OTIS Problems' Solutions

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Contents

1 A - Geometry

A.1) Let O be the center of this very circle. Now fix two chords and let them pass through a common point P. Let their midpoints be M and N respectively. Now it is clear as a day that $\angle OMP = \angle ONP = 90$. Now repeat this same argument for any other two arbitrary chords (sharing a point in common) and we should get that

$$\angle O[\text{insert line 1's midpoint here}][\text{insert the common point here}] =$$

$$\angle O[\text{insert line 2's midpoint here }][\text{insert the common point here}] = 90.$$

This implies the problem and we are done. [1]

A.2) Let the intersection of the two give circles (other than W) be P. Now we observe that $\angle XHW = \angle XPW = 90$ which implies XHP collinear and similarly, using the fact that P is the miquel point, we establish that YHP are collinear..

A.3) The angle condition implies tangency of BP and CQ with (PSR) and (QSR) respectively and so obviously they intersect somewhere, it is A. Now $AP = AQ$ and so $AP^2 = AQ^2$, which implies A lies on the radax of (PSR) and (QSR). Now assume that PQRS is cyclic, then RS is the radical axis and according to our current progress A must lies on it, but this means A also lies on BC which is simply not possible.

[1]Naming a point (and so a few more) here would make it easier to read, indeed, but it just does not feel like I should name points here.

A.5) Invert the diagram at E. Now, since they were tangent, we get that the circumcircles are all lines; and hence WXYZ is a rectangle. Now since it is a rectangle, inverting back gives us a cyclic quadrilateral.

A.7) Assume that $AB \nparallel CD$, they obviously intersect at some point. Let their intersection be K. Now note that Q_1 and Q_2 are isogonal conjugates of P with respect to the triangles AKD and KBC, which means they lie on the reflection of KP, which in turn implies that they are collinear. Now assume that $Q_1Q_2 \parallel AB$ but $Q_1Q_2 \nparallel CD$ at the same time, then AB and CD, again, intersect at a point, call it K. From the above, we know that K, Q_1, Q_2 are collinear. Which is similarly true when $Q_1Q_2 \parallel CD$, but $Q_1Q_2 \nparallel AB$. So we are done.[2]

A.8) Notice that P,Q and the midpoint of arc BC with A are collinear, because of a very well known theorem. Let the triangle ABC's circumcenter be O'. Extend PQ to meet (ABC) at $M \neq Q$, where M is the midpoint of arc BC. Extend AO to meet (ABC) at $M' \neq A$. Now we notice that $\angle OAQ = \angle M'AQ = \angle O'MQ = \angle M'MQ$, and since $O'Q = O'M, \angle M'MQ = \angle O'MQ = \angle O'QM = \angle O'QM = \angle OQP$. So we are done. APOQ is cyclic and this implies OP=OQ and so the problem.

2 B - Inequalities

B.1) We get $a^4 + b^4 + c^4 \geq (a+b+c)abc$ by simplifying the given and now that in turn is $a^4 + b^4 + c^4 \geq a^2bc + ab^2c + abc^2$. Now consider $2a^4 + b^4 + c^4$. By AM-GM,

$$\sum_{cyc} \frac{2a^4 + b^4 + c^4}{4} \geq a^2bc.$$

Summing it up yields the result.

B.2)We will assume that the given below claim is true that is proved later on but for now, assuming that $(\frac{ab}{c} \frac{bc}{a} \frac{ac}{b})^2 \geq 3(a^2 + b^2 + c^2)$,

$$\frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} \geq \sqrt{3}.$$

This means the minimum is at $\frac{ab}{c} = \frac{bc}{a} = \frac{ac}{b} = \frac{\sqrt{3}}{3}$.

[2]This solution is the same as before (exact same wording) and the previous solution had some of my friend, Jason's, effort involved in the process of drawing of the diagram.

Claim. $(\frac{ab}{c}\frac{bc}{a}\frac{ac}{b})^2 \geq 3(a^2+b^2+c^2)$

Proof.

$$(\frac{ab}{c}\frac{bc}{a}\frac{ac}{b})^2 \geq 0$$

and so

$$(\frac{ab}{c})^2 + (\frac{bc}{a})^2(\frac{ac}{b})^2 + 2(a^2+b^2+c^2) \geq 0.$$

Now

$$\sum_{cyc}(\frac{ab}{c})^2 + (\frac{bc}{a})^2 \geq \sum_{cyc} 2b^2$$

and summing these up implies the result.

And we are done.

B.4)The given condition is basically

$$\sum_{cyc} a^2 + ab \leq 2.$$

And so now

$$2(\sum_{cyc}\frac{ab+1}{(a+b)^2}) \geq \sum_{cyc}\frac{2ab + \sum_{cyc} a^2 + ab}{(a+b)^2}.$$

Which, after rearranging the terms of the RHS is soemthing like

$$2(\sum_{cyc}\frac{ab+1}{(a+b)^2}) \geq \sum_{cyc}\frac{(a+b)^2 + (a+c)(b+c)}{(a+b)^2},$$

which is basically

$$3 + \sum_{cyc}\frac{(a+c)(b+c)}{(a+b)^2}.$$

Now we take the AM-GM of

$$\sum_{cyc}\frac{(a+c)(b+c)}{(a+b)^2} \geq 3\sqrt[3]{\frac{(a+c)(b+c)}{(a+b)^2}\frac{(a+b)(a+c)}{(b+c)^2}\frac{(a+b)(c+b)}{(a+c)^2}} = 3(1)$$

and so we get that, finally,

$$2(\sum_{cyc}\frac{ab+1}{a+b)^2}) \geq \sum_{cyc}\frac{(a+b)^2+(a+c)(b+c)}{(a+b)^2} = 3+\sum_{cyc}\frac{(a+c)(b+c)}{(a+b)^2} \geq 3+3 = 6,$$

which now follows the result.

B.5) Sub in $x = 1/a, y = 1/b, z = 1/c$. Simplifying the given with the sub we get that $\sum_{cyc} \frac{x^2}{y+z} \geq \frac{3}{2}$. But

$$\sum_{cyc} \frac{x^2}{y+z} \geq \frac{(x+y+z)^2}{y+z+x+y+z+x}$$

by titu. Now,

Claim. $x + y + z \geq 3$

Proof. By AM-GM, $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ or $x + y + z \geq 3$.

subbing this back into $\sum_{cyc} \frac{x^2}{y+z} \geq \frac{(x+y+z)^2}{y+z+x+y+z+x}$ concludes the proof. To be more precise, the above is the same as $\sum_{cyc} \frac{x^2}{y+z} \geq \frac{(x+y+z)^2}{2(x+y+z)}$ or $\sum_{cyc} \frac{x^2}{y+z} \geq \frac{(x+y+z)}{2} \geq 3/2$.

3 C-Others

C.1) Let P(x,y) denote the assertion. P(0,y):

$$f(f(y)) = y$$

which is an involution and so we get bijectivity. Now P(0,0) gives f(0)=0. P(x,0):

$$f(xf(x)) = f(x)^2$$

but P(f(x),0):

$$f(xf(x)) = x^2$$

and so we get that

$$f(x)^2 = x^2$$
$$\implies f(x) = \pm x.$$

Now suppose $f(x_0) = x_0 \text{and} f(x_1) = -x_1$ for some $x_0, x_1 \in$ f's domain. Then,

$$P(x_0, x_1) : f(x_0^2 - x_1) = x_0^2 + x_1$$

which is absurd unless one of them is zero.

C.3) We start with a very bold claim that literally kills the problem.
The claim is the following: The rabbit can increase the distance between the hunter and itself from d to $\sqrt{d^2 + 1/2}$ in 4d moves, irrespective of the previous play.
Now, assuming that the claim there is true, without the loss of generality after the first n moves where, let the distance be d ($n > d$). The rabbit can now

increase the distance between itself and the hunter by using whatever the tactic we will read about below from d to $\sqrt{d^2+1/2}$ in exactly 4d moves. The following table is basically what we want to say:

S.No.	Moves(at least)	Distance	moves less than 500d?
1	4d	d	yes
2	$4\sqrt{d^2+1/2}$	$\sqrt{d^2+1/2}$	yes
3	$4\sqrt{d^2+1}$	$\sqrt{d^2+1}$	yes

After the third numbered move, the rabbit will keep on increasing the distance and already $\sqrt{d^2+1/2} < \sqrt{d^2+1}$ and so we only have to prove that $4\sqrt{d^2+1} < 500d$ but that is obvious.

Proof of the claim: Let the initial distance between the rabbit and hunter be d. Call the point rabbit is in R and the point the hunter is in H. Now, create two line segments RX and RY with equal distance, namely n where n is the actual number of moves made for when the distance between them became d; such that XY=2. Now wlog the rabbit hops all the way to X. The hunter's device must follow and as the hunter must follow the rabbit, it must go somewhere near the circle of radius one around the rabbit whose point in the initial line is distance n away from its initial position H. Call it P. So far, we have that HR=d lying in that order on a line segment l and RX=RY=n. We also have that HP=n as the rabbit must have taken n hops. Also, name the intersection of XY with line l M. Now, look at the distance PX. $PX^2 = 1 + PM^2$
$= 1 - (RM - RP)^2 = 1 + (\sqrt{RX^2 - 1} - RP)^2$
$= 1 + (\sqrt{n^2 - 1} - (n-d))^2$
$\geq 1 + ((n - 1/n) - (n-d))^2$
$= 1 + (d - 1/n)^2 \geq d^2 + 1/2$ whenever either d=0 or $n \geq 4d$ which is easy to notice once we solve the quadratic inequality. This solves the problem. The rabbit can repeat this process in every 4d moves where d is the distance.