# FES-ISL Collection

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A collection of some functional equation problems.

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**Problem 1 (IMO 1998).** Determine the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ for all $m, n \in \mathbb{N}$ such that

$$f(n^2 f(m)) = m(f(n))^2.$$

**Problem 2 (IMO 1999).** Find all functions $f : \mathbb{R} \to \mathbb{R}$ for all $x, y \in \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1.$$

**Problem 3 (IMO 2000).** Find all pairs of fucntions $f : \mathbb{R} \to \mathbb{R}$; $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + g(y)) = xf(y) - yf(x) + g(x) \quad \text{for all } x, y \in \mathbb{R}.$$

**Problem 4 (IMO 2001).** Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$$

for all $x, y$.

**Problem 5 (IMO 2002).** Find all functions $f$ from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real $x, y$.

**Problem 6 (IMO 2002).** Find all functions $f$ from the reals to the reals such that

$$(f(x)+f(z))(f(y)+f(t))=f(xy-zt)+f(xt+yz)$$

for all real $x,y,z,t$.

**Problem 7 (IMO 2003).** Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ that satisfy the following conditions:

- $f(xyz)+f(x)+f(y)+f(z)=f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x,y,z\in\mathbb{R}^+$;
- $f(x)<f(y)$ for all $1\le x<y$.

**Problem 8 (IMO 2004).** Find all polynomials $f$ with real coefficients such that for all reals $a,b,c$ such that $ab+bc+ca=0$ we have the following relations

$$f(a-b)+f(b-c)+f(c-a)=2f(a+b+c).$$

**Problem 9 (IMO 2004).** Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the equation

$$f(x^2+y^2+2f(xy))=(f(x+y))^2.$$

for all $x,y\in\mathbb{R}$.

**Problem 10 (IMO 2005).** We denote by $\mathbb{R}^+$ the set of all positive real numbers.

Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ which have the property:

$$f(x)f(y)=2f(x+yf(x))$$

for all positive real numbers $x$ and $y$.

**Problem 11 (IMO 2005).** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)+f(x)f(y)=f(xy)+2xy+1$ for all real numbers $x$ and $y$.

**Problem 12 (IMO 2007).** Consider those functions $f:\mathbb{N}\mapsto\mathbb{N}$ which satisfy the condition

$$f(m+n)\ge f(m)+f(f(n))-1$$

for all $m,n\in\mathbb{N}$. Find all possible values of $f(2007)$.

**Problem 13 (IMO 2007).** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f(x + f(y)) = f(x + y) + f(y)$ for all pairs of positive reals $x$ and $y$. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

**Problem 14 (IMO 2007).** Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime $p$, the number $f(m+n)$ is divisible by $p$ if and only if $f(m) + f(n)$ is divisible by $p$.

**Problem 15 (IMO 2008).** Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so $f$ is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers $w, x, y, z$, satisfying $wx = yz$.

**Problem 16 (IMO 2008).** Let $f : \mathbb{R} \to \mathbb{N}$ be a function which satisfies $f\left(x + \frac{1}{f(y)}\right) = f\left(y + \frac{1}{f(x)}\right)$ for all $x$, $y \in \mathbb{R}$. Prove that there is a positive integer which is not a value of $f$.

**Problem 17 (IMO 2009).** Determine all functions $f$ from the set of positive integers to the set of positive integers such that, for all positive integers $a$ and $b$, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

**Problem 18 (IMO 2009).** Let $f$ be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers $x$ and $y$ such that

$$f(x - f(y)) > yf(x) + x$$

**Problem 19 (IMO 2009).** Find all functions $f$ from the set of real numbers into the set of real numbers which satisfy for all $x$, $y$ the identity

$$f(xf(x + y)) = f(yf(x)) + x^2$$

**Problem 20 (IMO 2009).** Let $f$ be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers $a$, $b$. Prove that there exist infinitely many primes $p$ such that $p$ divides $f(c)$ for some positive integer $c$.

**Problem 21 (IMO 2010).** Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than $a$.

**Problem 22 (IMO 2010).** Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$ :

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

**Problem 23 (IMO 2010).** Suppose that $f$ and $g$ are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer $n$.

**Problem 24 (IMO 2010).** Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

**Problem 25 (IMO 2011).** Find all functions $f$ from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real $x, y$.

**Problem 26 (IMO 2011).** Find all functions $f$ from the reals to the reals such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real $x, y, z, t$.

**Problem 27 (IMO 2012).** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a, b, c$ that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

**Problem 28 (IMO 2012).** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

and $f(-1) \neq 0$.

**Problem 29 (IMO 2012).** Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let $f^m$ be $f$ applied $m$ times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let $k_n$ be the smallest such $k$. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

**Problem 30 (IMO 2013).** Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$; (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x)+f(y)$; (iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

**Problem 31 (IMO 2013).** Let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation

$$f(f(f(n))) = f(n + 1) + 1$$

for all $n \in \mathbb{Z}_{\geq 0}$.

**Problem 32 (IMO 2013).** Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers $m$ and $n$.

**Problem 33 (IMO 2013).** Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying

$$f\left(\frac{f(x) + a}{b}\right) = f\left(\frac{x + a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$.

**Problem 34 (IMO 2014).** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers $m$ and $n$.

**Problem 35 (IMO 2014).** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2$$

for all $n \in \mathbb{Z}$.

**Problem 36 (IMO 2015).** Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

**Problem 37 (IMO 2015).** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers $x$ and $y$.

**Problem 38 (IMO 2015).** Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \mapsto 2\mathbb{Z} + 1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$

for every $x, y \in \mathbb{Z}$.

**Problem 39 (IMO 2015).** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\dots f}_{n}(m)\dots))$. Suppose that $f$ has the following two properties:

(i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$; (ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \dots$ is periodic.

**Problem 40 (IMO 2015).** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer $k$, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called $k$-good if $\gcd(f(m) + n, f(n) + m) \le k$ for all $m \ne n$. Find all $k$ such that there exists a $k$-good function.

**Problem 41 (IMO 2016).** Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

**Problem 42 (IMO 2016).** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and for all $x, y \in \mathbb{R}$,

$$f(x+y)^2 = 2f(x)f(y) + \max\left\{f(x^2+y^2), f(x^2) + f(y^2)\right\}.$$

**Problem 43 (IMO 2016).** Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers $m$ and $n$, the integer $f(m)+f(n)-mn$ is nonzero and divides $mf(m)+nf(n)$.

**Problem 44 (IMO 2017).** Let $S$ be a finite set, and let $\mathcal{A}$ be the set of all functions from $S$ to $S$. Let $f$ be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of $S$ under $f$. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every $g$ in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

**Problem 45 (IMO 2017).** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers $x$ and $y$,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

**Problem 46 (IMO 2017).** A function $f : \mathbb{R} \to \mathbb{R}$ has the following property:

$$\text{For every } x, y \in \mathbb{R} \text{ such that } (f(x)+y)(f(y)+x) > 0, \text{ we have } f(x)+y = f(y)+x.$$

Prove that $f(x) + y \leq f(y) + x$ whenever $x > y$.

**Problem 47 (IMO 2018).** Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2f(y)^2) = f(x)^2f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

**Problem 48 (IMO 2018).** Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n) | f(m)+f(n)$ for all pairs $m, n$ of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of $f$.

**Problem 49 (IMO 2019).** Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a$ and $b$,

$$f(2a) + 2f(b) = f(f(a+b)).$$

**Problem 50 (IMO 2019).** Let $\mathbb{Z}$ be the set of integers. We consider functions $f:\mathbb{Z}\to\mathbb{Z}$ satisfying

$$f\left(f(x+y)+y\right)=f\left(f(x)+y\right)$$

for all integers $x$ and $y$. For such a function, we say that an integer $v$ is f-rare if the set

$$X_v=\{x\in\mathbb{Z}:f(x)=v\}$$

is finite and nonempty. (a) Prove that there exists such a function $f$ for which there is an $f$-rare integer. (b) Prove that no such function $f$ can have more than one $f$-rare integer.

**Problem 51 (IMO 2019).** Find all functions $f:\mathbb{Z}_{>0}\to\mathbb{Z}_{>0}$ such that $a+f(b)$ divides $a^2+bf(a)$ for all positive integers $a$ and $b$ with $a+b>2019$.

**Problem 52 (IMO 2020).** Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ satisfying

$$f^{a^2+b^2}(a+b)=af(a)+bf(b)$$

for all integers $a$ and $b$

**Problem 53 (IMO 2020).** Let $R^+$ be the set of positive real numbers. Determine all functions $f:R^+\to R^+$ such that for all positive real numbers $x$ and $y$:

$$f(x+f(xy))+y=f(x)f(y)+1$$

$Ukraine$ cause why not.

**Problem 54 (IMO 2020).** Determine all functions $f$ defined on the set of all positive integers and taking non-negative integer values, satisfying the three conditions:

$(i)$ $f(n)\neq 0$ for at least one $n$; $(ii)$ $f(xy)=f(x)+f(y)$ for every positive integers $x$ and $y$; $(iii)$ there are infinitely many positive integers $n$ such that $f(k)=f(n-k)$ for all $k<n$.

**Problem 55 (IMO 2021).** Determine all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$(f(a)-f(b))(f(b)-f(c))(f(c)-f(a))=f(ab^2+bc^2+ca^2)-f(a^2b+b^2c+c^2a)$$

for all real numbers $a$, $b$, $c$.