $\pi(a \mid s, u)$ is an assignment to the probability of taking action a in state (s, u). For any policy, the underlying dynamics are Markovian under full observation of states and transitions: $s_t \perp\!\!\!\perp \mathcal{H}_{t-2} \mid (s_{t-1}, u_{t-1}), a_{t-1}$.

In the off-policy evaluation setting, we consider the case where the observational data are generated under an unknown *behavior policy* π_b, while we are interested in evaluating the (known) *evaluation policy* π_e, which only depends on the observed state, $\pi_e(a \mid s, u) = \pi_e(a \mid s)$. Both policies are assumed stationary (time invariant). The observational dataset does not have full information and comprises solely of observed states and actions, that is, $(s_0, a_0), \ldots, (s_t, a_t)$.[1] Thus, since the action also depends on the unobserved state u_t, we have that transition to next states are *confounded* by u_t.

Notationally, we reserve s, u (respectively, s', u') for the random variables representing state (respectively, next state) and we refer to realized observed state values (respectively, next observed state values) using j (respectively, k). We assume that $\mathcal{S}$ is a discrete state space, while $\mathcal{U}$ may be general.

We next discuss regularity conditions on the MDP structure which ensure ergodicity and that the limiting state-action occupancy frequencies exist. We assume that the Markov chain induced by π_e and any π_b is a positive Harris chain, so the stationary distribution exists.

Assumption 1 (Ergodic MDP)**.** The MDP M is ergodic: the Markov chains induced by π_b and π_e is positive Harris recurrent.

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[1] Our model differs from typical POMDPs [12], since rewards are a function of observed state, as we clarify in the related work, Section 8.

In this work, we focus on the infinite-horizon setting. Let $p_\pi^{(t)}(s)$ be the distribution of state s_t when executing policy π, starting from initial state s_0 drawn from an initial distribution over states. Then the average *state-action-next-state* visitation distribution exists, and under Assumption 1 the (long-run average) *value* of a stationary policy π_e is given by an expectation with respect to the marginalized state visitation distribution:

$$p_{\pi_e}^\infty(s, u, a, s', u') = \lim_{T \to \infty} \tfrac{1}{T} \sum_{t=0}^T p_{\pi_e}^{(t)}(s, u, a, s', u'), \qquad R_e = \mathbb{E}_{s \sim p_{\pi_e}^\infty}[\Phi(s)].$$

We similarly define the marginalized total-, unobserved-, and observed-state occupancy distributions as $p_\pi^\infty(s, u)$, $p_\pi^\infty(u)$, and $p_\pi^\infty(s)$, given by appropriately marginalizing the above. Notice we assumed that the reward only depends on the *observed* state[2].

Notationally, $\mathbb{E}$ denotes taking expectations over the joint stationary occupancy distribution of the behavior policy, where self-evident. We denote p_e^∞, p_b^∞ for visitation distributions induced under π_e, π_b. Since at times it is useful to distinguish between expectation over the marginalized occupancy distribution $p_b^\infty(s, a, s')$, and total expectation over full-information transitions $p_b^\infty(s, u, a, s', u')$, we include additional subscripts on the expectation whenever this is clarifying.

If we were able to actually run the MDP using the policy π_e, which is only a function of s, the dynamics would be Markovian with the marginalized transition probabilities:

$$p(k \mid j, a) := \sum_{u', u''} p(k, u'' \mid j, u', a) p_e^\infty(u' \mid j)$$

Note that $p(k \mid j, a)$ is *not* identifiable from the observational data collected under π_b. We analogously define (partially) marginalized transition probabilities $p(k \mid j, u', a)$.

3 Off-policy evaluation under unobserved confounding

In the following, we first discuss a *population* viewpoint, computing expectations with respect to the true marginalized stationary occupancy distribution $p_b^\infty(s, a, s')$ (identifiable from the data). We discuss finite-sample considerations in Section 6.

Given sample trajectories generated from π_b, the goal of off-policy evaluation is to estimate R_e, the value of a known (observed-state-dependent) evaluation policy $\pi_e(a \mid s)$. The full-information stationary density ratio is $w(s, u)$. If $w(s, u)$ *were* known, we could use it to estimate policy value under π_e using samples from π_b by a simple density ratio argument:

$$w(s, u) = \frac{p_e^{(\infty)}(s, u)}{p_b^{(\infty)}(s, u)}, \qquad R_e = \mathbb{E}[w(s, u)\Phi(s)]$$

From observational data, we are only able to estimate the *marginalized* behavior policy, $\pi_b(a \mid s) = \mathbb{E}[\pi_b(a \mid s, u) \mid s]$, which is insufficient for identifying the policy value or the true marginalized transition probabilities.

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[2]This does not preclude, however, dependence on action: if we are given observed-state-action reward function $\Phi'(s, a)$, we may simply define $\Phi(s) = \sum_a \pi_e(a \mid s)\Phi'(s, a)$, since $\pi_e(a \mid s)$ is assumed known. Then R_e gives π_e's value with respect to the given observed-state-action reward function.
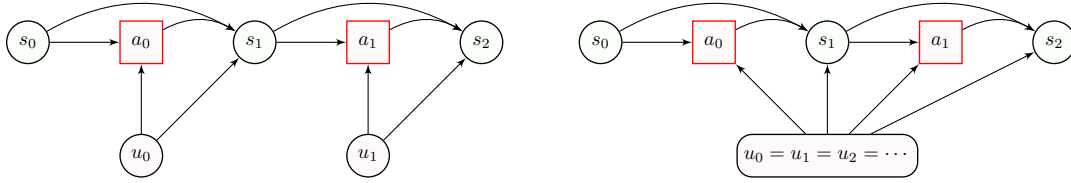
Figure 1: Two causal models satisfying Assumption 2.

3.1 Model restrictions on unobserved confounding

To make progress, we introduce restrictions on the underlying dynamics of the unobserved confounder, u, under which we will conduct our evaluation. In particular, we will seek to compute the range of all values of R_e that match our data, encapsulated in $p_b^\infty(s, a, s')$, and the following structural assumptions.

Assumption 2 (Memoryless Unobserved Confounding).

$$\frac{p_e^{(\infty)}(s,u)}{p_b^{(\infty)}(s,u)} = \frac{p_e^{(\infty)}(s,u')}{p_b^{(\infty)}(s,u')} \qquad \forall s \in \mathcal{S}, u, u' \in \mathcal{U}$$

Lemma 1. *Assumption 2 holds if the MDP transitions satisfy*

$$p(j, u' \mid k, u, a) = p(j, \tilde{u}' \mid k, u, a), \ \forall u', \tilde{u}' \in \mathcal{U}.$$

Assumption 2 essentially requires no time-varying confounders, *i.e.*, confounders that are influenced by past actions. This assumption may appear strong but it is necessary: if confounders could be time-varying and the dependence on them may be arbitrary, we may need to be "exponentially conservative" in accounting for them (or even "infinitely conservative" in the infinite-horizon case)[3].

For example, Assumption 2 holds if there is just some baseline unobserved confounder for each individual, or if the unobserved confounder is exogenously drawn at each timestep (which satisfies the sufficient condition of Lemma 1). These examples are shown in Figure 1. A combination of a baseline unobserved confounder as well as exogenous confounders at each timestep is also allowed. Such MDPs would satisfy Assumption 2 for *any* two policies. In healthcare, baseline confounders such as socio-economic status or risk/toxicity preferences, confound actions by affecting access to doctors and treatments, or by affecting choices between intensive or conservative treatments. For example, [40] identifies psychosocial factors and underlying comorbidities as baseline (unobserved) confounders in assessing sequential HIV treatment. Lemma 1 also holds if a confounded behavior policy arises from agents optimizing upon exogenous realizations of private information, as in econometric discrete choice models.

Under Assumption 2, we simply define $w(s) = w(s, u)$ as it does not depend on u. Note that $w(s)$ remains unidentifiable even under Assumption 2.

3.2 Sensitivity model

Next, we introduce a sensitivity model to control the level of assumed dependence of the behavior policy on the unobserved confounders. Sensitivity analysis allows a practitioner to

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[3]We discuss this in Appendix A.1 of the appendix in considering a finite-horizon off-policy estimate.

assess how conclusions might change for reasonable ranges of Γ. Following [2, 14] we phrase this as lower and upper bounds on the (unknown) inverse behavior policy, $l(a \mid s), m(a \mid s)$:[4]

$$\beta(a \mid s, u) := \pi_b(a \mid s, u)^{-1}, \qquad l(a \mid s) \leq \beta(a \mid s, u) \leq m(a \mid s) \quad \forall a, s, u. \tag{1}$$

The set $\mathcal{B}$ consists of all functions $\beta(a \mid s, u)$ that satisfy Eq. (1). This ambiguity set is motivated by a sensitivity model used in causal inference, which restricts how far propensities can vary pointwise from the nominal propensities [34] and which has also been used in the logged bandit setting [14]. Given a sensitivity parameter $\Gamma \geq 1$ that controls the amounts of allowed confounding, the marginal sensitivity model posits the following odds-ratio restriction:

$$\Gamma^{-1} \leq \frac{(1-\pi_b(a|s))\pi_b(a|s,u)}{\pi_b(a|s)(1-\pi_b(a|s,u))} \leq \Gamma, \quad \forall a, s, u. \tag{2}$$

Eq. (2) is equivalent to saying that Eq. (1) holds with

$$l(a \mid s) = \Gamma/(\pi_b(a \mid s)) + 1 - \Gamma, \qquad m(a \mid s) = 1/(\Gamma\pi_b(a \mid s)) + 1 - 1/\Gamma.$$

Lastly, β functions which are themselves valid inverse probability distributions must satisfy the next-state conditional constraints:

$$\mathbb{E}_{s,u,a,s'\sim p_b^\infty}\left[\frac{\mathbb{I}[a=a']}{\pi_b(a'|s,u)} \mid s' = k\right] = p_b^\infty(k \mid a') \quad \forall k, a' \tag{3}$$

We let $\tilde{\mathcal{B}}$ denote the set of all functions $\beta(a \mid s, u)$ that satisfy *both* Eqs. (1) and (3).

3.3 The partially identified set

Given the above restrictions, we can define the set of partially identified evaluation policy values. To do so, suppose we are given a target behavior policy π_e, the observed stationary distribution $p_b^\infty(s, a, s')$, and bounds $l(a \mid s), m(a \mid s)$ on β. We are then concerned with what w could be, given the degrees of freedom that remain. So, we define the following set for what values w can take:[5]

$$\Theta = \left\{ \frac{p_e^{(\infty)}(s, u)}{p_b^{(\infty)}(s, u)} : \begin{array}{l} M \text{ is an MDP,} \\ M \text{ satisfies Assumptions 1 and 2 with respect to } \pi_b \text{ and} \\ \text{the given } \pi_e, \\ \pi_b(a \mid s, u) \text{ is a stationary policy with } \beta \in \mathcal{B} \text{ and} \\ p_b^\infty(s, a, s') \text{ as given} \end{array} \right\}$$

We are then interested in determining the largest and smallest that R_e can be. That is, we are interested in

$$\underline{R}_e = \inf_{w \in \Theta} \mathbb{E}[w(s)\Phi(s)], \quad \overline{R}_e = \sup_{w \in \Theta} \mathbb{E}[w(s)\Phi(s)]. \tag{4}$$

Notice that this is equivalent to computing the *support function* of Θ at $-\Phi$ and Φ with respect to the L_2 inner product defined by $p_b^\infty(s)$, $\langle f, g \rangle = \mathbb{E}[f(s)g(s)] = \sum_j p_b^\infty(j)f(j)g(j)$. The support function of a set $\mathcal{S}$ is $\psi(v) = \sup_{s \in \mathcal{S}} \langle v, s \rangle$ [32].

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[4]Our approach immediately generalizes to any linearly representable ambiguity set.

[5]Note that, as defined, Θ is a set of functions of (s, u) but because we enforce Assumption 2, all members are constant with respect to u for each s; we therefore often implicitly refer to it as a set of functions of s alone.

4 Characterizing the partially identified set

In this section we derive a linear program to check membership in Θ for any given w.

The partially identified estimating equation We begin by showing that w is uniquely characterized by an estimating equation characterizing its stationarity, but where some parts of the equation are not actually known.

Lemma 2. *Suppose Assumptions 1 and 2 hold. Then* $w(s) = \frac{p_e^{(\infty)}(s,u)}{p_b^{(\infty)}(s,u)}$ *$\forall s, u$ if and only if*

$$\mathbb{E}[\pi_e(a \mid s)w(s)\beta(a \mid s, u) \mid s' = k] = w(k) \quad \forall k, \tag{5}$$

$$\mathbb{E}[w(s)] = 1. \tag{6}$$

The forward implication of Lemma 2 follows from Theorem 1 of [17] applied to the state variable (s, u) after recognizing that $w(s, u)$ only depends on s under Assumption 2 and marginalizing out u'. The backward implications of Lemma 2 follows from the recurrence of the aggregated MDP obtained from the transform $(s, u) \mapsto s$ [16]. A complete proof appears in the appendix.

Fortunately, Eqs. (5) and (6) exactly characterize w. Unfortunately, Eq. (5) involves two unknowns: $\beta(a \mid s, u)$ and the distribution $p_b^\infty(s, u, a, s')$ with respect to which the expectation is taken. In that sense, the estimated equation is only partially identified. Nonetheless, this allows to make progress toward a tractable characterization of Θ.

Marginalization We next show that when optimizing over $\beta \in \mathcal{B}$, the sensitivity model can be reparametrized with respect to *marginal weights* $g_k(a \mid j)$ (in the following, j, a, k are generic indices into $\mathcal{S}, \mathcal{A}, \mathcal{S}$, respectively):

$$g_k(a \mid j) := \sum_{u'} \frac{p_b^{(\infty)}(j, u', a \mid k)}{p_b^\infty(j, a \mid k)}\beta(a \mid j, u') = \left(\sum_{u'} \pi_b(a \mid j, u')\frac{p_b^\infty(u' \mid j)p(k \mid j, u', a)}{p(k \mid j, a)}\right)^{-1}$$

Note that the values of the $g_k(a \mid j)$ weights are *not* equivalent to the confounded $\pi_b(a \mid s)^{-1}$: the difference is exactly the *variability* in the underlying full-information transition probabilities $p(k \mid j, u', a)$. We will show that $g_k(a \mid j) \in \tilde{\mathcal{B}}$ satisfies the following constraints, where Eq. (7) corresponds to Eq. (3):

$$l(a \mid j) \le g_k(a \mid j) \le m(a \mid j), \qquad \forall j, a, k$$

$$p_b^\infty(k \mid a) = \sum_j p_b^\infty(j, a, k)g_k(a \mid j), \qquad \forall k, a \tag{7}$$

Reparametrization with respect to $g_k(a \mid j)$ follows from an optimization argument, recognizing the symmetry of optimizing a function of unknown realizations of u with respect to an unknown conditional visitation density. Crucially, reparametrization improves the scaling of the number of nonconvex bilinear variables from the number of samples or trajectories, $O(NT)$, to $O(|\mathcal{S}|^2|\mathcal{A}|)$.

Unlike sensitivity models in causal inference, it is possible that the partial identification set is empty, $\Theta = \emptyset$, even if its associated sensitivity model $\tilde{\mathcal{B}}$ is nonempty in the space of weights.

In Appendix B.1 of the appendix, we explain this further by studying the state-action-state polytope [20], the set of all valid visitation distributions achievable by some policy. The next result summarizes that imposing constraints on the *marginalized* joint distribution $p_b^\infty(s, a, s')$ is insufficient to ensure the full (unobserved) joint distribution corresponds to a valid MDP stationary distribution.

Proposition 1. *The implementable implications of the marginalized state-action and marginalized state-action-state polytopes are:* $p_b^\infty(k \mid a) = \sum_j p_b^\infty(j, a, k) g_k(a \mid j), \forall k \in \mathcal{S}, a \in \mathcal{A}.$

Proposition 1 justifies our restrictions on $\tilde{\mathcal{B}}$. However, it also implies that further imposing constraints on $\tilde{\mathcal{B}}$ *cannot* ensure compatibility of $g_k(a \mid j)$ for the observed $p_b^\infty(s, a, s')$ (and therefore Θ), where compatibility is the requirement that $p_b^\infty(s, a, s')$ is stationary for $g_k(a \mid j)$.

Feasibility Linear Program We next show that $w \in \Theta$ can be expressed using the linear program $F(w)$ that minimizes the L1 norm of residuals of the estimating equation of Lemma 2, for a given w, over the sensitivity model $g \in \tilde{\mathcal{B}}^6$:

$$F(w) := \min_{g \in \tilde{\mathcal{B}}} \sum_k \left| \sum_{j,a} p_b^{(\infty)}(j, a \mid k) w(j) \pi_e(a \mid j) g_k(a \mid j) - w(k) \right|. \tag{8}$$

Proposition 2 (Feasibility Linear Program).

$$w \in \Theta \iff F(w) \leq 0, \ \mathbb{E}[w(s)] = 1 \tag{9}$$

In relation to Proposition 1, the analysis for Eq. (9) shows that it is exactly the partially identified estimating equation, Lemma 2, which enforces compatibility such that combining the restrictions on $\tilde{\mathcal{B}}$ and Lemma 2 verifies membership of w in Θ. A consequence of Proposition 2 is sharpness of the partially identified interval $[\underline{R}_e, \overline{R}_e]$; each point in the interval corresponds to some policy value.

Theorem 1 (Sharpness). $\{\mathbb{E}[w(s)\Phi(s)] : w \in \Theta\} = [\underline{R}_e, \overline{R}_e].$

5 Optimizing over the partially identified set

Eq. (9) suggests computing $\underline{R}_e, \overline{R}_e$ by solving

$$\inf/\sup \ \{\mathbb{E}[w(s)\Phi(s)] \colon F(w) \leq 0, \ \mathbb{E}[w(s)] = 1\}. \tag{10}$$

The restriction $F(w) = 0$ implicitly encodes an optimization over g, resulting in a hard nonconvex bilevel optimization problem. We first show that the structure of the feasibility program admits reformulation as a disjunctive program.

Proposition 3. *Eq. (10) can be reformulated as a disjunctive program (hence a finite linear program).*

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[6]Linearity of $F(w)$ also holds if an instrument function is used to convert the conditional moment equality to an unconditional one, as in Eqn. 10 [17], and as we use in Section 5 and Proposition 4.

(a) Algorithm 1: Nonconvex nonconvex-projected gradient descent

> **Input:** step size η_0, exponent $\kappa \in (0, 1]$, initial iterate g_0, number of iterations N
> **for** $k = 0, \ldots, N - 1$ **do**
> $\quad \eta_k \leftarrow \eta_0 t^{-k}$
> $\quad w^*_{g_k} \in \underset{\mathbb{E}[w(s)]=1}{\arg\min} \{\|A(g_k)w\|_1\}$
> $\quad \tilde{g}_k \in \underset{g \in \tilde{\mathcal{B}}}{\arg\min} \{\|g - g_k\|_1 : A(g)w^*_{g_k} = 0\}$
> $\quad g_{k+1} \leftarrow \text{Proj}_{\mathcal{B}}(g_k + \eta_t \nabla_{\tilde{g}_k}(\varphi^T \tilde{A}(g)^{-1}v))$
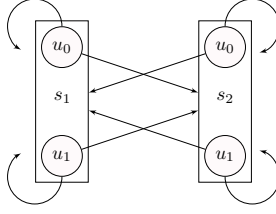> **end for** $\quad$ Return g_k with the best loss.
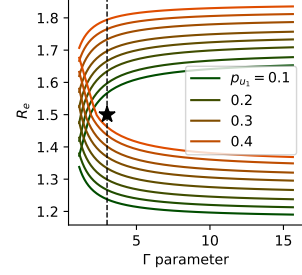

Figure 2: Confounded random walk.

Figure 3: Varying transition models on (s, u). $\star$ is true policy value.

However, the size of the resulting program is super-exponential in the cardinality of the state space, hence not practical. In Appendix E we also discuss the computational intractability of lifted SDP relaxations, which incur $O(|\mathcal{S}|^9)$ iteration complexity. Therefore, for small state spaces, we suggest to solve Eq. (10) directly via Gurobi.

Nonconvex nonconvex-projected gradient method We next develop a more practical optimization approach based on non-convex first-order methods. First we restate the estimating equation Eq. (5) for a fixed g as a matrix system. To evaluate expectations on the unconditional joint distribution, we introduce instrument functions $\phi_s, \phi_{s'} \in \mathbb{R}^{|\mathcal{S}| \times 1}$, random (row) vectors which are one-hot indicators for the state random variable s, s' taking on each value, $\phi_s = \begin{bmatrix} \mathbb{I}[s = 0] & \ldots & \mathbb{I}[s = |\mathcal{S}|] \end{bmatrix}$. Let $A(g) = \mathbb{E}[\phi_{s'}(\pi_e(a \mid s)g_{s'}(a \mid s)\phi_s - \phi_{s'})^{\top}]$ and $b_s = p_b^{\infty}(s)$. Let ψ be the set of $g \in \tilde{\mathcal{B}}$ that admit a feasible solution to the estimating equation for some $w \in \Theta$:

$$\psi := \{g \in \tilde{\mathcal{B}} : \exists\, w \geq 0 \text{ s.t. } A(g)w = 0, b^{\top}w = 1\} \tag{11}$$

Define $\tilde{A}(g)$ by replacing the last row of $A(g)$ by b and let $v = (0, \ldots, 0, 1) \in \mathbb{R}^{|\mathcal{S}|}$.

Proposition 4. *If $g \in \psi$ then $\tilde{A}(g)$ is invertible. Moreover, $\Theta = \{\tilde{A}(g)^{-1}v \; : \; g \in \psi\}$.*

Proposition 4 suggests computing $\underline{R}_e, \overline{R}_e$ by solving

$$\inf/\sup \; \{\varphi^T \tilde{A}(g)^{-1}v \; : \; g \in \psi\}, \tag{12}$$

where $\varphi_s = \Phi(s)p_b^{\infty}(s)$. This optimization problem has both a non-convex objective and a non-convex feasible set, but it has small size. As a way to approximate $\underline{R}_e, \overline{R}_e$, we propose a gradient descent approach to solving Eq. (12) in Algorithm 1. Since the feasible set is itself non-convex, we use an approximate projection that corrects each g iterate to a feasible point but may not be a projection. This is based on taking alternating projection steps on $w^*_{g_k} \in \arg\min\{\|A(g_k)w\|_1 : \mathbb{E}[w(s)] = 1\}$ and $\tilde{g}_k \in \arg\min\{\|g - g_k\|_1 : A(g)w^*_{g_k} = 0, g \in \tilde{\mathcal{B}}\}$; each of these is a linear program with $|\mathcal{S}|$ or $|\mathcal{S}|^2|\mathcal{A}|$ many variables, respectively.

Remark 2. Since the tabular setting is a special case of linear function approximation for w, our approach directly handles the case where $w = \theta^{\top}s$ is a linear function of the state, but further requires well-specification. See Appendix D.1 for detail and discussion of additional challenges.
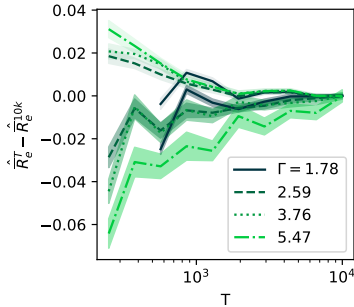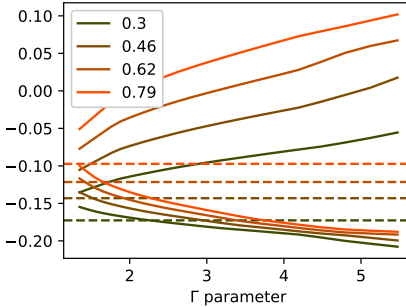
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Figure 4: Statistical consistency

Figure 5: Varying evaluation η mixture weight from uniform to $\pi_b^{*,u=0}$.
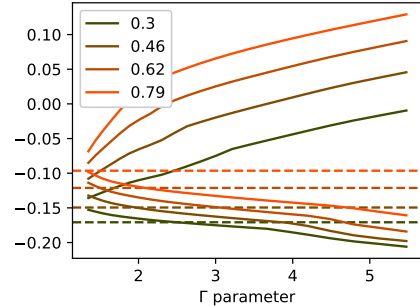
Figure 6: Varying from uniform to $\pi_b^{*,\mathcal{S}}$.

6 Consistency

The above analysis considered the population setting, but in practice we use the empirical state-action occupancy distribution, $\hat{p}_b^\infty(s, a, s')$. Define $\hat{\overline{R}}_e, \hat{\underline{R}}_e$ as the corresponding values when we solve Eq. (4) with this estimate in place of $p_b^\infty(s, a, s')$. We establish consistency of the estimated bounds.

Theorem 3 (Consistency). *If $\hat{p}_b^\infty(s, a, s') \to p_b^\infty(s, a, s')$, then $\hat{\overline{R}}_e \to \overline{R}_e$, $\hat{\underline{R}}_e \to \underline{R}_e$.*

Since the empirical distributions satisfy $\hat{p}_b^\infty(s, a, s') \to_p p_b^\infty(s, a, s')$ [see, *e.g.*, 20], Theorem 3 and the continuous mapping theorem would then together imply that $\hat{\overline{R}}_e \to_p \overline{R}_e$, $\hat{\underline{R}}_e \to_p \underline{R}_e$. Since the perturbation of p_b^∞ to $\hat{p}_b^\infty$ introduces perturbations to the *constraint matrix* of the LP, to prove this result we leverage a general stability analysis due to [31, Theorem 1].

7 Empirics

Illustrative example: Confounded random walk We introduce a simple example in Figure 2. Figure 2 satisfies a sufficient condition of Lemma 1 this is graphically denoted by arrows from (s, u) tuples to s' states. The confounded random walk is parametrized by the transition probabilities under action $a = 1$: $P(s_1 \mid s_1, u_1, a = 1) = p_{u_1}$, $P(s_1 \mid s_1, u_2, 1) = \frac{1}{2} - p_{u_2}$, where u is generated exogenously upon transiting a state s. Transitions are antisymmetric under action $a = 2$, $P(s_1 \mid s_1, u_1, 1) = \frac{1}{2} - p_{u_1}$, $P(s_2 \mid s_1, u_2, 1) = p_{u_2}$. Then, a stationary policy that is uniform over actions generates a random walk on $\mathcal{S}$. See Appendix E for a full description.

In Fig. 3, we vary the underlying transition model, varying $p_{u_1} = p_{u_2}$ on a grid $[0.1, 0.45]$, and we plot the varying bounds with action-marginal control variates. The true underlying behavior policy takes action $a = 1$ with probability $\pi(a = 1 \mid s_1, u_1) = \pi(a = 1 \mid s_2, u_1) = \frac{1}{4}$ (and the complementary probability when $u = u_2$), modeling the setting where a full-information behavior policy is correlated with the full-information transitions. The behavior policy on $\mathcal{S}$ appears to be uniform; using these confounded estimates results in biased estimates of the transition probabilities. As we vary the transition model in Fig. 3, note that

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the true policy value when $\Phi = [1, 2]$ is 1.5, indicated by $\star$, and it is within the bounds for large enough $\Gamma = 3$, uniformly over different data-generating processes.

3x3 confounded windy gridworld. We introduce unobserved confounding to a 3x3 simple windy gridworld environment, depicted in Figure 7 in the appendix [33]. The the agent receives $\Phi(s) = 1$ reward at a goal state but $\Phi(s) = -0.3$ at hazard states (shaded red). We assume a binary unobserved confounder $u \in \{0, 1\}$ that represents "wind strength". Transitions in the action direction succeed in that direction with probability $p = 0.8$, otherwise with probability 0.1 the agent goes east or west. However, when $u = 1$, the "westward wind" is strong, but if the agent takes action "east", the agent instead stays in place (otherwise the agent transitions west). The wind is generated exogenously from all else in the environment. An optimal full-information behavior policy (agent with wind sensors) varies depending on u_t by taking the left action, to avoid the penalty states. This models the setting where unobserved confounding arises due to exogenous private information.

In Figure 4, we study the finite-sample properties of the bounds estimator, plotting $\overline{\hat{R^T}}_e - \overline{\hat{R_e^{10k}}}$ for differing trajectory lengths on a logarithmic grid, $T \in [250, 10000]$, and standard errors averaged over 50 replications. (We plot the difference in order to normalize by an estimate of the large-sample limit). The bounds converge; optimizing over larger Γ values tends to increase finite sample bias.

We illustrate bounds obtained by our approach in Figure 5 for evaluation policies which are mixtures to $\pi_b^{*,u=0}$, a suboptimal policy that is optimal for the transitions when $u = 0$ (no wind), and in Figure 6, $\pi^{*\mathcal{S}}$, a policy that is optimal on $\mathcal{S}$ given the true marginalized transition probabilities (which are unknown to the analyst, but known in this environment). We display the bounds as we range mixture weights on the non-uniform policy from 0.3 to 0.8. We display in a dashed line, with the same color for corresponding mixture weight η, the true value of R_e.

8 Related work, discussion, and conclusions

Off-policy evaluation in RL. We build most directly on a line of recent work in off-policy policy evaluation which targets estimation of the stationary distribution density ratio [6, 8, 13, 17], which can be highly advantageous for variance reduction compared to stepwise importance sampling.

Sensitivity analysis in the batch setting in causal inference. Sensitivity analysis is a rich area in causal inference. A related work in approach is [14], which builds on [2] and considers robust off-policy evaluation and learning in the *one-decision-point* setting. In Appendix A.1, we discuss an extension of their or other inverse-weight robust approaches to this setting and the inherent challenges of such an approach with long horizons in introducing "exponential robustness". The identification approach in this work is very different: the partial identification region is only identified *implicitly* as the solution region of an estimating equation. Unlike [14], [4] consider an *identifiable* setting where we are given a well-specified latent variable model and propose a minimax balancing solution. Finally, [15, 39] study bounds for conditional

average treatment effects in the *one-decision-point* setting. [40] consider sensitivity analysis under assumptions on the outcome functions for a marginal structural model.

Off-policy evaluation in RL with unobservables. Various recent work considers unobserved confounders in RL. [23] considers identification of counterfactuals of trajectories in an POMDP and SCM model. [35] study off-policy evaluation in the POMDP setting, proposing a "decoupled POMDP" and leveraging the identification result of [21], viewing previous and future states as negative controls. [19] propose a "deconfounded RL method" that builds on the deep latent variable approach of [18] . [41] uses partial identification bounds to narrow confidence regions on the transition matrix to warm start the UCRL algorithm of [10].

These generally consider a setting with sufficient assumptions or data to render policy values *identifiable*, where in the general observational setting they are unidentifiable. Specifically, [35] require an invertibility assumption that implies in a sense that we have a proxy observation for *every* unobserved confounder, [19] assume a well-specified latent variable model, also requiring that every unobserved confounder is reflected in the proxies, and [41] consider an online setting where additional experimentation can eventually identify policy value. Our approach is complementary to these works: we focus on the time-homogeneous, infinite-horizon case, and are agnostic to distributional or support assumptions on u. Our structural assumption on u's dynamics (Assumption 2) is also new.

In contrast to POMDPs in general, which *emphasize* the hidden underlying state and its statistical recovery, our model is distinct in that we focus on rewards as functions of the observed state. In contrast to robust MDPs, one may derive a corresponding ambiguity set on transition matrices (see Lemma 4 in the appendix), but it is generally *non-rectangular* because the ambiguity set does not decompose as a product set over states, which leads to a NP-hard problem in the general case [38]. See Appendix A.2 of the appendix for a fuller discussion.

Conclusions and future work Our work establishes partial identification results for policy evaluation in infinite-horizon RL under unobserved confounding. We showed that the set of all policy values that agree with both the data and a marginal sensitivity model can be expressed in terms of whether the value of a linear program is non-positive and developed algorithms that assess the minimal- and maximal-possible values of a policy. Further algorithmic improvements are necessary in order to extend our results to infinite state spaces. Finally, an important next step is to translate the partial identification bounds to robust policy learning.

Broader Impact

As a contribution to offline RL, our work is of particular importance for RL in the context of social and medical sciences, where experimentation is limited and observational data must be used. The validity of the no-unobserved-confounders assumption is of particular importance to applied research in these fields since the presence of unobserved confounding can bias standard evaluations that assume the issue away and, unchecked, this may potentially hide harms done by the policy being evaluated or learned. Our work is the first step in developing offline RL algorithms that directly address this real, practical issue, and its primary purpose is to directly deal with such biases in the data.

That said, it is well understood that there is generally a gap between theory and practice in RL as it is applied to very complex and large-scale systems. It is therefore important to keep in mind practical heuristics, stopgaps, and approximations from applied RL when translating this work into practice. The systematic investigation of the use of these in the context of confounded offline RL and more extensive experimental studies in larger environments may require additional future work.

Moreover, there are several general potential dangers to be cognizant of when applying any offline RL tool in practice. First, if the observational data is not representative of the population, that is, there is a covariate shift in the state distribution, then the evaluation will reflect these biases and correspondingly be unrepresentative, under-emphasizing value to some parts of the population and over-emphasizing value to others. More generally, even without covariate shift, here we focused on evaluation of *average* welfare, which may average the harms to some and the benefits to others; it may therefore be important in some applications to also conduct auxiliary evaluations on certain protected subgroups to ensure equal impact, which can be done by segmenting the data. Third, it is possible that any offline RL approach may be applied inappropriately when the assumptions are not met – here we dealt directly with how to deal with violations of unconfoundedness assumptions but there may still be other assumptions such as our Assumption 1 – and this will mislead any evaluation or learning. For example, concerns about violations of absolute continuity of importance weights (overlap) are especially relevant in the offline RL setting. Therefore, we strongly recommend considering such offline RL and, in particular the sensitivity analyses we developed herein, as a way to inform further investigation, additional data collection, and/or investment in a randomized trial, rather than as an outright replacement for any of these. That said, offline evaluation, especially robust evaluation as we propose herein, is crucial for assessing policies *before* even trialling them, where they may effect actual negative impacts on study populations.

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Outline of the appendix

- Appendix A contains omitted discussion and comparison to other frameworks.

- Appendix B contains proofs regarding the model and partial identification thereof.

- Appendix C discusses optimization and algorithms.

- Appendix D proves statistical consistency.

- Appendix E contains additional empirics and computational discussion.

A   Additional context

A.1   Off Policy Policy Evaluation: Relationship to finite-horizon case

To aid comparison to the off-policy policy evaluation literature, we describe an approach to robustness under unobserved confounding which might be pursued in the finite-horizon case, which does not leverage stationarity. Such an approach would bound the density ratio product of true behavior policy weights $\prod_{t \in [H]} \frac{\pi_e(a_t|s_t)}{\pi_b(a_t|s_t,u_t)}$ relative to the product of nominal inverse propensity weights, $\prod_{t \in [H]} \frac{\pi_e(a_t|s_t)}{\pi_b(a_t|s_t)}$ (including the moment restrictions according to it being a valid density ratio). It is apparent by factorizing the joint distribution that the true density ratio product would identify the policy value.

$$
\begin{aligned}
R_e =& \mathbb{E}[\sum_{h=1}^{H} r_h \mid a_{1:H} \sim \pi_e] \\
=& \mathbb{E}_b \left[\left(\frac{p^{(0)}(s_0, u_0)}{p^{(0)}(s_0, u_0)} \prod_{t \in [H]} \frac{\pi_e(a_t \mid s_t) p(r_t \mid s_t, a_t) p(s_{t+1}, u_{t+1} \mid s_t, u_t, a_t)}{\pi_b(a_t \mid s_t, u_t) p(r_t \mid \mathcal{H}_t) p(s_{t+1}, u_{t+1} \mid s_t, u_t, a_t)} \right) \sum_{h=1}^{H} r_h \right] \\
=& \mathbb{E}_b \left[\prod_{t \in [H]} \frac{\pi_e(a_t \mid s_t)}{\pi_b(a_t \mid s_t, u_t)} \sum_{h=1}^{H} r_h \right]
\end{aligned}
$$

While optimizing bounds on the range of the product of importance-sampling weights, $\prod_{t \in [H]} \frac{\pi_e(a_t|s_t)}{\pi_b(a_t|s_t,u_t)}$ may be tractable using geometric programming [5], enforcing the moment restrictions on density ratios (as in Eq. (7)) introduces further difficulty (nonconvex equality constraints). The core difficulty is that considering an uncertainty set which decomposes as a product set over timesteps may be too conservative to be useful in practice.

Further, in the infinite horizon setting, the divergence of the robust density ratio products grows with the horizon length. This is immediate when considering a relaxation of the sharp uncertainty set, i.e. optimizing the density ratio within bounds without enforcing moment constraints that must be satisfied by inverse probabilities by a geometric programming reformulation [5]; verifying similar properties in the sharp setting with additional nonlinear constraints may require additional analysis.

A.2 Related work and relation to POMDPs and robust MDPs

Contrast to POMDPs. In contrast to POMDPs in general, which *emphasize* the hidden underlying state, our model is distinct in that we focus on rewards as functions of the observed state. The unobserved confounder is therefore a "*nuisance*" confounder which prevents us from estimating policy value, rather than the true underlying state to recover. In settings where unobserved confounders are of concern in observational data in causal inference, typically it is unclear whether or not a latent variable model such as those underlying POMDPs indeed generalizes to the time of deployment: assuming so corresponds to a structural assumption about the environment.

Contrast to Robust MDPs. Robust MDPs, representing a *model-based* approach, consider policy evaluation or improvement over an ambiguity set of the *transition probabilities* [9, 22, 38]. Alternatively, some approaches build confidence regions from concentration inequalities [26, 37] and restrict recommendations within them. [25] improve performance guarantee bounds for state aggregation in MDPs; but in their setting they are able to sample additional full-information transitions unlike our fully-observational data setting The difficulty in applying the robust MDP framework using an ambiguity set on transition matrices suggested from Lemma 4 (in the appendix) is *non-rectangularity* because the ambiguity set does not decompose as a product set over states, which leads to a NP-hard problem in the general case [38].

B Proofs: Model and Partial Identification

Proof of Lemma 1. This is apparent from a recursive definition of $w(j, u)$:

$$
\begin{aligned}
w(s, u) &= \lim_{T \to \infty} \frac{\sum_{t=1}^{T} \sum_{a,k,u'} p(j, u \mid k, u', a) p_e^{t-1}(k, u', a)}{\sum_{n=1}^{T} \sum_{a,k,u'} p(j, u \mid k, u', a) p_b^{t-1}(k, u', a)} \\
&= \lim_{T \to \infty} \frac{\sum_{t=1}^{T} \sum_{a,k,u'} p(j, u'' \mid k, u', a) p_e^{t-1}(k, u', a)}{\sum_{n=1}^{T} \sum_{a,k,u'} p(j, u'' \mid k, u', a) p_b^{t-1}(k, u', a)} = w(s, u'')
\end{aligned}
$$

$\square$

Proof of Lemma 2. Clearly, by assumption of Markovian dynamics on the full information state space, $w(s, u)$ solves the estimating equation (state-action flow equations) on the space of $\mathcal{S} \times \mathcal{U}$,

$$
\mathbb{E}[\pi_e(a \mid s) w(s, u) \beta(a \mid s, u) \mid s' = k] = w(k, v) \quad \forall k \in \mathcal{S}, v \in \mathcal{U} \tag{13}
$$
$$
\mathbb{E}[w(s, u)] = 1. \tag{14}
$$

We will proceed to show that under Assumption 2, the projected $\tilde{w}(k)$ defined as $\tilde{w}(k) := w(k, v)$ equivalently solves the estimating equation on the observed state space $\mathcal{S}$.

The forward implication of Lemma 2 follows from Theorem 1 of Liu et al. [17] applied to the state variable (s, u) after recognizing that $w(s, u)$ only depends on s under Assumption 2

and marginalizing out u'.

$$\tilde{w}(k) = \frac{p_e^\infty(k)}{p_b^\infty(k)} = w(k,v) = \mathbb{E}_{(s,u),a,(s',u')\sim p_b^\infty}[w(s,u)\pi_e(a \mid s)\beta(a \mid s,u) \mid s' = k, u' = v]$$

$$= \mathbb{E}_{(s,u),a,s'\sim p_b^\infty}[w(s,u)\pi_e(a \mid s)\beta(a \mid s,u) \mid s' = k] \qquad \text{by Assumption 2}$$

$$= \mathbb{E}_{(s,u),a,s'\sim p_b^\infty}[\tilde{w}(s)\pi_e(a \mid s)\beta(a \mid s,u) \mid s' = k]$$

Verifying that $\tilde{w}(k)p_b^{(\infty)}(k)$ satisfies conditions on the invariant measure on s:

$$\tilde{w}(k) = \frac{1}{p_b^{(\infty)}(k)}\sum_{j,a}\tilde{w}(j)\sum_u p(k \mid j,u,a)p_b^{(\infty)}(s,u)\pi_b(a \mid s,u)\frac{\pi(a \mid s)}{\pi_b(a \mid s,u)}, \forall k$$

$$= \frac{1}{p_b^{(\infty)}(k)}\sum_{j,a}\tilde{w}(j)\pi(a \mid j)\sum_u p_b^{(\infty)}(j,u)p(k \mid j,u,a), \forall k$$

Therefore,

$$\tilde{w}(k)p_b^{(\infty)}(k) = \sum_{j,a}\tilde{w}(j)p_b^{(\infty)}(j)\pi(a \mid j)p(k \mid j,a),$$

so we conclude that $\tilde{w}(s) \propto p_e^{(\infty)}$, the stationary distribution on $\mathcal{S}$ induced by π_e.

Finally, we argue the reverse implication; uniqueness of the solution of $\tilde{w}(s)$. Uniqueness is a consequence of the positive recurrence assumption (Assumption 1) on the full-information MDP on $\mathcal{S} \times \mathcal{U}$. Note that by definition of recurrence, recurrence on the full-observation state space of the Markov process induced under π implies recurrence of the Markov process induced under π on its marginalized transitions $p(k \mid j,a)$. Recurrence requires that starting from any state j, u in the recurrent class, the number of visits of the chain to the state is infinite. Clearly, if this is satisfied by the full-information transition matrix, this is also satisfied for the aggregated recurrent class corresponding to marginalized transitions.

Therefore, the stationary distribution exists, and is unique on $\mathcal{S}$, under the marginalized transition matrix induced by π_e. The solution to the invariant measure flow equations on $\mathcal{S}$ satisfies that $\tilde{w}(s)p_b^\infty(s) \propto p_e^\infty(s)$; and only $\tilde{w}(s)$ satisfies this requirement. $\qquad\square$

B.1 Observable Implications, Membership Oracle, and Sharpness

In this section, we introduce the state-action polytope and the state-action-state frequency polytope and deduce the observable implications which lead to Proposition 1, the main membership certificate result of Proposition 2, and Theorem 1.

For any infinite-horizon MDP with transition matrix on $\mathcal{S} \times \mathcal{U}$, the stationary dynamics impose restrictions on the unknown full-information state-action-state visitation distribution, $p_b^\infty(s,u,a,s',u')$, and its observable marginalization $p_b^\infty(s,a,s')$. These restrictions are encapsulated as the full-information *state-action polytope* (SAP), which is the set of all limiting state-action occupancy probabilities achievable under any policy, and the closely related *state-action-state* polytope (SASP) [20, 29]. Marginalizing the full-information constraints with respect to $p_b^\infty(u \mid s)$ leads to the marginalized versions mSAP and SASP.

Proof of Proposition 1. To study the observational implications of SAP, SASP (e.g. the implications of the full-information polytopes which are additionally enforceable as constraints on $\tilde{\mathcal{B}}$), we study the marginalized versions of both the state-action polytope and the state-action-state frequency polytope under the behavior policy as studied in [20, 29]. Typically the extremal analysis of the state-action polytope in the infinite-horizon case characterizes the structure of the optimal policy. We simply focus on its properties as a characterization of all the possible limiting state-action frequencies under *any* stationary policy.

While [20] studies the asymptotic inclusion of more general policies such as non-stationary policies, we focus on the case of stationary policies for simplicity. While we state the following analysis for discrete state and action spaces (e.g. discrete u), the discussion of Altman & Shwartz [1, Sec. 4] provides regularity results on the set of limiting state-action measures for the continuous state case with continuous u. One sufficient condition for the case of continuous u is that, indexing x, y into abstract state tuples on $\mathcal{S} \times \mathcal{U}$, for the transition probability density defined as $P_{x,K}^{\pi} := \sum_{y \in K} P_{x,y}^{\pi}$, with $P_{x,y}^{\pi} = \mathbb{P}(s_{t+1} = y \mid s_t = x)$, given any $\epsilon > 0$ there exist a finite set $K(\epsilon)$ and an integer $N(\epsilon)$ such that for all $x \in X$ and $g \in U(S)$, $[(P^{\pi})^{N(\epsilon)}]_{x,K(\epsilon)} \geq 1 - \epsilon$.

First, we introduce SAP, SASP as studied in [20].

Definition 1 (State-action polytope)**.** Given an MDP, the **state-action polytope** SAP is defined as the set of vectors x in $\Delta^{\mathcal{S} \times \mathcal{A}}$ that satisfy

$$\sum_{j,u'} \sum_{a} p(k, u'' \mid j, u', a) p_b^{\infty}(j, u', a) = \sum_{a'} p_b^{\infty}(k, u'', a'), \qquad \forall s' \tag{15}$$

where $p_b^{\infty}(j, u', a) \in \Delta^{\mathcal{S} \times \mathcal{A}}$ is the limiting expected state-action frequency vector under policy π. This constraint can be understood as a "flow conservation" constraint satisfied by any full-information joint distribution $p_b^{\infty}(s, u, a, s', u')$.

Definition 2 (State-action-state polytope)**.** The **state-action-state** frequency polytope, SASP, is the set of vectors in $\Delta^{\mathcal{S} \times \mathcal{A} \times \mathcal{S}}$ which satisfy conformability of state transitions under transition probabilities; and marginalization:

$$p_b^{\infty}(j, u', a, k, u'') = p(k, u'' \mid j, u', a) \sum_{\tilde{k}, \tilde{u}''} p_b^{\infty}(j, u', a, \tilde{k}, \tilde{u}''), \qquad \forall j, u', a, (k, u'') \tag{16}$$

$$\sum_{j,u'} \sum_{a} p_b^{\infty}(j, u', a, k, u'') = \sum_{a'} \sum_{\tilde{k}, \tilde{u}''} p_b^{\infty}(k, u'', a', \tilde{k}, \tilde{u}''), \qquad \forall (k, u'') \tag{17}$$

Lemma 3.1 of [20] states that the two sets are equivalent: if $p_b^{\infty}(s, u, a) \in$ SAP, and furthermore under the transformation

$$p_b^{\infty}(j, u', a, k, u'') = p_b^{\infty}(j, u', a) p(k, u'' \mid j, u', a),$$

then $p_b^{\infty}(j, u', a, k, u'') \in$ SASP. Every element of SASP can be generated in this manner from some element of SAP. It is this construction that leads to the "information loss" stated formally in the next result which characterizes the marginalized versions of Eqs. (15) to (17) in terms of the marginalized weight of the reparametrization, $g_k(a \mid j)$.

Lemma 3. *The marginalized version of Eq. (15) is*

$$p_b^\infty(k) = \sum_a \sum_j p_b^\infty(j) p(k \mid j, a) g_k(a \mid j)^{-1}, \ \forall k \in \mathcal{S} \tag{18}$$

The marginalized version of Eq. (16) enforces conformability of $g_k(a \mid j)$ for the true non-identifiable transitions,

$$p(k \mid j, a) = p_b^\infty(a, k \mid j) g_k(a \mid j), \ \forall j, a, k \tag{19}$$

and of Eq. (17) is the conditional control variate,

$$p_b^\infty(k \mid a) = \sum_j p_b^\infty(j, a, k) g_k(a \mid j), \ \forall k, a \tag{20}$$

Proof of Lemma 3. **Marginalizing Eq. (16):**
Starting from the compatibility restriction with the observed empirical state-action frequencies: $\sum_u p_b^\infty(j, u) \pi_b(a \mid j, u) p(k, u' \mid j, u, a) = p_b^\infty(j, a, k, u')$ and marginalizing over u':

$$\sum_u p_b^\infty(j, u) \pi_b(a \mid j, u) p(k \mid j, u, a) = \sum_u p_b^\infty(j, a, k)$$

$$p(k \mid j, a) p_b^\infty(j) \sum_u p_b^\infty(u \mid j) \pi_b(a \mid j, u) \frac{p(k \mid j, u, a)}{p(k \mid j, a)} = p_b^\infty(j, a, k)$$

$$p(k \mid j, a) g_k(a \mid j)^{-1} = p_b^\infty(a, k \mid j)$$

where $p(k \mid j, a) = \sum_u p(k \mid j, u, a) p_b^\infty(u \mid j)$ and $p_b^\infty(a, k \mid j) = \frac{p_b^\infty(j,a,k)}{p_b^\infty(j)}$.

This leads to the conformability requirement of $g_k(a \mid j)$ for the marginal transition probabilities with respect to the observational joint distribution.

$$p(k \mid j, a) = p_b^\infty(a, k \mid j) g_k(a \mid j) \tag{21}$$

which can also be derived as a marginalization of Eq. (16):

$$\sum_{u,u'} p_b^\infty(j, u, a, k, u') = \sum_{u,u'} p(k, u' \mid j, u, a) \sum_{\tilde{k},\tilde{u}'} p_b^\infty(j, u, a, \tilde{k}, \tilde{u}'), \qquad \forall j, k$$

Marginalizing Eq. (17):
Recall that from the forward decomposition of joint distribution with respect to the transition from $j, u \to k, u'$, but conditioning on current state and action, we have that:

$$p_b^\infty(k, u' \mid a, j, u) = \frac{p_b^\infty(j, u, a, k, u')}{\pi_a(j, u) p_b^\infty(j, u)}$$

Then marginalize the definition of $p_b^\infty(k, u' \mid a)$ over u' to obtain $p_b^\infty(k \mid a)$:

$$\sum_{u'} p_b^\infty(k, u' \mid a) = \sum_{u'} \sum_{j,u} p_b^\infty(k, u' \mid a, j, u) p_b^\infty(j, u) = \sum_{u'} \sum_{j,u} \frac{p_b^\infty(j, u, a, k, u') p_b^\infty(j, u)}{\pi(a \mid j, u) p_b^\infty(j, u)}$$

$$= \sum_j p_b^\infty(j, a, k) \sum_u \frac{p_b^\infty(j, u, a \mid k)}{p_b^\infty(j, a \mid k) \pi(a \mid j, u)}$$

$$= \sum_j p_b^\infty(j, a, k) g_k(a \mid j)$$

Therefore:

$$p_b^\infty(k \mid a) = \sum_j p_b^\infty(j, a, k) g_k(a \mid j), \forall k, a$$

$\square$

In order to interpret *which* of these constraints are observable implications and which are ultimately informative, we next leverage a structural characterization that $g_k(a \mid j)$ can be interpreted as the function which renders the transition probabilities conformable to the joint distribution. Its proof is of independent interest in establishing the relationship to robust MDPs. We introduce the *biased* marginalized transition probabilities $\tilde{p}(s' \mid s, a)$, which would be obtained from naive estimation from the observational joint distribution:

$$\tilde{p}(s' \mid s, a) := \frac{p_b^\infty(s, a, s')}{p_b^\infty(s) \mathbb{E}[\pi_b(a \mid s, u) \mid s]} = p_b^\infty(s)^{-1} \sum_u \frac{p_b^\infty(s, u, a, s')}{\mathbb{E}[\pi_b(a \mid s, u) \mid s]}$$

These are biased estimates because they do not appropriately account for the transitions under the true $\pi_b(s, u)$ policy, only its marginalization over s.

Lemma 4.

$$\tilde{p}(k \mid j, a) \pi_b(a \mid j) = p(k \mid j, a) g_k(a \mid j)$$

Proof of Lemma 4. With full information, the transition probabilities could be estimated as

$$p(s', u' \mid s, u, a) = \frac{p_b^\infty(s, u, a, s', u')}{\pi_b(a \mid s, u) p_b^\infty(s, u)}$$

and similarly, the marginalized transition probabilities as $\frac{p_b^\infty(s, u, a, s')}{\pi_b(a \mid s, u) p_b^\infty(s, u)} = p(s' \mid s, u, a)$.

A model-based perspective would partially identify the transition matrix under π_e, deduce the bounds of $p(s' \mid s, a)$ relative to $\tilde{p}(s' \mid s, a)$. Note that under Assumption 2,

$$p(s' \mid s, a) = \sum_u p(s' \mid s, u, a) p_b^\infty(u) = \sum_u \frac{p_b^\infty(s, u, a, s') p_b^\infty(u)}{\pi_b(a \mid s, u) p_b^\infty(s, u)} = p_b^\infty(s)^{-1} \sum_u \frac{p_b^\infty(s, u, a, s')}{\pi_b(a \mid s, u)} \tag{22}$$

and further the distribution on unobserved confounders is independent of the policy, $p_b^\infty(u) = p_e^\infty(u)$.

$$\tilde{p}(s' \mid s, a) = \frac{p_b^\infty(s, a, s')}{p_b^\infty(s) \mathbb{E}[\pi_b(a \mid s, u) \mid s]} = p_b^\infty(s)^{-1} \sum_u \frac{p_b^\infty(s, u, a, s')}{\mathbb{E}[\pi_b(a \mid s, u) \mid s]} \tag{23}$$

Combining Equations (22) and (23) yields the statement of the lemma.

$\square$

Lemma 4 shows that the constraints in Equations (18) and (19) are uninformative: further restricting $\tilde{p}(s' \mid a, s)$ within the given range of $p(s' \mid a, s)$ is redundant. Another interpretation is that $g_k(a \mid j)$ are precisely the weights which render the observed stationary occupancy distribution $p_b^\infty(s, a, s')$ conformable under the *unobserved* true marginal transition probabilities.

Proposition 1 follows from Lemmas 3 and 4.

$\square$

Proof of Proposition 2. We verify that optimizing over $F(w) = 0 \iff w \in \Theta, \mathbb{E}[w] = 1$ by first showing the reparametrization of $F(w)$ with respect to g, and then leveraging the characterization of Proposition 1 and sharpness argument to verify that $F(w) = 0 \iff w \in \Theta, \mathbb{E}[w] = 1$.

Step 1: Proving the reparametrization of $F(w)$ with respect to $g_k(a \mid j)$, and reformulating $g_k(a \mid j)$.

We first expand the sample expectations for the estimating equation of Lemma 2:

$$\sum_j \sum_{i=1}^N \sum_{t=0}^T \sum_{a,u} \frac{\mathbb{I}[(s_t^{(i)} = j, u_t^{(i)} = u), a_t^{(i)} = a, s_{t+1}^{(i)} = k]}{p(s_{t+1}^{(i)} = k)} \left(w(k) - w(j)\beta^{(i)}(a \mid j, u)\pi_e(a \mid j) \right) = 0, \forall k$$

Rewrite as an expectation with respect to the observational (identifiable) joint distribution $p_b^\infty(j, a \mid k)$, taking limits as $T \to \infty, N \to \infty$ and multiplying by $\frac{p_b^\infty(j,a|k)}{p_b^\infty(j,a|k)} = 1$:

$$w(k) - \sum_j w(j) \sum_a \pi_e(a \mid j) p_b^\infty(j, a \mid k) \underbrace{\sum_u \frac{p_b^\infty(j, u, a \mid k)}{p_b^\infty(j, a \mid k)} \beta(a \mid j, u)}_{g_k(j,a;W)} = 0, \forall k$$

Therefore, dependence on β arises only through the marginalized weight $g_k(a \mid j)$:

$$g_k(a \mid j) = \sum_u \frac{p_b^{(\infty)}(j, u, a \mid k)}{p_b^\infty(j, a \mid k)} \beta(a \mid j, u)$$

However, the marginalized weight depends on the unknown data-generating joint distribution $p_b^{(\infty)}(j, u, a \mid k)$; and so it is unclear how to optimize over it. Next, we show that *optimizing over $g_k(a \mid j)$, which are the unknown inverse weights $\beta(a \mid j, u)$ convolved with an unknown density, is almost *equivalent* to optimizing over the set of weights $\mathcal{B}$, up to a moment constraint on ensuring that the implied full-information transition probabilities are valid probability distributions, e.g. that $\sum_k p(k \mid j, u, a) = 1$. We first we show that it is equivalent to optimize over $g_k(a \mid j)$ over the same bounds, though this may not enforce the restriction $\sum_k p(k \mid j, u, a) = 1$. In the next step, we will argue that this restriction to valid transition probabilities is enforced by the feasibility of g for the estimating equation.

Define

$$\tilde{\mathcal{B}}' := \{g \in \mathbb{R}_+^{|\mathcal{S}||\mathcal{A}||\mathcal{S}|} : \exists \beta \in \mathcal{B} \text{ such that } g_k(a \mid j) = \sum_u \frac{p_b^{(\infty)}(j, u, a \mid k)}{p_b^{(\infty)}(j, a \mid k)} \beta\}$$

as the ambiguity region of $g_k(a \mid j)$ induced by restrictions on $\beta \in \mathcal{B}$. We show how to identify elements $\tilde{\beta} \in \tilde{\mathcal{B}}'$ with corresponding elements $\beta \in \mathcal{B}$. Although $p_b^{(\infty)}(j, u, a \mid k)$ is not identifiable from observed data, its marginalization over u, $p_b^{(\infty)}(j, a \mid k)$, is identifiable, so we can partially identify $\tilde{\mathcal{B}}'$ as follows:

$$\tilde{\mathcal{B}}' = \left\{ g \in \mathbb{R}_+^{|\mathcal{S}||\mathcal{A}||\mathcal{S}|} : \begin{array}{c} g_k(a \mid j) = \sum_u \frac{p_b^{(\infty)}(j,u,a|k)}{p_b^{(\infty)}(j,a|k)} \beta(a \mid j, u) \\ \sum_u p_b^{(\infty)}(j, u, a \mid k) = p_b^{(\infty)}(j, a \mid k) \\ 0 \le p_b^{(\infty)}(j, u, a \mid k) \le 1 \\ \beta \in \mathcal{B} \end{array} \right\}$$

A simple reparametrization with respect to $q_{j,u,a|k} := \frac{p_b^{(\infty)}(j,u,a|k)}{p_b^{(\infty)}(j,a|k)}$ shows that optimizing over $\tilde{\mathcal{B}}'$ is equivalent to optimizing over elements of $\mathcal{B}$ averaged by unknown weights on the simplex. In the following, suppress dependence of $q_{j,u,a|k}\beta(a \mid j, u)$ on a, j for brevity, and let $q_{j,\cdot,a,|k}$ denote the vector .

$$\tilde{\mathcal{B}}' = \left\{ q_{j,\cdot,a,|k}^\top \beta \ : \ \beta \in \mathcal{B}; \quad q_{j,\cdot,a,|k}^\top 1 = 1, \ 0 \le q_{j,\cdot,a,|k} \le 1, \quad \forall j, a, k \right\}$$

In particular this suggests that $g_k'(a \mid j) \in \mathcal{B}$ since by convexity of $\mathcal{B}$, we can map $g_k'(a \mid j) \in \tilde{\mathcal{B}}'$ to some $\beta \in \mathcal{B}$; q are the convex combination weights. In the other direction, clearly any $\beta' \in \mathcal{B}$ is realizable by a q which is a Dirac measure which selects β', so $\beta' \in \tilde{\mathcal{B}}'$.

Lastly, we directly verify the control variate property (Eq. (7) corresponding to Eq. (3)) that $\sum_j \sum_k p_b^{(\infty)}(j, a \mid k) p_b(k) g_k(a \mid j) = 1$:

$$\sum_j p_b^{(\infty)}(j, a, k) g_k(j, a) = \sum_{j,u} p_b^{(\infty)}(j, a \mid k) p_b^\infty(k) \frac{p_b^{(\infty)}(j, u, a \mid k)}{p_b^{(\infty)}(j, a \mid k)} \beta(a \mid j, u)$$

$$= \sum_{j,u} \frac{p_b^{(\infty)}(j, u) \pi_b(a \mid j, u) p(k \mid j, u, a)}{p_b^\infty(k)} p_b^\infty(k) \beta(a \mid j, u)$$

$$= \sum_{j,u} p_b^\infty(k, j, u \mid a)$$

so that we verify the action-marginal control variate, $\sum_k \sum_j p_b^{(\infty)}(j, a, k) g_k(a \mid j) = 1$. Note that this further implies Eq. (7):

$$\sum_j p_b^{(\infty)}(j, a, k) g_k(j, a) = \sum_{j,u} p_b^\infty(k, j, u \mid a) = p_b^\infty(k \mid a)$$

Finally, to help interpret $g_k(a \mid j)$, we may further simplify and observe that

$$g_k(a \mid j) = \sum_u \frac{p_b^{(\infty)}((j,u), a \mid k)}{p_b^{(\infty)}(j, a \mid k)} \beta(a \mid j, u) = \frac{\sum_u p_b^\infty(u \mid j) p(k \mid j, u, a)}{\sum_u p_b^\infty(u \mid j) p(k \mid j, u, a) \pi_b(a \mid j, u)}$$

$$= \frac{p(k \mid j, a)}{\sum_u p_b^\infty(u \mid j) p(k \mid j, u, a) \pi_b(a \mid j, u)}$$

$$= \frac{1}{\sum_u \pi_b(a \mid j, u) \frac{p_b^\infty(u \mid j) p(k \mid j, u, a)}{p(k \mid j, a)}}$$

Note that in this step, we did *not* require the extremal $g_k(a \mid j)$ to be achieved by valid unobserved transition probabilities such that $\sum_k p(k \mid j, u, a) = 1$. To do so directly would introduce technical difficulties in effectively requiring a bilinear formulation. Instead, we have shown this one-to-one correspondence of $\tilde{\mathcal{B}}'$ with $\beta \in \tilde{\mathcal{B}}$ occurs for the marginalized weights, which are not further required to correspond to valid adversarial transition probability distribution. In the next steps, we argue that feasibility of $g \in \tilde{\mathcal{B}}'$ for $F(w) \leq 0$ ensures this compatibility with valid transition matrices.

Step 2: Proving $F(w) \leq 0, \mathbb{E}[w] = 1 \implies w \in \Theta$:
Proposition 1 shows that the specification of $\tilde{\mathcal{B}}$ exhausts the observable implications of the sharp full information polytope of all limiting state-action-state occupancy probabilities. It remains to show that w is feasible for some $g_k(a \mid j) \in \mathcal{B}$ iff $g_k(a \mid j)$ satisfies Eq. (19), $p(k \mid j, a) = p_b^\infty(a, k \mid j) g_k(a \mid j)$, $\forall j, a, k$.

First we show that $F(w) \leq 0, \mathbb{E}[w] = 1$ implies $w \in \Theta$. Suppose g is feasible for the estimating equation:

$$w(k) - \sum_j w(j) \sum_a \pi_e(a \mid j) p_b^\infty(j, a \mid k) g_k(a \mid j) = 0, \forall k \tag{24}$$

Next we verify that $g_k(a \mid j)$ satisfying Eq. (19) is feasible for the estimating equation for w, Eq. (24). Note that $g_k a \mid j$ corresponding to underlying transitions which do not satisfy $\sum_k p(k \mid j, u, a) = 1$ cannot satisfy Eq. (19). By Bayes' rule and conformability of $g_k(a \mid j)$ for $p(k \mid j, a)$,

$$p_b^\infty(j, a \mid k) = \frac{p(k \mid j, a) p_b^\infty(j) g_k(a \mid j)^{-1}}{p_b^\infty(k)}, \tag{25}$$

so that we can verify the estimating equation holds with respect to the true marginalized transition dynamics under π_e:

$$w(k) p_b^\infty(k) - \sum_j w(j) p_b^\infty(j) \sum_a \pi_e(a \mid j) p(k \mid j, a) = 0, \forall k$$

Markovianness of the induced MDP under the *true* marginal transition probabilities $p(k \mid j, a)$ (follows since $p(k \mid j, a)$ corresponds to the $p^\infty(u)$-occupancy-weighted aggregation to $\mathcal{S}$ under Assumption 2), then $w(k) p_b^\infty(k)$ is proportional to the invariant measure on $\mathcal{S}$.

Step 3: Proving $w \in \Theta \implies F(w) = 0, \mathbb{E}[w] = 1$**:**

Next we show the other direction, that w feasible for Equation (24) for some $g_k(a \mid j)$ implies that $g_k(a \mid j)$ satisfies Equation (19). This direction follows once we identify $w(s)p_b^\infty(s)$ feasible for Equation (24) uniquely with $p_e^\infty(s)$, which follows from Assumption 1. By Equation (25), feasibility implies that $w(s)p_b^\infty(s)$ satisfies compatibilty under π_e for $g_k(a \mid j)$. Uniqueness of the density ratio implies that compatibility must also hold under π_b. $\quad\square$

Proof of Theorem 1. Theorem 1 is a consequence of Proposition 1 and that computing the support function of a set (e.g. optimizing an arbitrary linear objective over Θ) is equivalent to optimizing over the convex hull of Θ [32], conv(Θ). Convexity of the interval and of conv(Θ) yields sharpness. $\quad\square$

Relationship between Lemma 4 and robust MDPs

Remark 4. We can define an ambiguity set for marginal transition probabilities on $\mathcal{S}$ for s, a state-action pairs, $\mathcal{P}_{s' \mid s, a}$:

$$P(\cdot \mid s, a) \in \mathcal{P}_{s' \mid s, a} := \left\{ P_{sa} \in \Delta^{|\mathcal{S}|} : \exists\, \beta \in \mathcal{B} \text{ such that } P_{sa}(k) = \sum_u \beta \cdot \frac{p_b^\infty(s, u, a, k)}{p_b^\infty(s)}, \forall k \in \mathcal{S} \right\}$$

By an analogous construction as in Proposition 2, without additional restrictions on the variation of the unobserved joint visitation distribution $p_b^\infty(s, u, a, s')$,

$$\mathcal{P}_{s' \mid s, a} := \left\{ P_{sa} \in \Delta^{|\mathcal{S}|} : \exists\, g \in \tilde{\mathcal{B}} \quad \text{s.t.} \quad P(s' \mid s, a) = g_{s'}(a \mid s) \cdot \tilde{P}(s' \mid s, a) \right\}$$

However, due to the restrictions on $g_k(a \mid j)$ corresponding to valid probability distributions, as well as the restrictions that $P(s' \mid s, a)$ corresponds to valid probability distributions, the feasible implicit ambiguity set on transition sets is not merely the union of $\mathcal{P}_{s' \mid s, a}$ for all s, a. Instead, the valid ambiguity set combines the restrictions induced by the bounds assumptions of β over $P_{sa} \in [\Delta^{|\mathcal{S}|}]^{|\mathcal{S}| \times |\mathcal{A}|}$, with $P_{s' \mid s, a} = p(s' \mid s, a)$, and the observable implications of Proposition 1.

$$\mathcal{P}_{\text{feas}} = \left\{ P \in [\Delta^{|\mathcal{S}|}]^{|\mathcal{S}| \times |\mathcal{A}|} : \begin{array}{ll} \exists\, g \in \mathcal{B}, w \quad \text{s.t.} & \\ P(k \mid j, a) = g_k(a \mid j) \cdot \pi(a \mid j)\tilde{p}(k \mid j, a), & \forall j, a, k \\ p_b^\infty(k \mid a) = \sum_j p_b^\infty(j, a, k)g_k(a \mid j), & \forall k, a \\ \mathbb{E}[\pi_e(a \mid s)w(s)g_k(a \mid s) \mid s' = k] = w(k) \; \forall k, & \\ \mathbb{E}[w(s)] = 1 & \end{array} \right\} \quad (26)$$

Notably, the inverse probability restrictions on $g_k(a \mid j)$ render such an ambiguity set *non-rectangular* over states and actions, while the requirement of compatibility (enforced by the estimating equations) additionally introduces nonconvexity.

C Proofs: Optimization and algorithms

Proof of Proposition 3. A feasibility oracle for Θ, for a given w, is given by checking the *existence* of $g \in \mathcal{W}$ satisfying the moment condition. For brevity, let $m_k(w, g)$ denote the kth moment restriction:

$$m_k(w, g) := \sum_{j,a} p_b^{(\infty)}(j, a \mid k) w(j) \pi_e(a \mid j) g_k(a \mid j) - w(k)$$

$$F(w) := \min_{g \in \tilde{\mathcal{B}}} \sum_{k} |m_j(w, g)|$$

We will generate a disjunctive reformulation of $F(w)$ by appealing to a different lifting of the ℓ_1 norm that enumerates the possible sign patterns on $\{-1, 1\}^{|\mathcal{S}|}$; the next lemma briefly verifies equivalence.

Lemma 5.

$$\min\{\sum_j \lambda_j m_k(w, g) \colon g \in \tilde{\mathcal{B}}\} = 0, \forall \lambda_i \in \{-1, 1\}^p \iff \min\{\sum_j |m_k(w, g)| : g \in \tilde{\mathcal{B}}\} = 0$$

We next briefly introduce the disjunctive programming framework.

Preliminaries for disjunctive programs First we introduce disjunctive programs with generic notation [3]. A disjunctive program optimizes over the union of polyhedra. A disjunctive program is of the form: $\min\{cx | Ax \geqslant a_0, x \geqslant 0, x \in L\}$ where the logical conditions $x \in L$ can be represented by the disjunctive normal form or the conjunctive normal form,

$$\{Ax \geqslant a_0, \quad x \geqslant 0; \bigvee_{i \in Q_j} (d^i x \geqslant d_{i0}), \quad j \in S\},$$

or equivalently, to make the conjunctions apparent,

$$\begin{bmatrix} Ax \geq a_0 \\ x \geq 0 \end{bmatrix} \wedge [\bigvee_{i \in Q_0} (d^i x \geqslant d_{i0})] \wedge \cdots \wedge [\bigvee_{i \in Q_{|S|}} (d^i x \geqslant d_{i0})]$$

The linear programming equivalent of a disjunctive normal form is given by Theorem 2.1 of [3]. It generically provides the linear programming formulation of a disjunctive form, $F = \{x \in R^n | \bigvee_{h \in Q} (A^h x \geqslant a_0^h, x \geqslant 0)\}$, as

$$\text{clconv } F = \left\{ x \in R^n | \begin{array}{ll} x = \sum_{h \in Q^*} \xi^h, & \\ A^h \xi^h - a_0^h \xi_0^h \geqslant 0, & h \in Q^* \\ \sum_{h \in Q^*} \xi_0^h = 1, & (\xi^h, \check{\xi}_0^h) \geqslant 0, h \in Q^* \end{array} \right\},$$

where Q^* is the restriction of Q to nonempty disjunctions.

Reformulating as a disjunctive program. Using Lemma 5, we can rewrite (10), where $X(\mathcal{B})$ denotes the set of extreme points of polytope $\mathcal{B}$:

$$\max\left\{\mathbb{E}[w(s)\Phi(s)]\colon \ \forall \lambda \in \{-1,1\}^{|\mathcal{S}|}, \ \exists \ g \in X(\tilde{\mathcal{B}}) \text{ s.t. } \sum_k \lambda_k m_k(w,g) = 0\right\}$$

and therefore in the disjunctive syntax,

$$\max\left\{\mathbb{E}[w(s)\Phi(s)]\colon \ \bigwedge_{\lambda \in \{-1,1\}^{|\mathcal{S}|}} \left(\bigvee_{g \in X(\tilde{\mathcal{B}})} \text{ s.t. } \lambda_k m_k(w,g) = 0\right)\right\}$$

Next, by applying the distributive property of conjunctions and disjunctions ([3]), we can express the conjunctive form into the disjunctive form, and then apply Theorem 2.1 to obtain the corresponding linear programming representation. Note that this operation generates exponentially many unions of disjunctions (each also exponential in cardinality of state space), and therefore admits a overall superexponentially-sized program. $\qquad\square$

Proof of Lemma 5. For brevity, denote

$$P(\lambda) := \min\{\sum_j \lambda_j m_k(w,g)\colon g \in \tilde{\mathcal{B}}\}$$

$$P(\ell_1) := \min\{\sum_j |m_k(w,g)| \colon g \in \tilde{\mathcal{B}}\}$$

so that the lemma can be stated as:

$$P(\lambda) = 0, \forall \lambda \in \{-1,1\}^p \iff P(\ell_1) = 0$$

First we argue $\impliedby$: Suppose not, where $\{\min \sum_j |m_k(w,g)| = 0, \forall j\}$ is true (and achieved by some w^*) but there is some λ^* such that $\min\{\sum_j \lambda_j^* m_k(w,\lambda)\colon a \le w \le b\} > 0$. But g^* was feasible for $P(\lambda^*)$, and $P(\ell_1) \le 0$ implies that $m_k(w,g^*) = 0, \forall j$, so we could further reduce the value of $P(\lambda^*)$ at feasible g^*, contradicting optimality of λ^* at a strictly positive value.

In the other direction, $\implies$, suppose by way of contradiction that $P(\lambda) \le 0, \ \forall \lambda$ but $P(\ell_1) > 0$ (equivalently, that g is infeasible for $\mathcal{B}$). Let $\lambda_{\ell_1}^*$ be the particular sign pattern which achieves strict nonnegativity at optimality for $P(\ell_1)$. Then, the optimal value of $P(\lambda)$ at this particular choice of $\lambda_{\ell_1}^*$ is also strictly positive, $P(\lambda_{\ell_1}^*) = P(\ell_1) > 0$ since it induces the same optimization objective over the same feasible set of weights, $\mathcal{B}$.

By uniqueness of w for feasible values of the unknown weights g_k (by properties of ergodicity and uniqueness of the stationary distribution), it is not a relaxation to optimize over different weights for each λ value. $\qquad\square$

Proof of Proposition 4. We recall the notation which encodes the estimating equation as the matrix A: we introduce the instrument functions $\phi_s, \phi_{s'} \in \mathbb{R}^{|\mathcal{S}|\times 1}$, random (row) vectors which are one-hot indicators for the state random variable s, s' taking on each value, $\phi_s = \begin{bmatrix}\mathbb{I}[s=0] & \cdots & \mathbb{I}[s=|\mathcal{S}|]\end{bmatrix}$. Let $A(g) = \mathbb{E}[\phi_{s'}(\pi_e(a \mid s)g_{s'}(a \mid s)\phi_s - \phi_{s'})^\top]$ and $b_s = p_b^\infty(s)$.

The set of $g \in \tilde{\mathcal{B}}$ that admit a feasible solution to the estimating equation for some $w \in \Theta$ is
$\psi := \{g \in \tilde{\mathcal{B}} \colon \exists\, w \geq 0 \text{ s.t. } A(g)w = 0, b^\top w = 1\}$

A has rank $|\mathcal{S}| - 1$ *if* g is feasible since satisfying the conformability constraint Equations (16) and (20) implies linear dependence on the rows of A:

$$\sum_k \Big(w_k - \sum_{j,a} p_b^\infty(a, j \mid k)\pi_e(a \mid j)g_k(a \mid j)w_j\Big) = 0$$

Define $\tilde{A}(g)$ by replacing the last row of $A(g)$ by b and let $v = (0, \ldots, 0, 1)$. Then $w = \tilde{A}^{-1}v$ which results in the following program, with $v = \begin{bmatrix} 0_{|\mathcal{S}|-1} & 1 \end{bmatrix}^\top$:

$$\inf \ / \ \sup \left\{ \varphi^\top \tilde{A}^{-1} v \ : \ g \in \psi \right\}$$

Partial derivatives of the matrix-valued function of $g_k(a \mid j)$ follow from the matrix chain rule, where $J_{j,k}$ is a one-hot matrix with a 1 in the j, k entry and 0 everywhere else:

$$\begin{aligned}
\frac{\partial \varphi^\top \tilde{A}^{-1} v}{\partial g_k(a \mid j)} &= \sum_{i,j} \frac{\partial \varphi^\top \tilde{A}^{-1} v}{\partial \tilde{A}_{i,j}} \frac{\partial \tilde{A}_{i,j}}{\partial g_k(a \mid j)} \\
&= -\sum_{i,j} \tilde{A}_{i,j}^{-\top} \varphi v^\top \tilde{A}_{i,j}^{-\top} (\mathbb{I}[j \neq |\mathcal{S}|]\pi_{a,j}^e \overset{\infty_b}{p}_{j,a,k} J_{j,k}) \\
&= -(\mathbb{I}[j \neq |\mathcal{S}|]\pi_{a,j}^e \overset{\infty_b}{p}_{j,a,k})(\tilde{A}^{-\top} \varphi w^\top)_{i,j}
\end{aligned}$$

$\square$

D   Proof of Theorem 3 (statistical consistency)

Consistency follows from stability of the optimization problem in terms of the deviations of empirical probabilities from their population values. The former is a result from variational analysis/stability analysis of linear programs, which is non-standard because the perturbations occur in the constraint matrix *coefficients*. The latter is simply the convergence of empirical probabilities.

We prove statistical consistency by considering a support function estimate that discretizes the space (restricting attention to feasible w). We apply a stability analysis argument to argue consistency for every fixed w in the discretization. Since w is in a compact set (the $\mathcal{S}$ simplex), a covering argument over the solution space provides a bound via a union bound over elements of the discretization. Lastly, we bound the approximation error arising from the discretization. While the discretization approach provides a statistical consistency result (in the limit as $n \to \infty$ we assume the discretization grows finer), it is a tool of the analysis but not the algorithmic proposal.

Preliminaries: Stability Analysis   Consistency follows from stability of the optimization problem in terms of the deviations of empirical probabilities from their population values. The former is a result from variational analysis/stability analysis of linear programs, which

is non-standard because the perturbations occur in the constraint matrix *coefficients*. The latter is simply the convergence of empirical probabilities.

Stability analysis establishes convergence in Hausdorff distance between $\hat{\Theta}$, Θ, the partial identification set obtained from optimizing the sample estimating equation vs. from optimizing the population estimating equation. The Hausdorff distance between two sets $A, B \subset \mathbb{R}^d$ is $d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$ where $d(a, B) = \inf_{b \in B} \|a - b\|$; e.g. it measures the furthest distance from an arbitrary point in one of the sets to its closest neighbor in the other set.

The main stability analysis result we use is Theorem 1 of [31]. To help keep the presentation of the theorem self-contained, we state some preliminary notation. The paper considers the general case of a system of linear inequalities, where A is a continuous linear operator from X into Y which are real Banach spaces, and K is a nonempty closed convex cone in Y. We study

$$Ax \leq_K b, \forall x \in C \tag{27}$$

with $C \subseteq X$ a convenience set to represent *unperturbed* constraints. We want to ascertain the *stability region* of the solution set G, which implies that for *each* $x_0 \in G$, for some positive number β, and for any continuous linear operator $A' \colon X \mapsto Y$ and any $b' \in Y$, the distance from x_0 to the solution set of the perturbed system,

$$A'x \leq_K b', \tag{28}$$

is bounded by $\beta\rho(x_0)$, with $\rho(x)$ being the residual vector,

$$\rho(x) := d(b' - A'x, K) := \inf\{\|b' - A'x - k\| \mid k \in K\}.$$

For a more concise statement of the main stability analysis result, we introduce the augmented operator with an auxiliary dimension to homogenize the system $Q \colon X \times \mathbb{R} \mapsto Y$: for finite-dimensional systems of linear equations, this is the usual homogenization.

$$Q(\begin{bmatrix} x \\ \xi \end{bmatrix}) = \begin{cases} \begin{bmatrix} A & -b \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + K & , \begin{bmatrix} x & \xi \end{bmatrix}^\top \in P, \\ \infty & \begin{bmatrix} x & \xi \end{bmatrix}^\top, \notin P \end{cases}$$

Now, under this notation, $x \in C$ satisfies $Ax \leq_K b, \forall x \in C$ iff $0 \in Q(\begin{bmatrix} x \\ \xi \end{bmatrix})$. The result will leverage properties of linear operators as special cases of convex processes (which are themselves multivalued functions between two linear spaces): If T carries X into Y, with X, Y normed linear spaces, then the *inverse* of T is T^{-1}, which is defined for $y \in \mathcal{Y}$ by

$$T^{-1}y := \{x \mid y \in Tx\}.$$

T is closed if $\text{gph}(T) := \{(x, y) \mid y \in Tx\}$ is closed on product space $X \times Y$. The norm of T is operator norm.

This approach then allows us to identify another linear operator which parametrizes the *perturbation* $\Delta(\begin{bmatrix} x \\ \xi \end{bmatrix})$ defined analogously to Q but with $(A' - A)x - (b' - b)\xi + K$. The size

of the perturbation is measured by the operator norm of this system, and a crude bound is $\|\Delta\| \leq \|A' - A\| + \|b' - b\|$; we also have that $\rho(x) \leq \|\Delta\| \max\{1, \|x\|\}$.

Assumption 3. Regularity: $b \in \text{int}\{A(C) + K\}$ and singular otherwise.

The required regularity assumption is similar to strict consistency of [32], which states that $0 \in \text{int}(\text{dom})(G)$, e.g. there exists u, v such that all inequality constraints $f < 0$ hold strictly.

Finally, having introduced the homogenized system Q and the perturbation Δ, we state the required theorem: $Q' = Q + \Delta$ is the perturbed augmented system.

Theorem 5 (Linear system stability (Theorem 1, [31]))**.** *Suppose that the system Eq. (27) is regular. Then Q is surjective, $\|Q^{-1}\| < +\infty$, and if $\|Q^{-1}\| \|\Delta\| < 1$, then is also regular (hence solvable), with $\|Q'^{-1}\| \leqq \|Q^{-1}\| / (1 - \|Q^{-1}\| \|\Delta\|)$. Further, if G' denotes the solution set of Eq. (28), then for any $x \in C$ with $\|Q'^{-1}\| \rho(x) < 1$ we have*

$$d(x, G') \leqq \left[\frac{\|Q'^{-1}\| \rho(x)}{1 - \|Q'^{-1}\| \rho(x)} \right] (1 + \|x\|) \tag{29}$$

We now use these results to prove consistency.

Proof. Proof of Theorem 3

We consider the proposed support function estimator which conducts a grid search over an ϵ-covering, $\mathcal{E}_w$ covering $\mathbb{R}_+^{|\mathcal{S}|}$ such that $\mathcal{E}_w$ is the smallest set satisfying that $\min_{w':} \|w - w'\|_1 \leq \epsilon \ \forall w$ s.t. $w^\top \vec{1}$. For simplicity, we consider the self-normalized version of the estimator that constrains $w^\top \vec{1} = 1$: convergence for the stochastic constraint $\mathbb{E}[w] = 1$ holds by additionally conditioning on the event $\|p_b^\infty(s)\|_1 \leq \epsilon$ under fast geometric convergence of the stationary distribution.

The discretization-based support function estimator is:

$$\hat{\bar{R}}_e = \max \left\{ \mathbb{E}_n[w(s)\Phi(s)] \ : \ w \in \mathcal{E}_w, \ w \in \hat{\Theta}, w^\top \vec{1} = 1 \right\}$$

$$\text{where } \hat{\Theta} := \{w \in \mathcal{E}_w \colon \sum_{j,a} \hat{h}_{j,a,k}(w) g_k(a \mid j) - w(k) = 0, \ \forall k \in \mathcal{S}; g \in \tilde{\mathcal{B}}\}$$

$$\hat{h}_{j,a,k}(w) := \hat{p}_b^{(\infty)}(j, a \mid k) w(j) \pi_e(a \mid j).$$

We will bound the approximation error

$$\left\| \hat{\bar{R}}_e - \overline{R}_e \right\| \leq \left\| \hat{\bar{R}}_e - \hat{\tilde{R}}_e \right\| + \left\| \hat{\tilde{R}}_e - \tilde{\bar{R}}_e \right\| + \left\| \tilde{\bar{R}}_e - \overline{R}_e \right\|$$

with respect to the intermediary terms,

$$\hat{\tilde{R}}_e = \max \left\{ \mathbb{E}_n[w(s)\Phi(s)] \ : \ w \in \mathcal{E}_w, \ w \in \Theta, w^\top \vec{1} = 1 \right\}$$

$$\tilde{\bar{R}}_e = \max \left\{ \mathbb{E}[w(s)\Phi(s)] \ : \ w \in \mathcal{E}_w, w \in \Theta, w^\top \vec{1} = 1 \right\}$$

$$\overline{R}_e = \max \left\{ \mathbb{E}[w(s)\Phi(s)] \ : \ w \in \Theta, w^\top \vec{1} = 1 \right\}$$

Bounding $\left\|\hat{\bar{R}}_e - \hat{\tilde{R}}_e\right\|$ follows by verifying stability of the feasibility problem $\{\sum_{j,a} \hat{h}_{j,a,k}(w)g_k(a \mid j) - w(k) = 0, \forall k \in \mathcal{S}\}$ for every such value of w, taking a union bound over the covering, and then applying stability of w for a given feasible g to bound the objective values.

$G(w), \hat{G}(w)$ correspondingly denote the solution sets (in the space of g) of the feasibility program, for a fixed w vector. We first bound $\left\|\hat{\bar{R}}_e - \hat{\tilde{R}}_e\right\|$ by bounding the distance between the solution set for $G(w), \hat{G}(w)$ for a fixed w. We then apply the bound for every w on a covering of the $|\mathcal{S}|$ simplex.

To map the problem quantities to the stability analysis notation, let the set of unperturbed constraints be $C := \{x \in \mathcal{B}\}$. Observe that $\|Q'^{-1}\| = \max_{\|y\| \leq 1}\{\|x\| : Q'x = y, x \in C\} \leq |\mathcal{S}|^2|\mathcal{A}|\nu$ because the *unperturbed* constraints include bounds constraints on g.

Bounding $\|\Delta\|$ proceeds by observing that since the perturbation matrix to the coefficients comprises of terms

$$\sum_{j,a} w(j)\pi_e(a \mid j)(\hat{p}_b^\infty(j, a \mid k') - p_b^\infty(j, a \mid k')),$$

applying a crude bound that $w(j), \pi_e(a \mid j) \leq 1$, it is sufficient to bound the operator norm of perturbations as

$$\|\Delta\|_1 \leq \|\hat{p}_b^\infty(s, a \mid s') - p_b^\infty(s, a \mid s')\|_\infty.$$

Consistency of the empirical state-action probabilities for the population state-action probabilities yields the result, e.g. that $\hat{p}_b^\infty(s, a, s') \to_p p_b^\infty(s, a, s')$, and $\hat{p}_b^\infty(s) \to_p p_b^\infty(s)$. (See [20] for quantitative rates). Consider n large enough so that the condition $\|Q'^{-1}\| \rho(x) < \frac{1}{2}$ holds. Then,

$$d(\hat{x}, G) \leq \left[\frac{\|Q'^{-1}\|\rho(x)}{1 - \|Q'^{-1}\| \rho(x)}\right](1 + \|x\|) \leq 2(1 + |\mathcal{S}|^2|\mathcal{A}|\nu)^2 \|\hat{p}_b^\infty(s, a \mid s') - p_b^\infty(s, a \mid s')\|_\infty$$

(30)

for any $\hat{x} \in \hat{G}$, with probability 1; and analogously for $d(x, \hat{G})$. The above bound holds for a fixed w.

We next bound $\left\|\hat{\bar{R}}_e - \hat{\tilde{R}}_e\right\|$. Taking a union bound over finitely many $w \in \mathcal{E}_w$ so that the bound (30) holds uniformly over $\mathcal{E}_w$, there exists n_1 large enough such that

$$\max_{w \in \mathcal{E}_w} d(\hat{G}(w), G(w)) \leq \delta.$$

(31)

This in turn suggests finite-time identification of the feasible set, $\hat{\Theta}(\mathcal{E}_w) = \Theta(\mathcal{E}_w)$, so that $\hat{\bar{R}}_e = \hat{\tilde{R}}_e$.

We next bound $\left\|\hat{\tilde{R}}_e - \tilde{\tilde{R}}_e\right\|$. For every ϵ', by geometric convergence of the stationary distribution $\hat{p}_b^\infty(s) - p_b^\infty(s)$ and taking a union bound over $w \in \mathcal{E}_w$ to establish uniform convergence, there exists n_2 large enough so that:

$$\left\|\hat{\tilde{R}}_e - \tilde{\tilde{R}}_e\right\| \leq \max_{w \in \Theta}\{\mathbb{E}_n[w(s)\Phi(s)] - \mathbb{E}[w(s)\Phi(s)]\} \leq \epsilon'.$$

Lastly, $\left\|\tilde{\bar{R}}_e - \bar{R}_e\right\|$ is bounded by the uniform approximation error which is satisfied by the definition of the covering, and bounded state rewards $\Phi(s)$. Therefore, we have that for some $n \geq \max\{n_1, n_2\}$,

$$\left\|\hat{\bar{R}}_e - R_e\right\| \leq \epsilon' + \epsilon$$

Therefore we obtain statistical consistency as we take the discretization width $\epsilon \to 0$ as $n \to \infty$.

$\square$

D.1 Linear function approximation

Note that since the tabular case is a special case of linear function approximation of the state space, our approach also handles the case where $w = \theta^\top s$ is a linear parameter of the state observation.

We introduce some simplifications. $\Psi_k^{i:t,t+1} = (\phi(s_{t+1}^i)\phi(s_t^i)^\top)_k$ is the kth row vector (resp. $\Psi_k^{t,t}$) and

$$\bar{\phi}(s) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \phi(s_t^i) = \mathbb{E}_N \mathbb{E}_T[\phi(s)]$$

$$\bar{\Psi}^{t+1,t+1} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \phi(s_{t+1}^i)\phi(s_t^i)^\top$$

The corresponding finite-sample feasibility oracle for the case $w = \theta^\top s$, drawing on the least-squares representation of Proposition 4, is:

$$F(\theta) := \min_{g \in \mathcal{B}, z \geq 0} \sum_k z_k$$

$$\text{s.t. } z_k \geq \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\pi_{i,t}^e \Psi_k^{i:t,t+1}\theta \cdot g_{i,t} - \Psi_k^{i:t+1,t+1}\theta\right),, \qquad \forall k \in [d]$$

$$z_k \geq -\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\pi_{i,t}^e \Psi_k^{i:t,t+1}\theta \cdot g_{i,t} - \Psi_k^{i:t+1,t+1}\theta\right), \qquad \forall k \in [d]$$

$$l_{i,t} \leq g_{i,t} \leq m_{i,t}, \quad \forall i \in [N], t \in [T]$$

$$\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbb{I}[A_{it} = a']g_{i,t} = 1, \forall a'$$

To obtain bounds, using $\mathbb{E}_N \mathbb{E}_T$ as shorthand notation for empirical expectations over trajectories and timesteps within trajectories, one then solves:

$$\inf/\sup \ \{\mathbb{E}_N \mathbb{E}_T[\Phi(s)]^\top \theta \colon F(w) \leq 0, \ \mathbb{E}_N \mathbb{E}_T[\phi(s)]^\top \theta\}. \tag{32}$$

since $\mathbb{E}_N \mathbb{E}_T[w(s)] = \mathbb{E}_N \mathbb{E}_T[\phi(s)^\top \top \theta] = \mathbb{E}_N \mathbb{E}_T[\phi(s)]^\top \theta$.

Finally we remark on additional difficulties in the linear function approximation setting that are not encountered in the tabular setting. The computational burden increases because the reparametrization argument of Proposition 2 crucially relied on the discrete distribution of S. In practice, introducing a number of bilinear variables which grows with n performs quite poorly in comparison to the re-parametrization. Furthermore, the approach *requires* well-specification of the linear function class for w, since we effectively require realizability of a linear parameter for some set of possible inverse propensity weights. Handling model misspecification via a feasibility relaxation for the approximation error of the function class may therefore tradeoff sharpness; we leave this for future work.

E   Additional Empirics and Details

Confounded gridworld   In Figure 7 we include a depiction of the 3x3 gridworld as described in the main text describing the reward structure in greater detail. (If an action moves an agent into a wall, it simply remains in place).

SDP relaxation   One possibility is to consider the standard SDP relaxation for nonconvex quadratic programs [24]. Let $x = \begin{bmatrix} g \\ w \end{bmatrix}$, and P_k be the matrix that generates the quadratic form $\sum_{a,j} w(j) p_b^\infty(s,a,s) g_k(a \mid j) \pi_e(a \mid j)$, e.g. we have that $P^k_{(k,a,j),|\mathcal{S}|^2|\mathcal{A}|+j} = \pi_e(a \mid j) p_b^\infty(s,a,s), \forall a,j$. Then a standard relaxation gives that the solution to the following semidefinite program is a lower bound:

$$\max \left\{ \mathbb{E}[w(s)\Phi(s)] \colon \operatorname{Tr}(XP_k) - w(k)p_b^\infty(k) = 0, \forall k; \ g \in \tilde{\mathcal{B}}, \mathbb{E}_b[w(s)] = 1, \begin{bmatrix} X & x \\ x^\top & 0 \end{bmatrix} \succeq 0 \right\}$$

However, the lifting results in a $(|\mathcal{S}|^2|\mathcal{A}|)$ square matrix; given that under mild assumptions, typical SDPs can be solved in $O(n^2 m^{5/2} \log(\epsilon^{-1}))$ where n is dimension of the vector variable and m is dimension of the matrix variable, we incur an intractable $O(|\mathcal{S}|^9 |\mathcal{A}|^{9/2})$ scaling overall.

Details for confounded random walk   Note that the stationary distribution under π_b is, for s_1, s_2:

$$\left(\frac{-1 + (1 - 2\pi_{s_2 u_1})(p_{u_1} + p_{u_2}) + \pi_{s_2 u_1}}{-1 + (\pi_{s_2 u_2} - \pi_{s_1 u_1})(1 - 2p_{u_1} - 2p_{u_2})}, \frac{p_{u_1} + p_{u_2} + \pi_{s_1 u_1}(1 - 2p_{u_1} - 2p_{u_2})}{1 + (\pi_{s_1 u_1} - \pi_{s_2 u_1})(1 - 2p_{u_1} - 2p_{u_2})} \right)$$

Fig. 2a vs. global optimization   We compare the results from using Fig. 2a (nonconvex projected gradient descent) vs. solving to full optimality by Gurobi. We find that empirically, storing previous solutions and imposing monotonicity (e.g. for any Γ, taking the max of all previous returned values and the computed values) helps stabilize the optimization.

Computational details   A complete description of the data collection process, including sample size.
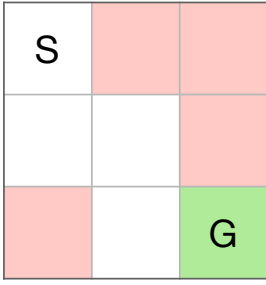
Figure 7: 3x3 gridworld.
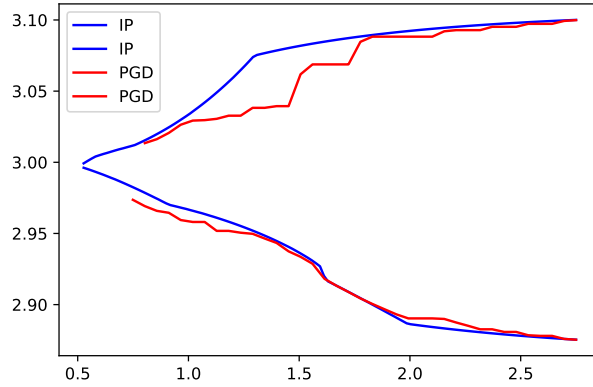$\Phi(s)$ of green is 1, red -0.3.

Figure 8: Comparison of global optimization and Fig. 2a.

- Data: bounds computed based on a trajectory with 40000 steps, and a grid of 25 linearly spaced Γ values from $\log(\Gamma) \in [0.1, 1.7]$ (equivalently, $\Gamma \in [1.10, 5.47]$).

- Experiments were run on a Macbook Pro with 16gb RAM.

- Packages: Python (numpy/scipy/pandas), Gurobi Version 9