Who are you, dear f(x)?

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1 Introduction

We all must have, at least once, suffered through an algebraic equation. They usually look something like

$$2x + 3 = 45,$$

in our case of past middle-schoolers at least. We can easily solve that equation by just manipulating both of the sides, namely LHS and RHS a little bit. We must get $x = 21$. Now what I do not expect you to have suffered through before is with a **Diophantine Equation**. Woah! What does that word even mean, sir fancy pants? Well, they are basically like the normal ones except for n variables they have $\leq n - 1$ equations to support (remember that we can solve for equations with n variables and n equations using simple algebraic manipulation, we learnt this in middle school!). One of such is

$$5x + 13y = 100$$

and there are several ways to solve this. We will, however interesting it is, not discuss about that. We simply observe that (x,y)=(7,5) works by hit and trial. Now what if I tell you anything of the form (x,y)=(20-13k, 5k) works? Well, we shall try. I recommend you to go through the cases where (x,y)=(20,0); (-6,10).

Exercise 1.

Why does $(x, y) = (20 - 13k, 5k)$ work?

Now the thing is such equations may posses infinitely many solutions. Why did I bring that example up here? It is to show you that equations can exist of different types. Now similarly another one of those is called **functional equation**. We are asked to find all functions satisfying some given relation(s). What are functions? They are basically something you can imagine as a machine that takes some specific type of input and gives out a specific type of output. We call the input here the domain and the output here the range, well its type I mean. I will leave an example of a function: Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ for all reals such that $f(x) + f(y) = f(x + y)$. This equation is very popular and is known as the cauchy's equation. We shall solve this question further on but for now: the first one after f is called the domain and the second one is range. Domain is all the function can take in and range is all it can give out. One of the examples is that both of them can take in $\frac{1}{2}$ but had the range been $\mathbb{N}$, the only thing the function could give out would have been Natural numbers. Now I will give an actual example and we shall solve that; we will not discuss all of the methods in this section, though.

Exercise 2.

Find all functions such that $f : \mathbb{R} \to \mathbb{R}$, $f(xy) = x^2 f(y)$, and $f(x) = -f(-x)$.

We simply observe that $-f(xy) = f(-xy) = f((-x)y) = (-x)^2 f(y) = x^2 f(y) = f(xy)$. It follows that $f(xy) = 0$ (Why?). Now taking $y = 1$ we see that $f(x) = 0$ for all reals x. And thus, this equation has only one solution: $f(x) = 0$.

Functional equations can be intimidating just because they are so unlike other problems. However, once you learn a few basic tricks, you will find even the hard ones are pretty approachable.

2 Come on! Show yourself twice, you shy girl!

Why is this section's name so weird? Hmm, I wonder too. Mood.

Anyways, continuing with the learning part: When you start working on a functional equation, it is always a good idea to plug in small values like x = 0 and see what comes out. However, this will not usually be enough to solve the whole problem. You will need to move on to bigger values and then it becomes helpful to have a plan.
In general, I would say you should start by looking for multiple ways to get a single term to show up. Think about the functional equation as a giant system of equations, and try to find a way to make some things cancel out.
Stratergy 1. Try to cancel things out by subbing in good stuff ~~Not Rabneet's ganja, though.~~

Example 1.

Find all functions for all $x, y \in \mathbb{R}$ such that $f : \mathbb{R} \to \mathbb{R}$ $f(x^2 - y^2) = (x-y)(f(x) + f(y))$.

Hmmmmmmm, what is this? It seems so weird, doesn't it? Don't you think the x and y are almost everywhere are kind of interchangeable... are they really though? How about we interchange them and see if they are?
We sub in $P(x, y) \to P(y, x)$ where P(x,y) denotes what is in the place of x,y for example P(0,0) means we must have

$$f(0^2 - 0^2) = (0-0)((f(0) + f(0)) = 0.$$

Now subbing that good stuff ~~again, not Rabneet's ganja please~~ we get

$$P(y^2 - x^2) = (y - x)(f(y) + f(x)).$$

Does that help? Hmmm... we can not tell yet. How about we try to get something like $f(x) = -f(-x)$? Why would that work? Well, if we have that then we must have $f(x^2 - y^2) = -f(y^2 - x^2)$. Motivation is simply the fact that this is a very well-known fact and it actually turns out to be very good here. This particular thingy with $f(x) = -f(-x)$ is called an **odd function.** Now how do we actually get that? Well, first of all we might have to do something for the squares there-for we only need a simple variable with degree one. That motivates us to try

$$P(x, y) \to P(\sqrt{x}, 0) : f(x) = (\sqrt{x}) \cdot (f(\sqrt{x}) + f(0)).$$

Now doing the same with y gives us

$$P(x, y) \to P(0, \sqrt{x}) : f(-x) = (-\sqrt{x}) \cdot (f(\sqrt{x}) + f(0)).$$

These both imply $f(x) = -f(-x)$ and we are done. Now all we are left to do is actually finish this off.
The next idea is similar: can we change things in a way that we somehow make use of the symmetry? We try to think of that and get that y=-z might work. This leads us to doing

$$P(x, y) \to P(x, -z) : f(x^2 - z^2) = (x+z) \cdot (f(x) + f(-z)) = (x+z) \cdot (f(x) - f(z)).$$

But we already know from the original equation that

$$P(x, y) \to P(x, z) : f(x^2 - z^2) = (x - z) \cdot (f(x) + f(z)).$$

Equating both of the right hand sides leads us to $xf(z) = zf(x)$. Now we have a simple finish: let f(1)=c, a constant and then we have f(x)=cx for z=1. Now we check that this works and we are done.

Example 2.

Find all functions for all $x, y \in \mathbb{R}$ such that

$$f(x^2 + y) = f(x^{27} + 2y) + f(x^4).$$

Hmmmmm, can we do anything that will make lots of terms go away? There's actually a very artificial choice that will do wonders. I will remind you to think of it before reading on. We want to cancel stuff So we do the most blithely stupid thing possible. Do you see that $x^2 + y$ and $x^{27} + 2y$ up there? Let's make them equal in the rudest way possible:

$$x^2 + y = x^{27} + 2y \to y = x^2 - x^{27}.$$

Yes. I am that evil. Plugging in this choice of y, this gives us $f(x^4) = 0$, so f is zero on all nonnegatives. All that remains is to get f zero on all reals. The easiest way to do this is put y = 0 since this won't hurt the already positive x^2 and x^4 terms there. QED.

3 Too lengthy; I want a summary

- **Always seek for almost-symmetry:** If you have something like we did in the example above, all you have to do is get a nice property and then try exchanging the variables.

- **Changing just a little:** Can you change one variable so as to alter the equation only slightly? If so, compare with what you started with.

- **Cancel stuff:** Make them fight their own kind. Such a devil you are.

- **Define different functions:** this is a hint for problem number 8 and nothing else. You must define the function there as $g(x) = f(x) - f(0)$. You will have to solve the problem completely to see why this thing works.

4 Injectivity, Surjectivity

Injectivity. What does this term even mean? Are we all getting injections togeether? Is it going to be painful? Am i scared of injections? So many questions... well, I don't know if we will get injections soon enough but what I

do know is that injective functions are nothing but functions that are defined as to be only one of their kind... hey Bhabani, what does that mean? Hmm... it is weird calling myself by my own name but for the sake of the handout: it means for each functions $f(a)$ that is equal to $f(b,)$ $a = b$. More formally, $f(a) = f(b) \longleftrightarrow a = b$.
Surjectivity. This means that for every $x \in$ domain there is a $y \in$ codomain such that $f(x) = y$.
Bijectivity. This means both of those given properties are true for a given function f.

5 Practice Problems

> The only way to get better at maths is by doing it.
> By Somebody, probably.

Before you get on solving the problems here, do remember that this is not school maths. This will be a lot harder and you might actually end up solving none of them. That is fine. What is not fine is when you give up without even trying so hard... I will try to discuss all of the problems given below in the class except the ones I give as homework. If you are having trouble with any of the problems given below, do contact me either through whatsapp (you can find my number in the group) or through my email id mentioned. This will take time and it should... in fact if it does not, you indeed are a super human! Go contact NASA ASAP! We need super humans like you, though I hope you will not be the reason the earth gets destroyed. If you are being able to solve all of the problems with a single thought, do consider registering yourself in **The Mathheroes Association** you might actually get ranked C, though.

Problems now. Again: The only way to get better at maths is by doing it. For motivation, in case you feel like dying after trying some of the problems below, click here. Also, I have not yet solved all of the problems mentioned here myself so yeah, it will be a journey for all of us including me. I should have probably solved that beforehand given it will be a lot of trouble explaining some certain properties of functions if I use any...

- Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x+y)) = f(y) + x$ and it is $\forall x, y \in \mathbb{R}$.

- Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)+y) = f(f(x)-y) + 4f(x)y$.

- Let $f(x)$ be a real-valued function defined for all positive x, satisfying $f(x+y) = f(xy)$ for all positive x, y. Prove that f is a constant function.

- Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds
$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$
where $\lfloor a \rfloor$ is greatest integer not greater than a.

- Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + y) = f(x) + f(y)$ for all x,y $\in \mathbb{Q}$. [a].

[a] You might need induction for this one so I recommend googling induction up or alternatively you can click here Induction

- Can we solve the above problem if we change the range to $\mathbb{R}$?

- Can we solve that problem if we change both domain and range to $\mathbb{R}$?

- Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x) + f(y) = 2f(\frac{x+y}{2})$. [a]

[a] Hint: $-f(0) - f(0) = -2f(0)$

- Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - y^2) = xf(x) - yf(y)$.

- Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(xy) = f(x)f(y) - f(x + y) + 1$ and $f(1) = 2$.

6 Hints

These are serially organized as per the problems. Do not look at these unless you have tried hard enough.

- What happens if we swap x and y?

- Try to make a sub to cancel the two of them. $y = \frac{x^2 - f(x)}{2}$. Now don't fall for a trap here.

- Choose x and y such that xy is fixed and then vary x+y. You can prove thst $f(x) = f(y)$ for $x \leq y\sqrt{2}$.

- Put $x = y = 0$. Then $f(0) = 0$ or $\lfloor f(0) \rfloor = 1$.

  • If $\lfloor f(0) \rfloor = 1$, putting $y = 0$ we get $f(x) = f(0)$, that is f is constant. Substituing in the original equation we find $f(x) = 0$. • If $f(0) = 0$, putting $x = y = 1$ we get $f(1) = 0$ or $\lfloor f(1) \rfloor = 1$. For $f(1) = 0$, we set $x = 1$ to find $f(y) = 0\ \forall y$, which is a solution. For $\lfloor f(1) \rfloor = 1$, setting $y = 1$ yields $f(\lfloor x \rfloor) = f(x),\ (*)$.

  Putting $x = 2, y = \frac{1}{2}$ to the original we get $f(1) = f(2)\lfloor f(\frac{1}{2}) \rfloor$. However, from $(*)$ we have $f(\frac{1}{2}) = f(0) = 0$, so $f(1) = 0$ which contradicts the fact $\lfloor f(1) \rfloor = 1$.

  So, $f(x) = 0,\ \forall x$.

- Fix $f(1)$ and then try to get all the natural number values. Get that $f(x) = -f(-x)$. Now prove for rationals.

- No. E.g is of $\mathbb{Q}[\sqrt{2}]$

- No.

- Try the sub $g(x) = f(x) - f(0)$. Try $P(x, y) = (x, 0)0$

- Get f odd. Get $f(0) = 0$. f is additive. $f(x + 1)^2 = (x + 1)f(x + 1)$ and then expand and fix $f(1)$. $f(x) = cx$

- $f(0) = 1$ by completing the square. $f(-1) = 0$. $f(x) = f(x+1)-1$. Induct to get $f(x) = x + 1$ for all negative x's. Note that $f(x) = f(x + 1) - 1$ implies that $f(x + q) = f(x) + q$ by induction for all x and integers q. For a rational number $\frac{p}{q}$, put x=p/q and y=q in the original equation to get $f(p) = f(p/q)f(q) - f(p/q + q) + 1 = f(p/q)f(q) - [f(p/q) + q] + 1$ and then get that $1 + p = f(p/q)(q + 1) - f(p/q) - q + 1$ to conclude.