Polynomials

Bhabanishankar Rath

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> There's something about Algebra, I just can't figure it out. Polynomials, derivatives, quadratic equations, I see no absolute value in them. A bunch of irrational numbers with square roots and exponential functions. I'm still trying to see through the horizontal and vertical blurred lines. This all reminds me of Y I left my X.
>
> — Charmaine J Forde

§1 Introduction

Note. This unit is only an example of what I meant by "producing handouts." In no way is a teacher forced to use Latex like I am using here and nor is he/she expected to cover it in this format only. They have their freedom.

Polynomials are expressions with one or more terms with a non-zero coefficient. A polynomial can have more than one term. An algebraic expression

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_0 x^0.$$

Example 1.1

Some of the examples of polynomials are given below:

- $x^2 + 3x + 9$
- x+y
- $x^4 + 98x + 9$
- 69
- $x^2 + x + 1$

Some terms relating to the polynomials:

- **Term.** Each expression in a polynomial is called a term. To identify the number of terms, the trick I use is this: count how many number of "things" are there before each plus or minus sign or simply add 1 to how many *plus* + *minus* are there.

Example 1.2
$x^3 + 3x^2 + 5x - 8 = 0$.
Using our first trick, there is a x^3 before a plus, a $3x^2$ before the next plus, a $5x$ before the next plus, and finally a 8 before the last minus and therefore, this polynomial has 4 terms.

- **Constant Number vs Variable.** A variable is something that changes its value while a constant number does not.

Example 1.3
4 is a constant number since its value is always 4 but x is not since we can substitute *anything* in the place of x.

- **Coefficient.** This is the constant number before a variable in a polynomial.

Example 1.4
In the polynomial $2x^2 + 3x + 6$, the coefficient of x^2 is 2 since 2 is the constant number before x^2 and similarly 3 for x etc.

- **Like and Unlike terms.** The terms with the same power and the same variable are called like terms and the terms with not the same power and not the same variable are called unlike terms.

Example 1.5
x and $2x$ are like terms but $2x$ and $3z$ are not.

- **Types of Polynomial.** These are divided accordingly to the number of terms. A **monomial** is the one with 1 term; a **binomial** is the one with 2 terms; a **trinomial** is the one with 3 terms.

Example 1.6
$x + 2$ is a binomial;
x is a monomial;
$x + 3x^2 + 98$ is a trinomial.

- **Power/degree of a Polynomial.** The highest power of any given polynomial is called the power/degree of the polynomial.

Example 1.7
The power of $2x^2 + 4x^3 + 3x^7 + 68$ is 7. A polynomial of the degree/power 0 is called a **Constant Polynomial.**

Example 1.8
3 is a constant polynomial since it is equivalent to $3x^0$.

- **Roots or Zeroes of a Polynomial.** Simply put, the zeroes or the roots of a polynomial are those numbers after substituting which in the place of x makes the polynomial 0.

Example 1.9

The zero of $P(x) = 4x + 4$ is -1 since $P(-1) = 4(-1) + 4 = 0$.

The zeros of $P(x) = x^2 - 3x + 2$ are 1 and 2 since $P(1) = 1^2 - 3(1) + 2 = 2^2 - 3(2) + 2 = 0$

§2 Theorems

This section is about some of the most important theorems in the field of polynomials.

§2.1 Remainder Theorem

Theorem 2.1

If p(x) is any polynomial having degree greater than or equal to 1 and if it is divided by the linear polynomial x − a, then the remainder is p(a).

Proof of this theorem is beyond the scope of this handout, however you can contact me if you are interested.

Example 2.2

Find the remainder when $x^3 - 4x^2 - 7x + 10$ is divided by $x - 2$.

We will simply use the remainder theorem. According to that, the answer should be $P(a = 2) = 2^3 - 4(2^2) - 7(2) + 10 = -12$.

Exercise 2.3. Verify that this is indeed the reminder by long polynomial division.

Theorem 2.4

x − c is a factor of the polynomial p(x), if p(c) = 0. Also, conversely, if x − c is a factor of p(x), then p(c) = 0.

This comes straight from the factor theorem. Proving this is one of your homework problems.

Example 2.5

Find the factors of the polynomial $P(x) = x^2 + 2x - 15$.

We simply use the factor theorem. According to that, the numbers at which our polynomial becomes zero should be our solution. We use **middle term splitting** here. This is also an example of middle-term splitting.
Observe that $x^2 + 2x - 15$ is equivalent to

$$x^2 + 5x - 3x - 15$$

which could be written as

$$x(x+5) - x(x+5)$$

or

$$(x+5)(x-3)$$

which means those two are our factors since $P(-5) = 0$ and $P(3) = 0$.

Exercise 2.6. Verify the solution above.

§3 Algebraic Identities

Algebraic Identities are something which are true for all values of the variables used. We will not prove any of them but some of their proofs are homework. We will be using them in the homework problems too.

- $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
- $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$
- $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$
- $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$
- $(x+a)(x+b) = x^2 + (a+b)x + ab$
- $(x+y)^2 = x^2 + y^2 + 2xy$
- $(x-y)^2 = x^2 + y^2 - 2xy$
- $a^2 - b^2 = (a-b)(a+b)$

Example 3.1

Evaluate the following:

- $(399)^2$
- $(0.98)^2$
- Expand $(a-b)^3$
- Find $(2^3 + 3^3 + 5^3 - 90)$

Not complete solutions. 1 is really easy, and so is 2. The third bit is literally copy pasting the formula. The fourth bit is realizing that $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$.

§4 Practice Problems

The Law speaks: you are cast out. You are un-dwarf. I AM A WITNESS! Angarthing in *The Hammer of Thursagan*,
from *The Battle for Wesnoth*

Problem 4.1. 9♣ Factor the polynomial

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3.$$

Problem 4.2. 5♣ Prove all of the identities.

Problem 4.3. 9♣ Let a, b, c be real numbers. Prove that

$$a^3 + b^3 + c^3 = (a+b+c)^3 \quad \text{if and only if} \quad a^5 + b^5 + c^5 = (a+b+c)^5.$$

Problem 4.4. 2♣ Evaluate $(10099)^2$.

Problem 4.5. 3♣ Compute the value of $9x^2 + 4y^2$ if $xy = 6$ and $3x + 2y = 12$.

Problem 4.6. 2♣ Find the value of the polynomial $P(x) = 5x - 4x^2 + 3$ at x = 2 and x = –1.

Problem 4.7. 3♣ Calculate the perimeter of a rectangle whose area is $25x^2 - 35x + 12$.

Problem 4.8. 3♣ Find the value of $x^3 + y^3 + z^3 - 3xyz$ if $x^2 + y^2 + z^2 = 83$ and $x + y + z = 15$

Problem 4.9. 2♣ If $a + b + c = 15$ and $a^2 + b^2 + c^2 = 83$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Problem 4.10. 5♣ If $x - \frac{1}{x} = 4$, then find the values of $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$.