How to use the pumping lemma?

Think of playing a 2 person game

- <u>Role 1:</u> ***You*** claim that the language cannot be regular
- <u>Role 2:</u> An ***adversary*** who claims the language is regular
- You show that the adversary's statement will lead to a contradiction that implyies pumping lemma *cannot* hold for the language.
- You win!!

How to use the pumping lemma? (The Steps)

1. (you) L is not regular.
2. (adv.) Claims that L is regular and gives you a value for N as its P/L constant
3. (you) Using N, choose a string $w \in L$ s.t.,
   1. $|w| \geq N$,
   2. Using w as the template, construct other words w_k of the form xy^kz and show that at least one such $w_k \notin L$

      => this implies you have successfully broken the pumping lemma for the language, and hence that the adversary is wrong.

   (Note: In this process, you may have to try many values of k, starting with k=0, and then 2, 3, .. so on, until $w_k \notin L$)

Note: This N can be anything (need not necessarily be the #states in the DFA. It's the adversary's choice.)

Example of using the Pumping Lemma to prove that a language is not regular

Let L_{eq} = {w | w is a binary string with equal number of 1s and 0s}

- Your Claim: L_{eq} is not regular
- Proof:
  - By contradiction, let L_{eq} be regular → adv.
  - P/L constant should exist → adv.
    - Let N = that P/L constant
  - Consider input $w = 0^N 1^N$ → you
    (your choice for the template string)
  - By pumping lemma, we should be able to break $w=xyz$, such that: →you
    1) $y \neq \varepsilon$
    2) $|xy| \leq N$
    3) For all $k \geq 0$, the string xy^kz is also in L

Template string $w = 0^N1^N = \underbrace{00\ldots}_{N}\underbrace{011\ldots1}_{N}$

Proof…

- Because $|xy| \leq N$, xy should contain only 0s
  - (This and because $y \neq \varepsilon$, implies $y=0^+$)
- Therefore x can contain *at most* N-1 0s
- Also, all the N 1s must be inside z
- By (3), any string of the form $xy^kz \in L_{eq}$ for all $k \geq 0$
- <u>Case k=0:</u> xz has at most N-1 0s but has N 1s
- Therefore, $xy^0z \notin L_{eq}$
- This violates the P/L (a contradiction)

→ you

Setting k=0 is referred to as **<u>"pumping down"</u>**

Setting k>1 is referred to as **<u>"pumping up"</u>**

Another way of proving this will be to show that if the #0s is arbitrarily pumped up (e.g., k=2), then the #0s will become exceed the #1s

Exercise 2

Prove $L = \{0^n 1 0^n \mid n \geq 1\}$ is not regular

Note: This n is not to be confused with the pumping lemma constant N. That *can* be different.

In other words, the above question is same as proving:

- $L = \{0^m 1 0^m \mid m \geq 1\}$ is not regular

Example 3: Pumping Lemma

Claim: L = { 0^i | i is a perfect square} is not regular

- Proof:
  - By contradiction, let L be regular.
  - P/L should apply
  - Let *N* = P/L constant
  - Choose $w=0^{N^2}$
  - By pumping lemma, w=xyz satisfying all three rules
  - By rules (1) & (2), y has between 1 and N 0s
  - By rule (3), any string of the form xy^kz is also in L for all $k \geq 0$
  - Case k=0:
    - #zeros (xy^0z) = #zeros (xyz) - #zeros (y)
    - $N^2 - N \leq$ #zeros (xy^0z) $\leq N^2 - 1$
    - $\mathbf{(N-1)^2} < N^2 - N \leq$ #zeros (xy^0z) $\leq N^2 - 1 < \mathbf{N^2}$
    - $xy^0z \notin L$
    - But the above will complete the proof ONLY IF N>1.
    - … (proof contd.. Next slide)

Example 3: Pumping Lemma

- (proof contd…)
  - If the adversary pick N=1, then $(\mathbf{N-1})^2 \leq N^2 - N$, and therefore the #zeros(xy^0z) could end up being a perfect square!
  - This means that pumping down (i.e., setting k=0) is not giving us the proof!
  - So lets try pumping up next…
- Case k=2:
  - #zeros (xy^2z) = #zeros (xyz) + #zeros (y)
  - $N^2 + 1 \leq$ #zeros (xy^2z) $\leq N^2 + N$
  - $\mathbf{N^2} < N^2 + 1 \leq$ #zeros (xy^2z) $\leq N^2 + N < (\mathbf{N+1})^2$
  - $xy^2z \notin L$
  - (Notice that the above should hold for all possible N values of N>0. Therefore, this completes the proof.)