Solving Isls

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1 Introduction

A small introduction to what this place is and why this exists: This is a file to maintain all of my write-ups of every IMO Shortlist or Longlist problems I ever solve. I will also try to maintain a list of all IMO Shortlist and Longlist problems I have seen and/or solved here. To find where I attempt these problems at, please join the discord server *here*.

2 Algebra

Problem. 2004 A4

Find all functions f from the reals to the reals such that

$$(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

Solution.

- p(0,0,0,0) gives $f(0) = 2f(0)^2$ which gives wither $f(0) = 0$ or $f(0) = 1/2$ I am too lazy to write up for why $f(0) = 1/2$ implies $f(x) = 1/2$ and why $f(0) = 0$ gives $f(x) = 0$ as a solution but assuming the next solution is not constant, we work on $f(0) = 0$.

- Putting $z = t = 0$ gives the $f(xy) = f(x)f(y)$. Now $f(1) = 1 or 0$, say if it's 0 then it is easy to get that it implies that $f(x) = 0$.

- Now given we are done with that let's just take $f(1) = 1$, we get f(z)=f(-z) for x=0, y=t=1.

- Also get that $f(x^2) = f(x)^2$.

- Now using that with $f(x^2) = f(x)^2$, we observe that $f(x) \geq 0$ and I forgot how I finished but I do remember getting $f(x) = x^2$ as the solution. And we are done.

Problem. 2002 A1

Find all functions f from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real x, y.

Solution.

- $f(f(0) + y) = f(f(y))$, after plugging x=0 and y=y.
- f is bijective because of the 2x there.
- Note that the first pointer implies $f(0) + y = f(y)$.
- $f(s + y) = f(f(y))$ that is $y + s = f(y)$.
- Which simply means $f(x) = x + s$ for some constant s.

3 Combinatorics

Problem. 2020 C8

Players A and B play a game on a blackboard that initially contains 2020 copies of the number 1 . In every round, player A erases two numbers x and y from the blackboard, and then player B writes one of the numbers $x + y$ and $|x - y|$ on the blackboard. The game terminates as soon as, at the end of some round, one of the following holds: (1) one of the numbers on the blackboard is larger than the sum of all other numbers; (2) there are only zeros on the blackboard. Player B must then give as many cookies to player A as there are numbers on the blackboard. Player A wants to get as many cookies as possible, whereas player B wants to give as few as possible. Determine the number of cookies that A receives if both players play optimally.

Problem. 2020 C3

There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem. 2019 C4

On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.
At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.
After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.
Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

Problem. 2019 C7

There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game. In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps: (a) Bob chooses an integer k with $1 \leq k \leq 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$. (b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group. Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

Problem. 2017 C5

A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order: The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1. The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

Problem. 2015 C4

Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:
(i) A player cannot choose a number that has been chosen by either player on any previous turn. (ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn. (iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game. The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Problem. 2014 C8

A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared. Determine all possible first moves of the first player after which he has a winning strategy.

Problem. 2013 C8

Players A and B play a "paintful" game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened. Decide whether there exists a strategy for player A to win in a finite number of moves.

Problem. 2012 C4

Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules: (a) On every move of his B passes 1 coin from every box to an adjacent box. (b) On every move of hers A chooses several coins that were not involved in B's previous move and are in different boxes. She passes every coin to an adjacent box. Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Problem. 2012 C6

The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.
At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.
After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:
1. If $n \geq 2^k$, then B can guarantee a win. 2. For all sufficiently large k, there exists an integer $n \geq (1.99)^k$ such that B cannot guarantee a win.

Problem. 2009 C1

Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins. (a) Does the game necessarily end? (b) Does there exist a winning strategy for the starting player?

Problem. 2009 C5

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Problem. 2004 C5

A and B play a game, given an integer N, A writes down 1 first, then every player sees the last number written and if it is n then in his turn he writes $n+1$ or $2n$, but his number cannot be bigger than N. The player who writes N wins. For which values of N does B win?

Problem.

Problem.

Problem.

Problem.

ISLs Solved or Seen list. Seen is marked as S, solved is marked as Y, a problem that I already saw as an example but needs to get revisited is marked with S/P, S/N is I do not want to see it again and solved with enough hints to reconsider is Y/P. I think I better go over all of the problems with Y/P without hints once I am done with all of the basic theory required aka after summer vacation this is the first thing I will do

- 2004 G1 Y
- 2017 G1 Y
- 2017 N1 S
- 1972 P1 Y
- 2000 A3 Y/P
- 1999 A5 S
- 1994 A3 S/R
- 1990 L25 S
- 1992 A6 S
- 1982 P1 S
- 1978 P3 S
- 2001 A1 S/N
- 1996 A8 S
- 1988 P3 S/N
- 1977 P6 S/N
- 2010 G1 S
- 2006 P1 Y/P
- 2013 P4 Y/P
- 1985 P1 Y/P
- 2008 P1 Y/P
- 1995 P1 Y/P
- 2000 P1 Y/P
- 2009 P2 Y/P

- 2000 G3 S/P
- 2006 G3 Y
- 2001 G1 Y/P
- 1994 C1 S
- 2004 C5 S
- 2009 C5 S
- 2009 C1 S
- 1994 C6 Y/P