

June 1, 2020

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We, the authors of the paper “Smoothness-Adaptive Contextual Bandits,” hereby certify that the paper and the authors meet all eligibility criteria for the APS student paper competition. In particular, we affirm that

1. Ahmadreza Momeni has been a PhD candidate on or after January 1, 2020
2. The research has been conducted while Ahmadreza Momeni was a student
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4. There is only one entrant among the authors of the paper, who is Ahmadreza Momeni, and he has not been a co-author for any other paper submitted to the competition.
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7. Ahmadreza Momeni will be available to present the work Sunday afternoon Nov 8 2020 at the INFORMS National Meeting in National Harbor, MD (assuming coronavirus issues do not disrupt the conference).

Sincerely,

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Smoothness-Adaptive Contextual Bandits

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We study a non-parametric multi-armed bandit problem with stochastic covariates, where a key complexity driver is the smoothness of payoff functions with respect to covariates. Previous studies have focused on deriving minimax-optimal algorithms in cases where it is a priori known how smooth the payoff functions are. In practice, however, the smoothness of payoff functions is typically not known in advance, and misspecification of smoothness may severely deteriorate the performance of existing methods. In this work, we consider a framework where the smoothness of payoff functions is not known, and study when and how algorithms may adapt to unknown smoothness. First, we establish that designing algorithms that adapt to unknown smoothness of payoff functions is, in general, impossible. However, under a self-similarity condition (which does not reduce the minimax complexity of the dynamic optimization problem at hand), we establish that adapting to unknown smoothness is possible, and further devise a general policy for achieving smoothness-adaptive performance. Our policy infers the smoothness of payoffs throughout the decision-making process, while leveraging the structure of non-adaptive off-the-shelf policies. We establish that for problem settings with either differentiable or non-differentiable payoff functions this policy matches (up to a logarithmic scale) the regret rate that is achievable when the smoothness of payoffs is known a priori.

Key words: Contextual multi-armed bandits, Hölder smoothness, self-similarity, non-parametric confidence intervals, non-parametric estimation, experiment design

1. Introduction

A well-studied dynamic optimization framework that captures the trade-off between new information acquisition (exploration) and optimization of payoffs based on available information (exploitation), is the multi-armed bandit (MAB) framework, originated by the work of Thompson (1933) and Robbins (1952). An important generalization of this framework where the decision maker also has access to covariates that can be informative about the effectiveness of different actions, is typically referred to as the contextual MAB problem (Woodroffe 1979).¹ The contextual MAB framework has been applied for analyzing sequential experimentation in many application domains, including pricing (e.g., Cohen et al. 2016, Qiang and Bayati 2016, Ban and Keskin 2019, Bastani et al. 2019, Javanmard and Nazerzadeh 2019, Wang et al. 2019), product recommendations (e.g., Chu et al. 2011, Chandrashekar et al. 2017, Bastani et al. 2018, Agrawal et al. 2019, Gur and Momeni 2019, Kallus and Udell 2020), and healthcare (e.g., Tewari and Murphy 2017, Chick et al. 2018, Zhou et al. 2019, Bastani and Bayati 2020).

Following Woodroffe (1979), most of the analysis of contextual MAB problems assumes a parametric (usually linear) model for the payoff functions that are associated with different actions; see, e.g., Goldenshluger and Zeevi (2013) and Bastani and Bayati (2020) for some notable results. Recently, however, there has been a growing interest in studying *non-parametric* contextual MAB formulations, which make less structural assumptions, are typically more robust, and can be applied to a more general class of problems, especially when less is known about the structure of payoff functions. One of the main findings of this line of work is that, in non-parametric MAB formulations, the smoothness of the payoff functions is a key driver of the difficulty of the dynamic optimization problem at hand (Rigollet and Zeevi 2010, Perchet and Rigollet 2013, Hu et al. 2019). Qualitatively, the smoother the payoff functions are, the further one may extrapolate payoff patterns over the covariate space—and the less exploration one requires in order to guarantee good performance.

We next illustrate this phenomenon through the problem of artwork selection on Netflix. When Netflix recommends a title, it also needs to select an image to display along with the recommendation. Different images may induce different probabilities to play the movie. Given the personal viewing history of the user, for each recommended title Netflix aims to select imagery that maximizes the probability of playing that title. A simple version of this problem is described in Chandrashekar et al. (2017) where two different artworks are available for the movie *Good Will Hunting* (see the top parts of Figure 1).

In the bottom plots of Figure 1 we illustrate two different scenarios of how the probability to play the title changes as a function of a single context for two artwork options: image A and image B.² Let x^*

¹ Other terms that are used in the literature include bandit problem with side observations (Wang et al. 2005), and associative bandit problem (Strehl et al. 2006).

² For example, this context may equal a normalized difference between the viewer’s romance and comedy scores, measures that Netflix computes based on the viewing history of users; see Chandrashekar et al. (2017) for more details.

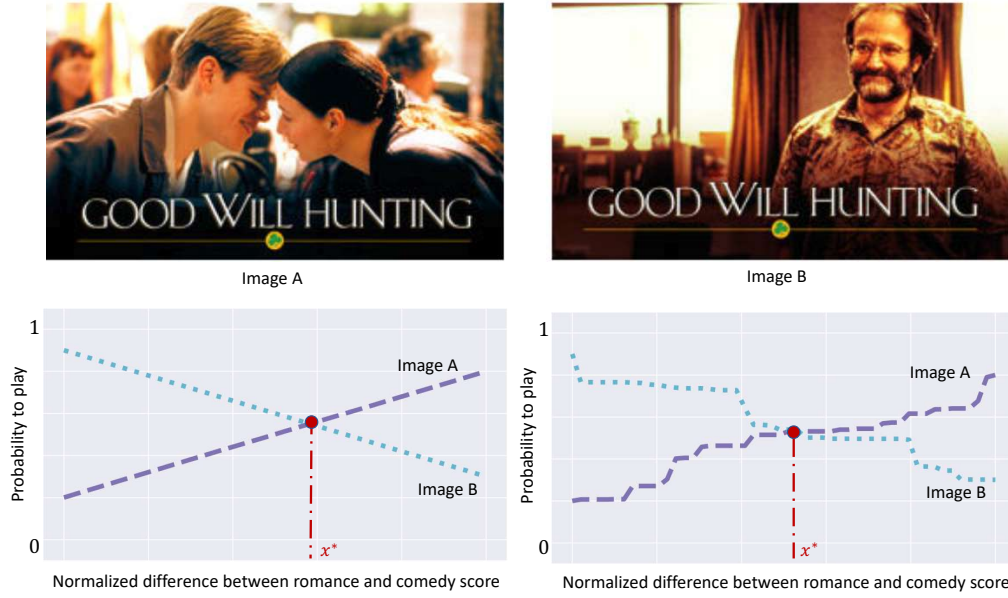


Figure 1 Top: Example of artwork selection on Netflix for recommending the movie *Good Will Hunting* (for details and discussion see Chandrashekar et al. 2017); Bottom: The probability of users to play the recommended title as a function of the normalized difference of their romance and comedy score (context) when either image A (dashed line) or image B (dotted line) is shown, in two different scenarios: (Bottom Left) users’ behavior changes linearly as a function of the context; Bottom Right: users’ behavior changes more abruptly as a function of the context. In each case x^* denotes the context at which the optimal imagery switches.

denote the context at which the optimal imagery switches. Particularly, for contexts that belong to the interval $[0, x^*]$ image B is the optimal imagery to display; otherwise, the optimal artwork selection is image A. In the scenario illustrated on the bottom-left part of Figure 1, users’ behavior is smooth and changes linearly with respect to the context. In this case, any observation of a user’s behavior, even when the context is not close to x^* , is informative and can be utilized for estimating the probability lines and the crossing point x^* . In contrast, the bottom-right part of Figure 1 depicts a scenario where the probability to play each title changes more abruptly as a function of the context. In this case, observations with contexts that are not close to x^* are less informative and cannot be easily utilized for estimating the crossing point x^* . As a result, the second scenario requires more experimentation over the context space in order to determine optimal decision regions. When payoffs are not monotone or smooth functions of the contexts, optimal decisions regions might be non-convex and complex to identify, and required experimentation rates further increase.

Previous studies of non-parametric contextual MAB problems typically assume prior knowledge of the worst-case smoothness of payoff functions. A standard approach is to assume that payoff functions are (β, L) -Hölder (see Definition 2.1) for some *known* parameters β and L , and develop policies that

are predicated on this assumption. For example, Rigollet and Zeevi (2010) and Perchet and Rigollet (2013) develop minimax rate-optimal algorithms when payoff functions are assumed to be Lipschitz or “rougher” (that is, when $0 < \beta \leq 1$); more recently, Hu, Kallus, and Mao (2019) extended this analysis to the “smoother” case (where $\beta > 1$).

In practice, however, the class of functions to which payoff functions belong is often unknown, and misspecification of smoothness may cause significant deterioration in the performance of existing methods (see Example 2.1 in §2.2). While underestimating the smoothness of payoff functions leads to excessive and unnecessary experimentation, overestimating the smoothness might lead to insufficient experimentation; both cases may result in poor performance relative to the one that could have been achieved with accurate information on the smoothness. The focus of this paper is in studying when and how algorithms may *adapt* to unknown smoothness, in the sense of achieving, without prior knowledge of the smoothness of payoffs, the best performance that is achievable when smoothness is a priori known.

Main contributions. Our contributions are in (1) formulating a non-parametric contextual MAB problem where the smoothness of payoff functions is a priori unknown; (2) analyzing the complexity of adapting to smoothness, and establishing that smoothness-adaptivity is in general impossible; and (3) identifying a self-similarity condition that makes it possible to achieve smoothness-adaptivity, and devising a general policy that leverages this condition to guarantee rate-optimality without prior information on the smoothness of payoffs. More specifically, our contribution is along the following lines.

(1) *Modeling.* We formulate a non-parametric contextual MAB problem where the smoothness of payoff functions is a priori unknown: the payoff functions are assumed to belong to a Hölder class of functions with some unknown Hölder exponent. We identify a policy as *smoothness-adaptive* if for any problem instance it guarantees the optimal regret rate as a function of the Hölder exponent that characterizes that instance, up to a multiplicative term that is poly-logarithmic in the horizon length and a multiplicative constant that may depend on other problem parameters (such as the dimension of the covariate space); see Definition 2.3. In that sense, smoothness-adaptive policies guarantee (up to a logarithmic factor) the minimax regret that characterizes the achievable performance when the smoothness parameter is a priori known. Our formulation allows for any arbitrary range of this smoothness parameter, and thus captures a large variety of real-world phenomena.

(2) *Impossibility of adaptation.* We establish a lower bound on the best achievable performance when two different classes of payoff functions (characterized by two different smoothness exponents) are considered simultaneously. Through this lower bound we show that adaptively achieving rate-optimal performance uniformly over different classes of smooth payoff functions is impossible. In that sense, adapting to unknown smoothness carries non-trivial cost in sequential experimentation. This is despite the fact that smoothness-adaptive estimation of non-parametric functions is possible (see, e.g., Lepskii

1992). Thus, this impossibility result highlights the fundamental difference between the complexities of non-parametric function estimation and non-parametric contextual MAB.

(3) *Smoothness-adaptivity and policy design under self-similar payoffs.* To advance beyond the general impossibility of adapting to unknown smoothness, we turn to consider smoothness-adaptivity when payoff functions are *self-similar*. Self-similarity has been used for studying adaptivity problems in the statistics literature; for example, while constructing smoothness-adaptive confidence bands is impossible in general (Low 1997), it becomes possible under a self-similarity assumption (Giné and Nickl 2010).

While we establish that self-similarity does not reduce the minimax complexity of the problem at hand, we show that, when payoffs are self-similar, it is possible to design smoothness-adaptive policies. We further devise a general policy that, under self-similarity, guarantees rate-optimal performance without prior information on the smoothness of payoffs. Our policy efficiently estimates the smoothness of self-similar payoff functions throughout the sequential decision process, while leveraging the structure of effective off-the-shelf non-adaptive policies that are designed to perform well under accurate smoothness specification. We establish that, when paired with off-the-shelf input policies that guarantee the optimal regret rate under accurate smoothness specification, our policy guarantees (up to a logarithmic factor) the latter regret rate without any prior information on the smoothness of payoffs. We demonstrate our approach by leveraging non-adaptive policies designed for payoff functions that are at most Lipschitz-smooth (Perchet and Rigollet 2013) and at least Lipschitz-smooth (Hu et al. 2019) to guarantee rate-optimal performance without prior information on the underlying payoff smoothness.

1.1. Related literature

Parametric and non-parametric approaches in contextual MAB. Most of the literature on contextual bandits assume parametric payoff functions. Some researchers have studied this setting when contexts are independently drawn from identical distribution. For example, Goldenshluger and Zeevi (2013), Bastani et al. (2017), and Bastani and Bayati (2020) consider linear payoff functions. On the other hand, Langford and Zhang (2008) and Dudik et al. (2011) study the problem of finding the best mapping from contexts to arms among a finite set of hypotheses. In addition, Wang et al. (2005) considers a general relationships between the parameters of payoff functions and contexts. In contrast to these studies, some other papers consider settings with contexts that are selected by an *adversary*; see Bubeck and Cesa-Bianchi (2012) and references therein.

In addition to these parametric approaches, the contextual MAB problem has also been addressed from a *non-parametric* point of view to account for general relationships between covariates and mean rewards. Yang and Zhu (2002), which initiated this line of research, combined an ϵ -greedy-type policy with non-parametric estimation methods such as nearest neighbors to achieve *strong consistency*. This solution concept ensures that the total reward collected by the agent is almost surely asymptotically equivalent to those obtained by always pulling the best arm. Following this work, stronger results

have been established for the regret rate. Rigollet and Zeevi (2010) introduced the UCBogram policy which decomposes the covariates space into bins and follows a traditional UCB policy in each bin separately. Perchet and Rigollet (2013) improved upon this result by introducing the Adaptively Binned Successive Elimination (ABSE) policy, which implements an increasing refinement of the covariate space and achieves the minimax regret rate. Recently, Hu et al. (2019) extended this framework to the case of smooth differentiable functions. Finally, Reeve et al. (2018) proposes a kNN-UCB policy that achieves the minimax regret rate and also adapts to the intrinsic dimension of data. All these studies, however, assume that the smoothness of the payoff functions is known a priori.

Adaptive non-parametric methods. For the general theory on adaptive non-parametric estimation we refer the readers to Lepskii (1992). This line of research includes various approaches: For example, Donoho and Johnstone (1994), Donoho et al. (1995), and Juditsky (1997) deploy techniques based on wavelets, Lepski et al. (1997) proposes a kernel-based method, and Goldenshluger and Nemirovski (1997) develops a method that is based on local polynomial regression.

A related line of research studies the construction of adaptive non-parametric confidence intervals. The work of Low (1997) showed that, in general, it is impossible to construct adaptive confidence bands simultaneously over different classes of Hölder functions; for recent results on the impossibility of adaptive confidence intervals, see Armstrong and Kolesár (2018) and references therein. Following that work, several studies have focused on identifying conditions under which adaptive confidence band construction is feasible. A well-studied condition is the one of *self-similarity*, which was first used in this context by Picard and Tribouley (2000) using wavelet methods for point-wise purposes. Later on, self-similarity was also used by Giné and Nickl (2010) to construct confidence bounds over finite intervals. Aside from these two works, self-similarity has also been used in a variety of other non-parametric problems and applications, including high-dimensional sparse signal estimation (Nickl and van de Geer 2013), binary regression (Mukherjee and Sen 2018), and L_p -confidence sets (Nickl and Szabó 2016), etc.

In a contextual MAB setting, Qian and Yang (2016) were the first to consider a self-similarity condition for establishing performance guarantees without precise smoothness knowledge in a non-differentiable case (with $0 < \beta \leq 1$). We note that even in that case the method they provide is not smoothness-adaptive in the sense that is defined in the current paper (particularly, the optimality gap they suggest scales exponentially with the context dimension). In that respect, one may view the current paper as further grounding self-similarity as an important condition for adapting to unknown smoothness through: (i) establishing that, in general, smoothness-adaptive policy design is impossible without imposing additional conditions; (ii) showing that self-similarity assumption does not reduce the minimax complexity of the problem; and (iii) showing that self-similarity can be leveraged in a new way that allows the design of smoothness-adaptive policies for a general class of problems.

2. Problem formulation

We next formulate the non-parametric contextual MAB problem with unknown smoothness. §2.1 includes main modelling assumptions. In §2.2 we discuss and illustrate the performance reduction that is caused by misspecifying the smoothness in existing methods. In §2.3 we formalize the adaptivity notion that is used as a policy design goal in the analysis that will follow.

Reward and feedback structure. Let $\mathcal{K} = \{1, 2\}$ be a set of actions (arms) and let $\mathcal{T} = \{1, \dots, T\}$ denote a sequence of decision epochs.³ At each time period $t \in \mathcal{T}$, a decision maker observes a context $X_t \in [0, 1]^d$ that is realized according to an unknown distribution \mathbf{P}_X , and then selects one of the two actions. When selecting an action $k \in \mathcal{K}$ at time $t \in \mathcal{T}$, a reward $Y_{k,t} \sim \mathbf{P}_{Y|X}^{(k)}$ is realized and observed such that $Y_{k,t} \in \{0, 1\}$, where $\mathbf{P}_{Y|X}^{(k)}$ denotes the payoff distribution conditional on the context X_t and the selected action k . Equivalently, the rewards $Y_{k,t}$ may be expressed as follows:

$$Y_{k,t} = f_k(X_t) + \epsilon_{k,t},$$

where $f_k(X_t) = \mathbb{E}[Y_{k,t} | X_t]$ and $\epsilon_{k,t}$ is a random variable such that $\mathbb{E}[\epsilon_{k,t} | X_t] = 0$. The conditional distributions $\mathbf{P}_{Y|X}^{(k)}$ and the payoff functions f_k are assumed to be unknown.

Admissible policies. Let U be a random variable defined over probability space $(\mathbb{U}, \mathcal{U}, \mathbf{P}_u)$. Let $\pi_t : [0, 1]^{d \times t} \times [0, 1]^{t-1} \times \mathbb{U} \rightarrow \mathcal{K}$ for $t = 1, 2, 3, \dots$ be a sequence of measurable functions given by

$$\pi_t = \begin{cases} \pi_1(X_1, U) & t = 1, \\ \pi_t(X_t, \dots, X_1, Y_{t-1}, \dots, Y_1, U) & t = 2, 3, \dots \end{cases}.$$

(We abuse notation by also denoting the action at time t by $\pi_t \in \mathcal{K}$.) The mappings $\{\pi_t; t = 1, \dots, T\}$ and the distribution \mathbf{P}_u together define the class of admissible policies, denoted by Π .

Performance. For a problem instance $\mathbf{P} = (\mathbf{P}_X, \mathbf{P}_{Y|X}^{(1)}, \mathbf{P}_{Y|X}^{(2)})$ let $\pi^*(\mathbf{P}) = (\pi_t^*(\mathbf{P}); t = 1, 2, \dots)$ denote the oracle rule, which under knowledge of the problem instance \mathbf{P} (including the functions f_k), prescribes at each period t the best action given the realized context X_t ; that is, $\pi_t^*(\mathbf{P}) = \arg \max_{k \in \mathcal{K}} f_k(X_t)$ for all $t \in \mathcal{T}$. The performance of a policy $\pi = \{\pi_t; t = 1, \dots, T\}$ is measured in terms of expected regret relative to the oracle performance:

$$\mathcal{R}^\pi(\mathbf{P}; T) := \mathbb{E}^\pi \left[\sum_{t=1}^T f_{\pi_t^*(\mathbf{P})}(X_t) - f_{\pi_t}(X_t) \right].$$

A prominent characteristic of a given problem instance \mathbf{P} that directly impacts achievable regret, is the smoothness with which the payoffs functions f_1 and f_2 , and correspondingly, the conditional distributions $\mathbf{P}_{Y|X}^{(k)}$ and $\mathbf{P}_{Y|X}^{(2)}$, vary over the covariate space. This characteristic is formulated in §2.1, along with other key model assumptions.

³ We focus here on a setting with two actions only to simplify exposition and analysis; we expect all the results to hold for any action set of finite cardinality.

2.1. Model assumptions

We next detail our main model assumptions, which are conventional in the non-parametric contextual MAB literature (see, e.g., Perchet and Rigollet 2013). Our first model assumption addresses the smoothness of payoff functions. Before advancing it, we first formalize how payoff functions can change as a function of the covariates using Hölder smoothness. For any multi-index $s = (s_1, \dots, s_d) \in \mathbb{N}^d$ and any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define $|s| = \sum_{i=1}^d s_i$, $s! = s_1! \dots s_d!$, $x^s = x_1^{s_1} \dots x_d^{s_d}$, and $\|x\| = (x_1^2 + \dots + x_d^2)^{\frac{1}{2}}$. Let D^s denote the differential operator $D^s := \frac{\partial^{s_1 + \dots + s_d}}{\partial x_1^{s_1} \dots \partial x_d^{s_d}}$. Let $\beta > 0$. Denote by $\lfloor \beta \rfloor$ the maximal integer that is strictly less than β , e.g., $\lfloor 1 \rfloor = 0$. For any $x \in \mathbb{R}^d$ and any $\lfloor \beta \rfloor$ times continuously differentiable function $g(\cdot)$ on \mathbb{R}^d , we denote by g_x its Taylor expansion of degree $\lfloor \beta \rfloor$ at point x : $g_x(x') := \sum_{|s| \leq \lfloor \beta \rfloor} \frac{(x-x')^s}{s!} D^s g(x)$.

DEFINITION 2.1. The Hölder class of functions $\mathcal{H}_{\mathcal{X}}(\beta, L)$ for the parameters $\beta > 0$ and $L > 0$ and the set $\mathcal{X} \subseteq \mathbb{R}^d$ is defined as the set of functions $f : \mathcal{X} \rightarrow \mathbb{R}$ that are $\lfloor \beta \rfloor$ times continuously differentiable, and for any $x, x' \in \mathcal{X}$, satisfy the following inequality:

$$|f(x') - f_x(x')| \leq L \|x - x'\|_{\infty}^{\beta}.$$

Furthermore, let $\mathcal{H}_{\mathcal{X}}(\beta) := \bigcup_{0 \leq L < \infty} \mathcal{H}_{\mathcal{X}}(\beta, L)$. We drop the indication \mathcal{X} whenever $\mathcal{X} = [0, 1]^d$.

ASSUMPTION 2.1 (Smoothness). The payoff functions f_k , $k \in \mathcal{K}$, belong to the Hölder class of functions $\mathcal{H}(\beta, L)$ for some $L > 0$ and $\beta \in [\underline{\beta}, \bar{\beta}]$ with $0 < \underline{\beta} \leq 1$.

Our second assumption requires the distribution of contexts to be bounded from above and away from zero. Consequently, in every region of the covariate space, sufficiently many samples can be collected in order to estimate the payoff functions.

ASSUMPTION 2.2 (Covariate distribution). The distribution \mathbf{P}_X is equivalent to the Lebesgue measure on $[0, 1]^d$, that is, there exist constants $0 < \underline{\rho} \leq \bar{\rho}$ such that p_X , the density of \mathbf{P}_X , satisfies $\underline{\rho} \leq p_X(x) \leq \bar{\rho}$ for all $x \in [0, 1]^d$.

Our third assumption, known as the *margin* condition, captures the interplay between the payoff functions and the covariate distribution.

ASSUMPTION 2.3 (Margin condition). There exist some $\alpha > 0$ and $C_0 > 0$ such that

$$\mathbf{P}_X \{0 < |f_1(X) - f_2(X)| \leq \delta\} \leq C_0 \delta^{\alpha}, \quad \forall 0 < \delta \leq 1.$$

The mass of covariates near the decision boundary is a key complexity driver: the larger the parameter α , the faster this mass shrinks near the boundary, and the hardness of the problem reduces. Together, the above three assumptions characterize the general class of problems that we consider.

DEFINITION 2.2 (CLASS OF PROBLEMS). For any $\beta \geq 0$ and $\alpha \geq 0$, we denote by $\mathcal{P}(\beta, \alpha, d) = \mathcal{P}(\beta, L, \alpha, C_0, \underline{\rho}, \bar{\rho})$ the class of problems $\mathbf{P} = (\mathbf{P}_X, \mathbf{P}_{Y|X}^{(1)}, \mathbf{P}_{Y|X}^{(2)})$ that satisfy Assumption 2.1 for β and $L > 0$, Assumption 2.2 for some $\bar{\rho} \geq \underline{\rho} > 0$, and Assumption 2.3 for α and some $C_0 > 0$.

It is worth noting the relation between the smoothness condition and the margin condition. The smoothness of payoff functions also determines how they might change near the decision boundary, which affects the mass of contexts in that region. That is, smooth payoff functions (large β) implies more mass of contexts near the decision boundary (small α). This relationship is formalized in the following proposition, which is a simple extension of Proposition 3.1 in Perchet and Rigollet (2013).

PROPOSITION 2.1 (Margin condition and smoothness). *Assume that Assumption 2.1 holds with parameters (β, L) , and that Assumption 2.3 holds with parameter α . Then, the following statements hold:*

1. *If $\alpha \cdot \min\{1, \beta\} > 1$ then, a given action is always or never optimal; the oracle policy π^* dictates pulling only one of the actions all the time;*
2. *If $\alpha \cdot \min\{1, \beta\} \leq 1$ then, there exists problem instances in $\mathcal{P}(\beta, \alpha, d)$ with non-trivial oracle policies.*

Based on this proposition, when $\alpha > \frac{1}{\min\{1, \beta\}}$, the problem becomes equivalent to the classical stochastic MAB problem without covariates. Hence, we will assume $0 < \alpha \leq \frac{1}{\min\{1, \beta\}}$ in the rest of the paper.

2.2. The cost of smoothness misspecification

We next demonstrate the loss that might be incurred by existing policies when the smoothness is misspecified. When the problem instance belongs to $\mathcal{P}(\beta, \alpha, d)$, the minimax regret is:

$$\inf_{\pi \in \Pi} \sup_{P \in \mathcal{P}(\beta, \alpha, d)} \mathcal{R}^\pi(P; T) = \Theta\left(T^{\zeta(\beta, \alpha, d)}\right), \quad \text{where } \zeta(\beta, \alpha, d) = 1 - \frac{\beta(1 + \alpha)}{2\beta + d}. \quad (2.1)$$

This characterization was established by Rigollet and Zeevi (2010) and Perchet and Rigollet (2013) in the case $\beta \leq 1$. With further assumptions on the regularity of the decision regions, Hu et al. (2019) establish a similar characterization for $\beta > 1$, up to an additional multiplicative term of $(\log T)^{\frac{2\beta + d}{2\beta}}$ that appears in their upper bound.

Perchet and Rigollet (2013) provide the ABSE policy and establish that, when tuned with the correct smoothness parameter, it guarantees the minimax regret rate in (2.1) whenever $\beta \leq 1$. The design of the ABSE policy and the performance it achieves are nevertheless predicated on accurate knowledge of smoothness. The following example demonstrates that when the smoothness parameter is misspecified, the ABSE policy cannot guarantee rate-optimality anymore.

EXAMPLE 2.1 (COST OF SMOOTHNESS MISSPECIFICATION FOR ABSE). Fix a smoothness parameter $0 < \beta \leq 1$ and a margin parameter $\alpha \leq \frac{1}{\beta}$. Let $\text{ABSE}(\hat{\beta})$ denote the ABSE policy tuned by a misspecified smoothness parameter $0 < \hat{\beta} \leq 1$. Then, there exist constants $\underline{C}^{\text{ABSE}}$ and T_0 independent of T such that for all $T \geq T_0$, the followings hold:

1. If $\hat{\beta} < \beta \leq 1$, then $\sup_{P \in \mathcal{P}(\beta, \alpha, d)} \mathcal{R}^{\text{ABSE}(\hat{\beta})}(P; T) \geq \underline{C}^{\text{ABSE}} T^{\zeta(\hat{\beta}, \alpha, d)}$;
2. If $0 < \beta < \hat{\beta}$, then $\sup_{P \in \mathcal{P}(\beta, \alpha, d)} \mathcal{R}^{\text{ABSE}(\hat{\beta})}(P; T) \geq \underline{C}^{\text{ABSE}} T$.

When smoothness is underestimated, the worst-case regret rate is equal to the minimax regret rate over the class of problems with “rougher” payoff functions, and when smoothness is overestimated, the worst-case regret is linear in the horizon length. Similar results can be obtained for other policies proposed in the literature, including for the case $\beta > 1$; in §3 we provide a broad impossibility result that generalizes this example.

2.3. The smoothness-adaptive property

Next, we formalize a notion of adaptivity as our policy design goal. We say that a policy is smoothness-adaptive if for any problem instance, it achieves the optimal regret rate as a function of the Hölder exponent β that characterizes that instance, up to a multiplicative term that is poly-logarithmic in the horizon length and a multiplicative constant that may depend on other problem parameters.⁴

DEFINITION 2.3 (SMOOTHNESS-ADAPTIVE POLICIES). Fix two Hölder exponents $\underline{\beta} < \bar{\beta}$, and dimension d . Define:

$$\mathcal{P}^{\text{all}} := \mathcal{P}^{\text{all}}(\underline{\beta}, \bar{\beta}, d) = \bigcup_{\underline{\beta} \leq \beta \leq \bar{\beta}} \bigcup_{0 < \alpha \leq 1 \vee \frac{1}{\bar{\beta}}} \mathcal{P}(\beta, \alpha, d).$$

Given a family of problem instances $\mathcal{P} \subseteq \mathcal{P}^{\text{all}}$, a policy $\pi \in \Pi$ is said to be smoothness-adaptive if for any $\underline{\beta} \leq \beta \leq \bar{\beta}$ and $0 < \alpha \leq \frac{1}{\min\{1, \bar{\beta}\}}$, there exists some function $\iota(\beta, \underline{\beta}, \bar{\beta}, \alpha) > 0$ independent of d , and some constant $\bar{C} > 0$ such that

$$\sup_{\mathbf{P} \in \mathcal{P} \cap \mathcal{P}(\beta, \alpha, d)} \mathcal{R}^{\pi}(\mathbf{P}; T) \leq \bar{C} (\log T)^{\iota(\beta, \underline{\beta}, \bar{\beta}, \alpha)} T^{\zeta(\beta, \alpha, d)},$$

where the function $\zeta(\beta, \alpha, d)$ is given in (2.1).

Smoothness-adaptive policies guarantee (up to a logarithmic factor) the minimax regret that characterizes the achievable performance when smoothness parameters are a priori known.

3. Impossibility of costless adaptation to smoothness

In this section, we discuss the possibility of adapting to the smoothness of payoff functions. The objective we consider is to design policies that are smoothness-adaptive (see Definition 2.3), that is, achieve the rate of convergence detailed in (2.1) without prior knowledge of the best smoothness parameter β that characterizes all the payoff functions $\{f_k\}$. Our first key result, however, shows that this is impossible.

In the following analysis we consider a setting with a pair of smoothness parameters $0 < \beta < \gamma$, for which we know that \mathbf{P} is β -smooth, i.e., $\mathbf{P} \in \mathcal{P}(\beta, \alpha, d)$, but we do not know whether \mathbf{P} is also γ -smooth, i.e., whether $\mathbf{P} \in \mathcal{P}(\gamma, \alpha, d)$. We show that there exist pairs (β, γ) such that any admissible policy π that (nearly) achieves the optimal regret rate over the smoother class $\mathcal{P}(\gamma, \alpha, d)$, cannot simultaneously (nearly) achieve optimal rates over the rougher one. Therefore, without imposing additional requirements over the class of problems $\mathcal{P}(\beta, \alpha, d)$, no admissible policy can be *smoothness-adaptive*.

⁴ We note that a similar property has been suggested and analyzed in the context of adaptive confidence bands; see, e.g., Nickl and van de Geer (2013).

THEOREM 3.1 (Impossibility of adapting to smoothness). *Fix two Hölder exponents $0 < \beta < \gamma$ and some margin parameter $0 < \alpha \leq \max\{1, \frac{1}{\gamma}\}$. Then, for any horizon length $T \geq T_0$, with T_0 independent of T , and any admissible policy $\pi \in \Pi$ that achieves rate-optimal regret $\mathcal{O}(T^{\zeta(\gamma, \alpha, d)})$ over $\mathcal{P}(\gamma, \alpha, d)$, there exists a constant $\underline{C} > 0$ independent of T such that the followings hold:*

1. (At most Lipschitz-smooth) If $0 < \beta < \gamma \leq 1$ then,

$$\sup_{P \in \mathcal{P}(\beta, \alpha, d)} \mathcal{R}^\pi(P; T) \geq \underline{C} T^{1 - \frac{d}{\alpha(2\beta + d - \alpha\beta)}} [T^{\zeta(\gamma, \alpha, d)}]^{-\frac{d}{2\beta + d - \alpha\beta}};$$

2. (At least Lipschitz-smooth) If $\beta = 1 < \gamma$ then,

$$\sup_{P \in \mathcal{P}(1, \alpha, d)} \mathcal{R}^\pi(P; T) \geq \underline{C} T^{1 - \frac{1}{2\alpha}} [T^{\zeta(\gamma, \alpha, d)}]^{-\frac{1}{2}}.$$

Theorem 3.1 establishes a lower bound on the achievable performance over a class of problems as a function of the performance over another class of problems with smoother payoff functions. This lower bound depends on the smoothness parameters of the two considered classes of payoff functions and also the margin parameter. As the examples below illustrate, Theorem 3.1 implies that there exist pairs of smoothness parameters across which adaptivity is impossible without further assumptions.

EXAMPLE 3.1 (AT MOST LIPSCHITZ-SMOOTH). Part 1 of Theorem 3.1 can be simplified as follows:

$$\sup_{P \in \mathcal{P}(\beta, \alpha, d)} \mathcal{R}^\pi(P; T) \geq \underline{C}' T^{1 - \frac{d(\alpha+1)(2\gamma+d-\alpha\gamma)}{\alpha(2\gamma+d)(2\beta+d-\alpha\beta)}},$$

for some constant $\underline{C}' > 0$. Thus, if $\gamma = \frac{15}{100}$, $\beta = \frac{\gamma}{2}$, $\alpha = \frac{99}{100\gamma}$ and $d = 1$, the optimal regret rate with knowledge of smoothness over $\mathcal{P}(\beta, \alpha, 1)$ is $\mathcal{O}(T^{0.504348})$, while Part 1 of Theorem 3.1 establishes a lower bound of order $\Omega(T^{0.58})$ if the policy π achieves rate-optimal performance over $\mathcal{P}(\gamma, \alpha, 1)$.

EXAMPLE 3.2 (AT LEAST LIPSCHITZ-SMOOTH). Part 2 of Theorem 3.1 can be simplified as follows:

$$\sup_{P \in \mathcal{P}(1, \alpha, d)} \mathcal{R}^\pi(P; T) \geq \underline{C}' T^{1 - \frac{(2+\alpha-\alpha^2)\gamma+(\alpha+1)d}{2\alpha(2\gamma+d)}},$$

for some constant $\underline{C}' > 0$. Thus, if $\gamma > 1$, $\alpha = 1$ and $d = 1$, the optimal regret rate with knowledge of smoothness over $\mathcal{P}(1, 1, 1)$ is $\mathcal{O}(T^{\frac{1}{3}})$, while Part 2 of Theorem 3.1 establishes a lower bound of order $\Omega(T^{\frac{\gamma}{2\gamma+1}})$ if the policy π achieves rate-optimal performance over $\mathcal{P}(\gamma, 1, 1)$. Since $\frac{\gamma}{2\gamma+1} > \frac{1}{3}$ for any $\gamma > 1$, no policy can be simultaneously rate-optimal over both $\mathcal{P}(1, 1, 1)$ and $\mathcal{P}(\gamma, 1, 1)$, for $\gamma > 1$.

We note that Theorem 3.1 rules out adaptivity across some, but not necessarily all, pairs of smoothness parameters $0 < \beta < \gamma$. Understanding whether there exist some pairs over which adaptivity is possible—and, more broadly, providing a comprehensive characterization of adaptive rates over mixtures of Hölder classes—would be of considerable interest. In the current paper, however, we leave these questions to future work. In the next sections we turn our focus to payoff functions that are self-similar and show that—in this case—there exist policies that are smoothness-adaptive with considerable generality.

Key ideas in the proof of Theorem 3.1. The proof of Theorem 3.1 adapts to our framework ideas of identifying a worst-case nature “strategy,” while devising a novel construction of instances to reduce the problem to one of hypothesis testing. The proof of Part 1 of the theorem is detailed in §7; the proof of Part 2 is deferred to the appendix, together with the proofs of all subsequent results. We next illustrate the key ideas of the proof, focusing on Part 1, in the special case of $\alpha = \frac{1}{\gamma}$ and $d = 1$; the construction of the worst-case instance in this setting is depicted in Figure 2.

Fix a parameter $\Delta \leq \frac{1}{4}$. First, consider a nominal problem instance in $\mathcal{P}(\gamma, \alpha, 1)$, such that the first action’s payoff function is $\frac{1}{2}$ for every context, except for the interval $[0, 2\Delta^{\frac{1}{\gamma}}]$, where it has a “downward bump” and reaches its minimum, $\frac{1}{2} - \Delta$, and the second action’s payoff function is $\frac{1}{2}$ everywhere. Furthermore, for each $1 \leq m \leq M := \lfloor \Delta^{\frac{1}{\gamma} - \frac{1}{\beta}} \rfloor$, consider a problem instance in $\mathcal{P}(\beta, \alpha, 1)$ such that the payoff functions are equal to the aforementioned payoff functions everywhere except for the interval $I_m := [2(m-1)\Delta^{\frac{1}{\beta}}, 2m\Delta^{\frac{1}{\beta}}]$, where the first action’s payoff function has an “upward bump” and reaches its maximum, $\frac{1}{2} + \Delta$, as depicted in Figure 2. That is, for the problem m , the first action is optimal over some segment of I_m . To meet its performance guarantees over $\mathcal{P}(\gamma, \alpha, 1)$, in at least one of the intervals I_m , the number of times π selects action 1 must be “small.” We denote one such interval by I_{m^*} . Using this observation along with the fact that one can differentiate between the nominal problem described above and problem instance m^* only based on the outcomes of action 1 in the interval I_{m^*} , one may show that any admissible policy cannot distinguish between these two problem instances with strictly positive probability. This causes such policy not to select action 1 almost half of the times in which realized contexts belong to the interval I_{m^*} . Interval I_{m^*} contains a segment over which the first action is optimal for the problem m^* , which guarantees the regret bound stated in the theorem for a carefully selected value for the parameter Δ .

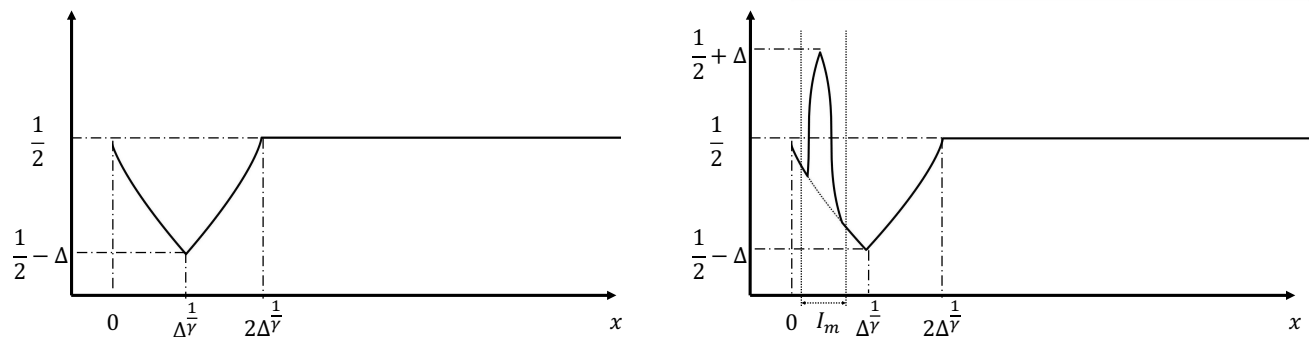


Figure 2 Description of the worst-case instance constructed in the proof of Theorem 3.1. Left: The first arm’s payoff function for the problem in $\mathcal{P}(\gamma, \alpha, 1)$; Right: The first arm’s payoff function for the problem in $\mathcal{P}(\beta, \alpha, 1)$.

4. Self-similar payoffs

In this section we first adapt in §4.1 a *self-similarity* condition that appears in the literature on non-parametric confidence bands (e.g., Picard and Tribouley 2000 and Giné and Nickl 2010), and then show in §4.2 that the assumption that payoff functions are self-similar does not reduce the minimax regret complexity of the problem at hand. Later on, in §5, we will show that self-similarity makes it possible to guarantee rate-optimality without prior knowledge of the payoff smoothness, and devise a general policy for achieving smoothness-adaptive performance.

4.1. The self-similarity assumption

Before introducing the self-similarity condition we first advance some relevant notation. For a given function $f(\cdot)$ and non-negative integers l and p , define $\mathbf{\Gamma}_l^p f(\cdot; U)$ to be the $L_2(\mathbf{P}_X)$ -projection of the function $f(\cdot)$ to the class of polynomial functions of degree at most p over the hypercube U . Formally, for any $x \in U$ we define:

$$\mathbf{\Gamma}_l^p f(x; U) := g(x), \quad \text{s.t.} \quad g = \arg \min_{q \in \text{Poly}(p)} \int_U |f(u) - q(u)|^2 K\left(\frac{x-u}{h}\right) p_X(u | U) du, \quad (4.1)$$

where we use kernel $K(\cdot) = \mathbb{1}\{\|\cdot\|_\infty \leq 1\}$ and bandwidth $h = 2^{-l}$, and $\text{Poly}(p)$ is the class of polynomials of degree at most p . Next, we formalize the notion of self-similarity using the projection $\mathbf{\Gamma}_l^p f$. For an integer $l \geq 0$, let $\mathcal{B}_l := \{\mathbf{B}_m, m = 1, \dots, 2^{ld}\}$ be a re-indexed collection of the hypercubes:

$$\mathbf{B}_m = \mathbf{B}_m := \left\{ x \in [0, 1]^d : \frac{\mathbf{m}_i - 1}{2^l} \leq x_i \leq \frac{\mathbf{m}_i}{2^l}, i \in \{1, \dots, d\} \right\},$$

for $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_d)$ with $\mathbf{m}_i \in \{1, \dots, 2^l\}$.

DEFINITION 4.1 (SELF-SIMILARITY). A set of payoff functions $\{f_k\}_{k \in \mathcal{K}}$ is said to be self-similar if $f_k \in \mathcal{H}(\beta)$, $k \in \mathcal{K}$, for some $\beta \in [\underline{\beta}, \bar{\beta}]$, and there exist some positive integer l_0 and some constant $b > 0$ such that for any $l \geq l_0$ and integer $\lfloor \beta \rfloor \leq p \leq \lceil \bar{\beta} \rceil$,

$$\max_{\mathbf{B} \in \mathcal{B}_l} \max_{k \in \mathcal{K}} \sup_{x \in \mathbf{B}} |\mathbf{\Gamma}_l^p f_k(x; \mathbf{B}) - f_k(x)| \geq b 2^{-l\beta}.$$

One may view the self-similarity condition as complementing the Hölder smoothness (Assumption 2.1) in the following way. On the one hand, **Hölder smoothness implies an upper bound on the estimation bias of payoff functions at every point** (*estimation bias* refers to the absolute difference between the value of a function and the expected value of its estimator, using, e.g., local polynomial regression). On the other hand, the above self-similarity condition effectively **implies a global lower bound on the estimation bias**. More precisely, self-similarity implies that for a set of β -smooth payoffs, the estimation bias is guaranteed to be at least of order h^β for bandwidth $h > 0$. This provides an opportunity to estimate the smoothness of payoff functions by “comparing” estimation variance and bias (for further details see discussion in §5.2.1). The next example illustrates a set of self-similar payoff functions.

EXAMPLE 4.1 (SELF-SIMILAR PAYOFFS). Fix some $\beta \leq 1 = \bar{\beta}$ and let $f_1(x) = x^\beta$ and $f_2(x) = \frac{1}{2}$ for all $x \in [0, 1]$. Then, $f_k \in \mathcal{H}(\beta), k \in \mathcal{K}$. Furthermore, for any $l \geq 1$ and $p = 0$, one has

$$\max_{\mathbf{B} \in \mathcal{B}_l} \max_{k \in \mathcal{K}} \sup_{x \in \mathbf{B}} |\mathbf{\Gamma}_l^p f_k(x; \mathbf{B}) - f_k(x)| \geq \left| \frac{1}{2^{-l}} \int_0^{2^{-l}} f_1(x) dx - f_1(0) \right| = \frac{1}{\beta + 1} 2^{-l\beta} =: b 2^{-l\beta}.$$

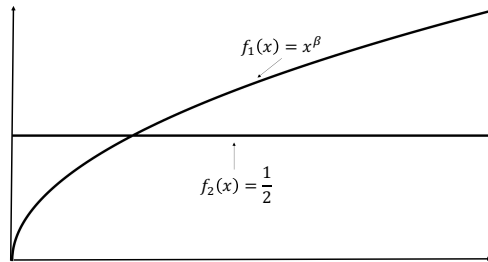


Figure 3 An instance of the self-similar set of payoff functions given in Example 4.1.

Figure 3 depicts the payoff functions detailed in Example 4.1. We next provide the self-similarity assumption, followed by a formulation of the class of problem instances with self-similar payoff functions, which is a subset of the more general class from Definition 2.2.

ASSUMPTION 4.1 (Self-similar payoffs). *The set of payoff functions $\{f_k\}_{k \in \mathcal{K}}$ is self-similar.*

DEFINITION 4.2 (CLASS OF PROBLEMS WITH SELF-SIMILAR PAYOFFS). For any $\underline{\beta} \leq 1$, $\bar{\beta} > \underline{\beta}$, $\beta \in [\underline{\beta}, \bar{\beta}]$, and $\alpha \geq 0$, we define by $\mathcal{P}^{\text{ss}}(\underline{\beta}, \bar{\beta}, \beta, \alpha, d) := \{\mathbf{P} \in \mathcal{P}(\beta, \alpha, d) : \mathbf{P} \text{ satisfies Assumption 4.1}\}$ the class of problems with self-similar payoffs.

4.2. Minimax complexity with self-similar payoffs

While Example 4.1 illustrates a set of particularly simple payoff functions, we note that the class of self-similar payoffs is quite general and includes many different payoff structures. In fact, we next show that the minimax complexity of the dynamic optimization problem at hand does not reduce by introducing the self-similarity condition.

We establish this result by constructing regret lower bounds that are of the same order as in (2.1). To do so, we design worst-case instances consisting of payoff functions that satisfy Assumption 4.1. More precisely, we show that worst-case instances developed in Rigollet and Zeevi (2010) for the case of $\beta \leq 1$ and in Hu et al. (2019) for the case of $\beta \geq 1$ can essentially be constructed using self-similar payoffs.

For consistency with the setting in Hu et al. (2019) that allows a more general structure for the support of the covariate distribution in the case of $\beta \geq 1$, we denote by $\tilde{\mathcal{P}}^{\text{ss}}(\underline{\beta}, \bar{\beta}, \beta, \alpha, d)$ the class of problems with self-similar payoffs where the covariate density p_X has a compact support $\mathcal{X} \subseteq [0, 1]^d$; for further details see Appendix A.3.

THEOREM 4.1 (Self-similarity assumption does not reduce the minimax complexity).

Fix some non-integer Hölder exponent $\beta \in [\underline{\beta}, \bar{\beta}]$, and some margin parameter $\alpha > 0$ such that $\alpha \leq \max\{1, \frac{1}{\beta}\}$ and $\alpha\beta \leq d$. Then, there exist $T_0, \underline{C} > 0$ such that for any horizon length $T \geq T_0$ and any admissible policy $\pi \in \Pi$, then followings hold:

1. (At most Lipschitz-smooth) If $\beta \leq 1$ then,

$$\sup_{P \in \mathcal{P}^{\text{ss}}(\underline{\beta}, \bar{\beta}, \beta, \alpha, d)} \mathcal{R}^\pi(P; T) \geq \underline{C} T^{1 - \frac{\beta(\alpha+1)}{2\beta+d}}.$$

2. (At least Lipschitz-smooth) If $\beta \geq 1$ then,

$$\sup_{P \in \tilde{\mathcal{P}}^{\text{ss}}(\underline{\beta}, \bar{\beta}, \beta, \alpha, d)} \mathcal{R}^\pi(P; T) \geq \underline{C} T^{1 - \frac{\beta(\alpha+1)}{2\beta+d}}.$$

Theorem 4.1 establishes that requiring payoff functions to be self-similar does not reduce the minimax (regret) complexity, and therefore implies that the minimax complexity of the problem under self-similar payoffs (Assumption 4.1) is still as stated in (2.1). Nevertheless, in the next section we establish that under self-similar payoffs one may design policies that are smoothness-adaptive, and essentially guarantee that minimax regret rate without prior information on the smoothness of payoff functions.

5. Adaptivity to smoothness

In this section, we first detail in §5.1 the main result of the section, establishing that under self-similar payoffs one may guarantee smoothness-adaptive performance. This result is based on providing a *Smoothness-Adaptive Contextual Bandits* (SACB) policy, and establishing that this policy is smoothness-adaptive. In §5.2 we then provide a detailed description of the SACB policy and discuss its key components.

5.1. Smoothness-adaptive performance with self-similar payoffs

We next detail the main results of this section. We show that the *Smoothness-Adaptive Contextual Bandits* (SACB) policy (that is detailed in §5.2, see Algorithm 1) is smoothness-adaptive under Assumption 4.1.

The key idea of the SACB policy lies in observing that local polynomial regression estimation of any function $f(\cdot)$ cannot largely deviate from the projection $\mathbf{\Gamma}_l^p f$ with high probability. That is, Assumption 4.1 is key in establishing that for a set of Hölder smooth payoff functions, not only is the estimation bias bounded from above, but also this bias cannot shrink fast (see further discussion and analysis in Appendix C). This suggests an opportunity to estimate the smoothness of the payoff functions by appropriately examining the estimation bias against its variance over the unit cube.

The SACB policy adaptively integrates a smoothness estimation sub-routine with some collection of non-adaptive policies $\{\pi_0(\beta_0)\}_{\beta_0 \in [\underline{\beta}, \bar{\beta}]}$ that are rate-optimal under accurate tuning of the smoothness parameter. The estimation sub-routine of SACB consists of three steps: (i) collecting samples over the covariate space; (ii) estimating the payoff functions; and (iii) conducting a hypothesis test. After the

estimation sub-routine is terminated, the produced estimate $\hat{\beta}_{\text{SACB}}$ is used to choose the corresponding rate-optimal non-adaptive policy $\pi_0^{(\hat{\beta}_{\text{SACB}})}$. The following result characterizes the quality of the smoothness estimation of the SACB policy.

THEOREM 5.1 (Smoothness estimation under self-similarity). *Suppose Assumption 2.1 holds for some $L > 0$ and $\beta \in [\underline{\beta}, \bar{\beta}]$, and Assumption 4.1 holds for some $b > 0$ and $l_0 \geq 0$. Then, there exists $T_0 > 0$ independent of T such that for any horizon length $T \geq T_0$ the SACB policy detailed in Algorithm 1, run with tuning parameter $\gamma > 0$, computes an estimate of β , denoted by $\hat{\beta}_{\text{SACB}}$, by time step $t = \left\lceil \frac{4}{\underline{\rho}} (\log T)^{\frac{2d}{\underline{\beta}} + 4} T^{\frac{(\beta + d - 1)}{(2\bar{\beta} + d)}} \right\rceil$ with probability at least $1 - 2T^{C_1} \exp(-C_2 T^{C_3})$, such that:*

$$\mathbb{P} \left\{ \hat{\beta}_{\text{SACB}} \in \left[\beta - \frac{3(2\bar{\beta} + d)^2 \log_2 \log T}{(\underline{\beta} + d - 1) \log_2 T}, \beta \right] \right\} \geq 1 - C_4 (\log T)^{\frac{d}{\underline{\beta}}} T^{-\gamma^2 C_5 + C_6},$$

where the constants C_1, C_2, C_3, C_4, C_5 , and C_6 depend only on $\underline{\beta}, \bar{\beta}, b, L, \underline{\rho}, \bar{\rho}$, and d .

The proof of Theorem 5.1 follows from Propositions 5.1 and 5.2 that will be advanced in §5.2.1 for analyzing the performance of the smoothness estimation subroutine in the SACB policy. Theorem 5.1 implies that **the error of the smoothness estimate grows linearly with the covariate dimension and decays as a function of the time horizon at a rate of $\frac{\log \log T}{\log T}$** . This characterization of the smoothness estimation is leveraged in the next theorem to establish that, when coupled with appropriate off-the-shelf non-adaptive policies, the SACB policy guarantees optimal regret rate up to poly-logarithmic terms, and smoothness-adaptive performance as stated in Definition 2.3.

THEOREM 5.2 (Smoothness-adaptive policy under self-similarity). *Let π be the SACB policy detailed in Algorithm 1, and let $\{\pi_0(\beta_0)\}_{\beta_0 \in [\underline{\beta}, \bar{\beta}]}$ be a set of non-adaptive policies such that if initialized with the true smoothness parameter, for any $\underline{\beta} \leq \beta_0 \leq \bar{\beta}$, $\alpha \leq \frac{1}{\min\{1, \beta_0\}}$, and $T \geq 1$, it satisfies*

$$\sup_{P \in \mathcal{P}(\beta_0, \alpha, d)} \mathcal{R}^{\pi_0(\beta_0)}(P; T) \leq \bar{C}_0 (\log T)^{\iota_0(\beta_0, \alpha, d)} T^{\zeta(\beta_0, \alpha, d)},$$

for some $\iota_0(\beta_0, \alpha, d)$ and a constant $\bar{C}_0 > 0$ that is independent of T , where the function $\zeta(\beta_0, \alpha, d)$ is given in (2.1). Consider a problem instance $P \in \mathcal{P}^{\text{ss}}(\underline{\beta}, \bar{\beta}, \beta, \alpha, d)$ with $\underline{\beta} \leq \beta \leq \bar{\beta}$ and $\alpha \leq \frac{1}{\min\{1, \beta\}}$. Then, there exist γ_0 , and $\bar{C} > 0$ such that for any tuning parameter $\gamma \geq \gamma_0$ and horizon length:

$$\mathcal{R}^\pi(P; T) \leq \bar{C} (\log T)^{\frac{3d(\alpha+1)(2\bar{\beta}+d)^2}{(2\bar{\beta}+d)(\beta+d)(\underline{\beta}+d-1)} + \iota_0\left(\beta - \frac{3(2\bar{\beta}+d)^2 \log_2 \log T}{(\underline{\beta}+d-1) \log_2 T}, \alpha, d\right)} T^{\zeta(\beta, \alpha, d)}.$$

The proof of Theorem 5.2 follows from observing that, with high probability, the number of time periods that are required in order to generate the smoothness estimate (and thus the regret that is incurred throughout the smoothness estimation process) is “small” relative to the optimal regret rate, and from

plugging the lower confidence bound established in Theorem 5.1 for the smoothness estimate into the regret rate of the non-adaptive policy π_0 .

When the policy π_0 that is deployed in the **SACB** policy is rate-optimal in the sense that $\iota_0(\beta, \alpha, d) = 0$ then, the resulting **SACB** policy is smoothness-adaptive according to Definition 2.3. More precisely, the adaptation cost for the **SACB** policy is poly-logarithmic in the horizon length with the degree $\frac{3d(\alpha+1)(2\bar{\beta}+d)^2}{(2\bar{\beta}+d)(\beta+d)(\bar{\beta}+d-1)}$, which is bounded from above for any dimension d , and hence, can be replaced by some function $\iota(\beta, \underline{\beta}, \bar{\beta}, \alpha)$ independent of d as required in Definition 2.3. We next demonstrate this for the cases of at most Lipschitz-smooth and at least Lipschitz-smooth payoffs.

5.1.1. Rate-optimality with at most Lipschitz-smooth payoffs When the estimated smoothness in the **SACB** policy is less than 1, that is, $\hat{\beta}_{\text{SACB}} \leq 1$, one may deploy the *Adaptively Binned Successive Elimination* (**ABSE**) policy from Perchet and Rigollet (2013) as the input non-adaptive policy π_0 to guarantee rate-optimal performance without prior information on the smoothness. This is formalized by the following Corollary.

COROLLARY 5.1 (Rate optimality with at most Lipschitz-smooth payoffs). *Consider the setting in Theorem 5.2, and suppose $\pi_0(\beta_0) = \text{ABSE}(\min(1, \beta_0))$ then,*

$$\mathcal{R}^\pi(\mathbf{P}; T) \leq \bar{C} T^{\zeta(\beta, \alpha, d)} (\log T)^{\frac{3d(\alpha+1)(2\bar{\beta}+d)^2}{(2\bar{\beta}+d)(\beta+d)(\bar{\beta}+d-1)}} \quad \forall \beta \in [\underline{\beta}, 1]. \quad (5.1)$$

The **ABSE** policy from Perchet and Rigollet (2013) relies on the knowledge of β , and achieves the rate-optimal regret of order $T^{\zeta(\beta, \alpha, d)}$ for any problem instance with $0 < \beta \leq 1$. The **SACB** policy resulting from deploying **ABSE** as an input policy when $\hat{\beta}_{\text{SACB}} \leq 1$ is *smoothness-adaptive* in the regime of smooth non-differentiable payoff functions with the adaptation penalty $(\log T)^{\frac{3d(\alpha+1)(2\bar{\beta}+d)^2}{(2\bar{\beta}+d)(\beta+d)(\bar{\beta}+d-1)}}$.

5.1.2. Rate-optimality with at least Lipschitz-smooth payoffs When the estimated smoothness in the **SACB** policy is larger than 1, that is, $\hat{\beta}_{\text{SACB}} > 1$, one may deploy the **SmoothBandit** policy from Hu et al. (2019) as the non-adaptive input policy π_0 . The **SmoothBandit** policy relies on the following additional assumption on the regularity of decision regions.

ASSUMPTION 5.1 (Regularity). *Let $\mathcal{Q}_k := \{x \in [0, 1]^d : (-1)^{k-1}(f_1(x) - f_2(x)) \geq 0\}$, $k \in \mathcal{K}$, be the optimal decision regions. Then, each \mathcal{Q}_k , is a non-empty (c_0, r_0) -regular set, where a Lebesgue measurable set \mathcal{S} is said to be (c_0, r_0) -regular if for all $x \in \mathcal{S}$, $\lambda[\mathcal{S} \cap \text{Ball}_2(x, r)] \geq c_0 \lambda[\text{Ball}_2(x, r)]$, where $\text{Ball}_2(x, r)$ is the Euclidean ball of radius r centered around x and $\lambda[\cdot]$ denotes the Lebesgue measure.*

Under Assumption 5.1, the resulting **SACB** policy guarantees rate-optimal performance without prior information on the smoothness. This is formalized by the following Corollary.

COROLLARY 5.2 (Rate optimality with at least Lipschitz-smooth payoffs). *Consider the setting in Theorem 5.2, and suppose $\pi_0(\beta_0) = \text{SmoothBandit}(\max(1, \beta_0))$ for $\beta_0 \geq 1$ then, if the decision regions associated with \mathbf{P} satisfy the regularity condition in Assumption 5.1 one has,*

$$\mathcal{R}^\pi(\mathbf{P}; T) \leq \bar{C} T^{\zeta(\beta, \alpha, d)} (\log T)^{\frac{3d(\alpha+1)(2\bar{\beta}+d)^2}{(2\bar{\beta}+d)(\bar{\beta}+d)(\bar{\beta}+d-1)} + \frac{2\bar{\beta}+d}{2\bar{\beta}}} \quad \forall \beta \in [1, \bar{\beta}]. \quad (5.2)$$

The **SmoothBandit** policy relies on the knowledge of β and achieves the near-optimal regret of order $\mathcal{O}\left((\log T)^{\frac{2\bar{\beta}+d}{2\bar{\beta}}} T^{\zeta(\beta, \alpha, d)}\right)$ for any problem instance with $\beta \geq 1$. The **SACB** policy, when paired with **SmoothBandit** as its non-adaptive input policy, guarantees near-optimality without prior knowledge of the smoothness in the regime of differentiable payoff functions, incurring the adaptation penalty $(\log T)^{\frac{3d(\alpha+1)(2\bar{\beta}+d)^2}{(2\bar{\beta}+d)(\bar{\beta}+d)(\bar{\beta}+d-1)}}$.

We note that the upper and lower bounds established in this regime with prior knowledge of the smoothness are separated by a factor $(\log T)^{\frac{2\bar{\beta}+d}{2\bar{\beta}}}$ which is exponential in d . If the upper bound of Hu et al. (2019) is indeed optimal in the sense that the above factor cannot be removed then, Corollary 5.2 establishes that the resulting **SACB** policy is smoothness-adaptive in the sense of Definition 2.3. Otherwise, if another non-adaptive policy could be shown to eliminate the above factor and achieve the lower bound of order $\Omega\left(T^{\zeta(\beta, \alpha, d)}\right)$ then, it could be deployed to construct a smoothness-adaptive **SACB** policy.

We conclude this subsection by noting that Corollaries 5.1 and 5.2 demonstrate that through the **SACB** policy one could achieve rate-optimality without prior knowledge of the smoothness parameter β in each of the two smoothness regimes that have been studied in the literature; that is $\beta \leq 1$ in, e.g., Perchet and Rigollet (2013), and $\beta \geq 1$ in Hu et al. (2019). However, it is important to note that the **SACB** policy does not require prior knowledge of the regime in which the smoothness parameter lies in order to achieve rate-optimality. This is formalized by the following remark.

REMARK 5.1 (RATE OPTIMALITY WITH GENERAL SMOOTHNESS). Consider the setting in Theorem 5.2, and suppose $\pi_0(\beta_0) = \text{ABSE}(\beta_0)$ for $\beta_0 \leq 1$ and $\pi_0(\beta_0) = \text{SmoothBandit}(\beta_0)$ for $\beta_0 > 1$. Then, for $\beta \leq 1$ one recovers the same regret bound as in (5.1), and for $\beta > 1$, under Assumption 5.1, one recovers the same regret bound as in (5.2).

5.2. The SACB policy

The *Smoothness-Adaptive Contextual Bandits* (**SACB**) policy adaptively integrates a smoothness estimation sub-routine with a non-adaptive off-the-self policy that is rate optimal under prior knowledge of the smoothness. The smoothness estimation sub-routine consists of three consecutive steps: (i) collecting samples in different regions of the covariate space; (ii) estimating the payoff functions; and (iii) examining a hypothesis test over the estimated functions. The policy repeats these steps until the smoothness estimation sub-routine is terminated. Afterwards, the smoothness of the payoff functions is estimated based on the results of the hypothesis tests.

After the estimation sub-routine is terminated, the estimate $\hat{\beta}_{\text{SACB}}$ is used as an input to a non-adaptive off-the-shelf policy that is designed to perform well under accurate tuning of the smoothness. We next formalize the SACB policy, and then discuss the estimation subroutine in §5.2.1.

Algorithm 1: Smoothness-Adaptive Contextual Bandits (SACB)

```

1 Input: Set of non-adaptive policies  $\{\pi_0(\beta_0)\}_{\beta_0 \in [\underline{\beta}, \bar{\beta}]}$ , horizon length  $T$ , minimum and maximum smoothness
   exponents  $\underline{\beta}$  and  $\bar{\beta}$ , and a tuning parameter  $\gamma$ 
2 Initialize:  $l \leftarrow \left\lceil \frac{(\underline{\beta}+d-1) \log_2 T}{(2\bar{\beta}+d)^2} \right\rceil$  and  $\xi^{(\text{B})} \leftarrow 0$ ,  $N_k^{(\text{B})} \leftarrow 0$  for all  $\text{B} \in \mathcal{B}_l$  and  $k \in \mathcal{K}$ 
3 for  $t = 1, \dots$  do
4   Determine the bin in which the current covariate is located:  $\text{B} \in \mathcal{B}_l$  s.t.  $X_t \in \text{B}$ 
5   Alternate between the arms:  $\pi_t \leftarrow 1 + \mathbb{1}\{N_1^{(\text{B})} > N_2^{(\text{B})}\}$ 
6   Update the counters:  $N_k^{(\text{B})} \leftarrow N_k^{(\text{B})} + \mathbb{1}\{\pi_t = k\} \forall k \in \mathcal{K}$ 
7   if  $N_1^{(\text{B})} + N_2^{(\text{B})} \geq 2 \times 2^{r^{(\text{B})}}$  and  $r^{(\text{B})} \leq \bar{r}$  then
8     if  $\xi^{(\text{B})} = 0$  and  $\sup_{k \in \mathcal{K}, x \in \mathcal{M}^{(\text{B})}} \left| \hat{f}_k^{(\text{B}, r^{(\text{B})})}(x; j_1^{(\text{B})}) - \hat{f}_k^{(\text{B}, r^{(\text{B})})}(x; j_2^{(\text{B})}) \right| > \frac{\gamma(\log T)^{\frac{d}{2\underline{\beta}} + \frac{1}{2}}}{2^{r^{(\text{B})}/2}}$  ;    /* see (5.4) */
9     then
10      Record  $r_{\text{last}}^{(\text{B})} : r_{\text{last}}^{(\text{B})} \leftarrow r^{(\text{B})}$ ; Raise the flag:  $\xi^{(\text{B})} \leftarrow 1$ 
11      Collect double number of samples in the next round:  $r^{(\text{B})} \leftarrow r^{(\text{B})} + 1$ ; Reset the counters:
12       $N_k^{(\text{B})} \leftarrow 0 \forall k \in \mathcal{K}$ 
13      if  $[\xi^{(\text{B}')} = 1 \text{ or } r^{(\text{B}')} > \bar{r}]$  for all  $\text{B}' \in \mathcal{B}_l$  then
14        Record  $T_{\text{SACB}} : T_{\text{SACB}} \leftarrow t$ 
15        break
16 Estimate the smoothness:  $\hat{\beta}_{\text{SACB}} \leftarrow \frac{1}{2l} \left[ \min_{\text{B} \in \mathcal{B}_l} r_{\text{last}}^{(\text{B})} - \left( \frac{2d}{\underline{\beta}} + 4 \right) \log_2 \log T \right]$ 
17 Choose the corresponding non-adaptive policy  $\pi_0 \leftarrow \pi_0(\min[\max[\underline{\beta}, \hat{\beta}_{\text{SACB}}], \bar{\beta}])$ 
18 for  $t = T_{\text{SACB}} + 1, \dots, T$  do
19    $\pi_t \leftarrow \pi_0(X_t)$ 

```

5.2.1. Smoothness estimation under the SACB policy

Sampling. In the SACB policy, we consider the partition of the unit cube corresponding to \mathcal{B}_l with $l = \left\lceil \frac{(\underline{\beta}+d-1) \log_2 T}{(2\bar{\beta}+d)^2} \right\rceil$. For each bin (hypercube) $\text{B} \in \mathcal{B}_l$, we collect samples for both actions in multiple rounds. Define the maximum round index as follows: $\bar{r} := \lceil 2l\bar{\beta} + (\frac{2d}{\underline{\beta}} + 4) \log_2 \log T \rceil$. At every round $r \in \{1, \dots, \bar{r}\}$, we collect 2^r samples for each action by alternating between them every time the context belongs to B . If for some $\text{B} \in \mathcal{B}_l$ we reach \bar{r} before the smoothness estimation sub-routine is terminated, we continue alternating between the arms every time the context belongs B . We denote by T_{SACB} the time step at which the smoothness estimation sub-routine is terminated.

Estimation. We briefly review the local polynomial regression method based on the analysis in Audibert and Tsybakov (2007); further analysis can be found in Appendix C. Let $\mathcal{D} = \{(X_i, Y_i)\}_{i=1}^n$ be a set of n i.i.d. pairs $(X_i, Y_i) \in \mathcal{X} \times \mathbb{R}$, distributed according to a joint distribution P . Denote by μ the marginal density of X_i 's and define the regression function $\eta(x) := \mathbb{E}[Y|X=x]$. To estimate the value of the function η at any point $x \in \mathcal{X}$, the local polynomial regression method is defined as follows.

DEFINITION 5.1 (LOCAL POLYNOMIAL REGRESSION). Fix a set of pairs $\mathcal{D} = \{(X_i, Y_i)\}_{i=1}^n$, a point $x \in \mathbb{R}^d$, a bandwidth $h > 0$, an integer $p > 0$ and the kernel function $K(\cdot) = \mathbb{1}\{\|\cdot\|_\infty \leq 1\}$. Define by $\hat{\theta}_x(u; \mathcal{D}, h, p) = \sum_{|s| \leq p} \xi_s u^s$ a polynomial of degree p on \mathbb{R}^d that minimizes

$$\sum_{i=1}^n \left(Y_i - \hat{\theta}_x(X_i - x; \mathcal{D}, h, p) \right)^2 K\left(\frac{X_i - x}{h}\right). \quad (5.3)$$

The local polynomial estimator $\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p)$ of the value $\eta(x)$ of the regression function $f(\cdot)$ at point x is defined by: $\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) := \hat{\theta}_x(0; \mathcal{D}, h, p)$ if (5.3) has a unique minimizer, and $\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) := 0$ otherwise.

Denote by $X_{k,1}^{(\mathbf{B},r)}, X_{k,2}^{(\mathbf{B},r)}, \dots$ and $Y_{k,1}^{(\mathbf{B},r)}, Y_{k,2}^{(\mathbf{B},r)}, \dots$ the successive covariates and outcomes when action k is selected in \mathbf{B} at round r , respectively. Denote by $\mathcal{D}_k^{(\mathbf{B},r)} := \left\{ \left(X_{k,\tau}^{(\mathbf{B},r)}, Y_{k,\tau}^{(\mathbf{B},r)} \right) \right\}_{\tau=1}^{2^r}$ the corresponding set of pairs. Define the two bandwidth exponents: $j_1^{(\mathbf{B})} := l$, and $j_2^{(\mathbf{B})} := l + \lceil \frac{1}{\underline{\beta}} \log_2 \log T \rceil$. Let $\tilde{l} := \lceil \frac{\bar{\beta}l}{\underline{\beta}} + \frac{\log_2 \log T}{\underline{\beta}} \rceil \vee \lceil (1 + \bar{\beta})l + \log_2 \log T \rceil$. For every bin \mathbf{B} define the mesh points:

$$\mathcal{M}^{(\mathbf{B})} := \left\{ x = \left(\frac{m_1}{2^{\tilde{l}}}, \dots, \frac{m_d}{2^{\tilde{l}}} \right) : x \in \mathbf{B}, m_i \in \{1, \dots, 2^{\tilde{l}}\} \text{ for } i \in \{1, \dots, d\} \right\}.$$

For every mesh point $x \in \mathcal{M}^{(\mathbf{B})}$, we form two separate estimates of the payoff functions using local polynomial regression of degree $\lfloor \bar{\beta} \rfloor$:

$$\hat{f}_k^{(\mathbf{B},r)}(x; j) := \hat{\eta}^{\text{LP}}(x; \mathcal{D}_k^{(\mathbf{B},r)}, 2^{-j}, \lfloor \bar{\beta} \rfloor), j \in \{j_1^{(\mathbf{B})}, j_2^{(\mathbf{B})}\}. \quad (5.4)$$

Hypothesis test. At the end of each sampling round r in bin \mathbf{B} , we check whether the difference between the estimation using the two bandwidths exponents $j_1^{(\mathbf{B})}$ and $j_2^{(\mathbf{B})}$ exceeds a pre-determined threshold. Formally, for a tuning parameter γ , we check whether the following holds:

$$\sup_{k \in \mathcal{K}, x \in \mathcal{M}^{(\mathbf{B})}} \left| \hat{f}_k^{(\mathbf{B},r)}(x; j_1^{(\mathbf{B})}) - \hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})}) \right| \geq \frac{\gamma (\log T)^{\frac{d}{2\underline{\beta}} + \frac{1}{2}}}{2^{r/2}}, \quad (5.5)$$

The left hand side of (5.5) is driven by two terms: the estimation bias of $\hat{f}_k^{(\mathbf{B},r)}(x; j_1^{(\mathbf{B})})$, which is potentially larger due to a larger bandwidth; and the standard deviation of $\hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})})$, which is potentially larger since, on average, it is based on less samples. **The right hand side of (5.5), however, is proportional to the standard deviation of the estimate $\hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})})$.** That is, by examining (5.5), we are detecting the number of samples that is required for the estimation bias of $\hat{f}_k^{(\mathbf{B},r)}(x; j_1^{(\mathbf{B})})$ to dominate the standard deviation of $\hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})})$, which, as we will see, is dependent on the smoothness of the payoff functions. This dependence allows one to infer the smoothness of payoff functions with a good precision with high probability. Denote by $r_{\text{last}}^{(\mathbf{B})}$ the smallest round index for which (5.5) holds in bin \mathbf{B} (upon this event, we set the flag $\xi^{(\mathbf{B})} = 1$). If (5.5) never holds in \mathbf{B} , we simply set $r_{\text{last}}^{(\mathbf{B})} = \bar{r}$.

The quantity $r_{\text{last}}^{(\text{B})}$ closely relates to the smoothness of the payoff functions. In what follows, we show that $\min_{\text{B} \in \mathcal{B}_l} r_{\text{last}}^{(\text{B})} \approx 2l\beta$ with high probability; this relation stems from $2^{r_{\text{last}}^{(\text{B})}}$ essentially being the minimal number of samples required for the bias and standard deviation to be balanced for hypercube B under our procedure (in the sense of equation 5.5).

We next develop high-probability bounds for $r_{\text{last}}^{(\text{B})}$; following the above connection, these bounds are used for establishing the smoothness estimate in (5.6), as well as Theorem 5.1. The next proposition provides a high-probability lower bound for $r_{\text{last}}^{(\text{B})}$ for all the bins $\text{B} \in \mathcal{B}_l$.

PROPOSITION 5.1 (High-probability lower bound for $r_{\text{last}}^{(\text{B})}$). *Suppose that Assumption 2.1 holds for some $L > 0$ and $\beta \in [\underline{\beta}, \bar{\beta}]$. Then, there exist constants \underline{C}_r , C_7 , C_8 , and C_9 such that for all $T \geq 1$,*

$$r_{\text{last}}^{(\text{B})} < \underline{C}_r + 2l\beta + \left(\frac{d}{\underline{\beta}} + 1\right) \log_2 \log T$$

for some $\text{B} \in \mathcal{B}_l$, with probability less than $C_7 (\log T)^{\frac{d}{2}} T^{-\gamma^2 C_8 + C_9}$, where the constants C_7 , C_8 , and C_9 depend only on $\underline{\beta}, \bar{\beta}, L, \underline{\rho}, \bar{\rho}$, and d , and \underline{C}_r depends only on $\underline{\beta}, \bar{\beta}, L, \underline{\rho}$, and $\bar{\rho}$.

The proof of Proposition 5.1 is based on the discussion provided after (5.5). Since the payoff functions belong to $\mathcal{H}(\beta, L)$, their estimation bias is bounded in each bin $\text{B} \in \mathcal{B}_l$. This implies that when the number of samples is “small”, the left hand side of (5.5) is dominated by the standard deviation of $\hat{f}_k^{(\text{B}, r)}(x; j_2^{(\text{B})})$, which is proportional to the right hand side of (5.5), with high probability. The next result complements Proposition 5.1 by providing a high-probability upper bound for $\min_{\text{B} \in \mathcal{B}_l} r_{\text{last}}^{(\text{B})}$.

PROPOSITION 5.2 (High-probability upper bound for $\min_{\text{B} \in \mathcal{B}_l} r_{\text{last}}^{(\text{B})}$). *Suppose that Assumption 2.1 holds for some $L > 0$ and $\beta \in [\underline{\beta}, \bar{\beta}]$, and that Assumption 4.1 holds for some $b > 0$ and $l_0 \geq 0$ then, there exist some $\text{B} \in \mathcal{B}_l$ and some constants \bar{C}_r , C_{10} , and C_{11} such that for all $T \geq 1$,*

$$r_{\text{last}}^{(\text{B})} > \bar{C}_r + 2l\beta + \left(\frac{d}{\underline{\beta}} + 3\right) \log_2 \log T$$

with probability less than $C_{10} T^{-\gamma^2 C_{11}}$, where the constants C_{10} and C_{11} depend only on $\underline{\beta}, \bar{\beta}, L, b, \underline{\rho}, \bar{\rho}$, and d , and \bar{C}_r depends only on $\underline{\beta}, \bar{\beta}, L, b, \underline{\rho}$, and $\bar{\rho}$.

The proof of Proposition 5.2 is again based on the discussion provided after (5.5). Since the set of payoff functions is self-similar, the estimation bias of the estimate $\hat{f}_k^{(\text{B}, r)}(x; j_1^{(\text{B})})$ remains “large” in at least one of the bins $\text{B} \in \mathcal{B}_l$ and for one of the arms, which implies that for that specific bin and arm, if the number of samples is “large” enough then, the left hand side of (5.5) is dominated by the aforementioned bias and eventually exceed the right hand side of (5.5) with high probability.

Based on Proposition 5.1 and 5.2, we estimate the smoothness of the problem as follows:

$$\hat{\beta}_{\text{SACB}} = \frac{1}{2l} \left[\min_{\text{B} \in \mathcal{B}_l} r_{\text{last}}^{(\text{B})} - \left(\frac{2d}{\underline{\beta}} + 4\right) \log_2 \log T \right]. \quad (5.6)$$

Note that in order to avoid costly estimation errors this estimate is designed to be less than β with high probability, which is commonly referred to as “undersmoothing” in the construction of confidence intervals; see, e.g., Bickel and Rosenblatt (1973), Hall (1992), Picard and Tribouley (2000), and Giné and Nickl (2010).

We conclude this section with a discussion on the low sample complexity of our smoothness estimation sub-routine relative to the optimal regret rates. In order to achieve rate-optimality one is required to estimate the smoothness β with precision of order $\frac{1}{\log T}$. Broadly speaking, our proposed estimation sub-routine (i) collects n independent samples from payoff functions that are Holder-smooth and self similar; (ii) partitions the unit cube to hypercubes of side-length h ; and (iii) estimates the payoff functions in each hypercube using local polynomial regression. The resulting estimation bias is of order h^β . However, using our proposed sub-routine, one may evaluate the estimation bias as ch^β for some finite c . This results in an estimate of β of the form $\frac{\log(ch^\beta)}{\log h} = \beta + \frac{c}{\log h}$. Hence, to achieve precision of order $\frac{1}{\log T}$, it suffices to have $h = T^p$ for some p . In order for the estimation sub-routine to perform well, one requires the estimation bias ($\approx h^\beta$) and the estimation standard deviation ($\approx \frac{1}{\sqrt{nh^d}}$) to be balanced, that is, n should be of order T^q for some q . Finally, one can make q arbitrarily small such that n is not large relative to the optimal regret rate.

6. Concluding remarks

In this paper, we studied the problem of adapting to unknown smoothness of payoff functions in a non-parametric contextual MAB setting. First, we showed that, in general, it is impossible to achieve rate-optimal performance simultaneously over different classes of payoff functions in the following sense: There exist some pairs of smoothness parameters (γ, β) such that no policy can simultaneously attain optimal regret rates over the problems $\mathcal{P}(\gamma, \alpha, d)$ and $\mathcal{P}(\beta, \alpha, d)$. This implies that, in general, one might incur non-trivial adaptation cost when the smoothness of payoffs is a priori unknown.

We overcome the impossibility of adaptation by leveraging a self-similarity condition (which does not reduce the minimax complexity of the problem). We devised a general policy based on: (i) inferring the smoothness of the payoff functions using observations that are collected throughout the decision-making process; and (ii) using effective non-adaptive policies as off-the-shelf input policies. We showed that this approach allows one to guarantee the best regret rate that is achievable given the underlying smoothness exponent β that characterizes the problem instance, without requiring prior knowledge of that smoothness. Our policy is *smoothness-adaptive*, in the sense of achieving that rate up to a multiplicative term that is poly-logarithmic in the horizon length and a multiplicative constant that may depend on other problem parameters.

Avenues for future research. Our study presents several new research directions. One open question is whether our impossibility statement holds for any pair of Hölder exponents (γ, β) . More precisely, it is

left to understand whether there is any policy that can achieve rate optimal performance simultaneously over two different problem instances characterized by different smoothness parameters, without additional assumptions such as the one of self-similarity. If not, it would be desirable to extend our analysis to establish impossibility of adaptation for any pair of Hölder exponents (γ, β) .

Another path is to study how tight the lower bound provided in Theorem 3.1 is. In other words, for pairs of smoothness parameters (γ, β) over which impossibility of adaptation is established, can one design a MAB policy that achieves rate-optimal performance over problem instances characterized by γ , and incurs the regret rate provided in Theorem 3.1 for problem instances characterized by β ?

Another interesting question is whether there exists any assumption weaker than that of self-similarity that allows for designing smoothness-adaptive policies. If not, a natural direction would be to study the adaptation cost that one has to incur with respect to the self-similarity constant b in Definition 4.1.

7. Proof of Part 1 of Theorem 3.1

In this section, we describe the proof of Part 1 of Theorem 3.1. The proof follows the next steps. In Step 1, we discuss some notations and definitions including the definition of inferior sampling rate. In Step 2, we leverage a result from Rigollet and Zeevi (2010) to connect regret and inferior sampling rate, which enables one to simplify analysis by focusing on the inferior sampling rate throughout the proof.

In Step 3, which is a key step of the proof, we reduce the problem at hand to a hypothesis testing problem by introducing a novel construction of a set of problem instances. This set consists of a nominal problem instance with smoothness parameter γ and some other problem instances with smoothness parameter β , each of which differs from the nominal one only over a specific region of the covariate space. These problem instances are designed to connect between the amount of exploration and the ability to identify the correct smoothness parameter. **This construction is designed for showing that if a policy achieves rate-optimal performance over smooth problems, it is likely to under-explore in “rougher” problems, and hence, not being able to differentiate between the two.**

In Step 4, we verify that the aforementioned problem instances satisfy the margin condition. In Step 5, we show that with high probability, the number of contexts that belong to the regions mentioned in Step 3 grow linearly with respect to the time horizon and the volume of the regions. In Steps 6 and 7, we show that since the policy is rate optimal for γ -smooth problems, it cannot distinguish between the nominal problem and at least one of the β -smooth problems. In Step 8, we lower bound the inferior sampling rate due to not being able to identify the correct smoothness parameter.⁵ In Step 9, we revert back the lower bound on inferior sampling rate to a lower bound on regret.

⁵ Some high-level ideas in Steps 7 and 8 are adopted from the proof of Theorem 3 in Locatelli and Carpentier (2018).

Step 1 (Preliminaries). For any policy π and decision horizon T , let $\mathcal{S}^\pi(\mathbf{P}; T)$ be the *inferior sampling rate* defined as

$$\mathcal{S}^\pi(\mathbf{P}; T) := \mathbb{E}^\pi \left[\sum_{t=1}^T \mathbb{1} \{f_{\pi_t^*}(X_t) \neq f_{\pi_t}(X_t)\} \right]. \quad (7.1)$$

Fix a covariate distribution \mathbf{P}_X . For any policy π and function $f: [0, 1]^d \rightarrow [0, 1]$, denote by $\mathcal{S}^\pi(f; T)$ the inferior sampling rate of π when \mathbf{P}_X is the covariate distribution, $\mathbb{E}[Y_{1,t} | X_t] = f(X_t)$, and $\mathbb{E}[Y_{2,t} | X_t] = \frac{1}{2}$. Notably, the oracle policy π_f^* is given by $\pi_f^*(x) = 2 - \mathbb{1} \{f(x) \geq \frac{1}{2}\}$. We further denote by $\mathbb{P}_{\pi, f}$ and $\mathbb{E}_{\pi, f}$ the corresponding probability and expectation. Finally, for any Hölder exponent $\beta > 0$ and margin parameter $\alpha > 0$, define:

$$\mathcal{R}_{\beta, \alpha}^\pi(T) := \sup_{\mathbf{P} \in \mathcal{P}(\beta, \alpha, d)} \mathcal{R}^\pi(\mathbf{P}; T); \quad \mathcal{S}_{\beta, \alpha}^\pi(T) := \sup_{\mathbf{P} \in \mathcal{P}(\beta, \alpha, d)} \mathcal{S}^\pi(\mathbf{P}; T).$$

Fix $T \geq 1$, two Hölder exponents $0 < \beta < \gamma \leq 1$, a margin parameter $0 \leq \alpha \leq \frac{1}{\gamma}$, a positive Lipschitz constant L , and positive constants $\underline{\rho}, \bar{\rho}$ such that \mathbf{P}_X satisfies Assumption 2.2 with parameters $\underline{\rho}, \bar{\rho}$.

Step 2 (From regret to inferior sampling rate). The following lemma implies that it suffices to first analyze inferior sampling rate and then, revert the result back to regret.

LEMMA 7.1 (Rigollet and Zeevi 2010, Lemma 3.1). *For any $\alpha > 0$ under the margin condition in Assumption 2.3, one has*

$$\mathcal{S}^\pi(\mathbf{P}; T) \leq C_{sr} T^{\frac{1}{\alpha+1}} [\mathcal{R}^\pi(\mathbf{P}; T)]^{\frac{\alpha}{\alpha+1}},$$

for any policy π and some positive constant C_{sr} .

By Lemma 7.1, we have $\mathcal{S}_{\beta, \alpha}^\pi(T) \leq C_{sr} T^{\frac{1}{\alpha+1}} [\mathcal{R}_{\beta, \alpha}^\pi(T)]^{\frac{\alpha}{\alpha+1}}$. Note that when π is rate-optimal over $\mathcal{P}(\gamma, \alpha, d)$, Lemma 7.1 implies that for some constants $C_r, C_s > 0$, one has:

$$\mathcal{R}_{\gamma, \alpha}^\pi(T) \leq C_r T^{1 - \frac{\gamma(1+\alpha)}{2\beta+d}} =: \mathcal{R}_{\gamma, \alpha}^*(T); \quad \mathcal{S}_{\gamma, \alpha}^\pi(T) \leq C_s T^{1 - \frac{\gamma\alpha}{2\gamma+d}} =: \mathcal{S}_{\gamma, \alpha}^*(T).$$

Step 3 (Constructing problem instances). In this step we reduce our problem to a hypothesis testing problem. In order to do so, we first construct some problem instances. Defining $M := \lceil \Delta^{\alpha - \frac{d}{\beta}} \rceil$ and $C_\phi := \frac{L}{2^{2+2\beta}}$, fix the parameter $\Delta > 0$ such that

$$\frac{64C_\phi^2 \Delta^2 \mathcal{S}_{\gamma, \alpha}^*(T)}{3M} = \frac{1}{2}.$$

This selection of Δ implies that for large enough T one has $C_\phi \Delta \leq \frac{1}{4}$. For any $0 < \kappa \leq 1$, define the functions $\tilde{\psi}_\kappa$ and $\hat{\psi}_\kappa$ as follows:

$$\tilde{\psi}_\kappa(x) := \begin{cases} |1 - \|x\|_\infty|^\kappa & \text{if } 0 \leq \|x\|_\infty \leq 1; \\ 0 & \text{o.w.}; \end{cases} \quad \hat{\psi}_\kappa(x) := \begin{cases} |1 - \|x\|_\infty|^\kappa & \text{if } 0 \leq \|x\|_\infty \leq 1; \\ -|\|x\|_\infty - 1|^\kappa & \text{if } 1 \leq \|x\|_\infty \leq 2; \\ -1 & \text{o.w.} \end{cases}$$

Note that $\tilde{\psi}_\kappa \in \mathcal{H}_{\mathbb{R}^d}(\kappa, 1)$ and $\hat{\psi}_\kappa \in \mathcal{H}_{\mathbb{R}^d}(\kappa, 2)$. The following two lemmas (proved in Appendix E) are the main tools to analyze the smoothness of the payoff functions that we construct in this step.

LEMMA 7.2 (Scaling and smoothness). Suppose $f \in \mathcal{H}_{\mathbb{R}^d}(\beta, L)$ for some $0 < \beta \leq 1$ and $L > 0$, and define the function g such that $g(x) = C^{-\beta} f(Cx)$ for all $x \in \mathbb{R}^d$ and some $C > 0$. Then, $g \in \mathcal{H}_{\mathbb{R}^d}(\beta, L)$.

LEMMA 7.3 (Min/Max and smoothness). Suppose $f, g \in \mathcal{H}_{\mathcal{X}}(\beta, L)$ for some $\mathcal{X} \subseteq \mathbb{R}^d$, $0 < \beta \leq 1$ and $L > 0$, and define the functions $h_1 := \max(f, g)$ and $h_2 := \min(f, g)$. Then, $h_1, h_2 \in \mathcal{H}_{\mathcal{X}}(\beta, L)$.

Define a hypercube $H_0 := [0, 2\Delta^{\frac{\alpha}{d}}]^d$ with a center $q_0 := (\Delta^{\frac{\alpha}{d}}, \Delta^{\frac{\alpha}{d}}, \dots, \Delta^{\frac{\alpha}{d}}) \in \mathbb{R}^d$. Define the function

$$\phi_0(x) := \frac{1}{2} - C_\phi \cdot \min \left\{ \Delta, \Delta^{\frac{\alpha\gamma}{d}} \cdot \tilde{\psi}_\gamma \left(\Delta^{-\frac{\alpha}{d}} [x - q_0] \right) \right\}.$$

By Lemmas 7.2 and 7.3, $\phi_0 \in \mathcal{H}(\gamma, L)$ since $C_\phi \leq L$. Consider the grid G that partitions the hypercube H_0 into M disjoint hypercubes $(H_m)_{m \in \{1, \dots, M\}}$. Let $q_m \in \mathbb{R}^d, m \in \{1, \dots, M\}$, be the center of the hypercube H_m . Let \tilde{H}_m be the hypercube of side-length $l := \frac{\Delta^{\frac{\alpha}{d}}}{2M^{\frac{d}{d}}}$ centered around q_m . Note that $\tilde{H}_m \subset H_m$ and that the side-length of H_m is $2l$. Define the functions $\phi_m, m \in \{1, \dots, M\}$, as follows:

$$\phi_m(x) := \max \left\{ \phi_0(x), \frac{1}{2} + C_\phi \cdot \Delta \cdot \hat{\psi}_\beta (2l^{-1} [x - q_m]) \right\}.$$

Since $C_\phi = \frac{L}{2^{2+2\beta}}$ and $\Delta \leq 2^{\beta+1} l^\beta$ for large enough T , by Lemmas 7.2 and 7.3 one has that $\phi_m \in \mathcal{H}(\beta, L)$ for $1 \leq m \leq M$.

Step 4 (Verifying the margin condition). By examining different cases of parametric values, we verify that the margin condition is satisfied with parameters α and $C_0 := 2^d 3d \bar{\rho} C_\phi^{-\alpha}$ when $f_1 = \phi_m$ and $f_2 = \frac{1}{2}$ for all $0 \leq m \leq M$. For space considerations, we defer the analysis of this step to Appendix A.1.

Step 5 (Desirable event). For $m \in \{1, \dots, M\}$, define $Q_m := \sum_{t=1}^T \mathbb{1} \{X_t \in \tilde{H}_m\} =: \sum_{t=1}^T Z_{m,t}$ to be the number of times periods at which the realized contexts belong to the hypercube \tilde{H}_m . Define $\mathcal{A} := \{\exists m \in \{1, \dots, M\} : Q_m < \underline{\rho} T l^d\}$ to be the event where Q_m is less than $\underline{\rho} T l^d$ for at least one value of $m \in \{1, \dots, M\}$. Note that

$$\mathbb{P} \{ \mathcal{A} \} \leq \sum_{m=1}^M \mathbb{P} \{ Q_m < \underline{\rho} T l^d \}.$$

In order to bound each of the summands on the right hand side of the above inequality, one may apply Bernstein's inequality in the following lemma 7.4 to Q_m :

LEMMA 7.4 (Bernstein inequality). Let X_1, \dots, X_n be random variables with range $|X_i| \leq M$ and $\sum_{t=1}^n \text{Var} [X_t | X_{t-1}, \dots, X_1] = \sigma^2$. Let $S_n = X_1 + \dots + X_n$. Then for all $a \geq 0$

$$\mathbb{P} \{ S_n \geq \mathbb{E}[S_n] + a \} \leq \exp \left(- \frac{a^2/2}{\sigma^2 + Ma/3} \right).$$

Note that since $\mathbb{E} Z_{m,t} \geq 2\underline{\rho} l^d$, $|Z_{m,t}| \leq 1$, and $\text{Var} Z_{m,t} \leq \mathbb{E} Z_{m,t}^2 \leq 2\bar{\rho} l^d$, one obtains:

$$\begin{aligned} \mathbb{P} \{ \mathcal{A} \} &\leq M \exp (-\underline{\rho} T l^d / 5) \\ &\stackrel{(a)}{\leq} c_1 [\mathcal{S}_{\gamma, \alpha}^*]^{\frac{\alpha\beta-d}{2\beta+d-\alpha\beta}} \exp \left(-c_2 T [\mathcal{S}_{\gamma, \alpha}^*(T)]^{-\frac{d}{2\beta+d-\alpha\beta}} \right) \\ &\leq c_1 T^{\frac{\alpha\beta-d}{2\beta+d-\alpha\beta}} \exp \left(-c_3 T^{\frac{2\beta(2\gamma+d-\alpha\gamma)+\alpha d(\gamma-\beta)}{(2\gamma+d)(2\beta+d-\alpha\beta)}} \right) \stackrel{(b)}{\leq} c_4 T^{-3}, \end{aligned}$$

for large enough T and constants $c_1, c_2, c_3, c_4 > 0$, where (a) follows from the definition of M and l , and (b) holds by $\frac{2\beta(2\gamma+d-\alpha\gamma)+\alpha d(\gamma-\beta)}{(2\gamma+1)(2\beta+d-\alpha\beta)} > 0$ for $\alpha \leq \frac{1}{\gamma}$. For any problem instance \mathbf{P} and horizon length T , denote the inferior sampling rate of π when the event \mathcal{A} does not occur by

$$\bar{\mathcal{S}}^\pi(\mathbf{P}; T) := \mathbb{E}^\pi \left[\sum_{t=1}^T \mathbb{1} \{f_{\pi_t^*}(X_t) \neq f_{\pi_t}(X_t)\} \mid \bar{\mathcal{A}} \right].$$

Define $\bar{\mathcal{S}}_{\gamma, \alpha}^\pi(T) := \sup_{\mathbf{P} \in \mathcal{P}(\gamma, \alpha, d)} \bar{\mathcal{S}}^\pi(\mathbf{P}; T)$. Note that

$$(1 - \mathbb{P}\{\mathcal{A}\}) \bar{\mathcal{S}}^\pi(\mathbf{P}; T) \leq \mathcal{S}^\pi(\mathbf{P}; T) \leq \bar{\mathcal{S}}^\pi(\mathbf{P}; T) + T\mathbb{P}\{\mathcal{A}\},$$

which implies that

$$|\bar{\mathcal{S}}_{\gamma, \alpha}^\pi(T) - \mathcal{S}_{\gamma, \alpha}^\pi(T)| \leq c_4 T^{-2}. \quad (7.2)$$

For the rest of the proof, all probabilities and expectations will be computed conditional on $\bar{\mathcal{A}}$.

Step 6 (Selecting a single problem with smoothness β). Let $N_{m, T} := \sum_{t=1}^T \mathbb{1} \{\pi_t = 1, X_t \in H_m\}$ denote the number of times policy π selects arm 1 when realized covariates belong to the hypercube H_m . By definition, $\mathbb{E}_{\pi, \phi_0}^T \left[\sum_{m=1}^M N_{m, T} \mid \bar{\mathcal{A}} \right] \leq \bar{\mathcal{S}}_{\gamma, \alpha}^\pi(T)$, implying that there exists some $m^* \in \{1, \dots, M\}$ such that

$$\mathbb{E}_{\pi, \phi_0}^T [N_{m^*, T} \mid \bar{\mathcal{A}}] \leq \frac{\bar{\mathcal{S}}_{\gamma, \alpha}^\pi(T)}{M} \leq \frac{\mathcal{S}_{\gamma, \alpha}^\pi(T)}{M} + c_4 T^{-2},$$

where the last inequality holds by (7.2).

Step 7 (Likelihood of distinguishing between different smoothness parameters). We show that policy π cannot distinguish between ϕ_0 and ϕ_{m^*} with a strictly positive probability. For any set of samples $\{(\pi_t, X_t, Y_{\pi_t, t})\}_{t=1}^T$, define the log-likelihood ratio $L_{m, T} = L_{m, T} \left(\{(\pi_t, X_t, Y_{\pi_t, t})\}_{t=1}^T \right)$ for $m \in \{1, \dots, M\}$ as:

$$\begin{aligned} L_{m, T} &:= \sum_{t=1}^T \log \left(\frac{\mathbb{P}_{\pi, \phi_0} \{Y_{\pi_t, t} \mid \pi_t, X_t\}}{\mathbb{P}_{\pi, \phi_m} \{Y_{\pi_t, t} \mid \pi_t, X_t\}} \right) \\ &\leq \sum_{t=1}^T \mathbb{1} \{\pi_t = 1, X_t \in H_m\} \cdot \left[Y_{\pi_t, t} \log \left(\frac{\phi_0(X_t)}{\phi_m(X_t)} \right) + (1 - Y_{\pi_t, t}) \log \left(\frac{(1 - \phi_0(X_t))}{(1 - \phi_m(X_t))} \right) \right] \\ &\leq \sum_{t=1}^T \mathbb{1} \{\pi_t = 1, X_t \in H_m\} \cdot \left[Y_{\pi_t, t} \frac{(\phi_0(X_t) - \phi_m(X_t))}{\phi_m(X_t)} + (1 - Y_{\pi_t, t}) \frac{(\phi_m(X_t) - \phi_0(X_t))}{(1 - \phi_m(X_t))} \right] \\ &= \sum_{t=1}^T \mathbb{1} \{\pi_t = 1, X_t \in H_m\} \cdot \frac{(Y_{\pi_t, t} - \phi_m(X_t))(\phi_0(X_t) - \phi_m(X_t))}{\phi_m(X_t)(1 - \phi_m(X_t))}, \end{aligned}$$

where the last inequality follows from $\log(1+x) \leq x$ for all $x > 0$. By taking expectations of the above inequality and conditioning on the event $\bar{\mathcal{A}}$ for $m = m^*$, one obtains:

$$\begin{aligned} \mathbb{E}_{\pi, \phi_0} [L_{m^*, T} \mid \bar{\mathcal{A}}] &\leq \mathbb{E}_{\pi, \phi_0} \left[\sum_{t=1}^T \mathbb{1} \{\pi_t = 1, X_t \in H_{m^*}\} \cdot \frac{(Y_{\pi_t, t} - \phi_{m^*}(X_t))(\phi_0(X_t) - \phi_{m^*}(X_t))}{\phi_{m^*}(X_t)(1 - \phi_{m^*}(X_t))} \mid \bar{\mathcal{A}} \right] \\ &= \mathbb{E}_{\pi, \phi_0} \left[\sum_{t=1}^T \mathbb{E} \left[\mathbb{1} \{\pi_t = 1, X_t \in H_{m^*}\} \cdot \frac{(Y_{\pi_t, t} - \phi_{m^*}(X_t))(\phi_0(X_t) - \phi_{m^*}(X_t))}{\phi_{m^*}(X_t)(1 - \phi_{m^*}(X_t))} \mid X_t \right] \mid \bar{\mathcal{A}} \right] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{\pi, \phi_0} \left[\sum_{t=1}^T \mathbb{1} \{ \pi_t = 1, X_t \in H_{m^*} \} \cdot \frac{(\phi_0(X_t) - \phi_{m^*}(X_t))^2}{\phi_{m^*}(X_t)(1 - \phi_{m^*}(X_t))} \middle| \bar{\mathcal{A}} \right] \\
&\stackrel{(a)}{\leq} \frac{64C_\phi^2 \Delta^2}{3} \mathbb{E}_{\pi, \phi_0} \left[\sum_{t=1}^T \mathbb{1} \{ \pi_t = 1, X_t \in H_{m^*} \} \middle| \bar{\mathcal{A}} \right] \\
&= \frac{64C_\phi^2 \Delta^2}{3} \mathbb{E}_{\pi, \phi_0} [N_{m^*, T} \mid \bar{\mathcal{A}}] \stackrel{(b)}{\leq} \frac{64C_\phi^2 \Delta^2 \mathcal{S}_{\gamma, \alpha}^*(T)}{3M} + c_4 T^{-2} \stackrel{(c)}{\leq} 1,
\end{aligned} \tag{7.3}$$

for large enough T , where: (a) follows from $C_\phi \Delta \leq \frac{1}{4}$; (b) follows from the definition of m^* ; and (c) holds by the definition of Δ .

Step 8 (Lower bound on inferior sampling). Let $\tilde{N}_{m, T} := \sum_{t=1}^T \mathbb{1} \{ \pi_t = 1, X_t \in \tilde{H}_m \}$ denote the number of times policy π selects arm 1 when realized covariates belong to the hypercube \tilde{H}_m . We next use two lemmas in order to show that with a strictly positive probability one has $\tilde{N}_{m^*, T} < \frac{\rho T l^d}{2}$ conditional on the event $\bar{\mathcal{A}}$ under problem m^* , implying that π selects an inferior arm at least $\frac{\rho T l^d}{2}$ times. The first lemma is a simple variation of Lemma 2.6 in Tsybakov (2008) and is proved for completeness in Appendix E; the second lemma is a straightforward extension of Lemma 19 in Kaufmann et al. (2016).

LEMMA 7.5 (Hypothesis testing error probability). *Let ρ_0, ρ_1 be two probability distributions supported on \mathcal{X} , with ρ_0 absolutely continuous with respect to ρ_1 . Then, for any measurable function $\Psi : \mathcal{X} \rightarrow \{0, 1\}$:*

$$\mathbb{P}_{\rho_0} \{ \Psi(X) = 1 \} + \mathbb{P}_{\rho_1} \{ \Psi(X) = 0 \} \geq \frac{1}{2} \exp(-\text{KL}(\rho_0, \rho_1)).$$

LEMMA 7.6 (Log-likelihood ratio and historical events). *For any event $\mathcal{E} \in \mathcal{F}_t^- = \sigma(\pi_1, X_1, Y_{\pi_1, 1}, \dots, \pi_T, X_T, Y_{\pi_T, T})$ and an arbitrary event \mathcal{A} , one has*

$$\mathbb{E}_{\pi, \phi_0} [L_{m, T} \mid \mathcal{E}, \mathcal{A}] \geq \log \left(\frac{\mathbb{P}_{\pi, \phi_0} \{ \mathcal{E} \mid \mathcal{A} \}}{\mathbb{P}_{\pi, \phi_{m^*}} \{ \mathcal{E} \mid \mathcal{A} \}} \right).$$

Denote by ρ_0 and ρ_m the distributions of $\tilde{N}_{m, T}$ under the problems 0 and m conditional on the event $\bar{\mathcal{A}}$. Define the test function $\Psi(x) = \mathbb{1} \left\{ x \geq \frac{\rho T l^d}{2} \right\}$. With this selection of ρ_0 , ρ_m , and Ψ , Lemma 7.5 yields:

$$\mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_{m^*, T} \geq \frac{\rho T l^d}{2} \middle| \bar{\mathcal{A}} \right\} + \mathbb{P}_{\pi, \phi_{m^*}} \left\{ \tilde{N}_{m^*, T} < \frac{\rho T l^d}{2} \middle| \bar{\mathcal{A}} \right\} \geq \frac{1}{2} \exp(-\text{KL}(\rho_0, \rho_{m^*})).$$

To establish a lower bound on the right hand side of the above inequality, we note that:

$$\begin{aligned}
\mathbb{E}_{\pi, \phi_0} [L_{m^*, T} \mid \bar{\mathcal{A}}] &= \sum_{s=1}^T \mathbb{E}_{\pi, \phi_0} [L_{m^*, T} \mid \bar{\mathcal{A}}, \tilde{N}_{m^*, T} = s] \mathbb{P}_{\pi, \phi_0} \{ \tilde{N}_{m^*, T} = s \mid \bar{\mathcal{A}} \} \\
&\geq \sum_{s=1}^T \log \left(\frac{\mathbb{P}_{\pi, \phi_0} \{ \tilde{N}_{m^*, T} = s \mid \bar{\mathcal{A}} \}}{\mathbb{P}_{\pi, \phi_{m^*}} \{ \tilde{N}_{m^*, T} = s \mid \bar{\mathcal{A}} \}} \right) \mathbb{P}_{\pi, \phi_0} \{ \tilde{N}_{m^*, T} = s \mid \bar{\mathcal{A}} \} = \text{KL}(\rho_0, \rho_{m^*}),
\end{aligned}$$

where the inequality follows from Lemma 7.6. The last two inequalities, along with (7.3), yield

$$\mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_{m^*, T} \geq \frac{\rho T l^d}{2} \middle| \bar{\mathcal{A}} \right\} + \mathbb{P}_{\pi, \phi_{m^*}} \left\{ \tilde{N}_{m^*, T} < \frac{\rho T l^d}{2} \middle| \bar{\mathcal{A}} \right\} \geq \frac{1}{2} \exp(-1).$$

Next, we show that $\mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_{m^*, T} \geq \frac{\rho T l^d}{2} \mid \bar{\mathcal{A}} \right\}$ is small. We apply Markov's inequality to obtain:

$$\begin{aligned} \mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_{m^*, T} \geq \frac{\rho T l^d}{2} \mid \bar{\mathcal{A}} \right\} &\leq \frac{\mathbb{E}_{\pi, \phi_0} \left[\tilde{N}_{m^*, T} \mid \bar{\mathcal{A}} \right]}{\frac{\rho T l^d}{2}} \stackrel{(a)}{\leq} \frac{\frac{\mathcal{S}_{\gamma, \alpha}^*(T)}{M} + c_4 T^{-2}}{\frac{\rho T l^d}{2}} \\ &\stackrel{(b)}{\leq} \frac{2^d [\mathcal{S}_{\gamma, \alpha}^*]^{1 + \frac{\alpha \beta}{2\beta + d - \alpha \beta}} + c_4 l^{-d} T^{-2}}{\frac{\rho T}{2}} \leq c_5 T^{\frac{\alpha d(\beta - \gamma)}{(2\beta + d - \alpha \beta)(2\gamma + d)}} \stackrel{(c)}{\leq} \frac{1}{4} \exp(-1), \end{aligned}$$

for large enough T and some constant $c_5 > 0$, where: (a) follows from the definition of m^* and (7.2); (b) holds due to the definition of l and M ; and (c) holds due to the fact that $\frac{\alpha d(\beta - \gamma)}{(2\beta + d - \alpha \beta)(2\gamma + d)} < 0$ since $\alpha \leq \frac{1}{\gamma}$. The last two displays yield that for large enough T , one has $\mathbb{P}_{\pi, \phi_{m^*}} \left\{ \tilde{N}_{m^*, T} < \frac{\rho T l^d}{2} \mid \bar{\mathcal{A}} \right\} \geq \frac{1}{4e}$. By definition, when event $\bar{\mathcal{A}}$ holds, at least $\frac{\rho T l^d}{2}$ times realized contexts belong to the hypercube \tilde{H}_m , that is, for some constant $c_6 > 0$, one has:

$$\bar{\mathcal{S}}^\pi(T; \phi_{m^*}) \geq \frac{\rho T l^d}{2} \mathbb{P}_{\pi, \phi_{m^*}} \left\{ \tilde{N}_{m^*, T} < \frac{\rho T l^d}{2} \mid \bar{\mathcal{A}} \right\} \geq \frac{\rho T l^d}{8e} \geq c_6 T [\mathcal{S}_{\gamma, \alpha}^*(T)]^{-\frac{d}{2\beta + d - \alpha \beta}}. \quad (7.4)$$

Step 9 (From inferior sampling rate to regret). Note that $\mathcal{S}_{\beta, \alpha}^\pi(T) \geq \bar{\mathcal{S}}^\pi(\phi_{m^*}; T)$. That is, by putting together (7.4), (7.2), and Lemma 7.1, one obtains

$$\mathcal{R}_{\beta, \alpha}^\pi(T) \geq \underline{C} T^{1 - \frac{d}{\alpha(2\beta + d - \alpha \beta)}} [\mathcal{R}_{\gamma, \alpha}^*(T)]^{-\frac{d}{2\beta + d - \alpha \beta}},$$

for some constant $\underline{C} > 0$. This concludes the proof. ■

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Appendix A: Proofs of main results

A.1. Verifying the margin condition in Step 4 of the proof of Theorem 3.1.

- For $m = 0$ and $\delta \leq C_\phi \Delta$, one has

$$\begin{aligned}
 \mathbf{P}_X \left\{ 0 < |\phi_0(X) - \frac{1}{2}| \leq \delta \right\} &\leq \bar{\rho} \int_{H_0} \mathbb{1} \left\{ C_\phi \Delta^{\frac{\alpha\gamma}{d}} \cdot \tilde{\psi}_\gamma \left(\Delta^{-\frac{\alpha}{d}} [x - q_0] \right) \leq \delta \right\} dx \\
 &\leq 2^d \bar{\rho} \Delta^\alpha \int_{[0,1]^d} \mathbb{1} \left\{ \tilde{\psi}_\gamma(x) \leq \delta C_\phi^{-1} \Delta^{-\frac{\alpha\gamma}{d}} \right\} dx \\
 &\leq 2^d \bar{\rho} \Delta^\alpha \left[1 - \int_{[0,1]^d} \mathbb{1} \left\{ \|x\|_\infty \leq 1 - \delta^{\frac{1}{\gamma}} C_\phi^{-\frac{1}{\gamma}} \Delta^{-\frac{\alpha}{d}} \right\} dx \right] \\
 &\leq 2^d \bar{\rho} \Delta^\alpha \left[1 - \left(1 - \delta^{\frac{1}{\gamma}} C_\phi^{-\frac{1}{\gamma}} \Delta^{-\frac{\alpha}{d}} \right)^d \right] \\
 &\leq 2^d \bar{\rho} \Delta^\alpha \left[d \delta^{\frac{1}{\gamma}} C_\phi^{-\frac{1}{\gamma}} \Delta^{-\frac{\alpha}{d}} \right] \stackrel{(a)}{\leq} 2^d d \bar{\rho} C_\phi^{-\alpha} \Delta^{\frac{1}{\gamma} - \frac{\alpha}{d}} \delta^\alpha \leq 2^d d \bar{\rho} C_\phi^{-\alpha} \delta^\alpha, \tag{A.1}
 \end{aligned}$$

where (a) holds since $\alpha \leq \frac{1}{\gamma}$.

- For $m = 0$ and $\delta > C_\phi \Delta$, one has

$$\mathbf{P}_X \left\{ 0 < |\phi_0(X) - \frac{1}{2}| \leq \delta \right\} \leq 2^d \bar{\rho} \Delta^\alpha \leq 2^d \bar{\rho} C_\phi^{-\alpha} \delta^\alpha.$$

- For $1 \leq m \leq M$ and $\delta \leq C_\phi \Delta$, one has

$$\begin{aligned}
 \mathbf{P}_X \left\{ 0 < |\phi_m(X) - \frac{1}{2}| \leq \delta \right\} &\leq \mathbf{P}_X \left\{ 0 < C_\phi \cdot \Delta \cdot |1 - 2l^{-1} \|x - q_m\|_\infty|^\beta \leq \delta, X \in \tilde{H}_m \right\} \\
 &\quad + \mathbf{P}_X \left\{ 0 < C_\phi \cdot \Delta \cdot |2l^{-1} \|x - q_m\|_\infty - 1|^\beta \leq \delta, X \in H_m \setminus \tilde{H}_m \right\} \\
 &\quad + \mathbf{P}_X \left\{ 0 < |\phi_0(X) - \frac{1}{2}| \leq \delta \right\}. \tag{A.2}
 \end{aligned}$$

Next, we analyze each term separately. One has

$$\begin{aligned}
 \mathbf{P}_X \left\{ 0 < C_\phi \cdot \Delta \cdot |1 - 2l^{-1} \|x - q_m\|_\infty|^\beta \leq \delta, X \in \tilde{H}_m \right\} \\
 &\leq \bar{\rho} \int_{\tilde{H}_m} \mathbb{1} \left\{ C_\phi \cdot \Delta \cdot |1 - 2l^{-1} \|x - q_m\|_\infty|^\beta \leq \delta \right\} dx \\
 &\leq \bar{\rho} \int_{\tilde{H}_m} \mathbb{1} \left\{ \|x - q_m\|_\infty \geq \frac{l}{2} \left(1 - C_\phi^{-\frac{1}{\beta}} \Delta^{-\frac{1}{\beta}} \delta^{\frac{1}{\beta}} \right) \right\} dx \\
 &= \bar{\rho} 2^{-d} l^d \left[1 - \left(1 - C_\phi^{-\frac{1}{\beta}} \Delta^{-\frac{1}{\beta}} \delta^{\frac{1}{\beta}} \right)^d \right] \\
 &\stackrel{(a)}{\leq} \bar{\rho} 2^{-d} l^d \left[d \delta^{\frac{1}{\beta}} C_\phi^{-\frac{1}{\beta}} \Delta^{-\frac{1}{\beta}} \right] \stackrel{(b)}{\leq} d \bar{\rho} 2^{-2d} C_\phi^{-\alpha} \delta^\alpha, \tag{A.3}
 \end{aligned}$$

where (a) follows from the inequality $(1 - x)^r \geq 1 - rx$ for $0 \leq x \leq 1, r \geq 1$, and (b) holds by $\alpha \leq \frac{1}{\gamma}$. Similarly,

$$\begin{aligned}
 \mathbf{P}_X \left\{ 0 < C_\phi \Delta |2l^{-1} \|x - q_m\|_\infty - 1|^\beta \leq \delta, X \in H_m \setminus \tilde{H}_m \right\} \\
 &\leq \bar{\rho} \int_{H_m \setminus \tilde{H}_m} \mathbb{1} \left\{ C_\phi \Delta |2l^{-1} \|x - q_m\|_\infty - 1|^\beta \leq \delta \right\} dx \\
 &\leq \bar{\rho} \int_{H_m \setminus \tilde{H}_m} \mathbb{1} \left\{ \|x - q_m\|_\infty \leq \frac{l}{2} \left(1 + C_\phi^{-\frac{1}{\beta}} \Delta^{-\frac{1}{\beta}} \delta^{\frac{1}{\beta}} \right) \right\} dx \\
 &= \bar{\rho} 2^{-d} l^d \left[\left(1 + C_\phi^{-\frac{1}{\beta}} \Delta^{-\frac{1}{\beta}} \delta^{\frac{1}{\beta}} \right)^d - 1 \right] \\
 &\stackrel{(a)}{\leq} \bar{\rho} l^d C_\phi^{-\frac{1}{\beta}} \Delta^{-\frac{1}{\beta}} \delta^{\frac{1}{\beta}} \stackrel{(b)}{\leq} \bar{\rho} C_\phi^{-\alpha} \delta^\alpha, \tag{A.4}
 \end{aligned}$$

where (a) follows from the inequality $(1+x)^r \leq 2^r x + 1$ for $0 \leq x \leq 1, r \geq 1$, and (b) holds by $\alpha \leq \frac{1}{\gamma}$. Putting together (A.1), (A.2), (A.3), and (A.4), yields for $\delta \leq C_\phi \Delta$:

$$\mathbf{P}_X \left\{ 0 < |\phi_m(X) - \frac{1}{2}| \leq \delta \right\} \leq C_0 \delta^\alpha.$$

- The case $1 \leq m \leq M$ and $\delta > C_\phi \Delta$ can be analyzed similar to the case $m = 0$ and $\delta > C_\phi \Delta$.

A.2. Proof of Part 2 of Theorem 3.1

The proof follows similar lines of argument as in the proof of Part 1 of Theorem 3.1.

Step 1 (Preliminaries). Fix time horizon length $T \geq 1$, and some Hölder exponent $\gamma > 1$, some margin parameter $0 \leq \alpha \leq 1$, some positive Lipschitz constants L , and some positive constants $\underline{\rho}, \bar{\rho}$ such that \mathbf{P}_X , the covariate distribution, satisfies Assumption 2.2 with parameters $\underline{\rho}, \bar{\rho}$.

Step 2 (From regret to inferior sampling rate). By Lemma 7.1, we have $\mathcal{S}_{\beta, \alpha}^\pi(T) \leq C_{sr} T^{\frac{1}{\alpha+1}} [\mathcal{R}_{\beta, \alpha}^\pi(T)]^{\frac{\alpha}{\alpha+1}}$. Note that by the assumption that π is rate-optimal over $\mathcal{P}(\gamma, \alpha, d)$ and Lemma 7.1, one has

$$\mathcal{R}_{\gamma, \alpha}^\pi(T) \leq C_r T^{1 - \frac{\gamma(1+\alpha)}{2\beta+d}} =: \mathcal{R}_{\gamma, \alpha}^*(T), \quad \mathcal{S}_{\gamma, \alpha}^\pi(T) \leq C_s T^{1 - \frac{\gamma\alpha}{2\gamma+d}} =: \mathcal{S}_{\gamma, \alpha}^*(T),$$

for some constants $C_r, C_s > 0$.

Step 3 (Constructing problem instances). We will reduce our problem to a hypothesis testing problem. To do so, we construct some problem instances first. Define the parameter $\Delta > 0$ such that

$$\frac{64C_\phi^2 \Delta^2 \mathcal{S}_{\gamma, \alpha}^*(T)}{3} = \frac{1}{2},$$

where we define $C_\phi := \frac{L}{2^{2\beta}}$. Note that the definition of Δ implies that for large enough T , one has $C_\phi \Delta \leq \frac{1}{4}$. Define the function:

$$\phi_0(x) := \frac{1}{2} - C_\phi \cdot \left(\frac{1}{2} - x_1\right).$$

Note that $\phi_0 \in \mathcal{H}(\gamma, L)$ since $C_\phi \leq L$. Define the hypercube $H := [\frac{1}{2} - \Delta, \frac{1}{2}] \times [0, 1]^{d-1}$, with a center $q_0 := (\frac{1-\Delta}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^d$, and the function:

$$\phi_1(x) := \phi_0(x) + 2C_\phi \cdot \Delta \cdot \tilde{\psi}(2\Delta^{-1}[x - q_0]).$$

For any $0 < \kappa \leq 1$, define the functions $\tilde{\psi}$:

$$\tilde{\psi}(x) := \begin{cases} |1 - |x_1|| & \text{if } |x_1| \leq 1 \\ 0 & \text{o.w.} \end{cases}.$$

Note that by Lemmas 7.2 and 7.3, $\phi_0 \in \mathcal{H}(1, L)$ since $C_\phi \leq \frac{L}{2^{2\beta}}$.

Step 4 (Verifying the margin condition). We verify that the margin condition is satisfied with parameters α and $C_0 := \frac{5\bar{\rho}}{2C_\phi}$ when $f_1 = \phi_m$ and $f_2 = \frac{1}{2}$ for all $0 \leq m \leq 1$.

- For $m = 0$ and $0 < \delta \leq 1$, one has

$$\mathbf{P}_X \left\{ 0 < |\phi_0(X) - \frac{1}{2}| \leq \delta \right\} \leq \frac{2\bar{\rho}\delta}{C_\phi} \leq \frac{2\bar{\rho}\delta^\alpha}{C_\phi}.$$

- For $m = 1$ and $0 < \delta \leq 1$, one has

$$\mathbf{P}_X \left\{ 0 < |\phi_m(X) - \frac{1}{2}| \leq \delta \right\} \leq \frac{5\bar{\rho}\delta}{2C_\phi} \leq \frac{5\bar{\rho}\delta^\alpha}{2C_\phi}.$$

Step 5 (Desirable event). Note that for $x \in \tilde{H} := [\frac{1}{2} - \frac{2\Delta}{3}, \frac{1}{2}] \times [0, 1]^{d-1}$, the first arm is optimal under the problem $m = 1$. Define $Q := \sum_{t=1}^T \mathbb{1}\{X_t \in \tilde{H}\} =: \sum_{t=1}^T Z_t$ to be the number of times contexts fall into the hypercube \tilde{H}_m during the entire time horizon. Define the event

$$\mathcal{A} := \left\{ Q < \frac{2}{3}\rho T\Delta \right\}$$

to be the event on which the number of contexts that have fallen into the hypercube \tilde{H} is less than $\frac{2}{3}\rho T\Delta$. In order to bound $\mathbb{P}\{\mathcal{A}\}$, one can apply Bernstein's inequality in Lemma 7.4 to Q by noting that $\mathbb{E}Z_t \geq \frac{2}{3}\rho\Delta$, $|Z_t| \leq 1$, and $\text{Var}Z_t \leq \mathbb{E}Z_t^2 \leq \frac{2}{3}\rho\Delta$ to obtain

$$\mathbb{P}\{\mathcal{A}\} \leq \exp\left(-\frac{2}{3}\rho T\Delta/5\right) \stackrel{(a)}{\leq} \exp\left(-c_1 T[\mathcal{S}_{\gamma,\alpha}^*(T)]^{-\frac{1}{2}}\right) \leq \exp\left(-c_1 T^{\frac{(1+\frac{\alpha}{2})\gamma+\frac{d}{2}}{(2\gamma+d)}}\right) \leq c_2 T^{-3},$$

for large enough T and constants $c_1, c_2 > 0$, where (a) follows from the definition of Δ .

For any problem instance \mathbf{P} and time horizon T , denote by

$$\bar{\mathcal{S}}^\pi(\mathbf{P}; T) := \mathbb{E}^\pi \left[\sum_{t=1}^T \mathbb{1}\{f_{\pi_t^*}(X_t) \neq f_{\pi_t}(X_t)\} \mid \bar{\mathcal{A}} \right]$$

the inferior sampling rate of π when the event \mathcal{A} fails, and let $\bar{\mathcal{S}}_{\gamma,\alpha}^\pi(T) := \sup_{\mathbf{P} \in \mathcal{P}(\gamma,\alpha,d)} \bar{\mathcal{S}}^\pi(\mathbf{P}; T)$. Note that

$$(1 - \mathbb{P}\{\mathcal{A}\}) \bar{\mathcal{S}}^\pi(\mathbf{P}; T) \leq \mathcal{S}^\pi(\mathbf{P}; T) \leq \bar{\mathcal{S}}^\pi(\mathbf{P}; T) + T\mathbb{P}\{\mathcal{A}\},$$

which implies

$$|\bar{\mathcal{S}}_{\gamma,\alpha}^\pi(T) - \mathcal{S}_{\gamma,\alpha}^\pi(T)| \leq c_2 T^{-2}. \quad (\text{A.5})$$

For the rest of the proof probabilities and expectations will be computed conditional on the event $\bar{\mathcal{A}}$.

Step 6 (Likelihood of distinguishing different smoothness parameters). In this step, we will show that policy π cannot distinguish between ϕ_0 and ϕ_1 with a strictly positive probability. For any set of samples $\{(\pi_t, X_t, Y_{\pi_t,t})\}_{t=1}^T$, define the log-likelihood ratio $L_T = L_T(\{(\pi_t, X_t, Y_{\pi_t,t})\}_{t=1}^T)$ as:

$$\begin{aligned} L_T &:= \sum_{t=1}^T \log \left(\frac{\mathbb{P}_{\pi,\phi_0}\{Y_{\pi_t,t} \mid \pi_t, X_t\}}{\mathbb{P}_{\pi,\phi_1}\{Y_{\pi_t,t} \mid \pi_t, X_t\}} \right) \\ &\leq \sum_{t=1}^T \mathbb{1}\{\pi_t = 1, X_t \in H\} \cdot \left[Y_{\pi_t,t} \log \left(\frac{\phi_0(X_t)}{\phi_1(X_t)} \right) + (1 - Y_{\pi_t,t}) \log \left(\frac{(1 - \phi_0(X_t))}{(1 - \phi_1(X_t))} \right) \right] \\ &\leq \sum_{t=1}^T \mathbb{1}\{\pi_t = 1, X_t \in H\} \cdot \left[Y_{\pi_t,t} \frac{(\phi_0(X_t) - \phi_1(X_t))}{\phi_m(X_t)} + (1 - Y_{\pi_t,t}) \frac{(\phi_1(X_t) - \phi_0(X_t))}{(1 - \phi_1(X_t))} \right] \\ &= \sum_{t=1}^T \mathbb{1}\{\pi_t = 1, X_t \in H\} \cdot \frac{(Y_{\pi_t,t} - \phi_1(X_t))(\phi_0(X_t) - \phi_1(X_t))}{\phi_1(X_t)(1 - \phi_1(X_t))}, \end{aligned}$$

where the last inequality follows from $\log(1+x) \leq x$ for all $x > 0$. Taking expectation conditional on the event $\bar{\mathcal{A}}$, one obtains:

$$\begin{aligned} \mathbb{E}_{\pi,\phi_0}[L_T \mid \bar{\mathcal{A}}] &\leq \mathbb{E}_{\pi,\phi_0} \left[\sum_{t=1}^T \mathbb{1}\{\pi_t = 1, X_t \in \tilde{H}\} \cdot \frac{(Y_{\pi_t,t} - \phi_1(X_t))(\phi_0(X_t) - \phi_1(X_t))}{\phi_1(X_t)(1 - \phi_1(X_t))} \mid \bar{\mathcal{A}} \right] \\ &\leq \mathbb{E}_{\pi,\phi_0} \left[\sum_{t=1}^T \mathbb{E} \left[\mathbb{1}\{\pi_t = 1, X_t \in \tilde{H}_1\} \cdot \frac{(Y_{\pi_t,t} - \phi_1(X_t))(\phi_0(X_t) - \phi_1(X_t))}{\phi_1(X_t)(1 - \phi_1(X_t))} \mid X_t \right] \mid \bar{\mathcal{A}} \right] \\ &= \mathbb{E}_{\pi,\phi_0} \left[\sum_{t=1}^T \mathbb{1}\{\pi_t = 1, X_t \in \tilde{H}\} \cdot \frac{(\phi_0(X_t) - \phi_1(X_t))^2}{\phi_1(X_t)(1 - \phi_1(X_t))} \mid \bar{\mathcal{A}} \right] \\ &\stackrel{(a)}{\leq} \frac{64C_\phi^2\Delta^2}{3} \mathbb{E}_{\pi,\phi_0} \left[\sum_{t=1}^T \mathbb{1}\{\pi_t = 1, X_t \in \tilde{H}\} \mid \bar{\mathcal{A}} \right] \\ &\leq \frac{64C_\phi^2\Delta^2\mathcal{S}_{\gamma,\alpha}^*(T)}{3} + c_2 T^{-2} \stackrel{(b)}{\leq} 1, \end{aligned} \quad (\text{A.6})$$

for large enough T , where (a) follows from $C_\phi\Delta \leq \frac{1}{4}$, and (b) follows from the definition of Δ .

Step 7 (Lower bound on inferior sampling). Let $\tilde{N}_T := \sum_{t=1}^T \mathbb{1} \left\{ \pi_t = 1, X_t \in \tilde{H} \right\}$ be the number of times policy π pulls arm 1 when covariates fall into the hypercube \tilde{H} . We will show that with a strictly positive probability one has $\tilde{N}_T < \frac{\rho T \Delta}{3}$ conditional on the event $\bar{\mathcal{A}}$. This will imply that policy π makes at least $\frac{\rho T \Delta}{3}$ number of mistakes under the problem $m = 1$.

Denote by ρ_0 and ρ_1 the distribution of \tilde{N}_T under the problems $m = 0$ and $m = 1$ conditional on the event $\bar{\mathcal{A}}$. Define the test function $\Psi(x) = \mathbb{1} \left\{ x \geq \frac{\rho T \Delta}{3} \right\}$. With this choice of ρ_0 , ρ_1 , and Ψ , one can apply Lemma 7.5 to obtain

$$\mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_T \geq \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\} + \mathbb{P}_{\pi, \phi_1} \left\{ \tilde{N}_T < \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\} \geq \frac{1}{2} \exp(-\text{KL}(\rho_0, \rho_1)).$$

In order to lower bound the right hand side of this inequality, we note that

$$\begin{aligned} \mathbb{E}_{\pi, \phi_0} [L_T \mid \bar{\mathcal{A}}] &= \sum_{s=1}^T \mathbb{E}_{\pi, \phi_0} [L_T \mid \bar{\mathcal{A}}, \tilde{N}_T = s] \mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_T = s \mid \bar{\mathcal{A}} \right\} \\ &\geq \sum_{s=1}^T \log \left(\frac{\mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_T = s \mid \bar{\mathcal{A}} \right\}}{\mathbb{P}_{\pi, \phi_1} \left\{ \tilde{N}_T = s \mid \bar{\mathcal{A}} \right\}} \right) \mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_T = s \mid \bar{\mathcal{A}} \right\} = \text{KL}(\rho_0, \rho_1), \end{aligned}$$

where the inequality follows from Lemma 7.6. The last two displays along with (A.6) yield

$$\mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_T \geq \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\} + \mathbb{P}_{\pi, \phi_1} \left\{ \tilde{N}_T < \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\} \geq \frac{1}{2} \exp(-1).$$

To show that $\mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_T \geq \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\}$ is small, we apply Markov's inequality:

$$\begin{aligned} \mathbb{P}_{\pi, \phi_0} \left\{ \tilde{N}_T \geq \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\} &\leq \frac{\mathbb{E}_{\pi, \phi_0} [\tilde{N}_T \mid \bar{\mathcal{A}}]}{\frac{\rho T \Delta}{3}} \stackrel{(a)}{\leq} \frac{\mathcal{S}_{\gamma, \alpha}^*(T) + c_4 T^{-2}}{\frac{\rho T \Delta}{3}} \\ &\leq \frac{c_3 [\mathcal{S}_{\gamma, \alpha}^*]^{\frac{1}{2}} + c_2 l^{-d} T^{-2}}{\frac{\rho T}{2}} \leq c_4 T^{-\frac{1}{2}} \leq \frac{1}{4} \exp(-1) \end{aligned}$$

for large enough T and some constant $c_3, c_4 > 0$, where (a) follows from (A.5). The last two displays yield that for large enough T , one has

$$\mathbb{P}_{\pi, \phi_1} \left\{ \tilde{N}_T < \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\} \geq \frac{1}{4e}.$$

Note that by definition, when the event $\bar{\mathcal{A}}$ holds, at least $\frac{2\rho T \Delta}{3}$ number of contexts fall into the hypercube \tilde{H} , that is,

$$\bar{\mathcal{S}}^\pi(T; \phi_1) \geq \frac{\rho T \Delta}{3} \mathbb{P}_{\pi, \phi_1} \left\{ \tilde{N}_T < \frac{\rho T \Delta}{3} \mid \bar{\mathcal{A}} \right\} \geq \frac{\rho T \Delta}{12e} \geq c_5 T [\mathcal{S}_{\gamma, \alpha}^*(T)]^{-\frac{1}{2}}, \quad (\text{A.7})$$

for some constant $c_5 > 0$.

Step 8 (From inferior sampling rate to regret). Note that $\mathcal{S}_{1, \alpha}^\pi(T) \geq \bar{\mathcal{S}}^\pi(\phi_1; T)$. That is, by putting together (A.7), (A.5), and Lemma 7.1, one obtains

$$\mathcal{R}_{1, \alpha}^\pi(T) \geq \underline{C} T^{1 - \frac{1}{2\alpha}} [\mathcal{R}_{\gamma, \alpha}^*(T)]^{-\frac{1}{2}},$$

for some constant $\underline{C} > 0$. This concludes the proof. ■

A.3. Proof of Theorem 4.1

The following lemma characterizes a general class of self-similar payoff functions for any non-integer smoothness parameter $\beta \in [\underline{\beta}, \bar{\beta}]$.

LEMMA A.1. *Fix dimension d , some positive non-integer β and some $\bar{\beta} \geq \beta$. Consider some set of payoff functions $\{f_k\}_k$ such that $f_k \in \mathcal{H}(\beta)$, $k \in \mathcal{K}$. Suppose $f_1(x) = a + bx_1^\beta$ for $x_1 \in [0, c]$ where a, b and $0 \leq c \leq 1$ are some constants. Then, the set of payoff functions $\{f_k\}_k$ is self-similar as in Definition 4.1.*

Proof. It suffices to show that for any non-negative integer p , one has

$$\max_{B \in \mathcal{B}_l} \max_{k \in \mathcal{K}} \sup_{x \in B} |\Gamma_l^p f_k(x; B) - f_k(x)| \geq b' 2^{-l\beta},$$

for any $l \geq l_0 = \lceil \log \frac{1}{c} \rceil$ and some $b' > 0$. Fix some $l > l_0$. Let $B_0 := [0, 2^{-l}]^d$. One has

$$\max_{B \in \mathcal{B}_l} \max_{k \in \mathcal{K}} \sup_{x \in B} |\Gamma_l^p f_k(x; B) - f_k(x)| \geq |\Gamma_l^p f_1(0; B_0) - f_1(0)| = b |\Gamma_l^p g(0; B_0)|, \quad (\text{A.8})$$

where $g(x) = x^\beta$. By Part 1 of Lemma B.1, one has $\Gamma_l^p g(0; B_0) = e_1^\top B^{-1} W$, where

$$e_1 = (\mathbb{1}\{s=0\})_{s \in \{0,1,\dots,p\}}, \quad B = \left(\frac{1}{s_1 + s_2 + 1} \right)_{s_1, s_2 \in \{0,1,\dots,p\}}, \quad W = \left(\frac{2^{-l\beta}}{s + \beta + 1} \right)_{s \in \{0,1,\dots,p\}}.$$

By Cramer's rule for linear matrix equations, one has

$$\Gamma_l^p g(0; B_0) = \frac{\det(B_1)}{\det(B)} 2^{-l\beta}, \quad (\text{A.9})$$

where

$$B_1 = \begin{pmatrix} \frac{1}{\beta+1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{p+1} \\ \frac{1}{\beta+2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{p+2} \\ \frac{1}{\beta+3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{p+3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{\beta+p+1} & \frac{1}{p+2} & \frac{1}{p+3} & \cdots & \frac{1}{2p+1} \end{pmatrix}.$$

Note that one can rewrite both matrices B and B_1 as follows

$$B = \left(\frac{1}{u_i + w_j} \right)_{1 \leq i, j \leq p+1}, \quad u_i = i, w_j = j - 1;$$

$$B_1 = \left(\frac{1}{u'_i + w'_j} \right)_{1 \leq i, j \leq p+1}, \quad u'_i = i, w'_j = \beta \mathbb{1}\{j=1\} + (j-1) \mathbb{1}\{j>1\}.$$

The next theorem shows that the determinants of B and B_1 are non-zero.

THEOREM A.1 (Cauchy double alternant determinant). *For any set of indeterminates $\{u_i\}_{1 \leq i \leq n}$ and $\{v_j\}_{1 \leq j \leq n}$ such that $u_i + v_j \neq 0$, $\forall i, j \in \{1, \dots, n\}$, one has*

$$\det \left(\frac{1}{u_i + w_j} \right)_{1 \leq i, j \leq n} = \frac{\prod_{1 \leq i < j \leq n} (u_i - u_j)(w_i - w_j)}{\prod_{1 \leq i \neq j \leq n} (u_i + w_j)}.$$

Hence, putting together A.8 and A.9 yields that for any integer $l > l_0$,

$$\max_{B \in \mathcal{B}_l} \max_{k \in \mathcal{K}} \sup_{x \in B} |\Gamma_l^p f_k(x; B) - f_k(x)| \geq b \frac{\det(B_1)}{\det(B)} 2^{-l\beta}.$$

This concludes the proof. ■

Using Lemma A.1, one can adjust the lower bound arguments in Rigollet and Zeevi (2010) and Hu et al. (2019) in order to establish the same lower bounds for optimal regret when payoff functions are self-similar. We provide here the proof of the second part of the theorem; the proof of the first part is very similar, except for using Theorem 4.1 in Rigollet and Zeevi (2010) instead of Theorem 3 in Hu et al. (2019). First, we define the class of problems of interest.

DEFINITION A.1. For any $\beta \geq 0$ and $\alpha \geq 0$, we denote by $\tilde{\mathcal{P}}(\beta, \alpha, d) = \tilde{\mathcal{P}}(\beta, L, \alpha, C_0, \underline{\rho}, \bar{\rho})$ the class of problems $\mathbf{P} = (\mathbf{P}_X, \mathbf{P}_{Y|X}^{(1)}, \mathbf{P}_{Y|X}^{(2)})$ that satisfy Assumption 2.1 for β and $L > 0$, Assumption 2.3 for α and some $C_0 > 0$, and the following assumption regarding covariate distribution: the covariate density p_X has a compact support $\mathcal{X} \subseteq [0, 1]^d$ and $\underline{\rho} \leq p_X(x) \leq \bar{\rho}$ for some $\underline{\rho} \geq \bar{\rho} > 0$ and $x \in \mathcal{X}$. Furthermore, For any $\underline{\beta} \leq 1$, $\bar{\beta} > \underline{\beta}$, $\beta \in [\underline{\beta}, \bar{\beta}]$, and $\alpha \geq 0$, we define by $\tilde{\mathcal{P}}^{\text{ss}}(\underline{\beta}, \bar{\beta}, \beta, \alpha, d) := \left\{ \mathbf{P} \in \tilde{\mathcal{P}}(\beta, \alpha, d) : \mathbf{P} \text{ satisfies Assumption 4.1} \right\}$ the corresponding class of problems with self-similar payoffs.

In their Theorem 3, Hu et al. (2019) construct a problem instance $\mathbf{P}^* \in \tilde{\mathcal{P}}(\beta, \alpha, d)$ such that $\mathcal{R}^\pi(\mathbf{P}^*; T) \geq CT^{1 - \frac{\beta(\alpha+1)}{2\beta+d}}$ for some constant $C > 0$. Let $\{f_k^*\}_k$ be the set of payoff functions of \mathbf{P}^* . Define the set of payoff functions $\{f_k^{**}\}_k$ such that

$$f_k^{**}(x) := \begin{cases} \frac{1+L_1x_1^\beta}{2} & \text{if } 0 \leq x_1 \leq \frac{1}{8}, \\ \frac{1+L_1u(x_1)x_1^\beta}{2} & \text{if } \frac{1}{8} \leq x_1 \leq \frac{1}{4}, \\ \frac{1}{2} & \text{if } \frac{1}{4} \leq x_1 \leq \frac{1}{2}, \\ L_1f_k^*(g(x)) & \text{if } \frac{1}{2} \leq x_1 \leq 1, \end{cases}$$

where $L_1 > 0$ is some constant and we define

$$g(x) := \begin{pmatrix} 2x_1 - 1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{pmatrix}, \quad u(x_1) := \frac{\int_{x_1}^{\frac{1}{4}} \exp\left(\frac{-1}{|s-\frac{1}{8}||s-\frac{1}{4}|}\right) ds}{\int_{\frac{1}{8}}^{\frac{1}{4}} \exp\left(\frac{-1}{|s-\frac{1}{8}||s-\frac{1}{4}|}\right) ds}.$$

Now, we show that $f_k^{**} \in \mathcal{H}(\beta)$, $k \in \mathcal{K}$. Note that $u(x_1)$ is infinitely differentiable over $[\frac{1}{8}, \frac{1}{4}]$ and $x_1^\beta \in \mathcal{H}(\beta)$. Hence, by the following lemma, $u(x_1)x_1^\beta \in \mathcal{H}_{[\frac{1}{8}, \frac{1}{4}]}(\beta)$.

LEMMA A.2. Suppose $f, g \in \mathcal{H}_X(\beta, L)$ for some $X \subseteq [0, 1]$, $\beta > 0$, and $L > 0$, and define the function $h := f \cdot g$ as the product of f and g . Then, $h \in \mathcal{H}(\beta, L')$ for some $L' > 0$.

Furthermore, any derivative of f_k^{**} up to degree $\lfloor \beta \rfloor$ exists for $x_1 \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}\}$. Hence, $f_k^{**} \in \mathcal{H}(\beta)$. One can also make $L_1 > 0$ small enough so that $f_k^{**} \in \mathcal{H}(\beta, L)$. Finally, by Lemma A.1, the set of payoff functions $\{f_k^{**}\}_k$ is self-similar. Now, let \mathbf{P}^{**} be a problem instance that is the same as \mathbf{P}^* except for its payoff functions that are $\{f_k^{**}\}_k$. One can perform a similar analysis as in the proof of Theorem 3 in Hu et al. (2019) in order to show that $\mathcal{R}^\pi(\mathbf{P}^{**}; T) \geq CT^{1 - \frac{\beta(\alpha+1)}{2\beta+d}}$. This concludes the proof. ■

A.4. Proof of Proposition 5.1

Let $\tilde{r} := \lfloor 2\log_2(\frac{\gamma}{2C_{14}}) + 2l\beta + (\frac{d}{\beta} + 1)\log_2 \log T \rfloor$ where the constant C_{14} was introduced in Proposition C.1. We will prove the result by bounding the following probability

$$\begin{aligned} & \mathbb{P} \left\{ \exists r \leq \tilde{r} : \sup_{k \in \mathcal{K}, x \in \mathcal{M}^{(B)}} \left| \hat{f}_k^{(B,r)}(x; j_1^{(B)}) - \hat{f}_k^{(B,r)}(x; j_2^{(B)}) \right| \geq \frac{\gamma(\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{r/2}} \right\} \\ & \leq \sum_{r \in [\tilde{r}]} \sum_{k \in \mathcal{K}} \sum_{x \in \mathcal{M}^{(B)}} \mathbb{P} \left\{ \left| \hat{f}_k^{(B,r)}(x; j_1^{(B)}) - \hat{f}_k^{(B,r)}(x; j_2^{(B)}) \right| \geq \frac{\gamma(\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{r/2}} \right\}. \end{aligned} \quad (\text{A.10})$$

Note that by the triangle inequality,

$$\left| \hat{f}_k^{(B,r)}(x; j_1^{(B)}) - \hat{f}_k^{(B,r)}(x; j_2^{(B)}) \right| \leq \left| f_k(x) - \hat{f}_k^{(B,r)}(x; j_1^{(B)}) \right| + \left| f_k(x) - \hat{f}_k^{(B,r)}(x; j_2^{(B)}) \right|.$$

That is,

$$\begin{aligned} & \mathbb{P} \left\{ \left| \hat{f}_k^{(\mathbf{B},r)}(x; j_1^{(\mathbf{B})}) - \hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})}) \right| \geq \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{r/2}} \right\} \\ & \leq \mathbb{P} \left\{ \left| f_k(x) - \hat{f}_k^{(\mathbf{B},r)}(x; j_1^{(\mathbf{B})}) \right| \geq \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{1+r/2}} \right\} \\ & + \mathbb{P} \left\{ \left| f_k(x) - \hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})}) \right| \geq \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{1+r/2}} \right\}. \end{aligned} \quad (\text{A.11})$$

Note that since when $r \leq \tilde{r}$ one has $\frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{1+r/2}} \geq C_{14} 2^{-\beta j_1^{\mathbf{B}}} \geq C_{14} 2^{-\beta j_2^{\mathbf{B}}}$, one can apply Proposition C.1 to bound the two terms on the right hand side of above inequality. Namely, one can apply Proposition C.1 with $n = 2^r$, $\underline{\mu} = \frac{\underline{\rho}}{\rho 2^{-d\bar{l}}}$, $\bar{\mu} = \frac{\bar{\rho}}{\rho 2^{-d\bar{l}}}$, $\delta = \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{1+r/2}}$, and $h = 2^{-j_1^{\mathbf{B}}}$ for the first term and $h = 2^{-j_2^{\mathbf{B}}}$ for the second term to obtain

$$\begin{aligned} & \mathbb{P} \left\{ \left| f_k(x) - \hat{f}_k^{(\mathbf{B},r)}(x; j_1^{(\mathbf{B})}) \right| \geq \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{1+r/2}} \right\} \leq \tilde{C}_{12} T^{-\gamma^2 \tilde{C}_{13}}, \\ & \mathbb{P} \left\{ \left| f_k(x) - \hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})}) \right| \geq \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{1+r/2}} \right\} \leq \tilde{C}_{12} T^{-\gamma^2 \tilde{C}_{13}}, \end{aligned}$$

where the constants $\tilde{C}_{12}, \tilde{C}_{13}$ depend only on $L, \underline{\rho}, \bar{\rho}$, and d . These two inequalities along with (A.10) and (A.11) imply

$$\begin{aligned} & \mathbb{P} \left\{ \exists r \leq \tilde{r} : \sup_{k \in \mathcal{K}, x \in \mathcal{M}^{(\mathbf{B})}} \left| \hat{f}_k^{(\mathbf{B},r)}(x; j_1^{(\mathbf{B})}) - \hat{f}_k^{(\mathbf{B},r)}(x; j_2^{(\mathbf{B})}) \right| \geq \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{r/2}} \right\} \leq 2^{1-l\bar{d}} |\mathcal{M}^{(\mathbf{B})}| \tilde{r} \tilde{C}_{12} T^{-\gamma^2 \tilde{C}_{13}} \\ & \leq C_7 2^{-l\bar{d}} (\log T)^{\frac{d}{2}} T^{-\gamma^2 C_8 + C_9}, \end{aligned}$$

where the constants C_7, C_8, C_9 depend only on $\underline{\beta}, \bar{\beta}, L, \underline{\rho}, \bar{\rho}$, and d . The results follows by applying union bound over $\mathbf{B} \in \mathcal{B}_l$. This concludes the proof. \blacksquare

A.5. Proof of Proposition 5.2

By Assumption 4.1, there exists at least one bin $\tilde{\mathbf{B}} \in \mathcal{B}_l$, an arm $\tilde{k} \in \mathcal{K}$, and a point $\hat{x} \in \tilde{\mathbf{B}}$ such that

$$\left| \mathbf{\Gamma}_{j_1^{(\mathbf{B})}}^0 f_k(\hat{x}; \tilde{\mathbf{B}}) - f_k(\hat{x}) \right| = \left| \mathbf{\Gamma}_l^{[\bar{\beta}]} f_k(\hat{x}) - f_k(\hat{x}; \tilde{\mathbf{B}}) \right| \geq b 2^{-l\bar{\beta}}. \quad (\text{A.12})$$

Let $\tilde{x} = \arg \min_{x \in \mathcal{M}^{(\mathbf{B})}} \|x - \hat{x}\|_\infty$ (if there is more than one minimizer we choose the one with the minimum L_1 -norm). Note that $\|\tilde{x} - \hat{x}\|_\infty \leq 2^{-\bar{l}}$, which along with the assumption $f_{\tilde{k}} \in \mathcal{H}(\underline{\beta}, L)$ implies that

$$|f_{\tilde{k}}(\tilde{x}) - f_{\tilde{k}}(\hat{x})| \leq L \|\tilde{x} - \hat{x}\|_\infty^{\underline{\beta}} \leq L 2^{-\bar{l}\underline{\beta}} \leq \frac{L}{\log T} 2^{-l\bar{\beta}}. \quad (\text{A.13})$$

In addition, by Lemma B.1, one has

$$\left| \mathbf{\Gamma}_l^{[\bar{\beta}]} f_{\tilde{k}}(\hat{x}; \tilde{\mathbf{B}}) - \mathbf{\Gamma}_l^{[\bar{\beta}]} f_{\tilde{k}}(\tilde{x}; \tilde{\mathbf{B}}) \right| \leq \kappa_0 2^l \|\hat{x} - \tilde{x}\|_\infty \leq \kappa_0 2^{l-\bar{l}} \leq \frac{\kappa_0}{\log T} 2^{-l\bar{\beta}}, \quad (\text{A.14})$$

where κ_0 was introduced in Lemma B.1. Let $\hat{r} := \lceil 2 \log_2(\frac{4\gamma}{L \wedge \kappa_0}) + 2l\bar{\beta} + (\frac{d}{\underline{\beta}} + 3) \log_2 \log T \rceil$. One has

$$\mathbb{P} \left\{ r_{\text{last}}^{(\tilde{\mathbf{B}})} > \hat{r} \right\} \leq \mathbb{P} \left\{ \left| \hat{f}_{\tilde{k}}^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) - \hat{f}_{\tilde{k}}^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_2^{(\tilde{\mathbf{B}})}) \right| < \frac{\gamma (\log T)^{\frac{d}{2\beta} + \frac{1}{2}}}{2^{\hat{r}/2}} \right\}. \quad (\text{A.15})$$

Note that by the triangle inequality,

$$\left| \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_2^{(\tilde{\mathbf{B}})}) \right| \geq \left| f_k(\tilde{x}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) \right| - \left| f_k(\tilde{x}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_2^{(\tilde{\mathbf{B}})}) \right|. \quad (\text{A.16})$$

Note that since one has $\frac{\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{1+\hat{r}/2}} \geq C_{14}2^{-\beta j_2^{\tilde{\mathbf{B}}}}$, one can apply Proposition C.1 to show that second term on the right hand side of above inequality is “small” with high probability. Namely, one can apply Proposition C.1 with $n = 2^{\hat{r}}$, $\underline{\mu} = \frac{\rho}{\rho 2^{-d\bar{l}}}$, $\bar{\mu} = \frac{\bar{\rho}}{\bar{\rho} 2^{-d\bar{l}}}$, $\delta = \frac{\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{1+\hat{r}/2}}$, and $h = 2^{-j_2^{\tilde{\mathbf{B}}}}$ to obtain

$$\mathbb{P} \left\{ \left| f_k(\tilde{x}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_2^{(\tilde{\mathbf{B}})}) \right| \geq \frac{\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{1+\hat{r}/2}} \right\} \leq \tilde{C}_{12} T^{-\gamma^2 \tilde{C}_{13}}, \quad (\text{A.17})$$

where the constants $\tilde{C}_{12}, \tilde{C}_{13}$ depend only on $\bar{\beta}, L, \underline{\rho}, \bar{\rho}$, and d . Now, we show that the first term on the right hand side of (A.15) cannot get “small” with high probability. One can write

$$\left| f_k(\tilde{x}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) \right| \geq \left| f_k(\tilde{x}) - \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\tilde{x}; \tilde{\mathbf{B}}) \right| - \left| \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\tilde{x}; \tilde{\mathbf{B}}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) \right|. \quad (\text{A.18})$$

The first term corresponds to bias and the second term corresponds to stochastic error. Note that by (A.12), (A.13), and (A.14), one has

$$\begin{aligned} \left| f_k(\tilde{x}) - \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\tilde{x}; \tilde{\mathbf{B}}) \right| &\geq \left| f_k(\hat{x}) - \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\hat{x}; \tilde{\mathbf{B}}) \right| - |f_k(\tilde{x}) - f_k(\hat{x})| - \left| \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\hat{x}; \tilde{\mathbf{B}}) - \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\tilde{x}; \tilde{\mathbf{B}}) \right| \\ &\geq b2^{-l\beta} - \frac{L}{\log T} 2^{-l\beta} - \frac{\kappa_0}{\log T} 2^{-l\beta} \geq \frac{L \wedge \kappa_0}{2 \log T} 2^{-l\beta} \geq \frac{2\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{\hat{r}/2}} \end{aligned} \quad (\text{A.19})$$

for large enough $T \geq T_0(L, b, \underline{\rho}, \bar{\rho}, d)$. In order to bound the second term on the right hand side of (A.18), we apply Proposition C.2, with $n = 2^{\hat{r}}$, $\delta = \frac{\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{1+\hat{r}/2}}$, and $h = 2^{-j_1^{\tilde{\mathbf{B}}}}$ to obtain

$$\mathbb{P} \left\{ \left| \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\tilde{x}; \tilde{\mathbf{B}}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) \right| \geq \frac{\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{1+\hat{r}/2}} \right\} \leq \tilde{C}_{15} T^{-\gamma^2 \tilde{C}_{16}}, \quad (\text{A.20})$$

where the constants $\tilde{C}_{15}, \tilde{C}_{16}$ depend only on $\bar{\beta}, L, \underline{\rho}, \bar{\rho}$, and d . Putting together (A.15), (A.16), (A.18), and (A.19), one obtains

$$\begin{aligned} \mathbb{P} \left\{ r_{\text{last}}^{(\tilde{\mathbf{B}})} > \hat{r} \right\} &\leq \mathbb{P} \left\{ \left| \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\tilde{x}; \tilde{\mathbf{B}}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) \right| \geq \frac{\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{1+\hat{r}/2}} \right\} \\ &\quad + \mathbb{P} \left\{ \left| \mathbf{\Gamma}_{j_1^{(\tilde{\mathbf{B}})}}^{[\bar{\beta}]} f_k(\tilde{x}; \tilde{\mathbf{B}}) - \hat{f}_k^{(\tilde{\mathbf{B}}, \hat{r})}(\tilde{x}; j_1^{(\tilde{\mathbf{B}})}) \right| \geq \frac{\gamma(\log T)^{\frac{d}{2\bar{\beta}} + \frac{1}{2}}}{2^{1+\hat{r}/2}} \right\} \\ &\leq C_{10} T^{-\gamma^2 C_{11}}, \end{aligned}$$

where the last inequality follows from (A.17) and (A.20), and the constants C_{10}, C_{11} depend only on $\bar{\beta}, L, \underline{\rho}, \bar{\rho}$, and d . This concludes the proof. \blacksquare

A.6. Proof of Theorem 5.1

Note that for large enough T , one has

$$\begin{aligned} \mathbb{P} \left\{ \hat{\beta}_{\text{SACB}} \in [\beta - \frac{3(2\bar{\beta} + d)^2 \log_2 \log T}{(\bar{\beta} + d - 1) \log_2 T}, \beta] \right\} \\ \leq \mathbb{P} \left\{ 2l\beta + \left(\frac{d}{\bar{\beta}} + 1 \right) \log_2 \log T \leq r_{\text{last}}^{(\tilde{\mathbf{B}})} \leq 2l\beta + \left(\frac{d}{\bar{\beta}} + 4 \right) \log_2 \log T \right\} \\ \leq 1 - C_7 2^{ld} (\log T)^{\frac{d}{2}} T^{-\gamma^2 C_8 + C_9} - C_{10} T^{-\gamma^2 C_{11}}, \end{aligned}$$

where the last inequality follows from Propositions 5.1, and 5.2, and the constants $C_7, C_8, C_9 > 0$ were introduced in Proposition 5.1, and the constants $C_{10}, C_{11} > 0$ were introduced in Proposition 5.2.

Next, we show that with high probability, $T_{\text{SACB}} \leq \frac{4}{\underline{\rho}} (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{(\beta+d-1)}{(2\bar{\beta}+d)}}$ $=: \bar{T}_{\text{SACB}}$. Note that the smoothness estimation sub-routine terminates when all the bins $\mathbf{B} \in \mathcal{B}_l$ have reached round $\bar{r} = \lceil 2l\bar{\beta} + (\frac{2d}{\underline{\beta}} + 4) \log_2 \log T \rceil$. That is, T_{SACB} is less than the time step by which $2 \sum_{r=\bar{r}}^{\bar{r}} 2^r$ contexts have realized in each $\mathbf{B} \in \mathcal{B}_l$. Note that

$$\sum_{r=\bar{r}}^{\bar{r}} 2^r \leq 2^{\bar{r}+1} \leq 2 (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{2\bar{\beta}(\beta+d-1)}{(2\bar{\beta}+d)^2}}.$$

Let $\bar{N}^{(\mathbf{B})} := \sum_{t=1}^{\bar{T}_{\text{SACB}}} Z_t$ be the number of contexts that have realized in \mathbf{B} by $t = \bar{T}_{\text{SACB}}$, where Z_t 's are i.i.d. Bernoulli random variables with $\mathbb{E}[Z_t] \geq \underline{\rho} T^{-\frac{d(\beta+d-1)}{(2\bar{\beta}+d)^2}}$ and $\text{Var}(Z_t) \leq \mathbb{E}[Z_t^2] \leq \bar{\rho} T^{-\frac{d(\beta+d-1)}{(2\bar{\beta}+d)^2}}$. Applying Bernstein's inequality in Lemma 7.4 to $\bar{N}^{(\mathbf{B})}$ with $a = 2 (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{2\bar{\beta}(\beta+d-1)}{(2\bar{\beta}+d)^2}}$ yields:

$$\begin{aligned} \mathbb{P} \left\{ \bar{N}^{(\mathbf{B})} < 2 (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{2\bar{\beta}(\beta+d-1)}{(2\bar{\beta}+d)^2}} \right\} &\leq \exp \left(-\frac{a^2}{2\bar{T}_{\text{SACB}} \text{Var}(Z_t) + a} \right) \\ &\leq \exp \left(-\frac{\underline{\rho}}{4\bar{\rho} + 2\underline{\rho}} (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{2\bar{\beta}(\beta+d-1)}{(2\bar{\beta}+d)^2}} \right), \end{aligned}$$

and, by the union bound:

$$\begin{aligned} \mathbb{P} \{ T_{\text{SACB}} > \bar{T}_{\text{SACB}} \} &\leq \sum_{\mathbf{B} \in \mathcal{B}_l} \mathbb{P} \left\{ \bar{N}^{(\mathbf{B})} < 2 (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{2\bar{\beta}(\beta+d-1)}{(2\bar{\beta}+d)^2}} \right\} \\ &\leq 2T^{\frac{d(\beta+d-1)}{(2\bar{\beta}+d)^2}} \exp \left(-\frac{\underline{\rho}}{4\bar{\rho} + 2\underline{\rho}} (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{2\bar{\beta}(\beta+d-1)}{(2\bar{\beta}+d)^2}} \right). \end{aligned}$$

This concludes the proof. ■

A.7. Proof of Theorem 5.2

The regret incurred by the SACB policy up to $t = \lfloor \bar{T}_{\text{SACB}} \rfloor$ is bounded by

$$\begin{aligned} \mathbb{E}^\pi \left[\sum_{t=1}^{\lfloor \bar{T}_{\text{SACB}} \rfloor} f_{\pi_t^*}(X_t) - f_{\pi_t}(X_t) \right] &\leq T \cdot \mathbb{P} \{ T_{\text{SACB}} > \bar{T}_{\text{SACB}} \} + \bar{T}_{\text{SACB}} \\ &\stackrel{(a)}{\leq} 2T^{1+\frac{d(\beta+d-1)}{(2\bar{\beta}+d)^2}} \exp \left(-\frac{\underline{\rho}}{4\bar{\rho} + 2\underline{\rho}} (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{2\bar{\beta}(\beta+d-1)}{(2\bar{\beta}+d)^2}} \right) \\ &\quad + \frac{4}{\underline{\rho}} (\log T)^{\frac{2d}{\underline{\beta}}+4} T^{\frac{(\beta+d-1)}{(2\bar{\beta}+d)}} \\ &\stackrel{(b)}{=} o \left(T^{1-\frac{\beta(\alpha+1)}{2\bar{\beta}+d}} \right), \end{aligned} \tag{A.21}$$

where (a) follows from Theorem 5.1 and (b) holds by $\frac{(\beta+d-1)}{(2\bar{\beta}+d)} \leq 1 - \frac{\beta(\alpha+1)}{2\bar{\beta}+d}$ for any $\underline{\beta} \leq \beta \leq \bar{\beta}$ and $\alpha \leq \frac{1}{\min\{1, \beta\}}$. Define $\hat{\beta}_T := \beta - \frac{3(2\bar{\beta}+d)^2 \log_2 \log T}{(\beta+d-1) \log_2 T}$. The regret from $t = \lfloor \bar{T}_{\text{SACB}} \rfloor + 1$ to $t = T$ is bounded by

$$\begin{aligned} \mathbb{E}^\pi \left[\sum_{t=\lfloor \bar{T}_{\text{SACB}} \rfloor + 1}^T f_{\pi_t^*}(X_t) - f_{\pi_t}(X_t) \right] &\leq T \cdot \mathbb{P} \left\{ \hat{\beta}_{\text{SACB}} \notin [\hat{\beta}_T, \beta] \right\} + \bar{C}_0 (\log T)^{\iota_0(\hat{\beta}_T, \alpha, d)} T^{1-\frac{\hat{\beta}_T(\alpha+1)}{2\bar{\beta}_T+d}} \\ &\leq C_4 (\log T)^{\frac{d}{\underline{\beta}}} T^{-\gamma^2 C_5 + C_6 + 1 + \frac{d(\beta+d-1)}{(2\bar{\beta}+d)^2}} \\ &\quad + CT^{1-\frac{\beta(\alpha+1)}{2\bar{\beta}+d}} (\log T)^{\frac{3d(\alpha+1)(2\bar{\beta}+d)^2}{(2\bar{\beta}+d)(\beta+d)(\bar{\beta}+d-1)} + \iota_0(\hat{\beta}_T, \alpha, d)}, \end{aligned} \tag{A.22}$$

for some constant $C > 0$, where the last inequality follows from Corollary 5.1 and the constants C_4, C_5 , and C_6 were introduced in Theorem 5.1. Putting together (A.21) and (A.22) concludes the proof. ■

A.8. Proof of Corollary 5.1

The result follows from Theorem 5.2 and the fact that for any $\beta_0 \leq 1$:

$$\sup_{P \in \mathcal{P}(\beta_0, \alpha, d)} \mathcal{R}^{\text{ABSE}(\beta_0)}(P; T) = \mathcal{O}\left(T^{\zeta(\beta_0, \alpha, d)}\right).$$

A.9. Proof of Corollary 5.2

The result follows from Theorem 5.2 and since for any problem instance $P \in \mathcal{P}(\beta_0, \alpha, d)$, and decision regions which satisfy the regularity condition in Assumption 5.1, one has for any $\beta_0 \geq 1$:

$$\mathcal{R}^{\text{SmoothBandit}(\beta_0)}(P; T) = \mathcal{O}\left((\log T)^{\frac{2\beta_0+d}{2\beta_0}} T^{\zeta(\beta_0, \alpha, d)}\right).$$

A.10. Proof of Remark 5.1

Note that

$$\pi_0(\beta_0) = \begin{cases} \text{ABSE}(\beta_0) & \text{if } \beta_0 \leq 1; \\ \text{SmoothBandit}(\beta_0) & \text{if } \beta_0 > 1. \end{cases}$$

Furthermore, for any $\beta_0 \leq 1$

$$\sup_{P \in \mathcal{P}(\beta_0, \alpha, d)} \mathcal{R}^{\text{ABSE}(\beta_0)}(P; T) = \mathcal{O}\left(T^{\zeta(\beta_0, \alpha, d)}\right),$$

and for any $\beta_0 > 1$ and any problem instance $P \in \mathcal{P}(\beta_0, \alpha, d)$, and decision regions which satisfy the regularity condition in Assumption 5.1,

$$\mathcal{R}^{\text{SmoothBandit}(\beta_0)}(P; T) = \mathcal{O}\left((\log T)^{\frac{2\beta_0+d}{2\beta_0}} T^{\zeta(\beta_0, \alpha, d)}\right).$$

The result follows from applying Theorem 5.2 with

$$\iota_0(\beta_0, \alpha, d) := \begin{cases} 0 & \text{if } \beta_0 \leq 1; \\ \frac{2\beta_0}{2\beta_0+d} & \text{o.w.} \end{cases}$$

Appendix B: Properties of the $L_2(P_X)$ -projection

LEMMA B.1. Fix non-negative integers l and p , an bin U of side-length $2^{-l'}$, $l' \in \mathbb{R}_+$, and some point $x \in U$ and let $K(\cdot) = \mathbb{1}\{\|\cdot\|_\infty \leq 1\}$ and $h = 2^{-l}$. Let μ_0 , κ_0 , and L_0 be some constants that only depend on $p, \rho, \bar{\rho}$ (introduced in Assumption 2.2), and d . The following statements hold:

1. $\mathbf{\Gamma}_l^p f(x; U) = R^\top(0) B^{-1} W$, where we define the vector $R(u) := (u^s)_{|s| \leq p}$, the matrix $B := (B_{s_1, s_2})_{|s_1|, |s_2| \leq p}$, and the vector $W := (W_s)_{|s| \leq p}$ with elements

$$B_{s_1, s_2} := \int_{\mathbb{R}^d} u^{s_1 + s_2} K(u) p_X(x + hu | U) du, \quad W_s := \int_{\mathbb{R}^d} u^s f(x + hu) K(u) p_X(x + hu | U) du;$$

2. $\lambda_{\min}(B) \geq \mu_0 2^{dl'}$;
3. $|\mathbf{\Gamma}_l^p f(x; U) - \mathbf{\Gamma}_l^p f(\hat{x}; U)| \leq \kappa_0 h^{-1} \|\hat{x} - x\|_\infty$ for all $x, \hat{x} \in U$;
4. If $f \in \mathcal{H}(\beta, L)$ for $0 < \beta \leq p + 1$ then, $|\mathbf{\Gamma}_l^p f(x; U) - f(x)| \leq L_0 h^\beta$ for all $x \in U$.

Proof. Fix some $x \in U$. Let $\tilde{\theta}(u; p, l, U) := \sum_{|s| \leq p} \xi_s u^s$ be a polynomial of degree p on \mathbb{R}^d that minimizes

$$\begin{aligned} & \int_U \left| f(u) - \tilde{\theta}\left(\frac{u-x}{h}; p, l, U\right) \right|^2 K\left(\frac{u-x}{h}\right) p_X(u | U) du = \int_U f^2(u) K\left(\frac{u-x}{h}\right) p_X(u | U) du \\ & + \sum_{|s_1|, |s_2| \leq p} \xi_{s_1} \xi_{s_2} \int_U \left(\frac{u-x}{h}\right)^{s_1 + s_2} K\left(\frac{u-x}{h}\right) p_X(u | U) du \\ & - 2 \sum_{|s| \leq p} \xi_s \int_U f(u) \left(\frac{u-x}{h}\right)^s K\left(\frac{u-x}{h}\right) p_X(u | U) du, \end{aligned}$$

where $h = 2^{-l}$. Equivalently, $\tilde{\theta}(u; p, l, U)$ can be characterized by its vector of coefficients ξ that minimizes

$$\sum_{|s_1|, |s_2| \leq p} \xi_{s_1} \xi_{s_2} \int_{\mathbb{R}^d} u^{s_1+s_2} K(u) p_X(x+hu | U) du - 2 \sum_{|s| \leq p} \xi_s \int_{\mathbb{R}^d} f(u) u^s K(u) p_X(x+hu | U) du = \xi^\top B \xi - 2W^\top \xi, \quad (\text{B.1})$$

where we define the matrix $B := (B_{s_1, s_2})_{|s_1|, |s_2| \leq p}$ and the vector $W := (W_s)_{|s| \leq p}$ with elements

$$B_{s_1, s_2} := \int_{\mathbb{R}^d} u^{s_1+s_2} K(u) p_X(x+hu | U) du, \quad W_s := \int_{\mathbb{R}^d} f(u) u^s K(u) p_X(x+hu | U) du.$$

Note that if B is a positive definite matrix then, the minimizer of (B.1) is $\xi = B^{-1}W$, which implies the desired result: $\mathbf{\Gamma}_l^p f(x; U) = R^\top(0)B^{-1}W$. In order to show that this is indeed the case, we note that

$$\lambda_{\min}(B) = \min_{\|Z\|=1} Z^\top B Z = \int_U \left(\sum_{|s| \leq p} Z_s u^s \right)^2 K(u) p_X(x+hu | U) du \geq \frac{\rho 2^{dl'}}{\bar{\rho}} \int_A \left(\sum_{|s| \leq p} Z_s u^s \right)^2 du,$$

where $A = \{u \in \mathbb{R}^d : \|u\|_\infty \leq 1; x+hu \in U\}$. Note that

$$\lambda[A] \geq h^{-d} \lambda[\Xi(x, h) \cap U] \geq 2^{-d} h^{-d} \lambda[\Xi(x, h)] = 2^{-d} \lambda[\Xi(0, 1)]$$

. Let \mathcal{A} denote the class of compact subsets of $\Xi(0, 1)$ having the Lebesgue measure $2^{-d} \lambda[\Xi(0, 1)]$. Using the previous display, we obtain

$$\lambda_{\min}(B) \geq \frac{\rho 2^{dl'}}{\bar{\rho}} \min_{\|Z\| \leq 1; S \in \mathcal{A}} \int_S \left(\sum_{|s| \leq p} Z_s u^s \right)^2 du =: \frac{\rho 2^{dl'}}{\bar{\rho}} \tilde{\mu}_0. \quad (\text{B.2})$$

By the compactness argument, the minimum in the above expression exists, and is strictly positive.

In order to prove the last claim in the lemma, note that for any $\hat{x} \in U$,

$$\begin{aligned} |\mathbf{\Gamma}_l^p f(x; U) - \mathbf{\Gamma}_l^p f(\hat{x}; U)| &= \left| \tilde{\theta}(0; p, l, U) - \tilde{\theta}\left(\frac{\hat{x} - x}{h}; p, l, U\right) \right| \\ &= \left| \sum_{|s| \leq p, s \neq (0, \dots, 0)} \xi_s \left(\frac{\hat{x} - x}{h}\right)^s \right| \leq M h^{-1} \|\hat{x} - x\|_\infty \|\xi\|. \end{aligned}$$

Also, by (B.2), one has

$$\|\xi\| \leq \|B^{-1}W\| \leq \frac{2^{-dl'} \bar{\rho}}{\underline{\rho}} \tilde{\mu}_0^{-1} M^{\frac{1}{2}} \max_s |W_s|,$$

and

$$|W_s| = \left| \int_{\mathbb{R}^d} u^s f(x+hu) K(u) p_X(x+hu | U) du \right| \leq \int_{\mathbb{R}^d} K(u) p_X(x+hu | U) du \leq 2^{dl'}.$$

Putting together the above three displays, one obtains

$$|\mathbf{\Gamma}_l^p f(x; U) - \mathbf{\Gamma}_l^p f(\hat{x}; U)| \leq \bar{\rho} \underline{\rho}^{-1} \tilde{\mu}_0^{-1} M^{3/2} h^{-1} \|\hat{x} - x\|_\infty.$$

To prove the last part, define the vector $Z := (Z_s)_{|s| \leq p}$ with elements

$$Z_s := \frac{h^{|s|} f^{(s)}(x)}{s!} \cdot \mathbb{1}_{\{|s| \leq \lfloor \beta \rfloor\}}.$$

Note that

$$f(x) = R^\top(0)B^{-1}BZ.$$

As a result, one has

$$|f(x) - \mathbf{\Gamma}_l^p f(x; U)| = |R^\top(0)B^{-1}(BZ - W)| \leq \|B^{-1}\| \cdot \|BZ - W\| \leq \frac{2^{-dl'} \bar{\rho}}{\rho \bar{\mu}_0} M^{\frac{1}{2}} \max_s |(BZ)_s - W_s|,$$

where the last inequality follows from (B.2). Furthermore, one has

$$\begin{aligned} |(BZ)_s - W_s| &= \left| \int_{\mathbb{R}^d} u^s \left(\sum_{|s'| \leq \lfloor \beta \rfloor} \frac{(hu)^{s'} f^{(s)}(x)}{s'!} - f(x + hu) \right) K(u) p_X(x + hu | U) du \right| \\ &\leq \int_{\mathbb{R}^d} |u^s| \left| \sum_{|s'| \leq \lfloor \beta \rfloor} \frac{(hu)^{s'} f^{(s)}(x)}{s'!} - f(x + hu) \right| K(u) p_X(x + hu | U) du \\ &\leq \int_{\mathbb{R}^d} L h^\beta p_X(x + hu | U) du = L h^\beta 2^{dl'} \end{aligned}$$

where the last inequality follows from the assumption that $f \in \mathcal{H}(\beta, L)$. Putting the last two displays together, the result follows. This concludes the proof. \blacksquare

Appendix C: Proofs and analysis for the review of local polynomial regression

In this section of the appendix, we provide the proofs for our review of the local polynomial regression estimation method. Fix a set of pairs $\mathcal{D} = \{(X_i, Y_i)\}_{i=1}^n$, a point $x \in \mathbb{R}^d$, a bandwidth $h > 0$, an integer $p > 0$ and a kernel function $K : \mathbb{R}^d \rightarrow \mathbb{R}_+$. Define the matrix $Q := (Q_{s_1, s_2})_{|s_1|, |s_2| \leq p}$ and the vector $V := (V_s)_{|s| \leq p}$ with the elements

$$Q_{s_1, s_2} := \sum_{i=1}^n (X_i - x)^{s_1 + s_2} K\left(\frac{X_i - x}{h}\right), \quad V_s := \sum_{i=1}^n Y_i (X_i - x)^s K\left(\frac{X_i - x}{h}\right).$$

Also, define the matrix $U := (u^s)_{|s| \leq p}$. The next result from Audibert and Tsybakov (2007) provides a closed-form expression for local polynomial regression at any arbitrary point.

LEMMA C.1 (Audibert and Tsybakov 2007, Proposition 2.1). *If the matrix Q is positive definite, there exists a polynomial on \mathbb{R}^d of degree p minimizing (5.3). Its vector of coefficients is given by $\xi = Q^{-1}V$ and the corresponding local polynomial regression function at point x is given by*

$$\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) = U(0)^\top Q^{-1}V = \sum_{i=1}^n Y_i K\left(\frac{X_i - x}{h}\right) U(0)^\top Q^{-1}U(X_i - x).$$

The following simple extension of Theorem 3.2 in Audibert and Tsybakov (2007) will be one of the main tools to bound our estimation error in our proposed policy.

PROPOSITION C.1. *Let $\mathcal{D} = \{(X_i, Y_i)\}_{i=1}^n$ be a set of n i.i.d pairs $(X_i, Y_i) \in \mathcal{X} \times \mathbb{R}$. If the marginal density μ of X_i 's satisfies $\underline{\mu} \leq \mu(x) \leq \bar{\mu}$ for some $0 < \underline{\mu} \leq \bar{\mu}$ with a support \mathcal{X} that is a closed hypercube in \mathbb{R}^d of side-length $2^{-l}, l \geq 0$, and the function η belongs to the Hölder class of functions $\mathcal{H}_{\mathcal{X}}(\beta, L)$ for some $\beta, L > 0$ then, there exist constants $C_{12}, C_{13}, C_{14} > 0$ such that for any $0 < h < 2^{-l}$, any $C_{14}h^\beta < \delta$, any $n \geq 1$ and the kernel function $K(\cdot) = \mathbb{1}\{\|\cdot\|_\infty \leq 1\}$, the local polynomial estimator $\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p)$ satisfies*

$$|\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \eta(x)| \leq \delta$$

with probability at least $1 - C_{12} \exp(-C_{13} n h^d \underline{\mu}^2 \bar{\mu}^{-1} \delta^2)$ for all $x \in \mathcal{X}$. The constants C_1, C_2, C_3 depend only on p, d, L .

The next proposition states that local polynomial regression estimation of a function inside a hypercube cannot largely deviate from the $L_2(P_X)$ -projection of that function with high probability.

PROPOSITION C.2. *Fix a hypercube $U \subseteq (0, 1)^d$ with side-length $2^{-l'}$, $l' \in \mathbb{R}_+$. Let $\mathcal{D} = \{(X_i, Y_i)\}_{i=1}^n$ be a set of n i.i.d pairs $(X_i, Y_i) \in U \times \mathbb{R}$. If the marginal density μ of X_i 's satisfies $\mu(\cdot) = p_X(\cdot|U)$, where p_X is the density of a distribution \mathbf{P}_X that satisfies Assumption 2.2 then, there exist constants $C_{15}, C_{16}, C_{17} > 0$ such that for any $\delta < C_{17}$, any $n \geq 1$, $h = 2^{-l}$, $l \geq l'$, and the kernel function $K(\cdot) = \mathbb{1}\{\|\cdot\|_\infty \leq 1\}$, the local polynomial estimator $\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p)$ satisfies*

$$|\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \mathbf{\Gamma}_l^p \eta(x; U)| \leq \delta$$

with probability at least $1 - C_{15} \exp(-C_{16} n 2^{d(l'-l)} \delta^2)$ for all $x \in U$. The constants C_{15}, C_{16}, C_{17} depend only on $p, \bar{\rho}, \rho$, and d .

C.1. Proof of Proposition C.1

The proof is a simple extension of the proof of Theorem 3.2 in Audibert and Tsybakov (2007); however, we provide the proof for completeness. Fix $x \in \mathcal{X}$ and $\delta > 0$. Consider the matrices $B := (B_{s_1, s_2})_{|s_1|, |s_2| \leq p}$ and $\bar{B} := (\bar{B}_{s_1, s_2})_{|s_1|, |s_2| \leq p}$ with the elements

$$B_{s_1, s_2} := \int_{\mathbb{R}^d} u^{s_1 + s_2} K(u) \mu(x + hu) du, \quad \bar{B}_{s_1, s_2} := \frac{1}{nh^d} \sum_{i=1}^n \left(\frac{X_i - x}{h} \right)^{s_1 + s_2} K\left(\frac{X_i - x}{h} \right).$$

The smallest eigenvalue of \bar{B} satisfies

$$\begin{aligned} \lambda_{\min}(\bar{B}) &= \min_{\|W\|=1} W^\top \bar{B} W \\ &\geq \min_{\|W\|=1} W^\top B W + \min_{\|W\|=1} W^\top (\bar{B} - B) W \\ &\geq \min_{\|W\|=1} W^\top B W - \sum_{|s_1|, |s_2| \leq p} |\bar{B}_{s_1, s_2} - B_{s_1, s_2}|. \end{aligned} \quad (\text{C.1})$$

Define $\mathcal{X}_n := \{u \in \mathbb{R}^d : \|u\| \leq 1; x + hu \in \mathcal{X}\}$. For any vector W satisfying $\|W\|=1$, we obtain

$$W^\top B W = \int_{\mathbb{R}^d} \left(\sum_{|s| \leq p} W_s u_s \right)^2 K(u) \mu(x + hu) du \geq \mu \int_{\mathcal{X}_n} \left(\sum_{|s| \leq p} W_s u_s \right)^2 du.$$

Since \mathcal{X} is a closed hypercube and we have assumed that $h \leq l$, we get

$$\lambda[\mathcal{X}_n] \geq h^{-d} \lambda[\text{Ball}_2(x, h) \cap \mathcal{X}] \geq 2^{-d} h^{-d} \lambda[\text{Ball}_2(x, h)] \geq 2^{-d} \lambda[\text{Ball}_2(0, 1)],$$

where $\text{Ball}_2(x, h)$ is the Euclidean ball of radius h centered around x .

Let \mathcal{A} denote the class of all compact subsets of $\text{Ball}_2(0, 1)$ having the Lebesgue measure $2^{-d} \lambda[\text{Ball}_2(0, 1)]$. Using the previous display, we obtain

$$\min_{\|W\|=1} W^\top B W \geq \mu \min_{\|W\|=1; S \in \mathcal{A}} \int_S \left(\sum_{|s| \leq p} W_s u_s \right)^2 du =: 2c\mu \quad (\text{C.2})$$

By the compactness argument, the above minimum exists and is strictly positive.

For $i = 1, \dots, n$ and any multi-indices s_1, s_2 such that $|s_1|, |s_2| \leq p$, define

$$T_i^{(s_1, s_2)} := \frac{1}{h^d} \left(\frac{X_i - x}{h} \right)^{s_1 + s_2} K\left(\frac{X_i - x}{h} \right) - \int_{\mathbb{R}^d} u^{s_1 + s_2} K(u) \mu(x + hu) du.$$

We have $\mathbb{E}T_i^{(s_1, s_2)} = 0$, $|T_i^{(s_1, s_2)}| \leq 2h^{-d}$, and the following bound on the variance of $T_i^{(s_1, s_2)}$:

$$\begin{aligned} \text{Var}T_i^{(s_1, s_2)} &\leq \frac{1}{h^{2d}} \mathbb{E} \left[\left(\frac{X_i - x}{h} \right)^{2s_1 + 2s_2} K^2 \left(\frac{X_i - x}{h} \right) \right] \\ &\leq \frac{1}{h^d} \int_{\mathbb{R}^d} u^{2s_1 + 2s_2} K^2(u) \mu(x + hu | \mathcal{B}) du \\ &\leq \frac{\bar{\mu}}{h^d} \max_{j \leq p} \int_{\mathbb{R}^d} (1 + |u^{4j}|) K^2(u) du =: \frac{\kappa \bar{\mu}}{h^d}. \end{aligned}$$

From Bernstein's inequality, we get

$$\mathbb{P} \left\{ |\bar{B}_{s_1, s_2} - B_{s_1, s_2}| > \epsilon \right\} = \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n T_i^{(s_1, s_2)} \right| > \epsilon \right\} \leq 2 \exp \left(\frac{-nh^d \epsilon^2}{2\kappa \bar{\mu} + 4\epsilon/3} \right)$$

This inequality along with (C.1) and (C.2) imply that

$$\mathbb{P} \left\{ \lambda_{\min}(\bar{B}) \leq c\bar{\mu} \right\} \leq 2M^2 \exp \left(\frac{-nh^d M^{-4} c^2 \bar{\mu}^2}{2\kappa \bar{\mu} + 4M^{-2} c\bar{\mu}/3} \right), \quad (\text{C.3})$$

where M^2 is the number of elements in the matrix \bar{B} . In what follows assume that $\lambda_{\min}(\bar{B}) \geq c\bar{\mu}$. Therefore,

$$\mathbb{P} \left\{ |\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \eta(x)| \geq \delta \right\} \leq \mathbb{P} \left\{ \lambda_{\min}(\bar{B}) \leq c\bar{\mu} \right\} + \mathbb{P} \left\{ |\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \eta(x)| \geq \delta, \lambda_{\min}(\bar{B}) > c\bar{\mu} \right\}. \quad (\text{C.4})$$

We now evaluate the second term on the right hand side of the above inequality. Define the matrix $Z := (Z_{i,s})_{1 \leq i \leq n, |s| \leq p}$ with elements

$$Z_{i,s} := (X_i - x)^s \sqrt{K \left(\frac{X_i - x}{h} \right)}.$$

The s -th column of Z is denoted by Z_s , and we introduce $Z^{(\eta)} := \sum_{|s| \leq [\beta]} \frac{\eta^{(s)}(x)}{s!} Z_s$. Since $Q = Z^\top Z$ we get

$$\forall |s| \leq [\beta] : U^\top(0) Q^{-1} Z^\top Z = \mathbb{1} \{s = (0, \dots, 0)\},$$

hence $R^\top(0) Q^{-1} Z^\top Z^{(\eta)} = \eta(x)$. So we can write

$$\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \eta(x) = R^\top(0) Q^{-1} (V - Z^\top Z^{(\eta)}) = R^\top(0) \bar{B}^{-1} \mathbf{a},$$

where $\mathbf{a} := \frac{1}{nh^d} H (V - Z^\top Z^{(\eta)}) \in \mathbb{R}^M$ and H is a diagonal matrix $H := (H_{s_1, s_2})_{|s_1|, |s_2| \leq p}$ with elements $H_{s_1, s_2} := h^{-s_1} \mathbb{1} \{s_1 = s_2\}$. For $\lambda_{\min}(\bar{B}) > c\bar{\mu}$, one has

$$|\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \eta(x)| \leq \|\bar{B}^{-1} \mathbf{a}\| \leq \lambda_{\min}^{-1}(\bar{B}) \|\mathbf{a}\| \leq c^{-1} \bar{\mu}^{-1} M \max_s \|a_s\|, \quad (\text{C.5})$$

where a_s are the components of the vector \mathbf{a} given by

$$a_s = \frac{1}{nh^d} \sum_{i=1}^n [Y_i - \eta_x(X_i)] \left(\frac{X_i - x}{h} \right)^s K \left(\frac{X_i - x}{h} \right).$$

Note that $\eta_x(X_i)$ is the Taylor expansion of η at x and of degree $[\beta]$ (not necessarily p) evaluated at X_i .

Define:

$$\begin{aligned} T_i^{(s,1)} &:= [Y_i - \eta(X_i)] \left(\frac{X_i - x}{h} \right)^s K \left(\frac{X_i - x}{h} \right), \\ T_i^{(s,2)} &:= [\eta(X - i) - \eta_x(X_i)] \left(\frac{X_i - x}{h} \right)^s K \left(\frac{X_i - x}{h} \right). \end{aligned}$$

One has

$$|a_s| \leq \left| \frac{1}{n} \sum_{i=1}^n T_i^{(s,1)} \right| + \left| \frac{1}{n} \sum_{i=1}^n [T_i^{(s,2)} - \mathbb{E}T_i^{(s,2)}] \right| + |\mathbb{E}T_i^{(s,2)}|. \quad (\text{C.6})$$

Note that $\mathbb{E}T_i^{(s,1)} = 0$, $|T_i^{(s,1)}| \leq 2h^{-d}$, and

$$\begin{aligned}\text{Var}T_i^{(s,1)} &\leq \frac{1}{4h^d} \int_{\mathbb{R}^d} u^{2s} K^2(u) \mu(x+hu) du \leq \frac{\kappa \bar{\mu}}{4h^d}, \\ |T_i^{(s,2)} - \mathbb{E}T_i^{(s,2)}| &\leq Lh^{\beta-d} + L\kappa h^\beta \leq Ch^{\beta-d}, \\ \text{Var}T_i^{(s,2)} &\leq L^2 h^{2\beta-d} \int_{\mathbb{R}^d} |u^{2s}| K^2(u) \mu(x+hu) du \leq L^2 \bar{\mu} \kappa h^{2\beta-d}.\end{aligned}$$

From Bernstein's inequality, for $\epsilon_1, \epsilon_2 > 0$, we obtain

$$\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n T_i^{(s,1)} \right| \geq \epsilon_1 \right\} \leq 2 \exp \left(\frac{-nh^d \epsilon_1^2}{\kappa \bar{\mu}/2 + 4\epsilon_1/3} \right)$$

and

$$\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n [T_i^{(s,2)} - \mathbb{E}T_i^{(s,2)}] \right| \geq \epsilon_2 \right\} \leq 2 \exp \left(\frac{-nh^d \epsilon_2^2}{2L^2 \kappa \bar{\mu} h^{2\beta} + 2Ch^\beta \epsilon_2/3} \right).$$

Since also

$$|\mathbb{E}T_i^{(s,2)}| \leq Lh^\beta \int_{\mathbb{R}^d} |u^s| K^2(u) \mu(x+hu) du \leq L\kappa \bar{\mu} h^\beta$$

we get, using (C.6), that if $3L\kappa \bar{\mu} h^\beta c^{-1} \underline{\mu}^{-1} M \leq \delta \leq 1$ the following inequality holds

$$\begin{aligned}\mathbb{P} \left\{ |a_s| \geq \frac{c\mu\delta}{M} \right\} &\leq \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n T_i^{(s,1)} \right| > \frac{c\mu\delta}{3M} \right\} + \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n [T_i^{(s,2)} - \mathbb{E}T_i^{(s,2)}] \right| > \frac{c\mu\delta}{3M} \right\} \\ &\leq 4 \exp(-Cnh^d \underline{\mu}^2 \bar{\mu}^{-1} \delta^2).\end{aligned}$$

Combining this inequality with (C.3), (C.4), and (C.5), one has

$$\mathbb{P} \{ |\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \eta(x)| \geq \delta \} \leq C_{12} \exp(-C_{13} nh^d \underline{\mu}^2 \bar{\mu}^{-1} \delta^2)$$

for $3L\kappa \bar{\mu} h^\beta c^{-1} \underline{\mu}^{-1} M \leq \delta$ (for $\delta > 1$, this inequality is obvious since $\eta, \hat{\eta}^{\text{LP}}$ take values in $[0, 1]$). The constants C_{12}, C_{13} do not depend on the density μ , on its support \mathcal{X} and the point $x \in \mathcal{X}$. This concludes the proof. \blacksquare

C.2. Proof of Proposition C.2

Fix a bin $U \subseteq (0, 1)^d$ with side-length $2^{-l'}$, $l' \in \mathbb{R}_+$. Consider the matrix $B := (B_{s_1, s_2})_{|s_1|, |s_2| \leq p}$ and the vector $W := (W_s)_{|s| \leq p}$ with elements

$$B_{s_1, s_2} := \int_{\mathbb{R}^d} u^{s_1+s_2} K(u) \mu(x+hu) du, \quad W_s := \int_{\mathbb{R}^d} u^s \eta(x+hu) K(u) \mu(x+hu) du,$$

as well as the matrix $\bar{B} := (\bar{B}_{s_1, s_2})_{|s_1|, |s_2| \leq p}$ and the vector $\bar{W} := (\bar{W}_s)_{|s| \leq p}$ with elements

$$\bar{B}_{s_1, s_2} := \frac{1}{nh^d} \sum_{i=1}^n \left(\frac{X_i - x}{h} \right)^{s_1+s_2} K \left(\frac{X_i - x}{h} \right), \quad \bar{W}_s := \frac{1}{nh^d} \sum_{i=1}^n Y_i \left(\frac{X_i - x}{h} \right)^s K \left(\frac{X_i - x}{h} \right).$$

By Lemmas B.1 and C.1, one has

$$\begin{aligned}|\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \mathbf{\Gamma}_l^p \eta(x)| &= |U(0)^\top Q^{-1} V - U(0)^\top B^{-1} W| \\ &= |U(0)^\top \bar{B}^{-1} \bar{W} - U(0)^\top B^{-1} W| \\ &\leq |U(0)^\top B^{-1} (\bar{W} - W)| + |U(0)^\top (\bar{B}^{-1} - B^{-1}) \bar{W}| =: J_1 + J_2.\end{aligned}$$

That is,

$$\mathbb{P} \{ |\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \mathbf{\Gamma}_l^p \eta(x; U)| \geq \delta \} \leq \mathbb{P} \{ J_1 \geq 3\delta/4 \} + \mathbb{P} \{ J_2 \geq \delta/4 \}. \quad (\text{C.7})$$

First, we analyze J_1 . Note that

$$J_1 \leq \|B^{-1}(\bar{W} - W)\| \leq \lambda_{\min}^{-1}(B) \|\bar{W} - W\| \leq \mu_0^{-1} 2^{-dl'} \|\bar{W} - W\| \leq \mu_0^{-1} 2^{-dl'} M \max_s |\bar{W}_s - W_s|, \quad (\text{C.8})$$

where the third inequality follows from $\lambda_{\min}(B) \geq \mu_0 2^{dl'}$ by Lemma B.1, and M is the number of elements in the vector W . Define:

$$T_i^{(s)} := \frac{1}{h^d} Y_i \left(\frac{X_i - x}{h} \right)^s K \left(\frac{X_i - x}{h} \right) - \int_{\mathbb{R}^d} \eta(x + hu) u^s K(u) p_X(x + hu | U) du.$$

We have $\mathbb{E}[T_i^{(s)}] = 0$, $|T_i^{(s)}| \leq 2h^{-d}$, and

$$\mathbb{V}\text{ar}[T_i^{(s)}] \leq \frac{1}{h^{2d}} \mathbb{E} \left[\left(\frac{X_i - x}{h} \right)^{2s} K^2 \left(\frac{X_i - x}{h} \right) \right] \leq \frac{1}{h^d} \int_{\mathbb{R}^d} u^{2s} K^2(u) p_X(x + hu | U) du \leq \frac{2^{dl'}}{h^d}.$$

By Bernstein's inequality, we get

$$\mathbb{P}\{| \bar{W}_s - W_s | \geq \epsilon\} = \mathbb{P}\left\{ \left| \frac{1}{n} \sum_{i=1}^n T_i^{(s)} \right| > \epsilon \right\} \leq 2 \exp \left(\frac{-nh^d \epsilon^2}{2^{1+dl'} + 4\epsilon/3} \right).$$

Combining this inequality with (C.8), one obtains

$$\begin{aligned} \mathbb{P}\{J_1 \geq 3\delta/4\} &\leq \sum_{|s| \leq p} \mathbb{P}\left\{ |\bar{W}_s - W_s| \geq 3\mu_0 2^{dl'} M^{-1} \delta/4 \right\} \\ &= \sum_{|s| \leq p} \mathbb{P}\left\{ \left| \frac{1}{n} \sum_{i=1}^n T_i^{(s)} \right| \geq 3\mu_0 2^{dl'} M^{-1} \delta/4 \right\} \leq 2M \exp \left(\frac{-9\mu_0^2 M^{-2} 2^{d(l'-l)} n \delta^2 / 16}{2 + \mu_0 M^{-1} \delta} \right). \end{aligned} \quad (\text{C.9})$$

Now, we analyze J_2 . Note that

$$J_2 \leq \|(\bar{B}^{-1} - B^{-1}) \bar{W}\| \leq \|\bar{B}^{-1} - B^{-1}\| \|\bar{W}\| \leq M \|\bar{B}^{-1} - B^{-1}\| \cdot \max_s |\bar{W}_s| \leq M \|\bar{B}^{-1} - B^{-1}\| h^{-d}. \quad (\text{C.10})$$

Define $Z := \bar{B} - B$. One has

$$\lambda_{\max}(Z) \leq \sum_{|s_1|, |s_2| \leq p} |Z_{s_1, s_2}|.$$

Define

$$T_i^{(s_1, s_2)} := \frac{1}{h^d} \left(\frac{X_i - x}{h} \right)^{s_1 + s_2} K \left(\frac{X_i - x}{h} \right) - \int_{\mathbb{R}^d} u^{s_1 + s_2} K(u) p_X(x + hu | U) du.$$

We have $\mathbb{E}[T_i^{(s_1, s_2)}] = 0$, $|T_i^{(s_1, s_2)}| \leq 2h^{-d}$, and

$$\mathbb{V}\text{ar}[T_i^{(s_1, s_2)}] \leq \mathbb{E} \left[\frac{1}{h^{2d}} \left(\frac{X_i - x}{h} \right)^{2s_1 + 2s_2} K^2 \left(\frac{X_i - x}{h} \right) \right] = \frac{1}{h^d} \int_{\mathbb{R}^d} u^{2s_1 + 2s_2} K^2(u) p_X(x + hu | U) du \leq \frac{2^{dl'}}{h^d}.$$

By Bernstein's inequality, one obtains

$$\begin{aligned} \mathbb{P}\left\{ \lambda_{\max}(Z) \geq 2^{dl'} M^{-1} \mu_0^2 \delta / 8 \right\} &\leq \mathbb{P}\left\{ \sum_{|s_1|, |s_2| \leq p} |Z_{s_1, s_2}| \geq 2^{dl'} M^{-1} \mu_0^2 \delta / 8 \right\} \\ &\leq \sum_{|s_1|, |s_2| \leq p} \mathbb{P}\left\{ |Z_{s_1, s_2}| \geq 2^{dl'} M^{-3} \mu_0^2 \delta / 8 \right\} \\ &= \sum_{|s_1|, |s_2| \leq p} \mathbb{P}\left\{ \left| \frac{1}{n} \sum_{i=1}^n T_i^{(s_1, s_2)} \right| \geq h^{-d} M^{-3} \mu_0^2 \delta / 8 \right\} \\ &\leq 2M^2 \exp \left(\frac{-n 2^{d(l'-l)} M^{-6} \mu_0^4 \delta^2 / 64}{2 + M^{-3} \mu_0^2 \delta / 6} \right). \end{aligned}$$

By Lemma B.1, $\|B^{-1}\| \leq 2^{-dl'} \mu_0^{-1}$. That is, on the event $\{\lambda_{\max}(Z) \leq 2^{dl'} M^{-1} \mu_0^2 \delta / 8\}$, one has $\|B^{-\frac{1}{2}} Z B^{-\frac{1}{2}}\| \leq M^{-1} \mu_0 \delta / 8$ in which case if $M^{-1} \mu_0 \delta / 8 < \frac{1}{2}$, one obtains

$$\begin{aligned} \|\bar{B}^{-1} - B^{-1}\| &= \left\| B^{-\frac{1}{2}} \left(\left(I + B^{-\frac{1}{2}} Z B^{-\frac{1}{2}} \right)^{-1} - I \right) B^{-\frac{1}{2}} \right\| \\ &\leq \|B^{-1}\| \left\| \left(I + B^{-\frac{1}{2}} Z B^{-\frac{1}{2}} \right)^{-1} - I \right\| \\ &\leq 2^{-dl'} \mu_0^{-1} \sum_{j=1}^{\infty} \left\| B^{-\frac{1}{2}} Z B^{-\frac{1}{2}} \right\|^j \\ &\leq 2^{-dl'} \mu_0^{-1} \sum_{j=1}^{\infty} (M^{-1} \mu_0 \delta / 8)^j \leq 2^{-dl'} M^{-1} \delta / 4. \end{aligned}$$

This inequality along with (C.10) imply $J_2 \leq \delta / 4$. In other words,

$$\mathbb{P}\{J_2 \geq \delta / 4\} \leq \mathbb{P}\left\{\lambda_{\max}(Z) \geq 2^{dl'} M^{-1} \mu_0^2 \delta / 8\right\} \leq 2M^2 \exp\left(\frac{-n2^{d(l'-l)} M^{-6} \mu_0^4 \delta^2 / 64}{2 + M^{-3} \mu_0^2 \delta / 6}\right).$$

Combining this inequality with (C.7) and (C.9) gives

$$\mathbb{P}\left\{|\hat{\eta}^{\text{LP}}(x; \mathcal{D}, h, p) - \mathbf{\Gamma}_l^p \eta(x; U)| \geq \delta\right\} \leq C_{15} \exp\left(-C_{16} n 2^{d(l'-l)} \delta^2\right)$$

if $M^{-1} \mu_0 \delta / 8 < \frac{1}{2}$. This concludes the proof. \blacksquare

Appendix D: Auxiliary analysis for Section 2.1

D.1. Analysis of Part 1 of Example 2.1

Step 1. Following the proof of Theorem 4.1 in Rigollet and Zeevi (2010), we first construct a problem instance in $\mathcal{P}(\beta, \alpha, d)$. Define $M := \left\lceil 2^{-1} c_0^{-1} \left(\frac{2 \log 2}{T} \right)^{\frac{-\beta}{2\beta+d}} \right\rceil^{\frac{1}{\beta}}$ and let $\mathcal{B} := \{\mathbf{B}_m, m = 1, \dots, M^d\}$ be a re-indexed collection of the hypercubes

$$\mathbf{B}_m = \mathbf{B}_{\mathbf{m}} := \left\{ x \in [0, 1]^d : \frac{\mathbf{m}_i - 1}{2^l} \leq x_i \leq \frac{\mathbf{m}_i}{2^l}, i \in \{1, \dots, d\} \right\},$$

for $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_d)$ with $\mathbf{m}_i \in \{1, \dots, M\}$. Consider the regular grid $\mathcal{Q} = \{q_1, q_2, \dots, q_{M^d}\}$, where q_k denotes the center of bin $\mathbf{B}_k, k = 1, \dots, M^d$. Define $C := 2^{\beta-1} L \wedge \frac{1}{4}$ and let ϕ be defined as follows:

$$\phi(x) = \begin{cases} (1 - \|x\|_{\infty})^{\beta} & \text{if } \|x\|_{\infty} \leq 1 \\ 0 & \text{o.w.} \end{cases}.$$

Define $m := \lceil \mu M^{d-\alpha\beta} \rceil$, where $\mu \in (0, 1)$ is chosen small enough to ensure $m \leq M^d$. Define the payoff functions as follows:

$$f_1(x) = \frac{1}{2} + \sum_{j=1}^m M^{-\beta} C \phi(M[x - q_j]), \quad f_2(x) = \frac{1}{2},$$

and assume that covariates are distributed uniformly. Similar to the proof of Theorem 4.1 in Rigollet and Zeevi (2010), one can show that the margin condition and smoothness condition in Assumptions 2.3 and 2.1 are satisfied for the constructed problem instance.

Step 2. Next, we lower bound the regret of $\text{ABSE}(\hat{\beta})$ under the constructed problem instance. To do so, we use the same exact terminology and notation as in Perchet and Rigollet (2013); for the sake of brevity, we do not re-introduce the notation here. By construction, for all bins \mathbf{B} with $|\mathbf{B}| = 2^{-k}$, $k = 0, 1, \dots, k_0$, we have $\mathcal{I}_{\mathbf{B}} = \mathcal{K} = \{1, 2\}$. Define the event $\mathcal{W}_{\mathbf{B},s} := \{\mathcal{I}_{\mathbf{B}} \subseteq \mathbb{I}_{\mathbf{B},s}\} = \{\mathbb{I}_{\mathbf{B},s} = \mathcal{K}\}$ and $\mathcal{V}_{\mathbf{B}} := \bigcap_{\mathbf{B}' \in \mathcal{P}(\mathbf{B})} \mathcal{W}_{\mathbf{B}',t_{\mathbf{B}}}$. Let

$$\mathcal{A}_1 := \{\exists t \leq T; \exists \mathbf{B} \in \mathcal{L}_t; \exists s \leq l_{\mathbf{B}} : \mathbb{I}_{\mathbf{B},s} \neq \mathcal{K} \text{ and } |\mathbf{B}| \geq 2^{-k_0+1}\}$$

denote the event where one of the arms is eliminated in at least one of the bins at depth less than k_0 . One has:

$$\mathbb{P}\{\mathcal{A}_1\} \leq \sum_{k=1}^{k_0-1} \sum_{|\mathbf{B}|=2^{-k}} \mathbb{P}\{\mathcal{V}_{\mathbf{B}} \cap \bar{\mathcal{W}}_{\mathbf{B},t_{\mathbf{B}}}\}. \quad (\text{D.1})$$

Note that for any bin \mathbf{B} with $|\mathbf{B}| \geq 2^{-k_0+1}$, $|\bar{f}_{\mathbf{B}}^{(1)} - \bar{f}_{\mathbf{B}}^{(2)}| < c_0 |\mathbf{B}|^{\beta} \leq \frac{\epsilon_{\mathbf{B},l_{\mathbf{B}}}}{2}$. This implies that $\mathcal{W}_{\mathbf{B}}$ can only happen if either $\bar{f}_{\mathbf{B}}^{(1)}$ or $\bar{f}_{\mathbf{B}}^{(2)}$ does not belong to its respective confidence interval $[\bar{Y}_{\mathbf{B},s}^{(1)} \pm \epsilon_{\mathbf{B},s}]$ or $[\bar{Y}_{\mathbf{B},s}^{(2)} \pm \epsilon_{\mathbf{B},s}]$ for some $s \leq l_{\mathbf{B}}$. Therefore, since $-\bar{f}_{\mathbf{B}}^{(i)} \leq Y_s - \bar{f}_{\mathbf{B}}^{(i)} \leq 1 - \bar{f}_{\mathbf{B}}^{(i)}$,

$$\mathbb{P}\{\mathcal{V}_{\mathbf{B}} \cap \bar{\mathcal{W}}_{\mathbf{B},t_{\mathbf{B}}}\} \leq \mathbb{P}\left\{\exists s \leq l_{\mathbf{B}}; \exists i \in \mathcal{K} : \left|\bar{Y}_{\mathbf{B},s}^{(i)} - \bar{f}_{\mathbf{B}}^{(i)}\right| \geq \frac{\epsilon_{\mathbf{B},s}}{4}\right\} \leq \frac{4l_{\mathbf{B}}}{T|\mathbf{B}|^d}. \quad (\text{D.2})$$

Putting together (D.1) and (D.2), one obtains

$$\mathbb{P}\{\mathcal{A}_1\} \leq \sum_{k=1}^{k_0-1} \frac{4C_l 2^{-2\hat{\beta}k} \log(T 2^{(2\hat{\beta}+d)k})}{T 2^{-kd}} \leq \frac{4C_l 2^{-(2\hat{\beta}-d)k_0} \log(T^2)}{T} \leq 8C_l T^{\frac{-4\hat{\beta}}{2\hat{\beta}+d}} \log T. \quad (\text{D.3})$$

Step 3. Let $\tilde{c} := 2^{1-d-2\hat{\beta}} c_0^{-2} \log 2$ and define

$$\mathcal{A}_2 := \{\exists t \leq \tilde{c}T/2; \exists \mathbf{B} \in \mathcal{L}_t : |\mathbf{B}| \geq 2^{-k_0+1}\}$$

to be the event that for some $t \leq \tilde{c}T/2$ some bin at depth k_0 becomes live. Note that for a bin \mathbf{B} to become live by $t = \lfloor \tilde{c}T/2 \rfloor$, we need $l_{\mathbf{p}(\mathbf{B})}$ number of contexts to fall into its parent $\mathbf{p}(\mathbf{B})$ by $t = \lfloor \tilde{c}T/2 \rfloor$. Let $Z_{\mathbf{B},t} = \mathbb{1}\{X_t \in \mathbf{p}(\mathbf{B})\}$. Note that $|Z_t| \leq 1$, $\mathbb{E}Z_t = |\mathbf{B}|^d$, and $\text{Var}Z_{\mathbf{B},t} \leq \mathbb{E}Z_{\mathbf{B},t}^2 = |\mathbf{B}|^d$. Hence, one can apply the Bernstein's inequality in Lemma 7.4 to to $\sum_{t=1}^{\lfloor \tilde{c}T/2 \rfloor} Z_{\mathbf{B},t}$ for $|\mathbf{B}| = 2^{-k_0}$ to obtain

$$\begin{aligned} \mathbb{P}\{\mathcal{A}_2\} &\leq \sum_{|\mathbf{B}|=2^{-k_0}} \mathbb{P}\left\{\sum_{t=1}^{\lfloor \tilde{c}T/2 \rfloor} Z_{\mathbf{B},t} \geq l_{\mathbf{p}(\mathbf{B})}\right\} \\ &\stackrel{(a)}{\leq} \sum_{|\mathbf{B}|=2^{-k_0}} \mathbb{P}\left\{\sum_{t=1}^{\lfloor \tilde{c}T/2 \rfloor} Z_{\mathbf{B},t} \geq c_0^{-2} |\mathbf{p}(\mathbf{B})|^{-2\hat{\beta}}\right\} \\ &\leq 2^{k_0 d} \exp\left(-\frac{c_0^{-4} 2^{4(k_0-1)\hat{\beta}-1}/2}{\tilde{c}T 2^{(k_0-1)d-1} + c_0^{-2} 2^{2(k_0-1)\hat{\beta}-1}/3}\right) \\ &= 2^{k_0 d} \exp\left(-\frac{c_0^{-2} 2^{2(k_0-1)\hat{\beta}-1}/2}{\frac{1}{2} + 1/3}\right) \\ &\leq c_1 T^{\frac{-d}{2\hat{\beta}+d}} \exp\left(-c_2 T^{\frac{2\hat{\beta}}{2\hat{\beta}+d}}\right) \leq c_3 T^{-1}, \end{aligned} \quad (\text{D.4})$$

for some constants $c_1, c_2, c_3 > 0$, where (a) follows from $l_{\mathbf{B}} \geq c_0^{-2} |\mathbf{B}|^{-2\hat{\beta}}$ by the definition of $l_{\mathbf{B}}$.

Step 4. Let $S := \{x \in [0, 1]^d : f_1(x) \neq \frac{1}{2}\}$. Define the event

$$\mathcal{A}_3 := \left\{ \sum_{t=1}^{\lfloor \tilde{c}T/2 \rfloor} \mathbb{1}\{X_t \in S\} < \tilde{c}mM^{-d}T/4 \right\}.$$

Define $Z_t := \mathbb{1}\{X_t \in S\}$ and note that $|Z_t| \leq 1$, $\mathbb{E}Z_t = mM^{-d}$, and $\text{Var}Z_t \leq \mathbb{E}Z_t = mM^{-d}$. As a result we can apply the Bernstein's inequality in Lemma 7.4 to obtain

$$\mathbb{P}\{\mathcal{A}_3\} \leq \exp(-\tilde{c}mM^{-d}T/20) \leq \exp\left(-c_5T^{\frac{2\hat{\beta}+d-\alpha\hat{\beta}}{2\hat{\beta}+d}}\right) \stackrel{(a)}{\leq} \exp\left(-c_4T^{\frac{2\hat{\beta}}{2\hat{\beta}+d}}\right) \leq c_5T^{-1}, \quad (\text{D.5})$$

for some constants $c_4, c_5 > 0$, where (a) follows from the assumption that $\alpha \leq \frac{1}{\hat{\beta}} \leq \frac{1}{\beta}$.

Step 5. Note that on the event $\bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2$, the $\text{ABSE}(\hat{\beta})$ has not eliminated any arms over any region of the covariate space up to time $t = \lfloor \tilde{c}T/2 \rfloor$. On the other hand, on the event $\bar{\mathcal{A}}_3$, up to time $t = \lfloor \tilde{c}T/2 \rfloor$, at least $\tilde{c}mM^{-d}T/4$ number of contexts have fallen into S , where the first arm is strictly optimal. Recall the definition of the inferior sampling rate in (7.1). One has:

$$\begin{aligned} \mathcal{S}^{\text{ABSE}(\hat{\beta})}(\mathbf{P}; T) &\geq \mathbb{E}^\pi \left[\sum_{t=1}^T \mathbb{1}\{f_{\pi_t^*}(X_t) \neq f_{\pi_t}(X_t)\} \middle| \bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2 \cap \bar{\mathcal{A}}_3 \right] \mathbb{P}\{\bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2 \cap \bar{\mathcal{A}}_3\} \\ &\stackrel{(a)}{\geq} \tilde{c}mM^{-d}T/8 \left(1 - 8C_tT^{\frac{-4\hat{\beta}}{2\hat{\beta}+d}} \log T - c_3T^{-1} - c_5T^{-1} \right) \\ &\geq c_6T^{\frac{\alpha\hat{\beta}}{2\hat{\beta}+d}}, \end{aligned}$$

for some constant $c_6 > 0$, where (a) follows from (D.3), (D.4), and (D.5). Using this inequality along with Lemma 7.1, the result follows. \blacksquare

D.2. Analysis of Part 2 of Example 2.1

Step 1. let $\hat{k} := \left\lceil \frac{4}{2^{\hat{\beta}}} \vee \frac{1}{\beta - \hat{\beta}} \log_2 \left(40 \times 8^{\hat{\beta}} d^{\frac{\hat{\beta}}{2}} \right) \right\rceil$, $M := 2^{\hat{k}}$, and $\mathcal{B} := \{\mathbf{B}_m, m = 1, \dots, M^d\}$ be a re-indexed collection of the hypercubes

$$\mathbf{B}_m = \mathbf{B}_{\mathbf{m}} := \left\{ x \in [0, 1]^d : \frac{\mathbf{m}_i - 1}{2^l} \leq x_i \leq \frac{\mathbf{m}_i}{2^l}, i \in \{1, \dots, d\} \right\},$$

for $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_d)$ with $\mathbf{m}_i \in \{1, \dots, M\}$. Let $\tilde{x}_0 := \left(40d^{\frac{\hat{\beta}}{2}} M^{-\hat{\beta}} \right)^{\frac{1}{\hat{\beta}}}$, and define the function

$$\psi(x) := \begin{cases} -L|x_1 - \tilde{x}_0|^\beta + 2L\tilde{x}_0^\beta & \text{if } 0 \leq x_1 \leq \tilde{x}_0 \\ L|x_1 - \tilde{x}_0|^\beta + 2L\tilde{x}_0^\beta & \text{if } \tilde{x}_0 \leq x_1 \leq \frac{M^{-1}}{4} \\ L\left|x_1 + \tilde{x}_0 - \frac{M^{-1}}{2}\right|^\beta + 2L\tilde{x}_0^\beta & \text{if } \frac{M^{-1}}{4} \leq x_1 \leq \frac{M^{-1}}{2} - \tilde{x}_0 \\ -L\left|x_1 + \tilde{x}_0 - \frac{M^{-1}}{2}\right|^\beta + 2L\tilde{x}_0^\beta & \text{if } \frac{M^{-1}}{2} - \tilde{x}_0 \leq x_1 \leq \frac{M^{-1}}{2} \\ L\left|x_1 - \tilde{x}_0 - \frac{M^{-1}}{2}\right|^\beta & \text{if } \frac{M^{-1}}{2} \leq x_1 \leq \frac{M^{-1}}{2} + \tilde{x}_0 \\ -L\left|x_1 - \tilde{x}_0 - \frac{M^{-1}}{2}\right|^\beta & \text{if } \frac{M^{-1}}{2} + \tilde{x}_0 \leq x_1 \leq \frac{3M^{-1}}{4} \\ -L\left|x_1 + \tilde{x}_0 - \frac{3M^{-1}}{4}\right|^\beta & \text{if } \frac{3M^{-1}}{4} \leq x_1 \leq M^{-1} - \tilde{x}_0 \\ L\left|x_1 + \tilde{x}_0 - \frac{3M^{-1}}{4}\right|^\beta & \text{if } M^{-1} - \tilde{x}_0 \leq x_1 \leq M^{-1} \\ \min\left(|x - M^{-1}|^\beta + L\tilde{x}_0^\beta, \frac{1}{2}\right) & \text{if } M^{-1} \leq x_1 \leq 1 \end{cases}.$$

Note that $|\psi(x)| \leq \frac{1}{2}$. We define the payoff functions as $f_1(x) := \frac{1}{2} + \psi(x)$, $f_2(x) := 1$, and assume that covariates are distributed uniformly. One can show that the margin condition and smoothness condition in Assumptions 2.3 and 2.1 are satisfied for the constructed problem instance.

Step 2. Next, we lower bound the regret of $\text{ABSE}(\hat{\beta})$ under the constructed problem instance. To do so, we use the same exact terminology and notation as in Perchet and Rigollet (2013); for the sake of brevity, we do not re-introduce the notation here. Define the set of bins $\mathcal{Z} := \{\mathbf{B} \in \mathcal{B} : \exists x \in \mathbf{B} : 0 < x_1 < M^{-1}\}$ to include all the bins in \mathcal{B} for which their first coordinate is between 0 and M^{-1} . For any bin \mathbf{B} , define $\mathcal{W}_{\mathbf{B}} := \{\mathbb{I}_{\mathbf{B}, l_{\mathbf{B}}} = \{1\}\}$ to be the event that at the end of sampling in \mathbf{B} , the only remaining arm is arm 1. Also, let $\mathcal{V}_{\mathbf{B}} := \{1 \in \mathbb{I}_{\mathbf{B}, l_{\mathbf{B}}}\}$ be the event that arm 1 is not eliminated at the end of sampling in bin \mathbf{B} . Let $\mathcal{A}_1 := \bigcup_{\mathbf{B} \in \mathcal{Z}} \bar{\mathcal{W}}_{\mathbf{B}}$. Note that by construction for any bin $\mathbf{B} \in \mathcal{Z}$, $\bar{f}_{\mathbf{B}}^{(1)} - \bar{f}_{\mathbf{B}}^{(2)} = 10c_0|\mathbf{B}|^{\hat{\beta}}$. Furthermore, for any ancestors of the bins in \mathcal{Z} , one has $\bar{f}_{\mathbf{B}}^{(1)} - \bar{f}_{\mathbf{B}}^{(2)} > 0 \forall \mathbf{B} : |\mathbf{B}| > M^{-1}$. The last two observations yield that for any bin $\mathbf{B} \in \mathcal{Z}$,

$$\mathcal{W}_{\mathbf{B}} = \bigcap_{\mathbf{B}' \in \mathcal{P}(\mathbf{B})} \mathcal{A}_{\mathbf{B}'} \cap \mathcal{A}_{\mathbf{B}}.$$

Hence by (5.8), (5.9), (5.11) in Perchet and Rigollet (2013), for any bin $\mathbf{B} \in \mathcal{Z}$, one has

$$\mathbb{P}\{\mathcal{A}_1\} \leq \sum_{\mathbf{B} \in \mathcal{Z}} \mathbb{P}\{\bar{\mathcal{W}}_{\mathbf{B}}\} \leq \sum_{\mathbf{B} \in \mathcal{Z}} 8 \frac{\hat{k} \cdot l_{\mathbf{B}}}{TM^d} \leq c_1 \frac{\log T}{T}, \quad (\text{D.6})$$

for some constant $c_1 > 0$.

Step 3. Define

$$\mathcal{A}_2 := \left\{ \exists \mathbf{B} \in \mathcal{L}_{\lceil \frac{T}{2} \rceil} : |\mathbf{B}| \geq M^{-1} \right\}$$

to be the event that at time $t = \lceil \frac{T}{2} \rceil$ there exists any live bin with $|\mathbf{B}| \geq M^{-1}$. Note that by (5.3) in Perchet and Rigollet (2013), if $\hat{k} \cdot l_{\mathbf{B}} \leq c_2 \log T$, $c_2 > 0$, number of contexts fall inside each bin $\mathbf{B} \in \mathcal{B}$ then, there will be no live bin with $|\mathbf{B}| \geq M^{-1}$. For any bin $\mathbf{B} \in \mathcal{B}$, let $Z_{\mathbf{B},t} = \mathbb{1}\{X_t \in \mathbf{B}\}$. Note that $|Z_{\mathbf{B},t}| \leq 1$, $\mathbb{E}Z_{\mathbf{B},t} = |\mathbf{B}|^d$, and $\text{Var}Z_{\mathbf{B},t} \leq \mathbb{E}Z_{\mathbf{B},t}^2 = |\mathbf{B}|^d$. Hence, one can apply the Bernstein's inequality in Lemma 7.4 to $\sum_{t=1}^{\lceil \frac{T}{2} \rceil} Z_{\mathbf{B},t}$ to obtain

$$\mathbb{P}\{\mathcal{A}_2\} \leq \sum_{\mathbf{B} \in \mathcal{B}} \mathbb{P}\left\{ \sum_{t=1}^{\lceil \frac{T}{2} \rceil} Z_{\mathbf{B},t} < c_2 \log T \right\} \leq 2^{ld} \exp(-c_3 T) \leq c_4 T^{-1}, \quad (\text{D.7})$$

for some constants $c_3, c_4 > 0$.

Step 4. Let $S := \left\{ x \in [0, 1]^d : \frac{M^{-1}}{2} + \tilde{x}_0 < x_1 < M^{-1} - \tilde{x}_0 \right\}$. Note that $\frac{M^{-1}}{4} \leq \lambda(S) \leq \frac{M^{-1}}{2}$, where $\lambda(\cdot)$ denotes the Lebesgue measure. Define the event

$$\mathcal{A}_3 := \left\{ \sum_{t=\lceil cT/2 \rceil}^T \mathbb{1}\{X_t \in S\} < \frac{TM^{-1}}{16} \right\}.$$

Define $Z_t := \mathbb{1}\{X_t \in S\}$ and note that $|Z_t| \leq 1$, $\mathbb{E}Z_t = \lambda(S)$, and $\text{Var}Z_t \leq \mathbb{E}Z_t = \lambda(S)$. Hence, one can apply the Bernstein's inequality in Lemma 7.4 to obtain

$$\mathbb{P}\{\mathcal{A}_3\} \leq \exp(-c_5 T) \leq c_6 T^{-1}, \quad (\text{D.8})$$

for some constants $c_5, c_6 > 0$.

Step 5. Note that on the event $\bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2$, $\text{ABSE}(\hat{\beta})$ has eliminated arm 2 for all the bins $\mathbf{B} \in \mathcal{Z}$, which contains S . On the other hand, on the event $\bar{\mathcal{A}}_3$, from time step $t = \lceil T/2 \rceil$ to the end of the time horizon, at least $\frac{TM^{-1}}{16}$ number of contexts have fallen into S , where the second arm is strictly optimal. Recall the definition of the inferior sampling rate in (7.1). One has:

$$\begin{aligned} \mathcal{S}^{\text{ABSE}(\hat{\beta})}(\mathbf{P}; T) &\geq \mathbb{E}^{\pi} \left[\sum_{t=1}^T \mathbb{1}\{f_{\pi_t^*}(X_t) \neq f_{\pi_t}(X_t)\} \middle| \bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2 \cap \bar{\mathcal{A}}_3 \right] \mathbb{P}\{\bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2 \cap \bar{\mathcal{A}}_3\} \\ &\stackrel{(a)}{\geq} \frac{TM^{-1}}{16} \left(1 - c_1 \frac{\log T}{T} - c_4 T^{-1} - c_6 T^{-1} \right) \\ &\geq c_7 T, \end{aligned}$$

for some constant $c_7 > 0$, where (a) follows from (D.6), (D.7), and (D.8). Using this inequality along with Lemma 7.1, the result follows. \blacksquare

Appendix E: Proof of auxiliary lemmas

E.1. Proof of Lemma 7.2

LEMMA. Suppose $f \in \mathcal{H}_{\mathbb{R}^d}(\beta, L)$ for some $\mathcal{X} \in \mathbb{R}^d$, $0 < \beta \leq 1$ and $L > 0$, and define the function g such that $g(x) = C^{-\beta} f(Cx)$ for all $x \in \mathbb{R}^d$ and some $C > 0$. Then, $g \in \mathcal{H}_{\mathbb{R}^d}(\beta, L)$.

Proof. For any $x, y \in \mathbb{R}^d$, one has

$$|g(x) - g(y)| = C^{-\beta} |f(Cx) - f(Cy)| \leq C^{-\beta} L \|Cx - Cy\|_{\infty}^{\beta} = L \|x - y\|_{\infty}^{\beta}.$$

This concludes the proof. \blacksquare

E.2. Proof of Lemma 7.3

LEMMA. Suppose $f, g \in \mathcal{H}_{\mathcal{X}}(\beta, L)$ for some $\mathcal{X} \subseteq \mathbb{R}^d$, $0 < \beta \leq 1$ and $L > 0$, and define the functions $h_1 := \max(f, g)$ and $h_2 := \min(f, g)$. Then, $h_1, h_2 \in \mathcal{H}_{\mathcal{X}}(\beta, L)$.

Proof. We only prove the result for the function h_1 . A similar analysis can be used for h_2 . Fix some $x, y \in \mathcal{X}$. If $h_1(x) = f(x)$ and $h_1(y) = f(y)$, or $h_1(x) = g(x)$ and $h_1(y) = g(y)$ then, one has

$$|h_1(x) - h_1(y)| \leq L \|x - y\|_{\infty}^{\beta}.$$

Now suppose $h_1(x) = f(x)$ and $h_1(y) = g(y)$. Without loss of generality, assume that $f(x) \leq g(y)$ then, one has

$$|h_1(x) - h_1(y)| \leq |g(x) - g(y)| \leq L \|x - y\|_{\infty}^{\beta}.$$

The case $h_1(x) = f(x)$ and $h_1(y) = g(y)$ can be analyzed similarly. This concludes the proof. \blacksquare

E.3. Proof of Lemma 7.5

LEMMA. Let ρ_0, ρ_1 be two probability distributions supported on some set \mathcal{X} , with ρ_0 absolutely continuous with respect to ρ_1 . Then for any measurable function $\Psi : \mathcal{X} \rightarrow \{0, 1\}$, one has:

$$\mathbb{P}_{\rho_0}\{\Psi(X) = 1\} + \mathbb{P}_{\rho_1}\{\Psi(X) = 0\} \geq \frac{1}{2} \exp(-\text{KL}(\rho_0, \rho_1)).$$

Proof. Define \mathcal{B} to be the event that $\Psi(X) = 1$. One has

$$\mathbb{P}_{\rho_0}\{\Psi(X) = 1\} + \mathbb{P}_{\rho_1}\{\Psi(X) = 0\} = \mathbb{P}_{\rho_0}\{\mathcal{B}\} + \mathbb{P}_{\rho_1}\{\bar{\mathcal{B}}\} \geq \int \min\{d\rho_0, d\rho_1\} \geq \frac{1}{2} \exp(-\text{KL}(\rho_0, \rho_1)),$$

where the last inequality follows from Tsybakov 2008, Lemma 2.6. \blacksquare

E.4. Proof of Lemma A.2

LEMMA. Suppose $f, g \in \mathcal{H}_{\mathcal{X}}(\beta, L)$ for some $\mathcal{X} \subseteq [0, 1]$, $\beta > 0$, and $L > 0$, and define the function $h := f \cdot g$ as the product of f and g . Then, $h \in \mathcal{H}(\beta, L')$ for some $L' > 0$.

Proof. Note that h is $\lfloor \beta \rfloor$ times continuously differentiable. Hence, we only need to show that there exists some $L' > 0$ such that for any $x, x' \in \mathcal{X}$,

$$|h(x') - h_x(x')| \leq L' \|x - x'\|_{\infty}^{\beta}.$$

By the triangle inequality, one has

$$\begin{aligned}
|h(x') - h_x(x')| &\leq |(f \cdot g)_x(x') - f_x(x') \cdot g_x(x')| + |f_x(x') \cdot g(x') - f_x(x') \cdot g_x(x')| \\
&\quad + |f_x(x') \cdot g(x') - f(x') \cdot g(x')|.
\end{aligned} \tag{E.1}$$

Since $\mathcal{X} \subseteq [0, 1]$ and $f, g \in \mathcal{H}_{\mathcal{X}}(\beta, L)$, one has for some $L_1, L_2 > 0$:

$$|f_x(x') \cdot g(x') - f_x(x') \cdot g_x(x')| = |f_x(x')| \cdot |g(x') - g_x(x')| \leq L_1 \|x - x'\|_{\infty}^{\beta}; \tag{E.2}$$

$$|f_x(x') \cdot g(x') - f(x') \cdot g(x')| = |f_x(x') - f(x')| |g(x')| \leq L_2 \|x - x'\|_{\infty}^{\beta}. \tag{E.3}$$

Furthermore, let $\{a_s\}_{0 \leq s \leq \lfloor \beta \rfloor}$, $\{b_s\}_{0 \leq s \leq \lfloor \beta \rfloor}$, and $\{c_s\}_{0 \leq s \leq \lfloor \beta \rfloor}$ be the coefficients of the Taylor expansions $f_x(x')$, $g_x(x')$, and $h_x(x')$, respectively. Notably, $c_s = \sum_{s'=0}^s a_{s'} b_{s-s'}$. This equality implies that

$$|(f \cdot g)_x(x') - f_x(x') \cdot g_x(x')| = \left| \sum_{s=\lfloor \beta \rfloor+1}^{2\lfloor \beta \rfloor} \sum_{s'=0}^s a_{s'} b_{s-s'} (x - x')^s \right| \leq L_3 \|x - x'\|_{\infty}^{\beta}, \tag{E.4}$$

for some $L_3 > 0$. Then, the result follows from putting together (E.1), (E.2), (E.3), and (E.4). \blacksquare