Causal Inference Under Unmeasured Confounding With Negative Controls: A Minimax Learning Approach

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Abstract

We study the causal inference when not all confounders are observed and instead negative controls are available. Recent work has shown how negative controls can enable identification and efficient estimation of average treatment effects via two so-called bridge functions. In this paper, we consider a generalized average causal effect (GACE) with general interventions (discrete or continuous) and tackle the central challenge to causal inference using negative controls: the identification and estimation of the two bridge functions. Previous work has largely relied on completeness assumptions for identification and uniqueness assumptions for estimation, and mainly focused on estimating these functions parametrically. We provide a new identification strategy for GACE that avoids completeness, and propose new minimax-learning estimators for the (nonnunique) bridge functions that can accommodate general function classes such as Reproducing Kernel Hilbert spaces and neural networks and can provide theoretical guarantees even when the bridge functions are nonnunique. We study finite-sample convergence results both for estimating bridge function themselves and for the final GACE estimator. We do this under a variety of combinations of assumptions on the hypothesis and critic classes employed in the minimax estimator. Depending on how much we are willing to assume, we obtain different convergence rates. In some cases, we show that the GACE estimator may converge to truth even when our minimax bridge function estimators do not converge to any valid bridge function. And, in other cases, we show we can obtain semiparametric efficiency.

1 Introduction

Causal inference from observational data is a necessity in many fields where experimentation and randomization is limited. Even when experimentation is feasible, observational data can help support initial or supplementary investigations. Compared to experimental-intervention data, the key difficulty with observational data is confounding or endogeneity: correlations between observed actions A and outcomes Y that are not due to a causal relationship, as might be induced by common causes such as a healthy lifestyle leading to both selection into treatment and good health outcomes. A common identification strategy is to control for many baseline covariates and assume they fully account for all such common causes, termed unconfoundedness, ignorability, or exchangeability. However, in practice, it is dubious that all confounders are ever truly accounted for, casting doubt on any resulting conclusion.

When some confounders are unobserved, an alternative identification strategy is to use *negative controls* [Miao et al., 2018a,b, Tchetgen et al., 2020, Cui et al., 2020, Deaner, 2021, Shi et al., 2020]. Negative controls are observed covariates that have a more restricted relationship with the action and outcome: *negative control actions* do not directly impact the outcome of interest and *negative control outcomes* are not directly impacted by either the negative control actions or the primary action of interest. See Fig. 1 for a typical causal diagram. When these

*Alphabetical order.

negative control variables are sufficiently informative about the unmeasured confounders, there exist the so-called *bridge functions* that enable identification and estimation of causal quantities. These bridge functions are the analogues to the propensity and outcome regression functions one would use if all confounders were observed.

Learning these bridge functions, however, is a nontrivial task, as it no longer amounts to a regression problem as in the unconfounded case, which can be outsourced to standard machine learning methods. Cui et al. [2020] recently studied the semiparametric efficiency of the negative control method and proposed a doubly robust approach, but they focused on parametric estimates, which may be too restrictive in practice. Moreover, their analysis relied on the bridge functions' *uniqueness*, which may be dubious in practice and even refutable in many examples. It also required certain completeness assumptions. In this paper, we tackle these practical challenges to estimation with negative controls by relaxing such uniqueness and completeness assumptions and introducing new minimax estimators for the bridge functions that are amenable to general function approximation, in the spirit of agnostic machine learning. We catalog a variety of settings in which the functions are learnable and estimation with negative controls is practically feasible.

Our contributions are:

- We introduce a new identification method, which relaxes several previous assumptions including the uniqueness of bridge functions and certain completeness conditions. We also consider a more general setting than average-effect estimation, where we allow counterfactual actions to be possibly stochastic and the action space to be possibly continuous.
- We propose new estimators for bridge functions (even if nonunique) by introducing an adversarial critic function and formulating the learning problem as a minimax game. Our minimax approach accommodate the use of any type of flexible function class such as reproducing kernel Hilbert spaces (RKHS) and neural networks. Then, by plugging these bridge functions estimators into different estimating equations, we derive our final estimators.
- We provide finite sample convergences results under a variety of different assumptions (see Table 1). One important assumption is the well-specification of the hypothesis classes, which we call *realizability*. Another assumption, which we call *closedness*, ensures that the critic classes in our minimax estimators are sufficiently rich. Depending on how much of these we are willing to assume, we obtain different convergence rates for the final estimator. One surprising result is that when we assume realizability of both bridges and critics, our estimator is consistent even though the bridge functions themselves are not consistently estimated (see row I in Table 1).

2 Setup

We consider an action $A \in \mathcal{A}$ that can be discrete or continuous. We associate $\mathcal{A}$ with a base measure μ; *e.g.*, the counting measure if $\mathcal{A}$ is finite or Lebesgue measure if $\mathcal{A}$ is continuous. Let $Y(a)$ denote the real-valued counterfactual outcome that would be observed if the action were set to $a \in \mathcal{A}$ and $Y = Y(A)$ be the observed outcome corresponding to the actually observed action. Moreover, let $X \in \mathcal{X} \subseteq \mathbb{R}^d$ be a collection of observed covariates. For a given contrast function $\pi : \mathcal{A} \times \mathcal{X} \mapsto \mathbb{R}$, we are interested in estimating the *generalized average causal effect* (GACE):

$$J = \mathbb{E}\left[\int Y(a)\pi(a \mid X)\mathrm{d}\mu(a)\right]. \tag{1}$$

Example 1 (Average treatment effect)**.** Consider $\mathcal{A} = \{0, 1\}$. We are interested in the effect of "treatment" $A = 1$ compared to "control" $A = 0$. Letting $\pi(a \mid x) = 2a - 1$ we obtain $J = \mathbb{E}[Y(1) - Y(0)]$, known as the average treatment effect (ATE).

Example 2 (Target-population ATE)**.** In the same setting as Example 1 if $\kappa(x)$ is the Radon-Nikodym derivative of the distribution of X in some target population with respect to (wrt)

Table 1: Convergence rates of different GACE estimators under different assumptions (ignoring polylogs). The key identification assumptions, Assumptions 1 and 2, are always assumed. The sets $\mathbb{H}_0^{\text{obs}}, \mathbb{Q}_0^{\text{obs}}$ denote the sets of observed bridge functions (see Lemma 3). The function classes $\mathbb{H}, \mathbb{Q}'$ are used to construct the estimator $\hat{h}$ for the outcome bridge function $h_0 \in \mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}, \mathbb{H}'$ to construct the estimator $\hat{q}$ for the action bridge function $q_0 \in \mathbb{Q}_0^{\text{obs}}$. We here summarize our conclusions when all relevant function classes are Hölder balls of α-times differentiable functions over a d-dimensional domain. "Est." refers to which estimating equation is used for the final estimator (see Section 4.4). "Sta." refers to whether the bridge function estimators use stabilizers (see Section 4). "Uni." refers to whether we assume bridge functions are unique. The projection operators P_z and P_w are defined in Eq. (8), and $P_u : L_2(W, Z, A, X) \to L_2(U, A, X)$ is defined as $P_u(g(W, Z, A, X)) = \mathbb{E}[g(W, Z, A, X) \mid U, A, X]$.

	Main Assumptions	Est.	Rate wrt n	Sta.	Uni.	Note
I	$\mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q} \neq \emptyset, \mathbb{H}_0^{\text{obs}} \cap \mathbb{H}' \neq \emptyset$	IPW	$n^{-\min(\frac{1}{2}, \frac{\alpha}{d})}$	No	No	$\hat{h}, \hat{q}$ generally do not converge to any $h_0 \in \mathbb{H}_0^{\text{obs}}$, $q_0 \in \mathbb{Q}_0^{\text{obs}}$.
	$\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} \neq \emptyset, \pi\mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q}' \neq \emptyset$	REG				
	$\mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q} \neq \emptyset, \mathbb{H}_0^{\text{obs}} \cap \{h : h - \mathbb{H} \subseteq \mathbb{H}'\} \neq \emptyset$	DR				
	$\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} \neq \emptyset, \mathbb{Q}_0^{\text{obs}} \cap \{q : \pi(q - \mathbb{Q}) \subseteq \mathbb{Q}'\} \neq \emptyset$	DR				
II	$\mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q} \neq \emptyset, \pi P_w(\mathbb{Q} - \mathbb{Q}_0^{\text{obs}}) \subseteq \mathbb{H}'$	IPW or DR	$n^{-\frac{\alpha}{2\alpha+d}}$	Yes	No	$\lVert \pi P_w(\hat{q} - q_0) \rVert_2 \to 0$ for any $q_0 \in \mathbb{Q}_0^{\text{obs}}$.
			$n^{-\min(\frac{1}{4}, \frac{\alpha}{2d})}$	No		
III	$\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} \neq \emptyset, P_z(\mathbb{H} - \mathbb{H}_0^{\text{obs}}) \subseteq \mathbb{Q}'$	REG or DR	$n^{-\frac{\alpha}{2\alpha+d}}$	Yes	No	$\lVert P_z(\hat{h} - h_0) \rVert_2 \to 0$ for any $h_0 \in \mathbb{H}_0^{\text{obs}}$.
			$n^{-\min(\frac{1}{4}, \frac{\alpha}{2d})}$	No		
IV	Assumptions in rows II and III, $\lVert P_u(\hat{q}\pi - q_0\pi) \rVert_2 \leq \tau_1 \lVert P_w(\hat{q}\pi - q_0\pi) \rVert_2$, $\lVert P_u(\hat{h} - h_0) \rVert_2 \leq \tau_1 \lVert P_z(\hat{h} - h_0) \rVert_2$	DR	$\max(n^{-\frac{1}{2}}, \tau_1^2 n^{-\frac{2\alpha}{2\alpha+d}})$	Yes	No	Faster than row I if $\tau_1^2 = o(n^{\frac{\alpha(d-2\alpha)}{d(2\alpha+d)}})$.
	Assumptions in rows II and III, $\lVert \hat{q}\pi - q_0\pi \rVert_2 \leq \tau_2 \lVert P_w(\hat{q}\pi - q_0\pi) \rVert_2$, $\lVert \hat{h} - h_0 \rVert_2 \leq \tau_2 \lVert P_z(\hat{h} - h_0) \rVert_2$	DR	$\max(n^{-\frac{1}{2}}, \tau_2 n^{-\frac{2\alpha}{2\alpha+d}})$	Yes	Yes	Achieves efficiency if $\tau_2 = o(n^{\frac{2\alpha-d}{2(2\alpha+d)}})$.

the data distribution of X, then setting $\pi(a \mid x) = (2a-1)\kappa(x)$, J is the ATE on the target population.

Example 3 (Policy evaluation)**.** If $\pi(a \mid x)$ is a density on $\mathcal{A}$ for each x wrt μ, then J is the average outcome we experience when we follow the policy that assigns an action drawn from $\pi(\cdot \mid X)$ for an individual with covariates X [Tian, 2008, Dudik et al., 2014, Kennedy, 2019, Muñoz and Van Der Laan, 2012]. The measure μ is the Lebesgue measure when the action space is continuous, and is the counting measure when the action space is discrete. A special case of this is deterministic interventions where $\pi(\cdot \mid x)$ is Dirac at one action.[1]

We *do not* assume that the observed covariates X include all confounders that affect both the action and the potential outcomes, and instead there exist some *unmeasured* confounders $U \in \mathcal{U} \subseteq \mathbb{R}^{p_u}$ (discrete, continuous, or mixed):

$$Y(a) \not\perp A \mid X, \quad \text{but} \quad Y(a) \perp A \mid U, X.$$

If U were observed, we could identify the GACE J simply by controlling for both X, U. However, in this paper we assume that confounders U *cannot* be observed, in which case, the GACE J is generally *unidentifiable* from the distribution of the observed variables (Y, X, A) alone. To overcome the challenge of unmeasured confounding, in this paper we employ the

[1]When the action space is discrete, we can estimate both stochastic and deterministic policies. When the action space is continuous, *i.e.*, μ is Lebesgue, we only deal with stochastic polices in this paper since otherwise they do not have valid densities wrt μ.

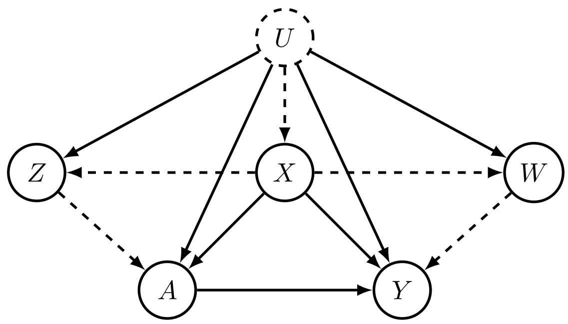

Figure 1: A typical causal diagram for negative controls. The dashed edges may be absent, and the dashed circle around U indicates that U is unobserved.

negative control framework proposed in Miao et al. [2018a], Cui et al. [2020], Miao et al. [2018b], Deaner [2021], Tchetgen et al. [2020]. This framework involves two additional types of observed variables: negative control actions $Z \in \mathcal{Z} \subseteq \mathbb{R}^{p_z}$ and negative control outcomes $W \in \mathcal{W} \subseteq \mathbb{R}^{p_w}$, which can be discrete, continuous, or mixed. These variables are called "negative" controls due to the assumed absence of certain causal effects: negative control actions cannot directly affect the outcome Y, and neither negative control actions Z nor the primary action A can affect the negative control outcomes W. Meanwhile, these variables are still relevant control variables as they are related to the unmeasured confounders. We can, in a sense, view them as proxies for the unmeasured confounders U.

Fig. 1 shows a typical causal diagram for this setting. To formalize our setting and allow for more generality, however, we will rely on potential outcome notation. Let $Y(a,z)$ and $W(a,z)$ denote the corresponding counterfactual outcomes one would observe had the primary action and negative control action taken value $(a,z) \in \mathcal{A} \times \mathcal{Z}$. We then formalize the negative control assumptions as follows.

Assumption 1 (Negative Controls). 1. Consistency: $Y = Y(A,Z), W = W(A,Z)$.

2. Negative control actions: $Y(a,z) = Y(a)$, $\forall a \in \mathcal{A}$.
3. Negative control outcomes: $W(a,z) = W$, $\forall a \in \mathcal{A}, z \in \mathcal{Z}$.
4. Latent unconfoundedness: $(Z,A) \perp (Y(a),W) \mid U, X, \quad \forall a \in \mathcal{A}$.
5. Overlap: $|\pi(a|x)/f(a|x,u)| < \infty, \quad \forall a \in \mathcal{A}, x \in \mathcal{X}, u \in \mathcal{U}$.

Here condition 1 encodes SUTVA and no interference across units [Imbens and Rubin, 2015]. Conditions 2 and 3 paraphrase the definition of negative controls in terms of the potential outcome notation: the negative control action Z cannot affect the primary outcome Y, and the negative control outcome W cannot be affected by either the primary action A or the negative control action Z. In particular, condition 2 ensures the potential outcome $Y(a)$ referred to in our estimand in Eq. (1) is well-defined. Condition 4 formalizes the assumption that the unmeasured variables U captures all common causes of (A,Z) and (Y,W) not included in X. Other causal diagrams other than Fig. 1 may also satisfy condition 4 (see table A.1 of Tchetgen et al., 2020). Condition 5 requires sufficient overlap between the contrast function π and the distribution of observed actions given both observed and unobserved confounders. This is a canonical assumption in causal inference and policy evaluation.

Our data consist of n independent and identically distributed (iid) observations of (Z,X,W,A,Y). Crucially, U is *unobserved*. And, our aim is to estimate the GACE J from these data.

Notation We let $\mathbb{E}$ denote expectations wrt (Z,X,W,A,Y), and $\mathbb{E}_n$ denote empirical average over the n observations thereof. For a function g of (z,x,w,a,y) (or a subset thereof) we often write g to mean the random variable $g(Z,X,W,A,Y)$. For O measurable with respect to (Z,X,W,A,Y), we let $L_2(O)$ denote the space of square-integrable functions of a variable O. Thus, *e.g.*, $L_2(W,A,X)$ and $L_2(Z,A,X)$ denote the space of square-integrable functions of

W, A, X and Z, A, X, respectively. For a function g, we let $\|g\|_2$ denote the norm in these spaces. For a vector θ, we let $\|\theta\|$ denote the Euclidean norm. We let $\|\cdot\|_\infty$ denote the sup norm of either a variable or function, and for a class of functions we let it denote the largest norm in the class. For subsets $\mathbb{A}, \mathbb{B}$ of a field, we define $\mathbb{A} + \mathbb{B} = \{a + b : a \in \mathbb{A}, b \in \mathbb{B}\}, \mathbb{A}\mathbb{B} = \{ab : a \in \mathbb{A}, b \in \mathbb{B}\}$. We call a subset S of a linear space symmetric if $-s \in S$ for any s in S, and we call S if star-shaped (around the origin) if $\alpha s \in S\ \forall s \in S, \alpha \in [0, 1]$. Finally, we often use $O(\cdot)$ notation to denote rates wrt n. When we use it in different ways, we explain the meaning.

3 Identifying GACE via Bridge Functions

In this section we discuss identification and define the bridge functions.

3.1 The Ideal Unconfounded Setting

If the unobserved confounders U *were* observed, then the GACE J could be identified, that is, it can be written as a function of the distribution of (Y, A, X, U). To illustrate this, define the regression function $k_0(a, u, x) = \mathbb{E}[Y \mid A = a, X = x, U = u]$, and define the generalized propensity score $f(a \mid u, x)$ as the conditional density of the distribution $A \mid X, U$ relative to the base measure μ [Hirano and Imbens, 2004]. Based on these two functions, the following lemma shows the identification of J if U *were* observed using three different formulations.

Lemma 1. *If $Y(a) \perp A \mid U, X$ and $|\pi(a|x)/f(a|x, u)| < \infty$ for any $a \in \mathcal{A}, x \in \mathcal{X}, u \in \mathcal{U}$, then*

$$J = \mathbb{E}[\phi_{\text{IPW}}(Y, A, U, X)] = \mathbb{E}[\phi_{\text{REG}}(Y, A, U, X)] = \mathbb{E}[\phi_{\text{DR}}(Y, A, U, X)],$$

where

$$\phi_{\text{IPW}}(y, a, u, x; f) = \frac{\pi(a|x)}{f(a|x, u)} y,$$

$$\phi_{\text{REG}}(y, a, u, x; k_0) = \int k_0(a', u, x)\pi(a'|x)\mathrm{d}\mu(a'),$$

$$\phi_{\text{DR}}(y, a, u, x; k_0, f) = \frac{\pi(a|x)}{f(a|x, u)}(y - k_0(a, u, x)) + \int k_0(a', u, x)\pi(a'|x)\mathrm{d}\mu(a').$$

Lemma 1 also suggests estimators for J if U were observed: we can first estimate the nuisance functions $k_0(a, u, x)$ and/or $f(a \mid u, x)$, and then estimate J by using any of the three estimating equations above with the estimated nuisance(s). The resulting three estimators are called the inverse propensity weighting (IPW) estimator, the regression-based (REG) estimator, and the doubly robust (DR) estimator, respectively [*e.g.*, Robins et al., 1994, Dudik et al., 2014].

3.2 The Negative-Control Setting

However, in this paper we deal with the setting where U is *unobserved*, so these estimators are infeasible. In particular, neither $k_0(a, u, x)$ nor $f(a \mid u, x)$ can be identified. Instead, we can use the negative controls Z, W to proxy these two functions via the *bridge functions* [Miao et al., 2018b, Cui et al., 2020].

Assumption 2 (Bridge functions)**.** There exist functions h_0 and q_0 s.t. $h_0 \in L_2(W, A, X)$ and $\pi q_0 \in L_2(Z, A, X)$ and the following holds almost surely:

$$\mathbb{E}[h_0(W, A, X) \mid A, U, X] = k_0(A, U, X), \tag{2}$$

$$\mathbb{E}[\pi(A \mid X)q_0(Z, A, X) \mid A, U, X] = \frac{\pi(A \mid X)}{f(A \mid U, X)}. \tag{3}$$

Assumption 2 implies that a function h_0 of the negative control outcomes W and another function q_0 of the negative control actions Z can play a similar role as the regression function k_0 and the generalized propensity score f, respectively (see Lemma 2 below). Here we call h_0 the *outcome bridge function* and q_0 the *action bridge function*. Note we only assume that such bridge functions exist, but they may be *nonunique*. In contrast, many previous negative control literature assume the uniqueness of bridge functions, either explicitly or as a consequence of other assumptions [*e.g.*, Miao et al., 2018b, Cui et al., 2020]. See more discussion in Section 7 and Appendix B.

The existence of bridge functions depends on the relationship between (Y, Z, W) and the unmeasured confounders U. Generally, such bridge functions exist only when the negative control proxies Z, W are *sufficiently informative* about the unmeasured confounders U.

Example 4 (Discrete setting)**.** Suppose the variables W, Z, A, U are all discrete variables with values w_i, z_j, u_s for $i = 1, \ldots, |\mathcal{W}|, j = 1, \ldots, |\mathcal{Z}|, s = 1, \ldots, |\mathcal{U}|$. Let $P(\mathbf{W} \mid \mathbf{U}, a, x)$ denote a $|\mathcal{W}| \times |\mathcal{U}|$ matrix whose (i, s)th element is $\mathbb{P}\left[W = w_i \mid U = u_s, A = a, X = x\right]$, $P(\mathbf{Z} \mid \mathbf{U}, a, x)$ a $|\mathcal{Z}| \times |\mathcal{U}|$ matrix whose (j, s)th element is $\mathbb{P}\left[Z = z_j \mid U = u_s, A = a, X = x\right]$, $\mathbb{E}\left[Y \mid \mathbf{U}, a, x\right]$ a $1 \times |\mathcal{U}|$ vector whose sth element is $\mathbb{E}\left[Y \mid U = u_s, A = a, X = x\right]$, $F(a \mid \mathbf{U}, x)$ a $|\mathcal{U}| \times |\mathcal{U}|$ diagonal matrix whose sth diagonal element is $f(a \mid u_s, x)$, and $\mathbf{e}$ a $|\mathcal{U}| \times 1$ vector of all ones.

With these notations, Eqs. (2) and (3) translates into the following linear equation system:

$$
\begin{aligned}
h_0^\top(\mathbf{W}, a, x) P(\mathbf{W} \mid \mathbf{U}, a, x) &= \mathbb{E}\left[Y \mid \mathbf{U}, a, x\right]. \\
q_0^\top(\mathbf{Z}, a, x) P(\mathbf{Z} \mid \mathbf{U}, a, x) F(a \mid \mathbf{U}, x) &= \mathbf{e}^\top,
\end{aligned}
$$

It is easy to show that if $P(\mathbf{W} \mid \mathbf{U}, a, x)$ and $P(\mathbf{Z} \mid \mathbf{U}, a, x)$ have full column rank (which implies that $|\mathcal{W}| \geq |\mathcal{U}|$ and $|\mathcal{Z}| \geq |\mathcal{U}|$) and $f(a \mid u, x) > 0$ for any $u \in \mathcal{U}$, then the linear equations system above have solutions, that is, the bridge functions exist. However, the solutions are generally *nonunique*, unless $|\mathcal{W}| = |\mathcal{Z}| = |\mathcal{U}|$ so that $P(\mathbf{W} \mid \mathbf{U}, a, x)$ and $P(\mathbf{Z} \mid \mathbf{U}, a, x)$ are invertible square matrices.

Example 5 (Linear Model)**.** Suppose (Y, W, Z, A) are generated from as follows:

$$
\begin{aligned}
Y &= \alpha_Y^\top U + \beta_Y^\top X + \gamma_Y^\top A + \omega_Y^\top W + \epsilon_Y, \\
Z &= \alpha_Z U + \beta_Z X + \gamma_Z A + \epsilon_Z, \\
W &= \alpha_W U + \beta_W X + \epsilon_W, \\
A &\sim \text{Unif}\left(\underline{\alpha}_A^\top U + \underline{\beta}_A^\top X, \overline{\alpha}_A^\top U + \overline{\beta}_A^\top X\right),
\end{aligned}
$$

where $\epsilon_Y, \epsilon_Z, \epsilon_W$ are independent mean-zero random noises that are also independent with (A, U, X).

Suppose that $\alpha_Z \in \mathbb{R}^{p_z \times p_u}, \alpha_W \in \mathbb{R}^{p_w \times p_u}$ both have full column rank. Then we can show that bridge functions h_0 and q_0 always exist:

$$
\begin{aligned}
h_0(W, A, X) &= (\theta_W + \omega_Y)^\top W + \left(\beta_Y - \theta_W^\top \beta_W\right) X + \gamma_Y A, \quad \forall \theta_W \text{ s.t. } \theta_W^\top \alpha_W = \alpha_Y^\top, \\
q_0(Z, A, X) &= \theta_Z^\top Z + \left(\overline{\beta}_A - \underline{\beta}_A - \theta_Z^\top \beta_Z\right) X - \theta_Z^\top \gamma_Z A, \quad \forall \theta_Z \text{ s.t. } \theta_Z^\top \alpha_Z = \overline{\alpha}_A^\top - \underline{\alpha}_A^\top.
\end{aligned}
$$

Obviously, the outcome bridge function h_0 is nonunique if $p_w > p_u$ and the action bridge function q_0 is nonunique if $p_z > p_u$.

Example 6. More generally, when we do not have parametric models for the data generating process as we do in Examples 4 and 5, we need to otherwise ensure the existence of solutions to conditional moment equations in Eqs. (2) and (3). These conditional moment equations define inverse problems known as Fredholm integral equations of the first kind. The existence of their solutions can be rigorously characterized by Picard's theorem [Kress, 2014, Carrasco et al., 2007]. Following Miao et al. [2018a], we show in Appendix B.1 that under some additional regularity conditions on the singular value decomposition of the linear operators associated with the Fredholm integral equations, the existence of solutions to Eqs. (2) and (3) can be ensured by the following completeness conditions: for any $g(U, A, X) \in L_2(U, A, X)$,

$$
\begin{aligned}
\mathbb{E}\left[g(U, A, X) \mid Z, A, X\right] = 0 \text{ only when } g(U, A, X) = 0, \\
\mathbb{E}\left[g(U, A, X) \mid W, A, X\right] = 0 \text{ only when } g(U, A, X) = 0.
\end{aligned}
$$

These completeness conditions mean that the negative controls Z, W have sufficient variability relative to the variability of the unobserved confounders U. In this paper we only explicitly rely on the minimal assumption of the existence of bridge functions, rather than such stronger completeness conditions and additional regularity conditions that might imply their existence.

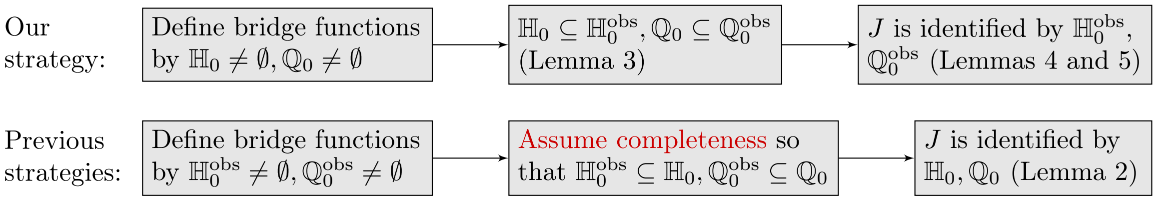

Figure 2: Different identification strategies. Without any assumptions, what we can identify from the data is $\mathbb{H}_0^{\text{obs}}, \mathbb{Q}_0^{\text{obs}}$ but not $\mathbb{H}_0, \mathbb{Q}_0$. See relevant discussions below Lemma 5.

Examples 4 and 5 illustrate that bridge functions, besides existing, are nonunique if the negative control proxies carry more information than the unmeasured confounders, namely, when the negative controls have more values than the unmeasured confounders in Example 4, or when the dimension of negative controls exceeds the dimension of the unmeasured confounders in Example 5. Since unmeasured confounders are unobserved in practice, we may tend to use as many negative control variables as possible to safeguard the existence of bridge functions. But this may also cause bridge functions to be nonunique. Therefore, the assumption of unique bridge functions imposed in previous literature [*e.g.*, Miao et al., 2018b, Cui et al., 2020] may be too strong. In contrast, our paper allows for nonunique bridge functions, and assumes uniqueness only when it is needed to derive stronger theoretical guarantee.

Assumption 2 can be equivalently stated as saying that the following partial identification sets of bridge functions are nonempty sets:

$$\begin{aligned}
&\mathbb{H}_0 = \{h \in L_2(W, A, X) : \mathbb{E}[Y - h(W, A, X) \mid A, U, X] = 0\} \neq \emptyset, \\
&\mathbb{Q}_0 = \{q : \pi q \in L_2(Z, A, X)\mathbb{E}[\pi(A \mid X)\left(q(Z, A, X) - 1/f(A \mid U, X)\right) \mid A, U, X] = 0\} \neq \emptyset.
\end{aligned} \tag{4}$$

Again, bridge functions may be nonunique so these two partial identification sets may each contain more than one element. The lemma below shows that *any* valid bridge function in these partial identification sets can identify J.

Lemma 2. *Let $O = (Y, W, Z, A, X)$ be the observed variables and $\mathcal{T} : L_2(W, A, X) \to L_2(W, X)$ be the linear operator defined by $(\mathcal{T}h)(w, x) = \int h(w, a, x)\pi(a|x)\mathrm{d}\mu(a)$. Under Assumptions 1 and 2, for any $h_0 \in \mathbb{H}_0$ and $q_0 \in \mathbb{Q}_0$,*

$$\begin{aligned}
J &= \mathbb{E}\left[\tilde{\phi}_{\text{IPW}}(O; q_0)\right] = \mathbb{E}\left[\tilde{\phi}_{\text{REG}}(O; h_0)\right] = \mathbb{E}\left[\tilde{\phi}_{\text{DR}}(O; h_0, q_0)\right], \\
\textit{where}\quad &\tilde{\phi}_{\text{IPW}}(O; q_0) = \pi(A \mid X)q_0(Z, A, X)Y, \\
&\tilde{\phi}_{\text{REG}}(O; h_0) = (\mathcal{T}h_0)(X, W), \\
&\tilde{\phi}_{\text{DR}}(O; h_0, q_0) = \pi(A \mid X)q_0(Z, A, X)\left(Y - h_0(W, A, X)\right) + (\mathcal{T}h_0)(X, W).
\end{aligned}$$

Note that the estimating equations in Lemma 2 simply replace the regression function $k_0(A, X, U)$ and the inverse propensity score weight $1/f(A \mid X, U)$ in Lemma 1 by the bridge functions $h_0(W, A, X)$ and $q_0(Z, A, X)$. Since the latter only depends on observed variables, as long as we can learn *any* pair of bridge functions, we can use the estimating equations in Lemma 2 to estimate J.

3.3 Learning Bridge Functions from Observed Data

Assumption 2 defines the bridge functions in terms of conditional moment equations[2] given the unobserved confounders U, so we cannot use it directly to learn the bridge functions from the observed data. Nevertheless, the following lemma shows that the bridge functions also satisfy analogous conditional moment equations based on only observed data.

[2]Equations (3) and (6) do not exactly fall into the usual conditional moment equation framework [*e.g.*, Ai and Chen, 2003], since they involve unknown density functions $f(A \mid U, X)$, $f(A \mid W, X)$, respectively. We still call it a conditional moment equation for simplicity, but estimating q_0 does require more care. See discussion below Lemma 6.

Lemma 3. *Under Assumptions 1 and 2, any $h_0 \in \mathbb{H}_0$ and $q_0 \in \mathbb{Q}_0$ satisfy that*

$$\mathbb{E}[Y - h_0(W, A, X) \mid Z, A, X] = 0, \tag{5}$$

$$\mathbb{E}[\pi(A \mid X)\,(q_0(Z, A, X) - 1/f(A \mid W, X)) \mid W, A, X] = 0. \tag{6}$$

The conditional moment equations in Eqs. (5) and (6) give rise to the following alternative sets of functions that we can possibly learn from observed data, whose elements we call *observed* bridge functions:

$$\mathbb{H}_0^{\text{obs}} = \{h \in L_2(W, A, X) : \mathbb{E}[Y - h(W, A, X) \mid Z, A, X] = 0\},$$

$$\mathbb{Q}_0^{\text{obs}} = \{q : \pi q \in L_2(Z, A, X), \mathbb{E}[\pi(A \mid X)\,(q(Z, A, X) - 1/f(A \mid W, X)) \mid W, A, X] = 0\}.$$

According to Lemma 3, $\mathbb{H}_0 \subseteq \mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0 \subseteq \mathbb{Q}_0^{\text{obs}}$. So, under Assumption 2, $\mathbb{H}^{\text{obs}} \neq \emptyset, \mathbb{Q}^{\text{obs}} \neq \emptyset$. The following lemma shows that, as long as some nominal bridge functions exist, *i.e.*, $\mathbb{H}_0 \neq \emptyset, \mathbb{Q}_0 \neq \emptyset$, then *any* pair of bridge functions in $\mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0^{\text{obs}}$ can *also* identify J even if they are not in $\mathbb{H}_0, \mathbb{Q}_0$.

Lemma 4. *Under Assumptions 1 and 2, for any $h_0 \in \mathbb{H}_0^{\text{obs}}$ and $q_0 \in \mathbb{Q}_0^{\text{obs}}$,*

$$J = \mathbb{E}\left[\tilde{\phi}_{\text{IPW}}(O; q_0)\right] = \mathbb{E}\left[\tilde{\phi}_{\text{REG}}(O; h_0)\right] = \mathbb{E}\left[\tilde{\phi}_{\text{DR}}(O; h_0, q_0)\right].$$

Lemma 4 suggests a straightforward way to estimate J: first estimate $h_0 \in \mathbb{H}_0^{\text{obs}}$ and $q_0 \in \mathbb{H}_0^{\text{obs}}$, and then estimate the GACE J by using any of the estimating equations above.

Interestingly, Lemma 4 holds for any bridge functions $h_0 \in \mathbb{H}_0^{\text{obs}}$ and $q_0 \in \mathbb{Q}_0^{\text{obs}}$, even if they violate the conditional moment equations in Eqs. (2) and (3), *i.e.*, it holds for $h_0 \in \mathbb{H}_0^{\text{obs}} \backslash \mathbb{H}_0$ and $q_0 \in \mathbb{Q}^{\text{obs}} \backslash \mathbb{Q}_0$. In other words, under Assumptions 1 and 2, even when the sets of bridge functions are unidentifiable (*i.e.*, $\mathbb{H}_0 \subsetneq \mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0 \subsetneq \mathbb{Q}_0^{\text{obs}}$), the GACE J can still be identifiable. This is possible due to the existence of *both* outcome bridge function $h_0 \in \mathbb{H}_0$ and action bridge function $q_0 \in \mathbb{Q}_0$ (*i.e.*, Assumption 2).[3] With the existence of both bridge functions, Lemma 4 follows as a consequence of the lemma below, which states that whether the REG and IPW estimating equations can identify J only depends on whether the bridge functions satisfy the observed-data conditional moment equations in Eqs. (5) and (6), instead of the unobserved counterparts in Eqs. (2) and (3).

Lemma 5. *Suppose that Assumptions 1 and 2 hold and fix any $h_0 \in \mathbb{H}_0^{\text{obs}}$ and $q_0 \in \mathbb{Q}_0^{\text{obs}}$. Then for any $h \in L_2(W, A, X)$ and $\pi q \in L_2(Z, A, X)$,*

$$\mathbb{E}\left[\tilde{\phi}_{\text{REG}}(O; h)\right] - J = \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}\left[h(W, A, X) - Y \mid Z, A, X\right]\right],$$

$$\mathbb{E}\left[\tilde{\phi}_{\text{IPW}}(O; q)\right] - J = \mathbb{E}\left[h_0(W, A, X) \mathbb{E}\left[\pi(A \mid X)\,(q(Z, A, X) - 1/f(A \mid W, X)) \mid W, A, X\right]\right].$$

Our proof of the identification of J via Lemma 5 is completely different from the identification results in previous literature [Cui et al., 2020, Deaner, 2021]. These previous literature start by assuming $\mathbb{H}_0^{\text{obs}} \neq \emptyset, \mathbb{Q}_0^{\text{obs}} \neq \emptyset$, and impose extra *completeness conditions* to ensure that any $h_0 \in \mathbb{H}_0^{\text{obs}}, q_0 \in \mathbb{Q}_0^{\text{obs}}$ must also satisfy that $h_0 \in \mathbb{H}_0, q_0 \in \mathbb{Q}_0$. Then they use $h_0 \in \mathbb{H}_0, q_0 \in \mathbb{Q}_0$ to identify causal estimands based on Lemma 2.[4] In contrast, we start with $\mathbb{H}_0 \neq \emptyset, \mathbb{Q}_0 \neq \emptyset$, which implies $\mathbb{H}_0^{\text{obs}} \neq \emptyset, \mathbb{Q}_0^{\text{obs}} \neq \emptyset$, and then show that we can directly use $h_0 \in \mathbb{H}_0^{\text{obs}}, q_0 \in \mathbb{Q}_0^{\text{obs}}$ to identify causal estimands based on Lemmas 4 and 5. In this way, we can achieve identification without assuming any completeness conditions. See Fig. 2 for an illustration for the difference in identification strategies and see Appendix A for more discussion. In Lemma 11 in Section 6.2, we will also show that Lemma 5 plays an important role in analyzing the convergence rates of our GACE estimators.

[3] We can actually weaken Assumption 2 in Lemma 4 further: we only need $\mathbb{H}_0 \neq \emptyset, \mathbb{Q}_0^{\text{obs}} \neq \emptyset$ for REG and DR estimating equations to identify J, and $\mathbb{Q}_0 \neq \emptyset, \mathbb{H}_0^{\text{obs}} \neq \emptyset$ for IPW and DR estimating equations to identify J. In contrast, if we followed the identification strategy in previous literature, then we would need $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ with an extra completeness condition for REG and DR estimating equations, and $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$ with another completeness condition for IPW and DR estimating equations. See discussion below Lemma 5 and in Appendix B.

[4] Miao et al. [2018b,a] only focus on the outcome bridge function, but their identification strategy is similar to these: they assume $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ along with an extra completeness condition in order to ensure that the REG estimating equation can identify the ATE.

4 Minimax Estimators of Bridge functions

Estimating bridge functions based on Eqs. (5) and (6) requires solving conditional moment equations, which is generally a difficult estimation problem. Estimation methods in the previous literature only focus on parametric methods [Cui et al., 2020] or sieve methods [Deaner, 2021]. In this paper, we propose to use minimax approaches to estimate the bridge functions, which enable the use of more flexible hypothesis classes such as neural networks and RKHS.

In this section, we introduce two minimax reformulations of the conditional moment equations in Eqs. (5) and (6). Each reformulation motivates an estimation strategy for bridge functions. Analogous reformulations have been also studied (separately) by previous literature in the context of estimation using instrument variables (IV). See Section 7 for the literature review.

4.1 Strategy I: Minimax Estimators without Stablizers

As motivation, consider a generic conditional estimating equation problem, $\mathbb{E}\left[\rho\left(g_0(O_1), O_1\right) \mid O_2\right] = 0$, where we wish to solve for the function g_0 and O_1, O_2 are two sets of random variables. Note that

$$
\begin{aligned}
\mathbb{E}\left[\rho\left(g_0(O_1), O_1\right) \mid O_2\right] = 0 &\iff \mathbb{E}\left[g'(O_2)\rho\left(g_0(O_1), O_1\right)\right] = 0, \quad \forall g' \in L_2(O_2), \\
&\iff \sup_{g' \in L_2(O_2)} \left(\mathbb{E}\left[g'(O_2)\rho\left(g_0(O_1), O_1\right)\right]\right)^2 = 0. \qquad (7)
\end{aligned}
$$

We next apply this observation to our Eqs. (5) and (6) in the following lemma. First, we define the linear operators $P_z : L_2(W, A, X) \to L_2(Z, A, X)$ and $P_w : L_2(Z, A, X) \to L_2(W, A, X)$ as follows:

$$
P_z(h) = \mathbb{E}\left[h(W, A, X) \mid Z, A, X\right], \quad P_w(q) = \mathbb{E}\left[q(Z, A, X) \mid W, A, X\right]. \qquad (8)
$$

Lemma 6. *Suppose Assumptions 1 and 2 hold, that $\mathbb{H} \subseteq L_2(W, A, X)$, $\pi\mathbb{Q} \subseteq L_2(Z, A, X)$ satisfy realizability in that $\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} \neq \emptyset$, $\mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q} \neq \emptyset$, and that $\mathbb{Q}' \subseteq L_2(Z, A, X)$, $\mathbb{H}' \subseteq L_2(W, A, X)$ satisfy closedness in that $P_z\left(\mathbb{H} - \mathbb{H}_0^{\text{obs}}\right) \subseteq \mathbb{Q}'$, $\pi P_w\left(\mathbb{Q} - \mathbb{Q}_0^{\text{obs}}\right) \subseteq \mathbb{H}'$. Then*

$$
\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} = \arg\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \left(\mathbb{E}\left[q(Z, A, X)\left(h(W, A, X) - Y\right)\right]\right)^2, \qquad (9)
$$

$$
\mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q} = \arg\min_{q \in \mathbb{Q}} \sup_{h \in \mathbb{H}'} \left(\mathbb{E}\left[\pi(A|X)q(X, A, Z)h(W, A, X) - (\mathcal{T}h)(W, X)\right]\right)^2. \qquad (10)
$$

This lemma motivates the following estimators for bridge functions

$$
\hat{h} \in \arg\min_{h \in \mathbb{H}} \max_{q \in \mathbb{Q}'} \left(\mathbb{E}_n[q(Z, A, X)\left(h(W, A, X) - Y\right)]\right)^2, \qquad (11)
$$

$$
\hat{q} \in \arg\min_{q \in \mathbb{Q}} \max_{h \in \mathbb{H}'} \left(\mathbb{E}_n[\pi(A|X)q(X, A, Z)h(W, A, X) - (\mathcal{T}h)(X, W)]\right)^2, \qquad (12)
$$

where $\mathbb{E}_n$ represents sample average based on observed data.

Our estimators $\hat{h}, \hat{q}$ can be viewed as solution to minimax games, where an adversarial player picks elements from function classes $\mathbb{Q}', \mathbb{H}'$ to form the most difficult marginal moments while our estimators minimize the violations of such marginal moments. We call $\mathbb{Q}', \mathbb{H}'$ the *critic classes* and elements therein *critic functions*, while we call $\mathbb{Q}, \mathbb{H}$ the *hypothesis classes*.

Note that although Eq. (6) involves the generalized propensity score $f(A \mid W, X)$, it does not appear in Eq. (12) at all. In this sense, our estimation method for $\hat{q}$ is different from a naïve application of Eq. (7) to Eq. (6), wherein we would first get a preliminary generalized propensity score estimator $\hat{f}(A \mid W, X)$ and then solve $\arg\min_{q \in \mathbb{Q}} \max_{h \in \mathbb{H}'} (\mathbb{E}_n[h(W, A, X)\pi(A|X)\{q_0 - 1/\hat{f}(A|W, X)\}])^2$. Instead, our estimator in Eq. (12) exploits the fact that $\mathbb{E}\left[h(W, A, X)\pi\left(A \mid X\right)/f(A \mid W, X)\right] = \mathbb{E}\left[(\mathcal{T}h)\left(X, W\right)\right]$. Thus it obviates the need to estimate the generalized propensity score before estimating the bridge functions, which is apparently more appealing. This fact also characterizes the difference in the estimation of h_0 and q_0. The estimation of h_0 is analogous to the nonparametric IV regression problem [Newey and Powell, 2003, Darolles et al., 2010], and the estimator Eq. (11) is analogous to minimax approaches thereto. In contrast, the estimation of q_0 requires additional considerations.

Example 1, Cont'd (Average treatment effect). Consider binary action $A \in \{0,1\}$ in Example 1. In this case, the conditional moment equation for the action bridge function q_0 is equivalent to

$$\mathbb{E}\left[\mathbb{I}\left[A=a\right] q_0(Z,a,X) - 1 \mid W, X\right] = 0, \quad a \in \{0,1\}.$$

Apparently, this equation does not explicitly depend on the propensity score either. Note that when $Z = W = \emptyset$, the action bridge function given by this conditional moment equation is exactly the inverse propensity score weight.

4.2 Strategy II: Minimax Estimators with Stablizers

Again considering the generic estimating equation, note that for any constant $\lambda > 0$,

$$\begin{aligned}&\mathbb{E}\left[\rho\left(g_0(O_1), O_1\right) \mid O_2\right] = 0 \iff \\ &0 = \frac{1}{4\lambda}\mathbb{E}\left[\left(\mathbb{E}\left[\rho\left(g_0(O_1), O_1\right) \mid O_2\right]\right)^2\right] = \sup_{g' \in L_2(O_2)} \mathbb{E}\left[g'(O_2)\rho\left(g_0(O_1), O_1\right)\right] - \lambda\|g'\|_2^2. \end{aligned} \tag{13}$$

We apply this observation to our Eqs. (5) and (6) in the following lemma.

Lemma 7. *Fix a constant $\lambda > 0$. Suppose Assumptions 1 and 2 hold, that $\mathbb{H} \subseteq L_2(W,A,X)$, $\pi\mathbb{Q} \subseteq L_2(Z,A,X)$ satisfy realizability in that $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H} \neq \emptyset$, $\mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q} \neq \emptyset$, and that $\mathbb{Q}' \subseteq L_2(Z,A,X)$, $\mathbb{H}' \subseteq L_2(W,A,X)$ are star-shaped[5] and satisfy closedness in that $P_z\left(\mathbb{H} - \mathbb{H}_0^{\mathrm{obs}}\right) \subseteq \mathbb{Q}'$, $\pi P_w\left(\mathbb{Q} - \mathbb{Q}_0^{\mathrm{obs}}\right) \subseteq \mathbb{H}'$.[6] Then,*

$$\begin{aligned}\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H} &= \arg\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X) - Y\right)\right] - \lambda\|q\|_2^2 \\ \mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q} &= \arg\min_{q \in \mathbb{Q}} \sup_{h \in \mathbb{H}'} \quad \mathbb{E}\left[\pi(A \mid X) q(X,A,Z) h(X,A,W) - (\mathcal{T}h)(X,W)\right] - \lambda\|h\|_2^2.\end{aligned}$$

This lemma motivates the following estimators for bridge functions

$$\hat{h} = \arg\min_{h \in \mathbb{H}} \max_{q \in \mathbb{Q}'} \mathbb{E}_n[\{h(X,A,W) - Y\}q] - \lambda\mathbb{E}_n[q^2], \tag{14}$$

$$\hat{q} = \arg\min_{q \in \mathbb{Q}} \max_{h \in \mathbb{H}'} \mathbb{E}_n[q(X,A,Z)\pi(A|X)h(X,A,W) - \mathcal{T}h(X,W)] - \lambda\mathbb{E}_n[h^2] \tag{15}$$

We call the terms $\lambda\mathbb{E}_n[q^2], \lambda\mathbb{E}_n[h^2]$ *stabilizers.* When critic classes $\mathbb{Q}'$ and $\mathbb{H}'$ are symmetric and λ is set to 0, the objective functions in Eqs. (14) and (15) are equivalent to their counterparts in Eqs. (11) and (12). Note that the stabilizers are quite different from regularizers. Regularizers typically introduce estimation bias, usually to deal with ill-posed inverse problems [*e.g.*, Knight and Fu, 2000, Carrasco et al., 2007]. Stabilizers on the other hand do not introduce bias and are merely a way to reformulate the conditional moment equation.

4.3 Examples of Bridge Function Estimators

In this part, we give examples of bridge function estimators based on three different critic function classes: linear class, RKHS, and neural networks. In particular, for the linear class and RKHS, the inner maximization problems in Eqs. (11), (12), (14) and (15) have a closed-form solution, so that the minimax problems can solved by standard optimization techniques such as stochastic gradient descent. In the following, we often omit the arguments of functions for brevity, but their meaning should be self-evident from the context.

[5] We can drop this assumption if $\lambda = \frac{1}{2}$.

[6] The assumptions that $\mathbb{H}'$ and $\mathbb{Q}'$ are star-shaped, $P_z\left(\mathbb{H} - \mathbb{H}_0^{\mathrm{obs}}\right) \subseteq \mathbb{Q}'$, and $\pi P_w\left(\mathbb{Q} - \mathbb{Q}_0^{\mathrm{obs}}\right) \subseteq \mathbb{H}'$ can also be replaced by assuming instead that $CP_z\left(\mathbb{H} - \mathbb{H}_0^{\mathrm{obs}}\right) \subseteq \mathbb{Q}'$, $C\pi P_w\left(\mathbb{Q} - \mathbb{Q}_0^{\mathrm{obs}}\right) \subseteq \mathbb{H}'$ for some $C > 0$.

4.3.1 Linear classes

Given basis functions $\phi(x,a,z), \psi(x,a,w)$ of dimensions d_1, d_2, respectively, consider

$$\mathbb{Q}' = \left\{(z,a,x) \mapsto \alpha_1^\top \phi(z,a,x) : \alpha_1 \in \mathbb{R}^{d_1}, \|\alpha_1\| \le c_1\right\}, \tag{16}$$

$$\mathbb{H}' = \left\{(w,a,x) \mapsto \alpha_2^\top \psi(w,a,x) : \alpha_2 \in \mathbb{R}^{d_2}, \|\alpha_2\| \le c_2\right\}. \tag{17}$$

Typical examples of basis functions include splines, polynomials, and wavelets [Chen, 2007]. When d_1, d_2 grow with n, these function classes are also called sieves. See also discussions in Example 10.

Minimax Estimators without Stablizers. It is easy to show that with linear critic classes, the inner maximum objectives in Eqs. (11) and (12) have closed-form expressions, and the resulting bridge function estimators are given as follows:

$$\hat{h} = \arg\min_{h\in\mathbb{H}} \ \left(\mathbb{E}_n[\{Y-h\}\phi]\right)^\top \left(\mathbb{E}_n[\{Y-h\}\phi]\right), \tag{18}$$

$$\hat{q} = \arg\min_{q\in\mathbb{Q}} \ \left(\mathbb{E}_n[q\pi\psi - \mathcal{T}\psi]\right)^\top \left(\mathbb{E}_n[\{q\pi\psi - \mathcal{T}\psi\}]\right). \tag{19}$$

If further $\mathbb{Q}$ and $\mathbb{H}$ are also linear classes (without norm constraints for simplicity), namely,

$$\pi\mathbb{Q} = \left\{(z,a,x) \mapsto \alpha_1^\top \tilde{\phi}(z,a,x) : \alpha_1 \in \mathbb{R}^{\tilde{d}_1}\right\}, \tag{20}$$

$$\mathbb{H} = \left\{(w,a,x) \mapsto \alpha_2^\top \tilde{\psi}(w,a,x) : \alpha_2 \in \mathbb{R}^{\tilde{d}_2}\right\}, \tag{21}$$

then we have

$$\hat{h} = \tilde{\psi}^\top \hat{\alpha}_2,\ \hat{\alpha}_2 = \{\mathbb{E}_n[\tilde{\psi}\phi^\top]\mathbb{E}_n[\phi\tilde{\psi}^\top]\}^+ \mathbb{E}_n[\tilde{\psi}\phi^\top]\mathbb{E}_n[Y\phi],$$

$$\hat{q} = \tilde{\phi}^\top \hat{\alpha}_1,\ \hat{\alpha}_1 = \{\mathbb{E}_n[\tilde{\phi}\psi^\top]\mathbb{E}_n[\psi\tilde{\phi}^\top]\}^+ \mathbb{E}_n[\tilde{\phi}\psi^\top]\mathbb{E}_n[\mathcal{T}\psi],$$

where A^+ is the Moore–Penrose inverse of a matrix A.

It may sometimes be useful to constrain the norms of coefficients in $\mathbb{Q}$ and $\mathbb{H}$, especially when using linear sieves with growing basis functions [Newey and Powell, 2003, Newey, 2013]. It is often more convenient to formulate this as Tikhonov regularization and simply penalize $\|\alpha_1\|^2$ and $\|\alpha_2\|^2$ in Eqs. (18) and (19). The solution remains closed form.

Minimax Estimators with Stablizers. The inner maximum objectives in Eqs. (14) and (15) may no longer admit closed-form solutions when we constrain the norms of α_1, α_2 in $\mathbb{H}', \mathbb{Q}'$. To circumvent this issue, we consider setting $c_1 = c_2 = \infty$ in the critic classes $\mathbb{H}', \mathbb{Q}'$ and regularizing coefficient norms in the inner maximization objectives.

Lemma 8. *Fix $c_1 = c_2 = +\infty$ in the critic function classes $\mathbb{Q}', \mathbb{H}'$ in Eqs.* (16) *and* (17)*. Consider the following estimators adapted from Eqs.* (14) *and* (15)*:*

$$\hat{h} = \arg\min_{h\in\mathbb{H}} \max_{q\in\mathbb{Q}'} \mathbb{E}_n[\{h(X,A,W) - Y\}q] - \lambda\mathbb{E}_n[q^2] - \gamma_1\|\alpha_1\|^2,$$

$$\hat{q} = \arg\min_{q\in\mathbb{Q}} \max_{h\in\mathbb{H}'} \mathbb{E}_n[q(X,A,Z)\pi(A|X)h(X,A,W) - \mathcal{T}h(X,W)] - \lambda\mathbb{E}_n[h^2] - \gamma_2\|\alpha_2\|^2$$

Then we have

$$\hat{h} = \arg\min_{h\in\mathbb{H}} \mathbb{E}_n[\{Y-h\}\phi]^\top \{\gamma_1 I + \lambda\mathbb{E}_n[\phi\phi]\}^{-1}\mathbb{E}_n[\{Y-h\}\phi],$$

$$\hat{q} = \arg\min_{q\in\mathbb{Q}} \mathbb{E}_n[q\pi\psi - \mathcal{T}\psi]^\top \{\gamma_2 I + \lambda\mathbb{E}_n[\psi\psi]\}^{-1}\mathbb{E}_n[\{q\pi\psi - \mathcal{T}\psi\}].$$

If further $\mathbb{Q}$ and $\mathbb{H}$ are linear classes as in Eqs. (20) and (21), then we have

$$\hat{h} = \tilde{\psi}^\top \hat{\alpha}_2,\ \hat{\alpha}_2 = \{\mathbb{E}_n[\tilde{\psi}\phi^\top]\{\gamma_1 I + \lambda\mathbb{E}_n[\phi\phi^\top]\}^{-1}\mathbb{E}_n[\phi\tilde{\psi}^\top]\}^+\mathbb{E}_n[\tilde{\psi}\phi^\top]\{\gamma_1 I + \lambda\mathbb{E}_n[\phi\phi^\top]\}^{-1}\mathbb{E}_n[Y\phi],$$

$$\hat{g} = \tilde{\phi}^\top \hat{\alpha}_1,\ \hat{\alpha}_1 = \{\mathbb{E}_n[\tilde{\phi}\psi^\top]\{\gamma_2 I + \lambda\mathbb{E}_n[\psi\psi^\top]\}^{-1}\mathbb{E}_n[\psi\tilde{\phi}^\top]\}^+\mathbb{E}_n[\tilde{\phi}\psi^\top]\{\gamma_2 I + \lambda\mathbb{E}_n[\psi\psi^\top]\}^{-1}\mathbb{E}_n[\mathcal{T}\psi].$$

Again, we can also add Tikhonov regularization on the coefficients of $\mathbb{H}, \mathbb{Q}$.

4.3.2 RKHS

When we use RKHS as the critic classes, the inner maximization problems in the two estimation strategies can also be computed in closed-form. Let us first introduce two kernel functions $k_z : (\mathcal{Z}, \mathcal{A}, \mathcal{X}) \times (\mathcal{Z}, \mathcal{A}, \mathcal{X}) \to \mathbb{R}$ and $k_w : (\mathcal{W}, \mathcal{A}, \mathcal{X}) \times (\mathcal{W}, \mathcal{A}, \mathcal{X}) \to \mathbb{R}$, and denote the induced RKHS as $\mathcal{L}_z$ and $\mathcal{L}_w$ with RKHS norms $\|\cdot\|_{\mathcal{L}_z}$ and $\|\cdot\|_{\mathcal{L}_w}$, respectively.

We consider the following critic function classes:

$$\mathbb{Q}' = \{q : q \in \mathcal{L}_z, \|q\|_{\mathcal{L}_z} \leq c_1\}, \quad \mathbb{H}' = \{h : h \in \mathcal{L}_w, \|h\|_{\mathcal{L}_w} \leq c_2\}. \tag{22}$$

Minimax Estimators without Stablizers. With RKHS critic function classes, we simplify the minimax estimation problems in Eqs. (11) and (12) in the following lemma.

Lemma 9. *For $\mathbb{Q}', \mathbb{H}'$ given in Eq. (22), Eqs. (11) and (12) are equivalent to*

$$\hat{h} = \operatorname*{arg\,min}_{h \in \mathbb{H}} \ (\psi_n(h))^\top K_{z,n} \psi_n(h), \tag{23}$$

$$\hat{q} = \operatorname*{arg\,min}_{q \in \mathbb{Q}} \ (\phi_n(q))^\top K_{w1,n} \phi_n(q) - 2(\phi_n(q))^\top K_{w2,n} \mathbf{1}_n, \tag{24}$$

where $K_{z,n}$, $K_{w1,n}$, $K_{w2,n}$ are $n \times n$ Gram matrices whose (i,j)th entry is $k_z((Z_i, A_i, X_i), (Z_j, A_j, X_j))$, $k_w((W_i, A_i, X_i), (W_j, A_j, X_j)$, $\mathbb{E}_{\pi(A_j|X_j)}[k_w((W_i, A_i, X_i), (W_j, A_j, X_j))]$, respectively, and $\psi_n(h)$, $\phi_n(q)$ are $n \times 1$ vectors whose ith elements are $(Y_i - h(X_i, A_i, Z_i))$, $q(X_i, A_i, Z_i)\pi(A_i|X_i)$, respectively, and $\mathbf{1}_n \in \mathbb{R}^n$ is the all-ones vector.

Minimax Estimators with Stablizers. Similar to the linear classes, the inner maximization problems in Eqs. (14) and (15) may no longer have closed-form solutions with the RKHS norm constraints in Eq. (22). To circumvent this issue, we can again set $c_1 = c_2 = \infty$ in Eq. (22), or equivalently set $\mathbb{Q}' = \mathcal{L}_z, \mathbb{H}' = \mathcal{L}_w$, and instead regularize the inner maximization objectives.

Lemma 10. *Consider the following estimators adapted from Eqs. (14) and (15):*

$$\hat{h} = \operatorname*{arg\,min}_{h \in \mathbb{H}} \max_{q \in \mathcal{L}_w,} \mathbb{E}_n[\{h(X, A, W) - Y\}q] - \lambda \mathbb{E}_n[q^2] - \gamma_1 \|q\|_{\mathcal{L}_w}, \tag{25}$$

$$\hat{q} = \operatorname*{arg\,min}_{q \in \mathbb{Q}} \max_{h \in \mathcal{L}_z} \mathbb{E}_n[q(X, A, Z)\pi(A|X)h(X, A, W) - \mathcal{T}h(X, W)] - \lambda \mathbb{E}_n[h^2] - \gamma_2 \|h\|_{\mathcal{L}_w}. \tag{26}$$

Then we have

$$\hat{h} = \operatorname*{arg\,min}_{h \in \mathbb{H}} \psi_n(h)^\top K_{z,n}^{1/2} \{\gamma_1 I + \lambda K_{z,n}\}^{-1} K_{z,n}^{1/2} \psi_n(h), \tag{27}$$

$$\hat{g} = \operatorname*{arg\,min}_{q \in \mathbb{Q}} \phi_n(q)^\top K_{w1,n}^{1/2} (\lambda K_{w1,n} + \gamma_2 I)^{-1} K_{w1,n}^{1/2} \phi_n(q) - 2\{\phi_n(q)^\top (\lambda K_{w1,n} + \gamma_2 I)^{-1} K_{w2,n} \mathbf{1}_n\}. \tag{28}$$

In all of Eqs. (23), (24), (27) and (28), computing the final bridge estimators only involves minimization problems whose objectives are convex in h or q and only depend on h or q via their evaluation at data points. If $\mathbb{H}, \mathbb{Q}$ are linear hypothesis classes like those in Eqs. (20) and (21), then $\hat{h}$ and $\hat{q}$ should also have closed form solutions. If $\mathbb{H}$ is a RKHS hypothesis classes with kernel k either with norm constraints or with norm regularizers, then the optimal solution to Eqs. (23) and (27) will have the form $\hat{h}(z, a, x) = \sum_{i=1}^n \alpha_i k((Z_i, A_i, X_i), (z, a, x))$, leading to a convex quadratic program in α with a closed-form solution. The same holds for Eqs. (24) and (28) if $\mathbb{Q}$ is a RKHS hypothesis class. If the hypothesis classes $\mathbb{H}, \mathbb{Q}$ are some more complex classes such as neural networks, then we can use stochastic gradient descent methods to solve for $\hat{h}, \hat{q}$, which have been shown to be highly successful in many nonconvex applications [Jain and Kar, 2017].

4.3.3 Neural Networks

When we use neural networks for $\mathbb{Q}, \mathbb{H}, \mathbb{Q}', \mathbb{H}'$, we obtain a non-convex minimax optimization. Several types of simultaneous SGD methods are known to be commonly used in this problem such as (simultaneous version of) Adam [Kingma and Ba, 2015], which is a variant of gradient descent with momentum and per-parameter adaptive learning rates. An improved approach is the Optimistic Adam [Daskalakis et al., 2018], which is an adaptation of Adam with additional negative momentum. A negative momentum term is known to be helpful for avoiding oscillations when solving minimax problems.

4.4 An Overview of the Estimation Theory for GACE

In Sections 4.1 and 4.2, we provide two types of minimax estimators $\hat{h}, \hat{q}$ for the bridge functions. Once we obtain these estimators, we can plug them into the estimating equations in Lemma 4 to construct the following estimators for the GACE J:

$$
\begin{aligned}
\hat{J}_{\mathrm{IPW}} &= \mathbb{E}_n\left[\tilde{\phi}_{\mathrm{IPW}}(O;\hat{q})\right] = \mathbb{E}_n[\pi(A \mid X)\hat{q}(Z,A,X)Y], \\
\hat{J}_{\mathrm{REG}} &= \mathbb{E}_n\left[\tilde{\phi}_{\mathrm{REG}}(O;\hat{h})\right] = \mathbb{E}_n\left[(\mathcal{T}\hat{h})(X,W)\right], \\
\hat{J}_{\mathrm{DR}} &= \mathbb{E}_n\left[\tilde{\phi}_{\mathrm{DR}}(O;\hat{h},\hat{q})\right] = \mathbb{E}_n\left[\pi(A \mid X)\hat{q}(Z,A,X)\left(Y - \hat{h}(W,A,X)\right) + (\mathcal{T}\hat{h})(X,W)\right].
\end{aligned}
$$

In Sections 5 and 6, we will show that GACE estimators based on different estimating equations and different minimax bridge functions estimators, either without stablizers (Section 4.1) or with stablizers (Section 4.2), have different theoretical properties. Each type of estimator has its merits depending on how much we are willing to assume.

In the rest of the paper, we will *always* assume Assumptions 1 and 2. We will show that two additional types of assumptions will play an important role in the theoretical guarantees for different estimators.

1. *Realizability*, which characterizes whether the hypothesis classes $\mathbb{H}, \mathbb{Q}$ or critic classes $\mathbb{H}', \mathbb{Q}'$ contain some observed bridge functions $h_0 \in \mathbb{H}_0^{\mathrm{obs}}, q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$, that is, whether $\mathbb{H} \cap \mathbb{H}_0^{\mathrm{obs}} \neq \emptyset$ and $\mathbb{Q} \cap \mathbb{Q}_0^{\mathrm{obs}} \neq \emptyset$, or $\mathbb{H}' \cap \mathbb{H}_0^{\mathrm{obs}} \neq \emptyset$ and $\mathbb{Q}' \cap \mathbb{Q}_0^{\mathrm{obs}} \neq \emptyset$.
2. *Closedness*, which characterizes whether the critic classes $\mathbb{H}', \mathbb{Q}'$ are rich enough so that the resulting minimax reformulation is equivalent to the original conditional moment equations. Specifically, the relevant assumptions are $P_z\left(\mathbb{H} - \mathbb{H}_0^{\mathrm{obs}}\right) \subseteq \mathbb{Q}'$ and/or $\pi P_w\left(\mathbb{Q} - \mathbb{Q}_0^{\mathrm{obs}}\right) \subseteq \mathbb{H}'$, as appear in Lemmas 6 and 7.

The theoretical results for our GACE estimators are organized as follows:

- In Section 5, we will derive finite-sample error bounds for GACE estimators based on minimax bridge function estimators without stabilizers (Section 4.1) under realizability assumptions for both hypothesis classes and critic classes.
- In Section 6 we will analyze error bounds when we use minimax bridge function estimators with stabilizers (Section 4.2), assuming the realizability assumption for one or both hypothesis classes and the closedness assumptions for one or both critic classes.
- In Section 6.3, we show that, when we additionally assume that bridge functions are unique and that the conditional moment equations in Eqs. (5) and (6) are not too ill-posed, we have that the resulting doubly robust estimator $\hat{J}_{\mathrm{DR}}$ is asymptotically normal with asymptotic variance equal to the semiparametric efficiency bound.

In Table 1, we list our results for different estimators under different assumptions, when specialized to Hölder function classes.

5 Finite Sample Analysis of Estimators without Stabilizers

In this section, we analyze different GACE estimators based on minimax estimation of bridge functions *without* stabilizers (Section 4.1). Thus, throughout this section, $\hat{h}, \hat{q}$ refer to the bridge function estimators in Eqs. (11) and (12). We also continue to assume Assumptions 1 and 2.

First, we present results for IPW and REG estimators under *only* realizability, *without* closedness.

Theorem 1 (Analysis of IPW and REG estimators)**.**

1. *If $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H}' \neq \emptyset, \mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q} \neq \emptyset$, then we have*

$$|\hat{J}_{\mathrm{IPW}} - J| \leq \sup_{q \in \mathbb{Q}} |(\mathbb{E} - \mathbb{E}_n)[q\pi Y]| + 4 \sup_{q \in \mathbb{Q}, h \in \mathbb{H}'} |(\mathbb{E} - \mathbb{E}_n)[-q\pi h + \mathcal{T}h]|.$$

2. *If $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H} \neq \emptyset, \pi\mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q}' \neq \emptyset$, then we have*

$$|\hat{J}_{\mathrm{REG}} - J| \leq \sup_{h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[\mathcal{T}h]| + 4 \sup_{q \in \mathbb{Q}', h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q(Y - h)]|.$$

Here we assume that $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H}' \neq \emptyset, \mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q} \neq \emptyset$ or $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H} \neq \emptyset, \pi\mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q}' \neq \emptyset$, which we termed *realizability* in Section 4.4. These conditions mean that one hypothesis class and its corresponding critic class are well specified, in the sense that they contain some valid (observed) bridge functions in $\mathbb{H}_0^{\mathrm{obs}}$ or $\mathbb{Q}_0^{\mathrm{obs}}$. Under these conditions, we can upper bound the estimation errors of $\hat{J}_{\mathrm{IPW}}$ and $\hat{J}_{\mathrm{REG}}$ by two empirical process terms. In Section 5.1, we will show that these terms converge to 0 as n goes to infinity for some common function classes of $\mathbb{H}, \mathbb{H}', \mathbb{Q}, \mathbb{Q}'$. Therefore, $\hat{J}_{\mathrm{IPW}}$ and $\hat{J}_{\mathrm{REG}}$ are consistent estimators for the GACE J.

Interestingly, the GACE estimators $\hat{J}_{\mathrm{REG}}, \hat{J}_{\mathrm{IPW}}$ may converge to the true GACE J even when the bridge function estimators $\hat{h}$ and $\hat{q}$ do not converge to any valid observed bridge functions in $\mathbb{H}_0^{\mathrm{obs}}$ and $\mathbb{Q}_0^{\mathrm{obs}}$. This is in stark contrast to the usual IPW or REG estimators in the unconfounded setting without using minimax estimation. We illustrate this phenomenon in the following simple example of the REG estimator.

Example 7. Suppose $\mathbb{H}_0^{\mathrm{obs}} = \{h_0\}$, $\mathbb{H} = \{a_1 + a_2 h_0 : a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}$, $\mathbb{Q}_0^{\mathrm{obs}} = \{q_0\}$, $\mathbb{Q}' = \{\pi q_0\}$. Then it is easy to show that minimizers for the population minimax objective in Eq. (9) are

$$\begin{aligned}\arg\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \quad (\mathbb{E}\left[q\left(h - Y\right)\right])^2 &= \{h \in \mathbb{H} : \mathbb{E}\left[\pi q_0 \left(h - Y\right)\right] = 0\} \\ &= \{a_1 + a_2 h_0 : a_1 = \mathbb{E}[\pi q_0 (1 - a_2) h_0] / \mathbb{E}[\pi q_0], a_2 \in \mathbb{R}\}.\end{aligned}$$

Notice that, unlike Lemma 6, the above is *not* equal to $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H}$, but this is not a contradiction, as here we do *not* have closedness. Therefore, the estimator $\hat{h}$ that minimizes the empirical analogue of the minimax objective above generally does not converge to h_0. Neverthelss, it is easy to show that $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H} \neq \emptyset, \pi\mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q}' \neq \emptyset$ are satisfied, and $|\hat{J}_{\mathrm{REG}} - J|$ converges to 0 as n goes to infinity. Later, in Theorem 5, we show that under closedness assumptions on $\mathbb{H}', \mathbb{Q}'$, the bridge function estimators in Eqs. (11) and (12) without stabilizers *do* converge to valid observed bridge functions in a suitable notion. But in this section, we do not require such convergence, yet we can still ensure our GACE estimators converge.

Next, we present an analogous result for the DR estimator.

Theorem 2 (Analysis of DR estimators)**.** *If $\mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q} \neq \emptyset, \mathbb{H}_0^{\mathrm{obs}} \cap \{h : h - \mathbb{H} \subseteq \mathbb{H}'\} \neq \emptyset$, then*

$$|\hat{J}_{\mathrm{DR}} - J| \leq \sup_{q \in \mathbb{Q}, h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q\pi Y - q\pi h + \mathcal{T}h]| + 4 \sup_{q \in \mathbb{Q}, h \in \mathbb{H}'} |(\mathbb{E} - \mathbb{E}_n)[-q\pi h + \mathcal{T}h]|. \quad (29)$$

If $\mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H} \neq \emptyset, \mathbb{Q}_0^{\mathrm{obs}} \cap \{q : \pi(q - \mathbb{Q}) \subseteq \mathbb{Q}'\} \neq \emptyset$, then

$$|\hat{J}_{\mathrm{DR}} - J| \leq \sup_{q \in \mathbb{Q}, h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q\pi Y - q\pi h + \mathcal{T}h]| + 4 \sup_{q \in \mathbb{Q}', h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q(Y - h)]|. \quad (30)$$

Theorem 2 suggests that as long as the hypothesis class $\mathbb{Q}$ or $\mathbb{H}$ is well-specified, and the (respective) associated critic class $\mathbb{H}'$ or $\mathbb{Q}'$ is rich enough relative to $\mathbb{H}$ or $\mathbb{Q}$, then $\hat{J}_{\mathrm{DR}}$ is consistent provided that these function classes have limited complexity to ensure uniform convergence of the empirical process terms above. In particular, $\hat{J}_{\mathrm{DR}}$ is consistent when either empirical process conditions on $\mathbb{Q}, \mathbb{H}'$ or on $\mathbb{H}, \mathbb{Q}'$ hold. Moreover, when $\mathbb{H} = \{0\}, \mathbb{Q}' = \{0\}$ so that $\hat{J}_{\mathrm{DR}}$ reduces to $\hat{J}_{\mathrm{IPW}}$, Eq. (29) exactly recovers the result for $\hat{J}_{\mathrm{IPW}}$ in Theorem 1. Analogously, when when $\mathbb{Q} = \{0\}, \mathbb{H}' = \{0\}$, Eq. (30) recovers the result for $\hat{J}_{\mathrm{REG}}$ in Theorem 1.

5.1 Convergence Rates of GACE Estimators for Common Function Classes

In this part, we analyze the convergence rates of $\hat{J}_{\mathrm{REG}}$ and $\hat{J}_{\mathrm{IPW}}$ by further bounding the error terms in Theorem 1 in terms of the complexity of some common function classes. For simplicity and brevity, we set $\mathbb{H}' = \mathbb{H}$ and $\mathbb{Q}' = \pi\mathbb{Q}$. The results can easily be specialized to each of $\hat{J}_{\mathrm{REG}}$ or $\hat{J}_{\mathrm{IPW}}$ when only the corresponding pair of hypothesis and critic classes are realizable. Each result we present also implies the same convergence rate for $\hat{J}_{\mathrm{DR}}$ if we simply modify the critic class realizability condition to be as in Theorem 2. We omit these corresponding results for $\hat{J}_{\mathrm{DR}}$ for brevity.

Example 8 (VC-subgraph classes)**.** VC-subgraph classes are function classes whose subgraph sets have bounded VC dimension [van der Vaart, 1998, Chapter 19]. For example, $\{\phi \mapsto \theta^\top \phi : \|\theta\|_2 \le 1, \theta \in \mathbb{R}^d\}$ has a VC-subgraph dimension $d+1$ [van der Vaart, 1998, Chapter 19].

Corollary 1. *Let $\mathbb{H}, \mathbb{Q}'$ be VC-subgraph classes with finite VC-subgraph dimensions $V(\mathbb{H}), V(\mathbb{Q}')$, respectively. Assume $\mathbb{H} \cap \mathbb{H}_0^{\mathrm{obs}} \neq \emptyset$, $\mathbb{Q}' \cap \pi\mathbb{Q}_0^{\mathrm{obs}} \neq \emptyset$, and $\|\mathbb{H}\|_\infty, \|\mathbb{Q}'\|_\infty, \|Y\|_\infty < \infty$. Then, letting $O(\cdot)$ be the order wrt $n, V(\mathbb{H}), V(\mathbb{Q}'), \delta$., with probability $1-\delta$, we have*

$$\max\{|\hat{J}_{\mathrm{REG}} - J|,\ |\hat{J}_{\mathrm{IPW}} - J|\} = O(\sqrt{(V(\mathbb{H}) + V(\mathbb{Q}') + 1 + \log(1/\delta))/n}).$$

Corollary 1 shows that $\hat{J}_{\mathrm{REG}}, \hat{J}_{\mathrm{IPW}}, \hat{J}_{\mathrm{DR}}$ have parametric convergence rates for VC-subgraph classes under realizability.

Example 9 (Nonparametric classes characterized by metric entropy)**.** Many common nonparametric classes, such as Hölder balls and Sobolev balls, cannot be characterized by VC-subgraph dimensions. Instead, their complexity can be characterized by their metric entropies [Wainwright, 2019]. For example, a Hölder ball $\mathbb{W}$ with smoothness level α and an input dimension d has metric entropy under infinity norm $\log \mathcal{N}(\varepsilon, \mathbb{W}, \|\cdot\|_\infty) = O(\varepsilon^{-d/\alpha})$.

Corollary 2. *Suppose $\log \mathcal{N}(\varepsilon, \mathbb{H}, \|\cdot\|_\infty) = O(\varepsilon^{-\beta})$, $\log \mathcal{N}(\varepsilon, \mathbb{Q}', \|\cdot\|_\infty) = O(\varepsilon^{-\beta})$ for a positive constants β. Further assume $\mathbb{H} \cap \mathbb{H}_0^{\mathrm{obs}} \neq \emptyset$, $\mathbb{Q}' \cap \pi\mathbb{Q}_0^{\mathrm{obs}} \neq \emptyset$, and $\|\mathbb{H}\|_\infty, \|\mathbb{Q}'\|_\infty, \|Y\|_\infty < \infty$. Then, letting $O(\cdot)$ be the order wrt n, δ, with probability $1-\delta$, we have*

$$\max\left\{|\hat{J}_{\mathrm{REG}} - J|,\ |\hat{J}_{\mathrm{IPW}} - J|\right\} = \begin{cases} O(n^{-1/2} + \sqrt{\log(1/\delta)/n}) & \beta < 2 \\ O(n^{-1/2}\log(n) + \sqrt{\log(1/\delta)/n}) & \beta = 2 \\ O(n^{-1/\beta} + \sqrt{\log(1/\delta)/n}) & \beta > 2 \end{cases}$$

Corollary 2 states that the convergence rates of $\hat{J}_{\mathrm{REG}}$ and $\hat{J}_{\mathrm{IPW}}$ are determined by the worse of the metric entropy of $\mathbb{H}$ and $\mathbb{Q}'$. This implies that when these two function classes are Donsker classes [van der Vaart, 1998], *i.e.*, $\beta < 2$, both estimators have parametric convergence rates. But if either function class is non-Donsker, *i.e.*, $\beta \ge 2$, then the estimators typically have slower convergence rates.

Example 10 (Linear Sieves)**.** It is often difficult to optimize over the infinitely dimensional nonparametric classes in Example 9, *e.g.*, the Hölder balls. Instead, we may consider using a sequence of more tractable function class with increasing complexity to approximate those nonparametric classes. These approximating classes are called sieves [Chen, 2007]. In particular, we may consider linear sieves for $\mathbb{H}$ and $\mathbb{Q}$ induced by basis functions $\{\psi_j(w,a,x)\}_{j=1}^{k_n}$ and $\{\phi_j(z,a,x)\}_{j=1}^{k_n}$, respectively, *e.g.*, splines, polynomials, wavelets, etc. For brevity, we consider both sieves having the same dimension k_n, which is increasing the sample size n. This leads to the classes

$$\begin{aligned}\mathcal{S}_{1,n} &= \left\{(w,a,x) \mapsto \sum_{j=1}^{k_n} \omega_j \psi_j(w,a,x) : \omega \in \mathbb{R}^{k_n}\right\}, \\ \mathcal{S}_{2,n} &= \left\{(z,a,x) \mapsto \sum_{j=1}^{k_n} \omega_j \phi_j(z,a,x) : \omega \in \mathbb{R}^{k_n}\right\}.\end{aligned} \tag{31}$$

For simplicity, assume that $\mathcal{W} \times \mathcal{Z} \times \mathcal{A} \times \mathcal{X} = [0,1]^{d_W + d_Z + d_A + d_X}$ with $d_W = d_Z$ and let $d = d_W + d_A + d_X$. Define $\Lambda^\alpha([0,1]^d)$ as the Hölder ball over $[0,1]^d$ with smoothness level α and finite radius. For brevity we consider the same smoothness level for both bridge functions; this can easily be relaxed. We require linear sieves in Eq. (31) to approximate these Hölder balls with small enough approximation errors:

$$\forall h \in \Lambda^\alpha([0,1]^d), \exists h_n \in S_{1,n}, \text{ s.t. } \|h - h_n\| = O\left(k_n^{-\alpha/d}\right), \tag{32}$$

$$\forall q \in \Lambda^\alpha([0,1]^d), \exists q_n \in S_{2,n}, \text{ s.t. } \|q - q_n\| = O\left(k_n^{-\alpha/d}\right). \tag{33}$$

Note the rate in Eqs. (32) and (33) is standard in sieve approximations [Chen, 2007].

Corollary 3. *Let $\mathbb{H} = \mathcal{S}_{1,n}$ and $\mathbb{Q}' = \mathcal{S}_{2,n}$ for $S_{1,n}, S_{2,n}$ in Eq.* (31) *satisfying the approximation conditions in Eqs.* (32) *and* (33)*. Assume $\mathbb{H}_0^{\text{obs}} \cap \Lambda^\alpha([0,1]^d) \neq \emptyset$, $\pi\mathbb{Q}_0^{\text{obs}} \cap \Lambda^\alpha([0,1]^d) \neq \emptyset$, and $\|\mathbb{H}\|_\infty, \|\mathbb{Q}'\|_\infty, \|Y\|_\infty < \infty$. Then with probability $1-\delta$,*

$$\max\left\{|\hat{J}_{\text{REG}} - J|,\ |\hat{J}_{\text{IPW}} - J|\right\} = O\left(n^{-\alpha/(2\alpha+d)} + \sqrt{\log(1/\delta)/n}\right).$$

Compared to Corollaries 1 and 2, Corollary 3 assumes realizability on $\Lambda^\alpha(([0,1]^d)$ rather than on $\mathbb{H}, \mathbb{Q}'$. Therefore, we cannot prove Corollary 3 by directly applying Theorem 1 that only considers stochastic errors. Instead, we need a more refined version of Theorem 1 that we develop in Appendix D, which additionally incorporates the bias terms due to the fact that the realizability conditions on $\mathbb{H}, \mathbb{Q}'$ may be violated. These bias terms are then upper-bounded by sieve approximation errors in Eqs. (32) and (33). We then balance the stochastic errors with the bias terms to get the convergence rate in Corollary 3.

Note that the rate in Corollary 3 is slower than the rate $\max(n^{-1/2}, n^{\alpha/d})$ in Corollary 2 when specialized to Hölder balls (*i.e.*, $\beta = \frac{d}{\alpha}$) under the same realizability assumption. This reflects the potential loss when we use linear sieves to approximate Hölder balls.

We can straightforwardly extend Corollaries 1 to 3 in a few aspects. First, all of the above finite-sample results can be extended to asymptotic results. For example, the conclusion in Corollary 1 can be used to show $\mathbb{E}[(\hat{J} - J)^2] = O(n^{-1})$ for $\hat{J} = \hat{J}_{\text{IPW}}$ or $\hat{J} = \hat{J}_{\text{REG}}$. Second, we have assumed that $\|\mathbb{H}\|_\infty, \|\mathbb{Q}'\|_\infty$, *i.e.*, envelopes of classes $\mathbb{H}$ and $\mathbb{Q}'$ are bounded functions. We can relax these assumptions by allowing square-integrable envelopes under additional regularity conditions. Finally, as mentioned, we can very easily consider $\mathbb{H} \neq \mathbb{H}'$ and $\pi\mathbb{Q} \neq \mathbb{Q}'$, assume different smoothness levels for the hypothesis and critic classes, and extend the results to the DR estimator.

6 Finite Sample Analysis of Estimators with Stabilizers

In the previous section, we studied the convergence of our GACE estimators based on minimax bridge function estimators *without* stabilizers (Section 4.1), by assuming the realizability of $\mathbb{H}, \mathbb{Q}'$ (or realizability of $\mathbb{H}', \mathbb{Q}$). In this section, we show that under a different set of assumptions, *i.e.*, closedness and realizability, bridge function estimators *with* stablizers converge to valid observed bridge functions at sufficiently fast rates in terms of projected mean-squared errors.[7] Then, we

[7] We can also show that under the same assumptions, bridge function estimators without stabilizers converge to valid observed bridge functions in projected mean-squared errors, albeit at slower rates. See Theorem 5.

establish the convergence rate of the resulting GACE estimators. Finally, by leveraging the fast convergence results with stablizers and additionally assuming that bridge functions are unique and the associated inverse problems are not too ill-posed, we can show that the resulting DR estimator is also semiparametrically efficient.

Throughout this section, $\hat{h}$, $\hat{q}$ refer to the minimax bridge function estimators in Eqs. (14) and (15), that is, the minimax bridge function estimators *with* stabilizers. We continue to assume Assumptions 1 and 2.

For any valid bridge function $h_0 \in \mathbb{H}_0^{\text{obs}}$ and $q_0 \in \mathbb{Q}_0^{\text{obs}}$, we quantify the estimation errors of bridge function estimators $\hat{h}$ and $\hat{q}$ by $\|P_z(\hat{h}-h_0)\|_2^2$ and $\|\pi P_w(\hat{q}-q_0)\|_2$, which we call projected mean-squared errors (projected MSE). Projected MSE measures how much $\hat{h}$ and $\hat{q}$ violate the conditional moment equations in Eqs. (5) and (6):

$$\|P_z(\hat{h} - h_0)\|_2^2 = \mathbb{E}\left[\left(\mathbb{E}[Y - \hat{h}(W, A, X) \mid Z, A, X]\right)^2\right], \tag{34}$$

$$\|\pi P_w(\hat{q} - q_0)\|_2^2 = \mathbb{E}\left[\left(\mathbb{E}[\pi(A \mid X)\left(\hat{q}(Z, A, X) - 1/f(A \mid W, X)\right) \mid W, A, X]\right)^2\right]. \tag{35}$$

Obviously, these estimation errors are invariant to the choice of h_0 and q_0 so they are particularly relevant when bridge functions are nonunique. Note that even when $\|P_z(\hat{h} - h_0)\|_2 \to 0$ and $\|\pi P_w(\hat{q} - q_0)\|_2 \to 0$, $\hat{h}$ and $\pi\hat{q}$ may not necessarily convergence to any fixed limits since $\mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0^{\text{obs}}$ need not be singletons.

To control these errors for our bridge function estimators, we further introduce an alternative function class complexity measure called the *critical radius* [Bartlett et al., 2005].

Definition 1 (Localized Rademacher Complexity and Critical Radius)**.** Given a class $\mathcal{G}$ of functions of variables O, its empirical localized Rademacher complexity with respect to the sample $\{O_1, \ldots, O_n\}$ and radius $\eta > 0$ is defined as

$$\mathcal{R}_n(\eta; \mathcal{G}) = \frac{1}{2^n} \sum_{\epsilon \in \{-1,+1\}^n} \left[\sup_{g \in \mathcal{G}: \mathbb{E}_n[g^2] \le \eta^2} \frac{1}{n} \sum_i \epsilon_i g(O_i)\right].$$

When $\eta = +\infty$, the quantity $\mathcal{R}_n(+\infty; \mathcal{G})$ is called the (global) Rademacher complexity. The critical radius of $\mathcal{G}$ is the smallest positive η that satisfies the inequality

$$\mathcal{R}_n(\eta; \mathcal{G}) \le \eta^2 / \|\mathcal{G}\|_\infty.$$

In the result below, we first bound the estimation error of the outcome bridge function estimator in terms of the critical radius of some special function classes.

Theorem 3 (Convergence rate of $\hat{h}$ with stabilizer)**.** *Assume $\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} \neq \emptyset$, and $\|\mathbb{H}\|_\infty, \|Y\|_\infty < \infty$. Fix some $h_0 \in \mathbb{H}_0^{\text{obs}} \cap \mathbb{H}$ and further assume the following conditions:*

1. *$\mathbb{Q}'$ is symmetric and star-shaped and it satisfies $P_z(\mathbb{H} - h_0) \subset \mathbb{Q}'$.*
2. *$\eta_{h,n}$ upper bounds the critical radii of $\mathbb{Q}'$ and the class $\mathcal{G}_h$ defined as*

$$\mathcal{G}_h := \left\{(w, z, a, x) \mapsto (h(w, a, x) - h_0(w, a, x))\, q(z, a, x) : h \in \mathbb{H}, q \in \mathbb{Q}'\right\}.$$

Then, letting $O(\cdot)$ be the order wrt $\lambda, \delta, n, \eta_{h,n}$, with probability $1 - \delta$, we have

$$\|P_z(\hat{h} - h_0)\|_2 = O\left(\left(1 + \lambda + \frac{1}{\lambda}\right)\left(\eta_{h,n} + \sqrt{(1 + \log(1/\delta))/n}\right)\right).$$

In Theorem 3, we assume the realizability condition for the hypothesis class $\mathbb{H}$, and the closedness condition $P_z(\mathbb{H} - h_0) \subset \mathbb{Q}'$. The latter closedness condition is invariant to choice of $h_0 \in \mathbb{H}_0^{\text{obs}} \cap \mathbb{H}$, by following Eq. (34). This closedness condition indicates that the critic class $\mathbb{Q}'$ is rich enough relative to the hypothesis class $\mathbb{H}$, so that solving the minimax formulation with stabilizers is equivalent to solving the corresponding conditional moment equation (see Lemma 7). Theorem 3 states that the convergence rate of outcome bridge function estimator $\hat{h}$ is determined by the critical radii $\eta_{h,n}$ of the critic class $\mathbb{Q}'$ and the class $\mathcal{G}_h$ induced by both $\mathbb{H}$ and $\mathbb{Q}'$. In Section 6.1, we will further bound $\eta_{h,n}$ for common function classes.

Analogously, we can also bound the estimation error of the action bridge function estimator under symmetric conditions.

Theorem 4 (Convergence rate of $\hat{q}$ with stabilizer)**.** *Assume $\mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q} \neq \emptyset$ and $\|\pi \mathbb{Q}\|_\infty$, $\|\pi(a|x)/f(a|x,w)\|_2 < \infty$. Fix some $q_0 \in \mathbb{Q}_0^{\text{obs}} \cap \mathbb{Q}$ and further assume the following conditions:*

1. *$\mathbb{H}'$ is symmetric and star-shaped, and it satisfies $\pi P_w(\mathbb{Q} - q_0) \subset \mathbb{H}'$.*
2. *$\eta_{q,n}$ upper bounds the critical radii of $\mathbb{H}'$ and the class $\mathcal{G}_q$ defined as*

$$\mathcal{G}_q \coloneqq \{(w,z,a,x) \mapsto \pi(a|x)\,(q(z,a,x) - q_0(z,a,x))\,h(w,a,x) : q \in \mathbb{Q}, h \in \mathbb{H}'\}.$$

Then, letting $O(\cdot)$ be the order wrt $\lambda, \delta, n, \eta_{q,n}$, with probability $1-\delta$, we have

$$\|\pi P_w(\hat{q} - q_0)\|_2 = O\left(\left(1 + \lambda + \frac{1}{\lambda}\right)\left(\eta_{q,n} + \sqrt{\{1 + \log(1/\delta)\}/n}\right)\right).$$

We finally remark that the use of stabiliziers in Eqs. (14) and (15) plays an important role in deriving the convergence rates in Theorems 3 and 4. Without stabilizers (*i.e.*, $\lambda = 0$), the error bounds in Theorems 3 and 4 are apparently vacuous since they scale with $1/\lambda$. Although we can use an alternative way to show the projected MSE convergence of bridge functions estimators that do not use stabilizers, *e.g.*, estimators in Eqs. (11) and (12), the resulting convergence rates will be slower than the rates of bridge functions using stabilizers.

Theorem 5 (Slow convergence rates of $\hat{h}, \hat{q}$ without stabilizers)**.** *Consider the bridge function estimators $\hat{h}, \hat{q}$ in Eqs. (11) and (12), i.e., without stabilizers (unlike the rest of this section and only for this theorem).*

1. *Suppose $\mathbb{H} \cap \mathbb{H}_0^{\text{obs}} \neq \emptyset$ and take some $h_0 \in \mathbb{H} \cap \mathbb{H}_0^{\text{obs}}$. If $P_z(\mathbb{H} - h_0) \subseteq \mathbb{Q}'$ and $\mathbb{Q}'$ is symmetric, then*

$$\|P_z(\hat{h} - h_0)\|_2 \leq 2\sqrt{\sup_{q \in \mathbb{Q}', h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[\{y - h\}q]|}.$$

2. *Suppose $\mathbb{Q} \cap \mathbb{Q}_0^{\text{obs}} \neq \emptyset$ and take some $q_0 \in \mathbb{Q} \cap \mathbb{Q}_0^{\text{obs}}$. If $\pi P_w(\mathbb{Q} - q_0) \subseteq \mathbb{H}'$, and $\mathbb{H}'$ is symmetric, then*

$$\|\pi P_w(\hat{q} - q_0)\|_2 \leq 2\sqrt{\sup_{q \in \mathbb{Q}, h \in \mathbb{H}'} |(\mathbb{E} - \mathbb{E}_n)[-q\pi h + \mathcal{T}h]|}.$$

Note that Theorem 5 also holds for estimators in Eqs. (14) and (15) with $\lambda = 0$, since these estimtors are equivalent to estimators in in Eqs. (11) and (12) when $\mathbb{H}', \mathbb{Q}'$ are symmetric (see discussions below Lemma 7). Crucially, the error bounds in Theorem 5 typically converge more slowly than error bounds in Theorems 3 and 4. For example, when we use VC-subgraph classes, the rates in Theorem 5 are $O(n^{-1/4})$, while the rates in Theorems 3 and 4 for bridge function estimators with stabilizers are $O(n^{-1/2})$, as we show below. (Recall that in Section 5.1 we showed $\hat{J}_{\text{REG}}$ and $\hat{J}_{\text{IPW}}$ still nonetheless converges at rate $O(n^{-1/2})$ in this setting.) As another example, when we use Hölder balls, the rates in Theorem 5 are $O(\max(n^{-1/4}, n^{-\alpha/2d}))$, while the rates in Theorems 3 and 4 are $O(n^{-\alpha/(2\alpha+d)})$.

6.1 Convergence Rates of Bridge Function Estimators for Common Function Classes

Theorems 3 and 4 show that the convergence rates of minimax estimators $\hat{h}, \hat{q}$ with stabilizers (Section 4.2) are determined by critical radii of related function classes. In this part, we further provide concrete convergence rates for these function classes. We focus on analyzing $\|P_z(\hat{h} - h_0)\|_2$ as an example. $\|\pi P_w(\hat{q} - q_0)\|_2$ can be analyzed analogously.

We first give the error bounds for VC-subgraph classes in Example 8 and nonparametric classes (*e.g.*, Hölder balls, Sobolev balls) in Example 9. The bounds are derived by directly employing Theorem 3 and bounding the critical radii of these function classes.

Corollary 4. *Assume that the conditions in Theorem 3 hold for some $h_0 \in \mathbb{H} \cap \mathbb{H}_0^{\text{obs}}$.*

1. *Suppose $\mathbb{H}$ and $\mathbb{Q}'$ are VC-subgraph classes with VC-subgraph dimensions $V(\mathbb{H}), V(\mathbb{Q}')$, respectively. Then, letting $O(\cdot)$ be the order wrt $n, \delta, V(\mathbb{H}), V(\mathbb{Q}')$, with probability $1-\delta$, we have*

$$\|P_z(\hat{h}-h_0)\|_2 = O(\sqrt{(V(\mathbb{Q}')+V(\mathbb{H}))\log n + 1 + \log(1/\delta)}/\sqrt{n}).$$

2. *Suppose $\mathbb{H}$ and $\mathbb{Q}'$ are classes whose metric entropy satisfies that for a constant $\beta > 0$,*

$$\max(\log \mathcal{N}(\varepsilon, \mathbb{H}, \|\cdot\|_\infty), \quad \log \mathcal{N}(\varepsilon, \mathbb{Q}', \|\cdot\|_\infty)) \le c_0(1/\varepsilon)^\beta.$$

Then, letting $O(\cdot)$ be the order wrt n, β, δ, with probability $1-\delta$, we have

$$\|P_z(\hat{h}-h_0)\|_2 = \begin{cases} O(n^{-1/(2+\beta)} + \sqrt{\log(1/\delta)/n}) & \beta < 2 \\ O(n^{-1/4}\log n + \sqrt{\log(1/\delta)/n}) & \beta = 2 \\ O(n^{-1/(2\beta)} + \sqrt{\log(1/\delta)/n}) & \beta > 2 \end{cases}$$

In the following corollary, we also prove error bounds for linear sieves as in Example 10.

Corollary 5. *Let $\mathbb{H} = \mathcal{S}_{1,n}$ and $\mathbb{Q}' = \mathcal{S}_{2,n}$ for $S_{1,n}, S_{2,n}$ in Eq. (31) satisfying the approximation conditions in Eqs. (32) and (33). Assume $\mathbb{H}_0^{\mathrm{obs}} \cap \Lambda^\alpha([0,1]^d) \neq \emptyset$, $\mathbb{H} \subseteq \Lambda^\alpha([0,1]^d)$. Additionally, suppose that for some $h_0 \in \mathbb{H}_0^{\mathrm{obs}} \cap \Lambda^\alpha([0,1]^d)$ we have $P_z(\Lambda^\alpha([0,1]^d) - h_0) \subset \Lambda^\alpha([0,1]^d)$. Then, letting $\tilde{O}(\cdot)$ be the order wrt n, α, δ up to poly logarithmic factors in n, with probability $1-\delta$, we have*

$$\|P_z(\hat{h}-h_0)\|_2 = \tilde{O}(n^{-\alpha/(2\alpha+d)} + \sqrt{\log(1/\delta)/n}).$$

In Corollary 5, we assume realizibility condition and closedness condition on the Hölder ball $\Lambda^\alpha([0,1]^d)$, rather than on the linear sieves $\mathbb{H}$ and $\mathbb{Q}'$. Therefore, we cannot directly apply Theorem 3 to prove Corollary 5, just like we cannot prove Corollary 3 by directly applying Theorem 1. Instead, we need to use a generalized version of Theorem 3 that can handle additional bias terms, which we establish in Appendix D, and then balance the bias terms with the stochastic errors shown in Theorem 3.

6.2 Analysis of the GACE Estimator

In this part, we bound the errors of GACE estimators based on results in Theorems 3 and 4. Before deriving these error bounds, we first note that according to Lemma 5, the bias of $\hat{J}_{\mathrm{REG}}$ and $\hat{J}_{\mathrm{IPW}}$ can be bounded by the projected MSE of bridge function estimators in Eqs. (34) and (35).

Lemma 11. *Under the assumptions in Lemma 5, for any $h_0 \in \mathbb{H}_0^{obs}$ and $q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$,*

$$\left|\mathbb{E}[\tilde{\phi}_{\mathrm{REG}}(O;\hat{h}) \mid \hat{h}] - J\right| \le \|\pi q_0\|_2 \|P_z(\hat{h}-h_0)\|_2,$$
$$\left|\mathbb{E}[\tilde{\phi}_{\mathrm{IPW}}(O;\hat{q}) \mid \hat{q}] - J\right| \le \|h_0\|_2 \|\pi P_w(\hat{q}-q_0)\|_2.$$

In the following theorem, we first derive the convergence rates of $\hat{J}_{\mathrm{REG}}$ and $\hat{J}_{\mathrm{IPW}}$ based on Lemma 11, Theorems 3 and 4.

Theorem 6. *1. Under the conditions of Theorem 3, if further $\emptyset \neq \pi\mathbb{Q}_0^{\mathrm{obs}} \subseteq L_2(Z,A,W)$, then with probability at least $1-\delta$,*

$$|\hat{J}_{\mathrm{REG}} - J| = O\left(\mathcal{R}_n(\infty; \mathcal{T}\mathbb{H}) + \eta_{h,n} + \sqrt{(\log(1/\delta))/n}\right). \tag{36}$$

2. *Under the conditions of Theorem 4 , if further $\emptyset \neq \mathbb{H}_0^{\mathrm{obs}} \subseteq L_2(X,A,W)$, then with probability at least $1-\delta$,*

$$|\hat{J}_{\mathrm{IPW}} - J| = O\left(\mathcal{R}_n(\infty; \pi\mathbb{Q}) + \eta_{q,n} + \sqrt{(\log(1/\delta))/n}\right). \tag{37}$$

Here $\mathcal{R}_n(\infty;\mathcal{T}\mathbb{H})$ and $\mathcal{R}_n(\infty;\pi\mathbb{Q})$ are the gloabl Rademacher complexity (see Definition 1) of function classes $\mathcal{T}\mathbb{H}$ and $\pi\mathbb{Q}$, respectively.

In the right hand sides of Eqs. (36) and (37), the global Radamacher complexity bound the variances due to plugging in estimted bridge functions rather than true ones, and the critical radii $\eta_{q,n}, \eta_{h,n}$ bound the plug-in bias. Typically, the critical radii $\eta_{q,n}, \eta_{h,n}$ dominate in these two error bounds. Therefore, the convergence rates of $\hat{J}_{\mathrm{REG}}, J_{\mathrm{IPW}}$ are typically the same as the convergence rates of bridge function estimators in terms of projected MSE (see Theorems 3 and 4). Notably, these convergence rates are often slower than the convergence rates in Theorem 1. For example, for nonparametric classes with a metric entropy exponent β (see Example 9), the convergence rates in Theorem 6 become $O\left(n^{-1/\max(2\beta,\beta+2)}\right)$ according to Corollary 4 statement 2, while the convergence rates in Theorem 1 become $O\left(n^{-1/\max(2,\beta)}\right)$ according to Corollary 2. This shows potential advantage of minimax bridge function estimators *without* stabilizers. However, we stress that Theorem 6 and Theorem 1 are based on quite different assumptions so they are not directly comparable. For example, Theorem 6 inherits the closedness assumptions for the critic classes from Theorems 3 and 4, while Theorem 1 requires realizability assumptions for the critic classes.

Next, we derive the convergence rates of the doubly robust estimator $\hat{J}_{\mathrm{DR}}$.

Theorem 7 (Analysis of DR estimators)**.** *Assume the conditions of Theorem 6 statement 1 hold and $\|\pi\mathbb{Q}\|_\infty < \infty$. Then, with probability $1-\delta$,*

$$|\hat{J}_{\mathrm{DR}} - J| = O\left(\mathcal{R}_n(\infty; \pi\mathbb{Q}\{y-\mathbb{H}\} + \mathcal{T}\mathbb{H}) + \eta_{h,n} + \sqrt{(1+\log(1/\delta))/n}\right).$$

Assume the conditions of Theorem 6 statement 2 hold and $\|\mathbb{H}\|_\infty < \infty$. Then, with probability $1-\delta$,

$$|\hat{J}_{\mathrm{DR}} - J| = O\left(\mathcal{R}_n(\infty; \pi\mathbb{Q}\{y-\mathbb{H}\} + \mathcal{T}\mathbb{H}) + \eta_{q,n} + \sqrt{(1+\log(1/\delta))/n}\right).$$

Again, the two error bounds above for $\hat{J}_{\mathrm{DR}}$ are typically dominated by $\eta_{h,n}$ and $\eta_{q,n}$, respectively. Thus Theorem 7 suggests that $\hat{J}_{\mathrm{DR}}$ is consistent when the projected MSE of either the outcome bridge function estimator $\hat{h}$ *or* the action bridge function estimator $\hat{q}$ vanishes to 0. In particular, if the projected MSE of either bridge function estimator converges at $\sqrt{n}$ rate, then $\hat{J}_{\mathrm{DR}}$ is $\sqrt{n}$-consistent. This result is reminiscent of the doubly robustness property in the unconfounded setting [*e.g.*, Bang and Robins, 2005].

6.3 Tighter Analysis of the Doubly Robust Estimator

In Theorem 9, we showed that the convergence rate of the doubly robust estimator $\hat{J}_{\mathrm{DR}}$ is determined by the projected MSE convergence rate of a single bridge function estimator, when conditions in either Theorem 3 *or* Theorem 4 are satisfied. In this section, we show that if conditions in *both* theorems are satisfied, and the related inverse problems are not too ill-posed, then the doubly robust estimator $\hat{J}_{\mathrm{DR}}$ can converge at a much faster rate than the bridge function estimators.

We first introduce measures of the ill-posedness of the inverse problems defined by conditional moment equations in Eqs. (5) and (6), relative to the hypothesis classes $\mathbb{H}, \mathbb{Q}$.

Definition 2. [Measures of Ill-posedness of Inverse Problems] Fix $h_0 \in \mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H}$ and $q_0 \in \mathbb{Q}_0^{\mathrm{obs}} \cap \mathbb{Q}$. Define the ill-posedness measures $\tau_{1,n}^{\mathbb{H}}, \tau_{1,n}^{\mathbb{Q}} \in \mathbb{R}_+$ and $\tau_{2,n}^{\mathbb{H}}, \tau_{2,n}^{\mathbb{Q}} \in \mathbb{R}_+$ as follows:

$$\tau_{1,n}^{\mathbb{H}} = \sup_{h\in\mathbb{H}} \frac{\|\mathbb{E}\left[h-h_0 \mid A,U,X\right]\|_2}{\|\mathbb{E}[h-h_0 \mid Z,A,X]\|_2}, \quad \tau_{1,n}^{\mathbb{Q}} = \sup_{q\in\mathbb{Q}} \frac{\|\mathbb{E}\left[\pi(q-q_0) \mid A,U,X\right]\|_2}{\|\mathbb{E}\left[\pi(q-q_0) \mid W,A,X\right]\|_2}. \tag{38}$$

$$\tau_{2,n}^{\mathbb{H}} = \sup_{h\in\mathbb{H}} \frac{\|h-h_0\|_2}{\|\mathbb{E}\left[h-h_0 \mid Z,A,X\right]\|_2}, \quad \tau_{2,n}^{\mathbb{Q}} = \sup_{q\in\mathbb{Q}} \frac{\|\pi(q-q_0)\|_2}{\|\mathbb{E}\left[\pi(q-q_0) \mid W,A,X\right]\|_2}. \tag{39}$$

In this definition, we follow the convention $\frac{0}{0} = 0$. Note the subscript n implies the possible (but not necessary) dependence on n, which is relevant if $\mathbb{H}$ and $\mathbb{Q}$ grow with n.

To interpret the ill-posedness measures above, let us focus on $\tau_{1,n}^{\mathbb{H}}$ and $\tau_{2,n}^{\mathbb{H}}$ as examples and note that $\mathbb{E}\left[h-h_0 \mid Z,A,X\right] = \mathbb{E}\left[\mathbb{E}\left[h-h_0 \mid A,U,X\right] \mid Z,A,X\right]$ according to Assumption 1 condition 4. This means that we have two different levels of inverse problems: the first involves inverting $\mathbb{E}\left[h-h_0 \mid A,U,X\right]$ from $\mathbb{E}\left[h-h_0 \mid Z,A,X\right]$, and the second involves inverting $h-h_0$ from $\mathbb{E}\left[h-h_0 \mid Z,A,X\right]$ for $h \in \mathbb{H}$. The degrees of ill-posedness of these two levels relative to the hypothesis class $\mathbb{H}$ are quantified by $\tau_{1,n}^{\mathbb{H}}$ and $\tau_{2,n}^{\mathbb{H}}$, respectively. Obviously, we have $1 \leq \tau_{1,n}^{\mathbb{H}} \leq \tau_{2,n}^{\mathbb{H}}$, which agrees with the fact that the second level of inverse problem is no less ill-posed than the first one. Note that $\tau_{1,n}^{\mathbb{H}}$ is invariant to the choice of $h_0 \in \mathbb{H}_0^{\text{obs}}$, since $\mathbb{E}\left[h_0 \mid Z,A,X\right] = \mathbb{E}\left[Y \mid Z,A,X\right]$ and $\mathbb{E}\left[h_0 \mid U,A,X\right] = \mathbb{E}\left[Y \mid U,A,X\right]$ for any $h_0 \in \mathbb{H}_0^{\text{obs}}$ according to Eqs. (2) and (5). So $\tau_{1,n}^{\mathbb{H}}$ may be finite even when the bridge function is not uniquely identified. In contrast, $\tau_{2,n}^{\mathbb{H}}$ is obviously $+\infty$ when $\mathbb{H}_0^{\text{obs}} \cap \mathbb{H}$ is not a singleton. Moreover, note that the definition of $\tau_{1,n}^{\mathbb{H}}$ and $\tau_{2,n}^{\mathbb{H}}$ depend on the choice of hypothesis class $\mathbb{H}$ (which can change with n). When specialized to linear sieves (Example 10), these measures are related to the sieve ill-posedness measures introduced in Blundell et al. [2003].

The ill-posedness measures in Definition 2 directly influence the convergence behavior of minimax bridge function estimators in different error measures. For example, if $\tau_{1,n}^{\mathbb{H}} < +\infty$ and $\tau_{2,n}^{\mathbb{H}} < +\infty$ for all n, then for the minimax estimator $\hat{h}$, the error bound on $\|P_z(\hat{h}-h_0)\|$ derived in Theorem 3 easily translate into bounds on $\|\mathbb{E}[\hat{h}-h_0 \mid A,U,X]\|_2$ (see Theorem 8) and $\|\hat{h}-h_0\|_2$, with additional inflation factors of $\tau_{1,n}^{\mathbb{H}}$ and $\tau_{2,n}^{\mathbb{H}}$, respectively. So larger $\tau_{1,n}^{\mathbb{H}}$ and $\tau_{2,n}^{\mathbb{H}}$, *i.e.*, more ill-posed inverse problems, correspond to slower convergence rates in terms of $\|\mathbb{E}[\hat{h}-h_0 \mid A,U,X]\|_2$ and $\|\hat{h}-h_0\|_2$. At the extreme, if these ill-posedness measures are infinite, then convergence in terms of $\|P_z(\hat{h}-h_0)\|$ does not necessarily imply convergence in terms of $\|\mathbb{E}[\hat{h}-h_0 \mid A,U,X]\|_2$ or $\|\hat{h}-h_0\|_2$. The ill-posedness measures depend both on the data generating process and on the choices of hypothesis classes $\mathbb{H}, \mathbb{Q}$. As is discussed in Blundell et al. [2003] for linear sieves, larger hypothesis classes typically correspond to bigger ill-posedness measures. Therefore, the choice of hypothesis classes $\mathbb{H}, \mathbb{Q}$ can considerably influence the convergence of bridge function estimators not only through their realizability and complexity (*e.g.*, the critical radii) but also their induced ill-posedness.

Next, we slightly revise our doubly robust estimator by cross-fitting the bridge function estimators, a technique that has been widely used in semiparametric estimation to remove restrictive Donsker conditions on hypothesis classes [*e.g.*, Zheng and van der Laan, 2011, Chernozhukov et al., 2018]. For simplicity, we focus on two-fold cross-fitting. It is straightforward to extend our results to cross-fitting with more folds.

Definition 3 (Cross-fitted DR Estimator)**.** Randomly split the whole sample into two halves denoted as $\mathcal{D}_0, \mathcal{D}_1$. Then for $j = 0, 1$, fit $\hat{h}^{(j)}, \hat{q}^{(j)}$ based on $\mathcal{D}_j$ according to Eqs. (14) and (15), respectively. Finally, redefine the doubly robust GACE estimator by

$$\hat{J}_{\text{DR}} = \frac{1}{n}\sum_{i=1}^{n} \pi(A_i|X_i)\hat{q}^{(I_i)}(Z_i,A_i,X_i)\{Y_i - \hat{h}^{(I_i)}(W_i,A_i,X_i)\} + (\mathcal{T}\hat{h}^{(I_i)})(W_i,X_i),$$

where $I_i = \mathbb{I}\left[i \in \mathcal{D}_0\right]$.

Now we derive the property of the cross-fitted DR estimator in terms of the ill-posedness measures $\tau_{1,n}^{\mathbb{H}}, \tau_{1,n}^{\mathbb{Q}}$ in Eq. (38), *without* assuming unique bridge functions.

Theorem 8. *[Tighter analysis of DR estimator without unique bridge functions] Suppose the conditions in Theorems 3 and 4 hold. Then with probability* $1-\delta$,

$$\|\mathbb{E}[\hat{h}-h_0 \mid A,U,X]\|_2 = O\left(\tau_{1,n}^{\mathbb{H}}\left(\eta_{h,n} + \sqrt{\{1+\log(1/\delta)\}/n}\right)\right), \tag{40}$$

$$\|\mathbb{E}\left[\pi\left(\hat{q}-q_0\right) \mid A,U,X\right]\|_2 = O\left(\tau_{1,n}^{\mathbb{Q}}\left(\eta_{q,n} + \sqrt{\{1+\log(1/\delta)\}/n}\right)\right), \tag{41}$$

$$|\hat{J}_{\text{DR}} - J| = O\left(\tau_{1,n}^{\mathbb{H}}\tau_{1,n}^{\mathbb{Q}}\left(\eta_{h,n} + \sqrt{\{1+\log(1/\delta)\}/n}\right)\left(\eta_{q,n} + \sqrt{\{1+\log(1/\delta)\}/n}\right)\right) \tag{42}$$

Theorem 8 shows that the doubly robust estimator $\hat{J}_{\text{DR}}$ converges faster than the bridge function estimators $\hat{h}, \hat{q}$ in terms of their error measures in Eqs. (40) and (41). In particular, if $\tau_{1,n}^{\mathbb{H}}\tau_{1,n}^{\mathbb{Q}}\eta_{h,n}\eta_{q,n} = o(n^{-1/2})$, *i.e.*, the inverse problems are not too ill-posed, then

$\hat{J}_{\text{DR}}$ is $\sqrt{n}$-consistent. To compare this with the result in Section 5, let us suppose that all relevant function classes are $\Lambda^{\alpha}\left([0,1]^d\right)$ as an example. In this case, the convergence rate here for the doubly robust estimtor based on bridge function estimators *with* stabilizers is $\max(\tau_{1,n}^{\mathbb{H}}\tau_{1,n}^{\mathbb{Q}}n^{-2\alpha/(2\alpha+d)}, n^{-1/2})$, while the convergence rate in Theorem 2 for the estimtor *without* stabilizers is $\max(n^{-\alpha/(2\alpha+d)}, n^{-1/2})$. Therefore, the estimator with stabilizers can achieve faster rate if the ill-posedness measures satisfy $\tau_{1,n}^{\mathbb{H}}\tau_{1,n}^{\mathbb{Q}} = o\left(n^{\alpha/(2\alpha+d)}\right)$. The convergence rate here can be faster primarily because the realizability and closedness assumptions in Theorems 3 and 4 and the assumptions of limited ill-posedness ensure that bridge function estimators converge to valid observed bridge functions in terms of error measures in Eqs. (40) and (41), while under only realizability assumptions in Theorem 2, the bridge function estimators may have no relation to the true bridge functions at all.

However, since bridge functions can be nonunique, the minimax estimators $\hat{h}, \hat{q}$ may not converge to any fixed limit, despite the fact that the error measures in Eqs. (40) and (41) vanish to 0. In this case, even if $\tau_{1,n}^{\mathbb{H}}\tau_{1,n}^{\mathbb{Q}}\eta_{h,n}\eta_{q,n} = o(n^{-1/2})$ so that $\hat{J}_{\text{DR}}$ is $\sqrt{n}$-consistent, certain stochastic equicontinuity term (*i.e.*, Eq. (58) in Appendix E.5) will contribute to nonnegligible but intractable asymptotic variance, so it is difficult to derive the asymptotic distribution of $\hat{J}_{\text{DR}}$. In the following lemma, we further assume that the hypothesis classes target *unique* bridge functions, and show conditions that enable $\hat{J}_{\text{DR}}$ to then have asymptotic normal distribution with a closed-form variance.

Theorem 9. *Suppose that conditions in Theorem 3 and Theorem 4 hold,* $\mathbb{H} \cap \mathbb{H}_0^{\text{obs}} = \{h_0\}$ *and* $\pi\mathbb{Q} \cap \pi\mathbb{Q}_0^{\text{obs}} = \{\pi q_0\}$. *Then*

$$\hat{J}_{\text{DR}} - J = \mathbb{E}_n[\tilde{\phi}_{\text{DR}}(O; h_0, q_0)] + O_p\left(\min(\tau_{2,n}^{\mathbb{H}}, \tau_{2,n}^{\mathbb{Q}})\eta_{q,n}\eta_{h,n}\right).$$

It follows that if $\min(\tau_{2,n}^{\mathbb{H}}, \tau_{2,n}^{\mathbb{Q}})\eta_{q,n}\eta_{h,n} = o(n^{-1/2})$, *then*

$$\sqrt{n}\left(\hat{J}_{\text{DR}} - J\right) \xrightarrow{d} \mathcal{N}\left(0, V_{\textit{eff}}\right),$$

where $V_{\textit{eff}} = \mathbb{E}\left[\tilde{\phi}_{\text{DR}}^2(O; h_0, q_0)\right]$.

In Theorem 9, when $\mathbb{H}, \mathbb{Q}$ can change with n, the assumptions of unique bridge functions should be understood in terms of their limiting spaces. For example, when $\mathbb{H}, \mathbb{Q}$ are linear sieves in Example 10, we can replace the uniqueness assumptions by $\mathbb{H}_0^{\text{obs}} \cap \Lambda^{\alpha}\left([0,1]^d\right) = \{h_0\}$ and $\pi\mathbb{Q}_0^{\text{obs}} \cap \Lambda^{\alpha}\left([0,1]^d\right) = \{\pi q_0\}$. Then we can easily prove the same conclusion given the sieve approximation conditions in Eqs. (32) and (33). Similar to Theorem 8, the convergence rate of $\hat{J}_{\text{DR}}$ in Theorem 9 is faster than the convergence rate of bridge function estimators $\hat{h}$ and $\hat{q}$ in L_2 error measures. However, although $\tau_{1,n}^{\mathbb{H}}\tau_{1,n}^{\mathbb{Q}}\eta_{h,n}\eta_{q,n} = o(n^{-1/2})$ suffices for $\sqrt{n}$-consistency of $\hat{J}_{\text{DR}}$ according to Theorem 8, Theorem 9 suggests that we need $\min(\tau_{2,n}^{\mathbb{H}}, \tau_{2,n}^{\mathbb{Q}})\eta_{q,n}\eta_{h,n} = o\left(n^{-1/2}\right)$ for the asymptotic normality. Therefore, by requiring uniqueness of bridge functions and restricting the ill-posedness of more difficult inverse problems, Theorem 9 proves a stronger asymptotic normality conclusion.

Finally, we remark that the asymptotic variance $V_{\textit{eff}}$ in Theorem 9 coincides with the semiparametric efficiency bound derived in Appendix C under additional regularity conditions. Our results thus extend the asymptotic efficiency results in Cui et al. [2020] who study estimating the average treatment effect with a binary action variable and a focus on parametric estimation of bridge functions.

7 Related Literature

Negative Controls. Miao et al. [2018a] first studied the identification of counterfactual distributions with negative controls under completeness conditions. Their results extend previous literature on identification with proxy variables in nonclassical measurement error models based on stronger assumptions [*e.g.*, Hu and Schennach, 2008, Kuroki and Pearl, 2014]. Shi et al.

[2020] focus on multiply robust estimation of the average treatment effect with negative controls when unmeasured confounders (U) as well as negative controls (Z, W) are categorical. Miao et al. [2018b] proposes to estimate the average treatment effect based on parametric models for a unique outcome bridge function (h_0 in Assumption 2) under completeness conditions similar to those assumed in Miao et al. [2018a]. The estimation and inference is carried out in the standard Generalized Method of Moments (GMM) framework [Hansen, 1982]. Tchetgen et al. [2020] further extend this method to longitudinal studies. Cui et al. [2020] derive the semiparametric efficiency bound for average treatment effect based on both the outcome bridge function and action bridge function, provided that these bridge functions uniquely exist and some completeness conditions also hold. Our work extends these previous literature in several aspects. First, our paper focuses on a generalized average causal effect with a general action space that can be either continuous or discrete, which generalizes the average treatment effect with discrete treatment studied in previous literature (see Example 1). Second, the assumptions in this paper are weaker than the aforementioned literature, in that we allow for nonunique bridge functions[8] and do not assume any completeness condition. See more detailed discussions in Appendices A and B. Third, our minimax estimation procedures accommodate both nonparametric models and parametric models for the bridge functions, while the aforementioned literature only consider parametric estimation.

Deaner [2021] studies the identification of counterfactual mean on the treated, based on both the outcome bridge function and the action bridge function. Notably, Deaner [2021] also allows the bridge functions to be nonunique but does require completeness conditions, albeit weaker than those assumed in Miao et al. [2018a], Cui et al. [2020]. Moreover, Deaner [2021] focuses on nonparametric estimation for the outcome bridge function based on penalized sieve minimum distance estimation [Chen and Pouzo, 2012, 2015], and estimate the final estimand based on an estimating equation analogous to $\tilde{\phi}_{\mathrm{REG}}$ in Lemma 4. Compared to Deaner [2021], the minimax estimation approach in our paper is more flexible and can leverage a wide array of machine learning methods such as RKHS methods and neural networks rather than just sieve methods. In addition, our paper studies estimators based on all three estimating equations, *i.e.*, $\tilde{\phi}_{\mathrm{IPW}}$, $\tilde{\phi}_{\mathrm{REG}}$, and $\tilde{\phi}_{\mathrm{DR}}$ in Lemma 4. Furthermore, we do not assume any completeness condition and instead we provide a new proof for identification in Lemma 4.

Instrumental variable estimation and minimax estimation Estimation of the outcome bridge function h_0 is closely related to the nonparametric instrumental variable (IV) regression problem [Newey, 2013]. Our paper nonetheless differs significantly from these previous literature. First, the conditional moment equation for the action bridge function q_0 in Eq. (6) is distinct from the IV conditional moment equation since it includes an unknown density, and naively applying the minimax reformulations will introduce the extra nuisance. See the discussions below Lemma 6. Second, our target estimand is the GACE J rather than bridge functions, so most of our analysis is substantially different from previous literature. For example, our new analysis shows that our GACE estimator with bridge functions estimated by the first minimax reformulation in Eq. (7) remains consistent even when the bridge function estimators are inconsistent (see Example 7). Despite of these differences, as we mentioned, the estimation of h_0 itself is similar to the IV regression problem. Therefore, we next summarize the relevant literature on IV regression.

The nonparametric IV problem is typically cast into the frame work of conditional moment equations [*e.g.*, Chen and Qiu, 2016, Ai and Chen, 2003]. One classical approach to this problem is a nonparametric analogue of the two-stage least squares (2SLS) method based on sieve estimators [Newey and Powell, 2003] or kernel density estimators [Darolles et al., 2010, Hall, 2005, Carrasco et al., 2007]. Another classical approach is to use sieves to convert the conditional moments into unconditional moments of of increasing dimension [*e.g.*, Chen, 2007, Ai and Chen, 2003, Chen and Qiu, 2016], and then combine all unconditional moments via standard GMM method [Hansen, 1982]. Later, Hartford et al. [2017], Singh et al. [2019] extend the two-stage approach by employing neural network density estimator or conditional mean embedding in

[8]Although Tchetgen et al. [2020], Shi et al. [2020], Miao et al. [2018a] briefly touch upon nonunique bridge functions in parametric models, they do not thoroughly investigate this in their estimation methods.

RKHS respectively in the first stage. Deaner [2021], Singh [2020] apply these methods for the estimation of outcome bridge functions. However, it remains unclear how to incorporate more general hypothesis classes while still providing rigorous theoretical guarantees for nonparametric estimators, on par with results in previous classical literature.

Recently, there have been intense interest in minimax approaches that reformulate the nonparametric IV regression problem via Eq. (7), Eq. (13), or other closely related variants. The resulting minimax formulation is more analogous to the M-estimation (or empirical risk minimization) framework predominant in machine learning and statistics, which naturally permits more general function classes. For example, Lewis and Syrgkanis [2018], Zhang et al. [2020] use the reformulation in Eq. (7), while Dikkala et al. [2020], Chernozhukov et al. [2020], Liao et al. [2020] employ the reformulation in Eq. (13). Moreover, Bennett et al. [2019], Bennett and Kallus [2020], Muandet et al. [2020] consider different variants that are closely related to the reformulation in Eq. (13). However, as we dicuss below Lemma 7, the distinction between the reformulations in Eq. (7) and Eq. (13) are subtle. Notably, Dikkala et al. [2020] provide a thorough analysis of the convergence rates of their minimax estimators. We build on their analysis technique when analyzing the bridge function estimators based on estimation strategy II in Section 4.2 (see Theorem 3).

Minimax estimators have been also employed in a variety of other contexts, such as the estimation of the causal effects [Wong and Chan, 2018, Chernozhukov et al., 2020, Hirshberg and Wager, 2019] and reinforcement learning policy evalution [Feng et al., 2019, Uehara et al., 2020, Yang et al., 2020], when all confounders are measured. Unlike these literature, our paper focuses on the more challenging setting with unmeasured confounders and addresses the confounding problem by using negative controls.

8 Conclusions and Future Work

In this paper we tackled a central challenge in doing causal inferences using negative controls: estimating the bridge functions. We developed an alternative identification strategy that eschewed both uniqueness and completeness assumptions that were imposed on bridge functions in previous approaches and that may be dubious in practice. We proposed new minimax estimators for the bridge functions that were amenable to general function approximation. We studied the behavior of these bridge function estimators and the resulting GACE estimators under a range of different assumptions. Depending on how much one is willing to assume, we showed different rates of convergence. In some settings, GACE estimators remained consistent even when bridge function estimators were not. In other settings, the GACE estimator was shown to achieve the semiparametric efficiency bound.

Our work can be extended to tackle complex estimation in other settings. For example, both Tennenholtz et al. [2020], Lee and Bareinboim [2021] study complex settings where a causal estimand is identified using a proxy or negative-control approach – the first a partially-observable reinforcement learning setting and the second a more general directed acyclic graph setting – but both focus on the setting of *discrete* data distributions for simplicity. By leveraging our work, which allows flexible hypothesis classes, we may be able to tackle these more complicated settings on continuous spaces with lax assumptions and strong guarantees.

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A Comparing Identification Strategies

Literature	Existence of Bridge Fun.	Uniqueness	Completeness Conditions
Miao et al. [2018b]	$\mathbb{H}_0 \neq \emptyset$	Yes	Assumption 5 condition 1
Shi et al. [2020]	$\mathbb{H}_0^{\text{obs}} \neq \emptyset$	Yes[9]	Assumption 5 condition 1
Cui et al. [2020]	$\mathbb{H}_0^{\text{obs}} \neq \emptyset$, $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$	Yes	Assumption 3 conditions 1-2, Assumption 5
Deaner [2021]	$\mathbb{H}_0^{\text{obs}} \neq \emptyset$, $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$	No	Assumption 3 conditions 1-2
Our paper	$\mathbb{H}_0 \neq \emptyset$, $\mathbb{Q}_0 \neq \emptyset$	No	No

Table 2: Assumptions for estimation in negative control literature. Note that Assumption 1 is assumed in all literature so we suppress it in this table. Here "Uniqueness" refers to whether bridge functions are assumed to be unique.

In Table 2, we list the assumptions assumed in different literature on negative controls. One notable difference between our paper and previous literature is that our paper does not assume any completeness condition but previous papers do. In this section, we describe the identification strategy in previous literature that relies on completeness conditions in detail, and compare it with our identification strategy via Lemmas 4 and 5.

We first recall the sets of bridge functions given by the conditional moment equations in Eqs. (2) and (3):

$$\mathbb{H}_0 = \{h \in L_2(W, A, X) : \mathbb{E}[Y - h(W, A, X) \mid A, U, X] = 0\},$$
$$\mathbb{Q}_0 = \{q : \pi q \in L_2(Z, A, X), \mathbb{E}[\pi(A \mid X)\,(q(Z, A, X) - 1/f(A \mid U, X)) \mid A, U, X] = 0\},$$

and the sets of bridge functions given by the observed data conditional moment equations in Eqs. (5) and (6):

$$\mathbb{H}_0^{\text{obs}} = \{h \in L_2(W, A, X) : \mathbb{E}[Y - h(W, A, X) \mid Z, A, X] = 0\},$$
$$\mathbb{Q}_0^{\text{obs}} = \{q : \pi q \in L_2(Z, A, X), \mathbb{E}[\pi(A \mid X)\,(q(Z, A, X) - 1/f(A \mid W, X)) \mid W, A, X] = 0\}.$$

In Lemma 2, we already show that any bridge functions in $\mathbb{H}_0$ and $\mathbb{Q}_0$ can identify the causal parameter. However, we can not directly learn functions in $\mathbb{H}_0$ and $\mathbb{Q}_0$ from the observed data, because they depend on the unmeasured confounders U. Instead, we can only learn functions in $\mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0^{\text{obs}}$ from the observed data. Although in Lemma 3, we already show that $\mathbb{H}_0 \subseteq \mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0 \subseteq \mathbb{Q}_0^{\text{obs}}$, the converse may not be true. In other words, functions in $\mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0^{\text{obs}}$ that we can learn from observed data may not necessarily belong to $\mathbb{H}_0$ and $\mathbb{Q}_0$. Thus without further conditions, we cannot use Lemma 2 for identification, since this lemma only applies to functions in $\mathbb{H}_0$ and $\mathbb{Q}_0$.

Previous literature handle this problem by assuming the following completeness conditions.

Assumption 3. Consider the following conditions:

1. For any $g(A, U, X) \in L_2(A, U, X)$, $\mathbb{E}[g(A, U, X) \mid W, A, X] = 0$ only when $g(A, U, X) = 0$.
2. For any $g(A, U, X) \in L_2(A, U, X)$, $\mathbb{E}[g(A, U, X) \mid Z, A, X] = 0$ only when $g(A, U, X) = 0$.

With these completeness conditions, we can show that $\mathbb{Q}_0^{\text{obs}} = \mathbb{Q}_0$ and $\mathbb{H}_0^{\text{obs}} = \mathbb{H}_0$, namely, any bridge functions that we can learn from the observed data (*i.e.*, functions in $\mathbb{Q}_0^{\text{obs}}$ and $\mathbb{H}_0^{\text{obs}}$) are indeed nominal bridge function (*i.e.*, functions in $\mathbb{Q}_0$ and $\mathbb{H}_0$).

Lemma 12. *Suppose Assumption 1 holds.*

1. *If Assumption 3 condition 1 further holds, then* $\mathbb{Q}_0^{obs} = \mathbb{Q}_0$.
2. *If Assumption 3 condition 2 further holds, then* $\mathbb{H}_0^{obs} = \mathbb{H}_0$.

[9] This paper only very briefly touches on nonuniqueness, and focuses on unique bridge functions in the estimation.

Therefore, by assuming completeness conditions in Assumption 3, previous literature can use Lemma 2 to identify the causal parameter by any $q_0 \in \mathbb{Q}_0^{\text{obs}} = \mathbb{Q}_0$ and/or $h_0 \in \mathbb{H}_0^{\text{obs}} = \mathbb{H}_0$.

Remark 1. Although it appears that Shi et al. [2020], Miao et al. [2018b] do not assume completeness conditions in Assumption 3, their identification strategy is actually the same as those do assume Assumption 3: they impose some other assumption that implicitly ensures $\mathbb{H}_0^{\text{obs}} = \mathbb{H}_0$, so that they can still use Lemma 2 to achieve identification. In Miao et al. [2018b], they assume that $\mathbb{H}_0 \neq \emptyset$, *i.e.*, $\mathbb{H}_0$ contains at least one function, and that $\mathbb{H}_0^{\text{obs}}$ is a singleton. Moreover, Lemma 2 implies $\mathbb{H}_0 \subseteq \mathbb{H}_0^{\text{obs}}$, *i.e.*, $\mathbb{H}_0$ contains at most one function, thus $\mathbb{H}_0$ must be identical to $\mathbb{H}_0^{\text{obs}}$. Shi et al. [2020] studies discrete negative controls and unmeasured confounders (see Example 4), and focuses on the setting where these variables have the same number of values, *i.e.*, $|\mathcal{W}| = |\mathcal{Z}| = |\mathcal{U}|$. In this case, they assume that the matrix $P(\mathbf{W} \mid \mathbf{Z}, a, x)$ is invertible for any $a \in \mathcal{A}, x \in \mathcal{X}$ (see Assumption 5 condition 1 below). It is easy to show that this condition implies that $P(\mathbf{U} \mid \mathbf{Z}, a, x)$ is also invertible, namely, Assumption 3 condition 2 holds. Thus, Shi et al. [2020] implicitly requires $\mathbb{H}_0^{\text{obs}} = \mathbb{H}_0$ as well.

In contrast, our paper assumes $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$ directly, which in turn implies $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ and $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$ according to Lemma 3. Then we prove identification based on functions in $\mathbb{H}_0^{\text{obs}}$ and $\mathbb{Q}_0^{\text{obs}}$ by Lemmas 4 and 5, without assuming any completeness conditions (see Fig. 2 for an illustration). Suprisingly, we show that identification can be even achieved by $h_0 \in \mathbb{H}_0^{\text{obs}} \setminus \mathbb{H}_0$ and $q_0 \in \mathbb{Q}_0^{\text{obs}} \setminus \mathbb{H}_0$ when $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$. Therefore, we do not need any completeness conditions to ensure $\mathbb{H}_0^{\text{obs}} = \mathbb{H}_0$ or $\mathbb{Q}_0^{\text{obs}} = \mathbb{Q}_0$.

Remark 2. In Appendix B.1, we also show another role of Assumption 3: condition 1 with some additional regularity conditions can ensure the existence of outcome bridge function h_0 that satisfies $\mathbb{E}[h_0(W, A, X) \mid A, U, X] = \mu(A, U, X)$, and condition 2 with similar regularity conditions can ensure the existence of action bridge function q_0 that satisfies $\mathbb{E}[\pi(A \mid X)q_0(Z, A, X) \mid A, U, X] = \frac{\pi(A|X)}{f(A|U,X)}$. In other words, completeness conditions 1 and 2 together with additional regularity conditions can ensure our Assumption 2, *i.e.*, $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$.

B Completeness Conditions

In Appendix A, we show that completeness conditions in Assumption 3 play an important role in the identification strategy in previous literature. In this section, we review some related completeness conditions, discuss about their relations, and further describe how our assumptions in Section 2 differ from those in previous literature. For completeness conditions in other settings, such as nonparametric instrumental variable models, see review and discussions in Hu and Shiu [2011], D'Haultfoeuille [2011], Darolles et al. [2010], Newey and Powell [2003]. Throughout this section, we always assume Assumption 1 so we suppress this assumption in all statements.

In the following assumptions, we list two other completeness conditions that also involve the unobserved confounders U. Although these conditions are not directly assumed in previous literature, we will show in Lemma 14 that they are implied by some other conditions assumed in previous literature.

Assumption 4. Consider the following conditions:

1. For any $g(W, A, X) \in L_2(W, A, X)$, $\mathbb{E}[g(W, A, X) \mid A, U, X] = 0$ only when $g(W, A, X) = 0$.
2. For any $g(Z, A, X) \in L_2(Z, A, X)$, $\mathbb{E}[g(Z, A, X) \mid A, U, X] = 0$ only when $g(Z, A, X) = 0$.

In the following lemma, we further show that completeness conditions 3 and 4 in Assumption 3 can ensure the uniqueness of bridge functions.

Lemma 13. *If completeness condition 1 in Assumption 4 holds, then $\mathbb{H}_0$ is either empty or a singleton. If completeness condition 2 in Assumption 4 holds, then $\mathbb{Q}_0$ is either empty or a singleton.*

Next, we introduce two completeness conditions that involve only observed variables.

Assumption 5. Consider the following conditions:

1. For any $g(W, A, X) \in L_2(W, A, X)$, $\mathbb{E}[g(W, A, X) \mid Z, A, X] = 0$ only when $g(W, A, X) = 0$.

2. For any $g(Z, A, X) \in L_2(Z, A, X)$, $\mathbb{E}[g(Z, A, X) \mid W, A, X] = 0$ only when $g(Z, A, X) = 0$.

In the following lemma, we further show the relationship among Assumptions 3 to 5.

Lemma 14. *Assume Assumption 1 holds.*

1. *If Assumptions 3 and 4 hold, then Assumption 5 holds.*
2. *If Assumption 5 holds, then Assumption 4 holds.*

Lemma 14 shows that Assumptions 3 and 4 are sufficient for Assumption 5, and Assumption 5 is sufficient for Assumption 4. Since Assumption 4 ensure unique bridge functions according to Lemma 13, assuming Assumption 5 [Cui et al., 2020] implicitly requires the bridge functions to be unique. However, our paper shows that this uniquenss requirement is too strong by Examples 4 and 5. Thus we intentionally avoid assuming uniqueness as much as possible.

Moreover, just like completeness conditions, conditions in Assumption 3 can ensure $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$ under additional regularity conditions (see Remark 2 and Lemmas 16 and 17), Assumption 5 can also ensure $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ and $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$ under similar regularity conditions. The latter is the starting point of identification strategy in Miao et al. [2018b,a], Cui et al. [2020]. In contrast, our paper starts with assuming $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$ directly without relying on any completeness conditions and additional regularity conditions.

We believe that our approach is desirable for several reasons. First, completeness conditions and the associated regularity conditions are generally difficult to interpret or verify [Newey and Powell, 2003], so it is appealing to get rid of these conditions. Second, as we show in Examples 4 and 5, confounding models that relate unmeasured confounders to negative controls may directly inform sufficient conditions for $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$, while it is less straightforward to do so for $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ and $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$. This is probably also one reason why previous literature often turn to opaque completeness conditions to ensure $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ and $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$ rather than motivate it from data generating processes. Third, requiring completeness conditions in Assumption 5 to ensure $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ and $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$ may result in stronger restrictions than requiring $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$. Take the discrete setting in Example 4 as an example. We already show that under the rank conditions in Example 4, $\mathbb{H}_0 \neq \emptyset$ and $\mathbb{Q}_0 \neq \emptyset$ need $|\mathcal{W}| \geq |\mathcal{U}|$ and $|\mathcal{Z}| \geq |\mathcal{U}|$. In contrast, if we assume completeness conditions in Assumption 5 to justify $\mathbb{H}_0^{\text{obs}} \neq \emptyset$ and $\mathbb{Q}_0^{\text{obs}} \neq \emptyset$, then we would require $|\mathcal{W}| = |\mathcal{Z}| = |\mathcal{U}|$.

Proposition 1. Consider the setting in Example 4. Assume Assumption 1 holds and matrices $P(\mathbf{W} \mid \mathbf{U}, a, x) \in \mathbb{R}^{|\mathcal{W}| \times |\mathcal{U}|}$, $P(\mathbf{U} \mid \mathbf{W}, a, x) \in \mathbb{R}^{|\mathcal{U}| \times |\mathcal{W}|}$, $P(\mathbf{Z} \mid \mathbf{U}, a, x) \in \mathbb{R}^{|\mathcal{Z}| \times |\mathcal{U}|}$, $P(\mathbf{U} \mid \mathbf{Z}, a, x) \in \mathbb{R}^{|\mathcal{U}| \times |\mathcal{Z}|}$ all have rank $|\mathcal{U}|$ for any $a \in \mathcal{A}, x \in \mathcal{X}$.

1. if condition 1 in Assumption 5 holds, then $|\mathcal{Z}| \geq |\mathcal{W}| = |\mathcal{U}|$.
2. if condition 2 in Assumption 5 holds, then $|\mathcal{W}| \geq |\mathcal{Z}| = |\mathcal{U}|$.

Note the rank conditions alone only require $|\mathcal{W}| \geq |\mathcal{U}|$ and $|\mathcal{Z}| \geq |\mathcal{U}|$.

B.1 Completeness Conditions and Existence of Bridge Functions

In this section, we show the existence of bridge functions in Eqs. (2) and (3) under the completeness conditions in Assumption 3 conditions 1 and 2 and some additional regularity conditions, using the singular value decomposition approach in Kress [2014], Miao et al. [2018a], Carrasco et al. [2007].

Characterize the linear operators associated with Eqs. (2) and (3). Let $K_{W|a,x}: L_2(W \mid A = a, X = x) \to L_2(U \mid A = a, X = x)$ and $K_{Z|a,x}: L_2(Z \mid A = a, X = x) \to L_2(U \mid A = a, X = x)$ be the linear operators defined as follows:

$$[K_{W|a,x}h](a, u, x) = \mathbb{E}[h(W, a, x) \mid A = a, U = u, X = x] = \int K(w, a, u, x)h(w, a, x)f(w \mid a, x)\mathrm{d}\mu(w),$$

$$[K_{Z|a,x}q](a, u, x) = \mathbb{E}[q(Z, a, x) \mid A = a, U = u, X = x] = \int K'(z, a, u, x)h(z, a, x)f(z \mid a, x)\mathrm{d}\mu(w),$$

where $K(w,a,u,x)$ and $K'(z,a,u,x)$ are the corresponding kernel functions defined as follows:

$$K(w,a,u,x) = \frac{f(w,u \mid a,x)}{f(u \mid a,x)f(w \mid a,x)}, \quad K'(z,a,u,x) = \frac{f(z,u \mid a,x)}{f(u \mid a,x)f(z \mid a,x)}.$$

Their adjoint operators $K^*_{W|a,x} : L_2(U \mid A=a, X=x) \to L_2(W \mid A=a, X=x)$ and $K^*_{Z|a,x} : L_2(U \mid A=a, X=x) \to L_2(Z \mid A=a, X=x)$ are given as follows:

$$[K^*_{W|a,x}g](w,a,x) = \int K(w,a,u,x)g(u,a,x)f(u \mid a,x)\mathrm{d}\mu(w) = \mathbb{E}\left[g(U,a,x) \mid W=w, A=a, X=x\right],$$

$$[K_{Z|a,x}g](z,a,x) = \int K'(z,a,u,x)g(a,u,x)f(u \mid a,x)\mathrm{d}\mu(w) = \mathbb{E}\left[g(U,a,x) \mid Z=z, A=a, X=x\right].$$

The existence of bridge functions is equivalent to existence of solutions to the following equations of the first kind:

$$[K_{W|a,x}h](a,u,x) = k_0(a,u,x), \quad [K_{Z|a,x}q](a,u,x) = 1/f(a \mid u,x), \quad \text{a.e. } u,a,x \text{ w.r.t } \mathbb{P}.$$

To ensure this existence, we further assume the following conditions.

Assumption 6. Assume that for almost every a, x:

1. $\iint f(w \mid u,a,x)f(u \mid w,a,x)\mathrm{d}\mu(w)\mathrm{d}\mu(u) < \infty$.
2. $\iint f(z \mid u,a,x)f(u \mid z,a,x)\mathrm{d}\mu(z)\mathrm{d}\mu(u) < \infty$.

According to Example 2.3 in Carrasco et al. [2007, P5659], Assumption 6 ensures that both $K_{W|a,x}$ and $K_{Z|a,x}$ are compact operators. Then by Theorem 2.41 in Carrasco et al. [2007, P5660], both $K_{W|a,x}$ and $K_{Z|a,x}$ admit singular value decomposition: there exist

$$\left(\lambda^j_{W|a,x}, \varphi^j_{W|a,x}, \psi^j_{W|a,x}\right)_{j=1}^{+\infty}, \quad \left(\lambda^j_{Z|a,x}, \varphi^j_{Z|a,x}, \psi^j_{Z|a,x}\right)_{j=1}^{+\infty}$$

with orthonormal sequences

$$\left\{\varphi^j_{W|a,x} \in L_2(W \mid a,x)\right\}, \left\{\varphi^j_{Z|a,x} \in L_2(Z \mid a,x)\right\}, \left\{\psi^j_{W|a,x} \in L_2(U \mid a,x)\right\}, \left\{\psi^j_{Z|a,x} \in L_2(U \mid a,x)\right\}$$

such that

$$K_{W|a,x}\varphi^j_{W|a,x} = \lambda^j_{W|a,x}\psi^j_{W|a,x}, \quad K_{Z|a,x}\varphi^j_{Z|a,x} = \lambda^j_{Z|a,x}\psi^j_{Z|a,x}.$$

Existence of bridge functions. Following Miao et al. [2018a], we use the Picard's Theorem [Kress, 2014, Theorem 15.18] to characterize the existence of solutions to equations of the first kind by the singular value decomposition of the associated operators.

Lemma 15 (Picard's Theorem)**.** *Let $K : H_1 \to H_2$ be a compact operator with singular system $(\lambda_j, \varphi_j, \psi_j)_{j=1}^{+\infty}$, and ϕ be a given function in H_2. Then the equation of the first kind $Kh = \phi$ have solutions if and only if*

1. *$\phi \in \mathcal{N}(K^*)^{\perp}$, where $\mathcal{N}(K^*) = \{h : K^*h = 0\}$ is the null space of the adjoint operator K^*.*
2. *$\sum_{n=1}^{+\infty} \lambda_n^{-2} |\langle\phi, \psi_n\rangle|^2 < +\infty$.*

In the following two lemmas, we show the existence of bridge functions under the completeness conditions.

Lemma 16. *Assume Assumption 3 condition 1, Assumption 6 condition 1 and the following conditions for almost all a, x:*

- *$k_0(a,U,x) \in L_2(U \mid A=a, X=x)$.*
- *$\sum_{j=1}^{+\infty} \left(\lambda^j_{W|a,x}\right)^{-2} \left\{\int k_0(a,u,x)\psi^j_{W|a,x}(u,a,x)\mathrm{d}\mu(u)\right\}^2 < \infty$.*

Then there exists function $h_0 \in L_2(W|A=a, X=x)$ for almost all a, x such that

$$\mathbb{E}\left[Y - h_0(W,A,X) \mid A,U,X\right] = 0.$$

Lemma 17. *Assume Assumption 3 condition 2 and Assumption 6 condition 2 and the following conditions for almost all a, x:*

- $\frac{\pi(a|x)}{f(a|U,x)} \in L_2(U \mid A = a, X = x)$.
- $\sum_{j=1}^{+\infty} \left(\lambda^j_{Z|a,x}\right)^{-2} \left\{\int \frac{\pi(a|x)}{f(a|u,x)} \psi^j_{Z|a,x}(u, a, x) \mathrm{d}\mu(u)\right\}^2 < \infty$.

Then there exists function q_0 such that

$$\mathbb{E}\left[\pi\left(A \mid X\right) q_0(Z, A, X) \mid A, U, X\right] = \frac{\pi\left(A \mid X\right)}{f(A \mid U, X)}$$

and $\pi(a|x)q_0(Z, a, x) \in L_2(Z|A = a, X = x)$ for almost all a, x.

C Semiparametric Efficiency Bound

In this section, we derive the semiparametric efficiency bound for $J = \mathbb{E}\left[\int Y(a)\pi(a|X)\mathrm{d}(a)\right]$ under a nonparametric model $\mathcal{M}_{np}$ in which bridge functions h_0 and q_0 are unrestricted. Crucially, our efficiency analysis follows Severini and Tripathi [2012] and does *not* require unique existence of bridge functions. Therefore, our analysis strictly generalizes the efficiency results in Cui et al. [2020]. When specialized to the average treatment effect with discrete treatments and unique bridge functions, our efficiency bound coincides with the bound in Theorem 3.1 of Cui et al. [2020].

We first recall that $P_z : L_2(W, A, X) \to L_2(Z, A, X)$ is the conditional expectation operator given by $P_z(g)(Z, A, X) \coloneqq \mathbb{E}[g(W, A, X) \mid Z, A, X]$, and $P_w : L_2(Z, A, X) \to L_2(W, A, X)$ is given by $P_w(g)(W, A, X) \coloneqq \mathbb{E}[g(Z, A, X) \mid W, A, X]$. For any $h \in L_2(W, A, X)$, define $P_{\mathcal{N}(P_z)}h$ and $P_{\mathcal{N}(P_z)^\perp}h$ as the projection of h onto the null space of the operator P_z and its orthogonal complement respectively. Similarly, we can define $P_{\mathcal{N}(P_w)}\pi q$ and $P_{\mathcal{N}(P_w)^\perp}\pi q$ for any $q \in L_2(Z, A, X)$.

As we show in Lemma 4, the GACE J can be identified by any bridge functions that satisfy Eqs. (5) and (6), namely, any bridge functions in $\mathbb{H}_0^{\mathrm{obs}}$ and $\mathbb{Q}_0^{\mathrm{obs}}$. Although these bridge functions may not be unique, the following lemma shows that their projections onto the orthogonal complement of the null space of the corresponding linear operator are always unique.

Lemma 18. *For any $h_0, h_0' \in \mathbb{H}_0^{obs}$ and $q_0, q_0' \in \mathbb{Q}_0^{obs}$,*

$$P_{\mathcal{N}(P_z)^\perp}h_0 = P_{\mathcal{N}(P_z)^\perp}h_0', \quad \pi P_{\mathcal{N}(P_w)^\perp}q_0 = \pi P_{\mathcal{N}(P_w)^\perp}q_0'.$$

Given Lemma 18, we can define $h_0^* = P_{\mathcal{N}(P_z)^\perp}\mathbb{H}_0^{\mathrm{obs}}$ and $\pi q_0^* = \pi P_{\mathcal{N}(P_w)^\perp}\mathbb{Q}_0^{\mathrm{obs}}$. If bridge functions happen to be unique, *i.e.*, $\mathbb{H}_0^{\mathrm{obs}} = \{h_0\}$ and $\pi\mathbb{Q}_0^{\mathrm{obs}} = \{\pi q_0\}$, then obviously $h_0^* = h_0$ and $\pi q_0^* = \pi q_0$.

In order to derive the efficiency bound, we further assume the following regularity conditions under the model $\mathcal{M}_{np}$.

Assumption 7. 1. Suppose

$$\begin{aligned}
&\mathbb{E}[(Y - h_0^*(W, A, X))\,\mathcal{T}h_0^*(W, X) \\
&\qquad + \pi(A \mid X)q_0^*(Z, A, X)\,(Y - h_0^*(W, A, X))^2 \mid Z, A, X] \in \mathrm{Range}(P_z), \quad (43)\\
&\mathbb{E}\left[(\pi(A \mid X))^2\, q_0^{*2}(Z, A, X)\,(\mathbb{E}\left[Y \mid W, Z, A, X\right] - h_0^*(W, A, X)) \mid W, A, X\right] \\
&-\mathbb{E}\left[\pi(A \mid X)q_0^*(Z, A, X)\mathbb{E}\left[\pi(A \mid X)q_0^*(Z, A, X)\,(\mathbb{E}\left[Y \mid W, Z, A, X\right] - h_0^*(W, A, X)) \mid W, X\right] \mid W, A, X\right] \\
&\qquad \in \mathrm{Range}\left(P_w\right). \quad (44)
\end{aligned}$$

2. For almost all $a \in \mathcal{A}, z \in \mathcal{Z}, x \in \mathcal{X}$,

$$\mathbb{E}\left[(\pi(A \mid X)q_0^*(Z, A, X))^2\right] < \infty, \quad \mathbb{E}\left[(\mathcal{T}h_0^*(W, X) - J)^2\right] < \infty,$$
$$\mathbb{E}\left[(Y - h_0^*(W, A, X))^2 \mid A = a, Z = z, X = x\right] < \infty.$$

In Assumption 7, Eq. (43) is ensured by the Assumption 10(3) in Cui et al. [2020], if the bridge functions are unique and J is average treatment effect for discrete treatments. We choose to assume Eq. (43) directly rather than conditions more similar to those in Cui et al. [2020] to avoid inverting some complicated operators in Cui et al. [2020]. Moreover, Eq. (44) is completely new. This condition is needed to ensure that the efficient influence function $\mathrm{EIF}(J)$ proposed in Theorem 10 satisfies the semiparametric restriction due to the existence of action bridge functions function q_0. Theorem 3.1 in Cui et al. [2020] appears to omit this restriction in their proof.

Theorem 10. *Under Assumptions 1, 2 and 7, the efficient influence function of J relative to the tangent space implied by $\mathcal{M}_{np}$ (i.e., $\mathcal{S}$ in in Eq. (67)) is the following:*

$$\mathrm{EIF}(J) = \pi(A \mid X)q_0^*(Z, A, X)[Y - h_0^*(W, A, X)] + \mathcal{T}h_0^*(W, X) - J.$$

The corresponding semiparametric efficiency bound of J is $V_{eff} = \mathbb{E}\left[\mathrm{EIF}^2(J)\right]$.

When the bridge functions happen to be unique, the efficient influence function and efficiency bound easily follow from Theorem 10.

Corollary 6. *Under the assumptions in Theorem 10, if further the bridge functions are unique, i.e., $\mathbb{H}_0^{obs} = \{h_0\}$ and $\pi\mathbb{Q}_0^{obs} = \{\pi q_0\}$, then the efficient influence function of J relative to the tangent space implied by $\mathcal{M}_{np}$ (i.e., $\mathcal{S}$ in in Eq. (67)) is the following:*

$$\mathrm{EIF}(J) = \pi(A \mid X)q_0(Z, A, X)[Y - h_0(W, A, X)] + \mathcal{T}h_0(W, X) - J.$$

The corresponding semiparametric efficiency bound of J is $V_{eff} = \mathbb{E}\left[\mathrm{EIF}^2(J)\right]$.

In Theorem 9, we show that our GACE estimator proposed in Section 6.3 can attain the efficiency bound in Corollary 6 when the bridge functions are unique.

D Finite Sample Results with Bias Terms

D.1 Extension of Theorem 1 and Theorem 2

We present a finite sample error bound when we use minimax estimators without stabilizers. Comparing to Theorem 1, the following theorems hold without assuming realizability assumptions.

Theorem 11. *For any $h \in \mathbb{H}_0^{\mathrm{obs}}, q \in \mathbb{Q}_0^{\mathrm{obs}}$,*

$$\begin{aligned}|\hat{J}_{\mathrm{IPW}} - J| &\le \sup_{q\in\mathbb{Q}} |(\mathbb{E} - \mathbb{E}_n)[q\pi Y]| + 4 \sup_{q\in\mathbb{Q}, h\in\mathbb{H}'} |(\mathbb{E} - \mathbb{E}_n)[-q\pi h + \mathcal{T}h]| + \\ &+ \inf_{h\in\mathbb{H}'} \sup_{q\in\mathbb{Q}} |\mathbb{E}[(q - q_0)\pi(h_0 - h)]| + \inf_{q\in\mathbb{Q}} \sup_{h\in\mathbb{H}'} |\mathbb{E}[(q_0 - q)\pi h]|.\end{aligned}$$

Theorem 12. *For any $h \in \mathbb{H}_0^{\mathrm{obs}}, q \in \mathbb{Q}_0^{\mathrm{obs}}$,*

$$\begin{aligned}|\hat{J}_{\mathrm{REG}} - J| &\le \sup_{h\in\mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[\mathcal{T}h]| + 4 \sup_{q\in\mathbb{Q}', h\in\mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q(Y - h)]| \\ &+ \inf_{h\in\mathbb{H}} \sup_{q\in\mathbb{Q}'} |\mathbb{E}[\{h_0 - h\}q]| + \inf_{q\in\mathbb{Q}'} \sup_{h\in\mathbb{H}} |\mathbb{E}[(q_0\pi - q)(h - h_0)|.\end{aligned}$$

In each theorem, our assumption is only the existence of bridge functions, *i.e.*, Assumption 2. The first term and the second term are variance terms, which converge to 0. The third term and the fourth term are bias terms, which may remain even if the sample size goes to infinity. In Theorem 1, we show that these bias terms are zero under additional realizability conditions.

The following is an extension of Theorem 2.

Theorem 13. *For any $h \in \mathbb{H}_0^{\mathrm{obs}}, q \in \mathbb{Q}_0^{\mathrm{obs}}$,*

$$
\begin{aligned}
|\hat{J}_{\mathrm{DR}} - J| \le & \sup_{q \in \mathbb{Q}, h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q\pi Y - q\pi h + \mathcal{T}h]| + 4 \sup_{q \in \mathbb{Q}, h \in \mathbb{H}'} |(\mathbb{E} - \mathbb{E}_n)[-q\pi h + \mathcal{T}h]| + \\
& + \inf_{q \in \mathbb{Q}} \sup_{h \in \mathbb{H}'} |\mathbb{E}[(q_0 - q)\pi h]| + \inf_{h \in \mathbb{H}'} \sup_{h' \in \mathbb{H}} \sup_{q \in \mathbb{Q}} |\mathbb{E}[(q - q_0)\pi(h_0 - h' - h)]|, \\
|\hat{J}_{\mathrm{DR}} - J| \le & \sup_{q \in \mathbb{Q}, h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q\pi Y - q\pi h + \mathcal{T}h]| + 4 \sup_{q \in \mathbb{Q}', h \in \mathbb{H}} |(\mathbb{E} - \mathbb{E}_n)[q(Y - h)]| \\
& + \inf_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} |\mathbb{E}[\{h_0 - h\}q]| + \inf_{q \in \mathbb{Q}'} \sup_{q' \in \mathbb{Q}} \sup_{h \in \mathbb{H}} |\mathbb{E}[(q_0\pi - q'\pi - q)(h - h_0)|.
\end{aligned}
$$

Again, in the theorem, the first and second terms are variance terms, and the third and fourth terms are bias terms, which may remain even if the sample size goes to infinity. In Theorem 2, we show that these bias terms are zero under additional realizability conditions.

D.2 Extension of Theorems 3 and 4

Theorem 14. *Suppose $\|\mathbb{H}\|_\infty, \|Y\|_\infty \le C_{\mathbb{H}}$. We take an arbitrary $h_0 \in \mathbb{H}_0^{\mathrm{obs}}$ and $h' \in \mathbb{H}$. Define $\epsilon_{h'} := \sup_{h \in \mathbb{H}} \inf_{q \in \mathbb{Q}'} \|q - P_z(h - h')\|$. Additionally, assume $\mathbb{Q}'$ is symmetric and star-shaped. Let η_h be an upper bound on the critical radius of $\mathbb{Q}'$ and $\mathcal{G}_h$, where*

$$\mathcal{G}_h := \{(x, a, w, z) \mapsto \{(h - h')(w, a, x)\}q(x, a, z); h \in \mathbb{H}, q \in \mathbb{Q}'\}.$$

Then, with probability $1 - \delta$,

$$\|P_z(\hat{h} - h_0)\|_2 \le \left(\left\{ 1 + \frac{\lambda^2 + C_{\mathbb{H}}^2}{\lambda} + C_{\mathbb{H}} \right\} \eta'_h + \epsilon_n + \frac{\|P_z(h_0 - h')\|_2}{\lambda \eta'_h} + \|P_z(h' - h_0)\|_2 \right)$$

where $\eta'_h = \eta_h + \sqrt{c_0 \log(c_1/\delta)/n}$ with some universal constants c_0, c_1, c_2.

When $P_z(\mathbb{H} - \mathbb{H}_0^{\mathrm{obs}}) \subset \mathbb{Q}', \mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H} \neq \emptyset$, we can take $h' = h_0 \in \mathbb{H}_0^{\mathrm{obs}} \cap \mathbb{H}$ and we have $\epsilon_{h'} = 0$, so that Theorem 14 reduces to Theorem 4.

Theorem 15. *Suppose $\|\pi\mathbb{Q}\|_\infty \le C_{\mathbb{Q}}$, $\|\pi(a|x)/f(a|x, u)\|_2 < \infty$. We take an arbitrary $q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$ and $q' \in \mathbb{Q}$. Define $\epsilon_n = \sup_{q \in \mathbb{Q}} \inf_{h \in \mathbb{H}'} \|h - P_w(q\pi - q'\pi)\|_2$. Additionally, assume $\mathbb{H}'$ is symmetric and star-shaped. Let η_q be an upper bound on the critical radius of $\mathbb{H}'$ and $\mathcal{G}_q$, where*

$$\mathcal{G}_q := \{(x, a, w, z) \mapsto \{(q - q')(x, a, z)\}\pi(a|x)h(x, a, w); q \in \mathbb{Q}, h \in \mathbb{H}'\}.$$

Then, with probability $1 - \delta$,

$$\|P_w(\hat{q}\pi - q_0\pi)\|_2 \le c_2 \left(\left\{ 1 + \frac{\lambda^2 + C_{\mathbb{Q}}^2}{\lambda} + C_{\mathbb{Q}} \right\} \eta'_q + \epsilon_n + \frac{\|P_w(q'\pi - q_0\pi)\|_2}{\lambda \eta'_q} + \|P_w(q'\pi - q_0\pi)\|_2 \right)$$

where $\eta'_q = \eta_q + \sqrt{c_0 \log(c_1/\delta)/n}$ with some universal constants c_0, c_1, c_2.

Similarly, when $\pi P_w(\mathbb{Q} - \mathbb{Q}_0^{\mathrm{obs}}) \subset \mathbb{H}', \mathbb{Q} \cap \mathbb{Q}_0^{\mathrm{obs}} \neq \emptyset$, Theorem 15 reduces to Theorem 3.

E Proofs

In all of the proofs, when we assume $\|Y\|_\infty, \|\mathbb{H}\|_\infty, \|\mathbb{Q}\|_\infty, \|\pi/f(a|x, w)\|_2$ are finite, we denote their upper bounds as $C_Y, C_{\mathbb{H}}, C_{\mathbb{Q}}, C_w$ respectively.

E.1 Proofs for Section 2

Proof of Lemma 1. We first prove the identification via ϕ_{REG}:

$$\begin{aligned}
J &= \mathbb{E}\left\{\int Y(a)\pi(a|X)\mathrm{d}\mu(a)\right\} \\
&= \mathbb{E}\left\{\int \mathbb{E}[Y(a)|U,X]\pi(a|X)\mathrm{d}\mu(a)\right\} && \text{(Tower property)} \\
&= \mathbb{E}\left\{\int \mathbb{E}[Y(a)|U,X,A=a]\pi(a|X)\mathrm{d}\mu(a)\right\} && (Y(a) \perp A|U,X) \\
&= \mathbb{E}\left\{\int \mathbb{E}[Y|U,X,A=a]\pi(a|X)\mathrm{d}\mu(a)\right\} = \mathbb{E}\left[\phi_{\mathrm{REG}}\left(Y,A,U,X;k_0\right)\right]. && \text{(Consistency)}
\end{aligned}$$

We next prove the identification via ϕ_{IPW}:

$$\begin{aligned}
J &= \mathbb{E}\left\{\int \mathbb{E}[Y|U,X,A=a]\pi(a|X)\mathrm{d}\mu(a)\right\} \\
&= \mathbb{E}\left\{\int \mathbb{E}[Y|U,X,A=a]\{\pi(a|X)/f(a|X,U)\} \times f(a|X,U)\mathrm{d}\mu(a)\right\} \\
&= \mathbb{E}\left\{\mathbb{E}[Y|U,X,A]\pi(A|X)/f(A|X,U)\right\} \\
&= \mathbb{E}\left\{Y\pi(A|X)/f(A|X,U)\right\} = \mathbb{E}\left[\phi_{\mathrm{IPW}}\left(Y,A,U,X;f\right)\right].
\end{aligned}$$

Finally, the identification via ϕ_{DR} follows from the fact that

$$\mathbb{E}\left[\phi_{\mathrm{DR}}\left(Y,A,U,X;k_0,f\right)\right] = \mathbb{E}\left[\phi_{\mathrm{REG}}\left(Y,A,U,X;k_0\right)\right] = J.$$

$\square$

Bridge functions in Example 5. Here we show the details of deriving bridge functions in Example 5.

We first derive the function $h_0(W,A,X)$. If $\theta_w^\top \alpha_W = \alpha_Y^\top$,then

$$\theta_W^\top W = \alpha_Y^\top U + \theta_W^\top \beta_W X + \theta_W^\top \epsilon_W \implies \alpha_Y^\top U = \theta_W^\top W - \theta_W^\top \beta_W X - \theta_W^\top \epsilon_W.$$

Therefore,

$$\begin{aligned}
Y &= (\theta_W + \omega_Y)^\top W + \left(\beta_Y^\top - \theta_W^\top \beta_W\right) X + \omega_W^\top W + \epsilon_Y - \theta_W^\top \epsilon_W \\
&= h_w(W,A,X) + \epsilon_Y - \theta_W^\top \epsilon_W.
\end{aligned}$$

It follows from the independence of ϵ_Y, ϵ_W with A,U,X that

$$\mathbb{E}\left[Y \mid A,U,X\right] = \mathbb{E}\left[h_0(W,A,X) \mid A,U,X\right] + \mathbb{E}\left[\epsilon_Y - \theta_W^\top \epsilon_W\right] = \mathbb{E}\left[h_0(W,A,X) \mid A,U,X\right].$$

Now we derive $q_0(Z,A,X)$. First note that $1/f(A \mid U,X) = 1/\left((\overline{\alpha}_A - \underline{\alpha}_A)U + \left(\overline{\beta}_A - \underline{\beta}_A\right)X\right)$. Because $\theta_Z^\top \alpha_Z = \overline{\alpha}_A^\top - \underline{\alpha}_A^\top$,

$$\theta_Z^\top Z = (\overline{\alpha}_A - \underline{\alpha}_A)^\top U + \theta_Z^\top \beta_Z X + \theta_Z^\top \gamma_Z A + \theta_Z^\top \epsilon_Z,$$

which means that

$$\begin{aligned}
q_0(Z,A,X) &= \theta_Z^\top Z + \left(\overline{\beta}_A - \underline{\beta}_A - \theta_Z^\top \beta_Z\right) X - \theta_Z^\top \gamma_Z A \\
&= (\overline{\alpha}_A - \underline{\alpha}_A)^\top U + \left(\overline{\beta}_A - \underline{\beta}_A\right)^\top X + \theta_Z^\top \epsilon_Z.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[q_0(Z,A,X) \mid A,U,X\right] = (\overline{\alpha}_A - \underline{\alpha}_A)^\top U + \left(\overline{\beta}_A - \underline{\beta}_A\right)^\top X + \mathbb{E}\left[\theta_Z^\top \epsilon_Z\right] = 1/f(A \mid U,X).$$

$\square$

Proof for Lemma 2. We derive the equations in Lemma 2 one by one.

First,

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\phi}_{\mathrm{IPW}}(O;q_0)\right] &= \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X)Y \mid A,U,X\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X) \mid A,U,X\right]\mathbb{E}\left[Y \mid A,U,X\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X)/f(A \mid U,X) \mid A,U,X\right]\mathbb{E}\left[Y \mid A,U,X\right]\right] \\
&= \mathbb{E}\left[\pi(A \mid X)Y/f(A \mid U,X)\right] \\
&= \mathbb{E}\left[\int \pi\left(a \mid X\right)\mathbb{E}\left[Y(a) \mid A=a,U,X\right]\mathrm{d}\mu(a)\right] \\
&= \mathbb{E}\left[\int \pi\left(a \mid X\right)\mathbb{E}\left[Y(a) \mid U,X\right]\mathrm{d}\mu(a)\right] = J.
\end{aligned}
$$

Here the second equality follows from $Y \perp Z \mid A,U,X$ and the third equality follows from the definition of q_0 according to Eq. (3).

Second,

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\phi}_{\mathrm{REG}}(O;h_0)\right] &= \mathbb{E}\left[\int \pi(a \mid X)h_0(W,a,X)\mathrm{d}\mu(a)\right] \\
&= \mathbb{E}\left[\int \pi(a \mid X)\mathbb{E}\left[h_0(W,a,X) \mid A=a,U,X\right]\mathrm{d}\mu(a)\right] \\
&= \mathbb{E}\left[\int \pi(a \mid X)\mathbb{E}\left[Y \mid A=a,U,X\right]\mathrm{d}\mu(a)\right] \\
&= J.
\end{aligned}
$$

Here the second equality follows from $W \perp A \mid U,X$ and the third equality follows from the definition of h_0 according to Eq. (2).

Finally,

$$
\begin{aligned}
\mathbb{E}\left[\tilde{\phi}_{\mathrm{DR}}(O;h_0,q_0)\right] &= \mathbb{E}\left[\tilde{\phi}_{\mathrm{REG}}(O;h_0,q_0)\right] + \mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X)\left(Y-h_0(W,A,X)\right)\right] \\
&= J + \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X)\left(Y-h_0(W,A,X)\right) \mid A,U,X\right]\right] \\
&= J + \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X) \mid A,U,X\right]\mathbb{E}\left[Y-h_0(W,A,X) \mid A,U,X\right]\right] \\
&= J.
\end{aligned}
$$

Here the third equality follows from that $Z \perp Y \mid A,U,X$ and the last equality follows from the definition of h_0 according to Eq. (2). □

Proof for Lemma 3. For any $h_0 \in \mathbb{H}_0$, we have that

$$
\mathbb{E}\left[Y-h_0(W,A,X) \mid Z,A,U,X\right] = \mathbb{E}\left[Y-h_0(W,A,X) \mid A,U,X\right] = 0,
$$

where the first equality holds because $(W,Y) \perp Z \mid A,U,X$. Therefore,

$$
\mathbb{E}[Y-h_0(W,A,X) \mid Z,A,X] = \mathbb{E}\left[\mathbb{E}\left[Y-h_0(W,A,X) \mid Z,A,U,X\right] \mid Z,A,X\right] = 0.
$$

For any $q_0 \in \mathbb{Q}_0$, we have that

$$
\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X) \mid W,A,U,X\right] = \mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X) \mid A,U,X\right] = \frac{\pi(A \mid X)}{f(A \mid U,X)},
$$

where the first equality follows from $Z \perp W \mid A,U,X$. Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X) \mid W,A,X\right] &= \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X) \mid W,A,U,X\right] \mid W,A,X\right] \\
&= \mathbb{E}\left[\frac{\pi(A \mid X)}{f(A \mid U,X)} \mid W,A,X\right].
\end{aligned}
$$

Then the conclusion follows from the fact that

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{f(A \mid U, X)} \mid W, A, X\right] &= \int \frac{1}{f(A \mid u, X)} f(u \mid W, A, X) \mathrm{d}(u) \\
&= \int \frac{f(W, A \mid u, X) f(u, X)}{f(A \mid u, X) f(W, A, X)} \mathrm{d}\mu(u) \\
&= \int \frac{f(W \mid u, X) f(u, X)}{f(W, A, X)} \mathrm{d}\mu(u) \\
&= \frac{1}{f(A \mid W, X)}.
\end{aligned}$$

Here the third equality follows from the fact that $W \perp A \mid U, X$.

In summary, the condition that $(W, Y) \perp Z \mid A, U, X$ ensures Eq. (5) holds, while the condition that $(Z, A) \perp W \mid U, X$ ensures Eq. (6) holds. □

Proof for Lemma 4. The conclusions that $\mathbb{E}\left[\tilde{\phi}_{\mathrm{REG}}\left(O; h_0\right)\right] = \mathbb{E}\left[\tilde{\phi}_{\mathrm{IPW}}\left(O; q_0\right)\right]$ for any $h_0 \in \mathbb{H}_0^{\mathrm{obs}}$ and $q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$ easily follows from Lemma 5 and that $h_0 \in \mathbb{H}_0^{\mathrm{obs}}$ and $q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$ satisfy Eqs. (5) and (6).

Note that for any $h_0 \in \mathbb{H}_0^{\mathrm{obs}}$ and $q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$,

$$\begin{aligned}
\mathbb{E}\left[\tilde{\phi}_{\mathrm{DR}}(O; h_0, q_0)\right] &= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X)\left(Y - h_0(W, A, X)\right)\right] + \mathbb{E}\left[(\mathcal{T} h_0)(X, W)\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}\left[Y - h_0(W, A, X) \mid Z, A, X\right]\right] + \mathbb{E}\left[\tilde{\phi}_{\mathrm{REG}}(O; q_0)\right] \\
&= \mathbb{E}\left[\tilde{\phi}_{\mathrm{REG}}(O;; h_0, q_0)\right] = J.
\end{aligned}$$

Here the second equation again follows from the definition of h_0 according to Assumption 2. □

Proof for Lemma 5. Before proving the conclusion, note that for any $h \in L_2(W, A, X)$ and $q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$,

$$\begin{aligned}
\mathbb{E}\left[q_0(Z, A, X) \pi(A \mid X) h(W, A, X)\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[q_0(Z, A, X) \pi(A|X) \mid W, A, X\right] h(W, A, X)\right]\right] \\
&= \mathbb{E}\left[\frac{\pi(A \mid X)}{f(A \mid W, X)} h(W, A, X)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{\pi(A \mid X)}{f(A \mid W, X)} h(W, A, X) \mid W, X\right]\right] \\
&= \mathbb{E}\left[\int \frac{\pi(a \mid X)}{f(a \mid W, X)} h(W, a, X) f(a \mid W, X) \mathrm{d}\mu(a)\right] \\
&= \mathbb{E}\left[(\mathcal{T} h)(W, X)\right].
\end{aligned}$$

Here the second equality follows because q_0 satisfies Eq. (6).

IPW We prove a stronger statement:

$$\mathbb{E}\left[\tilde{\phi}_{\mathrm{IPW}}(O; q)\right] - J = \mathbb{E}\left[\{\mathbb{E}\left[\pi(A \mid X) q(Z, A, X) \mid W, A, X\right] - \frac{\pi(A \mid X)}{f(A \mid W, X)}\} h_0(W, A, X)\right]$$

for any $h_0 \in \mathbb{H}_0^{\mathrm{obs}}$ assuming $\mathbb{Q}_0 \neq \emptyset$. First, by taking some element h_0 in $\mathbb{H}_0^{\mathrm{obs}}$, we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{\phi}_{\mathrm{IPW}}(O; q)\right] &= \mathbb{E}\left[\pi(A \mid X) q(Z, A, X) Y\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q(Z, A, X) \mathbb{E}\left[Y \mid Z, A, X\right]\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q(Z, A, X) h_0(W, A, X)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X) q(Z, A, X) \mid W, A, X\right] h_0(W, A, X)\right].
\end{aligned}$$

Moreover, by taking some element $q_0 \in \mathbb{Q}_0^{\text{obs}}$,

$$\begin{aligned}
J &= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) Y\right] && \text{(Lemma 2)} \\
&= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}[Y|A, X, W]\right] && \text{(Conditional independence)} \\
&= \mathbb{E}\left[\mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mid W, A, X\right] h_0(W, A, X)\right] \\
&= \mathbb{E}\left[\frac{\pi(A \mid X)}{f(A \mid W, X)} h_0(W, A, X)\right].
\end{aligned}$$

It follows that

$$\mathbb{E}\left[\tilde{\phi}_{\text{IPW}}(O; q)\right] - J = \mathbb{E}\left[\{\mathbb{E}\left[\pi(A \mid X) q(Z, A, X) \mid W, A, X\right] - \frac{\pi(A \mid X)}{f(A \mid W, X)}\} h_0(W, A, X)\right].$$

REG We prove a stronger statement:

$$\mathbb{E}\left[\tilde{\phi}_{\text{REG}}\left(O; h\right)\right] - J = \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}\left[h(W, A, X) - Y \mid Z, A, X\right]\right].$$

for some $q_0 \in \mathbb{Q}_0^{\text{obs}}$ assuming $\mathbb{H}_0 \neq \emptyset$. Note again that

$$\begin{aligned}
\mathbb{E}\left[\tilde{\phi}_{\text{REG}}(O; h)\right] &= \mathbb{E}\left[(\mathcal{T} h)(W, X)\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) h(W, A, X)\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}\left[h(W, A, X) \mid Z, A, X\right]\right].
\end{aligned}$$

Moreover, by taking some element $h_0 \in \mathbb{H}_0^{\text{obs}}$,

$$\begin{aligned}
J &= \mathbb{E}\left[(\mathcal{T} h_0)(W, X)\right] && \text{(Lemma 2)} \\
&= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) h_0(W, A, X)\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}\left[h_0(W, A, X) \mid Z, A, X\right]\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}\left[Y \mid Z, A, X\right]\right].
\end{aligned}$$

It follows that

$$\mathbb{E}\left[\tilde{\phi}_{\text{REG}}\left(O; h\right)\right] - J = \mathbb{E}\left[\pi(A \mid X) q_0(Z, A, X) \mathbb{E}\left[h(W, A, X) - Y \mid Z, A, X\right]\right].$$

$\square$

Proof for Lemma 6. Fix $h_0 \in \mathbb{H}_0^{\text{obs}}$ and $q_0 \in \mathbb{Q}_0^{\text{obs}}$. This conclusion follows from the fact that for any $h \in \mathbb{H}$, $h' \in L_2\left(W, A, X\right)$, $q \in \mathbb{Q}$, $q' \in L_2\left(Z, A, X\right)$,

$$\begin{aligned}
&\left(\mathbb{E}\left[q'(Z, A, X)\left(h(W, A, X) - Y\right)\right]\right)^2 \\
&\qquad \leq \mathbb{E}\left[\left(\mathbb{E}\left[\left(Y - h(W, A, X)\right) \mid Z, A, X\right]\right)^2\right] \\
&\left(\mathbb{E}\left[q(X, A, Z) h'(X, A, W) - (\mathcal{T} h')(X, W)\right]\right)^2 \\
&\qquad \leq \mathbb{E}\left[\left(\pi(A \mid X) \mathbb{E}\left[q(Z, A, X) - 1/f(A \mid X, W) \mid W, A, X\right]\right)^2\right],
\end{aligned}$$

and the two upper bounds are attained by

$$\begin{aligned}
q'(Z, A, X) &= \mathbb{E}\left[\left(h(W, A, X) - Y\right) \mid Z, A, X\right] \\
&= \mathbb{E}\left[h\left(W, A, X\right) - h_0(W, A, X) \mid Z, A, X\right] \in P_Z\left(\mathbb{H} - \mathbb{H}_0^{\text{obs}}\right) \subseteq \mathbb{Q}'
\end{aligned}$$

and

$$\begin{aligned}
h'(W, A, X) &= \pi(A \mid X) \mathbb{E}\left[q(Z, A, X) - 1/f(A \mid X, W) \mid W, A, X\right] \\
&= \pi(A \mid X) \mathbb{E}\left[q(Z, A, X) - q_0\left(Z, A, X\right) \mid W, A, X\right] \in \pi P_W\left(\mathbb{Q} - \mathbb{Q}_0^{\text{obs}}\right) \subseteq \mathbb{H}'.
\end{aligned}$$

$\square$

Proof for Lemma 7. We prove the first equation as an example, and the second equation can be proved analogously.

First, note that any $h_0 \in \mathbb{H}_0^{\text{obs}} \cap \mathbb{H}$ satisfies

$$\sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h_0(W,A,X)-Y\right)\right] - \lambda\|q\|^2 = \sup_{q \in \mathbb{Q}'} -\lambda\|q\|^2 = 0.$$

In addition, for any $h \in \mathbb{H}$,

$$\sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2 \geq \mathbb{E}\left[0 \times \left(h(W,A,X)-Y\right)\right] - \lambda\|0\|^2 = 0.$$

Thus we have

$$\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2 = 0$$

and

$$\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} \subseteq \arg\min_{h} \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2.$$

Second, take $\tilde{q} = c\mathbb{E}\left[h(W,A,X) - Y \mid Z,A,X\right]$ for $c = \min(1, 1/\lambda)$. Since $\mathbb{Q}'$ is star-shaped, we have $\tilde{q} \in \mathbb{Q}'$ and

$$\begin{aligned}
\mathbb{E}\left[\tilde{q}(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|\tilde{q}\|^2 &= (c-\lambda c^2)\mathbb{E}\left[\left(\mathbb{E}\left[h(W,A,X)-Y \mid Z,A,X\right]\right)^2\right] \\
&\leq \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2.
\end{aligned}$$

Thus for any $h \in \arg\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2$,

$$\begin{aligned}
0 &\leq (c-\lambda c^2)\mathbb{E}\left[\left(\mathbb{E}\left[h(W,A,X)-Y \mid Z,A,X\right]\right)^2\right] \\
&\leq \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2 = 0,
\end{aligned}$$

which implies that

$$h \in \arg\min_{h \in \mathbb{H}} (c-\lambda c^2)\mathbb{E}\left[\left(\mathbb{E}\left[h(W,A,X)-Y \mid Z,A,X\right]\right)^2\right] = \mathbb{H}_0^{\text{obs}} \cap \mathbb{H}.$$

It follows that

$$\arg\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2 \subseteq \mathbb{H}_0^{\text{obs}} \cap \mathbb{H}.$$

Therefore, we have

$$\mathbb{H}_0^{\text{obs}} \cap \mathbb{H} = \arg\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2.$$

Note that we do not need the star-shape condition when $\lambda = 1/2$. It is easy to show that

$$\begin{aligned}
&\sup_{q \in L_2(Z,A,X)} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2 \\
=&\sup_{q \in L_2(Z,A,X)} \quad \mathbb{E}\left[\mathbb{E}\left[(h(W,A,X)) - Y \mid Z,A,X\right] q(Z,A,X) - \lambda q^2(Z,A,X)\right] \\
=&\frac{1}{4\lambda}\mathbb{E}\left[\left(\mathbb{E}\left[h(W,A,X)-Y \mid Z,A,X\right]\right)^2\right],
\end{aligned}$$

where the supremum is achieved by $\pi(A \mid X)q(Z,A,X) = \frac{1}{2\lambda}\mathbb{E}\left[(h(W,A,X)-Y) \mid Z,A,X\right]$. In particular, when $\lambda = 1/2$, the supremum above is achieved by $\pi(A \mid X)q(Z,A,X) = \mathbb{E}\left[(h(W,A,X)-Y) \mid Z,A,X\right] \in P_Z\left(\mathbb{H} - \mathbb{H}_0\right) \subseteq \mathbb{Q}'$. This implies that

$$\begin{aligned}
\arg\min_{h \in \mathbb{H}} \sup_{q \in \mathbb{Q}'} \quad \mathbb{E}\left[q(Z,A,X)\left(h(W,A,X)-Y\right)\right] - \lambda\|q\|^2 &= \arg\min_{h \in \mathbb{H}} \frac{1}{2}\mathbb{E}\left[\left(\mathbb{E}\left[h(W,A,X)-Y \mid Z,A,X\right]\right)^2\right] \\
&= \mathbb{H} \cap \mathbb{H}_0^{\text{obs}}.
\end{aligned}$$

Obviously, if we assume $\cup_{t>}\left\{tP_z\left(\mathbb{H} - \mathbb{H}_0^{\text{obs}}\right)\right\} \subseteq \mathbb{Q}'$ instead of assuming $\mathbb{Q}'$ is star-shaped and $P_z\left(\mathbb{H} - \mathbb{H}_0^{\text{obs}}\right) \subseteq \mathbb{Q}'$, then we can prove the conclusion in a similar way.

∎

E.2 Proof of Section 4

Proof for Lemma 8. The conclusion follows easily from simple algebra. $\square$

Proof of Lemma 9. We derive Eq. (23) as follows:

$$\begin{aligned}
&\sup_{\|q\|_{\mathcal{L}_z}\leq 1} (\mathbb{E}_n[\{Y-h\}\langle q(\cdot), k_z((Z,A,X),\cdot)\rangle])^2 \\
&= \sup_{q\in\|q\|_{\mathcal{L}_z}\leq 1} (\langle q(\cdot), \mathbb{E}_n[\{Y-h\}k_z((Z,A,X),\cdot)]\rangle)^2 && \text{(Linearity)}\\
&= \langle \mathbb{E}_n[\{Y-h\}k_z((Z,A,X),\cdot)], \mathbb{E}_n[\{Y-h\}k_z((Z,A,X),\cdot)]\rangle && \text{(CS inequality)}\\
&= \frac{1}{n^2}\sum_{i,j}\{Y_i - h(X_j,A_i,Z_i)\}k_z((X_i,A_i,Z_i),(Z_j,A_j,X_j))\{Y_j - h(Z_j,A_j,X_j)\}.
\end{aligned}$$

Moreover, we can derive Eq. (24) as follows:

$$\begin{aligned}
&\sup_{\|h\|_{\mathcal{L}_w}\leq 1} (\mathbb{E}_n[q\pi\langle h, k_w\rangle - \mathcal{T}(\langle h(\cdot), k_w((W,A,X),\cdot)\rangle)])^2 \\
&= \sup_{\|h\|_{\mathcal{L}_w}\leq 1} (\langle h(\cdot), \mathbb{E}_n[q\pi k_w((W,A,X),\cdot) - \mathcal{T}k_w((W,A,X),\cdot))]\rangle)^2 && \text{(Linearity)}\\
&= \langle \mathbb{E}_n[q\pi k_w((W,A,X),\cdot) - \mathcal{T}k_w((W,A,X),\cdot)\rangle^2. && \text{(CS inequality)}
\end{aligned}$$

Here, the operator $\mathcal{T}k_w((x,a,w),\cdot) = \mathbb{E}_{\pi(a|x)}[k_w((x,a,w),\cdot)]$. Taking the term only related to q, the above is equal to

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i,j} q(W_i,A_i,X_i)\pi(A_i \mid X_i)k_w((W_i,A_i,X_i),(W_j,A_j,X_j))q(Z_j,A_j,X_j)\pi(A_j \mid X_j)\\
&-\frac{2}{n^2}\sum_{i,j} q(W_i,A_i,X_i)\pi(A_i \mid X_i)\mathcal{T}k_w((W_j,A_j,X_j),(W_i,A_i,X_i)).
\end{aligned}$$

$\square$

Proof of Lemma 10. From the representer theorem, an solution of the inner maximization in Eq. (25) should be $q^*(\cdot) = \sum_i \alpha_i k_w((Z_i,A_i,X_i),\cdot)$. Thus, this inner maximization problem can be reduced to solving

$$\begin{aligned}
&\max_{\alpha\in\mathbb{R}^n} \psi_n^\top K_{z,n}\alpha - \alpha^\top(\lambda K_{z,n}+\gamma_1 I)K_{z,n}\alpha \\
&= \max_{\tilde{\alpha}\in\mathbb{R}^n} \psi_n^\top K_{z,n}^{1/2}\tilde{\alpha} - \tilde{\alpha}^\top(\lambda K_{z,n}+\gamma_1 I)\tilde{\alpha} && (\tilde{\alpha} = K_{z,n}^{1/2}\alpha)\\
&= \frac{1}{4}\psi_n^\top K_{z,n}^{1/2}(\lambda K_{z,n}+\gamma_1 I)^{-1}K_{z,n}^{1/2}\psi_n,
\end{aligned}$$

where the last maximum is achieved by

$$\tilde{\alpha}^* = \frac{1}{2}(\lambda K_{z,n}+\gamma_1 I)^{-1}\psi_n.$$

From the representer theorem, a solution of the inner maximization problem in Eq. (26) should be $h^*(\cdot) = \sum_i \alpha_i k_w((X_i,A_i,W_i),\cdot)$. Thus, this inner maximization problem can be reduced to solving

$$\max_{\alpha\in\mathbb{R}^n} \phi_n^\top K_{w1,n}\alpha - \mathbf{1}_n^\top K_{w2,n}\alpha - \alpha^\top(\lambda K_{w1,n}+\gamma_2 I)K_{w1,n}\alpha.$$

Assuming $K_{w1,n}$ is a positive definite matrix, the optimization problem above is solved by

$$\hat{\alpha} = \frac{1}{2}(\lambda K_{w1,n}+\gamma I)^{-1}(\phi_n - K_{w1,n}^{-1}K_{w2,n}\mathbf{1}_n).$$

Thus, the resulting optimal value is

$$\frac{1}{4}\phi_n^\top K_{w1,n}^{1/2}(\lambda K_{w1,n}+\gamma I)^{-1}K_{w1,n}^{1/2}\phi_n - \frac{1}{2}\{\phi_n^\top(\lambda K_{w1,n}+\gamma I)^{-1}K_{w2,n}\mathbf{1}_n\}.$$

$\square$

E.3 Proofs for Section 5

We often use the following lemmas. Instead of proving Theorems 1 and 2, we prove the generalized versions, Theorems 11 and 12.

Lemma 19 (Dudley integral)**.**

$$\mathcal{R}_n(\infty;\mathcal{F}) \lesssim \inf_{\tau\geq 0}\left\{\tau + \int_{\tau}^{\sup_{f\in\mathcal{F}}\sqrt{\mathbb{E}_n[f^2]}}\sqrt{\frac{\log\mathcal{N}(\tau,\mathcal{F},\|\cdot\|_n)}{n}}\right\}.$$

Note $\sup_{f\in\mathcal{F}}\sqrt{\mathbb{P}_n[f^2]}$ is upper bounded by the envelope $\|\mathcal{F}\|_\infty$.

Lemma 20. *Covering number of VC-subgraph classes [van der Vaart, 1998, Lemma 19.15] For a VC class of functions $\mathcal{F}$ with measurable envelope function F and $r\geq 1$, one has for any probability measure Q with $\|F\|_{Q,r}>0$,*

$$\mathcal{N}(\epsilon\|F\|_{Q,r},\mathcal{F},L_r(Q)) \lesssim V(\mathcal{F})(4e)^{V(\mathcal{F})}\left(\frac{2}{\epsilon}\right)^{rV(\mathcal{F})}.$$

Proof of Theorem 11. We take elements h_0,q_0 s.t. $h_0\in\mathbb{H}_0^{\mathrm{obs}},q_0\in\mathbb{Q}_0^{\mathrm{obs}}$. This operation is feasible since $\mathbb{H}_0,\mathbb{Q}_0$ are non-empty from the assumption. (Note $\mathbb{H}_0\neq\emptyset,\mathbb{Q}_0\neq\emptyset$ imply $\mathbb{H}_0^{\mathrm{obs}}\neq\emptyset,\mathbb{Q}_0^{\mathrm{obs}}\neq\emptyset$).

First Step We define

$$J_{\mathrm{IPW}} \coloneqq \mathbb{E}[\hat{q}\pi Y].$$

Then,

$$|J_{\mathrm{IPW}}-\hat{J}_{\mathrm{IPW}}| \leq \sup_{q\in\mathbb{Q}}|(\mathbb{E}-\mathbb{E}_n)[q\pi Y]|.$$

In addition,

$$\begin{aligned}
|J_{\mathrm{IPW}}-J| &\leq |\mathbb{E}[(\hat{q}-q_0)\pi Y]| = |\mathbb{E}[\mathbb{E}[(\hat{q}-q_0)\pi Y|A,X,U]]| \\
&= |\mathbb{E}[(\hat{q}-q_0)\pi h_0]| && (Z\perp A,Y|X,U)\\
&\leq \sup_{h\in\mathbb{H}'}|\mathbb{E}[(\hat{q}-q_0)\pi h]| + \inf_{h\in\mathbb{H}'}|\mathbb{E}[(\hat{q}-q_0)\pi(h_0-h)]|\\
&= |\mathbb{E}[(\hat{q}-q_0)\pi\bar{h}]| + \sup_{q\in\mathbb{Q}}\inf_{h\in\mathbb{H}'}|\mathbb{E}[(q-q_0)\pi(h_0-h)]|,\ \bar{h}\coloneqq\operatorname*{arg\,max}_{h\in\mathbb{H}'}|\mathbb{E}[(\hat{q}-q_0)\pi h]|\\
&\leq |\mathbb{E}[(\hat{q}-q_0)\pi\bar{h}]| + \inf_{h\in\mathbb{H}'}\sup_{q\in\mathbb{Q}}|\mathbb{E}[(q-q_0)\pi(h_0-h)]|.
\end{aligned}$$

Second Step We define

$$f_q(q,h) = q\pi(-h)+\mathcal{T}h.$$

Note $\mathbb{E}[f_q(q,h)] = \mathbb{E}[(-q+q_0)\pi h]$. Next, we take

$$q' = \operatorname*{arg\,min}_{q\in\mathbb{Q}}\sup_{h\in\mathbb{H}'}|\mathbb{E}[f_q(q,h)]|,\ h^\dagger = \operatorname*{arg\,max}_{h\in\mathbb{H}'}|\mathbb{E}_n[f_q(\hat{q},h)]|.$$

Then,

$$\begin{aligned}
&|\mathbb{E}[f_q(\hat{q},h^\dagger)]| - |\mathbb{E}[f_q(q',h^\dagger)]|\\
&= |\mathbb{E}[f_q(\hat{q},h^\dagger)]| - |\mathbb{E}_n[f_q(\hat{q},h^\dagger)]| + |\mathbb{E}_n[f_q(\hat{q},h^\dagger)]| - |\mathbb{E}_n[f_q(q',h^\dagger)]| + |\mathbb{E}_n[f_q(q',h^\dagger)]| - |\mathbb{E}[f_q(q',h^\dagger)]|\\
&= |\mathbb{E}[f_q(\hat{q},h^\dagger)]| - |\mathbb{E}_n[f_q(\hat{q},h^\dagger)]| + |\mathbb{E}_n[f_q(q',h^\dagger)]| - |\mathbb{E}[f_q(q',h^\dagger)]| && (\text{Definition of } \hat{q})\\
&\leq 2|\sup_{q\in\mathbb{Q},h\in\mathbb{H}'}(\mathbb{E}-\mathbb{E}_n)[f_q(q,h)]|. && (h^\dagger\in\mathbb{H}',\hat{q}\in\mathbb{Q},q'\in\mathbb{Q})
\end{aligned}$$

Besides,

$$
\begin{aligned}
&|\mathbb{E}[f_q(\hat{q},\bar{h})]| - |\mathbb{E}[f_q(\hat{q},h^\dagger)]| \\
&\leq |\mathbb{E}[f_q(\hat{q},\bar{h})]| - |\mathbb{E}_n[f_q(\hat{q},\bar{h})]| + |\mathbb{E}_n[f_q(\hat{q},\bar{h})]| - |\mathbb{E}_n[f_q(\hat{q},h^\dagger)]| + |\mathbb{E}_n[f_q(\hat{q},h^\dagger)]| - |\mathbb{E}[f_q(\hat{q},h^\dagger)]| \\
&\leq |\mathbb{E}[f_q(\hat{q},\bar{h})]| - |\mathbb{E}_n[f_q(\hat{q},\bar{h})]| + |\mathbb{E}_n[f_q(\hat{q},h^\dagger)]| - |\mathbb{E}[f_q(\hat{q},h^\dagger)]| && \text{(Definition of } h^\dagger) \\
&\leq 2|\sup_{q\in\mathbb{Q},h\in\mathbb{H}'}(\mathbb{E}-\mathbb{E}_n)[f_q(q,h)]|. && (\hat{q}\in\mathbb{Q},\bar{h}\in\mathbb{H}',h^\dagger\in\mathbb{H}')
\end{aligned}
$$

Third Step Combining all results,

$$
\begin{aligned}
|J-\hat{J}_{\mathrm{IPW}}| &\leq |J_{\mathrm{IPW}}-\hat{J}_{\mathrm{IPW}}| + |J_{\mathrm{IPW}}-J| \\
&\leq \sup_{q\in\mathbb{Q}}|(\mathbb{E}-\mathbb{E}_n)[q\pi Y]| + |\mathbb{E}[f_q(\hat{q},\bar{h})]| + \inf_{h\in\mathbb{H}'}\sup_{q\in\mathbb{Q}}|\mathbb{E}[(q-q_0)\pi(h_0-h)]| \\
&\leq \sup_{q\in\mathbb{Q}}|(\mathbb{E}-\mathbb{E}_n)[q\pi Y]| + 4|\sup_{q\in\mathbb{Q},h\in\mathbb{H}'}(\mathbb{E}-\mathbb{E}_n)[f_q(q,h)]| + |\mathbb{E}[f_q(q',h^\dagger)]| \\
&+ \inf_{h\in\mathbb{H}'}\sup_{q\in\mathbb{Q}}|\mathbb{E}[(q-q_0)\pi(h_0-h)]| \\
&\leq \sup_{q\in\mathbb{Q}}|(\mathbb{E}-\mathbb{E}_n)[q\pi Y]| + 4|\sup_{q\in\mathbb{Q},h\in\mathbb{H}'}(\mathbb{E}-\mathbb{E}_n)[f_q(q,h)]| + \\
&+ \inf_{h\in\mathbb{H}'}\sup_{q\in\mathbb{Q}}|\mathbb{E}[(q-q_0)\pi(h_0-h)]| + \inf_{q\in\mathbb{Q}}\sup_{h\in\mathbb{H}'}|\mathbb{E}[(q-q_0)\pi h]|.
\end{aligned}
$$

□

Proof of Theorem 12. We take elements h_0, q_0 s.t. $h_0 \in \mathbb{H}_0^{\mathrm{obs}}, q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$. This operation is feasible since $\mathbb{H}_0, \mathbb{Q}_0$ are non-empty from the assumption.

First step We define

$$J_{\mathrm{REG}} \coloneqq \mathbb{E}[\mathcal{T}h].$$

Then,

$$|J_{\mathrm{REG}} - \hat{J}_{\mathrm{REG}}| \leq \sup_{h\in\mathbb{H}}(\mathbb{E}-\mathbb{E}_n)|[\mathcal{T}h]|.$$

In addition,

$$
\begin{aligned}
|J_{\mathrm{REG}}-J| &\leq |\mathbb{E}[\mathcal{T}(\hat{h}-h_0)| = |\mathbb{E}[q_0\pi(\hat{h}-h_0)| \\
&\leq \sup_{q\in\mathbb{Q}'}|\mathbb{E}[q(\hat{h}-h_0)| + \inf_{q\in\mathbb{Q}'}|\mathbb{E}[(q_0\pi-q)(\hat{h}-h_0)| \\
&\leq |\mathbb{E}[\bar{q}(\hat{h}-h_0)| + \sup_{h\in\mathbb{H}}\inf_{q\in\mathbb{Q}'}|\mathbb{E}[(q_0\pi-q)(h-h_0)|,\ \bar{q} \coloneqq \arg\max_{q\in\mathbb{Q}'}|\mathbb{E}[q(\hat{h}-h_0)| \\
&\leq |\mathbb{E}[\bar{q}(\hat{h}-h_0)| + \inf_{q\in\mathbb{Q}'}\sup_{h\in\mathbb{H}}|\mathbb{E}[(q_0\pi-q)(h-h_0)|.
\end{aligned}
$$

Second step We define

$$f_h(q,h) = \{y-h\}q.$$

Note $\mathbb{E}[f_h(q,h)] = \mathbb{E}[\{h_0-h\}q]$. Next, we take

$$h' = \arg\min_{h\in\mathbb{H}}\sup_{q\in\mathbb{Q}'}|\mathbb{E}[f_h(q,h)]|,\ q^\dagger = \arg\max_{q\in\mathbb{Q}'}\mathbb{E}_n[\{Y-h\}q].$$

Then,

$$
\begin{aligned}
&|\mathbb{E}[f_h(q^\dagger,\hat{h})]| - |\mathbb{E}[f_h(q^\dagger,h')]| \\
&= |\mathbb{E}[f_h(q^\dagger,\hat{h})]| - |\mathbb{E}_n[f_h(q^\dagger,\hat{h})]| + |\mathbb{E}_n[f_h(q^\dagger,\hat{h})]| - |\mathbb{E}_n[f_h(q^\dagger,h')]| + |\mathbb{E}_n[f_h(q^\dagger,h')]| - |\mathbb{E}[f_h(q^\dagger,h')]| \\
&\leq |\mathbb{E}[f_h(q^\dagger,\hat{h})]| - |\mathbb{E}_n[f_h(q^\dagger,\hat{h})]| + |\mathbb{E}_n[f_h(q^\dagger,h')]| - |\mathbb{E}[f_h(q^\dagger,h')]| && \text{(Definition of } \hat{h}) \\
&\leq 2\sup_{q\in\mathbb{Q}',h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[f_h(q,h)]|. && (q^\dagger\in\mathbb{Q}',\hat{h},h'\in\mathbb{H})
\end{aligned}
$$

Besides,

$$
\begin{aligned}
&|\mathbb{E}[f_h(\bar{q},\hat{h})]| - |\mathbb{E}[f_h(q^\dagger,\hat{h})]| \\
&= |\mathbb{E}[f_h(\bar{q},\hat{h})]| - |\mathbb{E}_n[f_h(\bar{q},\hat{h})]| + |\mathbb{E}_n[f_h(\bar{q},\hat{h})]| - |\mathbb{E}_n[f_h(q^\dagger,\hat{h})]| + |\mathbb{E}_n[f_h(q^\dagger,\hat{h})]| - |\mathbb{E}[f_h(q^\dagger,\hat{h})]| \\
&\leq |\mathbb{E}[f_h(\bar{q},\hat{h})]| - |\mathbb{E}_n[f_h(\bar{q},\hat{h})]| + |\mathbb{E}_n[f_h(q^\dagger,\hat{h})]| - |\mathbb{E}[f_h(q^\dagger,\hat{h})]| && \text{(Definition of } q^\dagger) \\
&\leq 2 \sup_{q\in\mathbb{Q}',h\in\mathbb{H}} |(\mathbb{E}-\mathbb{E}_n)[f_h(q,h)]|. && (\bar{q},q^\dagger\in\mathbb{Q}',\hat{h},h\in\mathbb{H})
\end{aligned}
$$

Third step Combining all results,

$$
\begin{aligned}
|J-\hat{J}_{\mathrm{REG}}| &\leq |J_{\mathrm{REG}}-\hat{J}_{\mathrm{REG}}| + |J_{\mathrm{REG}}-J| \\
&\leq \sup_{h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[\mathcal{T}h]| + |\mathbb{E}[f_h(\bar{q},\hat{h})]| + \inf_{q\in\mathbb{Q}'}\sup_{h\in\mathbb{H}}|\mathbb{E}[(q_0\pi-q)(h-h_0)| \\
&\leq \sup_{h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[\mathcal{T}h]| + 4\sup_{q\in\mathbb{Q}',h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[f_h(q,h)]| + |\mathbb{E}[f_h(q^\dagger,h')]| \\
&+ \inf_{q\in\mathbb{Q}'}\sup_{h\in\mathbb{Q}'}|\mathbb{E}[(q_0\pi-q)(h-h_0)| \\
&\leq \sup_{h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[\mathcal{T}h]| + 4\sup_{q\in\mathbb{Q}',h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[f_h(q,h)]| \\
&+ \inf_{h\in\mathbb{H}}\sup_{q\in\mathbb{Q}'}|\mathbb{E}[\{h_0-h\}q]| + \inf_{q\in\mathbb{Q}'}\sup_{h\in\mathbb{H}}|\mathbb{E}[(q_0\pi-q)(h-h_0)|.
\end{aligned}
$$

□

Proof of Theorem 13. We take elements h_0, q_0 s.t. $h_0 \in \mathbb{H}_0^{\mathrm{obs}}, q_0 \in \mathbb{Q}_0^{\mathrm{obs}}$. This operation is feasible since $\mathbb{H}_0, \mathbb{Q}_0$ are non-empty from the assumption.

First Statement First, we prove

$$
\begin{aligned}
|\hat{J}_{\mathrm{DR}}-J| \leq& \sup_{q\in\mathbb{Q},h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[q\pi Y - q\pi h + \mathcal{T}h]| + 4\sup_{q\in\mathbb{Q},h\in\mathbb{H}'}|(\mathbb{E}-\mathbb{E}_n)[-q\pi h+\mathcal{T}h]| + \\
&+ \inf_{q\in\mathbb{Q}}\sup_{h\in\mathbb{H}'}|\mathbb{E}[(q_0-q)\pi h]| + \inf_{h\in\mathbb{H}'}\sup_{h'\in\mathbb{H}}\sup_{q\in\mathbb{Q}}|\mathbb{E}[(q-q_0)\pi(h_0-h'-h)]|.
\end{aligned}
$$

We define

$$
J_{\mathrm{DR}} \coloneqq \mathbb{E}[\hat{q}\pi\{Y-\hat{h}\}+\mathcal{T}\hat{h}].
$$

Then,

$$
|J_{\mathrm{DR}}-\hat{J}_{\mathrm{DR}}| \leq \sup_{q\in\mathbb{Q},h\in\mathbb{H}}|(\mathbb{E}-\mathbb{E}_n)[q\pi\{Y-h\}+\mathcal{T}h]|.
$$

In addition,

$$
\begin{aligned}
|J_{\mathrm{DR}}-J| &\leq |\mathbb{E}[(\hat{q}-q_0)\pi(h_0-\hat{h})]| \\
&\leq \sup_{h\in\mathbb{H}'}|\mathbb{E}[(\hat{q}-q_0)\pi h]| + \inf_{h\in\mathbb{H}'}|\mathbb{E}[(\hat{q}-q_0)\pi(h_0-\hat{h}-h)]| \\
&= |\mathbb{E}[(\hat{q}-q_0)\bar{h}]| + \sup_{h'\in\mathbb{H}}\sup_{q\in\mathbb{Q}}\inf_{h\in\mathbb{H}'}|\mathbb{E}[(q-q_0)\pi(h_0-h'-h)]|,\ \bar{h}\coloneqq \operatorname*{arg\,max}_{h\in\mathbb{H}'}|\mathbb{E}[(\hat{q}-q_0)\pi h]| \\
&= |\mathbb{E}[(\hat{q}-q_0)\bar{h}]| + \inf_{h\in\mathbb{H}'}\sup_{h'\in\mathbb{H},q\in\mathbb{Q}}|\mathbb{E}[(q-q_0)\pi(h_0-h'-h)]|.
\end{aligned}
$$

The rest of the proof is the same as Theorem 11.

Second Statement Second, we prove

$$|\hat{J}_{\mathrm{DR}} - J| \leq \sup_{q\in\mathbb{Q},h\in\mathbb{H}} |(\mathbb{E}-\mathbb{E}_n)[q\pi Y - q\pi h + \mathcal{T}h]| + 4\sup_{q\in\mathbb{Q}',h\in\mathbb{H}} |(\mathbb{E}-\mathbb{E}_n)[q(Y-h)]|$$
$$+ \inf_{h\in\mathbb{H}}\sup_{q\in\mathbb{Q}'} |\mathbb{E}[\{h_0 - h\}q]| + \inf_{q\in\mathbb{Q}'}\sup_{h\in\mathbb{H},q'\in\mathbb{Q}} |\mathbb{E}[(q_0\pi - q' - q)(h-h_0)|.$$

We define

$$J_{\mathrm{DR}} \coloneqq \mathbb{E}[\hat{q}\pi\{Y-\hat{h}\} + \mathcal{T}\hat{h}].$$

Then,

$$|J_{\mathrm{DR}} - \hat{J}_{\mathrm{DR}}| \leq \sup_{q\in\mathbb{Q},h\in\mathbb{H}} |(\mathbb{E}-\mathbb{E}_n)[q\pi\{Y-h\}+\mathcal{T}h]|.$$

In addition,

$$\begin{aligned}
|J_{\mathrm{DR}} - J| &\leq |\mathbb{E}[(\hat{q}-q_0)\pi(h_0-\hat{h})]| \\
&\leq \sup_{q\in\mathbb{Q}'} |\mathbb{E}[q(\hat{h}-h_0)| + \inf_{q\in\mathbb{Q}'} |\mathbb{E}[(q_0\pi - \hat{q}\pi - q)\hat{h} - h_0)| \\
&\leq |\mathbb{E}[\bar{q}(\hat{h}-h_0)| + \sup_{h\in\mathbb{H},q'\in\mathbb{Q}}\inf_{q\in\mathbb{Q}'} |\mathbb{E}[(q_0\pi - q'\pi - q)(h-h_0)|,\ \bar{q} \coloneqq \arg\max_{q\in\mathbb{Q}'} |\mathbb{E}[q(\hat{h}-h_0)| \\
&\leq |\mathbb{E}[\bar{q}(\hat{h}-h_0)| + \inf_{q\in\mathbb{Q}'}\sup_{h\in\mathbb{H},q'\in\mathbb{Q}} |\mathbb{E}[(q_0\pi - q'\pi - q)(h-h_0)|.
\end{aligned}$$

The rest of the proof is the same as Theorem 12. □

Proof for Theorem 1. The conclusion for $\hat{J}_{\mathrm{IPW}}$ directly follows from Theorem 11 and the conclusion for $\hat{J}_{\mathrm{REG}}$ directly follows from Theorem 12. □

Proof of Corollary 1. We define three function classes for the analysis:

$$\begin{aligned}
\mathbb{A}_1 &= \{(x,a,z,y)\mapsto q(x,a,z)y : q\in\mathbb{Q}'\}, \\
\mathbb{A}_2 &= \{(x,a,z,w,y)\mapsto q(x,a,z)h(x,a,w) : q\in\mathbb{Q}', h\in\mathbb{H}\}, \\
\mathbb{A}_3 &= \{(x,a,w)\mapsto (\mathcal{T}h)(x,w)\in\mathbb{H}\}.
\end{aligned}$$

Then, from Wainwright [2019, Theorem 4.10] and Theorem 1, we have

$$|\hat{J} - J| \leq c\{\mathcal{R}_n(\infty;\mathbb{A}_1) + \mathcal{R}_n(\infty;\mathbb{A}_2) + \mathcal{R}_n(\infty;\mathbb{A}_3) + \sqrt{\log(1/\delta)/n}\}.$$

where c is some universal constant. First, it is seen

$$\begin{aligned}
\mathcal{R}_n(\infty;\mathbb{H}\mathbb{Q}') &\leq \mathcal{R}_n(\infty; 0.25\left\{(\mathbb{Q}'-\mathbb{H})^2) + (\mathbb{H}+\mathbb{Q}')^2\right\}) \\
&\leq 0.25\{\mathcal{R}_n(\infty;(\mathbb{Q}'-\mathbb{H})^2) + \mathcal{R}_n(\infty;(\mathbb{H}+\mathbb{Q}')^2)\} \\
&\leq 0.5(C_{\mathbb{Q}}+C_{\mathbb{H}})\{\mathcal{R}_n(\infty;(\mathbb{Q}'-\mathbb{H})) + \mathcal{R}_n(\infty;(\mathbb{H}+\mathbb{Q}'))\} \\
&\qquad\qquad \text{(Contraction property [Mendelson, 2002])} \\
&\leq (C_{\mathbb{Q}}+C_{\mathbb{H}})\{\mathcal{R}_n(\infty;\mathbb{Q}') + \mathcal{R}_n(\infty;\mathbb{H})\}.
\end{aligned}$$

Besides, from the definition of VC-dimension,

$$V(\mathbb{A}_1) \leq C_Y V(\mathbb{Q}'),\ V(\mathbb{A}_3) = V(\mathcal{T}\mathbb{H}) \leq V(\mathbb{H}).$$

Thus,

$$|\hat{J} - J| \leq c(\mathcal{R}_n(\infty;\mathbb{Q}') + \mathcal{R}_n(\infty;\mathbb{H}) + \sqrt{\log(1/\delta)/n}).$$

Here, from Dudley integral (Lemma 19), by using the covering number of the VC-subgraph class is explicitly calculated (Lemma 20), we have

$$\mathcal{R}_n(\infty;\mathbb{Q}') = O(\sqrt{V(\mathbb{Q}')/n}),\ \mathcal{R}_n(\infty;\mathbb{H}) = O(\sqrt{V(\mathbb{H})/n}).$$

Then,

$$|\hat{J} - J| \leq c(\sqrt{V(\mathbb{Q}')/n} + \sqrt{V(\mathbb{H})/n} + \sqrt{\log(1/\delta)/n}).$$

□

Proof of Corollary 2. From the proof of Corollary 1, we have

$$|\hat{J} - J| \leq c(\mathcal{R}_n(\infty; \mathbb{Q}') + \mathcal{R}_n(\infty; \mathbb{H}) + \sqrt{\log(1/\delta)/n}).$$

for some constant c. Then, from Dudley integral, the statement is immediately concluded. $\square$

Proof of Corollary 3. WLOS, we prove the result of $\hat{J}_{\mathrm{IPW}}$. By taking some element in $h_0 \in \mathbb{H}_0^{\mathrm{obs}} \cap \Lambda^\alpha([0,1]^d), q_0 \in \pi\mathbb{Q}_0^{\mathrm{obs}} \cap \Lambda^\alpha([0,1]^d)$. Then, from Theorem 11, with $1-\delta$, we have

$$|\hat{J}_{\mathrm{IPW}} - J| \leq c\{\sqrt{k_n/n} + \inf_{h\in\mathbb{H}'} \sup_{q\in\mathbb{Q}'} |\mathbb{E}[(q - q_0\pi)(h_0 - h)]| + \inf_{q\in\mathbb{Q}'} \sup_{h\in\mathbb{H}'} |\mathbb{E}[(q_0\pi - q)h]| + \sqrt{\log(1/\delta)/n}\}.$$

From the assumptions (32) and (33),

$$\inf_{h\in\mathbb{H}'} \sup_{q\in\mathbb{Q}} |\mathbb{E}[(q - q_0\pi)(h_0 - h)]| \leq 2C_{\mathbb{Q}} \inf_{h\in\mathbb{H}'} \mathbb{E}[|h_0 - h|] = O(k_n^{-p/d}), \qquad (\|\mathbb{Q}'\|_\infty \leq C_{\mathbb{Q}'})$$
$$\inf_{q\in\mathbb{Q}} \sup_{h\in\mathbb{H}'} |\mathbb{E}[|(q_0\pi - q)h]| \leq C_{\mathbb{H}} \inf_{q\in\mathbb{Q}'} \mathbb{E}[|q - q_0\pi|] = O(k_n^{-p/d}).. \qquad (\|\mathbb{H}\|_\infty \leq C_{\mathbb{H}})$$

In the end, the final error becomes

$$O(k_n^{-p/d} + \sqrt{k_n/n} + \sqrt{\log(1/\delta)/n}),$$

where the second and third term are statistical error terms, which is derived in the proof of Corollary 1. Balancing these two terms, the final error is $O(n^{-p/(2p+d)})$. $\square$

E.4 Proofs for Section 6

Instead of proving Theorems 3 and 4, we prove Theorems 14 and 15. We often use the following lemmas.

Lemma 21. *Wainwright [2019, Corollary 14.3] Let $\mathcal{N}(\tau; \mathbb{B}_n(\delta; \mathcal{F}), \|\cdot\|_n)$ denote the τ-covering number of $\mathbb{B}_n(\delta; \mathcal{F}) := \{f | \|f\|_n \leq \delta\}$. Then, the critical inequality of the empirical version is satisfied for any $\delta > 0$ s.t.*

$$\frac{1}{\sqrt{n}} \int_{\delta^2/(2\|\mathcal{F}\|_\infty)}^{\delta} \sqrt{\log \mathcal{N}(t, \mathbb{B}_n(\delta; \mathcal{F}), \|\cdot\|_n)} \mathrm{d}t \leq \frac{\delta^2}{\|\mathcal{F}\|_\infty}.$$

Lemma 22 (Theorem 14.1, Wainwright [2019])**.** *Given a star-shaped and b-uniformly bounded function class $\mathcal{G}$, let η_n be any positive solution of the inequality $R_s(\eta; \mathcal{G}) \leq \eta^2/b$. We call this solution to the critical radius of $\mathcal{G}$. Then, for any $t \geq \eta_n$, we have*

$$|\|g\|_n^2 - \|g|_2^2| \leq 0.5\|g\|_2^2 + 0.5t^2, \forall g \in \mathcal{G}.$$

Next, consider a function class $\mathcal{F} : X \to \mathbb{R}$ with loss $l : \mathbb{R} \times Z \to \mathbb{R}$.

Lemma 23 (Lemma 7 [Foster and Syrgkanis, 2019])**.** *Assume $\sup_{f\in\mathcal{F}} \|f\|_\infty \leq c$ and pick any $f^* \in \mathcal{F}$. Define η_n be solution to*

$$\mathcal{R}_n(\eta; \mathrm{star}(\mathcal{F} - f^*)) \leq \eta^2/c.$$

Moreover, assume that the loss $l(\cdot,\cdot)$ is L-Lipschitz in the first argument. Then, for $\eta = \eta_n + \sqrt{c_0 \log(c_1/\delta)/n}$ with some universal constants c_0, c_1, with $1-\delta$,

$$|(\mathbb{E}_n[l(f(x), z)] - \mathbb{E}_n[l(f^*(x), z)]) - (\mathbb{E}[l(f(x), z)] - \mathbb{E}[l(f^*(x), z)])| \lesssim L\eta_n(\|f - f^*\|_2 + \eta_n).$$

Proof of Theorem 14. Define

$$\Phi(h, q) = \mathbb{E}[\{y - h(x, a, w)\}q(x, a, z)]$$
$$\Phi_n(h, q) = \mathbb{E}_n[\{y - h(x, a, w)\}q(x, a, z)]$$
$$\Phi^\lambda(h, q) = \mathbb{E}[\{y - h(x, a, w)\}q(x, a, z)] - \lambda\|q\|_n^2$$
$$\Phi_n^\lambda(h, q) = \mathbb{E}_n[\{y - h(x, a, w)\}q(x, a, z)] - \lambda\|q\|_2^2.$$

where $\|q\|_n^2 = \{\mathbb{E}[q^2]\}^{1/2}$. From Lemma 22, we have

$$\forall q \in \mathbb{Q}, |\|q\|_n - \|q\|_2^2| \leq 0.5\|q\|_2^2 + \eta_n^2 \tag{45}$$

for our choice of $\eta_n := \eta_{h,n} + \sqrt{c_0 \log(c_1/\delta)/n}$ noting $\eta_{h,n}$ upper bounds the critical radius of $\mathbb{Q}$.

Part 1 By definition of the estimator $\hat{h}$ and the assumption $h' \in \mathbb{H}$, we have

$$\sup_{q\in\mathbb{Q}} \Phi_n^\lambda(\hat{h}, q) \leq \sup_{q\in\mathbb{Q}} \Phi_n^\lambda(h', q). \tag{46}$$

From Lemma 23, with probability $1-\delta$, we have

$$\forall q \in \mathbb{Q} : |\Phi_n(h', q) - \Phi(h', q)| \lesssim C_1\{\eta_n\|q\|_2 + \eta_n^2\}. \tag{47}$$

Here, we use $l(a_1, a_2) := a_1 a_2, a_1 = q(s, a), a_2 = y - h(x, a, w)$ is C_1-Lipschitz with respect to a_1 noting $y - h(x, a, w)$ is in $[-C_1, C_1]$ with some constants $C_1 := C_Y + C_{\mathbb{H}}$.

$$|l(a_1, a_2) - l(a_1', a_2)| \leq C_1|a_1 - a_1'|.$$

Thus,

$$\begin{aligned}
\sup_{q\in\mathbb{Q}} \Phi_n^\lambda(h', q) &\leq \sup_{q\in\mathbb{Q}}\{\Phi_n(h', q) - \lambda\|q\|_n^2\} && \text{definition}\\
&\leq \sup_{q\in\mathbb{Q}}\{\Phi(h', q) + cC_1\{\eta_n\|q\|_2 + \eta_n^2\} - \lambda\|q\|_n^2\} && \text{From Eq. (47)}\\
&\leq \sup_{q\in\mathbb{Q}}\{\Phi(h', q) + cC_1\{\eta_n\|q\|_2 + \eta_n^2\} - 0.5\lambda\|q\|_2^2 + \lambda\eta_n^2\} && \text{From Eq. (45)}\\
&\leq \sup_{q\in\mathbb{Q}}\{\Phi(h', q) - 0.25\lambda\|q\|_2^2 + cC_1\{\eta_n\|q\|_2 + \eta_n^2\} - 0.25\lambda\|q\|_2^2 + \lambda\eta_n^2\}\\
&\leq \sup_{q\in\mathbb{Q}}\{\Phi(h', q) - 0.25\lambda\|q\|_2^2 + c(\lambda + C_1^2/\lambda + C_1)\eta_n^2\}.
\end{aligned}$$

In the last line, we use a general inequality $a, b > 0$:

$$\sup_{q\in\mathbb{Q}}(a\|q\|_2 - b\|q\|_2^2) \leq a^2/4b.$$

Moreover,

$$\begin{aligned}
\sup_{q\in\mathbb{Q}} \Phi_n^\lambda(\hat{h}, q) &= \sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h}, q) - \Phi_n(h', q) + \Phi_n(h', q) - \lambda\|q\|_n^2\}\\
&\geq \sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h}, q) - \Phi_n(h', q) - 2\lambda\|q\|_n^2\} + \inf_{q\in\mathbb{Q}}\{\Phi_n(h', q) + \lambda\|q\|_n^2\}\\
&= \sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h}, q) - \Phi_n(h', q) - 2\lambda\|q\|_n^2\} + \inf_{-q\in\mathbb{Q}}\{\Phi_n(h', -q) + \lambda\|q\|_n^2\}\\
&= \sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h}, q) - \Phi_n(h', q) - 2\lambda\|q\|_n^2\} + \inf_{-q\in\mathbb{Q}}\{-\Phi_n(h', q) + \lambda\|q\|_n^2\}\\
&= \sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h}, q) - \Phi_n(h', q) - 2\lambda\|q\|_n^2\} - \sup_{-q\in\mathbb{Q}}\{\Phi_n(h', q) - \lambda\|q\|_n^2\}\\
&= \sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h}, q) - \Phi_n(h', q) - 2\lambda\|q\|_n^2\} - \sup_{-q\in\mathbb{Q}}\{\Phi_n^\lambda(h', q)\}\\
&= \sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h}, q) - \Phi_n(h', q) - 2\lambda\|q\|_n^2\} - \sup_{q\in\mathbb{Q}}\{\Phi_n^\lambda(h', q)\}.
\end{aligned}$$

Here, we use $\mathbb{Q}$ is symmetric. Therefore,

$$\begin{aligned}
\sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h},q)-\Phi_n(h',q)-2\|q\|_n^2\} &\leq \sup_{q\in\mathbb{Q}}\{\Phi_n^{\lambda}(\hat{h},q)\}+\sup_{q\in\mathbb{Q}}\{\Phi_n^{\lambda}(h',q)\} \\
&\leq 2\sup_{q\in\mathbb{Q}}\{\Phi_n^{\lambda}(h',q)\} \qquad \text{(From (46))} \\
&\leq 2\sup_{q\in\mathbb{Q}}\{\Phi(h',q)-0.25\lambda\|q\|_2^2+c(\lambda+C_1^2/\lambda+C_1)\eta_n^2\} \\
&\leq 2\sup_{q\in\mathbb{Q}}\{\|P_z(h_0-h')\|_2\|q\|_2-0.25\lambda\|q\|_2^2+c(\lambda+C_1^2/\lambda+C_1)\eta_n^2\} \\
&\leq c\{\|P_z(h_0-h')\|_2^2/\lambda+(\lambda+C_1^2/\lambda+C_1)\eta_n^2\}.
\end{aligned}$$

Part 2 We fix $q'\in\mathbb{Q}$. Then, we define

$$q_h := \operatorname*{arg\,min}_{q\in\mathbb{Q}}\|q-P_z(h'-h)\|.$$

Suppose $\|q_{\hat{h}}\|_2 \geq \eta_n$, and let $r=\eta_n/\{2\|q_{\hat{h}}\|\}\in[0,0.5]$. Then, noting $\mathbb{Q}$ is star-convex, $\mathbb{Q}$ is symmetric,

$$\sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h},q)-\Phi(h',q)-2\lambda\|q\|_n^2\}\geq r\{\Phi_n(\hat{h},q_{\hat{h}})-\Phi(h',q_{\hat{h}})\}-2\lambda r^2\|q_{\hat{h}}\|_n^2.$$

Here, we use $rq_{\hat{h}}\in\mathbb{Q}$. Then,

$$\begin{aligned}
r^2\|q_{\hat{h}}\|_2 &\lesssim r^2\{1.5\|q_{\hat{h}}\|_2^2+0.5\eta_n^2\} && \text{From Eq. (45)} \\
&\lesssim \eta_n^2. && \text{From definition of } r.
\end{aligned}$$

Therefore,

$$\sup_{q\in\mathbb{Q}}\{\Phi_n(\hat{h},q)-\Phi(h',q)-2\lambda\|q\|_n^2\}\geq r\{\Phi_n(\hat{h},q_{\hat{h}})-\Phi(h',q_{\hat{h}})\}-2\lambda\eta_n^2.$$

Observe that

$$\Phi_n(h,q_h)-\Phi_n(h',q_h)=\mathbb{E}_n[\{h'-h\}q_h(s,a)].$$

Therefore, from Lemma 23, noting η_n upper bounds the critical radius of $\mathrm{star}(\mathcal{G}_h)$, $\forall h\in\mathbb{H}$,

$$\begin{aligned}
&|\Phi_n(h,q_h)-\Phi_n(h',q_h)-\{\Phi(h,q_h)-\Phi(h',q_h)\}| \\
&\leq (\eta_n\|\{h-h'\}q_{\hat{h}}\|_2+\eta_n^2) \\
&\lesssim (C_1\eta_n\|q_{\hat{h}}\|_2+\eta_n^2).
\end{aligned}$$

In the last line, we use $\|h-h'\|_\infty\lesssim C_1$. We invoke Lemma 23 treating $l(a_1,a_2), a_1=(h-h')q$.
Thus,

$$\begin{aligned}
r\{\Phi_n(\hat{h},q_{\hat{h}})-\Phi_n(h',q_{\hat{h}})\} &\geq r\{\Phi(\hat{h},q_{\hat{h}})-\Phi(h',q_{\hat{h}})\}-cr(C_1\eta_n\|q_{\hat{h}}\|_2+\eta_n^2) \\
&\geq r\{\Phi(\hat{h},q_{\hat{h}})-\Phi(h',q_{\hat{h}})\}-crC_1\eta_n\|q_{\hat{h}}\|_2-c0.5\eta_n^2 \quad \text{(Definition of } r) \\
&\overset{(a)}{=} r\mathbb{E}[P_z(-\hat{h}+h')q_{\hat{h}}]-crC_1\eta_n\|q_{\hat{h}}\|_2-c0.5\eta_n^2 \\
&= \frac{\eta_n}{2\|q_{\hat{h}}\|_2}\{\mathbb{E}[P_z(-\hat{h}+h')q_{\hat{h}}]-cC_1\|q_{\hat{h}}\|_2\eta_n\}-0.5\eta_n^2 \\
&\geq 0.5\eta_n\{\|P_z(-\hat{h}+h')\|_2-2\epsilon_n\}-c(1+C_1)\eta_n^2.
\end{aligned}$$

For (a), we use

$$\Phi(\hat{h},q_{\hat{h}})-\Phi(h',q_{\hat{h}})=\mathbb{E}[\{-\hat{h}+h'\}q_{\hat{h}}]=\mathbb{E}[P_z(-\hat{h}+h')q_{\hat{h}}].$$

For (b), we use

$$\begin{aligned}\frac{\mathbb{E}[P_z(-\hat h+h')q_{\hat h}]}{\|q_{\hat h}\|_2} &= \frac{\mathbb{E}[\{-q_{\hat h}+q_{\hat h}+P_z(-\hat h+h')\}q_{\hat h}]}{\|q_{\hat h}\|_2}\\ &\geq \frac{\|q_{\hat h}\|_2^2-\|\{-q_{\hat h}+P_z(-\hat h+h')\}\|_2\|q_{\hat h}\|_2}{\|q_{\hat h}\|_2}\\ &= \|q_{\hat h}\|_2-\|\{-q_{\hat h}+P_z(-\hat h+h')\}\|_2\\ &= \|q_{\hat h}\|_2-\epsilon_n \geq \|P_z(-\hat h+h')\|_2-2\epsilon_n.\end{aligned}$$

Combining all results Thus, $\|q_{\hat h}\|_2 \leq \eta_n$ or

$$\eta_n\{\|P_z(\hat h-h')\|_2-\epsilon_n\}-(1+C_1+\lambda)\eta_n^2 \lesssim \frac{\|P_z(h_0-h')\|_2^2}{\lambda}+(\lambda+C_1^2/\lambda+C_1)\eta_n^2.$$

Therefore, we have

$$\|P_z(\hat h-h')\|_2 \lesssim \|q_{\hat h}\|_2+\|P_z(\hat h-h')-q_{\hat h}\|_2 \leq \eta_n+\epsilon_n$$

or

$$\|P_z(\hat h-h')\|_2 \lesssim (1+\lambda+C_1^2/\lambda+C_1)\eta_n+\frac{\|P_z(h_0-h')\|_2^2}{\eta_n\lambda}+\epsilon_n.$$

Thus, from triangle inequality,

$$\|P_z(h_0-\hat h)\|_2 \lesssim (1+\lambda+C_1^2/\lambda+C_1)\eta_n+\frac{\|P_z(h_0-h')\|_2^2}{\eta_n\lambda}+\epsilon_n+\|P_z(h_0-h')\|_2.$$

□

Proof of Theorem 15. Define

$$\begin{aligned}\Phi(q,h) &= \mathbb{E}[-q(x,a,z)\pi(a|x)h(x,a,w)+\mathbb{E}_{a\sim\pi(a|x)}[h(x,a,w)]]\\ \Phi_n(q,h) &= \mathbb{E}_n[-q(x,a,z)\pi(a|x)h(x,a,w)+\mathbb{E}_{a\sim\pi(a|x)}[h(x,a,w)]]\\ \Phi^\lambda(q,h) &= \mathbb{E}[-q(x,a,z)\pi(a|x)h(x,a,w)+\mathbb{E}_{a\sim\pi(a|x)}[h(x,a,w)]]-\lambda\mathbb{E}[h^2]\\ \Phi_n^\lambda(q,h) &= \mathbb{E}_n[-q(x,a,z)\pi(a|x)h(x,a,w)+\mathbb{E}_{a\sim\pi(a|x)}[h(x,a,w)]]-\lambda\mathbb{E}_n[h^2].\end{aligned}$$

From Lemma 22, we have

$$\forall h\in\mathbb{H},\ |\|h\|_n^2-\|h\|_2^2| \leq 0.5\|h\|_2^2+\eta_n^2 \tag{48}$$

for our choice of $\eta_n := \eta_{q,n}+\sqrt{c_0\log(c_1/\delta)/n}$ noting $\eta_{q,n}$ upper bounds the critical radius of $\mathbb{H}$.

First Part By definition of $\hat q$ and $q'\in\mathbb{Q}$, we have

$$\sup_{h\in\mathbb{H}}\Phi_n^\lambda(\hat q;h)=\sup_{h\in\mathbb{H}}\Phi_n^\lambda(q';h).$$

From Lemma 23, with probability $1-\delta$, we have

$$\forall h\in\mathbb{H}, |\Phi_n(q',h)-\Phi(q',h)| \lesssim C_2\{\eta_n\|h\|_2+\eta_n^2\},\ C_2=C_{\mathbb{Q}}+C_w^{1/2}, \tag{49}$$

where $C_{\mathbb{Q}}$ is an upper bound of the function class $\{q(s,a,z)\pi(a|x)\}$. This is derived as follows. First, we use the loss $l(a_1,a_2)=a_1a_2, a_1=-q(x,a,z)\pi(a|x), a_2=h(x,a,w)$ is $C_{\mathbb{Q}}$-Lipschitz in a_2. This implies

$$|\mathbb{E}_n[-q(x,a,z)\pi(a|x)h(x,a,w)]-\mathbb{E}[-q(x,a,z)\pi(a|x)h(x,a,w)]| \lesssim C_{\mathbb{Q}}\{\eta_n\|h\|_2+\eta_n^2\}.$$

Similarly,

$$|\mathbb{E}_n[\mathcal{T}h] - \mathbb{E}[\mathcal{T}h]| \lesssim \{\eta_n \|\mathcal{T}h\|_2 + \eta_n^2\} \lesssim \{\eta_n C_w^{1/2}\|h\|_2 + \eta_n^2\}.$$

Combining the above twos, Eq. (49) is derived.

Thus,

$$\begin{aligned}
\sup_{h\in\mathbb{H}} \Phi_n^{\lambda}(q',h) &= \sup_{h\in\mathbb{H}}\{\Phi_n(q',h) - \lambda\|h\|_n^2\} \\
&\leq \sup_{h\in\mathbb{H}}\{\Phi(q',h) + cC_2(\eta_n\|h\|_2 + \eta_n^2) - \lambda\|h\|_n^2\} \\
&\leq \sup_{h\in\mathbb{H}}\{\Phi(q',h) + cC_2(\eta_n\|h\|_2 + \eta_n^2) - 0.5\lambda\|h\|_2^2 + \lambda\eta_n^2\} \qquad \text{(Use Eq. (48))} \\
&\leq \sup_{h\in\mathbb{H}}\{\Phi(q',h) - 0.25\lambda\|h\|_2^2 + cC_2(\eta_n\|h\|_2 + \eta_n^2) - 0.25\lambda\|h\|_2^2 + \lambda\eta_n^2\} \\
&\leq \sup_{h\in\mathbb{H}}\{\Phi(q',h) - 0.25\lambda\|h\|_2^2 + c\{C_2^2/\lambda + \lambda + C_2\}\eta_n^2\}.
\end{aligned}$$

In the last line, we use a general inequality $a, b > 0$,

$$a\|h\|_2 - b\|h\|_2^2 \leq a^2/4b.$$

Moreover,

$$\begin{aligned}
&\sup_{h\in\mathbb{H}} \Phi_n^{\lambda}(\hat{q},h) \\
&= \sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h) - \Phi(q',h) + \Phi(q',h) - \|h\|_n^2 - \|\mathcal{T}h\|_n^2\} \\
&\geq \sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h) - \Phi(q',h) - 2\|h\|_n^2 - 2\|\mathcal{T}h\|_n^2\} + \inf_{h\in\mathbb{H}}\{\Phi(q',h) - \|h\|_n^2 - \|\mathcal{T}h\|_n^2\} \\
&= \sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h) - \Phi(q',h) - 2\|h\|_n^2 - 2\|\mathcal{T}h\|_n^2\} + \inf_{-h\in\mathbb{H}}\{\Phi(q',-h) - \|h\|_n^2 - \|\mathcal{T}h\|_n^2\} \\
&= \sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h) - \Phi(q',h) - 2\|h\|_n^2 - 2\|\mathcal{T}h\|_n^2\} - \sup_{-h\in\mathbb{H}}\{\Phi(q',h) + \|h\|_n^2 + \|\mathcal{T}h\|_n^2\} \\
&= \sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h) - \Phi(q',h) - 2\|h\|_n^2 - 2\|\mathcal{T}h\|_n^2\} - \sup_{h\in\mathbb{H}}\{\Phi^{\lambda}(q',h)\}.
\end{aligned}$$

Here, we use $\mathbb{H}$ is symmetric. Therefore,

$$\begin{aligned}
\sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h) - \Phi(q',h) - 2\|h\|_n^2\} &\leq \sup_{h\in\mathbb{H}}\{\Phi_n^{\lambda}(\hat{q},h)\} + \sup_{h\in\mathbb{H}}\{\Phi^{\lambda}(q',h)\} \\
&\leq 2\sup_{h\in\mathbb{H}}\{\Phi_n^{\lambda}(q',h)\} \\
&\leq \sup_{h\in\mathbb{H}}\{\Phi(q',h) - 0.25\lambda\|h\|_2^2 + c\{C_2^2/\lambda + \lambda + C_2\}\eta_n^2\} \\
&\leq \sup_{h\in\mathbb{H}}\{\|P_w(q'-q_0)\|_2\|h\|_2 - 0.25\lambda\|h\|_2^2 + c\{C_2^2/\lambda + \lambda + C_2\}\eta_n^2\} \\
&\leq c\{\|P_w(q'-q_0)\|_2/\lambda + C_2^2/\lambda + \lambda + C_2\}\eta_n^2\}.
\end{aligned}$$

Second Part We fix q'. We define

$$h_q \coloneqq \operatorname*{arg\,min}_{h\in\mathbb{H}} \|h - P_w\{(q-q')\pi\}\|_2.$$

Suppose $\|h_{\hat{q}}\| \geq \eta_n$, and let $r = \eta_n/\{2\|h_{\hat{q}}\|_2\} \in (0, 0.5]$. Then, noting $\mathbb{H}$ is star-convex and symmetric,

$$\sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h) - \Phi_n(q',h) - 2\lambda\|h\|_n^2\} \geq r\{\Phi_n(\hat{q},h_{\hat{q}}) - \Phi_n(q',h_{\hat{q}})\} - 2\lambda r^2\|h_{\hat{q}}\|_n^2.$$

Here, we use $rh_{\hat{q}} \in \mathbb{H}$. Then, from Lemma 22,

$$\begin{aligned}
r^2\|h_{\hat{q}}\|_n^2 &\leq cr^2\{\|h_{\hat{q}}\|_2^2 + \eta_n^2\} \\
&\leq c\eta_n^2. \qquad \text{(Definition of } r\text{)}
\end{aligned}$$

Therefore,

$$\sup_{h\in\mathbb{H}}\{\Phi_n(\hat{q},h)-\Phi_n(q',h)-2\lambda\|h\|_n^2\}\geq r\{\Phi_n(\hat{q},h_{\hat{q}})-\Phi_n(q',h_{\hat{q}})\}-c\lambda\eta_n^2.$$

Observe that

$$\Phi_n(q,h_q)-\Phi_n(q',h_q)=\mathbb{E}_n[(-q(x,a,z)+q'(x,a,z))\pi(a|x)h_q(a,x,y)].$$

Therefore, from Lemma 23, noting η_n upper bounds the critical radius of $\mathcal{G}_q$, setting $l(a_1,a_2)=a_1a_2, a_1=(q-q')\pi h_q$, we have $\forall q\in\mathbb{Q}$,

$$\begin{aligned}&|\Phi_n(q,h_q)-\Phi_n(q',h_q)-\{\Phi(q,h_q)-\Phi(q',h_q)\}|\\&\leq c(\eta_n\mathbb{E}[\{(q'(x,a,z)-q(x,a,z))\pi(a|x)h_q(a,x,y)\}^2]^{1/2}+\eta_n^2)\\&\leq c(\eta_n\|h_q\|_2C_2+\eta_n^2).\end{aligned}$$

Here, we use $\|(q'(x,a,z)-q(x,a,z))\pi(a|x)\|_\infty\leq C_2$. Then,

$$\begin{aligned}&r\{\Phi(\hat{q},h_{\hat{q}})-\Phi(q',h_{\hat{q}})\}\\&\geq r\{\Phi(\hat{q},h_{\hat{q}})-\Phi(q',h_{\hat{q}})\}-rc(\eta_n\|h_q\|_2C_2+\eta_n^2)\\&\geq r\{\Phi(\hat{q},h_{\hat{q}})-\Phi(q',h_{\hat{q}})\}-rc(\eta_n\|h_q\|_2C_2)-0.5c\eta_n^2\\&\overset{(a)}{=}r\mathbb{E}[P_w\{-q'+\hat{q}\}(a,x,w)h_q(a,x,w)]-rc(\eta_n\|h_q\|_2C_2)-0.5c\eta_n^2\\&=\frac{\eta_n}{2\{\|h_{\hat{q}}\|_2\}}(\mathbb{E}[P_w\{-q'+\hat{q}\}(a,x,w)h_q(a,x,w)]-c\eta_nC_2\|h_{\hat{q}}\|_2)-0.5c\eta_n^2\\&\overset{(b)}{\geq}0.5\eta_n(\|P_w\{(-q'+\hat{q})\pi\}\|_2-2\epsilon_n)-c\{1+C_2\}\eta_n^2.\end{aligned}$$

For (a), we use

$$\begin{aligned}\Phi(q,h_q)-\Phi(q',h_q)&=\mathbb{E}[\{-q(x,a,z)+q'(x,a,z)\}\pi(a|x)h_q(a,x,w)]\\&=\mathbb{E}[\{\mathbb{E}[-q(x,a,z)+q'(x,a,z)|x,a,w]\}\pi(a|x)h_q(a,x,w)]\\&=\mathbb{E}[P_w\{q'-q\}(a,x,w)h_q(a,x,w)].\end{aligned}$$

For (b), we use

$$\begin{aligned}\frac{\mathbb{E}[\pi P_w\{-q'+\hat{q}\}h_q]}{\|h_{\hat{q}}\|_2}&=\frac{\mathbb{E}[[-h_{\hat{q}}+h_{\hat{q}}+\pi P_w\{-q'+\hat{q}\}]h_q]}{\|h_{\hat{q}}\|_2}\\&=\frac{\|h_{\hat{q}}\|_2^2-\|h_{\hat{q}}\|_2\|-h_{\hat{q}}+\pi P_w\{-q'+\hat{q}\}\|_2}{\|h_{\hat{q}}\|_2}\\&\geq\|h_{\hat{q}}\|_2-\epsilon_n\geq\|\pi P_w\{-q'+\hat{q}\}\|_2-2\epsilon_n.\end{aligned}$$

Combining all results Thus, $\|h_{\hat{q}}\|\leq\eta_n$ or

$$\eta_n\{\|P_w(\pi\hat{q}-\pi q')\|_2-\epsilon_n\}-(1+C_2+\lambda)\eta_n^2\lesssim\|\pi P_w(q'-q_0)\|_2^2/\lambda+\{C_2^2/\lambda+\lambda+C_2\}\eta_n^2$$

Therefore,

$$\|P_w(\pi\hat{q}-\pi q')\|_2\lesssim\|h_{\hat{q}}-\pi P_w(\pi\hat{q}-\pi q')\|_2+\|h_{\hat{q}}\|_2\leq\eta_n+\epsilon_n.$$

and

$$\|P_w(\pi\hat{q}-\pi q')\|_2\lesssim\|\pi P_w(q'-q_0)\|_2^2/\lambda\eta_n+\{1+C_2^2/\lambda+\lambda+C_2\}\eta_n+\epsilon_n.$$

Finally, from triangle inequality,

$$\|P_w(\pi\hat{q}-\pi q_0)\|_2\lesssim\|\pi P_w(q'-q_0)\|_2^2/\lambda\eta_n+\{1+C_2^2/\lambda+\lambda+C_2\}\eta_n+\epsilon_n+\|P_w(\pi q_0-\pi q')\|_2.$$

$\square$

Proof of Theorem 5. We use notation in the proof of Theorem 11. Then,

$$\begin{aligned}
\mathbb{E}[\{P_z(\hat{h}-h_0)\}^2] &\leq \sup_{q\in\mathbb{Q}'} \mathbb{E}[P_z(\hat{h}-h_0)q] && (P_z(\mathbb{H}-h_0)\subseteq \mathbb{Q}') \\
&\leq \sup_{q'\in\mathbb{Q}'} \mathbb{E}[\{\hat{h}-h_0\}q] \\
&\leq \sup_{q'\in\mathbb{Q}'} |\mathbb{E}[\{\hat{h}-h_0\}q]| && (\mathbb{Q} \text{ is symmetric}) \\
&\leq 4 \sup_{q\in\mathbb{Q}', h\in\mathbb{H}} |(\mathbb{E}-\mathbb{E}_n)[\{y-h\}q]|. && (\text{See the argument in Theorem 11})
\end{aligned}$$

We use notation in the proof of Theorem 12. Then,

$$\begin{aligned}
\mathbb{E}[\{\pi P_w(\hat{q}-q_0)\}^2] &\leq \sup_{h\in\mathbb{H}'} \mathbb{E}[\pi P_w(\hat{q}-q_0)h] && (\pi P_w(\mathbb{Q}-q_0)\subseteq \mathbb{H}') \\
&\leq \sup_{h\in\mathbb{H}'} \mathbb{E}[(\hat{q}-q_0)h] \\
&\leq \sup_{h\in\mathbb{H}'} |\mathbb{E}[(\hat{q}-q_0)h]| && (\mathbb{H}' \text{ is symmetric}) \\
&\leq 4 \sup_{q\in\mathbb{Q}, h\in\mathbb{H}'} |(\mathbb{E}-\mathbb{E}_n)[-q\pi h+\mathcal{T}h]|. && (\text{See the argument in Theorem 12})
\end{aligned}$$

□

Proof of Corollary 4.

VC subgraph clases We define the empirical L^2-norm as $\|f(\cdot)\|_{n,2} = \{1/n\sum f(s_i,a_i)^2\}^{1/2}$, L^∞-norm as $\|f(\cdot)\|_{n,\infty} = \max_{1\leq i\leq n}|f(s_i,a_i)|$. Then, we have

$$\begin{aligned}
&\log\mathcal{N}(t,(\mathbb{H}-h_0)\mathbb{Q}',\|\cdot\|_{n,\infty}) \\
&\leq \log\mathcal{N}(t,\mathbb{H}\mathbb{Q}',\|\cdot\|_{n,\infty}) + \log\mathcal{N}(t,h_0\mathbb{Q}',\|\cdot\|_{n,\infty}) \\
&\leq \log\mathcal{N}(t/\{2C_{\mathbb{H}}\},\mathbb{Q}',\|\cdot\|_{n,\infty}) + \log\mathcal{N}(t/\{2C_{\mathbb{Q}}\},\mathbb{H},\|\cdot\|_{n,\infty}) + \log\mathcal{N}(t/C_{\mathbb{H}},\mathbb{Q}',\|\cdot\|_{n,\infty}).
\end{aligned}$$

With the above in mind,

$$\begin{aligned}
&\int_0^\eta \sqrt{\frac{\log\mathcal{N}(t,(\mathbb{H}-h_0)\mathbb{Q}',\|\cdot\|_{n,2})}{n}}\mathrm{d}(t) \\
&\leq \int_0^\eta \sqrt{\frac{\log\mathcal{N}(t,(\mathbb{H}-h_0)\mathbb{Q}',\|\cdot\|_{n,\infty})}{n}}\mathrm{d}(t) && (\|\cdot\|_{n,2}\leq\|\cdot\|_{n,\infty}) \\
&\leq \int_0^\eta \sqrt{\frac{2\log\mathcal{N}(t/2C_{\mathbb{H}},\mathbb{Q}',\|\cdot\|_{n,\infty})}{n}} + \sqrt{\frac{\log\mathcal{N}(t/2C_{\mathbb{Q}},\mathbb{H},\|\cdot\|_{n,\infty})}{n}}\mathrm{d}(t) \\
&\leq \int_0^\eta \sqrt{\frac{2\log\mathcal{N}(t/2C_{\mathbb{H}},\mathbb{Q}',\sqrt{n}\|\cdot\|_{n,2})}{n}} + \sqrt{\frac{\log\mathcal{N}(t/2C_{\mathbb{Q}},\mathbb{H},\sqrt{n}\|\cdot\|_{n,2})}{n}}\mathrm{d}(t). \\
& && (\|\cdot\|_{n,\infty}\leq\|\cdot\|_{n,2}\sqrt{n}) \\
&= O\left(\int_0^\eta \sqrt{\max(V(\mathbb{H}),V(\mathbb{Q}')\log(1/t)}\mathrm{d}(t)\frac{\log n}{\sqrt{n}}\right) && ([\text{van der Vaart, 1998, Lemma 19.15}]) \\
&= O\left(\sqrt{\max(V(\mathbb{H}),V(\mathbb{Q}')}\frac{\log n}{\sqrt{n}}\eta\log(1/\eta)\right).
\end{aligned}$$

Then, the critical inequality in Lemma 21 becomes

$$O\left(\sqrt{\max(V(\mathbb{H}),V(\mathbb{Q})}\frac{\log n}{\sqrt{n}}\eta\log(1/\eta)\right)\leq\eta^2.$$

This is satisfied with

$$\eta_h = O\left(\sqrt{\max(V(\mathbb{H}),V(\mathbb{Q})}\frac{\log n}{\sqrt{n}}\log\left(\sqrt{\max(V(\mathbb{H}),V(\mathbb{Q})}\frac{\log n}{\sqrt{n}}\right)\right).$$

Nonparametric models We have

$$
\begin{aligned}
&\log \mathcal{N}(t,(\mathbb{H}-h_0)\mathbb{Q}',\|\cdot\|_{n,\infty})\\
&\leq \log \mathcal{N}(t,\mathbb{H}\mathbb{Q}',\|\cdot\|_{n,\infty})+\log \mathcal{N}(t,h_0\mathbb{Q}',\|\cdot\|_{n,\infty})\\
&\leq \log \mathcal{N}(t/\{2C_{\mathbb{H}}\},\mathbb{Q}',\|\cdot\|_{n,\infty})+\log \mathcal{N}(t/\{2C_{\mathbb{Q}}\},\mathbb{H},\|\cdot\|_{n,\infty})+\log \mathcal{N}(t/C_{\mathbb{H}},\mathbb{Q}',\|\cdot\|_{n,\infty}).
\end{aligned}
$$

With the above in mind,

$$
\begin{aligned}
&\int_0^\eta \sqrt{\frac{\log \mathcal{N}(t,(\mathbb{H}-h_0)\mathbb{Q}',\|\cdot\|_{n,2})}{n}}\mathrm{d}(t)\\
&\leq \int_0^\eta \sqrt{\frac{\log \mathcal{N}(t,(\mathbb{H}-h_0)\mathbb{Q}',\|\cdot\|_{\infty})}{n}}\mathrm{d}(t) && (\|\cdot\|_{n,2}\leq\|\cdot\|_\infty)\\
&\leq \int_0^\eta \sqrt{\frac{2\log \mathcal{N}(t/2C_{\mathbb{H}},\mathbb{Q}',\|\cdot\|_{\infty})}{n}}+\sqrt{\frac{\log \mathcal{N}(t/2C_{\mathbb{Q}},\mathbb{H},\|\cdot\|_{\infty})}{n}}\mathrm{d}(t)
\end{aligned}
$$

Then, the critical inequality in Lemma 21 becomes

$$
\begin{cases}
n^{-1/2}\eta^{1-0.5\beta} & \leq \eta^2,\ (\beta\geq 2)\\
n^{-1/2}\log(\eta) & \leq \eta^2, (\beta=2)\\
n^{-1/2}\eta^{-\beta+2} & \leq \eta^2, (\beta\leq 2).
\end{cases}
$$

By solving the above equation wrt η, the rate η_h is derived.

□

Proof of Corollary 5. We consider some $h'\in\mathbb{H}$ which specified later. From Theorem 14, for $h_0\in\mathbb{H}_0^{\mathrm{obs}}\cap\Lambda^\alpha([0,1]^d)$, we have

$$
\|P_w(\hat{h}-h_0)\|_2 = O\left(\eta_h'+\epsilon_n+\frac{\|P_w(h'-h_0)\|_2}{\eta_h'}+\|P_w(h'-h_0)\|_2\right).
$$

where $\eta_h'=\eta_h+c_0\sqrt{\log(c_1/\delta)}n$, and η_h is the maximum of critical radii of $\mathbb{Q}'$ and $\mathcal{G}_h$, and $\epsilon_n=\sup_{h\in\mathbb{H}}\inf_{q\in\mathbb{Q}'}\|q-P_w(h-h')\|_2$.

First, from Corollary 4,

$$
\eta_h' = O\left(\sqrt{k_n}\frac{\log n}{\sqrt{n}}\log(\sqrt{k_n}\frac{\log n}{\sqrt{n}})+\sqrt{(1+\log(1/\delta))/n}\right).
$$

Next, we take h' s.t.

$$
\|P_w(h'-h_0)\|_2\leq\|h_0-h'\|_\infty = O(k_n^{-p/d}).
$$

from the assumption. Besides, for any $g:=P_w(h-h_0)$, we have$g\in\Lambda^\alpha[0,1]^d)$. Thus, there exists $q\in\mathbb{Q}'$ s.t.

$$
\begin{aligned}
\|q-P_w(h-h')\|_2 &\leq \|q-P_w(h-h_0)\|_\infty+\|P_w(h'-h_0)\|_2\\
&=\|q-P_w(h-h_0)\|_\infty+O(k_n^{-p/d})=O(k_n^{-p/d}),
\end{aligned}
$$

which leads to $\epsilon_n=O(k_n^{-p/d})$.

Finally, the error bound is

$$
\|P_w(h_0-\hat{h})\|_2=\tilde{O}\left(\sqrt{k_n/n}+k_n^{-p/d}+\frac{k_n^{-2p/d}}{\sqrt{k_n/n}}\right).
$$

This is equal to $\tilde{O}(n^{-p/(2p+d)})$ by balancing the all terms. □

Proof of Lemma 11. Obvious from Lemma 5. □

Proof of Theorem 6.

IPW estimators The error is decomposed into the three terms:

$$\begin{aligned}|\hat{J}_{\mathrm{IPW}} - J| &= |\mathbb{E}_n[\hat{q}\pi y] - J| \\ &\leq |\{\mathbb{E}_n[(\hat{q} - q_0)\pi y] - \mathbb{E}[(\hat{q} - q_0)\pi y]\}| + |\{\mathbb{E}[\hat{q}\pi y] - \mathbb{E}[q_0\pi y]\}| + |\mathbb{E}_n[q_0\pi y] - J|. \quad (50)\end{aligned}$$

The first term in (50) is upper bounded by

$$2\sup_{q\in\mathbb{Q}} |(\mathbb{E}_n - \mathbb{E})[q\pi y]|.$$

The second term in (50) is

$$\begin{aligned}\mathbb{E}[\hat{q}\pi y] - \mathbb{E}[q_0\pi y] &= \mathbb{E}[\{(\hat{q} - q_0)\pi\}y] = \mathbb{E}[\{(\hat{q} - q_0)\pi\}\mathbb{E}[y|x, a, z]] \\ &= \mathbb{E}[\{(\hat{q} - q_0)\pi\}\mathbb{E}[h_0|x, a, z]] \qquad \text{(Use the assumption } \mathbb{H}_0 \neq \emptyset) \\ &= \mathbb{E}[\{(\hat{q} - q_0)\pi\}h_0] \\ &= \mathbb{E}[P_w\{(\hat{q} - q_0)\pi\}h_0].\end{aligned}$$

Thus, from CS inequality, this term is upper-bounded by

$$\|P_w\{(\hat{q} - q_0)\pi\}\|_2\|h_0\|_2.$$

The third term in (50) is upper-bounded by Bernstein inequality. This concludes

$$|\hat{J}_{\mathrm{IPW}} - J| \leq O(\mathcal{R}_n(\infty; \pi\mathbb{Q}) + \eta_q + \sqrt{(1 + \log(1/\delta))/n}). \quad (51)$$

REG estimators The error is decomposed into the three terms:

$$|\hat{J}_{\mathrm{REG}} - J| \leq |\{\mathbb{E}_n[\mathcal{T}(\hat{h} - h_0)] - \mathbb{E}[\mathcal{T}(\hat{h} - h_0)]\}| + |\{\mathbb{E}[\mathcal{T}\hat{h}] - \mathbb{E}[\mathcal{T}h_0]\}| + |\mathbb{E}_n[\mathcal{T}h_0] - J|. \quad (52)$$

The first term in Eq. (52) is upper bounded by

$$2|\sup_{h\in\mathbb{H}}(\mathbb{E}_n - \mathbb{E})[\mathcal{T}h]|.$$

The second term in Eq. (52) is upper-bounded as follows:

$$\begin{aligned}|\mathbb{E}[\mathcal{T}\hat{h}] - \mathbb{E}[\mathcal{T}h_0]| &= |\mathbb{E}[\pi(a|x)/f(a|x, w)\{\hat{h} - h_0\}]| \\ &= |\mathbb{E}[\mathbb{E}[\pi(a|x)q_0(a, x, z)|a, x, w]\{\hat{h} - h_0\}]| \qquad \text{(Use the assumption } \mathbb{Q}_0 \neq \emptyset) \\ &= |\mathbb{E}[\pi(a|x)q_0(a, x, z)\{\hat{h} - h_0\}]| \\ &= |\mathbb{E}[\pi(a|x)q_0(a, x, z)P_z\{\hat{h} - h_0\}]| \leq \|q_0\pi\|_2\|P_z\{\hat{h} - h_0\}\|_2.\end{aligned}$$

The third term in Eq. (52) is upper-bounded by Bernstein inequality. This concludes

$$|\hat{J}_{\mathrm{REG}} - J| \leq O(\mathcal{R}_n(\infty; \mathcal{T}\mathbb{H}) + \eta_h + \sqrt{(1 + \log(1/\delta))/n}). \quad (53)$$

□

Proof of Theorem 7. The proof is similar to Theorem 6. The error is decomposed into the three terms:

$$\begin{aligned}|\hat{J}_{\mathrm{DR}} - J| \leq\, & |\{\mathbb{E}_n[\{\pi\hat{q}\{y - \hat{h}\} + \mathcal{T}\hat{h}\} - \{\pi q_0\{y - h_0\} + \mathcal{T}h_0\}] - \mathbb{E}[\pi\hat{q}\{y - \hat{h}\} + \mathcal{T}\hat{h}] - \{\pi q_0\{y - h_0\} + \mathcal{T}h_0\}]\}| \\ &+ |\{\mathbb{E}[\{\pi\hat{q}\{y - \hat{h}\} + \mathcal{T}\hat{h}\}] - \mathbb{E}[\{\pi q_0\{y - h_0\} + \mathcal{T}h_0\}]\}| \\ &+ |\mathbb{E}_n[\{\pi q_0\{y - h_0\} + \mathcal{T}h_0\}] - J|. \quad (54)\end{aligned}$$

The first term is upper-bounded by

$$2|\sup_{h\in\mathbb{H}, q\in\mathbb{Q}}(\mathbb{E}_n - \mathbb{E})[\pi q\{y - h\} + \mathcal{T}h]|.$$

If the condition of Theorem 4 holds, the second term is upper-bounded by

$$\|P_w\{(\hat q - q_0)\pi\}\|_2 \sup_{h\in\mathbb{H}}\|h_0 - h\|_2.$$

If the condition of Theorem 3 holds, the second term is upper-bounded by

$$\|P_z\{\hat h - h_0\}\|_2 \sup_{q\in\mathbb{Q}}\|\{q_0 - q\}\pi\|_2.$$

The third term is upper-bounded by Bernstein's inequality. Then, $|\hat J_{\mathrm{DR}} - J|$ is upper-bunded as the statements,

□

E.5 Proofs for Section 6.3

Proof of Theorem 8. We define

$$\eta'_h = \eta_h + \sqrt{1+\log(1/\delta)/n},\ \eta'_q = \eta_q + \sqrt{1+\log(1/\delta)/n}.$$

Recall we use sample splitting. The estimator is defined as

$$\mathbb{E}_{n_1}[\phi(\hat q^{(0)}, \hat h^{(0)})] + \mathbb{E}_{n_0}[\phi(\hat q^{(1)}, \hat h^{(1)})].$$

where $\mathbb{E}_{n_1}[\cdot]$ is the empirical average over $\mathcal{D}_1$ and $\mathbb{E}_{n_0}[\cdot]$ is the empirical average over $\mathcal{D}_0$. We define

$$\phi(x,a,w,z,y;h,q) = \pi(a|s)q(s,a)\{y - h(s,a)\} + (\mathcal{T}h)(s,a).$$

We take arbitrary elements h_0, q_0 s.t. $h_0\in\mathbb{H}_0^{\mathrm{obs}}, q_0\in\mathbb{Q}_0^{\mathrm{obs}}$. Then, the bias is decomposed into the three terms:

$$\mathbb{E}_{n_1}[\phi(\hat q,\hat h)] - J = (\mathbb{E}_{n_1} - \mathbb{E})[\phi(\hat q,\hat h) - \phi(q_0,h_0)] + \mathbb{E}[\phi(\hat q,\hat h) - \phi(q_0,h_0)] + \mathbb{E}_{n_1}[\phi(q_0,h_0)] - J. \tag{55}$$

We omit (1) from $\hat q, \hat h$. Recall with probability $1-2\delta$, from the assumption (38), we have

$$\|P_z(\hat h - h_0)\|_2 \le O(\eta'_h),\ \|\mathbb{E}[\hat h(a,x,w) - y|a,x,u]\|_2 \le O(\tau^{\mathbb{H}}_{1,n}\eta'_h),$$
$$\|P_w(\hat q - q_0)\|_2 \le O(\eta'_q),\ \|\mathbb{E}[\pi(a|x)\{\hat q(a,x,z) - 1/f(a|x,w)\}|a,x,u]\|_2 \le O(\tau^{\mathbb{Q}}_{1,n}\eta'_q)$$

We always condition on this event in the following proof.

First Term With $1-\delta$, the first term in (55) is obviously $O(\sqrt{\log(1/\delta)/n_1})$ from Bernstein's inequality noting

$$\|\phi(x,a,r,z,w,y)\|_\infty \le C_{\mathbb{Q}}\{C_Y + C_{\mathbb{H}}\} + C_{\mathbb{H}}.$$

Second Term Next, we analyze the second term in (55) . By some algebra, this is equal to

$$\begin{aligned}
&\mathbb{E}[\phi(\hat q,\hat h) - \phi(q_0,h_0)|\mathcal{D}_0]\\
&= \mathbb{E}[\pi(\hat q - q_0)(y - h_0)|\mathcal{D}_0] + \mathbb{E}[\pi q_0\{-\hat h + h_0\} + \mathcal{T}(\hat h - h_0)|\mathcal{D}_0] + \mathbb{E}[\pi\{\hat q - q_0\}\{h_0 - \hat h\}|\mathcal{D}_0]\\
&= \mathbb{E}[\pi\{\hat q - q_0\}\{h_0 - \hat h\}|\mathcal{D}_0]\\
&= \mathbb{E}[\mathbb{E}[\pi\{\hat q - q_0\}|A,X,U]\mathbb{E}[\{h_0 - \hat h\}|A,X,U]|\mathcal{D}_0] && (Z\perp W|A,X,U)\\
&\le \|\mathbb{E}[\hat h(a,x,w) - y|a,x,u]\|_2\|\mathbb{E}[\pi(a|x)\{\hat q(a,x,z) - 1/f(a|x,w)\}|a,x,u]\|_2 && \text{(CS inequality)}\\
&= O(\tau^{\mathbb{H}}_{1,n}\tau^{\mathbb{Q}}_{1,n}\eta'_h\eta'_q).
\end{aligned}$$

Third Term With $1-\delta$, the third term in (55) is obviously $O(\sqrt{\log(1/\delta)/n_1})$ from Bernstein's inequality

Combining all terms With $1-\delta$,

$$|\hat{J}_{\mathrm{DR}}-J| = O(\sqrt{1+\log(1/\delta)/n}+\tau^{\mathbb{H}}_{1,n}\tau^{\mathbb{Q}}_{1,n}\eta'_h\eta'_q) = O(\tau^{\mathbb{H}}_{1,n}\tau^{\mathbb{Q}}_{1,n}\eta'_h\eta'_q).$$

$\square$

Proof of Theorem 9. We prove the following stronger statement. Denote

$$\|P_z(\hat{h}-h_0)\|_2 \le C_h\eta'_h,\ \|P_w(\hat{q}-q_0)\|_2 \le C_q\eta'_q.$$

With probability $1-(\delta+10\delta')$,

$$|\hat{J}_{\mathrm{DR}}-J| \le \sqrt{\frac{2\mathrm{var}[\pi q_0\{y-h_0\}+\mathcal{T}h_0]\log(1/\delta)}{n}} + \frac{2\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{Q}}C_Y+C_{\mathbb{H}}\}\log(1/\delta)}{3n} + \min(\tau^{\mathbb{H}}_{2,n},\tau^{\mathbb{Q}}_{2,n})C_hC_q\eta'_h\eta'_q \tag{56}$$

$$+\,c\left\{[\tau^{\mathbb{Q}}_{2,n}\eta'_qC_q\{C_Y+C_{\mathbb{H}}\}+\tau^{\mathbb{H}}_{2,n}C_h\eta'_h\{C_{\mathbb{Q}}+\sqrt{C_w}\}]\sqrt{\frac{\log(1/\delta')}{n_1}}+\frac{\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{Q}}C_Y+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}\right\}.$$

where c is a universal constant and

$$V_{\mathrm{eff}} = \mathrm{var}[\pi q_0\{y-h_0\}+\mathcal{T}h_0].$$

Then, the asymptotic statement is immediately given.

Recall we use sample splitting. The estimator is defined as

$$\mathbb{E}_{n_1}[\phi(\hat{q}^{(0)},\hat{h}^{(0)})]+\mathbb{E}_{n_0}[\phi(\hat{q}^{(1)},\hat{h}^{(1)})]$$

where $\mathbb{E}_{n_1}[\cdot]$ is the empirical average over $\mathcal{D}_1$ and $\mathbb{E}_{n_0}[\cdot]$ is the empirical average over $\mathcal{D}_0$. We define

$$\phi(s,a,r,s';h,q) = \pi(a|s)q(s,a)\{y-h(s,a)\}+(\mathcal{T}h)(s,a).$$

The bias is decomposed into the three terms:

$$\mathbb{E}_{n_1}[\phi(\hat{q},\hat{h})]-J = (\mathbb{E}_{n_1}-\mathbb{E})[\phi(\hat{q},\hat{h})-\phi(q_0,h_0)]+\mathbb{E}[\phi(\hat{q},\hat{h})-\phi(q_0,h_0)]+\mathbb{E}_{n_1}[\phi(q_0,h_0)]-J. \tag{57}$$

Recall with probability $1-2\delta'$, from the assumption (39), we have

$$\|(\hat{h}-h_0)\|_2 \le \tau^{\mathbb{H}}_{2,n}C_h\eta'_h,\ \|(\hat{q}-q_0)\|_2 \le \tau^{\mathbb{Q}}_{2,n}C_q\eta'_q.$$

We always condition on this event in the proof.

First Term We analyze the first term in (57). Note we have

$$(\mathbb{E}_{n_1}-\mathbb{E})[\phi(\hat{q},\hat{h})-\phi(q_0,h_0)|\mathcal{D}_0] \tag{58}$$
$$= (\mathbb{E}_{n_1}-\mathbb{E})[\pi(\hat{q}-q_0)(y-h_0)|\mathcal{D}_0] \tag{59}$$
$$+\,(\mathbb{E}_{n_1}-\mathbb{E})[\pi q_0\{-\hat{h}+h_0\}+\mathcal{T}(\hat{h}-h_0)|\mathcal{D}_0] \tag{60}$$
$$+\,(\mathbb{E}_{n_1}-\mathbb{E})[\pi\{\hat{q}-q_0\}\{h_0-\hat{h}\}|\mathcal{D}_0]. \tag{61}$$

From Bernstein's inequality, with probability $1-\delta'$, (59) is

$$\begin{aligned}
&|(\mathbb{E}_{n_1}-\mathbb{E})[\pi(\hat{q}-q_0)(y-h_0)|\mathcal{D}_0]| \\
&\le \sqrt{\frac{2\mathbb{E}[\{\pi(\hat{q}-q_0)\}^2\{y-h_0\}^2|\mathcal{D}_0]\log(1/\delta')}{n_1}}+\frac{2\|\pi(\hat{q}-q_0)(y-h_0)\|_\infty\log(1/\delta')}{3n_1} \\
&\le c\left\{\sqrt{\frac{(\tau^{\mathbb{Q}}_{2,n})^2C_q^2\eta_q^2\{C_Y^2+C_{\mathbb{H}}^2\}\log(1/\delta')}{n_1}}+\frac{C_{\mathbb{Q}}\{C_Y+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}.\right\}. \\
&\qquad\qquad (\|y\|_\infty \le C_y,\ \|h_0\|_\infty \le C_{\mathbb{H}}, \|\pi q\|_\infty \le C_{\mathbb{Q}})
\end{aligned}$$

From Bernstein's inequality, with probability $1-\delta'$, (60) is

$$
\begin{aligned}
&|(\mathbb{E}_{n_1}-\mathbb{E})[q_0\{-\hat{h}+h_0\}+\mathcal{T}(\hat{h}-h_0)|\mathcal{D}_0]| \\
&\leq c\left\{\sqrt{\frac{\{\mathbb{E}[\{\pi q_0\{-\hat{h}+h_0\}\}^2|\mathcal{D}_0]+\mathbb{E}[\{\mathcal{T}(\hat{h}-h_0)\}^2|\mathcal{D}_0]\}\log(1/\delta')}{n_1}}+\frac{\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}\right\} \\
&\leq c\left\{\sqrt{\frac{\{\mathbb{E}[\{-\hat{h}+h_0\}^2|\mathcal{D}_0]C_{\mathbb{Q}}^2+\mathbb{E}[\{\mathcal{T}(\hat{h}-h_0)\}^2|\mathcal{D}_0]\}\log(1/\delta')}{n_1}}+\frac{\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}\right\}. \\
&\qquad\qquad (\|\pi q_0\|_\infty\leq C_{\mathbb{Q}})
\end{aligned}
$$

Besides,

$$
\begin{aligned}
\|\mathcal{T}(\hat{h}-h_0)\|_2^2 &= \mathbb{E}[\{\mathbb{E}_\pi[\hat{h}-h_0]\}^2]\leq \mathbb{E}[\mathbb{E}_\pi[\{\hat{h}-h_0\}^2]] && \text{(Jensen's inequality)} \\
&= \mathbb{E}\left[\frac{\pi(A|X)}{f(A|X,W)}\{\hat{h}-h_0\}^2\right]\leq C_w\mathbb{E}[\{\hat{h}-h_0\}^2]. && (\|\pi(a|x)/f(a|x,w)\|_2\leq C_w)
\end{aligned}
$$

Then,

$$
\begin{aligned}
&|(\mathbb{E}_{n_1}-\mathbb{E})[q_0\{-\hat{h}+h_0\}+\mathcal{T}(\hat{h}-h_0)|\mathcal{D}_0]| \\
&\leq c\left\{\sqrt{\frac{(\tau^{\mathbb{H}}_{2,n})^2C_h^2\eta_h^2\{C_{\mathbb{Q}}^2+C_w\}\log(1/\delta')}{n_1}}+\frac{\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}\right\}. \qquad (\|\pi q_0\|_\infty\leq C_{\mathbb{Q}})
\end{aligned}
$$

The term (61) is similarly calculated. Therefore,

$$
\begin{aligned}
&|(\mathbb{E}_{n_1}-\mathbb{E})[\phi(\hat{q},\hat{h})-\phi(q_0,h_0)|\mathcal{D}_0]| \\
&\leq c\left\{[\tau^{\mathbb{Q}}_{2,n}\eta'_qC_q\{C_Y+C_{\mathbb{H}}\}+\tau^{\mathbb{H}}_{2,n}C_h\eta'_h\{C_{\mathbb{Q}}+\sqrt{C_w}\}]\sqrt{\frac{\log(1/\delta')}{n_1}}+\frac{\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{Q}}C_Y+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}\right\}
\end{aligned}
$$

Second Term Next, we analyze the second term in (57) . By some algebra, this is equal to

$$
\begin{aligned}
&\mathbb{E}[\phi(\hat{q},\hat{h})-\phi(q_0,h_0)|\mathcal{D}_0] \\
&=\mathbb{E}[\pi(\hat{q}-q_0)(y-h_0)|\mathcal{D}_0]+\mathbb{E}[\pi q_0\{-\hat{h}+h_0\}+\mathcal{T}(\hat{h}-h_0)|\mathcal{D}_0]+\mathbb{E}[\pi\{\hat{q}-q_0\}\{h_0-\hat{h}\}|\mathcal{D}_0] \\
&=\mathbb{E}[\pi\{\hat{q}-q_0\}\{h_0-\hat{h}\}|\mathcal{D}_0] \\
&=\mathbb{E}[\pi\{\hat{q}-q_0\}\mathbb{E}[\{h_0-\hat{h}\}|a,x,z]|\mathcal{D}_0] \\
&\leq \min(\|\{q_0-\hat{q}\}\pi\|_2\|P_z(h_0-\hat{h})\|_2,\|P_w\{q_0-\hat{q}\}\pi\|_2\|(h_0-\hat{h})\|_2) \\
&\leq \min(\tau^{\mathbb{H}}_{2,n},\tau^{\mathbb{Q}}_{2,n})C_hC_q\eta'_h\eta'_q. && \text{(CS inequality)}
\end{aligned}
$$

Combining all terms With $1-(\delta+10\delta')$,

$$
\begin{aligned}
&|\mathbb{E}_{n_1}[\phi(\hat{q},\hat{h})]+\mathbb{E}_{n_0}[\phi(\hat{q},\hat{h})]-J| \\
&\leq |\mathbb{E}_n[\phi(q_0,h_0)]-J|+\min(\tau^{\mathbb{H}}_{2,n},\tau^{\mathbb{Q}}_{2,n})C_hC_q\eta'_h\eta'_q+ \\
&+c\left\{[\tau^{\mathbb{Q}}_{2,n}\eta'_qC_q\{C_Y+C_{\mathbb{H}}\}+\tau^{\mathbb{H}}_{2,n}C_h\eta'_h\{C_{\mathbb{Q}}+\sqrt{C_w}\}]\sqrt{\frac{\log(1/\delta')}{n_1}}+\frac{\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{Q}}C_Y+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}\right\} \\
&\leq \sqrt{\frac{2\mathrm{var}[\pi q_0\{y-h_0\}+\mathcal{T}h_0]\log(1/\delta)}{n}}+\min(\tau^{\mathbb{H}}_{2,n},\tau^{\mathbb{Q}}_{2,n})C_hC_q\eta'_h\eta'_q+\frac{2\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{Q}}C_Y+C_{\mathbb{H}}\}\log(1/\delta)}{3n} \\
&+c\left\{[\tau^{\mathbb{Q}}_{2,n}\eta'_qC_q\{C_Y+C_{\mathbb{H}}\}+\tau^{\mathbb{H}}_{2,n}C_h\eta'_h\{C_{\mathbb{Q}}+\sqrt{C_w}\}]\sqrt{\frac{\log(1/\delta')}{n_1}}+\frac{\{C_{\mathbb{Q}}C_{\mathbb{H}}+C_{\mathbb{Q}}C_Y+C_{\mathbb{H}}\}\log(1/\delta')}{n_1}\right\}.
\end{aligned}
$$

□

E.6 Proofs for Appendix B

Proof for Lemma 12. According to Lemma 3, we have that $\mathbb{H}_0 \subseteq \mathbb{H}_0^{\text{obs}}$. For any $h_0 \in \mathbb{H}_0^{\text{obs}} \subseteq \mathbb{H}$,

$$\mathbb{E}\left[Y - h_0(W, A, X) \mid Z, A, X\right] = \mathbb{E}\left[\mathbb{E}\left[Y - h_0(W, A, X) \mid A, U, X\right] \mid Z, A, X\right] = 0.$$

By Assumption 3 condition 2, we have that $\mathbb{E}\left[Y - h_0(W, A, X) \mid A, U, X\right] = 0$, *i.e.*, $h_0 \in \mathbb{H}_0$. It follows that $\mathbb{H}_0^{\text{obs}} = \mathbb{H}_0$. Similarly, we can prove that $\mathbb{Q}_0^{\text{obs}} = \mathbb{Q}_0$ under Assumption 3 condition 1. ☐

Proof for Lemma 13. Consider two bridge functions $h_0, h_0' \in \mathbb{H}_0$:

$$\mathbb{E}\left[Y - h_0(W, A, X) \mid U, A, X\right] = \mathbb{E}\left[Y - h_0'(W, A, X) \mid U, A, X\right] = 0.$$

Thus we have

$$\mathbb{E}\left[h_0(W, A, X) - h_0'(W, A, X) \mid U, A, X\right] = 0.$$

Then Assumption 4 condition 1 implies that $h_0(W, A, X) = h_0'(W, A, X)$. Therefore, $\mathbb{H}_0$ is at most a singleton. We can similarly prove that $\mathbb{Q}_0$ is at most a singleton. ☐

Proof for Lemma 14. Under Assumption 1, we have that

$$\mathbb{E}\left[g(W, A, X) \mid Z, A, X\right] = \mathbb{E}\left[\mathbb{E}\left[g(W, A, X) \mid U, A, X\right] \mid Z, A, X\right],$$
$$\mathbb{E}\left[g(Z, A, X) \mid W, A, X\right] = \mathbb{E}\left[\mathbb{E}\left[g(Z, A, X) \mid U, A, X\right] \mid W, A, X\right].$$

We first prove statement 1. According to Assumption 3 condition 2, $\mathbb{E}\left[g(W, A, X) \mid Z, A, X\right] = 0$ if and only if $\mathbb{E}\left[g(W, A, X) \mid U, A, X\right] = 0$. Assumption 4 condition 1 further ensures that this holds if and only if $g(W, A, X) = 0$. In other words, Assumption 3 conditions 2 and Assumption 4 condition 1 are sufficient for Assumption 5 condition 1. Similarly, we can show that Assumption 3 conditions 1 and Assumption 4 condition 2 are sufficient for Assumption 5 condition 2.

Next, we prove statement 2. If $\mathbb{E}\left[g(W, A, X) \mid U, A, X\right] = 0$, then $\mathbb{E}\left[g(W, A, X) \mid Z, A, X\right] = 0$ as well. By Assumption 5 condition 1, this holds if and only if $g(W, A, X) = 0$. Therefore, Assumption 5 condition 1 is sufficient for Assumption 4 condition 1. Similarly, we can prove that Assumption 5 condition 2 is sufficient for Assumption 4 condition 2. ☐

Proof for Proposition 1. Fix $a \in \mathcal{A}, x \in \mathcal{X}$. Let $g(\mathbf{W}, a, x)$ be an $|\mathcal{W}| \times 1$ vector whose ith element is $g(w_i, a, x)$. Then condition 1 in Assumption 5 is equivalent to the following:

$$g^\top(\mathbf{W}, a, x) P(\mathbf{W} \mid \mathbf{Z}, a, x) = 0 \iff g(\mathbf{W}, a, x) = 0.$$

According to Assumption 1, we have $W \perp Z \mid U, A, X$. Thus

$$P(\mathbf{W} \mid \mathbf{Z}, a, x) = P(\mathbf{W} \mid \mathbf{U}, a, x) P(\mathbf{U} \mid \mathbf{Z}, a, x).$$

Therefore, we need

$$g^\top(\mathbf{W}, a, x) P(\mathbf{W} \mid \mathbf{U}, a, x) P(\mathbf{U} \mid \mathbf{Z}, a, x) = 0 \iff g(\mathbf{W}, a, x) = 0.$$

This requires the rank of $P(\mathbf{W} \mid \mathbf{U}, a, x) P(\mathbf{U} \mid \mathbf{Z}, a, x)$ to be $|\mathcal{W}|$. It follows that $|\mathcal{Z}| \geq |\mathcal{W}| = |\mathcal{U}|$.

Similarly, we can prove that condition 2 in Assumption 5 holds implies $|\mathcal{W}| \geq |\mathcal{Z}| = |\mathcal{U}|$. ☐

Proof for Lemma 16. We need to prove that the following equation of the first kind is solvable:

$$[K_{W|a,x} h](a, u, x) = k_0(a, u, x), \quad \text{a.e. } u, a, x \text{ w.r.t } \mathbb{P}.$$

Thus we only need to verify the assumptions in the Picard's Theorem in Lemma 15 with $K = K_{W|a,x}$ and $\phi = k_0$. Note that condition 2 in Lemma 15 is satisfied by our asserted assumptions. Thus we only need to show $k_0 \in \mathcal{N}\left(K_{W|a,x}^*\right)^\perp$.

Since $(K_{W|a,x}^* g)(w, a, x) = \mathbb{E}\left[g(U, a, x) \mid W = w, A = a, X = x\right]$. By Assumption 3 condition 1, $\mathcal{N}\left(K_{W|a,x}^*\right) = \{0\}$, which means that $\mathcal{N}\left(K_{W|a,x}^*\right)^\perp = \operatorname{dom} K_{W|a,x}^* = L_2(U \mid A = a, X = x)$. Therefore, $k_0 \in \mathcal{N}\left(K_{W|a,x}^*\right)^\perp$. ☐

Proof for Lemma 17. The proof is completely analogous to the proof for Lemma 16. ☐

E.7 Proofs for Appendix C

Proof for Lemma 18. By the definition of $\mathbb{H}_0^{\text{obs}}$, we have that

$$\mathbb{E}\left[Y - h_0(W,A,X) \mid Z,A,X\right] = \mathbb{E}\left[Y - h_0'(W,A,X) \mid Z,A,X\right] = 0,$$

which implies that

$$P_z\left(h_0 - h_0'\right) = \mathbb{E}\left[h_0(W,A,X) - h_0'(W,A,X) \mid Z,A,X\right] = 0.$$

Therefore, $h_0 - h_0' \in \mathcal{N}\left(P_z\right)$. It follows that

$$P_{\mathcal{N}(P_z)^\perp}\left(h_0 - h_0'\right) = 0.$$

Equivalently,

$$P_{\mathcal{N}(P_z)^\perp} h_0 = P_{\mathcal{N}(P_z)^\perp} h_0'.$$

Similarly, we can prove $\pi P_{\mathcal{N}(P_w)^\perp} q_0 = \pi P_{\mathcal{N}(P_w)^\perp} q_0'$. $\square$

Proof for Theorem 10. **Step I: deriving the tangent space.** First, consider regular parametric submodel $\mathcal{P}_t = \{f_t(y,w,z,a,x) : t \in \mathbb{R}^s\}$ with $f_0(y,w,z,a,x)$ equals the true density $f(y,w,z,a,x)$. The associated score function is denoted as $S(y,w,z,a,x) = \partial_t \log f_t(y,w,z,a,x)|_{t=0}$. The expectation w.r.t the distribution $f_t(y,w,z,a,x)$ is denoted by $\mathbb{E}_t$. We can similarly denote the score functions for any component of this parametric submodel. For example, the score function for $f_t(y,w \mid z,a,x)$ is denoted as $S(y,w \mid z,a,x) = \partial_t \log f_t(y,w \mid z,a,x)|_{t=0}$. It is easy to show that

$$S(Y,W,Z,A,X) = S\left(Z,A,X\right) + S\left(Y,W \mid Z,A,X\right), \mathbb{E}\left[S(Z,A,X)\right] = 0, \mathbb{E}\left[S(Y,W \mid Z,A,X) \mid Z,A,X\right] = 0.$$

Let h_t and q_t be curves into $\mathcal{N}\left(P_z\right)^\perp$ and $\mathcal{N}\left(P_w\right)^\perp$ respectively, such that $h_t|_{t=0} = h_0^*$ and $\pi q_t|_{t=0} = \pi q_0^*$, and

$$\mathbb{E}_t\left[Y - h_t(W,A,X) \mid Z,A,X\right] = 0 \tag{62}$$

$$\mathbb{E}_t\left[\pi(A \mid X) q_t(Z,A,X) \mid W,A,X\right] = \pi(A \mid X)/f_t(A \mid W,X). \tag{63}$$

Eq. (62) implies that

$$\partial_t \mathbb{E}_t\left[Y - h_t(W,A,X) \mid Z,A,X\right]|_{t=0} = 0,$$

which in turn implies that

$$\mathbb{E}\left[\left(Y - h_0^*(W,A,X)\right) S(W,Y \mid Z,A,X) \mid Z,A,X\right] = \mathbb{E}\left[\partial_t h_t(W,A,X)|_{t=0} \mid Z,A,X\right]. \tag{64}$$

This means that $S(W,Y \mid Z,A,X)$ must satisfy that $\mathbb{E}\left[\left(Y - h_0^*(W,A,X)\right) S(W,Y \mid Z,A,X) \mid Z,A,X\right] \in \text{Range}\left(P_z\right)$.

Similarly, Eq. (63) implies that

$$\begin{aligned}&\mathbb{E}\left[\pi(A \mid X) q_0^*(Z,A,X) S(Z \mid W,A,X) + \pi(A \mid X)\frac{S(A \mid W,X)}{f(A \mid W,X)} \mid W,A,X\right] \\ =& -\mathbb{E}\left[\partial_t\left(\pi(A \mid X) q_t(Z,A,X)\right)|_{t=0} \mid W,A,X\right],\end{aligned}$$

or equivalently

$$\mathbb{E}\left[\pi(A \mid X) q_0^*(Z,A,X) S(Z,A \mid W,X) \mid W,A,X\right] = -\mathbb{E}\left[\partial_t\left(\pi(A \mid X) q_t(Z,A,X)\right)|_{t=0} \mid W,A,X\right].$$

This in turn suggests that

$$\mathbb{E}\left[\pi(A \mid X) q_0^*(Z,A,X) S(Z,A \mid W,X) \mid W,A,X\right] \in \text{Range}\left(P_w\right) \tag{65}$$

Note that

$$
\begin{aligned}
S(z, a \mid w, x) &= \partial_t \log f_t(z, a \mid w, x)|_{t=0} = \partial_t \log f_t(w, z, a, x)|_{t=0} - \partial_t \log f_t(w, x)|_{t=0} \\
&= \frac{\partial_t \int f_t(y, w, z, a, x) \mathrm{d}y|_{t=0}}{f(w, z, a, x)} - \frac{\partial_t \iiint f_t(y, w, z, a, x) \mathrm{d}y \mathrm{d}z \mathrm{d}a|_{t=0}}{f(w, x)} \\
&= \frac{\int f(y, w, z, a, x) S(y, w, z, a, x) \mathrm{d}y}{f(w, z, a, x)} - \frac{\iiint f(y, w, z, a, x) S(y, w, z, a, x) \mathrm{d}y \mathrm{d}z \mathrm{d}a}{f(w, x)} \\
&= \mathbb{E}\left[S(Y, W, Z, A, X) \mid Z = z, A = a, W = w, X = x\right] - \mathbb{E}\left[S(Y, W, Z, A, X) \mid W = w, X = x\right].
\end{aligned}
$$

Thus we have

$$
\mathbb{E}\left[\pi(A \mid X) q_0^*(Z, A, X) \left(\mathbb{E}\left[S(Y, W, Z, A, X) \mid W, Z, A, X\right] - \mathbb{E}\left[S(Y, W, Z, A, X) \mid W, X\right]\right) \mid W, A, X\right] \in \operatorname{Range}(P_w). \tag{66}
$$

Therefore, all score vectors under $\mathcal{M}_{np}$ must lie in the following set $\mathcal{S}$:

$$
\begin{aligned}
\mathcal{S} = \Big\{ & S(Y, W, Z, A, X) = S(Z, A, X) + S(Y, W \mid Z, A, X) : \\
& S(Z, A, X) \in L_2(Z, A, X), S(Y, W \mid Z, A, X) \in L_2(Y, W \mid Z, A, X), \\
& \mathbb{E}\left[S(Z, A, X)\right] = 0, \mathbb{E}\left[S(Y, W \mid Z, A, X) \mid Z, A, X\right] = 0, \\
& \mathbb{E}\left[\pi(A \mid X) q_0^*(Z, A, X) \left(\mathbb{E}\left[S(Y, W, Z, A, X) \mid W, Z, A, X\right] - \mathbb{E}\left[S(Y, W, Z, A, X) \mid W, X\right]\right) \mid W, A, X\right] \in \operatorname{Range}(P_w), \\
& \mathbb{E}\left[\left(Y - h_0^*(W, A, X)\right) S(Y, W \mid Z, A, X) \mid Z, A, X\right] \in \operatorname{Range}(P_z) \Big\}.
\end{aligned} \tag{67}
$$

Step II: deriving a preliminary influence function. Denote the target parameter under distribution $f_t(y, w, z, a, x)$ as $J_t = \mathbb{E}_t\left[\mathcal{T} h_t(W, X)\right]$. We need to derive

$$
\partial_t J_t|_{t=0} = \mathbb{E}\left[\mathcal{T} h_0^*(W, X) S(W, X)\right] + \mathbb{E}\left[\partial_t \mathcal{T} h_t(W, X)|_{t=0}\right]
$$

Note that

$$
\begin{aligned}
\mathbb{E}\left[\partial_t \mathcal{T} h_t(W, X)|_{t=0}\right] &= \mathbb{E}\left[\pi(A \mid X) q_0^*(Z, A, X) \mathbb{E}\left[\partial_t h_t(W, A, X)|_{t=0} \mid Z, A, X\right]\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0^*(Z, A, X) \left(Y - h_0^*(W, A, X)\right) S(W, Y \mid Z, A, X)\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0^*(Z, A, X) \left(Y - h_0^*(W, A, X)\right) S(W, Y \mid Z, A, X)\right] \\
&\quad + \mathbb{E}\left[\pi(A \mid X) q_0^*(Z, A, X) \left(Y - h_0^*(W, A, X)\right) S(Z, A, X)\right] \\
&= \mathbb{E}\left[\pi(A \mid X) q_0^*(Z, A, X) \left(Y - h_0^*(W, A, X)\right) S(W, Y, Z, A, X)\right].
\end{aligned}
$$

Here the second equality follows from Eq. (64), and the third equality follows from the fact that $\mathbb{E}\left[Y - h_0^*(W, A, X) \mid Z, A, X\right] = 0$.

Moreover,

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{T} h_0^*(W, X) S(W, X)\right] &= \mathbb{E}\left[\mathcal{T} h_0^*(W, X) S(W, X)\right] + \mathbb{E}\left[\mathcal{T} h_0^*(W, X) S(Y, Z, A \mid W, X)\right] \\
&= \mathbb{E}\left[\mathcal{T} h_0^*(W, X) S(Y, W, Z, A, X)\right] \\
&= \mathbb{E}\left[\left(\mathcal{T} h_0^*(W, X) - J\right) S(Y, W, Z, A, X)\right].
\end{aligned}
$$

Here the first equality follows from the fact that $\mathbb{E}\left[S(Y, Z, A \mid W, X) \mid W, X\right] = 0$, and the last equality follows from $\mathbb{E}\left[S(Y, Z, A, W, X) \mid W, X\right] = 0$.

Therefore, we have that

$$
\partial_t J_t|_{t=0} = \mathbb{E}\left[\left(\pi(A \mid X) q_0^*(Z, A, X)[Y - h_0^*(W, A, X)] + \mathcal{T} h_0^*(W, X) - J\right) S(Y, W, Z, A, X)\right].
$$

This means that $\pi(A \mid X) q_0^*(Z, A, X)[Y - h_0^*(W, A, X)] + \mathcal{T} h_0^*(W, X) - J$ is a valid influence function for J under the model $\mathcal{M}_{np}$.

Step III: verifying efficient influence function. Now we verify that $\pi(A \mid X) q_0^*(Z, A, X)[Y - h_0^*(W, A, X)] + \mathcal{T} h_0^*(W, X) - J$ also belongs to $\mathcal{S}$ so that it is also the efficient influence function for J relative to the tangent space $\mathcal{S}$.

First, note that we can decompose this influence function in the following way:

$$\begin{aligned}\tilde{S}(Y,W,Z,A,X) &:= \pi(A \mid X)q_0^*(Z,A,X)[Y - h_0^*(W,A,X)] + \mathcal{T}h_0^*(W,X) - J \\ &= \tilde{S}(Z,A,X) + \tilde{S}(Y,W \mid Z,A,X),\end{aligned}$$

where

$$\begin{aligned}\tilde{S}(Z,A,X) &= \mathbb{E}\left[\mathcal{T}h_0^*(W,X) - J \mid Z,A,X\right], \\ \tilde{S}(Y,W \mid Z,A,X) &= \mathcal{T}h_0^*(W,X) - \mathbb{E}\left[\mathcal{T}h_0^*(W,X) \mid Z,A,X\right] + \pi(A \mid X)q_0^*(Z,A,X)[Y - h_0^*(W,A,X)].\end{aligned}$$

It is easy to show that $\mathbb{E}\left[\tilde{S}(Z,A,X)\right] = 0$ and $\mathbb{E}\left[\tilde{S}(Y,W \mid Z,A,X) \mid Z,A,X\right] = 0$. According to the Eq. (43), we have that

$$\mathbb{E}\left[(Y - h_0^*(W,A,X))\,\tilde{S}(Y,W \mid Z,A,X) \mid Z,A,X\right] \in \operatorname{Range}(P_z).$$

Next, it is easy to show that

$$\begin{aligned}&\mathbb{E}\left[\tilde{S}(Y,W,Z,A,X) \mid W,Z,A,X\right] - \mathbb{E}\left[\tilde{S}(Y,W,Z,A,X) \mid W,X\right] \\ =&\pi(A \mid X)q_0(Z,A,X)\left(\mathbb{E}\left[Y \mid W,Z,A,X\right] - h_0(W,A,X)\right) \\ -&\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X)\left(\mathbb{E}\left[Y \mid W,Z,A,X\right] - h_0(W,A,X)\right) \mid W,X\right]\end{aligned}$$

By the asserted Eq. (44), we have that

$$\begin{aligned}&\mathbb{E}\left[\pi(A \mid X)q_0^2(Z,A,X)\left(\mathbb{E}\left[Y \mid W,Z,A,X\right] - h_0(W,A,X)\right) \mid W,A,X\right] \\ -&\mathbb{E}\left[q_0(Z,A,X)\mathbb{E}\left[\pi(A \mid X)q_0(Z,A,X)\left(\mathbb{E}\left[Y \mid W,Z,A,X\right] - h_0(W,A,X)\right) \mid W,X\right] \mid W,A,X\right] \\ &\in \operatorname{Range}(P_w),\end{aligned}$$

which implies that

$$\mathbb{E}\left[q_0(Z,A,X)\left(\mathbb{E}\left[\tilde{S}(Y,W,Z,A,X) \mid W,Z,A,X\right] - \mathbb{E}\left[\tilde{S}(Y,W,Z,A,X) \mid W,X\right]\right) \mid W,A,X\right] \in \operatorname{Range}(P_w).$$

Finally, Assumption 7 condition 2 implies that $\mathbb{E}\left[\mathrm{EIF}^2(J)\right] < \infty$.
Therefore,

$$\mathrm{EIF}(J) = \pi(A \mid X)q_0^*(Z,A,X)[Y - h_0^*(W,A,X)] + \mathcal{T}h_0^*(W,X) - J \in \mathcal{S},$$

which means that $\mathrm{EIF}(J)$ is the efficient influence function of J and $\mathbb{E}\left[\mathrm{EIF}^2(J)\right]$ is the corresponding semiparametric efficiency bound, both relative to the tangent space $\mathcal{S}$. $\square$